AN INTRODUCTION

TO THE THEORY OF

INFINITE SERIES

BY

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PREFACE TO THE SECOND EDITION

THE second edition of this book consists largely of a reproduction of the first edition, with additional theorems and examples. The arrangement of the first seven chapters, as well as of Chapter IX., has undergone very little alteration: to the eighth chapter a discussion of the solution of linear differential equations of the second order has been added. Chapter X. of the first edition ("Complex Series and Products") has been broken up into two chapters, X. and XI., the first of these containing the general theory of complex series and products, and the second dealing with special series and functions. The principal new feature of the latter chapter is a discussion of elliptic function formulae.

Chapter XI. of the first edition ("Non-Convergent and Asymptotic Series") now becomes Chapter XII. Here the entire discussion of the theory of summable series, apart from the historical introduction, has been omitted, as Dr. Bromwich felt that an adequate account of the subject with its later developments would require more space than could be given to it in the present volume. The part of the chapter devoted to asymptotic series has been enlarged, and contains, among other new matter, an exposition of the asymptotic expansions of the Bessel functions. Room has also been found for a discussion of trigonometrical series, including Stokes's transformation and Gibbs's phenomenon.

The alterations in Appendix I. are slight, but Appendix II. has been expanded to make room for an account of Napier's invention of logarithms. Appendix III. ("Infinite Integrals and Gamma Functions") was originally written in connection with the discussion of summable series, and might therefore have been omitted. As, however, this Appendix contains much material not otherwise accessible in English text-books, it has been decided to include a verbatim reprint of it in this edition. The set of "Harder Miscellaneous Examples" has also been omitted, but some of these examples will be found in the collections of examples throughout the book.

Dr. Bromwich, unfortunately, has not been able to supervise the passing of the final proofs of the new edition through the press. When these came into my hands the entire book, except Appendix III., was already in type, and my share of the work has been confined to matters of detail. Many errors have been eliminated, and it is hoped that the work has not suffered seriously from the absence, at the final stages, of the guiding hand of the author.

THOMAS M. MACROBERT.

GLASGOW, *October*, 1925.

Note.—The numbering of some of the articles referred to in the Preface to the First Edition has been altered in the Second Edition: Arts. 19, 20, 23, 149, 150, 151, 163 become Arts. 21, 22, 20, 143, 144, 145, 161 and Art. 83 is now replaced by Arts. 86 and 87.

PREFACE TO THE FIRST EDITION

This book is based on courses of lectures on Elementary Analysis given at Queen's College, Galway, during each of the sessions 1902-1907. But additions have naturally been made in preparing the manuscript for press: in particular the whole of Chapter XI. and the greater part of the Appendices have been added. In selecting the subject-matter, I have attempted to include proofs of all theorems stated in Pringsheim's article, *Irrationalzahlen und Konvergenz unendlicher Prozesse*,* with the exception of theorems relating to continued fractions.

In Chapter I. a preliminary account is given of the notions of a limit and of convergence. I have not in this chapter attempted to supply arithmetic proofs of the fundamental theorems concerning the existence of limits, but have allowed their truth to rest on an appeal to the reader's intuition, in the hope that the discussion may thus be made more attractive to beginners. An arithmetic treatment will be found in Appendix I., where Dedekind's definition of irrational numbers is adopted as fundamental; this method leads at once to the monotonic principle of convergence (Art. 149), from which the existence of extreme limits † is deduced (Arts. 5, 150); it is then easy to establish the general principle of convergence (Art. 151).

In the remainder of the book free use is made of the notation and principles of the Differential and Integral Calculus; I have for some time been convinced that beginners should not attempt to study Infinite Series in any detail until after they have mastered the

* *Encyklopädie der Mathematischen Wissenschaften*, Bd. I., A, 3 and G, 3 (pp. 47 and 1121).

† Not only here, but in many other places, the proofs and theorems have been made more concise by a systematic use of these maximum and minimum limits.

differentiation and integration of the simpler functions, and the geometrical meaning of these operations.

The use of the Calculus has enabled me to shorten and simplify the discussion of various theorems (for instance, Arts. 11, 61, 62), and to include other theorems which must have been omitted otherwise (for instance, Arts. 45, 46, and the latter part of 83).

It will be noticed that from Art. 11 onwards, free use is made of the equation

$$\frac{d}{dx}(\log x)=\frac{1}{x},$$

although the limit of $(1+1/\nu)^\nu$ (from which this equation is commonly deduced) is not obtained until Art. 57. To avoid the appearance of reasoning in a circle, I have given in Appendix II. a treatment of the theory of the logarithm of a real number, starting from the equation

$$\log a=\int_1^a \frac{dx}{x}.$$

The use of this definition of a logarithm goes back to Napier, but in modern teaching its advantages have been overlooked until comparatively recently. An arithmetic proof that the integral represents a definite number will be found in Art. 163, although this fact would naturally be treated as axiomatic when the subject is approached for the first time.

In Chapter V. will be found an account of Pringsheim's theory of double series, which has not been easily accessible to English readers hitherto.

The notion of uniform convergence usually presents difficulties to beginners; for this reason it has been explained at some length, and the definition has been illustrated by Osgood's graphical method. The use of Abel's and Dirichlet's names for the tests given in Art. 44 is not strictly historical, but is intended to emphasise the similarity between the tests for uniform convergence and for simple convergence (Arts. 19, 20).

In obtaining the fundamental power-series and products constant reference is made to the principle of uniform convergence, and particularly to Tannery's theorems (Art. 49); the proofs are thus simplified and made more uniform than is otherwise possible. Considerable use is also made of Abel's theorem (Arts. 50, 51, 83) on the continuity of power-series, a theorem which, in spite of its importance, has usually not been adequately discussed in text-books.

Chapter XI. contains a tolerably complete account of the recently developed theories of non-convergent and asymptotic series; the treatment has been confined to the arithmetic side, the applications to function-theory being outside the scope of the book. As might be expected, a systematic examination of the known results has led to some extensions of the theory (see, for instance, Arts. 118-121, 123, and parts of 133).

The investigations of Chapter XI. imply an acquaintance with the convergence of infinite integrals, but when the manuscript was being prepared for printing no English book was available from which the necessary theorems could be quoted.* I was therefore led to write out Appendix III., giving an introduction to the theory of integrals; here special attention is directed to the points of similarity and of difference between this theory and that of series. To emphasise the similarity, the tests of convergence and of uniform convergence (Arts. 169, 171, 172) are called by the same names as in the case of series; and the traditional form of the Second Theorem of Mean Value is replaced by inequalities (Art. 166) which are more obviously connected with Abel's Lemma (Arts. 23, 80). To illustrate the general theory, a short discussion of Dirichlet's integrals and of the Gamma integrals is given; it is hoped that these proofs will be found both simple and rigorous.

The examples (of which there are over 600) include a number of theorems which could not be inserted in the text, and in such cases references are given to sources of further information.

Throughout the book I have made it my aim to keep in view the practical applications of the theorems to every-day work in analysis. I hope that most double-limit problems, which present themselves *naturally*, in connexion with integration of series, differentiation of integrals, and so forth, can be settled without difficulty by using the results given here.

Mr. G. H. Hardy, M.A., Fellow and Lecturer of Trinity College, has given me great help during the preparation of the book; he has

* While my book has been in the press, three books have appeared, each of which contains some account of this theory: Gibson's *Calculus* (ch. xxi., 2nd ed.), Carslaw's *Fourier Series and Integrals* (ch. iv.), and Pierpont's *Theory of Functions of a Real Variable* (chs. xiv., xv.).

read all the proofs, and also the manuscript of Chapter XI. and the Appendices. I am deeply conscious that the value of the book has been much increased by Mr. Hardy's valuable suggestions and by his assistance in the selection and manufacture of examples.

The proofs have also been read by Mr. J. E. Bowen, B.A., Senior Scholar of Queen's College, Galway, 1906-1907; and in part by Mr. J. E. Wright, M.A., Fellow of Trinity College, and Professor at Bryn Mawr College, Pennsylvania. The examples have been verified by Mr. G. N. Watson, B.A., Scholar of Trinity College, who also read the proofs of Chapter XI. and Appendix III. To these three gentlemen my best thanks are due for their careful work.

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CAMBRIDGE, *December*, 1907.

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CHAPTER I.

SEQUENCES AND LIMITS.

1. Infinite sequences: convergence and divergence.

Suppose that we have agreed upon some rule, or rules, by which we can associate a definite number a_n with any assigned positive integer n; then the set of numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots,$$

arranged so as to correspond to the set of positive integers $1, 2, 3, 4, \ldots, n, \ldots,$ will be called an *infinite sequence,* or simply a *sequence.* We shall frequently find it convenient to use the notation (a_n) to represent this sequence. The use of the word *infinite* simply means that *every* term in the sequence is followed by another term.

The rule defining the sequence may be expressed either by some formula (or formulae) giving a_n as an explicit function of n, or by some verbal statement which indicates how each term can be determined, either directly or from the preceding terms.

Ex. 1. If $a_n = 2n - 1$, we have the sequence of odd numbers $1, 3, 5, 7, \ldots$.

Ex. 2. If $a_n = 1/n$, we have the harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.

Ex. 3. The set consisting of the rational positive proper fractions, *arranged in order of magnitude,* is not a sequence. For if a is any fraction of the set, $\frac{1}{2}a$ also belongs to the set; and since $\frac{1}{2}a$ is less than a, we must place $\frac{1}{2}a$ before a. Thus there can be no *first* number of the set; and so this mode of arrangement does not lead to any correspondence between the set and the positive integers. It is, however, possible to arrange these fractions as a sequence, by adopting a different mode of arrangement; for example $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, etc., in which the fractions are arranged, first, according to the magnitude of their denominators, and, secondly, according to the magnitude of their numerators.

The most important sequences in the applications of analysis are those which tend to a limit.

The **limit of a sequence** (a_n) *is said to be* l, *if an index* m *can be found to correspond to every positive number* ϵ, *however small, such that*

$$l-\epsilon < a_n < l+\epsilon,$$

provided only that $n > m$.*

It is generally more convenient to contract these two inequalities into the single one

$$|l-a_n| < \epsilon,$$

where the symbol $|x|$ is used to denote the numerical value of x.

The following notations will be convenient abbreviations for the above property:

$$l = \lim_{n\to\infty} a_n; \quad \text{or } l = \lim a_n; \quad \text{or } a_n \to l;$$

the two latter being only used when there is no doubt as to what variable tends to infinity.

Amongst sequences having no limit it is useful to distinguish those with an infinite limit.

A sequence (a_n) *has* **an infinite limit**, *if, no matter how large the number* N *may be, an index* m *can be found such that*

$$a_n > N,$$

provided only that $n > m = m(N)$.

This property is expressed by the equations

$$\lim_{n\to\infty} a_n = \infty; \quad \text{or } \lim a_n = \infty; \quad \text{or } a_n \to \infty.$$

In like manner, we interpret the equations

$$\lim_{n\to\infty} a_n = -\infty, \quad \lim a_n = -\infty, \quad a_n \to -\infty.$$

In case the sequence (a_n) has a finite limit l, it is called *convergent* and is said to *converge to* l *as a limit*; if the sequence has an infinite limit, it is called *divergent*.†

---

* It is evident that m depends on ϵ, a fact which is often indicated by writing the inequality for n in the form

$$n > m = m(\epsilon).$$

It may happen that a_n involves another variable x, in which case we may write

$$n > m = m(x, \epsilon).$$

† Some writers regard *divergent* as equivalent to *non-convergent*; but it seems convenient to distinguish between sequences which tend to infinity as a limit and those which oscillate. We shall call the latter sequences *oscillatory* (Art. 5·2).

Ex. 1 *bis.* With $a_n = 2n - 1$ (the sequence of odd integers) we have $a_n \to \infty$; a *divergent* sequence. In general m will be the integral part of $\frac{1}{2}(N+1)$.

Ex. 2 *bis.* With $a_n = 1/n$ (the harmonic sequence) we have $a_n \to 0$; a *convergent* sequence. In general m will be the integral part of $1/\epsilon$.

Ex. 3 *bis.* If the sequence consists of the rational proper fractions arranged in any definite order no limit (finite or infinite) can exist. For, no matter how far we go in the sequence, there will always remain an unlimited number of terms as close to 0 as we please; and also an unlimited number as close to 1 as we please.

We shall find it convenient sometimes to represent a sequence graphically, indicating a term a_n by an ordinate (y) equal to a_n and an abscissa (x) equal to n; the sequence may then be pictured by joining the successive points with a broken straight line. In the case of a convergent sequence, the representative points lie wholly within a horizontal strip of width 2ϵ, after x exceeds a certain value; if the sequence is divergent, the points lie wholly above (or below) a certain level, after x has passed a certain value.

The graphical representation of the initial terms in the three sequences already considered is given below.

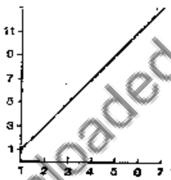

FIG. 1.

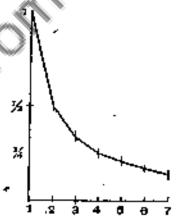

FIG. 2.

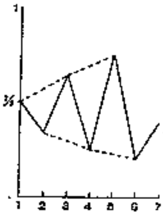

FIG. 3.

It will be seen at a glance that the few terms represented in the diagram shew that the first sequence is likely to diverge, the second to converge, and the third to oscillate (see Art. 5·2).

1·1. Notations for limits.

The arrow symbol $\to$ has been adopted in many recent books and papers as a convenient abbreviation for *tends to* or *approaches* (some limiting value); it was originally introduced for this purpose by the late Dr. J. G. Leathem.* It will be convenient to use the

* No. 1 of the Cambridge *Mathematical and Physical Tracts* (1st edition, 1905).

same symbol in a somewhat extended form, following a later suggestion of Dr. Leathem's. Suppose, for instance, that the sequences (a_n), (b_n), although not necessarily convergent, have the property that (a_n-b_n) tends to a definite limit l; this property is usually indicated by

$$a_n-b_n \to l,$$

but it will occasionally be more convenient to write this property in the form

$$a_n \to b_n+l, \qquad (\alpha)$$

particularly where we have to deal with a succession of such relations.

It is sometimes not possible to give quite such a precise statement as is contained in equation (α). Thus we may only know that a_n/b_n tends to a definite limit l'; this property can often be written more compactly in the form *

$$a_n \sim l'b_n, \qquad (\beta)$$

when a_n, b_n are somewhat complicated functions.

In cases when we know even less than is implied by (β), we may still be able to shew that, for n greater than some value n_0, $|a_n|$ is less than Kb_n, where (b_n) is a positive sequence, and K is independent of n; then we *may write*

$$a_n=O(b_n), \qquad (\gamma)$$

Or again, we may find that a_n/b_n tends to zero; and then a_n *is said to be of lower order than* b_n, *or in symbols,*

$$a_n=o(b_n). \qquad (\delta)$$

The use of the symbols O, o was suggested by Landau, and has proved of great use in modern investigations on the analytical side of prime-number theory.†

Ex. As an illustration, suppose that

$$a_n=\tfrac{1}{2}(n^2+2n+3)/(n-2).$$

Then we can write

$$a_n \to \tfrac{1}{2}n+2, \qquad (\alpha)$$

or

$$a_n \sim \tfrac{1}{2}n, \qquad (\beta)$$

or

$$a_n=O(n), \qquad (\gamma)$$

or

$$a_n=o(n^2), \qquad (\delta)$$

each line giving less precise information than the preceding.

* The use of the symbol $\sim$ to denote *difference* is nearly obsolete; and by means of the symbol $|a-b|$ to denote the numerical value of $a-b$, we can dispense with this older use of $\sim$. The modern use as defined in (β) is substantially due to du Bois Reymond.

† For a fuller account the reader may consult Prof. G. H. Hardy's tract, "Orders of Infinity," No. 12 of the *Cambridge Mathematical and Physical Tracts.*

Naturally it usually is easier to establish a result such as (β) rather than (α); and similarly (γ) than (β); or (δ) than (γ).

1·2. Notes on the definitions.

(1) The definition of a limit is often loosely stated as follows: *The sequence* (a_n) *approaches the limit* l, *if, by taking* n *large enough, we can make* $|l-a_n|$ *as small as we please.*

Such a definition does not exclude the possibility of *oscillation*, as may be seen from the sequence

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{7}, \tfrac{6}{7}, \ldots,$$

in which $a_n=2/(n+4)$ if n is even.

Here, by taking n large enough, we can find a term a_n which is as small as we please; but the sequence oscillates between 0 and 1, because $a_n=(n+1)/(n+3)$ when n is *odd*.

(2) *Infinity.**

It is to be remembered that the symbol ∞ and the terms *infinite, infinity, infinitely great,* etc., have purely conventional meanings in the present theory; in fact, anticipating the definitions of Art. 4, we may say that *infinity must be regarded as an upper limit which cannot be attained.* The statement that a set contains an infinite number of objects may be understood as implying that **no number** suffices to count the set.

Similarly, an equation such as $\lim a_n=\infty$ is merely a conventional abbreviation for the definition on p. 2.

In speaking of a divergent sequence (a_n), some writers use phrases such as: *The numbers* a_n **become** *infinitely great, when* n *increases without limit.* Of course this phrase is used as an equivalent for **tend** *to infinity*; but we shall avoid this practice in the sequel.

(3) It is evident that *the alteration of a* **finite** *number of terms in a sequence will not alter the limit.*

Ex. 1. The two sequences

$$1, 2, 3, 4, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \tfrac{1}{8}, \ldots$$
$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \tfrac{1}{8}, \ldots$$

have the same limit zero.

Further, it is evident that *the omission of* **any** *number of terms from a convergent or divergent sequence does not affect the limit; but such omission may change the character of an oscillatory sequence.*

* For a fuller account the reader should consult Art. 143 of Appendix I.

Ex. 2. Thus the sequences 1, 5, 9, 13, ... and 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, ... have the same limits as those considered in Exs. 1, 2 of Art. 1. But the omission of the alternate terms in the sequence

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{4}{5}, \ldots$$

changes it from an oscillatory to a convergent sequence.

(4) In a convergent sequence, *all*, *an infinity*, *a finite number* or *none* of the terms may be equal to the limit.

Examples of these four possibilities (in order) are given by:

$$a_n = 1; \quad a_n = \frac{1}{n}\sin(\tfrac{1}{2}n\pi); \quad a_n = \frac{1}{n} - \frac{3}{n^2} + \frac{2}{n^3}; \quad a_n = \frac{1}{n};$$

the limits of which are, in order, 1, 0, 0, 0.

(5) We shall usually employ ϵ to denote an arbitrarily small positive number; strictly speaking, the words *arbitrarily small*, or *no matter how small*, or *however small* (which are frequently added to ϵ) are redundant, but serve to emphasise the statement that the variable is *less* than ϵ.

We shall also use N (or G) to denote an arbitrarily great positive number; here again the adjectives *great* or *large* are unnecessary, but are usually added to emphasise the statement that the variable is *greater* than N.

By using ϵ to denote any positive number, we could dispense with N; but it avoids confusion to use two distinct symbols. However it is sometimes convenient to use $1/\epsilon$ for N.

(6) It is often convenient to shew that a sequence (a_n) tends to the limit l, by reasoning on the following lines:

Suppose that we can prove that

$$|a_n - l| < f(n, m), \quad \text{when } n > m,$$

and suppose further that, as $n \to \infty$, the function $f(n, m)$ tends to the limit $F(m)$, which can be made less than ϵ, by a suitable choice of $m = m(\epsilon)$.

Thus we can determine $n_0 > m$, where $n_0 = n_0(m, \epsilon) = n_0(\epsilon)$, and such that

$$|f(n, m) - F(m)| < \epsilon, \quad \text{if } n > n_0.$$

Hence

$$f(n, m) < 2\epsilon, \quad \text{if } n > n_0,$$

and so

$$|a_n - l| < 2\epsilon, \quad \text{if } n > n_0;$$

from which it is clear that $a_n \to l$.

A somewhat similar process can be applied if we know that (a_n) converges to some limit l, but we do not know the value of l; then, if we can shew that $|a_n - c| < \epsilon$, for all values of n greater than $m = m(\epsilon)$, *an index which does not occur in* l *or in* c, we shall have $l = c$.

For
$$l - c = \lim (a_n - c),$$
and so
$$|l - c| \leqq \epsilon, \quad \text{if } n > m(\epsilon);$$
now ϵ is arbitrarily small, and n is not present in l or c; so that the last inequality can hold only if $l = c$.

[Compare Exs. 3, 4 of Art. 2 below.]

(7) It should be observed that if (a_n), (b_n) are convergent sequences such that $a_n < b_n$, it may easily happen that
$$\lim a_n = \lim b_n.$$

For the difference $b_n - a_n$, although constantly positive, may converge to 0 as a limit. Thus the correct conclusion from the inequality $a_n < b_n$ is
$$\lim a_n \leqq \lim b_n.$$

2. Monotonic sequences; and conditions for their convergence.

A sequence in which $a_{n+1} \geqq a_n$ for all values of n is called an *increasing sequence*; and similarly, if $a_{n+1} \leqq a_n$ for all values of n, the sequence is called *decreasing*. Both increasing and decreasing sequences are included in the term *monotonic sequences*.

The first general theorem on convergence may now be stated:

A monotonic sequence has always a limit, either finite or infinite; the sequence is convergent provided that $|a_n|$ *is less than a number* A *independent of* n; *otherwise the sequence diverges.*

For the sake of definiteness, suppose that $a_{n+1} \geqq a_n$, and that a_n is constantly less than the fixed number A. Then, however small the positive number ϵ may be, it will be possible to find an index m such that $a_n < a_m + \epsilon$, if $n > m$; for, if not, it would be possible to select an *unlimited* sequence of indices $p, q, r, s, \ldots$, such that $a_p > a_1 + \epsilon$, $a_q > a_p + \epsilon$, $a_r > a_q + \epsilon$, $a_s > a_r + \epsilon$, etc.; and consequently, after going far enough* in the sequence $p, q, r, s, \ldots$, we should arrive at an index v such that $a_v > A$, contrary to hypothesis.

* The number of terms to be taken in the sequence $p, q, r, s, \ldots$ would be equal to the integer next greater than $(A - a_1)/\epsilon$.

Thus, if we employ the graphical representation described in the last article, we see that all the points to the right of the line $x = m$ will be within a strip of breadth ϵ; and that the breadth of the strip can be made as small as we please by going far enough to the right. From the graphical representation it appears intuitively obvious that the sequence approaches some limit, which cannot exceed A (but may be equal to this value). But inasmuch as intuition has occasionally led to serious blunders in mathematical reasoning, it is desirable to give a proof depending entirely on arithmetical grounds; such a proof will be found in Appendix I. Art. 143.

Ex. 1. As an example consider the increasing sequence

$$\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \ldots,$$

which is represented by the diagram below.

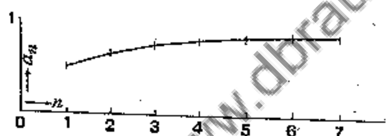

FIG. 4.

In this case we may take $A = 1$, and there is no difficulty in seeing that the limit of the sequence is equal to A; but of course we might have taken $A = 2$, in which case the limit would be less than A.

Ex. 2. A second example is given by the sequence $(1 + 1/n)^n$, which has for its first six terms the approximate values 2, 2·25, 2·37, 2·44, 2·49, 2·52.

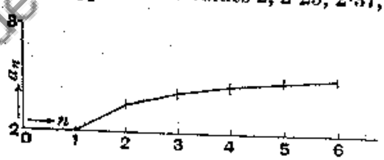

FIG. 5.

The reader is probably aware that this sequence always increases, but that its terms are always less than 3; the limit obtained is the number $e = 2{\cdot}71828\ldots$. A formal proof of the monotonic property is given in Appendix II. Art. 155; and the limit is evaluated in Art. 57.

Similarly $(1 - 1/n)^{-n}$ steadily decreases; starting with $n = 2$, the first six terms in this sequence are approximately 4, 3·38, 3·16, 3·05, 2·99, 2·94. The limit of this sequence is also equal to e.

But in case no number such as A can be found, so that, however great A may be, there is always an index m, such that

$a_m > A$, then it is plain that the sequence diverges to $+\infty$. For we have $a_n \geqq a_m > A$, if $n \geqq m$.

The reader will have no difficulty in modifying the foregoing work so as to apply to the case of a sequence which never increases, so that $a_{n+1} \leqq a_n$.

Ex. 3. Consider the sequence for which $a_n = r^n$.

If $0 < r < 1$, the sequence (a_n) steadily *decreases* but the terms are always positive; and consequently a_n approaches a definite limit l, such that $1 > l \geqq 0$. Thus we can find m to correspond to ϵ, so that

$$l < r^n < l + \epsilon, \qquad \text{if } n \geqq m = m(\epsilon).$$

Hence $$r^{n+1} < r(l+\epsilon);$$
and consequently $$l < r^{n+1} < r(l+\epsilon)$$
or $$l(1-r) < r\epsilon.$$

Since this inequality is true, however small ϵ may be, the limit l must be zero.

When $r > 1$, it follows from the last result that $1/r^n \to 0$, and hence we can determine m so that $1/r^n < \epsilon$, if $n > m = m(\epsilon)$.

Thus we find $$r^n > 1/\epsilon, \qquad \text{if } n > m,$$
and consequently $r^n \to \infty$. This result can also be established from the monotonic property of the sequence; or by direct reasoning, as in Ex. 1, Art. 6.

If r is *negative*, we have $$r^n = (-1)^n \cdot |r|^n$$
and so the behaviour of the sequence can be determined from the results already obtained.

Summing up, we conclude that:

If $-1 < r < 1$, $r^n \to 0$; if $r = 1$, $r^n = 1$; and if $r > 1$, $r^n \to \infty$.

In all other cases the sequence *oscillates*, and:

If $r < -1$, $r^{2n} \to \infty$, $r^{2n+1} \to -\infty$; if $r = -1$, $r^{2n} = 1$, $r^{2n+1} = -1$.

Ex. 4. Take next $$a_n = r^n/n!.$$

If r is positive, the sequence (a_n) decreases steadily, after n exceeds the value r; and since a_n is positive it follows that $a_n \to l \geqq 0$.

Now $$\frac{a_{2n}}{a_n} = \frac{r}{n+1} \cdot \frac{r}{n+2} \cdots \frac{r}{2n} < \frac{r}{2n} < \frac{1}{2}, \qquad \text{if } n > r.$$

Thus we can find m so that

$$\left.\begin{aligned} l < a_n < l + \epsilon \\ \text{and} \quad a_{2n} < \tfrac{1}{2} a_n \end{aligned}\right\}, \qquad \text{if } n > m = m(\epsilon).$$

Hence, proceeding as in Ex. 3, we obtain

$$l < \tfrac{1}{2}(l+\epsilon) \quad \text{or} \quad l < \epsilon.$$

Again it follows that the limit l must be zero.

When r is negative, we obtain the same result by writing

$$a_n = (-1)^n \cdot |r^n|/n!.$$

Thus for all values of r we have

$$\lim \frac{r^n}{n!} = 0.$$

3. General principle of convergence.

If a sequence is not monotonic, the condition that $|a_n|$ remains constantly less than a fixed number is by no means sufficient to ensure convergence; this may be seen at once from the sequence of rational proper fractions given in Example 3, Art. 1, for which $0 < a_n < 1$.

The necessary and sufficient condition for convergence is that it may be possible to find an index $m = m(\epsilon)$, corresponding to any positive number ϵ, such that

$$|a_n - a_m| < \epsilon$$

for all values of n greater than m.

Interpreted graphically, this implies that all points of the sequence which are to the right of $x = m$, lie within a strip of breadth 2ϵ. The statement is then almost intuitive, since the breadth of the strip can be made as small as we please, by going far enough to the right; an arithmetical proof will be found in Appendix I. Art. 143.

Ex. Consider the sequence

$$\tfrac{1}{2},\ 2,\ \tfrac{2}{3},\ \tfrac{3}{2},\ \tfrac{3}{4},\ \tfrac{4}{3},\ \tfrac{4}{5},\ \tfrac{5}{4},\ \tfrac{5}{6},\ \tfrac{6}{5},\ \ldots,$$

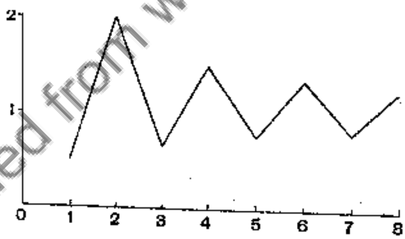

FIG. 6.

for which it is easily seen that the limit is 1. The diagram is as indicated; and it will be seen that m may be taken greater than or equal to $2/\epsilon$.

CAUTION. The reader must be warned not to regard the above test of convergence as equivalent to a condition sometimes given (even in books which are generally accurate), namely:

The necessary and sufficient condition for convergence is that

$$\lim_{n \to \infty} (a_{n+p} - a_n) = 0.$$

This condition is certainly necessary, but is NOT sufficient, *unless p is supposed to be an arbitrary function of n, which may tend towards infinity with n, in an arbitrary way.*

For example, suppose that $a_n = \log n$; then

$$\lim_{n\to\infty}(a_{n+p} - a_n) = \lim_{n\to\infty}\log(1 + p/n) = 0,$$

if p is any fixed number. But the sequence (a_n) is divergent, as may be seen from Appendix II. Art. 154.

The reader will have no difficulty in proving that *the elementary rules for calculating with limits are as follows*:

$$\lim(a_n \pm b_n) = \lim a_n \pm \lim b_n,$$

$$\lim(a_n \times b_n) = \lim a_n \times \lim b_n,$$

provided that the sequences (a_n), (b_n) are convergent.

$$\lim(a_n/b_n) = \lim a_n / \lim b_n,$$

provided that (a_n), (b_n) are convergent and that $\lim b_n$ is not zero. And generally that

$$\lim f(a_n, b_n, c_n, \ldots) = f(\lim a_n, \lim b_n, \lim c_n, \ldots),$$

where f denotes any combination of the four elementary operations, subject to conditions similar to those already specified.

If the functional symbol f contains other operations (such as extraction of roots), the equation above is sometimes a *theorem* (as for example when we assert that $\lim \sqrt{a_n} = \sqrt{l}$, if $\lim a_n = l$ and l is positive); but it may also be a *definition* of the right-hand side (as in the theory of irrational powers and indices, when developed from certain points of view). Thus $c^{\sqrt{2}}$, where c is positive, may be defined* as $\lim c^{a_n}$, when a_n tends to $\sqrt{2}$ through an appropriate sequence of rational indices (such as, $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$).

It is to be remembered that the limits on the left may be perfectly definite without implying the existence of $\lim a_n$ and $\lim b_n$. To illustrate this possibility, take $a_n = (-1)^n$, $b_n = (-1)^{n-1}(1 + 1/n)$.

Then $a_n + b_n = (-1)^{n-1}/n$ and $(a_n + b_n) \to 0$,

$a_n b_n = -(1 + 1/n)$ and $(a_n b_n) \to -1$,

$a_n / b_n = -n/(n+1)$ and $(a_n/b_n) \to -1$,

so that these three limits are quite definite, in spite of the non-existence of $\lim a_n$ and $\lim b_n$.

If (a_n) is convergent and $b_n \to 0$, *we cannot infer that* $a_n/b_n \to \infty$ without first proving that a_n/b_n has a *fixed sign*.

If $a_n \to 0$, and $b_n \to 0$, the quotient a_n/b_n may or may not have a limit (see Appendix I. Arts. 147-149).

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* For more details, the reader may consult Appendix I. Art. 142.

Thus with $a_n = 1$, $b_n = (-1)^n/n$, we see that $b_n \to 0$, but $a_n/b_n = (-1)^n n$ and so a_n/b_n *oscillates* between $-\infty$ and $+\infty$.

Again, with $a_n = 1/n$, $b_n = (-1)^n/n$, the value of a_n/b_n oscillates between -1 and $+1$.

When one of the sequences diverges (say $a_n \to \infty$) and the other converges (say to a *positive* limit) it is easy to see that

$$(a_n \pm b_n) \to \infty; \quad a_n b_n \to \infty; \quad a_n/b_n \to \infty; \quad b_n/a_n \to 0;$$

the only case of exception arising when $b_n \to 0$, and then the sequences $(a_n b_n)$ and (a_n/b_n) need special discussion.

Again, if both a_n and b_n diverge to ∞, we have

$$(a_n + b_n) \to \infty; \quad a_n b_n \to \infty;$$

but both $(a_n - b_n)$ and (a_n/b_n) have to be examined specially (see Appendix I. Art. 147).

If $a_n \to \infty$ and $b_n \to \infty$, there are three distinct alternatives with respect to the sequence (a_n/b_n), *assuming that it is convergent* *

$$\text{(i) } a_n/b_n \to 0; \quad \text{(ii) } a_n/b_n \to k > 0; \quad \text{(iii) } a_n/b_n \to \infty.$$

In case (i), a_n *diverges more slowly* than b_n, and $a_n = o(b_n)$; in case (iii) a_n *diverges more rapidly* than b_n. In case (ii) it is often convenient to use the notation (Art. 1·1)

$$a_n \sim kb_n,$$

when a_n, b_n are complicated expressions.

Rules are given in Appendix I. (Arts. 146-148) for the determination of $\lim (a_n/b_n)$ in a number of cases which are important in practical work.

4. Solution of numerical equations by means of sequences.† It is often possible to calculate a root of an equation of the type ‡

$$x = f(x),$$

by constructing a sequence (a_n) defined by

$$a_{n+1} = f(a_n).$$

If the sequence converges to a limit α, then $x = \alpha$ is a root of the original equation.

---

* Even when a_n and b_n are both monotonic, the sequence a_n/b_n need not converge (Appendix I., Art. 147, Ex. 4).

† An interesting example of this method has been given by Prof. W. B. Morton (*Phil. Mag.* (6), vol. 37, 1919, pp. 282, 283).

‡ Here $f(x)$ is any fairly simple function; in most practical applications it is usually possible to take the values of $f(x)$ directly from tables.

The construction of the sequence may be illustrated by drawing a succession of zig-zags between the line $y=x$ and the graph of $y=f(x)$. Two examples are sketched below; and the reader is advised to construct other typical figures for himself.

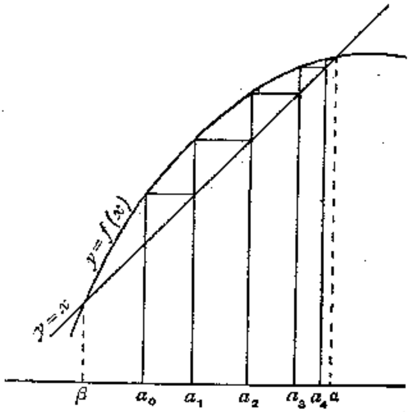

FIG. 6 (a).

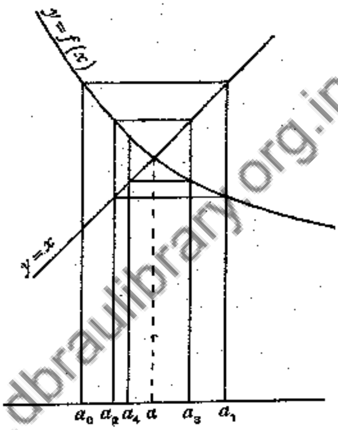

FIG. 6 (b)

It will be seen that in Fig. 6 (a), the sequence (a_n) converges towards the *larger* of the two roots indicated in the diagram; and this conclusion follows however close a_0 may be to the other root β. Of course if a_0 were to coincide *exactly* with β, we should find

$$a_1=f(a_0)=f(\beta)=\beta,$$

and so all the terms of the sequence would coincide with β.

But in working with tables, *exact* coincidence is usually impossible; and the smallest difference between a_0 and β will lead to the limit α for (a_n).

The last remark is illustrated by an example constructed by Lord Kelvin,* in connexion with Laplace's Theory of the Tides.

Taking $f(x)=6-1/x$, the values of α, β are the roots of $x^2-6x+1=0$; and so

$$\alpha=3+\sqrt{8}=5{\cdot}828427.$$
$$\beta=3-\sqrt{8}=0{\cdot}171573.$$

Taking $a_0=0{\cdot}1716$ (agreeing with β to four significant figures) it is calculated that

$$a_7=5{\cdot}8284=a_8=\alpha,$$

retaining four figures throughout the calculations.

* *Mathematical and Physical Papers*, vol. 4, pp. 244, 245.

In order that this method should lead to an effective solution, *it is generally necessary that $f'(x)$ should be numerically less than a positive constant k (less than unity) for all values of x within a range which includes α, the root to be calculated.*

The reader will find it easy to convince himself of the necessity for this condition, by drawing various typical diagrams; and then attempting to carry out the process as indicated in Figs. 6 (*a*), 6 (*b*).

To prove that the condition is sufficient we note that if a_n and α both belong to the range indicated, then, by the Mean-Value theorem,

$$|f(a_n) - f(\alpha)| < k\,|a_n - \alpha|.$$

Hence $$|a_{n+1} - \alpha| < k\,|a_n - \alpha|,$$

and so in general $|a_n - \alpha| < k^n |a_0 - \alpha|$, which tends to zero as n tends to infinity.

It follows also from the Mean-Value theorem that there is only *one* root of $x = f(x)$ within the range specified. For if there were two such roots α, β, we should have

$$|f(\beta) - f(\alpha)| < k\,|\beta - \alpha|,$$

which contradicts the hypotheses

$$f(\beta) = \beta \quad \text{and} \quad f(\alpha) = \alpha.$$

It is necessary, however, that a_0 should not be too far away from α; in fact, the above argument will only apply when a_0 falls within the range for which $|f'(x)| < k$.

On the other hand, it may happen that $f'(x)$ is numerically greater than N (greater than unity) for all values of x within a range which includes a root β of $x = f(x)$. We can then proceed similarly to find β, *by reversing the order of the sequence and taking the equation*

$$f(b_{n+1}) = b_n.$$

The numerical determination of b_{n+1} from b_n offers no difficulty when $f(x)$ is given by one of the ordinary mathematical tables.*

In the example quoted above from Lord Kelvin's *Papers*, we can obtain β by writing

$$6 - \frac{1}{b_{n+1}} = b_n \quad \text{or} \quad b_{n+1} = \frac{1}{6 - b_n}.$$

For instance, taking $$b_0 = 5{\cdot}8284,$$

so that b_0 agrees with α to four decimal places, it appears that

$$b_7 = 0{\cdot}1716 = b_8,$$

giving β to four decimal places.

To illustrate the application of the method, we may take the equation

$$x = 12 \log_{10} x,$$

* It might be troublesome in an example such as $f(x) = x^2 + 2/x$; but it would be easy to deal with, say, $f(x) = \log_{10} x$.

which leads to a figure of the type sketched in Fig. 6 (*a*). It is easy to see that the root α may be expected to lie between $x=13$, $x=14$: then, taking $a_0=13{\cdot}5$, and using the sequence

$$a_{n+1}=12\log_{10}a_n,$$

it will be found that $$a_3=13{\cdot}60=a_4,$$

which is accordingly the value of α to four significant figures. To obtain β, we use the sequence reversed, writing

$$\log_{10}b_{n+1}=\tfrac{1}{12}b_n,$$

and with $b_0=1{\cdot}4$, it appears that

$$b_4=1{\cdot}278=b_5,$$

which is therefore the root β to four figures.

It does not follow that convergence can be assured for *any* initial values of these sequences. For example, if a_0 is less than β, say $a_0=1{\cdot}2$, it will be found that a_2 is negative and the sequence breaks down. Again, if b_0 is greater than α, say $b_0=14$, it will be found that $b_4>100$, $b_5>10^8$; and so on.

It is therefore necessary as a rule to obtain first approximations to the roots by graphical or other rough methods, so as to avoid the risk of obtaining a divergent sequence owing to a faulty choice of the initial value.

It should also be noticed that in some cases the sequence given by

$$a_{n+1}=f(a_n)$$

leads to *two* values u, v, which are such that

$$v=f(u),\quad u=f(v).$$

An example with $f(x)=k^x$ is given in Ex. 11 (iv) at the end of this chapter. The reader will find it instructive to discuss this case by means of the curve $y=k^x$; although the method indicated on p. 13 leads more directly to the correct results.

5·1. Upper and lower limits of a sequence.

If a sequence (a_n) has a *greatest term** H, this term is called the *upper limit* of the sequence; and similarly, when there is a least term h, it is called the *lower limit*.

But if a sequence has no greatest term, it follows that no matter how large n may be, there is always a larger index p, such that $a_p \geqq a_n$. Further, there is an *infinite* number of such indices p; otherwise there would be a greatest term in the sequence; thus

* A sequence is said to have a *greatest term* H, when there is a term H which is greater than all but a finite number of the terms which are equal to H.

to make p definite we suppose p to be the *least* index satisfying the required condition. Hence the terms of the sequence which fall between a_n and a_p are all less than a_n and a_p.

Choose now a succession of values of p such that

$$a_{p_1} \geqq a_1, \quad a_{p_2} \geqq a_{p_1}, \quad a_{p_3} \geqq a_{p_2}, \quad \text{etc.}$$
$$p_1 > 1, \quad p_2 > p_1, \quad p_3 > p_2,$$

and for simplicity denote a_{p_r} by b_r. Then we have constructed an increasing sequence $b_1, b_2, b_3, \ldots$; and this sequence has a limit (Art. 2), either a finite number H or ∞. If $\lim b_n = H$, we can find m so that b_m lies between $H-\epsilon$ and H, no matter how small ϵ may be; and consequently we have

$$H-\epsilon < a_{p_r} < H, \quad \text{provided that } r \geqq m.$$

H is then called the upper limit of the sequence; and clearly H is greater than any number belonging to the sequence.

Similarly, if $\lim b_n = \infty$, we can find m so that

$$a_{p_r} > N, \quad \text{provided that } r \geqq m,$$

no matter how large N may be.

If the upper limit of the sequence is H, whether attained or not, the sequence has the two following properties:

(i) *No term of the sequence is greater than H.*

(ii) *At least one term of the sequence is greater than $H-\epsilon$, however small ϵ may be.**

But if the upper limit of the sequence is ∞, the sequence has the property:

An infinity of terms of the sequence exceed N, no matter how great N may be.

It is easy to modify these definitions and results so as to refer to the *lower limit* (h or $-\infty$).

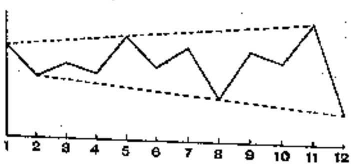

FIG. 7.

The diagram gives an indication of the mode of selecting the sub-sequences for H and h; these are represented by dotted lines.

* If H is not attained, there will be an infinite number of such terms.

Ex. 1. (Art. 1) $a_n = 2n - 1.$

Here we have $b_n = a_n$, and so the upper limit is ∞; $h = 1$, because 1 is the least number in the sequence.

Ex. 2. (Art. 1) $a_n = 1/n.$

Here $H = 1$, because this is the greatest number in the sequence; and h is seen to be 0, which is not actually attained by any number of the sequence.

Ex. 3. (Art. 1) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots.$

Here the sequence (b_n) is $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ and gives $H = 1$; and similarly h is found from the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ to be 0. These sequences are indicated in Fig. 3 of Art. 1 by the dotted lines.

5·2. Maximum and minimum limiting values of a sequence.

We have just seen in Art. 5·1, that any infinite sequence has an upper and a lower limit. Consider successively the sequences

$$a_1, a_2, a_3, a_4, a_5, \ldots,$$
$$a_2, a_3, a_4, a_5, \ldots,$$
$$a_3, a_4, a_5, \ldots,$$
$$a_4, a_5, \ldots,$$

and so on.

Let the corresponding upper and lower limits be denoted by H_1, h_1; H_2, h_2; H_3, h_3; H_4, h_4; and so on.

Then we may have $H_1 = a_1$, in which case a_1 must be the greatest term in the sequence (a_n), and so $H_1 \geqq H_2$; otherwise we shall have $H_1 = H_2$. Hence in all cases $H_1 \geqq H_2$. Thus

$$H_1 \geqq H_2 \geqq H_3 \geqq H_4 \geqq \ldots,$$

and so the sequence (H_n) is decreasing and tends to a limit G or $-\infty$ (Art. 2). Similarly $h_1 \leqq h_2 \leqq h_3 \leqq h_4 \leqq \ldots$, and therefore (h_n) has a limit g or $+\infty$. It may be noticed that G can only be $+\infty$, in case

$$H_1 = H_2 = H_3 = \ldots = +\infty;$$

and g can only be $-\infty$, if $h_1 = h_2 = h_3 = \ldots = -\infty$.

It is important to notice that *G, g can be obtained as the limits of two sub-sequences properly selected from* (a_n). For, either $H_1, H_2, H_3, \ldots$ *all* belong to the sequence (a_n), in which case the sub-sequence for G is coincident with (H_n); or else, after a certain stage, we have $H_m = H_{m+1} = H_{m+2} = \ldots = G$, and then H_m is itself the limit of a certain sub-sequence selected from (a_n), so that this same sub-sequence defines G. An exactly similar argument applies to g.

Again *no convergent sub-sequence selected from* (a_n) *can have a limit which is greater than* G *or less than* g. For since $\lim H_n = G$, we can find m so that $H_m \leqq G+\epsilon$, no matter how small ϵ is; but, by the definition of H_m, we have

$$a_n \leqq H_m, \quad \text{if } n \geqq m,$$

so that

$$a_n \leqq G+\epsilon, \quad \text{if } n \geqq m.$$

Thus, if l is the limit of any convergent sub-sequence selected from (a_n), we must have $l \leqq G+\epsilon$; and, as ϵ is arbitrarily small, this leads to $l \leqq G$. In like manner we prove that m' can be found to make $a_n \geqq g-\epsilon$, if $n \geqq m'$, and we deduce that $l \geqq g$.

The two properties just established justify us in calling G *the maximum limit and* g *the minimum limit of the sequence* (a_n); in symbols we write

$$G = \overline{\lim_{n\to\infty}}\, a_n = \overline{\lim}\, a_n, \quad g = \underline{\lim_{n\to\infty}}\, a_n = \underline{\lim}\, a_n.$$

The symbol $\overline{\underline{\lim}}\, a_n$ is used to denote either the maximum or minimum limit; thus the inequality $f < \overline{\underline{\lim}}\, a_n < F$ implies that

$$f < g \quad \text{and} \quad G < F.$$

If it happens that $G = \infty$, we have $H_n = \infty$, and consequently there must be an infinity of terms a_n greater than any assigned number N, however great; similarly when $g = -\infty$, there must be an infinity of terms less than $-N$. On the other hand, if $\lim H_n = -\infty$, it is easy to see that $\lim a_n = -\infty$; and similarly if $\lim h_n = +\infty$ we must have $\lim a_n = +\infty$.

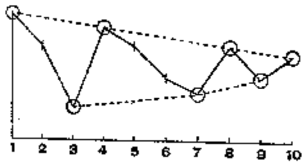

FIG. 8.

From what has been explained it is clear that *every sequence has a maximum and a minimum limit; and these limits are equal, if, and only if, the sequence converges.*

It is convenient to call sequences *oscillatory* when the maximum and minimum limits are unequal. We shall call these limits the *extreme limits* of the sequence, in case we wish to refer to both maximum and minimum limits.

It will be evident that *the maximum limit coincides with the upper limit, except when the latter is actually attained* by one or more terms of the sequence; and similarly for the minimum and lower limits.

The diagram gives an indication of the process defining G and g. The points H_n and h_n are marked with $\odot$ and are joined by dotted lines.

Ex. 1. In the sequence (Ex. 3, Art. 1)

$$\tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{1}{4},\ \tfrac{3}{4},\ \tfrac{1}{5},\ \tfrac{2}{5},\ \tfrac{3}{5},\ \tfrac{4}{5},\ \ldots$$

we have $H_n=1$, $h_n=0$; so that $G=1$, $g=0$.

Here it is plain that convergent sub-sequences can be selected to give *any* limit between the extreme limits. Thus

$$\tfrac{1}{3},\ \tfrac{2}{5},\ \tfrac{3}{7},\ \tfrac{4}{9},\ldots \text{ gives the limit } \tfrac{1}{2};$$

and

$$\tfrac{2}{3},\ \tfrac{5}{7},\ \tfrac{12}{17},\ \tfrac{29}{41},\ldots \text{ gives the limit } \frac{1}{\sqrt{2}};$$

the latter being the successive convergents of the continued fraction

$$\frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{2+}\ldots.$$

Ex. 2. With $2,\ -\tfrac{3}{2},\ \tfrac{4}{3},\ -\tfrac{5}{4},\ \tfrac{6}{5},\ldots\quad a_n=(-1)^{n-1}\left(1+\dfrac{1}{n}\right)$

we get $H_1=2,\quad H_2=H_3=\tfrac{4}{3},\quad H_4=H_5=\tfrac{6}{5},\ldots$

and $h_1=h_2=-\tfrac{3}{2},\quad h_3=h_4=-\tfrac{5}{4},\ldots,$

so that $G=1,\quad g=-1.$

Ex. 3. With $1,\ -2,\ 3,\ -4,\ 5,\ -6,\ldots\ a_n=(-1)^{n-1}n$

we find $H_n=\infty,\quad h_n=-\infty$

and so $G=\infty,\quad g=-\infty.$

In Exs. 2, 3 it will be seen that no sub-sequences can be found to converge to limits other than the extreme limits.

6. Sum of an infinite series; addition of two series.*

Suppose that a sequence $a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$ is given and that we deduce from this sequence a second $s_1,\ s_2,\ s_3,\ldots$ by addition, so that

$$s_1=a_1,\quad s_2=a_1+a_2,\quad s_3=a_1+a_2+a_3,$$
$$s_n=a_1+a_2+a_3+\ldots+a_n.$$

Then *if the sequence (s_n) is convergent and has the limit S, the infinite series*

$$a_1+a_2+a_3+\ldots=\sum_1^\infty a_n=\Sigma a_n$$

is called convergent; and S is called the sum of the series.

* Here, as in several places of the introductory matter, a sketch of the argument is given rather than a full discussion. Readers who wish for a more detailed account may consult §76 of Prof. G. H. Hardy's *Course of Pure Mathematics.*

It is, however, of fundamental importance to bear in mind that *S is a limit*; and accordingly care must be taken not to assume without proof that familiar properties of finite sums are necessarily true for limits such as S.

Similarly, *if the sequence (s_n) is divergent or oscillatory, the infinite series is said to diverge or to oscillate, respectively.*

Ex. 1. *The geometric series $1+r+r^2+r^3+\dots$ converges if r is numerically less than 1; it diverges if $r \geqq 1$; it oscillates if $r \leqq -1$.*

For, except when $r=1$, $s_n=(1-r^n)/(1-r)$; and when $r=1$, $s_n=n$.

Now, if
$$-1<r<1,$$
we can write
$$|r|=1/(1+a),$$
where a is positive. Then
$$|r|^n<1/(1+na)$$
using the binomial theorem for an integral index.

Thus
$$\lim_{n\to\infty} r^n=0,$$
a result which was obtained independently in Ex. 3, Art. 2.

Hence
$$S=\lim_{n\to\infty} s_n=1/(1-r), \quad \text{if } -1<r<1.$$

If $r \geqq 1$, it is obvious that $s_n \geqq n$, and accordingly
$$\lim_{n\to\infty} s_n=\infty,$$
so that the series diverges.

When r is less than -1, we have $r=-(1+b)$, where b is positive, and so
$$(-r)^n>1+nb.$$

Hence
$$s_n>(2+nb)/(2+b), \quad \text{if } n \text{ is odd,}$$
or
$$s_n<-nb/(2+b), \quad \text{if } n \text{ is even.}$$

Thus
$$\underline{\lim}\, s_n=-\infty, \quad \overline{\lim}\, s_n=+\infty,$$
and so the series oscillates between $-\infty$ and $+\infty$.

If $r=-1$,
$$s_n=1, \quad \text{if } n \text{ is odd,}$$
and
$$s_n=0, \quad \text{if } n \text{ is even.}$$

Thus the series oscillates between 0 and 1.

We have now justified all the statements of the enunciation.

If follows at once from the results stated in Art. 3 that if
$$S=a_1+a_2+a_3+\dots \text{ to } \infty$$
and
$$T=b_1+b_2+b_3+\dots \text{ to } \infty,$$
then
$$S\pm T=(a_1\pm b_1)+(a_2\pm b_2)+(a_3\pm b_3)+\dots.$$

The rule for multiplication of S, T does not follow quite so readily (see Art. 34).

It should be observed that the insertion of brackets in a series is equivalent to the selection of a sub-sequence from the sequence

(s_n); and since an oscillatory sequence always contains at least two convergent sub-sequences (those giving the extreme limits), it is evident that *an oscillatory series * can always be made to converge by grouping the terms in brackets; and, conversely, the removal of brackets may cause a convergent series to oscillate.*

Ex. 2. The series $1-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\dots$ oscillates between the values $\cdot 306\dots$ and $1\cdot 306\dots$; but the series $(1-\frac{1}{2})+(\frac{2}{3}-\frac{3}{4})+(\frac{4}{5}-\frac{5}{6})+\dots$ converges to the sum $\cdot 306\dots$, while $1-(\frac{1}{2}-\frac{2}{3})-(\frac{3}{4}-\frac{4}{5})-\dots$ converges to the sum $1\cdot 306\dots$. [$\cdot 306\dots = 1-\log 2 = \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots$, see Arts. 19, 24, 63.]

It is evident that when we are concerned only with determining whether a series is convergent or not, we may neglect any finite number of terms of the series; this is often convenient in order to avoid some irregularity of the terms, at the beginning of the series.

In particular, it is clear that the two series

$$a_1+a_2+a_3+\dots, \quad a_{m+1}+a_{m+2}+a_{m+3}+\dots,$$

are simultaneously convergent. The sum of the latter is often called *the remainder after m terms of the former.*

EXAMPLES.

Arts. 1-4.

1. If $a_{n+1}=\sqrt{(k+a_n)}$, where $k>0$, $a_1>0$, the sequence (a_n) is monotonic and converges to the positive root of the equation $x^2=x+k$.

2. If $a_{n+1}=k/(1+a_n)$, where $k>0$, $a_1>0$, the sequence (a_n) converges to the positive root of the equation $x^2+x=k$. The negative root is given by the sequence (b_n) where $b_{n+1}=(k/b_n)-1$.

3. If p, q, r, s are real and λ, μ are the roots of

$$(p-\lambda)(s-\lambda)-qr=0,$$

prove that

$$\frac{p-\lambda}{r}=\frac{q}{s-\lambda}=\alpha, \quad \text{and} \quad \frac{p-\mu}{r}=\frac{q}{s-\mu}=\beta,$$

where α, β are the roots of the equation $rx^2+(s-p)x-q=0$, and shew also that

$$\frac{\lambda}{\mu}+\frac{\mu}{\lambda}+2=\frac{(p+s)^2}{ps-qr}.$$

Deduce that, if $y=(px+q)/(rx+s)$, then

$$y-\alpha=\frac{\lambda(x-\alpha)}{rx+s}, \quad y-\beta=\frac{\mu(x-\beta)}{rx+s},$$

---

* It is understood that the limits of oscillation are finite; such a series as $1-2+3-4+5-6+\dots$ must be excluded.

so that the relation between x, y is equivalent to

$$\frac{y-\alpha}{y-\beta}=\frac{\lambda}{\mu}\left(\frac{x-\alpha}{x-\beta}\right).$$

If $\mu=\lambda$, prove that $(p-s)^2+4qr=0$, and that $s=\lambda-r\alpha$; deduce that now the relation is equivalent to the form

$$\frac{1}{y-\alpha}=\frac{1}{x-\alpha}+\frac{r}{\lambda}.$$

If λ, μ are complex so that $(p-s)^2+4qr<0$, we can write

$$\lambda/\mu=e^{-2i\theta},\quad \alpha=Ae^{i\omega},\quad \beta=Ae^{-i\omega},$$

so that $4\cos^2\theta=(p+s)^2/(ps-qr)$, and $p-s=2Ar\cos\omega$, $q=-rA^2$.

Then shew that $$y=\frac{A\{x\sin(\theta+\omega)-A\sin\theta\}}{x\sin\theta-A\sin(\theta-\omega)}.$$

4. Apply the formulae of Ex. 3 to discuss the sequence (a_n), where

$$a_{n+1}=\frac{pa_n+q}{ra_n+s};$$

and prove that $a_n\to\alpha$, when α, β are real and $|p-r\alpha|<|p-r\beta|$. Shew also that $a_n\to\alpha$, when $(p-s)^2+4qr=0$, so that $\beta=\alpha$.

If $p+s=0$, shew that a_n is always equal to one or other of two fixed values.

If $(p-s)^2+4qr<0$, shew that (a_n) has no definite limit; but that when θ/π is rational, (a_n) repeats itself in certain periods.

Discuss the same problem by considering the hyperbola $y=(px+q)/(rx+s)$ and the straight line $y=x$.

5. Simple examples of the types discussed in Ex. 4 may present themselves in connexion with topics which are often discussed with the aid of continued fractions. Two illustrations are given by:

(i) A conductor A is charged by successive contacts with a second conductor B, which is always re-charged to the amount E. If the charge left on A after the first contact is e, prove that the charge on A tends to the limit $Ee/(E-e)$.

(ii) A system of n convergent thin lenses, each of focal length f, is arranged on an axis, so that the distance between consecutive lenses is a; prove that the focal length of the system is equal to $f\sin\theta/\sin n\theta$, if $a=4f\sin^2\frac{1}{2}\theta$.

6. If $a_n\to 0$, prove that $b^{a_n}\to 1$, where b is any positive number; and deduce that if $a_n\to a$, $b^{a_n}\to b^a$. If $a_n\to\infty$, $b^{a_n}\to 0$ or ∞.

7. If $a_{n+1}=ka_n+la_{n-1}$, where k, l are positive, prove that a_n/α^n converges to the limit $(a_2-a_1\beta)/\alpha(\alpha-\beta)$, where α is the positive and β is the negative root of $x^2=kx+l$.

If $k+l=1$, prove that $a_n\to(a_2+la_1)/(1+l)$. In particular, if each term of a sequence is the arithmetic mean of the two preceding terms, the two sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ are separately monotonic and converge to the common value $\frac{1}{3}(2a_2+a_1)$.

Examine similarly the cases in which the geometric and harmonic means are taken.

8. If $a_{n+1}=\frac{1}{2}(a_n+b_n)$, $b_{n+1}=\sqrt{(a_{n+1}b_n)}$, where a_1, b_1 are positive, the sequences (a_n), (b_n) are monotonic and converge to a common limit. If $a_1=\cos\theta$, $b_1=1$, the common limit is $(\sin\theta)/\theta$; and if $a_1=\cosh u$, $b_1=1$, the common limit is $(\sinh u)/u$. [BORCHARDT.]

9. If $a_{n+1}=\frac{1}{2}(a_n+b_n)$, $a_{n+1}b_{n+1}=a_nb_n$, so that a_{n+1}, b_{n+1} are respectively the arithmetic and harmonic means of a_n, b_n, then the sequences (a_n), (b_n) are monotonic and converge to the common limit $\sqrt{(a_1b_1)}$, where a_1, b_1 are positive.

10. If $a_{n+1}=\frac{1}{2}(a_n+b_n)$, $b_{n+1}=\sqrt{(a_nb_n)}$, so that a_{n+1}, b_{n+1} are respectively the arithmetic and geometric means of a_n and b_n, then the sequences (a_n), (b_n) are monotonic and converge to a common limit l.

This limit was called by Gauss the arithmetico-geometric mean of a_1, b_1, and can be applied to calculate elliptic integrals by means of the formula —which follows from Landen's transformation—

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(a_1^2\cos^2\theta+b_1^2\sin^2\theta)^{\frac{1}{2}}}=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(a_n^2\cos^2\theta+b_n^2\sin^2\theta)^{\frac{1}{2}}}=\frac{\pi}{2l}$$

[The convergence of the sequences (a_n), (b_n) to l is usually quite rapid; for instance, with $a_1=\sqrt{2}=1\cdot41421356$ (p. 395), $b_1=1$, $a_4=b_4=1\cdot198140=l$.

This gives $$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}}=\frac{\sqrt{2}\pi}{2l}=1\cdot854075$$

and $$\int_0^1\frac{dx}{\sqrt{(1-x^4)}}=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1+\cos^2\theta)}}=\frac{\pi}{2l}=1\cdot3110288.$$

The method indicated enables us to calculate certain integrals very quickly.]

11. If $a_{n+1}=k^{a_n}$, $k>0$, a number of alternatives arise; we can write the condition in the form $\log a_{n+1}=\lambda a_n$, if $\lambda=\log k$. By means of the curve $y=\log x$ and the line $y=\lambda x$ we can prove that

(i) if $\lambda>1/e$, (a_n) is a divergent monotonic sequence;

(ii) if $0<\lambda<1/e$, the equation $\lambda x=\log x$ has two real roots α, β (say that $\alpha<\beta$); then the sequence (a_n) is monotonic, and $a_n\to\alpha$ if $a_1<\beta$; but if $a_1>\beta$, $a_n\to\infty$.

When λ is negative, the equation $\log x=\lambda x$ has *one* real root (α); but the sequence (a_n) will be seen to be no longer monotonic. To meet this difficulty we may write $\log\left(\log\frac{1}{a_{n+1}}\right)=\log(-\lambda)+\lambda a_{n-1}$ and use the curve $y=\log\left(\log\frac{1}{x}\right)$ and the line $y=\log(-\lambda)+\lambda x$. It can be proved that then the sequences (a_{2n}), (a_{2n+1}) are separately monotonic, and

(iii) if $-e<\lambda<0$, $a_n\to\alpha$;

(iv) if $\lambda<-e$, $a_{2n+1}\to u$, $a_{2n}\to v$, if $a_1<\alpha$; but $a_{2n}\to u$, $a_{2n+1}\to v$, if $a_1>\alpha$; and $a_n=\alpha$, if $a_1=\alpha$.

Here u, v are such that $u<\alpha<v$ and $k^u=v$, $k^v=u$.

This problem was discussed in the special case $a_1=k$ by Seidel (*Abhandlungen der k. Akad. der Wissensch. zu München*, Bd. 11, 1870), who was the first to point out the possibility of oscillation, in case (iv). Previously, Eisenstein (*Crelle's Journal für Math.*, Bd. 28, 1844, p. 49) had obtained the root α as a series proceeding in powers of λ; this series is the same as the one given in Art. 55·1, Ex. 4, below.

Arts. 5·1, 5·2.

12. The reader may find it instructive to determine the upper and lower limits, and also the extreme limiting values of the following sequences. The relations of the terms a_n to the limits should also be considered.

(1) $a_n = (-1)^n n/(2n+1)$. (2) $a_n = (-1)^n(n+1)/(2n+1)$.

(3) $a_n = n + (-1)^n(2n+1)$. (4) $a_n = 2n + 1 + (-1)^n n$.

13. In an oscillatory sequence there may be a finite number of limits derived from sub-sequences, all, some, or none of the limits being attained, as may be seen by considering:

(1) $a_n = \sin(\frac{1}{3}n\pi)$, which consists of the seven numbers $0, \pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}, \pm 1$ all repeated infinitely often.

(2) $a_n = \left(1 + \dfrac{1}{n}\right)\sin(\frac{1}{6}n\pi)$ has the same seven limits as in case (1), but only the value 0 is attained.

(3) $a_n = \left(1 + \dfrac{1}{2n}\right)\cos(\frac{1}{3}n\pi)$ has the four limits $\pm\frac{1}{2}, \pm 1$, but no term a_n is equal to any of these values.

There may also be a whole interval of limits (see Ex. 1, Art. 5·2); and an infinity of these limits may be attained. But it is then not possible for a_n to attain *all* the limits, for the set of points forming an interval are not countable (*i.e.* cannot be put into one-to-one correspondence with the set of positive integers), and therefore cannot form a sequence (a_n).

14. *Addition and subtraction of oscillatory sequences.*

If
$$\underline{\lim}\, a_n = k, \quad \overline{\lim}\, a_n = K$$
and
$$\underline{\lim}\, b_n = l, \quad \overline{\lim}\, b_n = L,$$
then
$$k + l \leqq \overline{\underline{\lim}}\,(a_n + b_n) \leqq K + L$$
and
$$k - L \leqq \overline{\underline{\lim}}\,(a_n - b_n) \leqq K - l.$$

For multiplication the results are not so simple; the reader will find some examples in the first edition of this book, but the space occupied is too great in comparison with the value of the results.

Art. 6.

15. The series
$$\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \dots$$
has the sum to n terms
$$\frac{1}{1-x} - \frac{1}{1-x^N}, \quad \text{if } N = 2^n.$$

Thus the infinite series converges except when $x = \pm 1$, and its sum is equal to $x/(1-x)$ when $|x| < 1$, or to $-1/(x-1)$ when $|x| > 1$. [De Morgan.]

16. The series

$$\frac{a_1}{1+a_1}+\frac{a_2}{(1+a_1)(1+a_2)}+\frac{a_3}{(1+a_1)(1+a_2)(1+a_3)}+\dots$$

can be summed to n terms. The infinite series converges if the terms a_n are all positive after a certain stage.

In particular (see Art. 38) the sum is 1 if the series Σa_n diverges; examples of which are given by $1/a_n = a+n$, $a+2n$, etc.

17. The infinite series

$$\frac{x}{1+x}+\frac{2x^2}{1+x^2}+\frac{4x^4}{1+x^4}+\frac{8x^8}{1+x^8}+\dots$$

converges to the sum $x/(1-x)$, if $|x|<1$.

18. Discuss the two series

$$1+2r+3r^2+4r^3+\dots,\qquad 1+3r+6r^2+10r^3+15r^4+\dots$$

on the same lines as the geometrical progression of Art. 6.

Miscellaneous.

19. If the sequence (a_n) is monotonic, prove that the same is true of the sequence whose nth term is

$$(a_1+a_2+\dots+a_n)/n,$$

and that these sequences vary in the same sense.

Compare similarly the sequences

$$(a_n/b_n) \quad\text{and}\quad (a_1+a_2+\dots+a_n)/(b_1+b_2+\dots+b_n),$$

where b_n is positive.

20. By taking $a_n = a^{n-1}(1-a)$ in Ex. 19, shew that the sequence $(1-a^n)/n$ is a decreasing sequence when a is positive. Deduce that

$$na^{n-1}(1-a) < 1-a^n < n(1-a), \qquad \text{if } 0<a<1,$$
$$na^{n-1}(a-1) > a^n-1 > n(a-1), \qquad \text{if } a>1.$$

Deduce that the same inequalities hold for fractional values of n, if $n>1$; and also if n is negative; but that the inequalities are reversed when $0<n<1$.

21. Deduce from the inequalities of Ex. 20 that $n(a^{\frac{1}{n}}-1)$ decreases as n increases, but remains positive (if $a>1$); prove also that this sequence converges to a limit $f(a)$, and that

$$f(b)-f(a)=f(b/a).$$

CHAPTER II.

SERIES OF POSITIVE TERMS.

7. If all the terms $(a_1, a_2, a_3, \ldots)$ of an infinite series Σa_n are *positive*, the sequence (s_n) steadily increases, where, as in Art. 6, we write

$$s_n = a_1 + a_2 + \ldots + a_n.$$

It follows from the principle of convergence for monotonic sequences (Art. 2) that the series Σa_n must be either convergent or divergent; that is, oscillation is impossible. It is therefore clear (from the same article) that:

(1) *The series Σa_n converges if s_n is less than some fixed number for all values of n.*

(2) *The series diverges if a value of n can be found so that s_n is greater than N, no matter how large N is.*

Ex. 1. Consider the series given by $a_n = 1/n\,!$, so that

$$s_n = 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \ldots + \frac{1}{n\,!}.$$

Compare s_n with the sum

$$\sigma_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}.$$

It is clear that $\quad 3\,! = 3 \,.\, 2 > 2^2$; $4\,! = 4 \,.\, 3 \,.\, 2 > 2^3$;

and so on, $\quad n\,! = n \,.\, (n-1) \ldots 3 \,.\, 2 > 2^{n-1}$.

Thus, from the third term onwards, every term in σ_n is greater than the corresponding term in s_n; and the first and second terms in the sums are equal. Thus

$$\sigma_n > s_n.$$

But
$$\sigma_n = \left(1 - \frac{1}{2^n}\right) \Big/ \left(1 - \frac{1}{2}\right) = 2 - \frac{1}{2^{n-1}} < 2,$$

so that
$$s_n < \sigma_n < 2.$$

Consequently the series Σa_n is convergent and its sum cannot exceed 2. If the sum is denoted by $e - 1$, as usual, we can prove similarly that

$$e - 1 - s_m < 1/\{m\,(m\,!)\}.$$

By direct calculation to 5 decimals we find that $1+s_7$ lies between 2·71822 and 2·71828 and that $1/\{7(7!)\}$ is less than ·00003, so that e lies between 2·7182 and 2·7183. Further calculations have shewn that

$$e = 2{\cdot}7182818285\ldots.$$

Ex. 2. Consider the harmonic series ($a_n = 1/n$), for which

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}.$$

Then arrange the sum s_n into groups thus:

$$s_n = \left(1+\frac{1}{2}\right) + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \left(\frac{1}{9}+\ldots+\frac{1}{16}\right)$$
$$+\left(\frac{1}{17}+\ldots+\frac{1}{32}\right) + \ldots + \left(\frac{1}{2^{m-1}+1}+\ldots+\frac{1}{2^m}\right),$$

where the last term in each group is a power of 2, and $n = 2^m$. Now compare s_n with the sum

$$\sigma_n = \left(\frac{1}{2}+\frac{1}{2}\right) + \left(\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) + \left(\frac{1}{16}+\ldots+\frac{1}{16}\right)$$
$$+\left(\frac{1}{32}+\ldots+\frac{1}{32}\right) + \ldots + \left(\frac{1}{2^m}+\ldots+\frac{1}{2^m}\right),$$

where the number of terms in each group is the same as in the corresponding group of s_n; but all the terms in any group of σ_n are equal to the last term of the group in s_n.

Then $$s_n > \sigma_n, \text{ by inspection.}$$

But each group in σ_n (after the first) is equal to $\frac{1}{2}$; for the r^{th} group contains 2^{r-1} terms each equal to $1/2^r$.

Hence $$\sigma_n = 1 + \tfrac{1}{2}(m-1) = \tfrac{1}{2}(m+1),$$

and so $$s_n > \tfrac{1}{2}(m+1).$$

Thus $s_n > N$, if $m \geqq 2N-1$; and consequently *the series diverges.*

Since all the terms a_n are positive, we need not stop to discuss s_n for cases when n is not a power of 2; of course if some terms in the series were negative, this would be necessary in order to make sure that the series could not oscillate.

If we take similarly

$$\Sigma_n = (1+1) + \left(\frac{1}{2}+\frac{1}{2}\right) + \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{8}+\ldots+\frac{1}{8}\right) + \ldots$$
$$+\left(\frac{1}{2^{m-1}}+\ldots+\frac{1}{2^{m-1}}\right) = 2\sigma_n$$

we can prove that $\Sigma_n > s_n$.

This gives $s_n < m+1$; and so the divergence is very slow. For instance, the sum to a million terms is less than 21, because *

$$2^{20} = (1024)^2 > 10^6.$$

---

* The results of Art. 11 shew that the sum of a million terms is given approximately by $$6 \log_e 10 + {\cdot}577\ldots = 14{\cdot}4 \text{ nearly}.$$

We note that since

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots+\frac{1}{2n-1} \geqslant \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n}\right),$$

the series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots$ is also divergent.

The method used here can easily be applied to discuss the two series

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots \quad \text{and} \quad \frac{1}{2\log 2}+\frac{1}{3\log 3}+\frac{1}{4\log 4}+\dots.$$

But the discussions in Art. 11 are as easy and have the advantage of being more easily remembered.

The method used in Ex. 2 can be put in the following rule (often called *Cauchy's test of condensation*):

The series Σa_n *converges or diverges with* $\Sigma N a_N$, *if* $N=2^n$ *and* $a_n \geqq a_{n+1}$; and it is easy to modify the proof given above so as to shew that we may take N as the integral part of k^n, where k is any number greater than 1.

(3) It is clear also, from the results of Art. 2, that *if we can find* n_1, *so that* $s_{n_1}-s_n > h$ (*where* h *is a fixed positive constant*), *no matter how large* n *may be, then the series must be divergent.*

For we can then select a succession of values $n_0, n_1, n_2, n_3, n_4, \dots$, such that

$$s_{n_1}-s_{n_0} > h, \quad s_{n_2}-s_{n_1} > h, \quad s_{n_3}-s_{n_2} > h, \quad s_{n_4}-s_{n_3} > h, \quad \text{etc.}$$

Thus, on adding, we find that

$$s_{n_r} > s_{n_0}+rh,$$

and therefore s_{n_r} can be made arbitrarily large by taking r sufficiently great; and so the series diverges in virtue of (2) above.

As an example, consider Ex. 2 above; we have then

$$s_{n_1}-s_n > (n_1-n)/n_1,$$

because $s_{n_1}-s_n$ contains (n_1-n) terms ranging from $1/(n+1)$ to $1/n_1$; and so, by taking $n_1=2n$, we get

$$s_{2n}-s_n > \tfrac{1}{2}.$$

(4) If S is the sum of a convergent series of *positive* terms, the sequence (s_n) *increases* to the limit S; the value of s_n cannot reach, and *a fortiori* cannot exceed S. Thus S must be greater than the sum of any number of terms, taken arbitrarily, in the series; for n can be chosen large enough to ensure that s_n includes all these terms.

On the other hand, any number smaller than S, (say $S-\epsilon$), has the property that we can find terms in the series whose sum exceeds $S-\epsilon$.

It is now clear that a series of positive terms remains convergent even if an *infinite* number of its terms are removed.

Also if a series can be proved to converge when its terms are grouped in brackets, it will still converge when the brackets are removed, *provided that all the terms are positive.*

8. Comparison test for convergence (of positive series).

If the series $c_1+c_2+c_3+\dots$ contains only positive terms and is convergent, and if another series $a_1+a_2+a_3+\dots$ has the property

$$0 \leqq a_n \leqq c_n$$

(at any rate for values of n greater than some fixed value), then Σa_n is also convergent.

For, if $a_n \leqq c_n$, when $n > m$, we have

$$a_{m+1}+a_{m+2}+\dots+a_n \leqq c_{m+1}+c_{m+2}+\dots+c_n < T,$$

if T is the sum $\sum_1^\infty c_n$.

Thus $$s_n < s_m + T;$$

so that s_n is less than a constant (independent of n), which establishes the convergence of Σa_n.

In case the inequality holds for all values of n, we have $s_n < T$; so that the sum cannot exceed T.

The condition that *all* the terms must be positive in Σa_n and Σc_n may be broken *if there are no negative terms after a certain stage.* For the convergence of the series will not be affected by the omission of a finite number of terms at the beginning of the series.

But *if there are negative terms left, however far we go in the series the Σc_n test is not sufficient.* For instance, take the series

$$1-\frac{1}{1.2}+\frac{1}{2}-\frac{1}{2.3}+\frac{1}{3}-\frac{1}{3.4}+\frac{1}{4}-\frac{1}{4.5}+\dots$$

and compare it with

$$1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\dots.$$

Every term in the second series is numerically greater than, or equal to, the corresponding term in the first series; and the second series converges to the sum 0. But the first series diverges; for in this series we find

$$1-\frac{1}{1.2}=\frac{1}{2},\quad \frac{1}{2}-\frac{1}{2.3}=\frac{1}{3},\quad \frac{1}{3}-\frac{1}{3.4}=\frac{1}{4},\quad \text{etc.,}$$

so that $$s_{2n}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n+1} \quad\text{and}\quad s_{2n+1}=s_{2n}+\frac{1}{n+1}$$

giving $\lim s_{2n+1}=\lim s_{2n}=\infty$. (Ex. 2, Art. 7.)

9. The comparison test may be stated in the following form, which is often easier to work with :

Let the series $\Sigma(1/C_n)$ *be convergent; then* Σa_n *will converge, provided that*

$$\overline{\lim}\,(a_n C_n)$$

is not infinite, if both series contain only positive terms.

For, when this condition is satisfied, we can find a constant G independent of n, such that

$$0 < a_n C_n < G.$$

Hence a_n is less than G/C_n, which is the general term of a convergent series.

It is useful to remark that *there is no need to assume the* existence *of the limit* $\lim(a_n C_n)$; this may be seen by considering the convergent series

$$\frac{1}{C_1}+\frac{2}{C_2}+\frac{1}{C_3}+\frac{2}{C_4}+\frac{1}{C_5}+\frac{2}{C_6}+\dots,$$

for which $a_n C_n$ is alternately equal to 1 and 2.

Further, the test is *sufficient* only and is not *necessary*; as we may see by taking $C_n = n!$ and $a_n = 1/2^{n-1}$; then $a_n C_n > n/2$, so that $\lim(a_n C_n) = \infty$. But Σa_n converges (see Ex. 1, Art. 6.)

The corresponding test for divergence runs :

Let the series $\Sigma(1/D_n)$ *be divergent, then* Σa_n *will diverge, provided that*

$$\underline{\lim}\,(a_n D_n) > 0,$$

both series containing only positive terms.

The proof is practically identical with the previous investigation, when the signs of inequality are reversed. We note also that the limit $\lim(a_n D_n)$ need not exist; and that the test is not *necessary*.

It follows immediately that the *following conditions are necessary but not sufficient* :

for convergence, $\underline{\lim}\,(a_n D_n) = 0$;

for divergence, $\overline{\lim}\,(a_n C_n) = \infty$.

But, in general there is no need for the limits of $(a_n D_n)$ or of $(a_n C_n)$ to exist; and *the condition,* $\lim(a_n D_n) = 0$, *sometimes given as necessary for convergence, is incorrect.*

Ex. Let $a_n = 1/n^2$, except when n is a squared integer, and let $a_n = 1/n^{\frac{3}{2}}$ when n is a square.

Thus the series is

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^{\frac{3}{2}}}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^{\frac{3}{2}}}+\dots.$$

If we take $D_n = n$, we find

$$\underline{\lim}\,(a_n D_n) = 0, \quad \overline{\lim}\,(a_n D_n) = \infty,$$

so that $\lim (a_n D_n)$ does not exist. But yet the series Σa_n converges, as will be seen in Art. 11.

It is easy to see that *if the terms a_n steadily decrease, the condition $\lim (na_n) = 0$ is necessary for convergence*; but even so, the general condition $\lim (a_n D_n) = 0$ is not necessary.

For if Σa_n is convergent, we can choose $m = m(\epsilon)$, so that

$$a_{m+1} + a_{m+2} + \dots + a_n < \epsilon, \quad \text{if } n > m.$$

Now each of these $(n-m)$ terms is greater than or equal to a_n, so that

$$(n-m)a_n < \epsilon, \quad \text{if } n > m.$$

But, since $a_n \to 0$, we can choose ν $(> m)$, so that $ma_n < \epsilon$, if $n > \nu$.

Thus $na_n < 2\epsilon$, if $n > \nu$, and consequently

$$\lim (na_n) = 0.$$

That this condition is not sufficient follows from Abel's example (Art. 11), $a_n = (n \log n)^{-1}$, which gives a divergent series, although $\lim (na_n) = 0$.

No condition such as $\lim (a_n D_n) = 0$ is necessary for convergence if D_n tends to ∞ more rapidly than n; and examples of convergent series for which $(a_n D_n)$ has no definite limit will be found in an article by Pringsheim (*Math. Annalen*, Bd. 35, p. 343). Of course, *if the limit exists*, its value must be zero for convergence; but convergence does not imply the existence of a limit for $(a_n D_n)$.

10. If the series Σa_n is compared with the geometric series Σr^n, we can infer Cauchy's test, which is theoretically of fundamental importance:

$$\textit{If } \overline{\lim}\, a_n^{\frac{1}{n}} < 1, \textit{ the series converges};$$

$$\textit{if } \overline{\lim}\, a_n^{\frac{1}{n}} > 1, \textit{ the series diverges}.$$

It is of great importance to remember that, in contrast with the ratio-tests of Art. 12, these conditions *both* relate to the *maximum* limiting value; and that the condition $\underline{\lim}\, a_n^{1/n} > 1$ is *not* necessary for divergence.

Further, to ensure divergence, it is *not* necessary that $a_n^{1/n}$ should be ultimately greater than unity, in spite of what is sometimes stated in text-books; and if $a_n^{1/n}$ oscillates between limits which include unity, the series *diverges*.

To prove these rules, suppose first that

$$\overline{\lim}\, a_n^{1/n} = G < 1.$$

Take any number ρ between G and 1; then we can find m so that

$$a_n^{1/n} < \rho < 1, \quad \text{if } n > m.$$

Hence, after the mth term, the terms of Σa_n are less than those of the convergent series $\Sigma \rho^n$; that is, Σa_n is convergent. And the remainder after p terms is less than $\rho^p/(1-\rho)$ provided that $p > m$.

But if $\overline{\lim}\, a_n^{1/n} > 1$, there will be an infinite sequence of values of n (say $n_1, n_2, n_3, \ldots$), such that

$$a_n^{1/n} > 1, \quad \text{if} \quad n = n_p;$$

and therefore

$$a_n > 1, \quad \text{if} \quad n = n_p.$$

Thus the sum Σa_n, taken from 1 to n_p, must be greater than p; and p may be taken as large as we please, so that Σa_n diverges.

We know from Art. 149 that $\overline{\lim}\, a_n^{1/n}$ lies between the extreme limits of (a_{n+1}/a_n); thus the series converges if $\overline{\lim}\,(a_{n+1}/a_n) < 1$, and diverges if $\underline{\lim}\,(a_{n+1}/a_n) > 1$. This shews that d'Alembert's test (Art. 12·2) is a deduction from Cauchy's.

But on the other hand, since we only know that $\overline{\lim}\, a_n^{1/n}$ falls between the extreme limits of (a_{n+1}/a_n), it is clear that *we cannot deduce Cauchy's test in its full generality from d'Alembert's.*

If we consider a power-series $\Sigma a_n x^n$ (in which a_n and x are supposed positive), Cauchy's test will give:

$$x < l, \quad \text{for convergence,} \quad \text{and } x > l, \quad \text{for divergence,}$$

where

$$1/l = \overline{\lim}\, a_n^{1/n}.$$

Thus $x = l$ gives an exact boundary between convergent and divergent series, supposing l to be different from zero and finite. If this maximum limit is 0, the condition for convergence is satisfied for all positive values of x; but if the maximum limit is ∞, the series will diverge for all values of x, except zero.

On the other hand, if we apply d'Alembert's test to the power-series, we can only infer that

$$x < g \quad \text{gives convergence,} \quad \text{and} \quad x > G \quad \text{gives divergence,}$$

where

$$g = \underline{\lim}\,(a_n/a_{n+1}) \quad \text{and} \quad G = \overline{\lim}\,(a_n/a_{n+1});$$

so that when g and G are unequal (as they may easily be), we can obtain no information as to the behaviour of the series if

$$g < x < G.$$

In spite of this theoretical objection, d'Alembert's test is sufficient to establish the region of convergence of the most useful power-series; and, on account of its simple character, this test (with its extensions in Art. 12·2) is of frequent use in ordinary work.

11. Second test for convergence; the logarithmic scale. Suppose that the terms of a positive series are arranged in order of magnitude, so that $a_n \geqq a_{n+1} > 0$.

If we write $f(n)=a_n$, it may happen that the function $f(x)$ is also definite for values of x which are not integers, and that $f(x)$ never increases with x. Then, if x lies between $(n-1)$ and n, it is plain that
$$a_{n-1} \geqq f(x) \geqq a_n > 0.$$

Thus, from the definition of an integral, we have
$$\int_{n-1}^{n} a_{n-1}\,dx \geqq \int_{n-1}^{n} f(x)\,dx \geqq \int_{n-1}^{n} a_n\,dx,$$
or
$$a_{n-1} \geqq \int_{n-1}^{n} f(x)\,dx \geqq a_n.$$

Write now $I_n=\int_1^n f(x)\,dx$, and we find, on addition for $n=1, 2, \ldots,$
$$a_1+a_2+\ldots+a_{n-1} \geqq I_n \geqq a_2+a_3+\ldots+a_n,$$
or
$$s_n-a_n \geqq I_n \geqq s_n-a_1.$$

Hence
$$a_1 \geqq s_n - I_n \geqq a_n > 0.$$

Further
$$(s_n-I_n)-(s_{n-1}-I_{n-1})=a_n-\int_{n-1}^{n} f(x)\,dx \leqq 0,$$

and therefore the sequence whose nth term is s_n-I_n *never increases*; and since its terms are contained between 0 and a_1, the sequence must have a limit (Art. 2) and
$$a_1 \geqq \lim\,(s_n-I_n) \geqq 0.$$

Thus, *the series Σa_n converges or diverges with the integral** $\int_1^{\infty} f(x)\,dx$; *if convergent, the sum of the series differs from the integral by less than a_1; if divergent, the limit of (s_n-I_n) nevertheless exists and lies between* 0 and a_1.†

For more details as to the connexions between s_n and I_n the reader may consult Art. 161 of Appendix II.

* The integral converges or diverges with the sequence (I_n); for further details see Appendix III.

† This test was originally given by Maclaurin (*Fluxions*, 1742, Art. 350), and was rediscovered by Cauchy; for an extension to other types of series, see Bromwich, *Proc. Lond. Math. Soc.*, vol. 6, and G. H. Hardy, *ibid.* vol. 9.

Ex. If $a_n = 1/n(n+1)$, $f(x) = 1/x(x+1)$, and $\int_1^\infty f(x)\,dx = \log 2$.

And $\sum_1^\infty a_n = 1$, which is contained between the values $\log 2$ and $\frac{1}{2} + \log 2$, in agreement with the general result.

The reader may find it instructive to consider the geometrical significance of these inequalities, in connexion with the curve $y = f(x)$. It is easy to see that $T_n = I_n - (s_n - a_1)$ represents the sum of the shaded area; while

$$U_n = (s_n - a_n) - I_n$$

represents the sum of the corresponding areas *above* the curve.

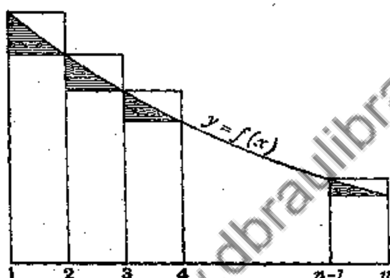

FIG. 9.

It is then obvious that the sequences (T_n) and (U_n) both increase with n, and since the sum of corresponding terms $(T_n + U_n)$ is equal to $(a_1 - a_n)$, it follows that each sequence has a positive limit (less than a_1).

Applications to special series.

(1) Consider $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$, where $a_n = n^{-p}$.

Here, if p is positive, the rule applies at once, and gives

$$f(x) = x^{-p}, \qquad \int_1^x f(x)\,dx = \frac{1}{1-p}(x^{1-p} - 1);$$

thus the integral to ∞ is convergent only if $p > 1$. Thus *the infinite series Σn^{-p} converges only if $p > 1$; and the sum is then contained between the values $1/(p-1)$ and $p/(p-1)$.*

If $p = 1$, the integral is equal to $\log x$, which shews that the harmonic series is divergent (see Art. 7); but we infer also that the limit

$$\lim_{n \to \infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n\right)$$

exists and lies between 0 and 1. This limit is *Euler's or Mascheroni's* *constant.* The value of the constant is 0·57721... (see Art. 106), and will be denoted usually by C.

[The notation γ is used in some works on analysis.]

In accordance with the notation explained in Art. 1·1, it is often convenient to write

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\to\log n+C.$$

The convergence of the series considered in Art. 9

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^{\frac{2}{3}}}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^{\frac{2}{3}}}+\ldots$$

can now be inferred.

For the first n terms are included in S_n+T_n, where

$$S_n=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2},$$

$$T_n=1+\frac{1}{4^{\frac{2}{3}}}+\frac{1}{9^{\frac{2}{3}}}+\frac{1}{16^{\frac{2}{3}}}+\ldots+\frac{1}{(n^2)^{\frac{2}{3}}},$$

and so the sum of these n terms is less than S_n+T_n.

Now by (1) $$S_n<2,\quad T_n<4,$$
and so $$S_n+T_n<6.$$

Hence the given series converges to a sum not greater than 6 (Arts. 2, 7).

(2) Consider $\frac{1}{2}(\log 2)^{-p}+\frac{1}{3}(\log 3)^{-p}+\frac{1}{4}(\log 4)^{-p}+\ldots,$

for which $$a_1=0 \quad\text{and}\quad a_n=n^{-1}(\log n)^{-p}.$$

Here $$f(x)=x^{-1}(\log x)^{-p},$$

and so $$\int_2^x f(x)\,dx=[(\log x)^{1-p}-(\log 2)^{1-p}]/(1-p)$$

or $$=\log(\log x/\log 2),\qquad \text{if } p=1.$$

Thus the given series converges if $p>1$ and diverges if $p\leqq 1$; it should be noted that if $p=1$, the divergence is very slow, the sum of a billion ($=10^{12}$) terms being less than 5.

(3) It can be proved similarly that if we omit a sufficient number of the early terms to ensure that all the logarithms are positive, and if

$$a_n=(n\log n)^{-1}(\log\log n)^{-p},$$

or $$(n\,.\log n\,.\log\log n)^{-1}[\log(\log\log n)]^{-p},$$

the series converges if $p>1$, diverges if $p\leqq 1$.

(4) Since $\int[F'(x)/F(x)]\,dx=\log[F(x)]$, it is clear that the two integrals

$$\int^{\infty}[F'(x)/F(x)]\,dx,\quad\text{and}\quad\int^{\infty}F'(x)\,dx$$

converge or diverge together; now if we suppose that $F'(x)$ is positive but decreases to zero as a limit, the same will be true of $F'(x)/F(x)$, because

$$\frac{d}{dx}\frac{F'(x)}{F(x)}=\frac{F''(x)}{F(x)}-\left\{\frac{F'(x)}{F(x)}\right\}^2,$$

and this is negative, because $F''(x)$ is negative, and $F(x)$ is supposed positive. Thus we deduce the result :

The series $\Sigma F'(n)/F(n)$ *converges or diverges according as the series* $\Sigma F'(n)$ *does.* Similarly, *when* $\Sigma F'(n)$ *is divergent, the series* $\Sigma F'(n)/\{F(n)\}^p$ *converges if* $p>1$, *but diverges if* $p \leqq 1$.

This result shews that the succession of series begun in 1, 2, 3 can be continued without stopping; but for ordinary work, the two types 1, 2 are sufficient.

The following results, which are independent of the Calculus, have a field of application substantially equivalent to (4):

Let (M_n) *denote an increasing sequence such that* $\lim M_n=\infty$; *then*

$$\Sigma(M_{n+1}-M_n)/M_n \quad and \quad \Sigma(M_{n+1}-M_n)/M_{n+1}$$

are divergent series, while $\Sigma(M_{n+1}-M_n)/M_n^{p-1}M_{n+1}$ *is convergent if* $p>1$.

For, if we take the sum of $(M_{n+1}-M_n)/M_n$ as n ranges from q to r, we see that its value is greater than $\sum_q^r(M_{n+1}-M_n)/M_{r+1}=(M_{r+1}-M_q)/M_{r+1}$ because in the summation $M_n<M_{r+1}$. We can choose r large enough to make $M_{r+1}\geqq 2M_q$; and so this sum is greater than $\frac{1}{2}$, no matter how large q may be. Thus the series diverges. (Art. 7 (3).)

Similarly, $\Sigma(M_{n+1}-M_n)/M_{n+1}$ is divergent.

If $p=2$, the third series reduces to $\Sigma\left(\frac{1}{M_n}-\frac{1}{M_{n+1}}\right)=\frac{1}{M_1}$, and so is convergent; thus if $p>2$, the terms are less than those of a convergent series, and so the only case left for discussion is given by $1<p<2$.

From Ex. 20, Ch. I., we have the inequality

$$1-c^k>k(1-c) \qquad \text{if } 0<k<1.$$

Thus, if we write

$$c=M_n/M_{n+1}, \quad k=p-1,$$

we get

$$1-\frac{M_n}{M_{n+1}}<\frac{1}{p-1}\left\{1-\left(\frac{M_n}{M_{n+1}}\right)^{p-1}\right\},$$

or

$$\frac{M_{n+1}-M_n}{M_n^{p-1}M_{n+1}}<\frac{1}{p-1}\left(\frac{1}{M_n^{p-1}}-\frac{1}{M_{n+1}^{p-1}}\right).$$

From this it is plain that the given series has its terms less than those a convergent series.

12·1. Ratio-tests for convergence.

The ratio-tests depend on the quotient a_n/a_{n+1}, obtained by division of two consecutive terms of the series; and in the case of many series of practical importance, this quotient is found to be simpler than the general term a_n. Then the following tests * will often lead to a rapid determination of the conditions for convergence of the series Σa_n.

If ΣD_n^{-1} is a divergent series and if

$$T_n = D_n \frac{a_n}{a_{n+1}} - D_{n+1},$$

then (C) Σa_n *is convergent, if* $\varliminf T_n > 0$,

(D) Σa_n *is divergent, if* $\varlimsup T_n < 0$.

In particular, if T_n tends to a definite limit l, then

(C) Σa_n *is convergent, if* $l > 0$,

(D) Σa_n *is divergent, if* $l < 0$.

For, if the minimum limit g is positive, and if h is any positive number less than g, an integer m can be found such that

$$T_n = D_n \frac{a_n}{a_{n+1}} - D_{n+1} > h, \quad \text{if } n \geqq m.$$

Thus $$a_n D_n - a_{n+1} D_{n+1} > h a_{n+1}, \quad \text{if } n \geqq m,$$
or adding, we have

$$a_m D_m - a_n D_n > h(a_{m+1} + a_{m+2} + \ldots + a_n).$$

Hence $$a_{m+1} + a_{m+2} + \ldots + a_n < a_m D_m / h,$$
and the last expression on the right *does not involve* n; so that $\sum_1^n a_n$ remains always less than a fixed number, and therefore Σa_n is convergent.

On the other hand, if the maximum limit is negative, all the expressions T_n must be negative after a certain stage: and thus we can find m, so that

$$D_n \frac{a_n}{a_{n+1}} - D_{n+1} < 0, \quad \text{if } n \geqq m,$$

or $$a_n D_n < a_{n+1} D_{n+1}, \quad \text{if } n \geqq m.$$

Hence $$a_n D_n > a_m D_m, \quad \text{if } n > m,$$

and so the terms of Σa_n are, after the mth, greater than those of the divergent series $(a_m D_m) \Sigma D_n^{-1}$. Thus Σa_n is also divergent.

* Originally due to Kummer; but arranged in the present form by Dini. See the historical note on p. 38.

The reader will notice that, in the discussion of the *convergent* case above no use is made of the property that ΣD_n^{-1} is a divergent series; and at first sight it may be expected that some advantage could be gained by stating the condition for convergence in the form

$$\underline{\lim}\, U_n > 0, \quad \text{where } U_n = f_n \frac{a_n}{a_{n+1}} - f_{n+1},$$

and f_n is *any* sequence of positive numbers.

However, if f_n is taken to be of the type C_n, where ΣC_n^{-1} is convergent, it will be evident that when $\underline{\lim}\, U_n > 0$, there is some value of m, such that

$$U_n = C_n \frac{a_n}{a_{n+1}} - C_{n+1} > 0, \quad \text{if } n \geqq m,$$

Thus

$$a_n C_n > a_{n+1} C_{n+1} > a_{n+2} C_{n+2} > \dots, \quad \text{if } n \geqq m,$$

and consequently, after a certain stage, $a_n C_n$ remains less than a fixed number K, so that $a_n < K/C_n$. Consequently the series ΣC_n^{-1} must converge *more slowly* than Σa_n, if the U_n-test is to be effective to establish convergence; and so we run the risk of introducing unnecessary restrictions by making an unsuitable choice for C_n.

For instance, if $\dfrac{a_n}{a_{n+1}} = \dfrac{n+\beta}{n}$, and if we choose $C_n = n^2$, we find that

$$U_n = n(n+\beta) - (n+1)^2 = (\beta - 2)\, n - 1.$$

Thus U_n tends to a positive limit only if $\beta > 2$; and the test would give convergence only when $\beta > 2$. But a reference to Ex. 2 below (with $\alpha = 0$) shews that the true condition for convergence is $\beta > 1$; and this can be deduced from the T_n form by taking $D_n = n$, which makes $T_n = \beta - 1 = l$.

A further reason for preferring the T_n form lies in the fact that the *same* function T_n is used to test for divergence as well as for convergence; and this advantage disappears if we introduce the U_n form in dealing with convergence.

Historical Note. Kummer himself gave the test in the form

$$\lim \left\{ \phi(n) - \phi(n+1) \frac{a_{n+1}}{a_n} \right\} > 0$$

for convergence, where $\phi(n)$ is an arbitrary sequence of *positive* numbers, subject to the restriction $\lim \phi(n) a_n = 0$, a condition which was proved to be superfluous by Dini. Dini also was the first to obtain the condition in the form given above, where the *same* expressions are used to test both for convergence and for divergence.* Further extensions have been given by Pringsheim.

* Some variations of the tests have been given by different writers; but Dini's are undoubtedly the most convenient in practice.

12·2. Special ratio-tests of importance.

It is easy to deduce from Art. 12·1 certain special tests which are of constant use in the applications of the theory.

(1) *d'Alembert's test.*

Let $D_n = D_{n+1} = 1$; then the conditions are

$$\text{(C)} \ \underline{\lim}\,(a_n/a_{n+1}) > 1; \qquad \text{(D)} \ \overline{\lim}\,(a_n/a_{n+1}) < 1.$$

This should be compared with Cauchy's test of Art. 10.

Ex. 1. If this test is applied to the series $1+2x+3x^2+4x^3+\dots$ we see that it converges if $x<1$, diverges if $x>1$; but the test gives no result if $x=1$, although the series is then obviously divergent.

(2) *Raabe's test*; to be tried when $\lim (a_n/a_{n+1}) = 1$.

Let $D_n = n$, then the conditions are

$$\text{(C)} \ \underline{\lim}\{n(a_n/a_{n+1}-1)\} > 1; \qquad \text{(D)} \ \overline{\lim}\{n(a_n/a_{n+1}-1)\} < 1.$$

Ex. 2. If we take

$$1+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\dots,$$

we find
$$n\left(\frac{a_n}{a_{n+1}}-1\right)=\frac{\beta-\alpha}{1+\alpha/n},$$

and so the series converges if $\beta > \alpha+1$, diverges if $\beta < \alpha+1$. If $\beta=\alpha+1$ the test fails, although the series may then be seen to diverge by comparison with $\Sigma 1/n$.

(3) If the limits used in (2) are both equal to 1, we must use more delicate tests, found by writing

$$D_n = n\log n, \quad n\log n\log(\log n), \quad \text{and so on.}$$

These functions are of the form $f(n)$, where $f(x)$ is continuous and $f''(x)$ tends to zero as x tends to infinity. Then Kummer's test becomes

$$\text{(C)} \ \underline{\lim}\,\rho_n > 0; \qquad \text{(D)} \ \overline{\lim}\,\rho_n < 0,$$

where
$$\frac{a_n}{a_{n+1}} = 1+\frac{f'(n)}{f(n)}+\frac{\rho_n}{f(n)}.$$

For we have

$$f(n+1)-f(n)-f'(n)=\int_0^1 [f'(n+x)-f'(n)]\,dx=\int_0^1 dx\int_0^x f''(n+t)\,dt.$$

Now we can find ν so that $|f''(\xi)|<\epsilon$, if $\xi>\nu$, and so the last integral is easily seen to be less than $\frac{1}{2}\epsilon$, if $n>\nu$. Thus

$$f(n+1)-f(n)-f'(n)\to 0, \quad \text{as } n\to\infty.$$

Writing $f(n+1)$ and $f(n)$ for D_{n+1} and D_n in Kummer's test, we are led at once to the form given above.

In particular, if $f(x)=x\log x$,

we find $$f'(x)=\log x+1, \qquad f''(x)=1/x;$$

thus we find *de Morgan's and Bertrand's first test,*

$$\text{(C)} \quad \underline{\lim}\, \rho_n > 1; \qquad \text{(D)} \quad \overline{\lim}\, \rho_n < 1,$$

where $$\frac{a_n}{a_{n+1}}=1+\frac{1}{n}+\frac{\rho_n}{n\log n}.$$

Their further tests, given by $f(x)=x\log x\log(\log x)$, etc., are of less importance.

(4) It is sometimes more convenient to replace the last test by the following:

$$\text{(C)} \quad \underline{\lim}\, \sigma_n > 1; \qquad \text{(D)} \quad \overline{\lim}\, \sigma_n < 1,$$

where $$\log\frac{a_n}{a_{n+1}}=\frac{1}{n}+\frac{\sigma_n}{n\log n}.$$

After a certain stage, we have $1 < a_n/a_{n+1} < 1+(2/n)$;

also $$0 < \xi - \log(1+\xi) = \int_0^\xi \frac{t}{1+t}\,dt < \tfrac{1}{2}\xi^2, \quad \text{if } \xi > 0;$$

thus we see that $0 < \rho_n - \sigma_n < 2(\log n)/n$, and so $\rho_n - \sigma_n \to 0$ (Art. 160).

(5) The most important cases in practical work admit of the quotient a_n/a_{n+1} being expressed in the form

$$\frac{a_n}{a_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^p}\right),$$

where μ is a constant, p an index greater than 1, which is usually equal to 2; and the notation O is explained in Art. 1·1.

It is then easy to see that d'Alembert's test fails: Raabe's test gives convergence if $\mu > 1$, divergence if $\mu < 1$. To discuss the case $\mu=1$, apply the test (3); then

$$\rho_n=O\left(\frac{n\log n}{n^p}\right)=O\left(\frac{\log n}{n^{p-1}}\right).$$

But $$\lim(\log n/n^{p-1})=0 \quad \text{(Art. 160)},$$

so that $$\lim \rho_n = 0 < 1.$$

Thus $\mu=1$ also gives divergence. We may sum up these results in the working rule (essentially due to Gauss in his investigations on the Hypergeometric series):

If it is possible to express the quotient a_n/a_{n+1} in the form

$$\frac{a_n}{a_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^p}\right) \quad (\text{where } p > 1),$$

the series Σa_n is divergent if $\mu \leqq 1$, convergent if $\mu > 1$.

If we apply the results of Art. 39, Ex. 4, to the quotient $na_n/(n+1)a_{n+1}$, it is not difficult to prove that, *when* a_n/a_{n+1} *can be expressed in the form above, the condition* $\lim(na_n)=0$ *is necessary and sufficient for convergence* (in contrast with the results for series in general, Art. 9).

Ex. 3. Consider the Hypergeometric Series

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\dots$$

By using d'Alembert's test this series is easily seen to converge if $0<x<1$, and to diverge if $x>1$. If $x=1$, consider

$$\frac{a_n}{a_{n+1}}=\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}=1+\frac{n(\gamma+1-\alpha-\beta)+\gamma-\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta},$$

which gives $\mu=\gamma+1-\alpha-\beta$, $p\geqq 2$, so that the series converges if $\gamma>\alpha+\beta$, and diverges if $\gamma\leqq\alpha+\beta$.

It will appear from Art. 50 that the series converges if $-1<x<0$; and from Art. 19 that it converges also for $x=-1$, if $\gamma+1>\alpha+\beta$.

Anticipating the results proved in Art. 42, we can reduce the majority of series covered by the ratio-tests to the type Σn^{-p}, discussed in Art. 11.

The method* will be easily understood by considering the following example:

Ex. 4. Suppose that

$$a_n=\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)\delta(\delta+1)\dots(\delta+n-1)}$$

which reduces to the Hypergeometric type when $\delta=1$.

From Art. 42 we see that as $n\to\infty$,

$$\alpha(\alpha+1)\dots(\alpha+n-1)\sim A(n^{\alpha-1}.n!),$$

where A is a certain constant depending on α.

Hence $$a_n\sim(ABn^{\alpha+\beta})/(CDn^{\gamma+\delta}),$$

or $$a_n\sim Kn^{\alpha+\beta-\gamma-\delta}.$$

Hence Σa_n converges if $\gamma+\delta-(\alpha+\beta)>1$,

and diverges if $\gamma+\delta-(\alpha+\beta)\leqq 1$.

It is easy to confirm these results by using Gauss's rule.

13. Notes on the ratio-tests.

It is to be noted that *d'Alembert's test does* **not** *ensure the convergence of a series if we only know that* $a_n/a_{n+1}>1$ *for all values of* n.

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* This method was suggested to me by Prof. A. E. Jolliffe.

For, if $\lim(a_n/a_{n+1})=1$, it will not be possible to find a number h such that

$$\frac{a_n}{a_{n+1}}-1>h>0, \quad \text{for } n \geqq m.$$

In particular, if $a_n=1/n$, $a_n/a_{n+1}=1+1/n>1$; and yet the series Σa_n is divergent.

Secondly, *it is not necessary for the convergence of the series Σa_n that a_n/a_{n+1} should have a definite limit.*

For it will be seen in Art. 26 that the *order* of the terms does not affect the convergence of a series of positive terms; but of course a change in the order may affect the value of $\lim a_n/a_{n+1}$.

Ex. 1. The series $\alpha+1+\alpha^3+\alpha^2+\alpha^5+\alpha^4+\ldots$ is a rearrangement of the geometric series $1+\alpha+\alpha^2+\alpha^3+\ldots$, and so is convergent if $0<\alpha<1$. But in this series the quotient a_n/a_{n+1} is alternately α and $1/\alpha^3$.

Ex. 2. The series $1+\alpha+\beta^2+\alpha^3+\beta^4+\alpha^5+\beta^6+\ldots$ is convergent if $0<\alpha<\beta<1$; as is plain by comparison with $1+\beta+\beta^2+\beta^3+\beta^4+\ldots$.

In this series we have

$$\lim \alpha^n/\beta^{n+1}=0, \quad \lim \beta^n/\alpha^{n+1}=\infty.$$

But even when the terms are arranged in order of magnitude, the convergence of the series Σa_n does not imply the existence of the limit of a_n/a_{n+1}.

Ex. 3. The series $1+\frac{1}{2}\alpha+\frac{1}{3}\alpha+\frac{1}{4}\alpha^2+\frac{1}{5}\alpha^2+\frac{1}{6}\alpha^3+\frac{1}{7}\alpha^3+\ldots$ has its terms arranged in order of magnitude, if $0<\alpha<1$; and it is then convergent, by comparison with $1+\alpha+\alpha+\alpha^2+\alpha^2+\alpha^3+\alpha^3+\ldots$.

But yet

$$\overline{\lim}\,(a_n/a_{n+1})=1/\alpha, \quad \underline{\lim}\,(a_n/a_{n+1})=1.$$

Thirdly, *if the quotient a_n/a_{n+1} has maximum and minimum limits which include unity, the whole scale of ratio-tests will fail.*

For, if $\overline{\lim}\,(a_n/a_{n+1})=G>1>g=\underline{\lim}\,(a_n/a_{n+1})$, we can take K, k such that

$$G>K>1>k>g,$$

and then a_n/a_{n+1} is greater than K for an infinite set of values of n, while it is less than k for a second infinite set of values.

If n belongs to the first set of values, we shall have

$$n(a_n/a_{n+1}-1)>n(K-1);$$

but if it belongs to the second set,

$$n(a_n/a_{n+1}-1)<-n(1-k).$$

Hence

$$\overline{\lim}\,n(a_n/a_{n+1}-1)=+\infty, \quad \underline{\lim}\,n(a_n/a_{n+1}-1)=-\infty,$$

and therefore Raabe's test fails entirely. It is easy to see that the failure extends to all the following tests.

If we apply Raabe's test to Ex. 3 above, we find

$$\overline{\lim}\, n(a_n/a_{n+1}-1) = +\infty, \quad \underline{\lim}\, n(a_n/a_{n+1}-1) = 1;$$

and passing to the next stage we get

$$\overline{\lim}\,(\log n)[n(a_n/a_{n+1}-1)-1] = +\infty, \quad \underline{\lim}\,(\log n)[n(a_n/a_{n+1}-1)-1] = 0,$$

so that the ratio-tests can give no information.

It will be seen from the foregoing remarks that the ratio-tests have a comparatively limited range of usefulness; and it may reasonably be asked, why should we trouble to introduce them at all, and not be content with the more general comparison-tests? The answer to this is that, in practice, the quotient a_n/a_{n+1} is often much simpler than a_n, and then it is easier to use the ratio-tests (if they apply) than any others.

14. Ermakoff's tests.*

The series $\Sigma f(n)$, in which $f(n)$ is subject to the conditions of Art. 11, is

$$\text{(i) convergent if } \overline{\lim_{x\to\infty}} \frac{e^x f(e^x)}{f(x)} < 1,$$

$$\text{(ii) divergent if } \underline{\lim_{x\to\infty}} \frac{e^x f(e^x)}{f(x)} > 1.$$

For, in the first case, if ρ is any number between the maximum limit and unity, we can find ξ so that

$$e^x f(e^x) < \rho f(x), \qquad \text{if} \quad x > \xi.$$

Thus $$\int_\xi^X e^x f(e^x)\,dx < \rho \int_\xi^X f(x)\,dx, \quad \text{if} \quad X > \xi,$$

or, changing the independent variable to e^x in the left-hand integral, we have†

$$\int_\zeta^Z f(x)\,dx < \rho \int_\xi^X f(x)\,dx,$$

where $$Z = e^X, \quad \zeta = e^\xi.$$

That is, $$(1-\rho)\int_\zeta^Z f(x)\,dx < \rho\left\{\int_\xi^X f(x)\,dx - \int_\zeta^Z f(x)\,dx\right\}$$

$$\text{or} \quad < \rho\left[\int_\xi^\zeta f(x)\,dx - \int_X^Z f(x)\,dx\right].$$

Or, again, since the last term in the bracket is positive (because $Z = e^X$ is greater than X), we have

$$(1-\rho)\int_\zeta^Z f(x)\,dx < \rho\int_\xi^\zeta f(x)\,dx.$$

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* *Bulletin des Sciences Mathématiques*, 1871, t. 2, p. 250.

† The reader is advised to use the geometrical representation of $\int f(x)\,dx$ as the area of the curve $y = f(x)$ when following out the argument.

As this inequality is true for any value of X greater than ξ, it is clear that the infinite integral $\int^{\infty} f(x)\,dx$ must converge; and, therefore, so also does the series $\Sigma f(n)$, by the integral test of Art. 11.

In the second case, ξ can be found so that

$$e^x f(e^x) \geqq f(x), \qquad \text{if} \quad x \geqq \xi.$$

As above, this gives

$$\int_{\zeta}^{Z} f(x)\,dx \geqq \int_{\xi}^{X} f(x)\,dx, \quad \text{if} \quad X > \xi,$$

or

$$\int_{X}^{Z} f(x)\,dx \geqq \int_{\xi}^{\zeta} f(x)\,dx, \quad \text{if} \quad X > \xi.$$

This indicates that the integral $\int^{\infty} f(x)\,dx$ is divergent, because, no matter how great X may be, a number $Z = e^X$ can be found such that $\int_X^Z f(x)\,dx$ is greater than a certain constant K; compare the argument of Art. 7 (3). Thus the series $\Sigma f(n)$ is divergent.

Ermakoff's tests include the whole of the logarithmic scale.

For example, consider

$$f(x) = 1/\{x \,.\, \log x \,.\, [\log(\log x)]^p\},$$

then

$$e^x f(e^x) = e^x/\{e^x \,.\, x \,.\, [\log x]^p\}.$$

Thus

$$e^x f(e^x)/f(x) = [\log(\log x)]^p/[\log x]^{p-1},$$

and so

$$\lim_{x\to\infty} e^x f(e^x)/f(x) = 0, \quad \text{if } p > 1,$$

or

$$= \infty. \quad \text{if } p \leqq 1. \quad \text{(Art. 160.)}$$

That is, the series $\Sigma f(n)$ converges if $p > 1$ and diverges if $p \leqq 1$.

It is easy to see that if $\phi(x)$ is a function which steadily increases with x, in such a way that $\phi(x) > x$, the proof above may be generalised to give Ermakoff's tests:

(i) convergence, if $\overline{\lim\limits_{x\to\infty}} \dfrac{\phi'(x) f(\phi(x))}{f(x)} < 1$,

(ii) divergence, if $\underline{\lim\limits_{x\to\infty}} \dfrac{\phi'(x) f(\phi(x))}{f(x)} > 1$.

15. Another sequence of tests.

Although the following sequence is of less importance than the ratio-tests in ordinary work, it is of theoretical interest, giving a continuation of Cauchy's test in Art. 10.

We have seen in Art. 11, that if $\Sigma F'(n)$ is divergent, $\Sigma F'(n)/[F(n)]^p$ converges only if $p > 1$. This gives the following test:

Σa_n converges if $\underline{\lim} \dfrac{\log[F'(n)/a_n]}{\log[F(n)]} > 1$

and diverges if $\overline{\lim} \dfrac{\log[F'(n)/a_n]}{\log[F(n)]} < 1$

where $F'(n)$ is positive but tends *steadily* to 0.

For, in the first case, as on previous occasions, we can find a value of $p > 1$ and an index m such that

$$\frac{\log[F'(n)/a_n]}{\log[F(n)]} > p, \quad \text{if } n > m,$$

or
$$a_n < F'(n)/[F(n)]^p, \quad \text{if } n > m.$$

This shews that Σa_n converges, by the principle of comparison.

But, in the second case, there is an index m such that

$$F'(n)/a_n \leqq F(n), \quad \text{if } n > m,$$

or
$$a_n \geqq F'(n)/F(n), \quad \text{if } n > m.$$

This shews that Σa_n diverges.

Special examples of this test are given by

(1) $F(n) = n$; and the function to examine is

$$\frac{\log(1/a_n)}{\log n}.$$

(2) $F(n) = \log n$; and the function is

$$\frac{\log(1/na_n)}{\log(\log n)}.$$

(3) $F(n) = \log(\log n)$; then the function is

$$\frac{\log\{1/(n \,.\, \log n \,.\, a_n)\}}{\log[\log(\log n)]}$$

and so on.

The test (1) can be transformed into another shape, first given by Jamet, in which the relation to Cauchy's test is easily recognised.

If we write $\lambda = \log(1/a_n)$, it is easy to see that

$$1 - \lambda/n < a_n^{\frac{1}{n}} < 1/(1 + \lambda/n),$$

so that
$$\lambda > n(1 - a_n^{\frac{1}{n}}) > \lambda/(1 + \lambda/n).$$

Thus we have
$$\overline{\lim}\, \frac{\lambda}{\log n} = \overline{\lim}\, \frac{n}{\log n}(1 - a_n^{\frac{1}{n}}),$$

provided that $\lim(\lambda/n) = 0$; and, if this condition is not satisfied, Cauchy's test will settle the question. So in all cases of practical interest, the test will be

$$\text{(C)} \quad \underline{\lim}\, \frac{n}{\log n}(1 - a_n^{\frac{1}{n}}) > 1, \qquad \text{(D)} \quad \overline{\lim}\, \frac{n}{\log n}(1 - a_n^{\frac{1}{n}}) < 1.$$

Similarly it can be proved that test (2) can be replaced by

$$\overline{\lim}\, \frac{1}{\log(\log n)}[n(1 - a_n^{\frac{1}{n}}) - \log n] \lesseqgtr 1.$$

For example, this form proves that the series $\Sigma\left(1 - \frac{x}{n}\log n\right)^n$ diverges if $0 \leqq x \leqq 1$, but converges for $x > 1$.

16. General notes on series of positive terms.

Although the rules which we have established are sufficient to test the convergence of all series which present themselves naturally in elementary analysis, yet it is impossible to frame any rule which will give a decisive test for an artificially constructed series. In other words, *whatever rule is given, a series can be invented for which the rule fails to give a decisive result.*

The following notes (1)–(3) and (8) shew how certain rules which appear plausible at first sight have been proved to be either incorrect or insufficient. Notes (4)–(7) shew that however slowly a series may diverge (or converge) we can always construct series which diverge (or converge) still more slowly; and thus no test of comparison can be sufficient for all series.

Other interesting questions in this connexion have been considered by Hadamard (*Acta Mathematica*, t. 18, 1894, p. 319, and t. 27, 1903, p. 177).

(1) Abel has pointed out that there *cannot* be a positive function $\phi(n)$ such that the two conditions

$$\text{(i)}\ \lim \phi(n)\,.\,a_n=0, \qquad \text{(ii)}\ \lim \phi(n)\,.\,a_n>0$$

are *sufficient*, the first for the convergence, the second for the divergence of any series Σa_n.

For, if so, $\Sigma[\phi(n)]^{-1}$ would diverge; and therefore, if

$$M_n=[\phi(1)]^{-1}+[\phi(2)]^{-1}+\ldots+[\phi(n)]^{-1},$$

the sequence M_n would be an increasing sequence tending to ∞.

Hence the series $\Sigma(M_n-M_{n-1})/M_n$ would diverge also (Art. 11); but

$$\phi(n)(M_n-M_{n-1})/M_n=1/M_n,$$

so that

$$\lim \phi(n)(M_n-M_{n-1})/M_n=0,$$

contradicting the first condition.

(2) Pringsheim has proved that there *cannot* be a positive function $\phi(n)$ tending to ∞, such that the condition

$$\lim \phi(n)\,.\,a_n \leqq G \qquad (G \geqq 0)$$

is *necessary* for the convergence of Σa_n. In fact, for *any* such function $\phi(n)$ and for *any* convergent series, the terms of the series can be so rearranged that

$$\overline{\lim}\ \phi(n)\,.\,a_n=\infty\,.$$

See *Math. Annalen*, Bd. 35, p. 344.

(3) Pringsheim has proved that there cannot be a positive function $\phi(n)$ such that the condition

$$\lim \phi(n)\,.\,a_n>0$$

is *necessary* for the divergence of Σa_n. In fact, for *any* function

$\phi(n)$ and *any* divergent series, the terms of the series can be so arranged that

$$\underline{\lim}\, \phi(n) \,.\, a_n = 0,$$

provided that the terms of the series tend to zero.

See *Math. Annalen*, Bd. 35, p. 358.

(4) Abel remarked that if Σa_n is divergent, a second series Σb_n can be found which is also divergent, but such that

$$\lim (b_n/a_n) = 0.$$

For, write $M_n = a_1 + a_2 + \ldots + a_n$, $b_n = a_n/M_n = (M_n - M_{n-1})/M_n$.

The series Σb_n diverges by Art. 11; and

$$\lim (b_n/a_n) = \lim (1/M_n) = 0.$$

(5) du Bois Reymond shewed that if Σa_n is convergent, a second convergent series Σb_n can be found which has the property

$$\lim (b_n/a_n) = \infty.$$

For, write $s_n = a_1 + a_2 + \ldots + a_n$, $s = \lim s_n$,

$$1/M_1 = s, \quad 1/M_{n+1} = s - s_n = a_{n+1} + a_{n+2} + \ldots \text{ to } \infty.$$

Then $M_n \to \infty$; and consequently the series Σb_n converges if

$$b_n = (M_{n+1} - M_n)/M_n^q M_{n+1} = a_n M_n^{1-q},$$

provided that q is positive (see Art. 11).

But if $q < 1$, it is evident that $b_n/a_n \to \infty$.

(6) Stieltjes shewed that if $u_1, u_2, u_3, \ldots$ is a *decreasing* sequence, tending to zero as a limit, a divergent series Σd_n can be found so that $\Sigma u_n d_n$ is convergent.

For, write $M_n = 1/u_n$; then if $d_n = (M_{n+1} - M_n)/M_{n+1}$ the series Σd_n is divergent (Art. 11). But

$$u_n d_n = \frac{M_{n+1} - M_n}{M_n M_{n+1}} = \frac{1}{M_n} - \frac{1}{M_{n+1}},$$

so that $\Sigma u_n d_n$ converges to the sum $1/M_1 = u_1$.

(7) Stieltjes also proved that if $v_1, v_2, v_3, \ldots$ is an *increasing* sequence, tending to infinity as a limit, a convergent series Σc_n can be found so that $\Sigma v_n c_n$ is divergent.

For, write $c_n = 1/v_n - 1/v_{n+1}$, which makes Σc_n a convergent series; then $v_n c_n = (v_{n+1} - v_n)/v_{n+1}$, so that $\Sigma v_n c_n$ is divergent.

(8) Finally, *even when the terms of the series Σa_n steadily decrease*, the following results have been found by Pringsheim:

However fast the series Σc_n^{-1} may *converge*, yet there are always *divergent* series Σa_n such that $\underline{\lim}\, c_n a_n = 0$.

However slowly $\phi(n)$ may increase to ∞ with n, there are always *convergent* series Σa_n, for which $\overline{\lim}\, n \,.\, \phi(n) \,.\, a_n = \infty$ (although $\lim n \,.\, a_n = 0$ by Art. 9).

See *Math. Annalen*, Bd. 35, pp. 347, 356.

EXAMPLES.

1. Test the convergence of the series Σa_n, where a_n is given by the following expressions:

$$\frac{1}{1+n^2},\quad \frac{1+n}{1+n^2},\quad \frac{1}{1+x^n},\quad \frac{1}{(\log n)^a},\qquad \text{(Arts. 8, 9)}$$

$$\frac{(n!)^2}{(2n)!}x^n,\quad \frac{n^4}{n!},\quad \frac{m(m+1)\dots(m+n-1)}{n!},\qquad \text{(Art. 12·2)}$$

$$\frac{1}{n^{a+b/n}},\quad \frac{n^q}{(n+1)^{p+q}},\quad \frac{1}{(\log n)^n},\quad \frac{1}{(\log n)^{\log n}},\quad \frac{1}{[\log(\log n)]^{\log n}},\quad \frac{1}{(\log n)^{\log(\log n)}},\qquad \text{(Art. 15)}$$

$$a^{1/n}-1. \quad \text{(Ex. 21, Ch. I., and Art. 11.)}$$

2. Prove that if $b-1 > a > 0$, the series

$$1+\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\dots$$

converges to the sum $(b-1)/(b-a-1)$.

Shew also that the sum of

$$\frac{a}{b}+2\frac{a(a+1)}{b(b+1)}+3\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\dots$$

is $a(b-1)/(b-a-1)(b-a-2)$, if $b-2 > a > 0$.

[If the first series is denoted by $u_0+u_1+u_2+\dots$, we get

$$(b+n)u_{n+1}=(a+n)u_n,$$

which gives $\qquad (b-a-1)u_{n+1}=(a+n)u_n-(a+n+1)u_{n+1}.$

Hence $(b-a-1)(s_n-u_0)=au_0-(a+n)u_n$ by addition. But when Σu_n is convergent we see that $\lim (nu_n)=0$ by Art. 9, since the terms steadily decrease. Hence $\lim s_n$ can be found.

The second series can be expressed as the difference between two series of the first type.]

3. Prove that the series

$$1+\frac{a+1}{b+1}+\frac{(a+1)(2a+1)}{(b+1)(2b+1)}+\frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\dots$$

converges if $b > a > 0$ and diverges if $a \geqq b > 0$.

4. Prove that the series

$$1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{7}+\dots$$

converges.

Shew also that

$$1+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+\dots$$

converges if $p>2$, and otherwise diverges (Art. 12. 2).

5. Prove that, if $0<q<1$, $\Sigma q^{n^2}x^n$ converges for any positive value of x.

6. Prove that $1+\frac{1}{2^\beta}+\frac{1}{3^\alpha}+\frac{1}{4^\beta}+\frac{1}{5^\alpha}+\dots$ converges if $\alpha>\beta>1$, but that the ratio of two consecutive terms oscillates between 0 and ∞ (Art. 10).

7. If Σa_n is a divergent series of positive terms and $f(x)$ is subject to the conditions of Art. 11, prove that $\Sigma a_n f(s_n)$ converges if $\int^\infty f(x)\,dx$ is convergent; and that $\Sigma a_n f(s_{n-1})$ diverges if the integral is divergent.

[DE LA VALLÉE POUSSIN.]

8. If a_{n+1}/a_n can be expressed as the quotient of two polynomials in n, $P(n)/Q(n)$, of the same degree k, whose highest term is n^k, and if the highest term in $Q(n)-P(n)$ is An^{k-1}, prove that Σa_n converges if $A>1$, diverges if $A \leqq 1$.

9. Test the convergence of the series Σa_n, where

$$a_n=(2-e)(2-e^{\frac{1}{2}})(2-e^{\frac{1}{3}})\dots(2-e^{\frac{1}{n}}).$$

10. Find limits for the sum

$$\sigma_n=\frac{n}{n^2}+\frac{n}{1+n^2}+\frac{n}{2^2+n^2}+\dots+\frac{n}{(n-1)^2+n^2}$$

by means of the integral

$$\int_0^n \frac{n\,dx}{x^2+n^2}=\int_0^1\frac{dt}{1+t^2},$$

and deduce that $\sigma_n\to\frac{1}{4}\pi$.

11. Prove that if p approaches zero through positive values,

$$\lim_{p\to 0} p\sum_1^\infty n^{-(1+p)}=1\,;$$

and that

$$\lim_{p\to 0}\left(\sum_1^\infty \frac{1}{n^{1+p}}-\frac{1}{p}\right)=C,$$

where C is Euler's constant. [DIRICHLET.]

[To prove the latter part, note that (as in Art. 11) if

$$f(\nu)=\sum_\nu^\infty \frac{1}{n^{1+p}}-\int_\nu^\infty\frac{dx}{x^{1+p}},$$

$f(\nu)$ is positive but less than $1/\nu$. The desired limit is that of

$$f(1)=1+\frac{1}{2^{1+p}}+\frac{1}{3^{1+p}}+\dots+\frac{1}{(\nu-1)^{1+p}}-\int_1^\nu\frac{dx}{x^{1+p}}+f(\nu).$$

If we now let $p\to 0$, we obtain the result

$$\overline{\lim_{p\to 0}}\, f(1)=1+\frac{1}{2}+\dots+\frac{1}{\nu}-\log\nu+\overline{\lim_{p\to 0}}\, f(\nu).$$

Now the right-hand side contains ν, which does not appear on the left; and if we make $\nu \to \infty$, the right-hand side $\to C$.

Thus
$$\overline{\lim_{p \to 0}} f(1) = C.$$
Accordingly the maximum and minimum limits of $f(1)$ are both equal to C; or in symbols
$$\lim_{p \to 0} f(1) = C.]$$

12. More generally, if $M_n = an + b_n$ (where $|b_n|$ is less than a fixed value, and M_n is never zero),
$$\lim_{p \to 0} ap \sum_1^\infty M_n^{-(1+p)} = 1,$$
and $\lim_{p \to 0}\left(\sum_1^\infty \frac{a^{1+p}}{M_n^{1+p}} - \frac{1}{p}\right)$ exists and is finite. [DIRICHLET.]

If M_n tends *steadily* to infinity with n, and
$$d_n = (M_{n+1} - M_n)/M_{n+1},$$
then
$$\lim_{p \to 0}\left(p \sum_1^\infty d_n M_n^{-p}\right) = 1, \qquad \text{if } M_{n+1}/M_n \to 1,$$
$$\text{or } = (1 - 1/c)/\log c, \qquad \text{if } M_{n+1}/M_n \to c > 1,$$
$$\text{or } = 0, \qquad \text{if } M_{n+1}/M_n \to \infty.$$
[PRINGSHEIM, *Math. Annalen*, Bd. 37.]

Interesting examples are given by $M_n = n^2$, 2^n, $n!$.

13. Utilise the Theorem of Art. 147 (Appendix) to shew that if (u_n) decreases steadily, the condition $\lim (nu_n) = 0$ is necessary (Art. 9) for the convergence of Σu_n, by writing
$$a_n = (S_n/u_n) - n, \quad b_n = 1/u_n,$$
so that
$$\frac{a_n}{b_n} = S_n - nu_n, \quad \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = S_{n-1}.$$
[CESÀRO.]

If $u_n = \left(1 - \frac{1}{n}\log n\right)^n$, prove that $\lim(nu_n) = 1$, and deduce the divergence of Σu_n (compare Art. 15).

14. If Σa_n, Σb_n are both convergent, so also is $\Sigma(a_n b_n)^{\frac{1}{2}}$. But Σa_n, Σb_n may *both* diverge and yet $\Sigma(a_n b_n)^{\frac{1}{2}}$ may converge; a fact illustrated by
$$1 + \frac{1}{2^3} + \frac{1}{3} + \frac{1}{4^3} + \frac{1}{5} + \frac{1}{6^3} + \dots \quad \text{and} \quad 1 + \frac{1}{2} + \frac{1}{3^3} + \frac{1}{4} + \frac{1}{5^3} + \frac{1}{6} + \dots.$$

If Σa_n converges, so also does $\Sigma(a_n a_{n+1})^{\frac{1}{2}}$; but the converse is not true, as may be seen from either of the two series just written down.

On the other hand, if (a_n) is *monotonic*, the convergence of $\Sigma(a_n a_{n+1})^{\frac{1}{2}}$ implies that of Σa_n. [PRINGSHEIM.]

15. Use the preceding example to prove that if Σa_n^2 is convergent, so also is $\Sigma a_n/n$.

16. If the function $f(x)$ is positive from $x = 0$ to ∞, and if the integral $\int_0^\infty \{f(x)\}^2 dx$ is convergent (at the upper limit), prove that the two integrals
$$\int_0^\infty f(x)\phi(x)\,dx \quad \text{and} \quad \int_0^\infty \{\phi(x)\}^2\,dx$$

are both convergent, where $\phi(x)$ is the "average" of $f(x)$ defined by

$$\phi(x)=\frac{1}{x}\int_0^x f(t)\,dt. \qquad \text{[Hardy.]}$$

For $\phi=f+(\phi-f)$, so that $\phi^2 \leqq 2f^2+2(\phi-f)^2$. Also $x\phi'+\phi=f$, and so we find that

$$\frac{d}{dx}(x\phi^2)=2f\phi-\phi^2.$$

Thus the previous inequality can be arranged in either of the forms

$$2f\phi \leqq 4f^2-\frac{d}{dx}(x\phi^2) \quad \text{or} \quad \phi^2 \leqq 4f^2-2\frac{d}{dx}(x\phi^2),$$

giving $$\int_0^X f\phi\,dx \leqq 2\int_0^X f^2\,dx-\tfrac{1}{2}X\{\phi(X)\}^2 \leqq 2\int_0^X f^2\,dx$$

and $$\int_0^X \phi^2\,dx \leqq 4\int_0^X f^2\,dx-2X\{\phi(X)\}^2 \leqq 4\int_0^X f^2\,dx.$$

From these inequalities the convergence is obvious.

17. If a_n is positive and the series $\Sigma a_n{}^2$ is convergent, prove that the series $\Sigma a_n b_n$ and $\Sigma b_n{}^2$ are both convergent, where b_n is the arithmetic mean of $a_1, a_2, \ldots, a_n$, so that $nb_n=a_1+a_2+\ldots+a_n$. [Hardy.]

[Apply the method of the last example, making a, b to correspond to f, ϕ, respectively, and using the identity $nb_n=a_n+(n-1)b_{n-1}$. It will be found that

$$\left(1-\frac{2}{n+1}\right)b_n{}^2 \leqq \left(4-\frac{2}{n}\right)a_n{}^2+2(B_{n-1}-B_n) < 4a_n{}^2+2(B_{n-1}-B_n),$$

where $$B_n=n^2b_n{}^2/(n+1).$$

Hence, on summing for 2, 3, ..., n, we find

$$\frac{1}{3}b_2{}^2+\frac{2}{4}b_3{}^2+\frac{3}{5}b_4{}^2+\ldots+\left(1-\frac{2}{n+1}\right)b_n{}^2 < 4(a_2{}^2+a_3{}^2+\ldots+a_n{}^2)+2(B_1-B_n)$$
$$< 4(a_1{}^2+a_2{}^2+\ldots+a_n{}^2)+b_1{}^2.$$

Multiplying by 2, we see that

$$b_1{}^2+b_2{}^2+b_3{}^2+\ldots+b_n{}^2 < 8(a_1{}^2+a_2{}^2+\ldots+a_n{}^2)+(3b_1{}^2+\tfrac{1}{3}b_2{}^2).$$

Hence $\Sigma b_n{}^2$ is convergent. The convergence of $\Sigma a_n b_n$ follows because

$$a_n b_n \leqq \tfrac{1}{2}(a_n{}^2+b_n{}^2).]$$

18. If $$\Sigma_n=1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1},$$

shew that $$\Sigma_n \to \tfrac{1}{2}(\log n+C)+\log 2,$$

where C is Euler's constant (see Art. 11).

19. By using Ex. 18 or otherwise, prove that

$$\sum_1^\infty [n(4n^2-1)]^{-1}=2\log 2-1, \quad \sum_1^\infty [n(9n^2-1)]^{-1}=\tfrac{3}{2}(\log 3-1).$$

20. Shew that, with the notation of Ex. 18,

$$\sum_1^\nu \frac{1}{n(36n^2-1)} = -3 + 3\Sigma_{3\nu+1} - \Sigma_\nu - \left(1 + \frac{1}{2} + \dots + \frac{1}{\nu}\right).$$

Deduce that $$\sum_1^\infty \frac{1}{n(36n^2-1)} = -3 + \tfrac{3}{2}\log 3 + 2\log 2.$$ [*Math. Trip.* 1905.]

21. Prove similarly that

$$\sum_1^\nu \frac{12n^2-1}{n(4n^2-1)^2} = \frac{1}{2\nu+1} - \frac{1}{(2\nu+1)^2} + 2\Sigma_\nu - \left(1 + \frac{1}{2} + \dots + \frac{1}{\nu}\right),$$

and that $$\sum_1^\infty \frac{12n^2-1}{n(4n^2-1)^2} = 2\log 2.$$ [*Math. Trip.* 1896.]

22. Shew that

$$\sum_1^\infty \frac{n}{(4n^2-1)^2} = \frac{1}{8}, \quad \sum_1^\infty \frac{1}{n(4n^2-1)^2} = \frac{3}{2} - 2\log 2.$$

23. Examine the convergence of $\Sigma x^{\phi(n)}$, where x is positive; in particular, if $\phi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, or if $\phi(n) = \log n$, prove that the series converges if $x < 1/e$. [Art. 15.]

24. Shew that

$$\sum_1^\infty \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_1^\infty \frac{n-1}{n!} = 1, \quad \sum_1^\infty \frac{2}{(t+n-1)(t+n)(t+n+1)} = \frac{1}{t(t+1)},$$

and $$\sum_1^\infty \frac{3}{(t+n-1)(t+n)(t+n+1)(t+n+2)} = \frac{1}{t(t+1)(t+2)}.$$

CHAPTER III.

SERIES IN GENERAL.

17. The only general test of convergence is simply a transformation of the condition for convergence of the sequence s_n (Art. 3); namely, that we must be able to find m, so that $|s_n - s_m| < \epsilon$, provided only that $n > m$. If we express this condition in terms of the series Σa_n, we get the form:

It must be possible to find m, corresponding to an arbitrary positive number ϵ, so that

$$|a_{m+1} + a_{m+2} + \ldots + a_{m+p}| < \epsilon,$$

no matter how large p may be.

It is an obvious consequence that in every convergent series *

$$\lim_{n \to \infty} a_n = 0, \quad \lim_{n \to \infty} (a_{n+1} + a_{n+2} + \ldots + a_{n+p}) = 0.$$

But these conditions are not sufficient unless p is allowed to take all possible forms of variation with n; and so they are not practically useful. However, it is sometimes possible to infer non-convergence by using a special form for p and shewing that then the limit is not zero (as in Art. 7 (3)).

We are obliged therefore to employ special tests, which suffice to shew that a large number of interesting series are convergent.

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* It is clear from the examples in Chapter II. that the condition $\lim a_n = 0$ does not exclude divergent series; but it does not even exclude *oscillatory* series. For consider

$$1 - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{1}{5} - \ldots,$$

where $\underline{\lim}\, s_n = 0$, $\overline{\lim}\, s_n = 1$, and yet the terms tend to zero.

18. A convergent series of positive terms remains convergent when each term a_n is multiplied by a factor v_n whose numerical value does not exceed a constant k.

For since Σa_n is convergent, the index m can be chosen so that $\sum_{m+1}^{m+p} a_n < \epsilon/k$, however small ϵ may be.

But
$$\left|\sum_{m+1}^{m+p} a_n v_n\right| \leqq \sum_{m+1}^{m+p} |a_n v_n|$$

and
$$|a_n v_n| = a_n |v_n| \leqq a_n k.$$

Thus
$$\left|\sum_{m+1}^{m+p} a_n v_n\right| \leqq k \sum_{m+1}^{m+p} a_n < \epsilon,$$

and therefore the series $\Sigma a_n v_n$ is convergent.

Two special cases of this theorem deserve mention :

(1) *A series $\Sigma \alpha_n$ is convergent, if the series of its absolute values $\Sigma|\alpha_n|$ is convergent.*

For here $a_n = |\alpha_n|$ and $v_n = \alpha_n/a_n = \pm 1$.

Such series are called *absolutely convergent.*

(2) *A series is convergent if its terms are numerically not greater than the corresponding terms of a convergent series of* positive *terms.*

The reader should observe that we cannot apply this method if an infinity of terms are negative in the series which is known to converge. An example is afforded by Ex. 2 below.

Ex. 1. If we take $a_n = n^{-p}$, we know from Art. 11 that Σa_n converges if $p > 1$. The present theorem enables us to deduce the convergence of the two series

$$\left.\begin{aligned} &1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \frac{1}{6^p} + \dots \\ &1 + \frac{1}{2^p} - \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} - \frac{1}{6^p} + \dots \end{aligned}\right\} \qquad p > 1.$$

It will appear from Arts. 19, 23 below that the first of these series converges, but the second diverges if $0 < p \leqq 1$.

Ex. 2. The series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$ obviously converges to the sum 0. Now take the factors (v_n) to be $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \dots$, so that $|v_n| \leqq 1$. The new series is

$$1 - \frac{1}{1.2} + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{3} - \frac{1}{3.4} + \frac{1}{4} - \frac{1}{4.5} + \dots.$$

The sum of the first $2n$ terms is

$$s_{2n}=\frac{2-1}{1.2}+\frac{3-1}{2.3}+\frac{4-1}{3.4}+\frac{5-1}{4.5}+\ldots+\frac{(n+1)-1}{n(n+1)}$$

$$=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n+1}\to\log n+C-1. \qquad \text{(Art. 11.)}$$

Thus $\lim s_{2n}=\infty$.

But $s_{2n-1}>s_{2n}$, and so also $\lim s_{2n-1}=\infty$.

Thus *the new series is divergent.*

The reason for the failure of the theorem is that the original series contains an infinity of negative terms; and that the series ceases to converge when these terms are made positive (Art. 11).

It is easy to see that the foregoing theorem can also be stated in the form:

An **absolutely** *convergent series remains convergent if each term is multiplied by a factor whose numerical value does not exceed a constant k.*

19. Alternating series.

Most series in common use are absolutely convergent; but a number of others can be proved to converge by the rule:

If the terms of a series $\Sigma(-1)^{n-1}v_n$ are alternately positive and negative, and never increase in numerical value, the series will converge, provided that the terms tend to zero as a limit.

For it is plain that

$$s_{2n}=(v_1-v_2)+(v_3-v_4)+\ldots+(v_{2n-1}-v_{2n}),$$

and since each of these brackets is positive (or at least not negative), the sequence of terms (s_{2n}) never decreases as n increases.

Also $\quad s_{2n+1}=v_1-(v_2-v_3)-(v_4-v_5)-\ldots-(v_{2n}-v_{2n+1}),$

and so the sequence (s_{2n+1}) never increases.

Further $\quad s_{2n}=s_{2n+1}-v_{2n+1}<v_1$

and $\quad s_{2n+1}=s_{2n}+v_{2n+1}>0.$

Hence, by Art. 2, the sequence (s_{2n}) has a limit not greater than v_1 and (s_{2n+1}) has a limit not less than 0. But these two limits must be equal since $\lim v_{2n+1}=0$, so that

$$\lim s_{2n}=\lim s_{2n+1}.$$

Hence the series converges to a sum lying between 0 and v_1.

Ex. 1. The series already mentioned in Art. 18, **Ex.** 1,

$$1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\frac{1}{5^p}-\frac{1}{6^p}+\ldots,$$

is now seen to converge, provided that $0<p\leqq 1$.

In the special case $p=1$ we get the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\dots,$$

which is easily seen to be equal to log 2. For, by Art. 11,

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n}\to\log(2n)+C,$$

and
$$2\left(\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2n}\right)\to\log n+C.$$

Thus
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots-\frac{1}{2n}\to\log 2.$$

The diagram indicates the first eight terms in the sequence (s_n) obtained from this series by addition; the dotted lines indicate the monotonic convergence of the two sequences (s_{2n}), (s_{2n+1}).

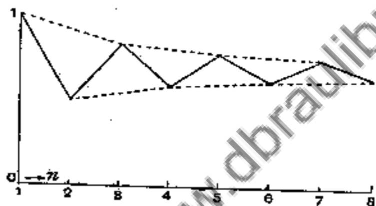


FIG 10.

It is obvious that *if the sequence* (v_n) *never increases, but approaches a limit* l, *not equal to zero, the series* $\Sigma(-1)^{n-1}v_n$ *will oscillate between two values whose difference is equal to* l; in fact by the previous argument we have $\lim s_{2n+1}=\lim s_{2n}+l$.

A special case of interest is given by the following test, which is similar to that of Art. 12·2:

If v_n/v_{n+1} *can be expressed in the form*

$$\frac{v_n}{v_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^p}\right),\quad p>1,$$

the series $\Sigma(-1)^{n-1}v_n$ *is convergent if* $\mu>0$, *oscillatory if* $\mu\leqq 0$.

For if $\mu>0$, after a certain stage we shall have

$$\frac{\mu}{n}>O\left(\frac{1}{n^p}\right),$$

so that $v_n>v_{n+1}$; and further (by Art. 39, Ex. 4), $\lim v_n=0$. But, on the other hand, if $\mu=0$, it is clear (from Art. 39) that $\lim v_n$ is not zero, and so the series must oscillate. And, if $\mu<0$, after a certain stage we shall have $v_n<v_{n+1}$, so that $\lim v_n$ cannot be zero, leading to oscillation again.

Ex. 2. Take the series

$$1-\frac{\alpha.\beta}{1.\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}-\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}+\dots.$$

Here $$\frac{v_n}{v_{n+1}}=\frac{n(\gamma+n-1)}{(\alpha+n-1)(\beta+n-1)};\ \mu=\gamma-\alpha-\beta+1.$$

So the series converges if $\gamma+1>\alpha+\beta$. It is also instructive to apply the method of Ex. 4, Art. 12·2.

It should be observed that if the positive and negative terms in the series form two *separately* decreasing sequences there is no reason to suppose that the theorem is still necessarily true; and in fact it is easy to construct examples of the failure, such as

$$1-\frac{1}{2}+\frac{1}{3^2}-\frac{1}{4}+\frac{1}{5^2}-\frac{1}{6}+\frac{1}{7^2}-\frac{1}{8}+\dots.$$

This is easily recognised as divergent; for the sum of the first n positive terms is less than

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots+\frac{1}{n^2},$$

and is therefore less than 2 (Art. 11). But the sum of the first n negative terms is

$$-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)\to-\frac{1}{2}(\log n+C);$$

and consequently the sum of the first $2n$ terms of the given series tends to $-\infty$ as its limit.

20. Abel's Lemma.

If the sequence (v_n) of positive terms never increases, the sum $\sum_1^p a_n v_n$ lies between Hv_1 and hv_1, where H and h are the upper and lower limits of the sums

$$a_1,\quad a_1+a_2,\quad a_1+a_2+a_3,\ \dots,\ a_1+a_2+\dots+a_p.$$

For, with the usual notation, we have

$$a_1=s_1,\quad a_2=s_2-s_1,\ \dots,\ a_p=s_p-s_{p-1}.$$

Thus

$$\sum_1^p a_n v_n=s_1v_1+(s_2-s_1)v_2+(s_3-s_2)v_3+\dots+(s_p-s_{p-1})v_p$$
$$=s_1(v_1-v_2)+s_2(v_2-v_3)+\dots+s_{p-1}(v_{p-1}-v_p)+s_pv_p. \quad \text{(A)}$$

Now the factors (v_1-v_2), (v_2-v_3), ..., $(v_{p-1}-v_p)$, v_p are *never negative*, and consequently the sum lies between

$$H(v_1-v_2)+H(v_2-v_3)+\dots+H(v_{p-1}-v_p)+Hv_p=Hv_1,$$

and

$$h(v_1-v_2)+h(v_2-v_3)+\dots+h(v_{p-1}-v_p)+hv_p=hv_1.$$

Hence
$$hv_1 < \sum_1^p a_n v_n < H v_1.$$

It follows that
$$\left|\sum_1^p a_n v_n\right| < K v_1,$$

where K is the greater of $|H|$ and $|h|$; that is, K is the upper limit of $|s_1|, |s_2|, \ldots, |s_p|$.

It is sometimes useful to obtain closer limits for $\Sigma a_n v_n$; suppose that H_m, h_m denote the upper and lower limits of $s_m, s_{m+1}, \ldots, s_p$ while H, h are those of $s_1, s_2, \ldots, s_{m-1}$. Then exactly the same argument gives

$$h(v_1 - v_m) + h_m v_m < \sum_1^p a_n v_n < H(v_1 - v_m) + H_m v_m.$$

We can deal with the case of $\Sigma a_n M_n$, where (M_n) is an increasing sequence, by writing $v_n = M_p - M_n$.

21. It is often convenient to infer the convergence of a series from one which is not absolutely convergent. For this purpose the following theorem may be used:

A convergent series Σa_n (which need not converge **absolutely**) *remains convergent if its terms are each multiplied by a factor u_n, provided that the sequence (u_n) is monotonic, and that $|u_n|$ is less than a constant k.* **(Abel's test.)**

Under these conditions (u_n) converges to a limit u; and let us write $v_n = u - u_n$ when (u_n) is an increasing sequence, but $v_n = u_n - u$ when (u_n) is decreasing. Then it is clear that *the sequence (v_n) never increases and converges to zero as a limit.* Now

$$a_n u_n = a_n u - a_n v_n, \quad \text{or} \quad a_n u + a_n v_n,$$

so that it will suffice to prove the convergence of $\Sigma a_n v_n$ in order to infer the convergence of $\Sigma a_n u_n$. But by Abel's lemma

$$\left|\sum_{m+1}^{m+p} a_n v_n\right| < K v_{m+1} < K v_1,$$

where K is the upper limit of the sums

$$|a_{m+1}|, \quad |a_{m+1} + a_{m+2}|, \quad |a_{m+1} + a_{m+2} + a_{m+3}|, \ \ldots,$$
$$|a_{m+1} + a_{m+2} + a_{m+3} + \ldots + a_{m+p}|.$$

Now, since Σa_n is convergent, m can be chosen so that $K \leqq \epsilon$, no matter how small ϵ is; thus $\left|\sum_{m+1}^{m+p} a_n v_n\right|$ is less than ϵv_1, and consequently $\Sigma a_n v_n$ is convergent.

The reader will observe that the series Σa_n is not subject to such stringent conditions as in Art. 18; but to counterbalance this, the factors v_n are subject to *more* stringent conditions.

Ex. 1. If we take the series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$ (already used in Ex. 2 of Art. 18) and employ the *monotonic* sequence of factors

$$0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{3}{4}, \dots,$$

we obtain the series

$$0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \dots,$$

which must therefore be convergent. To verify that this is the case, we observe that

$$s_{2n-1} = -\left(\frac{1}{2} - \frac{1}{2^2}\right) - \left(\frac{1}{3} - \frac{2}{3^2}\right) - \dots - \left(\frac{1}{n} - \frac{n-1}{n^2}\right)$$
$$= -\frac{1}{2^2} - \frac{1}{3^2} - \dots - \frac{1}{n^2}.$$

Thus $\lim s_{2n-1}$ exists (by Art. 11 (1)), and since $s_{2n} = s_{2n-1} - 1/(n+1)$, we have also $\lim s_{2n-1} = \lim s_{2n}$. That is, the series converges.

Ex. 2. From our present point of view, it is easy to see why the series in Ex. 2, Art. 18, does not converge; *the sequence of factors employed is not monotonic.*

Another important inference is that if the factors u_n depend in any way on a variable x (subject to the condition of forming a monotonic sequence), the remainder after m terms in the series $\Sigma a_n u_n$ is numerically less than $K(v_1 + |u|)$; and consequently the value of m, which makes this remainder less than ϵ, is *independent of* x, so long as $v_1 + |u|$ is finite.

This property may be expressed by saying that *the convergence of $\Sigma a_n u_n$ is uniform with respect to x.* (See Art. 44, below.)

A special case of this, which was the original object of Abel's lemma, is given by taking $u_n = x^n$, $0 < x \leqq 1$. Then $u = 0$, $v_1 = x \leqq 1$. [Art. 50.]

22. *If an oscillatory series Σa_n has finite maximum and minimum limiting values, it will become convergent if its terms are multiplied by a decreasing sequence (v_n) which tends to zero as a limit.* **(Dirichlet's test.)** *

Abel's lemma gives the inequality

$$\left| \sum_{m+1}^{m+p} a_n v_n \right| < \rho v_{m+1},$$

* It is practically certain that Abel knew of this test: the history is sketched briefly by Pringsheim (*Math. Annalen*, Bd. 25, p. 423, footnote). But to distinguish it clearly from the test of Art. 21, it seems better to use Dirichlet's name, following Jordan (*Cours d'Analyse*, t. 1, § 299).

where ρ is any number not less than the greatest of the sums

$$|a_{m+1}|,\quad |a_{m+1}+a_{m+2}|,\quad |a_{m+1}+a_{m+2}+a_{m+3}|,\quad \ldots,\quad |a_{m+1}+a_{m+2}+\ldots+a_{m+p}|.$$

It is sufficient to suppose ρ not less than each of the differences

$$|s_{m+1}-s_m|,\quad |s_{m+2}-s_m|,\quad \ldots,\quad |s_{m+p}-s_m|.$$

Now, if the extreme limits of s_n are both finite, we can find some constant* l, such that $|s_n|$ is not greater than l, for any value of n. Thus $|s_n-s_m|\leqq 2l$, and we may take $\rho=2l$.

We can now choose m so that $v_{m+1}<\epsilon/l$, and then

$$\left|\sum_{m+1}^{m+p} a_n v_n\right| < 2\epsilon,$$

proving that the series $\Sigma a_n v_n$ converges.

Ex. The series $\Sigma v_n \cos n\theta$, $\Sigma v_n \sin n\theta$ converge if θ is not 0 or a multiple of 2π.

For $$\sum_{m+1}^{m+p} \cos n\theta = \sin(\tfrac{1}{2}p\theta)\,.\,\cos\{m+\tfrac{1}{2}(p+1)\}\,\theta\,.\,\operatorname{cosec}\tfrac{1}{2}\theta$$

and $$\sum_{m+1}^{m+p} \sin n\theta = \sin(\tfrac{1}{2}p\theta)\,.\,\sin\{m+\tfrac{1}{2}(p+1)\}\,\theta\,.\,\operatorname{cosec}\tfrac{1}{2}\theta,$$

so that we can here take $\rho=|\operatorname{cosec}\tfrac{1}{2}\theta|$.

When $\theta=0$, the first series may be convergent or divergent according to the form of v_n; but the second series, being $0+0+0+\ldots$, converges to the sum 0.

If we take $\theta=\pi$, we return to the series $\Sigma(-1)^{n-1}v_n$ already discussed in Art. 19.

It is useful to note that these two series,

$$\Sigma v_n \cos n\theta,\quad \Sigma v_n \sin n\theta,$$

cannot converge absolutely, unless Σv_n is convergent; and if Σv_n converges, we could apply Art. 18 (2) without making use of Dirichlet's test at all.

To prove this statement, we note that

$$|v_n \cos n\theta| \geqq v_n \cos^2 n\theta,\quad |v_n \sin n\theta| \geqq v_n \sin^2 n\theta.$$

Further $$v_n \cos^2 n\theta = \tfrac{1}{2}v_n(1+\cos 2n\theta),$$

and $$v_n \sin^2 n\theta = \tfrac{1}{2}v_n(1-\cos 2n\theta).$$

Now, by what we have just proved, $\Sigma v_n \cos 2n\theta$ is convergent; and so the series of absolute values cannot converge, unless Σv_n converges.

* This constant l will be either the greatest value of $|s_n|$, or (if there is no greatest value) the greater of $|\overline{\lim}\, s_n|$ and $|\underline{\lim}\, s_n|$.

Further special cases are given by

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}\cos\theta+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}\cos 2\theta+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}\cos 3\theta+\ldots$$

and the corresponding series of sines. These both converge if $\gamma+1>\alpha+\beta$; see Ex. 2, Art. 19.

23. A curious theorem, to some extent a kind of converse of Art. 19, is due to Cesàro:

If a series $(\Sigma \pm v_n)$ is convergent, but not absolutely convergent, and if its terms are arranged in descending order of magnitude, the value of p_n/q_n cannot approach any other limit than unity; where p_n is the number of positive terms and q_n the number of negative terms in the first n terms of the series.

Remembering that $p_{r+1}-p_r$ is either 0 or 1, it is easy to see that the sum of the p_n positive terms is

$$p_1v_1+\sum_{r=1}^{n-1}(p_{r+1}-p_r)v_{r+1}$$
$$=p_1(v_1-v_2)+p_2(v_2-v_3)+\ldots+p_{n-1}(v_{n-1}-v_n)+p_nv_n.$$

On combining this with a similar formula for the sum of the q_n negative terms, we deduce that the sum of the first n terms is

$$s_n=(p_1-q_1)(v_1-v_2)+\ldots+(p_{n-1}-q_{n-1})(v_{n-1}-v_n)+(p_n-q_n)v_n.$$

Suppose now, if possible, that $(p_n-q_n)/n$ tends to a positive limit l; then, if l_1 is any positive number less than l, we can find an index m such that $(p_n-q_n)/n>l_1$, if $n\geqq m$.

Hence $$\sum_{m}^{n-1}(p_r-q_r)(v_r-v_{r+1})+(p_n-q_n)v_n$$

$$>l_1\left\{\sum_{m}^{n-1}r(v_r-v_{r+1})+nv_n\right\}>l_1\{mv_m+v_{m+1}+\ldots+v_n\}.$$

But, since the given series is not absolutely convergent, the series Σv_n is divergent; and consequently $(v_m+v_{m+1}+\ldots+v_n)$ can be made greater than N by taking n greater than (say) n_0. Hence, no matter how large N is, a value n_0 can be found so that

$$s_n>\sum_{1}^{m-1}(p_r-q_r)(v_r-v_{r+1})+l_1N,\quad \text{if } n>n_0;$$

hence s_n must tend to ∞ with n, contrary to hypothesis.

It follows similarly that $(p_n-q_n)/n$ cannot approach a negative limit; so that if $\lim(p_n-q_n)/n$ exists its value must be 0. Now $n=p_n+q_n$, and so if $\lim(p_n/q_n)$ exists, its value must be 1.

This proof is substantially the same as one given by Bagnera.*

Ex. The series $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\dots$ cannot converge.

As a verification, we note that the sum of $3n$ terms is certainly greater than

$$\frac{1}{3}+\frac{1}{3}-\frac{1}{3}+\frac{1}{6}+\frac{1}{6}-\frac{1}{6}+\frac{1}{9}+\frac{1}{9}-\frac{1}{9}+\dots+\frac{1}{3n}+\frac{1}{3n}-\frac{1}{3n}$$
$$=\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)\to\tfrac{1}{3}(\log n+C),$$

so that the series is divergent.

24. Transformation of slowly convergent alternating series.

Let us write $$v_n-v_{n+1}=Dv_n$$

and $$v_n-2v_{n+1}+v_{n+2}=Dv_n-Dv_{n+1}=D^2v_n, \text{ etc.}$$

Then, if $x\leqq 1$, we have

$$(1+x)(v_0-v_1x+v_2x^2-\dots)=v_0+xDv_0-x^2Dv_1+\dots,$$

and consequently $$\sum_0^\infty(-1)^n v_n x^n=\frac{v_0}{1+x}+y\{Dv_0-xDv_1+\dots\},$$

where $$y=x/(1+x).$$

Repeating this operation, we find

$$\sum_0^\infty(-1)^n v_n x^n$$
$$=\frac{1}{1+x}\{v_0+yDv_0+y^2D^2v_0+\dots+y^{p-1}D^{p-1}v_0\}+y^p\{D^pv_0-xD^pv_1+\dots\}.$$

It can be proved † that in all cases when the original series converges, the remainder term

$$y^p\{D^pv_0-xD^pv_1+\dots\}$$

tends to zero as p increases to infinity, at least when x is positive.

The cases of chief interest arise when $x=1$, and then we have

$$\sum_0^\infty(-1)^n v_n=\frac{1}{2}v_0+\frac{1}{4}(Dv_0)+\frac{1}{8}(D^2v_0)+\frac{1}{16}(D^3v_0)+\dots$$
$$+\frac{1}{2^p}(D^{p-1}v_0)+\frac{1}{2^p}[(D^pv_0)-(D^pv_1)+(D^pv_2)-\dots].$$

We can write down a simple expression for the remainder, if $v_n=f(n)$, where $f(x)$ is a function such that $f^{p}(x)$ has a fixed sign for all positive values of x, and steadily decreases in numerical value as x increases.

---

* Bagnera, *Bull. Sci. Math.* (2), t. 12, p. 227: Cesàro, *Rom. Acc. Lincei, Rend.* (4), t. 4, p. 133.

† For the case $x=1$, see L. D. Ames, *Annals of Mathematics*, series 2, vol. 3, p. 188.

For $$Dv_n = f(n) - f(n+1) = -\int_0^1 f'(x_1+n)\,dx_1,$$

and thus $$D^2 v_n = +\int_0^1 dx_1 \int_0^1 f''(x_1+x_2+n)\,dx_2,$$

and generally

$$D^p v_n = (-1)^p \int_0^1 dx_1 \int_0^1 dx_2 \ldots \int_0^1 f^p(x_1+x_2+\ldots+x_p+n)\,dx_p.$$

Thus the series $D^p v_0 - D^p v_1 + D^p v_2 - \ldots$ consists of a succession of *decreasing* terms, of alternate signs. Its sum is therefore less than $D^p v_0$ in numerical value by Art. 19; and consequently

$$\sum_0^\infty (-1)^n v_n = \frac{1}{2}v_0 + \frac{1}{4}(Dv_0) + \frac{1}{8}(D^2 v_0) + \ldots + \frac{1}{2^p}(D^{p-1}v_0) + R_p,$$

where $$|R_p| < \frac{1}{2^p}|D^p v_0|.$$

This result applies to any series of the type

$$1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \frac{1}{5^r} - \ldots, \quad \text{where } r > 0.$$

Here it is easy to see that $D^p v_n$ is always positive and decreases as n increases; it is a useful test of the accuracy of the work, in arithmetical calculations, to apply the transformation twice, starting first say at $\frac{1}{n^r}$ and secondly at $\frac{1}{(n+1)^r}$; if the results are substantially the same we may be satisfied that the work is correct.

Ex. 1. Take $r = \frac{1}{2}$; if we work to five decimals we get

$$s = 1 - \cdot 70711 + \cdot 57735 - \cdot 50000 + \cdot 44721 - s',$$

and we shall apply the transformation to s', whose first seven terms appear in the table below:

$v.$	$Dv.$	$D^2v.$	$D^3v.$	$D^4v.$	$D^5v.$
$6^{-\frac{1}{2}} = \cdot 40825$					
	3029				
$7^{-\frac{1}{2}} = \cdot 37796$		588			
	2441		169		
$8^{-\frac{1}{2}} = \cdot 35355$		419		62	
	2022		107		29
$9^{-\frac{1}{2}} = \cdot 33333$		312		33	
	1710		74		8
$10^{-\frac{1}{2}} = \cdot 31623$		238		25	
	1472		49		
$11^{-\frac{1}{2}} = \cdot 30151$		189			
	1283				
$12^{-\frac{1}{2}} = \cdot 28868$					

If we apply the transformation at the beginning of s' we get

$$\begin{aligned}
\cdot 20413 &= \tfrac{1}{2}(\cdot 40825)\\
757 &= \tfrac{1}{4}(\;3029)\\
73 &= \tfrac{1}{8}(\;588)\\
11 &= \tfrac{1}{16}(\;169)\\
2 &= \tfrac{1}{32}(\;62)\\
\hline
\cdot 21256 &
\end{aligned}$$

If we start from the second term of s' we get

$$\begin{aligned}
\cdot 18898 &= \tfrac{1}{2}(\cdot 37796)\\
610 &= \tfrac{1}{4}(\;2441)\\
52 &= \tfrac{1}{8}(\;419)\\
7 &= \tfrac{1}{16}(\;107)\\
1 &= \tfrac{1}{32}(\;33)\\
\hline
\cdot 19568 &
\end{aligned}$$

Now $\cdot 40825 - \cdot 19568 = \cdot 21257$, so that s' certainly is contained between 0·21256 and 0·21258.

But $s = 0{\cdot}81746 - s'$, so that $s = 0{\cdot}6049$ to four decimal places. If we used the original series, it would need over a hundred million terms to get this result.

Ex. 2. Similarly we may sum the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$.

To 6 decimals, the first 8 terms give 0·634524 and from the next 7 terms we get the table:

v.	Dv.	D^2v.	D^3v.	D^4v.	D^5v.	D^6v.
$9^{-1} = \cdot 111111$						
	11111					
$10^{-1} = \cdot 100000$		2020				
	9091		505			
$11^{-1} = \cdot 090909$		1515		156		
	7576		349		57	
$12^{-1} = \cdot 083333$		1166		99		24
	6410		250		33	
$13^{-1} = \cdot 076923$		916		66		
	5494		184			
$14^{-1} = \cdot 071429$		732				
	4762					
$15^{-1} = \cdot 066667$						

Thus the sum from the 9th term onwards is given by

(i)

$$\begin{aligned}
\cdot 055556 &= \tfrac{1}{2}(\cdot 111111)\\
2778 &= \tfrac{1}{4}(\;11111)\\
252 &= \tfrac{1}{8}(\;2020)\\
32 &= \tfrac{1}{16}(\;505)\\
5 &= \tfrac{1}{32}(\;156)\\
1 &= \tfrac{1}{64}(\;57)\\
\hline
\cdot 058624 &
\end{aligned}$$

or by (ii)

$$\begin{aligned}
\cdot 050000 &= \tfrac{1}{2}(\cdot 100000)\\
2273 &= \tfrac{1}{4}(\;9091)\\
189 &= \tfrac{1}{8}(\;1515)\\
22 &= \tfrac{1}{16}(\;349)\\
3 &= \tfrac{1}{32}(\;99)\\
\hline
\cdot 052487 &\\
111111 &\\
\hline
\cdot 058624 &
\end{aligned}$$

Thus the sum of the series is 0·634524 + 0·058624 = 0·693148, that is 0·69315 to five decimals.

To reach this degree of accuracy we should have to use over a hundred thousand terms of the original series.*

A number of other numerical examples will be found in the paper by Ames, just quoted.

Ex. 3. A physical application may be found in the theory of **Huygens' Zones in Physical Optics.**†

A reference to either of the authorities quoted will shew that we have there to sum a series of terms $v_0 - v_1 + v_2 - \ldots$, for which Dv_n is very small and $D^2 v_n$ has always the same sign. We have then

$$s = \Sigma(-1)^n v_n = \tfrac{1}{2}v_0 + \tfrac{1}{2}(Dv_0 - Dv_1 + \ldots).$$

Now if $D^2 v_n$ is positive we have $Dv_0 > Dv_1 > Dv_2 > \ldots$, and $\lim Dv_n = 0$, because the series in the bracket must converge if s does. Then we get

$$\tfrac{1}{2}v_0 < s < \tfrac{1}{2}(v_0 + Dv_0).$$

Similarly if $D^2 v_n$ is negative, we have $\tfrac{1}{2}v_0 > s > \tfrac{1}{2}(v_0 + Dv_0)$.

Thus the series can be represented by $\tfrac{1}{2}v_0$ to a very high degree of approximation, since Dv_0 is very small.

The transformation described above was first given by Euler, and the first proof of its accuracy is due to Poncelet. Kummer and Markoff have found other transformations for the same purpose; the latter's method includes Euler's as a special case. As an example of Markoff's, we may quote

$$\Sigma \frac{1}{n^3} = \Sigma(-1)^{n-1}\frac{[(n-1)!]^2}{(3n-2)!}\left[\frac{1}{(2n-1)^2} + \frac{5}{12n(3n-1)}\right],$$

13 terms of which give the sum correctly to 20 decimals.‡

To apply Euler's method to this example the reader may note that

$$\Sigma \frac{1}{n^3} = \frac{4}{3}\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \ldots\right).$$

The first ten terms of the series in the bracket give ·9011165, and if we apply Euler's method to the next six, we get ·0004262 for the value of the remainder: thus $\Sigma \frac{1}{n^3} = \frac{4}{3}(0·901427) = 1·202057$ to six places.

Similarly $\Sigma \frac{1}{n^2} = 2\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right)$.

* Of course the actual sum of the series is $\log 2 = ·69314718$. (Art. 19, Ex. 1; and Art. 63.)

† Schuster's *Optics*, § 46; Drude's *Optics*, ch. III. § 2; Schuster, *Phil. Mag.* (5th series), vol. 31, 1891, p. 85.

‡ *Comptes Rendus*, t. 109, 1889, p. 934; *Differenzenrechnung* (Leipzig, 1896), p. 178. For other references, consult Pringsheim (*Encyklopädie*, Bd. I. A. 3, § 37).

The sum of the first ten terms in the bracket is ·8179622, and Euler's method gives ·0045048 for the remainder.

Thus $\Sigma\frac{1}{n^2}=2(0{\cdot}822469)=1{\cdot}644934$. See also Art. 106, Exs. 5, 6.

EXAMPLES.

1. Prove that the series

$$\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}+\dots$$

converges for any value of x which makes none of the denominators zero; but that both the series

$$\frac{1}{x}+\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}+\frac{1}{x+4}-\frac{1}{x+5}+\dots$$

and

$$\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}-\frac{1}{x+4}-\frac{1}{x+5}+\dots$$

are divergent.

2. Prove that if Σna_n is convergent, so also is Σa_n. (Art. 21.)

3. If the series Σa_n is convergent and the sequence (M_n) steadily increases to ∞ with n, then (see Art. 20)

$$\lim\,(a_1M_1+a_2M_2+\dots+a_nM_n)/M_n=0. \qquad \text{[KRONECKER.]}$$

4. Prove that the series

$$a-a^{\frac{1}{2}}+a^{\frac{1}{3}}-a^{\frac{1}{4}}+a^{\frac{1}{5}}-a^{\frac{1}{6}}+\dots$$

oscillates, but can be made to converge to either of its two extreme limits by inserting brackets. On the other hand the series

$$(1-a)-(1-a^{\frac{1}{2}})+(1-a^{\frac{1}{3}})-(1-a^{\frac{1}{4}})+\dots$$

is convergent.

5. Shew that if a series converges, it is still convergent when any number of brackets are inserted, grouping the terms. And shew also that the converse is true, *if all the terms in the brackets are positive.*

6. Calculate, correctly to 20 decimals, the sum of the series

$$1+2x+2x^4+2x^9+2x^{16}+\dots$$

for $x=\pm\frac{1}{10}$, $\pm\frac{2}{10}$. How many terms would have to be taken, to calculate the sum for $x=\pm\frac{9}{10}$ to 3 decimals?

7. Shew that the series $a_1-a_2+a_3-a_4+\dots$ diverges if $a_n=\frac{1}{\sqrt{n}}+\frac{(-1)^{n-1}}{n}$ or if $a_n=1/[\sqrt{n}+(-1)^{n-1}]$; although the terms are alternately positive and negative and tend to zero as a limit.

8. If $|x|>1$, prove that

$$\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+\dots$$

converge to the sum $\frac{1}{x-1}$.

9. If $a_n \to a$ and $b_n \to b$, verify that if the series

$$a_1 b_1 + a_2(b_2 - b_1) + a_3(b_3 - b_2) + \dots$$

converges to the sum S, then the series

$$b_1(a_1 - a_2) + b_2(a_2 - a_3) + b_3(a_3 - a_4) + \dots$$

converges to the sum $S - ab$.

10. Prove that if the series

$$a_1 + a_2 + a_3 + \dots$$

is convergent, so also is

$$\tfrac{1}{2}(a_1 + a_2) + \tfrac{1}{2}(a_2 + a_3) + \tfrac{1}{2}(a_3 + a_4) + \dots,$$

and their sums differ by $\frac{1}{2}a_1$. Is the converse always true? Prove that the converse is certainly true when a_n is *positive*.

11. Discuss the series

$$\frac{x}{c_1} + \frac{x^2}{c_2^{\,2}} + \frac{x^3}{c_3^{\,3}} + \dots + \frac{x^n}{c_n^{\,n}} + \dots$$

$$\frac{1}{x - c_1} + \frac{1}{x - c_2}\frac{x}{c_2} + \dots + \frac{1}{x - c_n}\frac{x^{n-1}}{c_n^{\,n-1}} + \dots,$$

where $c_1, c_2, c_3, \dots$ is an increasing sequence tending to ∞.

12. Verify that

$$\sum_1^\infty \left(\frac{1}{x - c_n} + \frac{1}{c_n} + \frac{x}{c_n^{\,2}} + \dots + \frac{x^{n-1}}{c_n^{\,n}}\right)$$

is absolutely convergent if $|c_n|$ *steadily increases* to ∞, and x is not equal to any of the values $c_1, c_2, c_3, \dots$.

If $c_n = n^{\frac{1}{k}}$, where k is fixed, verify that

$$\sum_1^\infty \left(\frac{1}{x - c_n} + \frac{1}{c_n} + \frac{x}{c_n^{\,2}} + \dots + \frac{x^{r-1}}{c_n^{\,r}}\right)$$

is absolutely convergent if r is the integral part of k.

13. Shew that
$$\sum_1^\infty{}' \frac{1}{m^2 - n^2} = -\frac{3}{4m^2},$$
where m is an integer and the accented Σ means that $n = m$ is to be omitted from the summation.

[In fact the sum can be written in the form

$$\frac{1}{2m}\left\{\left(\frac{1}{m-1} + \frac{1}{m+1}\right) + \left(\frac{1}{m-2} + \frac{1}{m+2}\right) + \dots + \left(1 + \frac{1}{2m-1}\right)\right\}$$
$$-\frac{1}{2m}\left\{\left(1 - \frac{1}{2m+1}\right) + \left(\frac{1}{2} - \frac{1}{2m+2}\right) + \left(\frac{1}{3} - \frac{1}{2m+3}\right) + \dots \text{ to } \infty\right\}$$
$$= \frac{1}{2m}\left\{\left(1 + \frac{1}{2} + \dots + \frac{1}{2m-1}\right) - \frac{1}{m}\right\} - \frac{1}{2m}\left\{\left(1 + \frac{1}{2} + \dots + \frac{1}{2m}\right)\right\}.]$$

14. With the same notation as in Ex. 13, shew that

$$\sum_1^\infty{}' \frac{(-1)^{n-1}}{m^2 - n^2} = +\frac{3}{4m^2},$$

if m is even.

Find an expression for the sum when m is odd.

15. Discuss the convergence of the series whose general term is

$$\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)\frac{\sin n\theta}{n},$$

and also that of the series with $\cos n\theta$ in place of $\sin n\theta$. [Art. 22.]
[*Math. Trip.*, 1899.]

16. Apply Art. 22 to the series whose sums to n terms are $\sin(n+\frac{1}{2})^2\theta$, $\cos(n+\frac{1}{2})^2\theta$, and deduce that

$$\Sigma v_n \sin n^2\theta \,.\, \cos n\theta, \quad \Sigma v_n \sin n\theta \,.\, \sin n^2\theta$$

are convergent if v_n steadily decreases to zero. [HARDY.]

17. Shew that if v_n tends steadily to zero, in such a way that Σv_n is not convergent, then the series

$$\sum_{r=0}^{\infty}(a_1 v_{kr+1}+a_2 v_{kr+2}+\ldots+a_k v_{kr+k})$$

converges if (and only if) $a_1+a_2+\ldots+a_k=0$.

18. If the sequence (a_n) is convergent, prove that $\lim n(a_{n+1}-a_n)$ must either oscillate or converge to zero.

19. If Σa_n converges, and a_n steadily decreases to 0, $\Sigma n(a_n-a_{n+1})$ is convergent. If, in addition, $a_n-2a_{n+1}+a_{n+2}>0$, prove that

$$n^2(a_n-a_{n+1})\to 0.$$ [HARDY.]

20. Apply Euler's transformation to shew that

$$1+2^2x+3^2x^2+4^2x^3+5^2x^4+\ldots=\frac{1+x}{(1-x)^3}.$$

21. Utilise the result of Ex. 3, p. 65, to shew that the sum of the series

$$\frac{1}{2}-\frac{x}{1+x}+\frac{x^2}{1+x^2}-\frac{x^3}{1+x^3}+\ldots$$

tends to the limit $\frac{1}{4}$ as $x\to 1$.

[It is easy to see that (if $0<x<1$), Dv_n is positive and *decreases*; thus the sum lies between $\frac{1}{2}v_0=\frac{1}{4}$ and $\frac{1}{2}(v_0+Dv_0)=\dfrac{1}{2(1+x)}$.]

22. By taking $v_n=\log(a+n)$, shew, as on p. 63, that $D^p v_n$ is negative and steadily decreases; deduce that

$$\log a-p\log(a+1)+\frac{p(p-1)}{2\,!}\log(a+2)\ldots+(-1)^p\log(a+p)<0.$$

23. Shew, by Euler's method, that *

(i) $\Sigma 2/\{\omega J_1(\omega)\}$ differs from unity by less than 6×10^{-6};

(ii) $\Sigma 8/\{\omega^3 J_1(\omega)\}$ differs from unity by less than 7×10^{-6}.

Here the summation refers to the roots of $J_0(\omega)=0$ arranged in numerical order; and the functions $J_0(x)$, $J_1(x)$ are the Bessel Functions.

* See T. A. Lumsden, "A Certain Type of Fourier Bessel Series," *Proc. Lond. Math. Soc.* (2), vol. 22, 1924, p. 381.

CHAPTER IV.

ABSOLUTE CONVERGENCE.

25. It is a familiar fact that a finite sum has the same value, no matter how the terms of the sum are arranged. This property, however, is by no means universally true for infinite series; as an illustration, consider the series

$$s = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \ldots,$$

which we know is convergent (Art. 19, Ex. 1), and has a positive value S greater than $\frac{1}{2}$. Let us arrange the terms of this series so that each positive term is followed by two negative terms: the series then becomes

$$t = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots.$$

Now we have

$$\begin{aligned} t_{3n} &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \ldots + \left(\frac{1}{2n-1} - \frac{1}{4n-2}\right) - \frac{1}{4n} \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots + \frac{1}{4n-2} - \frac{1}{4n} \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}\right) \\ &= \tfrac{1}{2} s_{2n}. \end{aligned}$$

Thus
$$\lim_{n\to\infty} t_{3n} = \tfrac{1}{2} S,$$

and it is easily seen that $\lim t_{3n+1} = \lim t_{3n+2} = \lim t_{3n}$, so that the sum of the series t is $\frac{1}{2}S$.

Consequently, *this derangement of the terms in the series alters the sum of the series.*

In view of the foregoing example we naturally ask *under what conditions may we derange the terms of a series without altering its* *value?* It is to be observed that in the derangement we make a one-to-one correspondence between the terms of two series; so that every term in the first series occupies a perfectly definite place in the second series, and conversely. Thus, corresponding to any number (n) of terms in the first series, we can find a number (n') in the second series, such that the n' terms contain *all* the n terms (and some others); and conversely.

For instance, in the derangement considered above, the first $(2n+2)$ terms of s are all contained in the first $(3n+1)$ terms of t; and the first $3p$ terms of t are all contained in the first $4p$ terms of s.

Ex. More generally, suppose that a series t' is constructed from s by taking alternately α positive and β negative terms; then if $p=(\alpha+\beta)\nu$, we see that

$$t_p'=f(2\alpha\nu)-\tfrac{1}{2}f(\alpha\nu)-\tfrac{1}{2}f(\beta\nu),$$

where $$f(n)=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}.$$

It has been proved in Art. 11 that

$$f(n)\to\log n+C,$$

so that $$t_p'\to\log 2\alpha-\tfrac{1}{2}(\log\alpha+\log\beta).$$

Accordingly we see that $$T'=\tfrac{1}{2}\log(4\alpha/\beta),$$

while $S=\log 2$, corresponding to $\beta=\alpha$. Thus the alteration in the sum due to the derangement is $$T'-S=\tfrac{1}{2}\log(\alpha/\beta).$$

It will be noted that if $\alpha=1$, $\beta=2$, we obtain the series t; and that then the above formula gives $T=\tfrac{1}{2}\log 2=\tfrac{1}{2}S$.

26. A series of positive terms, if convergent, has a sum independent of the order of its terms; but if divergent it remains divergent, however its terms are deranged.

As above, denote the original series by s and the deranged series by t; and suppose first that s converges to the sum S. Then we can choose n, so that the sum s_n exceeds $S-\epsilon$, however small ϵ may be. Now, t contains *all* the terms of s (and if any term happens to be repeated in s, t contains it equally often); we can therefore find an index p such that t_p contains *all* the terms s_n. Thus we have found p so that t_p exceeds $S-\epsilon$, because all the terms in t_p-s_n are *positive or zero*. Now t contains no terms which are not present in s, so that, however great r may be, t_r cannot exceed S; and, combining these two conclusions, we get

$$S\geqq t_r>S-\epsilon,\quad \text{if } r\geqq p.$$

Consequently the series t converges to the sum S.

Secondly, if s is divergent, t cannot converge; for the foregoing argument shews that if t converges, s must also converge. Consequently t is divergent.

If we attempt to apply this argument to the two series considered in Art. 25,

$$s = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots, \quad t = 1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \dots,$$

we find that the terms in $t_p - s_n$ are partly *negative*. Thus we cannot prove that $t_r > S - \epsilon$; and as a matter of fact we see from Art. 25 that this inequality is inaccurate. Similarly, the argument used above fails to prove that $S \geqq t_r$, although this happens to be true here if $r > 1$.

It is now easy to prove that **if a series Σa_n is absolutely convergent, its sum S is not altered by derangement.**

Since the series $\Sigma|a_n|$ is convergent, we can find a value of n such that

$$|a_{n+1}| + |a_{n+2}| + |a_{n+3}| + \dots \text{ to } \infty < \epsilon.$$

Then suppose that t_p contains all of the terms s_n, as on p. 70 above; consequently if $r \geqq p$, the difference $t_r - s_n$ consists of a certain number of terms taken from s, the order of each term being greater than n. Thus, from the last inequality, we see that

$$|t_r - s_n| < \epsilon, \text{ if } r \geqq p.$$

Now, in virtue of the choice of n,

$$|S - s_n| = |a_{n+1} + a_{n+2} + a_{n+3} + \dots| < \epsilon.$$

Hence we have found p such that

$$|S - t_r| < 2\epsilon, \quad \text{if } r \geqq p,$$

and accordingly the series t is convergent and has S as its sum; that is, the sum is unaltered by the derangement.

Ex. 1. As an example, consider the series s:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots.$$

This is absolutely convergent by Art. 11; and therefore the series remains convergent, and has the same sum after any derangement. It is accordingly equal to the series t:

$$1 - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{3^2} - \frac{1}{6^2} - \frac{1}{8^2} + \frac{1}{5^2} - \frac{1}{10^2} - \frac{1}{12^2} + \dots,$$

where the law of derangement is the same as in the first example of Art. 25.

To illustrate the general theory, we note that here the first $2n$ terms in s are contained amongst the first $3n$ terms in t; and we find that

$$t_{3n} - s_{2n} = -\left\{\frac{1}{(2n+2)^2} + \frac{1}{(2n+4)^2} + \dots \frac{1}{(4n)^2}\right\}.$$

The sum in brackets { } consists of n terms, each less than $1/(2n)^2$; hence this sum is less than $1/(4n)$, and so tends to zero as n tends to infinity. From the general theory we can predict this property by using the remainder after $2n$ terms in the convergent series of positive terms,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\cdots.$$

Ex. 2. From our present point of view, we observe that the inequality between $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ and $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\ldots$ (Art. 25) is explained by the fact that these series are not absolutely convergent (Art. 7).

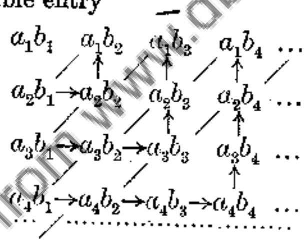
By way of contrast with **Ex. 1**, we note that here

$$t_{3n}-s_{2n}=-\left\{\frac{1}{2n+2}+\frac{1}{2n+4}+\cdots+\frac{1}{4n}\right\}.$$

Now the sum in brackets consists of n terms, each lying between $1/(2n)$ and $1/(4n)$; thus the sum lies between $\frac{1}{2}$ and $\frac{1}{4}$ for any value of n, and so *cannot* tend to zero as n tends to infinity.

27. Applications of absolute convergence.

Consider first *the multiplication of two absolutely convergent series* $A=\Sigma a_n$, $B=\Sigma b_n$. Write the terms of the product so as to form a table of double entry



It is easy to prove that AB is the sum of the series

(1) $a_1b_1+(a_2b_1+a_2b_2+a_1b_2)+(a_3b_1+a_3b_2+a_3b_3+a_2b_3+a_1b_3)+\ldots,$

where the order of the terms is the same as is indicated by the arrows in the table. For the sum to n terms of this series (1) is A_nB_n, if $A_n=a_1+a_2+\ldots+a_n$, $B_n=b_1+b_2+\ldots+b_n$.

Now $A'=\Sigma|a_n|$ and $B'=\Sigma|b_n|$ are convergent by hypothesis. Thus the series

(2) $$a_1b_1+a_2b_1+a_2b_2+a_1b_2+a_3b_1+\ldots,$$

obtained by removing the brackets from (1), is absolutely convergent, because the sum of the absolute values of any number of terms in (2) cannot exceed $A'B'$. Accordingly, (2) has the same sum AB as the series (1). Since (2) is absolutely convergent, we can arrange it in any order (by Art. 26) without changing the sum. Thus we may replace (2) by

(3) $$a_1b_1+a_2b_1+a_1b_2+a_3b_1+a_2b_2+a_1b_3+a_4b_1+\ldots,$$

following the order of the diagonals indicated in the diagram. Hence we find, on inserting brackets in (3),

$$AB = c_1 + c_2 + c_3 + \dots + \text{ to } \infty,$$

where $\quad c_1 = a_1 b_1, \quad c_2 = a_2 b_1 + a_1 b_2, \quad c_3 = a_3 b_1 + a_2 b_2 + a_1 b_3$

and $\quad c_n = a_n b_1 + a_{n-1} b_2 + \dots + a_1 b_n.$

For other results on the multiplication of series the reader should refer to Arts. 34, 35.

A second useful application of the theorem of Art. 26 is to justify the step of arranging a series $\Sigma a_n y^n$ in powers of x, where y is a polynomial in x; say $y = b_0 + b_1 x + \dots + b_k x^k$.

It is here sufficient to have $\Sigma \alpha_n \eta^n$ convergent where

$$\alpha_n = |a_n|, \quad \eta = \beta_0 + \beta_1 \xi + \dots + \beta_k \xi^k, \quad \beta_r = |b_r|, \quad \xi = |x|;$$

and from Art. 10, we see that this requires

$$\eta < \lambda, \quad \text{if } \lambda^{-1} = \overline{\lim}\, \alpha_n^{\frac{1}{n}}.$$

The last condition requires that $\beta_0 < \lambda$, and that ξ shall be less than some fixed value; and then the necessary derangement will certainly not alter the sum of the series.

In most of the ordinary cases $\lambda = 1$, and y is of the form $bx \pm x^2$; the condition is then

$$\xi^2 + \beta\xi < 1 \quad \text{or} \quad \xi < \tfrac{1}{2}[(4+\beta^2)^{\frac{1}{2}} - \beta].$$

In particular, if $\beta \leq 2$, it is enough to take $\xi < \sqrt{2} - 1$, which is certainly satisfied when $\xi < \frac{2}{5}$.

The beginner may be tempted to think that the condition $|y| < \lambda$ would be sufficient; but this is not correct. For we have to ensure the convergence of the series when $a_n y^n$ is written out at length, and every term is made positive in the *expanded* form.

As an illustration of this point, consider the series $1 + \Sigma (2x - x^2)^n$ which has the sum $[1 - (2x - x^2)]^{-1} = (1-x)^{-2}$, when $|2x - x^2| < 1$. This condition is satisfied by any value of x (except 1) lying between $1 \pm \sqrt{2}$; and in particular by $x = \frac{3}{2}$, because $2x - x^2$ is then $\frac{3}{4}$. But if the series is arranged in powers of x, we get

$$\begin{array}{l|l|l|l|l|l|l|l}
1+2x & -x^2 & & & & & & \\
 & +4x^2 & -4x^3 & +x^4 & & & & \\
 & & +8x^3 & -12x^4 & +6x^5 & -x^6 & & \\
 & & & +16x^4 & -32x^5 & +24x^6 & -8x^7 & +x^8 \\
 & & & & +\dots\dots & \dots\dots & \dots\dots & \dots\dots
\end{array}$$

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots,$$

which diverges if $x = \frac{3}{2}$.

Thus the condition $|2x-x^2|<1$ is not sufficient to allow the arrangement in powers of x. The condition found in the text above would be $|x|<\sqrt{2}-1$; and would be obtained from the expansion given at length, by making the negative coefficients positive; this leads to the series

$$1+2|x|+5|x|^2+12|x|^3+29|x|^4+\ldots$$

(in which $a_n=2a_{n-1}+a_{n-2}$).

As a matter of fact, the condition $|x|<\sqrt{2}-1$ is narrower than is necessary for the truth of the equation

$$1+\Sigma(2x-x^2)^n=1+\Sigma(n+1)x^n.$$

This equation is true if both series converge; although the proof does not follow from our present line of argument. It may be guessed that, in general, the condition found for ξ in the text is unnecessarily narrow; and this is certainly the case in a number of special applications. However, we are not here concerned with finding the widest limits for x; what we wish to shew is that the transformation is certainly legitimate when x is properly restricted.

In view of Riemann's theorem (Art. 28) it may seem surprising that the condition of absolute convergence gives an unnecessarily small value for ξ. However, a little consideration will shew that Riemann's theorem does not imply that *any* derangement of a non-absolutely convergent series will alter its sum; but that such a series can be made to have any value by means of a *special* derangement, which may easily be of a far more sweeping character than the derangement implied in arranging $\Sigma a_n y^n$ according to powers of x.

28. Riemann's Theorem.

If a series converges, but not absolutely, its sum can be made to have any arbitrary value by a suitable derangement of the series; it can also be made divergent or oscillatory.

Let x_p denote the sum of the first p positive terms and $-y_n$ the sum of the first n negative terms; then we are given that

$$\lim(x_p-y_n)=s, \quad \lim(x_p+y_n)=\infty,$$

where p, n tend to ∞ according to some definite relation. Hence

$$\lim_{p\to\infty} x_p=\infty, \quad \lim_{n\to\infty} y_n=\infty.$$

Suppose now that the sum of the series is to be made equal to σ; since $x_p\to\infty$ we can choose p_1 so that $x_{p_1}>\sigma$, and so that p_1 is the *smallest* index which satisfies this condition. Similarly we can find n_1 so that $y_{n_1}>x_{p_1}-\sigma$, and again suppose that n_1 is the *least* index consistent with the inequality.

Then, in the deranged series, we place first a group of p_1 positive terms, second a group of n_1 negative terms, keeping the terms in each group in their original order. Thus, if S_ν is the sum of ν terms, it is plain that

$$S_\nu < \sigma, \quad \text{if } \nu < p_1, \quad \text{but } S_\nu > \sigma, \quad \text{if } p_1 \leqq \nu < p_1 + n_1.$$

We now continue the process, placing third a group of $(p_2 - p_1)$ positive terms, where p_2 is the least index such that $x_{p_2} > y_{n_1} + \sigma$; and fourth, a group of $(n_2 - n_1)$ negative terms, where n_2 is the least index such that $y_{n_2} > x_{p_2} - \sigma$.

The method of construction can evidently be carried on indefinitely, and it is clear that if $p_r + n_r > \nu \geqq p_r + n_{r-1}$, $S_\nu - \sigma$ is positive, but cannot exceed the $(p_r + n_{r-1})$th term of the series; while if $p_{r+1} + n_r > \nu \geqq p_r + n_r$, $\sigma - S_\nu$ is positive, but does not exceed the $(p_r + n_r)$th term: for $S_\nu - \sigma$ changes sign at these terms.

Thus, since the terms of the series must tend to zero as ν increases, we have

$$\lim S_\nu = \sigma.$$

It is easy to modify the foregoing method so as to get a divergent or oscillatory series, by starting from a sequence (σ_r) which is either divergent or oscillatory and taking $p_1, n_1, \ldots$ in turn to be the first indices which satisfy the inequalities

$$x_{p_1} > \sigma_1, \quad y_{n_1} > x_{p_1} - \sigma_1, \quad x_{p_2} > y_{n_1} + \sigma_2, \quad y_{n_2} > x_{p_2} - \sigma_2,$$

and so on.

As a matter of fact, however, Riemann's process is quite out of the question with any actual series; and we have to adopt an entirely different method due to Pringsheim.*

Let $f(x)$ be a positive function, steadily decreasing to zero as x increases; and consider the series $\Sigma(-1)^{n-1} f(n)$, which converges, in virtue of Art. 19.

Here every positive term is followed by a negative term; and suppose that, in the deranged series, the first r terms contain p positive to n negative terms (so that $p + n = r$). Then the sum of these r terms is

$$\{f(1) - f(2) + \ldots - f(2n)\}$$
$$+ \{f(2n+1) + f(2n+3) + \ldots + f(2p-1)\},$$

where the second bracket contains $p - n$ terms, and so lies between

$$\nu f(2n) \quad \text{and} \quad \nu f(2n+2\nu), \quad \text{if } \nu = p - n.$$

Then the alteration in the sum is equal to the limit of this second bracket.

Suppose first that $nf(n)$ tends *steadily* to infinity with n, then $f(2n+2\nu)/f(2n)$ lies between 1 and $n/(n+\nu)$. Thus if we choose ν to be such a function of n that

$$\lim \nu f(2n) = l,$$

the change in the sum of the series is l, because then $\nu/n \to 0$.

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* *Math. Annalen*, Bd. 22, p. 455.

We have thus Pringsheim's first result:

If $nf(n)$ tends steadily to infinity the value of p requisite for an alteration l in the sum of the series is subject to the condition $\lim (p-n)f(2n)=l$.

For instance, taking the series $\Sigma(-1)^{n-1}n^{-\frac{1}{2}}$, we see that $(p-n)$ may be the integer nearest * to $l\sqrt{(2n)}$; or again with $\Sigma(-1)^{n-1}\frac{\log n}{n}$, $p-n$ may be the integer nearest * to $2ln/\log n$.

Next, if $\lim nf(n)$ is finite, say equal to g, it follows that, for any positive value of ϵ, however small, a value n_0 can be found such that

$$\frac{g-\epsilon}{x} < f(x) < \frac{g+\epsilon}{x}, \quad (x > n_0).$$

Let p be chosen so that $k = \lim (p)/n$.

It is easy to see, by an argument similar to that of Art. 11, that

$$\frac{1}{2n+1}+\frac{1}{2n+3}+\dots+\frac{1}{2p-1} \to \int_n^p \frac{dx}{2x} = \frac{1}{2}\log\frac{p}{n} \to \frac{1}{2}\log k.$$

Hence the alteration l is contained between the two values

$$\tfrac{1}{2}(g\pm\epsilon)\log k.$$

Thus, since ϵ is arbitrarily small, we must have

$$l = \tfrac{1}{2}g\log k.$$

Hence, *if $\lim nf(n)=g$, and if k is the limit of the ratio of the number of positive to the number of negative terms, the alteration l is given by $l=\frac{1}{2}g\log k$.*

In particular, since $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots=\log 2$ (Art. 19), we see that when this series is arranged so that kn positive terms correspond to n negative terms its sum is $\log 2+\frac{1}{2}\log k=\frac{1}{2}\log 4k$. (Compare Art. 25.)

To save space, we refer to § V. of Pringsheim's paper for the discussion of the more difficult case when $\lim nf(n)=0$.

EXAMPLES.

1. Criticise the following paradox:

$$\begin{aligned} &1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\dots \\ &=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\dots \\ &\quad -2(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+\dots) \\ &=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\dots-(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\dots) \\ &=0. \end{aligned}$$

2. If a transformation similar to that of Ex. 1 is applied to the series

$$1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\frac{1}{5^p}-\dots,$$

shew that (if $p<1$) we obtain the paradoxical result that the sum of the series is negative. But, if $p>1$, the result obtained is correct and expresses the sum s_1 of the given series in terms of the sum s of the corresponding series of positive terms by the formula

$$s_1=\left(1-\frac{1}{2^{p-1}}\right)s.$$

* It is not, of course, essential to take always the *nearest* integer, in order to satisfy the condition. But this is the simplest statement.

3. Apply a transformation similar to that of Ex. 1 to the series

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\dots,$$

and prove that the resulting series is

$$(1-\tfrac{2}{3})+(\tfrac{1}{3}-\tfrac{2}{7})+(\tfrac{1}{5}-\tfrac{2}{11})+\dots,$$

which converges to a sum *less* than that of the given series.

4. If we write $f(x)=1/x^2$, and $s=\sum_1^\infty f(n)$, shew that

$$f(1)+f(3)+f(5)+f(7)+f(9)+\dots=\tfrac{3}{4}s=\tfrac{1}{8}\pi^2,$$
$$f(1)+f(5)+f(7)+f(11)+f(13)+\dots=\tfrac{2}{3}s=\tfrac{1}{9}\pi^2,$$
$$f(1)-f(2)-f(4)+f(5)+f(7)-f(8)-f(10)+\dots=\tfrac{4}{9}s=\tfrac{2}{27}\pi^2.$$

[It is proved later in Art. 71 that $s=\frac{1}{6}\pi^2$.]

5. Prove that

$$1-\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{8}-\tfrac{1}{10}+\dots=\tfrac{2}{3}(1-\tfrac{1}{2}+\tfrac{1}{3}-\dots),$$
$$1+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{17}-\tfrac{1}{19}+\dots=\tfrac{4}{3}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\dots),$$

the two series on the left being found by omitting all multiples of 3 from those on the right.

6. Prove that

$$1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{4}+\dots=\tfrac{3}{2}\log 2,$$
$$1-\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{6}-\tfrac{1}{8}+\dots=\tfrac{1}{2}\log 2,$$
$$1-\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{6}-\tfrac{1}{8}+\tfrac{1}{3}-\tfrac{1}{10}-\tfrac{1}{12}-\tfrac{1}{14}-\tfrac{1}{16}+\dots=0.$$

7. Prove that $\sum_{-\infty}^{\infty}\dfrac{1}{x-n}$ is not a determinate number, but that

$$X=\frac{1}{x}+\sum_{-\infty}^{\infty}{}'\left(\frac{1}{n}+\frac{1}{x-n}\right)$$

is perfectly definite. Here x is supposed not to be an integer, and the accent implies that $n=0$ is to be omitted.

Shew that
$$\lim\left\{\sum_{-p}^{q}\left(\frac{1}{x-n}\right)\right\}-X=-\log k,$$

where p and q tend to ∞ in such a way that $\lim(q/p)=k$.

8. Find the product of the two series

$$1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\dots \quad\text{and}\quad 1-x+\frac{x^2}{2!}-\dots+(-1)^n\frac{x^n}{n!}+\dots.$$

9. Shew that if $s_n=a_0+a_1+a_2+\dots+a_n$, then

$$(\Sigma a_n x^n)/(1-x)=(\Sigma a_n x^n)(1+x+x^2+\dots)=\Sigma s_n x^n.$$

10. Any non-absolutely convergent series may be converted into an absolutely convergent series by the insertion of brackets. [See Art. 5.]

Any oscillating series may be converted into a convergent series by the insertion of brackets; and the brackets may be arranged so that the series has a sum equal to any of the limits of s_n.

11. In order that the value of a non-absolutely convergent series may remain unaltered after a certain change in the order of the terms, it is sufficient that the product of the displacement of the nth term by the greatest subsequent term may tend to zero as n increases to ∞.

[Borel, *Bulletin des Sci. Math.* (2), t. 14, 1890, p. 97.]

CHAPTER V.

DOUBLE SERIES.

29. Suppose an infinite number of terms arranged so as to form a network (or lattice) which is bounded on the left and above, but extends to infinity to the right and below, as indicated in the diagram:

$$\begin{aligned}&a_{1,1}+a_{1,2}+a_{1,3}+a_{1,4}+\dots\\+&a_{2,1}+a_{2,2}+a_{2,3}+a_{2,4}+\dots\\+&a_{3,1}+a_{3,2}+a_{3,3}+a_{3,4}+\dots\\+&a_{4,1}+a_{4,2}+a_{4,3}+a_{4,4}+\dots\\+&\dots\dots\dots\dots\dots\dots\end{aligned}$$

The first suffix refers to the row, the second suffix to the column in which the term stands.

Suppose next that a rectangle is drawn across the network so as to include the first m rows and the first n columns of the array of terms; and denote the sum of the terms contained within this rectangle by the symbol $s_{m,n}$. If $s_{m,n}$ *approaches a definite limit* s *as* m *and* n *tend to infinity at the same time* (*but* **independently**), *then* s *is called the sum of the double series represented by the array.**

In more precise form, this statement requires that it shall be possible to find an index μ, corresponding to an arbitrary positive number ϵ, such that

$$|s_{m,n}-s|<\epsilon, \quad \text{if } m,n>\mu.$$

By the last inequality is implied that m, n are subject to no other restriction than the condition of being greater than μ.

This property is also expressed by the equations

$$\lim_{m,n\to\infty} s_{m,n}=s, \quad \text{or} \quad \lim_{(m,n)} s_{m,n}=s.$$

* This definition is framed in accordance with the one adopted by Pringsheim (*Münchener Sitzungsberichte*, Bd. 27, 1897, p. 101; see particularly pp. 103, 140).

But the symbol $s_{m,n} \to s$ is not sufficient, unless some indication is added as to the mode of summation adopted; for it is often convenient to use other methods (see Arts. 30, 31) which may give values different from the above.

Since $$a_{m,n} = s_{m,n} - s_{m-1,n} - s_{m,n-1} + s_{m-1,n-1},$$

it follows that when $(s_{m,n})$ converges, we can find μ so that $|a_{m,n}| < \epsilon$, provided that *both* m and n are greater than μ: this of course does not imply that $a_{m,n}$ will tend to zero necessarily, when m and n tend to ∞ *separately*.

The equations

$$\lim_{m,n \to \infty} s_{m,n} = \infty \quad \text{or} \quad \lim_{(m,n)} s_{m,n} = \infty$$

imply that, given any positive number G, however large, we can find μ, such that

$$s_{m,n} > G, \quad \text{if } m, n > \mu;$$

and the double series is then said to *diverge* to ∞. We define similarly divergence to $-\infty$.

It is also possible that the double series may *oscillate*; and there is little difficulty in modifying the method of Art. 5 so as to establish the existence of *extreme limits* * for any double sequence $(s_{m,n})$; these may be denoted by

$$\varliminf_{(m,n)} s_{m,n} \quad \text{and} \quad \varlimsup_{(m,n)} s_{m,n}.$$

The general condition for convergence is simply that the sum of the terms between two rectangles m, n and p, q must be numerically less than ϵ, if m, n are greater than μ; or in symbols

$$|s_{p,q} - s_{m,n}| < \epsilon, \quad \text{if } p > m > \mu \text{ and } q > n > \mu,$$

where of course the value of μ will depend on ϵ. This condition is obviously *necessary*; and to see that it is *sufficient*, denote by σ_n the value of $s_{m,n}$ when $m = n$ (so that the rectangle is replaced by a *square*). Then our condition yields

$$|\sigma_q - \sigma_n| < \epsilon, \quad \text{if } q > n > \mu.$$

Hence σ_n approaches a limit s (Art. 3), and so we can find μ_1, such that

$$|s - \sigma_n| < \tfrac{1}{2}\epsilon, \quad \text{if } n > \mu_1.$$

Now the general condition gives also

$$|s_{p,q} - \sigma_n| < \tfrac{1}{2}\epsilon, \quad \text{if } p, q > n > \mu_2;$$

* The proof is given in full by Pringsheim, *Math. Annalen*, Bd. 53, 1900, pp. 294-301.

and so, if μ_3 is the greater of μ_1 and μ_2, and $n > \mu_3$, we find

$$|s_{p,q} - s| < \epsilon, \quad \text{if } p, q > n.$$

Ex. 1. *Convergence*: If $s_{m,n} = 1/m + 1/n$, $s = 0$ and $\mu \geqq 2/\epsilon$.

Ex. 2. *Divergence*: If $s_{m,n} = m + n$, the condition of divergence is satisfied.

Ex. 3. *Oscillation*: If $s_{m,n} = (-1)^{m+n}$, the extreme limits are -1 and $+1$.

30. Repeated series.

In addition to the mode of summation just defined it is often necessary to use the method of *repeated summation*; then we first form the sum of a *row* of terms in the diagram, and obtain $b_m = \sum_{n=1}^{\infty} a_{m,n}$, after which we sum $\sum_{m=1}^{\infty} b_m$.

This process gives a value which we denote by

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{m,n}\right) \text{ or } \sum_{(m)}\sum_{(n)} a_{m,n};$$

this is called *the sum by rows* of the double series.

In like manner we define the repeated sum

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{m,n}\right) \text{ or } \sum_{(n)}\sum_{(m)} a_{m,n},$$

which is called *the sum by columns* of the double series.

Each of these sums may be defined also as a *repeated limit*, thus:

$$\sum_{(m)}\sum_{(n)} a_{m,n} = \lim_{m\to\infty}(\lim_{n\to\infty} s_{m,n}) \text{ or } \lim_{(m)(n)} s_{m,n},$$

with a similar interpretation for the second repeated sum.

In dealing with a *finite* number of terms it is obvious that

$$s_{M,N} = \sum_{m=1}^{M}\left(\sum_{n=1}^{N} a_{m,n}\right) = \sum_{n=1}^{N}\left(\sum_{m=1}^{M} a_{m,n}\right).$$

But if a double series has the sum s in the sense of Art. 29, it is by no means necessarily true that we can infer

(1) $$s = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{m,n}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{m,n}\right);$$

for the single series formed by the rows and columns of the double series *need not converge at all, but may oscillate.*

That the rows and columns need not converge is shewn by the example $s_{m,n} = (-1)^{m+n}(1/m + 1/n)$, for which $s = 0$; but neither of the single limits

$$\lim_{m\to\infty} s_{m,n}, \quad \lim_{n\to\infty} s_{m,n}$$

exists at all.

Pringsheim has proved, however, that *if the rows and columns converge, and if the double series is convergent, then the equation* (1) *above is always true.*

In fact we have

$$|s_{m,n} - s| < \epsilon, \qquad \text{if } m, n > \mu,$$

so that
$$\left|\lim_{n\to\infty} s_{m,n} - s\right| \leqq \epsilon, \qquad \text{if } m > \mu;$$

since, by hypothesis, this single limit *exists*.

Hence
$$\lim_{m\to\infty} \left(\lim_{n\to\infty} s_{m,n}\right) = s.$$

In like manner we can prove the other half of equation (1).

When the double series is not convergent, the equation

$$(2) \qquad \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{m,n}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{m,n}\right)$$

is not necessarily valid whenever the two repeated series are convergent.

There is in fact no reason whatever for assuming that the equation

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} s_{m,n}\right) = \lim_{n\to\infty} \left(\lim_{m\to\infty} s_{m,n}\right)$$

is true whenever the repeated limits exist.

For instance, with $s_{m,n} = m/(m+n)$, we find

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} s_{m,n}\right) = 0, \quad \lim_{n\to\infty} \left(\lim_{m\to\infty} s_{m,n}\right) = 1.$$

From Pringsheim's theorem it is clear that the double series cannot converge (the rows and columns being supposed convergent) unless equation (2) is valid; but the truth of (2) is no reason for assuming the convergence of the double series.

For instance, with $s_{m,n} = mn/(m+n)^2$, we find

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} s_{m,n}\right) = 0 = \lim_{n\to\infty} \left(\lim_{m\to\infty} s_{m,n}\right).$$

But yet the double series cannot converge, since if $m = 2n$, $s_{m,n} = \frac{2}{9}$; while if $m = n$, $s_{m,n} = \frac{1}{4}$.

For some purposes it is useful to know that equation (2) is true, without troubling to consider the general question of convergence of the double series. In such cases, conditions may be used which will be found in the *Proceedings of the London Mathematical Society*; * the discussion of them here would go somewhat beyond our limits.

A further example, due to Arndt, of the possible failure of equation (2) may be added:

If we write $$a_{m,n}=\frac{1}{m+1}\left(\frac{m}{m+1}\right)^n-\frac{1}{m+2}\left(\frac{m+1}{m+2}\right)^n,$$

we find that $$s_{m,n}=\left(\frac{1}{2}-\frac{1}{2^{n+1}}\right)-\left[\frac{m+1}{m+2}-\left(\frac{m+1}{m+2}\right)^{n+1}\right].$$

Thus $$\lim_{m\to\infty}(\lim_{n\to\infty} s_{m,n})=-\tfrac{1}{2},$$

but $$\lim_{n\to\infty}(\lim_{m\to\infty} s_{m,n})=+\tfrac{1}{2}.$$

Other examples illustrating the general theory will be found at the end of the chapter. (See Exs. 1–6 and 10.)

31. Double series of positive terms.

In view of what has been proved in Art. 26, we may anticipate that if a series of positive terms converges to the sum s in any way, it will have the same sum if summed in any other way which includes all the terms. For, however many terms are taken, we cannot get a larger sum than s, but we can get as near to s as we please, by taking a sufficient number of terms. We shall now apply this general principle to the most useful special cases.

(1) *It is sufficient to consider squares only in testing a double series of positive terms for convergence.*

Write for brevity $s_{m,n}=\sigma_n$ when $m=n$; then plainly σ_n must converge to the limit s, if $s_{m,n}$ does so. Further, if σ_n converges to a limit s, so also will $s_{m,n}$. For then we can find μ so that σ_μ lies between s and $s-\epsilon$; but if m and n are greater than μ, we have

$$\sigma_{m+n}\geqq s_{m,n}\geqq\sigma_\mu,$$

so that $$s\geqq s_{m,n}>s-\epsilon.$$

Hence $s_{m,n}$ converges to the limit s.

The reader will find no difficulty in extending the argument to cases of *divergence*.

(2) *If more convenient for purposes of summation, we may replace the rectangles by any succession of curves † which tend to infinity in all directions.*

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* Bromwich, *Proc. Lond. Math. Soc.*, series 2, vol. 1, 1904, p. 176.

† These "curves" may consist, wholly or in part, of straight lines; and it is supposed that each curve encloses the *whole* of the preceding curve.

For, plainly, when the rectangles (and therefore the squares) give a sum s we can suppose any particular curve C_n to be contained between two of the squares and that the sides of these squares are p, q; thus if S_n is the sum for the curve C_n, we have, as in (1) above,

$$\sigma_p \leqq S_n \leqq \sigma_q \leqq s.$$

Further, since C_n is to tend to infinity in all directions, we can make p greater than μ by taking $n > n_0$, say.

Thus, since $\sigma_p > s - \epsilon$, because $p > \mu$,
we have also $s - \epsilon < S_n \leqq s$, if $n > n_0$,
and so $\lim S_n = s$.

In like manner, by enclosing a square between two of the curves, we can shew that if the curves give a sum s, so also do the squares (and therefore the rectangles, too, in virtue of (1) above).

A particular class of the curves used in (2) is formed by drawing diagonals, equally inclined to the horizontal and vertical sides of the network as indicated in the right-hand figure.

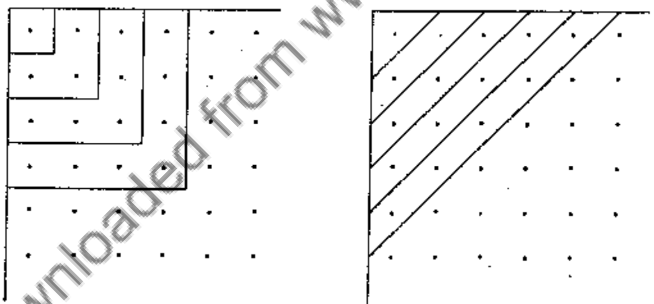

FIG. 11.

The summation by squares is indicated on the left. It should be noticed that *these two modes of summation give two methods of converting a double series into a single series.*

Thus, by squares, we are summing the series

$$a_{11} + (a_{21} + a_{22} + a_{12}) + (a_{31} + a_{32} + a_{33} + a_{23} + a_{13}) + \dots,$$

and by the diagonals we get

$$a_{11} + (a_{21} + a_{12}) + (a_{31} + a_{22} + a_{13}) + \dots .$$

Of course the equality between these two series is now seen to be a consequence of Art. 26; but we could not, without further proof, infer theorem (1) from that article since Art. 26 refers only to single and not to double series.

By combining Art. 26 with (1) above, it will be seen that:

(3) *No derangement of the (positive) terms of a double series can alter the sum, nor change divergence into convergence.*

It is also important to note that:

(4) *When the terms of the double series are positive, its convergence implies the convergence of all the rows and columns, and its sum is equal to the sums of the two repeated series.*

For, when the double series has the sum s, it is clear that $s_{m,n}$ cannot exceed s; and consequently the sum of any number of terms in a single row cannot be greater than s. Also, for any fixed value of m, $\lim\limits_{(n)} s_{m,n}$ exists and is not greater than s. Now we can find μ so that $s_{m,n} > s-\epsilon$, if m, n are greater than μ. Consequently

$$s \geqq \lim_{(n)} s_{m,n} > s-\epsilon, \quad \text{if} \quad m > \mu.$$

Hence

$$\lim_{(m)} [\lim_{(n)} s_{m,n}] = s,$$

or

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n} \right) = s.$$

In a similar way, we see that each column converges and that

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{m,n} \right) = s.$$

As a converse to (4), we have:

(5) *The terms being always positive, if either repeated series is convergent, so also is the other and also the double series; and the three sums are the same.*

For, suppose that

$$\lim_{(m)} [\lim_{(n)} s_{m,n}] = s,$$

then

$$\sigma_m \leqq s_{m,n}, \quad \text{if} \ n > m,$$

and so

$$\sigma_m \leqq \lim_{(n)} s_{m,n} \leqq s.$$

Hence by Art. 2 the sequence (σ_m) converges to a limit σ; and it then follows from (4) above that $s=\sigma$, and that the other repeated series has the same sum.

The reader will find little difficulty in modifying the proofs in (4) and (5) so as to cover the case of divergence.

(6) It is convenient to point out here that the proofs given in (4), (5) can be generalised at once to deal with a variety of *double-limit problems in which the variables are no longer restricted to be integers* (as in the case of double series).

To formulate the method more precisely, suppose that the variables are denoted by μ, ν (in place of m, n) and that the function is denoted by $s(\mu, \nu)$—corresponding to $s_{m,n}$; then *we assume in the first place that $s(\mu, \nu)$ steadily increases with both variables.* It is also convenient to define a function of a single variable $\sigma(\mu)$ by writing $\sigma(\mu) = s(\mu, \nu)$, where ν is expressed in terms of μ by any convenient relation,* such as ν/μ=const. or ν/μ^2=const.

Then $\sigma(\mu)$ takes the place of σ_m in the case of the double series; and it is proved as in (1), (2) above that if $\sigma(\mu)$ has a definite limit λ for any particular functional relation between μ, ν, then the double limit $\lim_{(\mu,\nu)} s(\mu, \nu)$ exists and is equal to λ; and also that λ is independent of the functional relation between μ, ν.

We can then state the two double-limit theorems:

(α) *Under the above restriction on $s(\mu, \nu)$ the existence of the double limit*

$$\lim_{(\mu,\nu)} s(\mu, \nu) = \lambda$$

implies the existence of the two repeated limits

$$\lim_{(\mu)} \left\{ \lim_{(\nu)} s(\mu, \nu) \right\}, \quad \lim_{(\nu)} \left\{ \lim_{(\mu)} s(\mu, \nu) \right\},$$

and these limits are equal to λ.

This is proved exactly as in (4); and we deduce

(β) *Under the same restriction on $s(\mu, \nu)$ the existence of either repeated limit implies the existence of the other repeated limit, and of the double limit; and these three limits are equal.*

Again the proof is exactly the same as in (5).

32. Tests for convergence of a double series of positive terms.

If we compare Art. 8 with (1) of the last article, we see that:

(1) *If the (positive) terms of a double series are less than those of another double series which is known to converge, the former converges.*

Similarly for divergence, with "greater" in place of "less."

* It is supposed that ν steadily increases with μ.

The most important type of convergent series * is given by $a_{m,n}=(C_mC_n)^{-1}$, where ΣC_ν^{-1} is a convergent single series; to see that this double series is convergent, we note that the sum σ_n contained in a square of side n is equal to

$$(C_1^{-1}+C_2^{-1}+\ldots+C_n^{-1})^2,$$

and therefore σ_n has a limit. Consequently the double series converges, by (1) of the last article.

Another useful form * is given by

$$a_{m,n}=(pC_p)^{-1} \quad \text{or} \quad (pD_p)^{-1},$$

where $p=m+n$; if ΣC_p^{-1} is convergent the double series converges, while $\Sigma\Sigma a_{m,n}$ diverges if ΣD_p^{-1} is divergent. To establish these results, take the sum by diagonals, as in (2) of the last article. We obtain in this way the single series

$$\tfrac{1}{2}C_2^{-1}+\tfrac{2}{3}C_3^{-1}+\tfrac{3}{4}C_4^{-1}+\tfrac{4}{5}C_5^{-1}+\ldots<\Sigma C_p^{-1}$$

or

$$\tfrac{1}{2}D_2^{-1}+\tfrac{2}{3}D_3^{-1}+\tfrac{3}{4}D_4^{-1}+\tfrac{4}{5}D_5^{-1}+\ldots>\tfrac{1}{2}\Sigma D_p^{-1},$$

from which the theorem becomes evident.

Ex. 1. $\Sigma m^{-\alpha}n^{-\beta}$ converges if $\alpha>1$, $\beta>1$.

Ex. 2. $\Sigma(m+n)^{-\alpha}$ converges if $\alpha>2$ and diverges if $\alpha\leqq 2$.

Ex. 3. If a, c are positive, (and $ac>b^2$, in case $b<0$), the series

$$\Sigma(am^2+2bmn+cn^2)^{-\lambda}$$

converges if $\lambda>1$ and diverges if $\lambda\leqq 1$; for we have

$$A(m+n)^2>am^2+2bmn+cn^2>2(b+\sqrt{ac})mn,$$

where A is the greatest of a, c, and $|b|$.

Thus the conditions of convergence or divergence follow from Exs. 1 and 2. See also Ex. 4 below.

The reader will have no difficulty in seeing that the following generalisation of Maclaurin's test (Art. 11) is correct:

(2) *If the function $f(x, y)$ is positive and steadily decreases to zero as x and y increase to infinity,*† *then the double series $\Sigma f(m, n)$ converges or diverges with the double integral*

$$\int^\infty\int^\infty f(x, y)\,dx\,dy.$$

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* Pringsheim, *Münchener Sitzungsberichte*, Bd. 27, pp. 146-150.

† That is, we suppose $f(\xi,\eta)\leqq f(x,y)$,

if $\xi\geqq x$ and $\eta\geqq y$.

However, nearly all cases of interest which come under the test (2) can be as easily tested by the following method, which depends only on a single integral:

(3) *If the positive function* $f(x, y)$ *has a lower limit* $g(\xi)$ *and an upper limit* $G(\xi)$ *when* $y=\xi-x$ *and* x *varies from* 0 *to* ξ, *and if* $\xi G(\xi)$, $\xi g(\xi)$ *tend steadily to zero as* $\xi\to\infty$, *then the double series* $\Sigma\Sigma f(m, n)$ *converges if the integral* $\int^\infty G(\xi)\xi\,d\xi$ *converges; but the series diverges if the integral* $\int^\infty g(\xi)\xi\,d\xi$ *diverges.**

For then the sum of the terms on the diagonal $x+y=n$ lies between $(n-1)g(n)$ and $(n-1)G(n)$; thus the series converges with $\Sigma(n-1)G(n)$, that is, with the integral $\int^\infty G(\xi)\xi\,d\xi$; but the series diverges with $\Sigma(n-1)g(n)$, that is, with the integral $\int^\infty g(\xi)\xi\,d\xi$.

Ex. 4. A particular case of (3) which has some interest is given by the double series $f(am^2+2bmn+cn^2)$, where $f(x)$ is a function which steadily decreases as its argument increases, and $am^2+2bmn+cn^2$ is subject to the same conditions as in Ex. 3 above.

If A is the greatest of a, $|b|$, c, it is evident that

$$ax^2+2bx(\xi-x)+c(\xi-x)^2$$

is less than $A[x^2+2x(\xi-x)+(\xi-x)^2]=A\xi^2$. When b is positive, we see in the same way that if B is the least of a, b, c, the expression is greater than $B\xi^2$. And if b is negative, we can put the expression in the form

$$[\{(a+c-2b)x+(b-c)\xi\}^2+(ac-b^2)\xi^2]/(a+c-2b)$$

and this is greater than $B\xi^2$, if $B=(ac-b^2)/(a+c-2b)$.

Hence $\quad g(\xi)=f(A\xi^2) \quad$ and $\quad G(\xi)=f(B\xi^2)$.

Thus the series converges if $\int^\infty f(B\xi^2)\xi\,d\xi$ converges; that is, if $\int^\infty f(x)\,dx$ is convergent. Similarly the series is seen to diverge when $\int^\infty f(x)dx$ is divergent.

This result confirms Ex. 3 above; and it shews also that when

$$f(x)=e^{-x} \quad \text{or} \quad 1/x(\log x)^{1+\alpha}, \quad (\alpha>0)$$

the series converges; on the other hand, the series diverges if

$$f(x)=1/x\log x.$$

* The use of a single integral for testing multiple series seems to be due to Riemann (*Ges. Werke*, 1876, p. 452); an alternative investigation is given by Hurwitz (*Math. Annalen*, Bd. 44, p. 83). The above form seems to be novel.

33. Absolutely convergent double series.

Just as in the theory of single series, we call the series $\Sigma a_{m,n}$ *absolutely convergent* if $\Sigma|a_{m,n}|$ is convergent.

The method used in Art. 26 can be applied at once to shew that the results proved in Art. 31 for double series of positive terms are still true for any absolutely convergent double series.*

In fact, to prove (1) we need only remark that if m, n are both greater than μ, the difference $|s_{m,n}-\sigma_\mu|$ is in general less than the corresponding difference, when all the terms are made positive; and it is proved in Art. 31 that this difference can be made less than ϵ by proper choice of μ. Similarly, in dealing with (2), we remark that $|S_n-\sigma_p|$ is less than the difference between the sums for the squares p, q when all the terms are made positive; and this difference can be made less than ϵ by proper choice of p.

No fresh proof is necessary for (3); but in dealing with (4), we note that the existence of $\lim_{(n)} s_{m,n}$ follows from the principle of absolute convergence (as applied to a single series), and the existence of the double sum s follows from (1). Then the difference

$$|s-\lim_{(n)} s_{m,n}|$$

is less than the sum of all terms of the *positive* series for which the first suffix exceeds m; and this sum can be made less than ϵ, if $m>\mu$.

The discussion in (5) is modified on exactly similar lines. But a complete formulation of the discussion of the general double limits in (6) is more troublesome; and it is better to consider each type of problem separately.

That these results are not necessarily true for non-absolutely convergent series may be seen by taking two simple examples:

(1) Consider first

$$\begin{array}{l} \;\;1+1+1+1+\ldots \\ +1-1-1-1-\ldots \\ +1-1+0+0+\ldots \\ +1-1+0+0+\ldots \\ +\ldots\ldots\ldots\ldots\ldots\ldots, \end{array}$$

where all the terms are 0 except in the first two rows and columns.

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* In this connexion the reader who has advanced beyond the elements of the subject should consult a paper by Hardy (*Proc. Lond. Math. Soc.* (2), vol. 1, 1903, p. 285).

Here $s_{m,n}=2$ if $m, n>1$, so that the series has the sum 2, according to Pringsheim's definition. But if we convert the double series into a single series by summing the diagonals (as in (2), Art. 31), we get

$$1+2+1+0+0+\ldots=4.$$

Obviously, too, the convergence of this series does not imply the convergence of the first two rows and columns (compare (4) Art. 31).

(2) Consider next the double series suggested by Cesàro:

$$\begin{aligned}&\frac{1}{2}-\frac{1}{4}+\frac{1}{4}-\frac{1}{8}+\frac{1}{8}-\frac{1}{16}+\frac{1}{16}-\ldots\\&+\frac{1}{2^2}-\frac{3}{4^2}+\frac{3}{4^2}-\frac{7}{8^2}+\frac{7}{8^2}-\frac{15}{16^2}+\frac{15}{16^2}-\ldots\\&+\frac{1}{2^3}-\frac{3^2}{4^3}+\frac{3^2}{4^3}-\frac{7^2}{8^3}+\frac{7^2}{8^3}-\frac{15^2}{16^3}+\frac{15^2}{16^3}-\ldots\\&+\frac{1}{2^4}-\frac{3^3}{4^4}+\frac{3^3}{4^4}-\frac{7^3}{8^4}+\frac{7^3}{8^4}-\frac{15^3}{16^4}+\frac{15^3}{16^4}-\ldots\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Here the sums of the rows in order are

$$\frac{1}{2},\ \frac{1}{2^2},\ \frac{1}{2^3},\ \frac{1}{2^4},\ \ldots,$$

and so the sum of all the rows is 1.

But the sums of the columns are

$$+1,\ -1,\ +1,\ -1,\ +1,\ -1,\ +1,\ \ldots,$$

proving that (5) of Art. 31 does not apply.

The second result is specially striking because each row converges absolutely (the terms being less than $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\ldots$), and secondly, the series formed by the sums of the rows is

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots,$$

which also converges absolutely.

But the justification for applying (5) of Art. 31 is that the double series still converges when all the terms are made positive, which is not the case here; since the sum of the first n columns then becomes equal to n.

The fact that the sum of a non-absolutely convergent double series may have different values according to the mode of summation has led Jordan* to frame a definition which admits only absolute convergence. Such restriction seems, however, unnecessary, pro-

* *Cours d'Analyse,* t. 1, p. 302; compare Goursat's *Analysis* (translation by Hedrick), vol. 1, p. 357.

vided that, when a non-absolutely convergent series is used, we do not attempt to employ theorems (1) to (5) of Art. 31 without special justification.

For example in Lord Kelvin's discussion of the force between two electrified spheres in contact, the repeated series

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{(-1)^{m+n}mn}{(m+n)^3}\right]$$

is used.* This series has the sum $\frac{1}{6}(\log 2-\frac{1}{4})$, and it has the same value if we sum first with respect to m. However, Pringsheim's sum does not exist but oscillates between limits † $\frac{1}{6}(\log 2-\frac{5}{8})$ and $\frac{1}{6}(\log 2+\frac{1}{8})$; while the diagonal series oscillates between $-\infty$ and $+\infty$.

34. A special example of deranging a double series is given by the *rule for multiplying single series* given in Art. 27 above.

Suppose we take the two single series $A=\Sigma a_n$, $B=\Sigma b_n$, and construct from them the double series $P=\Sigma a_m b_n$.

It is clear that P converges in Pringsheim's sense, provided that A, B converge; for we have

$$s_{m,n}=(a_1+a_2+\dots+a_m)(b_1+b_2+\dots+b_n),$$

so that

$$\lim_{m,n\to\infty} s_{m,n}=AB.$$

But for practical work in analysis it is generally necessary to convert the double series P into a single series; the one usually chosen being the sum by diagonals (see (2) Art. 31). This single series is Σc_n, where

$$c_n=a_1b_n+a_2b_{n-1}+\dots+a_nb_1.$$

It follows at once from Art. 33 that: *If the two series Σa_n, Σb_n are absolutely convergent, their product is equal to Σc_n, which is also absolutely convergent.*

For under these circumstances the double series is clearly absolutely convergent, because $\sum_{m,n}|a_m|\,.\,|b_n|$ converges to the sum $\Sigma|a_m|)\,.\,(\Sigma|b_n|)$; and $\Sigma|c_n|$ converges because the sum of any number of terms from $\Sigma|c_n|$ cannot exceed the product

$$(\Sigma|a_m|)\,.\,(\Sigma|b_n|).$$

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* Kelvin, *Reprint of Papers on Electrostatics and Magnetism*, § 140.

† Bromwich and Hardy, *Proc. Lond. Math. Soc.*, series 2, vol 2, 1904, p. 161 (see § 9).

If, however, one or both of A, B should not converge absolutely, we have **Abel's theorem**: *Provided that the series Σc_n converges, its sum is equal to the product AB.** For then, if we write

$$A_n = a_1 + a_2 + \dots + a_n, \quad B_n = b_1 + b_2 + \dots + b_n,$$

we find $\quad C_n = c_1 + c_2 + \dots + c_n = a_1 B_n + a_2 B_{n-1} + \dots + a_n B_1.$

Hence $\quad C_1 + C_2 + \dots + C_n = A_1 B_n + A_2 B_{n-1} + \dots + A_n B_1,$

or $$\frac{1}{n}(C_1 + C_2 + \dots + C_n) = \frac{1}{n}(A_1 B_n + A_2 B_{n-1} + \dots + A_n B_1).$$

Now (App. I. Art. 149), when $\lim C_n = C$, we have also

$$\lim \frac{1}{n}(C_1 + C_2 + \dots + C_n) = C;$$

and again (App. I. Art. 150.)

$$\lim \frac{1}{n}(A_1 B_n + A_2 B_{n-1} + \dots + A_n B_1) = AB.$$

Hence $$C = AB.$$

It should be observed that *the series Σc_n cannot diverge (if Σa_n and Σb_n are convergent), although it may oscillate.* For, if Σc_n is divergent, we should have $\lim C_n = \infty$, and therefore also

$$\lim \frac{1}{n}(C_1 + C_2 + \dots + C_n) = \infty,$$

by Art. 149; whereas this limit must be equal to AB. If Σc_n oscillates, it is clear from the article quoted that AB lies between the extreme limits of Σc_n; that in some cases Σc_n does oscillate (and that its extreme limits may be $-\infty$ and $+\infty$) is evident from Ex. 3 below; but in all cases the oscillation is of such a character † that

$$\lim \frac{1}{n}(C_1 + C_2 + \dots + C_n) = AB.$$

Ex. 1. Undoubtedly the cases of chief interest arise in the multiplication of power-series. Thus, if the two series

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots, \quad b_0 + b_1 x + b_2 x^2 + \dots$$

are both absolutely convergent for $|x| < r$ (see Art. 50), their product is given by

$$c_0 + c_1 x + c_2 x^2 + \dots,$$

* Pringsheim has proved, by a similar method, that if a double series is convergent, its sum is equal to the sum of the diagonal series, when the latter converges, provided that every row converges and also every column.

† Cesàro (to whom this result is due) calls such series *simply indeterminate*; the degree of indeterminacy being measured by the number of means which have to be taken before a definite value is obtained.

which is also absolutely convergent for $|x| < r$; where we have written

$$c_0 = a_0 b_0, \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0,$$

the notation being changed slightly from that used in the text.

Ex. 2. If we apply the rule to square the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$$

we have no reason (so far) to anticipate a convergent series; but we find the series

$$1 - \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{2^2} + \frac{1}{3}\right) - \ldots,$$

in which the general term is $(-1)^{n-1} w_n$, where

$$w_n = \frac{1}{1 \cdot n} + \frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \ldots + \frac{1}{n \cdot 1},$$

so that

$$(n+1) w_n = \left(1 + \frac{1}{n}\right) + \left(\frac{1}{2} + \frac{1}{n-1}\right) + \ldots + \left(\frac{1}{n} + 1\right)$$

$$= 2\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) \to 2(\log n + C) \text{ by Art. 11.}$$

Hence $$w_n \to 0.$$

Also $$(n+1) w_n - n w_{n-1} = 2/n.$$

Thus we get $$n(w_{n-1} - w_n) = w_n - 2/n > 0.$$

Accordingly (w_n) is a decreasing sequence and tends to zero as a limit.

Thus $\Sigma(-1)^{n-1} w_n$ is convergent (Art. 19), and therefore, by Abel's theorem,

$$\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)^2 = \frac{1}{2} - \frac{1}{3}\left(1 + \frac{1}{2}\right) + \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \frac{1}{5}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \ldots.$$

Of course this agrees with condition (iii) of Pringsheim's general theorem (Art. 35 below).

Ex. 3. But if we square the more general series

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots, \qquad (0 < p < 1),$$

we obtain $\Sigma(-1)^{n-1} w_n$, where

$$w_n = (1 \cdot n)^{-p} + [2(n-1)]^{-p} + \ldots + (n \cdot 1)^{-p}.$$

Now $$r(n+1-r) < n^2, \qquad \text{if } 0 < r < n+1,$$

so that $$[r(n+1-r)]^{-p} > n^{-2p}$$

and $$w_n > n^{1-2p}.$$

Consequently *if* $p \leqq \frac{1}{2}$, *the series* $\Sigma(-1)^{n-1} w_n$ *is oscillatory and the rule for multiplication fails,* in agreement with condition (iv) of Pringsheim's theorem (Art. 35). But if $p > \frac{1}{2}$, condition (iii) shews that the rule is correct.

35. Mertens has proved that *the series* Σc_n *will converge to the sum* AB, *provided that* one *of the series* Σa_n, Σb_n *is absolutely convergent.*

Suppose that Σa_n is absolutely convergent; and write $\alpha_n = |a_n|$, so that $\Sigma \alpha_n$ is convergent.

As in Art. 34 we find that

$$C_n = a_1 B_n + a_2 B_{n-1} + \dots + a_n B_1,$$

and $$A_n B = (a_1 + a_2 + \dots + a_n) B.$$

Thus $$A_n B - C_n = a_1 r_n + a_2 r_{n-1} + \dots + a_n r_1,$$

where $$r_n = B - B_n = b_{n+1} + b_{n+2} + \dots \text{ to } \infty,$$

so that r_n is the remainder after n terms in B.

Let m denote $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$, according as n is even or odd; and divide the last expression into two, thus:

$$A_n B - C_n = (a_1 r_n + a_2 r_{n-1} + \dots + a_m r_{n+1-m}) + (a_{m+1} r_{n-m} + a_{m+2} r_{n-m-1} + \dots + a_n r_1). \qquad (\omega)$$

In the first line of (ω), the suffix of every r is not less than m, because $n+1 \geqq 2m$; thus, in each of these terms we can write

$$|r| \leqq H_m,$$

where H_m is the upper limit of the sequence

$$|r_m|, \quad |r_{m+1}|, \quad |r_{m+2}|, \dots \text{ to } \infty.$$

Similarly, in the second line of (ω), we can write

$$|r| \leqq H_1.$$

It now follows that

$$|A_n B - C_n| < (\alpha_1 + \alpha_2 + \dots + \alpha_m) H_m + (\alpha_{m+1} + \alpha_{m+2} + \dots + \alpha_n) H_1. \qquad (\omega')$$

In this inequality we can allow m to tend to ∞; and then $H_m \to 0$ because the remainders $r_m, r_{m+1}, r_{m+2}, \dots$ all tend to zero. Further $(\alpha_1 + \alpha_2 + \dots + \alpha_m)$, being less than $\sum_1^\infty \alpha_n$, remains finite; and so $(\alpha_1 + \alpha_2 + \dots + \alpha_m) H_m \to 0$.

Again $(\alpha_{m+1} + \alpha_{m+2} + \dots + \alpha_n)$, being less than the remainder after m terms in $\Sigma \alpha_n$, tends to zero as m tends to ∞.

Thus, finally, both terms in (ω') must tend to zero; or

$$A_n B - C_n \to 0, \quad \text{as } n \to \infty.$$

But $A_n \to A$; and so we have

$$C_n \to AB,$$

which establishes the theorem as originally stated.

It must not, however, be supposed that the condition of Mertens is *necessary* for the convergence of Σc_n; in fact Pringsheim has established a large number of results on the multiplication of two series, neither of which converges absolutely. The simplest of these (including most cases of interest) is as follows.*

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* The following proof is based upon one published by Mr. Hardy (*Proc. Lond. Math. Soc.*, vol. 6, 1908, p. 410); it is considerably easier than the proof given in the first edition of this book, which was itself a simplified version of Pringsheim's method.

If u_n, v_n are positive and tend steadily to zero, we have the following alternative sets of conditions for the multiplication of the series

$$A=\Sigma(-1)^{n-1}u_n \quad \text{and} \quad B=\Sigma(-1)^{n-1}v_n$$

by the rule of Art. 34 :

(i) *it is necessary and sufficient that*

$$w_n=u_1v_n+u_2v_{n-1}+\ldots+u_nv_1 \to 0;$$

(ii) *it is necessary and sufficient that*

$$U_nv_n \to 0 \quad \textit{and} \quad V_nu_n \to 0,$$

where $$U_n=u_1+u_2+\ldots+u_n, \quad V_n=v_1+v_2+\ldots+v_n;$$

(iii) *it is sufficient (but not necessary) that* Σu_nv_n *should be convergent ;*

(iv) *it is necessary (but not sufficient) that* $nu_nv_n \to 0$.

To prove (i) we use the formula for A_nB-C_n given above ; in the present case

$$\alpha_1=u_1, \quad \alpha_2=u_2, \ \ldots \ \alpha_n=u_n, \ldots$$

and $$|r_n|=v_{n+1}-v_{n+2}+v_{n+3}-\ldots,$$

so that $$|r_n| \leqq v_n. \qquad \text{(Art. 19)}$$

Consequently, we have

$$|A_nB-C_n| \leqq u_1v_n+u_2v_{n-1}+\ldots+u_nv_1=w_n.$$

Thus the condition $w_n \to 0$ gives $C_n \to AB$; and since $C_n=(-1)^{n-1}w_n$, the condition $w_n \to 0$ is *necessary* to ensure convergence of the series ΣC_n; thus (i) is proved.

Again, since (v_n) is a decreasing sequence, we have

$$w_n>(u_1+u_1+\ldots+u_n)v_n=U_nv_n,$$

and similarly $$w_n>V_nu_n.$$

Also $$w_n<(u_1+u_2+\ldots+u_p)v_q+(v_1+v_2+\ldots+v_q)u_p,$$

where p, q are any two integers, such that $p+q=n$, because

$$v_n<v_{n-1}<\ldots<v_{q+1}<v_q,$$

and similarly $$u_n<u_{n-1}<\ldots<u_{p+1}<u_p.$$

Hence $$w_n<U_pv_p+V_pu_p, \qquad \text{when} \quad n=2p,$$

or $$<U_pv_{p+1}+V_{p+1}u_p, \qquad n=2p+1.$$

It is now clear that if $w_n \to 0$, U_nv_n and V_nu_n must also tend to zero, and conversely ; thus (ii) is proved.

If, as in (iii), Σu_nv_n is convergent, we can find m so that

$$u_mv_m+u_{m+1}v_{m+1}+\ldots+u_nv_n<\epsilon, \quad \text{if } n>m.$$

Consequently, since (v_n) is a decreasing sequence, we have

$$(u_{m+1}+u_{m+2}+\ldots+u_n)v_n<\epsilon,$$

or $$U_nv_n-U_mv_n<\epsilon, \quad \text{if } n>m.$$

We can now find a value n_0 such that

$$U_mv_n<\epsilon, \quad \text{if } n>n_0,$$

and so $$U_nv_n<2\epsilon, \quad \text{if } n>n_0.$$

Consequently $U_n v_n \to 0$; and similarly $V_n u_n \to 0$. Thus, by applying (ii), *the condition* (iii) *is sufficient.*

Finally, we have $w_n \geqq U_n v_n$ and $U_n \geqq n u_n$,

so that $w_n \geqq n u_n v_n$; and *the condition* $n u_n v_n \to 0$ *is necessary to make* $w_n \to 0$. Hence (iv) is proved.

Other results are due to Voss and Cajori, in addition to those found by Pringsheim. For references, see § 34 of Pringsheim's article in the *Encyklopädie*, Bd. I.; two other papers will be found in the *Trans. Amer. Math. Society*, vol. 2, 1901, pp. 25 and 404. Reference should also be made to the paper by Hardy quoted on p. 93, and to A. E. Jolliffe's results given in Exs. 18, 19 at the end of this chapter.

36. Substitution of a power-series in another power-series.

This operation gives another example of deranging a double series. Consider the series $z=f(y)=a_0+a_1y+a_2y^2+\dots$ and $y=b_0+b_1x+b_2x^2+\dots$; if convergent at all, they converge absolutely for $|y|<s$, $|x|<r$, say (see Art. 50). The question then arises whether the result of substituting the second series in the first and arranging in powers of x is ever convergent, and if so, for what values of x. It appears from Ex. 1, Art. 34, that the powers of y can be calculated by using the rule for the multiplication of series, and then z is equal to *the sum by rows* of the double series

$$\left.\begin{array}{l|l|l|l} a_0 & & & \\ +a_1b_0 & +\ a_1b_1x & +\ a_1b_2x^2 & +\dots \\ +a_2b_0^2 & +2a_2b_0b_1x & +\ a_2(b_1^2+2b_0b_2)x^2 & +\dots \\ +a_3b_0^3 & +3a_3b_0^2b_1x & +3a_3(b_0b_1^2+b_0^2b_2)x^2 & +\dots \\ +\dots\dots & \dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots & \dots\dots \end{array}\right\} \quad \dots\dots\dots(1)$$

If this double series is arranged according to powers of x, we are summing it by *columns*; these two sums are certainly equal if the double series still converges, after every term is made positive (Art. 31 (5) and Art. 33).

Write $|a_n|=\alpha_n$, $|b_n|=\beta_n$, $|x|=\xi$,

and then the new series is not greater than

$$\left.\begin{array}{l|l|l|l} \alpha_0 & & & \\ +\alpha_1\beta_0 & +\alpha_1\beta_1\xi & +\ \alpha_1\beta_2\xi^2 & +\dots \\ +\alpha_2\beta_0^2 & +2\alpha_2\beta_0\beta_1\xi & +\ \alpha_2(\beta_1^2+2\beta_0\beta_2)\xi^2 & +\dots \\ +\alpha_3\beta_0^3 & +3\alpha_3\beta_0^2\beta_1\xi & +3\alpha_3(\beta_0\beta_1^2+\beta_0^2\beta_2)\xi^2 & +\dots \\ +\dots\dots & \dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots & \dots\dots \end{array}\right\} \quad \dots(2)$$

Now this series, summed by rows, gives

$$\alpha_0+\Sigma\alpha_n(\beta_0+\Sigma\beta_m\xi^m)^n,$$

which converges, provided that $\beta_0+\Sigma\beta_m\xi^m < s$.

Take now any positive number less than r, say ρ, then the series $\Sigma\beta_m\rho^m$ is convergent, and consequently the terms $\beta_m\rho^m$ have a finite upper limit M. Thus our condition is satisfied if

$$\beta_0+\Sigma M(\xi/\rho)^m < s,$$

or if

$$\beta_0+M\xi/(\rho-\xi) < s.$$

Hence if $\beta_0 < s$, and $\xi < (s-\beta_0)\rho/(M+s-\beta_0)$, the series (2) of positive terms will converge. Consequently the derangement of the series (1) will not alter its sum. Thus *the transformation is permissible if the two conditions*

$$\text{(i) } |b_0| < s, \qquad \text{(ii) } |x| < (s-|b_0|)\rho/(M+s-|b_0|)$$

are satisfied (where $\rho < r$, $|b_n|\rho^n < M$). *In particular, if* $b_0=0$, *the conditions may be replaced by the one*

$$|x| < \rho s/(M+s).$$

If the series $z=\Sigma a_n y^n$ *converges for all values of* y, *it is evident that the condition* $|x| < r$ *is sufficient to justify the derangement.*

The case $b_0=0$ is of special interest in practice; and then the coefficient of x^n in the final series is not itself an infinite series, but terminates; a few of the coefficients are

$$c_0=a_0,\quad c_1=a_1b_1,\quad c_2=a_1b_2+a_2b_1^2,\quad c_3=a_1b_3+2a_2b_1b_2+a_3b_1^3,$$

and generally, if $n > 2$, c_n will contain the terms

$$a_1b_n+2a_2b_1b_{n-1}+\dots+a_nb_1^n.$$

Ex. Take
$$z=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\dots,$$
$$y=\mu(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\dots),$$
then the transformation is allowable, provided that $|x| < 1$, since z converges for all values of y, and $r=1$. The result is obviously of the form $1+\Sigma c_nx^n$, where c_n is a polynomial in μ, such that the term of highest degree is $\mu^n/n!$.

Assuming that $z=e^y$ and $y=\mu\log(1+x)$ (see Arts. 58, 62), we see that $z=(1+x)^\mu$. Thus c_n will vanish for $\mu=0, 1, 2, \dots, n-1$, because in these cases the series terminates before reaching x^n.

Hence, in general, $c_n=\dfrac{1}{n!}\mu(\mu-1)(\mu-2)\dots(\mu-n+1)$,

and so we obtain the binomial series (Arts. 61 and 96).

37. Non-absolutely convergent double series.

Almost the only general type of such series has been given by Hardy;* it corresponds to the type of series discussed in Arts. 21, 22. The theorem is the extension of Dirichlet's test, and runs:

If in a double series $\Sigma a_{m,n}$ the sum $s_{m,n}$ is numerically less than a constant C for all values of m, n, the double series $\Sigma a_{m,n}v_{m,n}$ converges, provided that the expressions

$$v_{m,n}-v_{m+1,n},\ \ v_{m,n}-v_{m,n+1},\quad v_{m,n}-v_{m+1,n}-v_{m,n+1}+v_{m+1,n+1}$$

are all positive and that $v_{m,n}$ tends to zero as either m or n tends to ∞.

In fact, just as in the proof of Abel's lemma (Art. 20), we can shew that under the given conditions for $v_{m,n}$, we have

$$\left|\sum_{\mu}^{M}\sum_{\nu}^{N} a_{m,n}v_{m,n}\right| < Hv_{\mu,\nu},\quad (M>\mu,\ N>\nu),$$

where H is an upper limit to

$$\left|\sum_{\mu}^{\xi}\sum_{\nu}^{\eta} a_{m,n}\right|,\quad (\xi=\mu,\ \mu+1,\ \ldots,\ M;\ \eta=\nu,\ \nu+1,\ \ldots,\ N).$$

But
$$\sum_{\mu}^{\xi}\sum_{\nu}^{\eta} a_{m,n}=s_{\xi,\eta}-s_{\xi,\nu-1}-s_{\mu-1,\eta}+s_{\mu-1,\nu-1},$$

so that if either μ or ν is 1, $H\leqq 2C$, and otherwise $H\leqq 4C$.

Now
$$\sum_{1}^{M}\sum_{1}^{N}-\sum_{1}^{\mu-1}\sum_{1}^{\nu-1}=\sum_{1}^{\mu-1}\sum_{\nu}^{N}+\sum_{\mu}^{M}\sum_{1}^{\nu-1}+\sum_{\mu}^{M}\sum_{\nu}^{N},$$

so that $$\left|\left(\sum_{1}^{M}\sum_{1}^{N}-\sum_{1}^{\mu-1}\sum_{1}^{\nu-1}\right)a_{m,n}v_{m,n}\right|<2C(v_{1,\nu}+v_{\mu,1}+2v_{\mu,\nu})<4C(v_{1,\nu}+v_{\mu,1}),$$

which can be made as small as we please by proper choice of μ, ν, because $v_{1,\nu}$ and $v_{\mu,1}$ both tend to 0; and so the double series $\Sigma a_{m,n}v_{m,n}$ converges.

An example is given by the series

$$a_{m,n}=\cos(m\theta+n\phi),\quad v_{m,n}=(m\alpha+n\beta)^{-p},$$

where α, β, p are positive.

For then
$$|s_{m,n}|\leqq 4\,|\operatorname{cosec}\tfrac{1}{2}\theta\operatorname{cosec}\tfrac{1}{2}\phi|$$

and
$$v_{m,n}-v_{m+1,n}=\int_m^{m+1}\frac{p\alpha\,dx}{(\alpha x+\beta n)^{p+1}}>0,$$

$$v_{m,n}-v_{m+1,n}-v_{m,n+1}+v_{m+1,n+1}=\int_m^{m+1}dx\int_n^{n+1}\frac{p(p+1)\alpha\beta\,dy}{(\alpha x+\beta y)^{p+2}}>0,$$

while $\lim v_{m,n}=0$.

* *Proc. Lond. Math. Soc.*, series 2, vol. 1, 1903, p. 124; vol. 2, 1904, p. 190.

EXAMPLES.

1. As examples of *double sequences* we take the following:

(1) $s_{m\cdot n} = \dfrac{1}{m} + \dfrac{1}{n}$; here the double and repeated limits exist and are all equal to 0.

(2) $s_{m,n} = (-1)^{m+n}\left(\dfrac{1}{m} + \dfrac{1}{n}\right)$; here the double limit is again 0, but the single and repeated limits do not exist, although we have

$$\lim_{m \to \infty}\left(\overline{\lim_{n \to \infty}}\, s_{m\cdot n}\right) = 0 = \lim_{n \to \infty}\left(\overline{\lim_{m \to \infty}}\, s_{m\cdot n}\right).$$

(3) $s_{m\cdot n} = mn/(m^3 + n^3)$; here the double and repeated limits are again all 0.

(4) $s_{m\cdot n} = m/(m+n)$; here the double limit does not exist, but we have

$$0 < s_{m\cdot n} < 1, \quad \text{and} \quad \underline{\lim}\, s_{m,n} = 0, \quad \overline{\lim}\, s_{m\cdot n} = 1,$$

because, however large μ may be, we can find values of m, n greater than μ, such that $s_{m\cdot n} < \epsilon$; and other values of m, n for which $s_{m\cdot n} > 1 - \epsilon$. But the repeated limits exist and are such that

$$\lim_{m \to \infty}\left(\lim_{n \to \infty} s_{m\cdot n}\right) = 0, \quad \lim_{n \to \infty}\left(\lim_{m \to \infty} s_{m\cdot n}\right) = 1.$$

Similar features present themselves in the sequences

$$s_{m\cdot n} = mn/(m^2 + n^2) \quad \text{and} \quad s_{m\cdot n} = 1/[1 + (m-n)^2]$$

(5) If $s_{m,n} = (-1)^m m^2 n^3/(m^3 + n^6)$, we have

$$\lim_{(m)}\left(\lim_{(n)} s_{m\cdot n}\right) = 0 = \lim_{(n)}\left(\lim_{(m)} s_{m\cdot n}\right);$$

but yet

$$\underline{\lim}\, s_{m\cdot n} = -\infty, \quad \overline{\lim}\, s_{m\cdot n} = +\infty;$$

as may be seen by taking $m = n^2$. Here it should be noticed that the limit of the single sequence given by putting $m = n$ exists and is equal to 0; although the double limit does not exist. [PRINGSHEIM.]

2. The double series given by

$$\begin{array}{llll}
(a_0 + b_0) + & (a_1 - b_0) & + a_2 + a_3 + a_4 + \dots \\
(-a_0 + b_1) + & (-a_1 - b_1) & - a_2 - a_3 - a_4 - \dots \\
\quad b_2 \quad - & \quad b_2 & + 0 + 0 + 0 + \dots \\
\quad b_3 \quad - & \quad b_3 & + 0 + 0 + 0 + \dots \\
\dots & \dots & \dots
\end{array}$$

gives the sum 0 in Pringsheim's sense, whatever may be the values of a_n, b_n. But the sum by rows is only convergent if Σa_n converges; and the sum by columns converges only if Σb_n is convergent. The sum by diagonals is $\lim (a_n + b_n)$, if this limit exists; and is otherwise oscillatory.

3. In the double series

$$\begin{array}{l} 1+2+4+8+\dots \\ -\tfrac{1}{2}-1-2-4-\dots \\ -\tfrac{1}{4}-\tfrac{1}{2}-1-2-\dots \\ -\tfrac{1}{8}-\tfrac{1}{4}-\tfrac{1}{2}-1-\dots \\ \dots\dots\dots\dots\dots \end{array}$$

every column converges to 0, but every row diverges. Of course Pringsheim's sum cannot exist; and the sum by diagonals is divergent.

4. The series given by

$$\begin{array}{r} 0+1+0+0+0+\dots \\ -1+0+1+0+0+\dots \\ 0-1+0+1+0+\dots \\ 0+0-1+0+1+\dots \\ 0+0+0-1+0+\dots \\ \dots\dots\dots\dots\dots \end{array}$$

has the sum 1 by rows; -1 by columns; 0 by diagonals; and naturally the double series cannot converge in Pringsheim's sense. In fact, $s_{m,n}$ is 0 if $m=n$, and is -1 if $m>n$, or $+1$ if $m<n$.

5. The series given by

$$\begin{array}{r} -2+1+0+0+0+\dots \\ +1-2+1+0+0+\dots \\ 0+1-2+1+0+\dots \\ 0+0+1-2+1+\dots \\ 0+0+0+1-2+\dots \\ \dots\dots\dots\dots\dots \end{array}$$

has the sum -1 both by rows and columns; and the diagonal sum oscillates between -2 and 0. There is no sum in Pringsheim's sense, because $s_{m,n}$ is -2 if $m=n$, and is otherwise -1.

6. The double series

$$\begin{array}{r} 2+0-1+0+0+0+\dots \\ 0+2+0-1+0+0+\dots \\ -1+0+2+0-1+0+\dots \\ 0-1+0+2+0-1+\dots \\ 0+0-1+0+2+0+\dots \\ 0+0+0-1+0+2+\dots \\ +\dots\dots\dots\dots\dots \end{array}$$

has the sum by rows $1+1+0+0+\dots=2$, and the same sum by columns; the sum by diagonals is $2+0+0+0+\dots=2$. Thus *these three sums are the same, but the series does not converge in Pringsheim's sense, since* $a_{n,n}=2$.

7. Prove that the multiplication rule for $\Sigma a_n x^n$, $\Sigma b_n x^n$ can be established by summing the double series

$$\begin{array}{r} a_0b_0+a_0b_1x+a_0b_2x^2+\dots \\ +a_1b_0x+a_1b_1x^2+\dots \\ +a_2b_0x^2+\dots \\ +\dots \end{array}$$

first by rows and secondly by columns.

8. Discuss the following paradox:

If we sum the double series of positive terms

$$\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\frac{1}{4\,.\,5}+\ldots$$
$$+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\frac{1}{4\,.\,5}+\ldots$$
$$+\frac{1}{3\,.\,4}+\frac{1}{4\,.\,5}+\ldots$$
$$+\frac{1}{4\,.\,5}+\ldots$$
$$+\ldots$$

first by rows and secondly by columns, we obtain $s+1=s$ or $1=0$, where

$$s=\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots.$$ [J. BERNOULLI.]

9. If a_n is positive and such that the series $\Sigma a_n{}^2$ is convergent, prove that the double series $\Sigma\Sigma a_m a_n/(m+n)$ is convergent. [HILBERT.] *

[Consider the terms given by the square $m=\mu$, $n=\mu$; on account of the symmetry (with respect to m and n), we consider first only the terms on the right of the diagonal $m=n$. Including the terms *on* the diagonal, we obtain

$$\sum_{n=1}^{\mu} a_n\left\{\sum_{m=1}^{n} a_m/(m+n)\right\}<\sum_{n=1}^{\mu} a_n b_n$$

if

$$b_n=(a_1+a_2+\ldots+a_n)/n.$$

Thus,

$$\sigma_\mu<2\sum_{n=1}^{\mu} a_n b_n<\sum_{n=1}^{\mu} a_n{}^2+\sum_{n=1}^{\mu} b_n{}^2,$$

where σ_μ is used in the sense of Art. 29.

Thus convergence follows from Art. 31 (1) and Ex. 17, Ch. II.]

10. If the double series $\Sigma\Sigma a_{m,n}$ is convergent in Pringsheim's sense, it does not follow (in contrast to the case of single series) that a constant C can be found such that $|s_{m,n}|<C$ for all values of m, n; this is seen by considering the series of Ex. 2, and supposing Σa_n, Σb_n to be divergent.

In like manner we cannot infer $|s_{m,n}|<C$ from the convergence of the sum by columns or by rows (see for instance Ex. 3).

11. The double series in which $a_{m,n}=(-1)^{m+n}/mn$ does *not* converge absolutely; but yet its sums by rows, columns and diagonals are equal to one another and to Pringsheim's sum. The common value is, in fact, $(\log 2)^2$.

Exactly similar results apply to the series in which $a_{m,n}=(-1)^{m+n}u_m v_n$, where the sequences (u_m), (v_n) steadily decrease to zero.

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* Hilbert's own proof is given by Weyl (*Singuläre Integralgleichungen*, Dissertation, Göttingen, 1908, p. 83); a somewhat simpler proof was given by Wiener (*Math. Annalen*, vol. 68). The connexion with Ex. 17, Ch. II., was suggested by Hardy (1919).

12. Consider the double series in which

$$a_{m,n} = 1/(m^2 - n^2) \qquad (m \gtrless n)$$

and
$$a_{m,n} = 0. \qquad (m = n)$$

Here we find
$$\left.\begin{aligned} \sum_{(m)} \sum_{(n)} &= -\tfrac{3}{4}\Sigma(1/m^2) \\ \sum_{(m)} \sum_{(n)} &= +\tfrac{3}{4}\Sigma(1/n^2). \end{aligned}\right\} \text{(Ex. 13, Chap. III.)}$$
and

[HARDY.]

13. Prove that
$$\sum_{m=-\nu}^{\nu} \sum_{n=-\nu}^{\nu} \frac{2x + m + n}{(x+m)^2 (x+n)^2}$$
tends to zero when ν tends to ∞, provided that all terms for which $m = n$ are omitted from the summation. [*Math. Trip.* 1895.]

14. If
$$a_{m,n} = \frac{m-n}{2^{m+n}} \frac{(m+n-1)!}{m!\,n!} \quad (m, n > 0)$$
and
$$a_{m,0} = 2^{-m}, \quad a_{0,n} = -2^{-n}, \quad a_{0,0} = 0,$$
then
$$\sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{m,n} \right) = -1, \quad \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} \right) = +1.$$

15. If $a_{m,n} = (-1)^{m+n} mn/(m+n)^2$, we have
$$\sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{m,n} \right) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} \right) = \tfrac{1}{8}(\log 2 - \tfrac{1}{4}) = l,$$
but the sum by diagonals oscillates between $-\infty$ and ∞: and Pringsheim's sum oscillates between $l - \frac{1}{16}$ and $l + \frac{1}{16}$.

[For the details of Exs. 2, 3, 10, 14, 15, see Bromwich and Hardy, *Proc. Lond. Math. Soc.* (2), vol. 2, 1904, p. 175.]

16. Prove that the product of the two series
$$1 + \frac{x}{1^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \dots, \quad 1 - \frac{x}{1^2} + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \dots$$
is equal to
$$1 - \frac{x^2}{1^2 \,.\, 2!} + \frac{x^4}{(2!)^2 \,.\, 4!} - \frac{x^6}{(3!)^2 \,.\, 6!} + \dots .$$

17. If
$$f(x, q) = 1 + q(x + 1/x) + q^4(x^2 + 1/x^2) + q^9(x^3 + 1/x^3) + \dots$$
$$= \sum_{-\infty}^{\infty} q^{n^2} x^n, \quad \text{where } |q| < 1,$$
and
$$g(x, q) = q^{\frac{1}{4}} x^{-\frac{1}{2}} f(x/q, q) = \sum_{-\infty}^{\infty} q^{\nu^2} x^{\nu}, \quad \nu = n - \tfrac{1}{2},$$
then
$$f(xy, q) \,.\, f(x/y, q) = f(x^2, q^2) \,.\, f(y^2, q^2) + g(x^2, q^2) \,.\, g(y^2, q^2),$$
$$g(xy, q) \,.\, g(x/y, q) = f(x^2, q^2) \,.\, g(y^2, q^2) + f(y^2, q^2) \,.\, g(x^2, q^2).$$

18. In Pringsheim's theorem of Art. 35, suppose that
$$u_n = 1/\phi(n), \quad v_n = 1/\psi(n),$$
where the functions $\phi(x)$, $\psi(x)$ tend steadily to infinity with x. Then prove that

(i) the conditions
$$\frac{1}{\phi(x)} \int_0^x \frac{d\xi}{\psi(\xi)} \to 0, \qquad \frac{1}{\psi(x)} \int_0^x \frac{d\xi}{\phi(\xi)} \to 0$$
are both necessary and sufficient;

(ii) the conditions

$$\phi'(x)\psi(x) \to \infty, \qquad \psi'(x)\phi(x) \to \infty$$

are sufficient (but not necessary). [A. E. JOLLIFFE.]

19. As special examples of Ex. 18, prove that

(i) if $\phi(x) = \log x$ and $\psi(x) = x\log(\log x)$,

the product-series $\Sigma(-1)^{n-1}w_n$ is convergent;

(ii) if $\phi(x) = x\log x$ and $\psi(x) = \log(\log x)$,

the product-series oscillates.

In cases (i) and (ii) the series $\Sigma u_n v_n$ is the same and is divergent (although $nu_n v_n \to 0$); and this indicates that no necessary and sufficient test can exist which depends only on the product $u_n v_n$. [A. E. JOLLIFFE.]

20. Verify that

$$\frac{1}{x(x+1)\dots(x+n)}$$
$$= \frac{1}{n!}\frac{1}{x} - \frac{1}{1!(n-1)!}\frac{1}{x+1} + \frac{1}{2!(n-2)!}\frac{1}{x+2} - \frac{1}{3!(n-3)!}\frac{1}{x+3} + \dots \pm \frac{1}{n!}\frac{1}{x+n},$$

and use Arts. 33, 57 to infer Prym's identity,

$$\frac{1}{x} + \frac{1}{x(x+1)} + \frac{1}{x(x+1)(x+2)} + \dots = e\left[\frac{1}{x} - \frac{1}{1!}\frac{1}{x+1} + \frac{1}{2!}\frac{1}{x+2} - \frac{1}{3!}\frac{1}{x+3} + \dots\right].$$

21. Shew that $\dfrac{1}{t^2} = \dfrac{1}{t(t+1)} + \dfrac{1}{t(t+1)(t+2)} + \dfrac{1 \cdot 2}{t(t+1)(t+2)(t+3)} + \dots$.

Hence convert $$\frac{1}{t^2} + \frac{1}{(t+1)^2} + \frac{1}{(t+2)^2} + \dots$$

into a double series, and transform it to

$$\frac{1}{t} + \frac{1}{2t(t+1)} + \frac{1 \cdot 2}{3t(t+1)(t+2)} + \frac{1 \cdot 2 \cdot 3}{4t(t+1)(t+2)(t+3)} + \dots.$$

Take $t = 10$ and so calculate $\Sigma 1/n^2$ to 7 decimal places. [STIRLING.]

22. Convert the series $\dfrac{x}{1+x^2} + \dfrac{x^2}{1+x^4} + \dfrac{x^3}{1+x^6} + \dots \quad (|x| < 1)$

into a double series, and deduce that it is equal to

$$\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \dots.$$

23. Shew that (if $|x| < 1$) Lambert's series,

$$\frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \dots = \sum_1^\infty \sum_1^\infty x^{mn},$$

and deduce that this series is equal to Clausen's series,

$$x\frac{1+x}{1-x} + x^4\frac{1+x^2}{1-x^2} + x^9\frac{1+x^3}{1-x^3} + x^{16}\frac{1+x^4}{1-x^4} + \dots.$$

Hence evaluate Lambert's series to five decimal places, for $x = \frac{1}{10}$ [$\cdot 122324$].

Shew that each of these series is also equal to

$$x + 2x^2 + 2x^3 + 3x^4 + 2x^5 + 4x^6 + \dots,$$

the coefficient of x^n being the number of divisors of n (1 and n included).

24. From Ex. 22, or directly, prove that, if $|x| < 1$,

$$\frac{x}{1+x^2}+\frac{x^3}{1+x^6}+\frac{x^5}{1+x^{10}}+\ldots=\frac{x}{1-x^2}-\frac{x^3}{1-x^6}+\frac{x^5}{1-x^{10}}-\ldots,$$

$$\frac{x}{1+x^2}-\frac{x^2}{1+x^4}+\frac{x^3}{1+x^6}-\ldots=\frac{x}{1+x}-\frac{x^3}{1+x^3}+\frac{x^5}{1+x^5}-\ldots$$

25. Shew that, if $|x| < 1$,

$$\frac{x}{1+x}-\frac{2x^2}{1+x^2}+\frac{3x^3}{1+x^3}-\ldots=\frac{x}{(1+x)^2}-\frac{x^2}{(1+x^2)^2}+\frac{x^3}{(1+x^3)^2}-\ldots,$$

$$\frac{x}{1-x^2}+\frac{3x^3}{1-x^6}+\frac{5x^5}{1-x^{10}}+\ldots=\frac{x(1+x^2)}{(1-x^2)^2}+\frac{x^3(1+x^6)}{(1-x^6)^2}+\frac{x^5(1+x^{10})}{(1-x^{10})^2}+\ldots.$$

[For the connexion between the series in Exs. 22–25 and elliptic functions, see Jacobi, *Fundamenta Nova*, § 40.]

26. If $|x| < 1$, shew that

$$\frac{x}{(1-x)^2}+\frac{x^2}{(1-x^2)^2}+\frac{x^3}{(1-x^3)^2}+\ldots=\sum_1^\infty \phi_n x^n,$$

where ϕ_n is the sum of the divisors of n (including 1 and n). Deduce that, if $\phi_{-1}=0=\phi_0$,

$$1+\frac{x}{(1+x)^2}+\frac{x^2}{(1+x+x^2)^2}+\ldots=\sum_0^\infty(\phi_{n+1}+\phi_{n-1}-2\phi_n)\,x^n.$$

[*Math. Trip.* 1899.]

27. If $|x| < 1$, prove that

$$f(1)\frac{x}{1-x}+f(2)\frac{x^2}{1-x^2}+f(3)\frac{x^3}{1-x^3}+\ldots=\sum_1^\infty\theta(n)\,x^n,$$

where $\theta(n)$ denotes the sum $\Sigma f(d)$ for all the divisors of n (including 1 and n).

In particular $\quad \dfrac{x}{1-x}-\dfrac{x^2}{1-x^2}-\dfrac{x^3}{1-x^3}+\dfrac{x^4}{1-x^4}-\dfrac{x^5}{1-x^5}+\dfrac{x^6}{1-x^6}-\ldots=\sum_1^\infty x^{n^2}.$

[LAGUERRE.]

28. Shew that in the special series of Art. 37, the repeated series also converge to the same sum as the double series; but the diagonal series may oscillate, for instance $\alpha=\beta=1$, $\theta=\phi=\pi$, $p=1$, gives for the diagonal series

$$\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{3}{4}-\tfrac{4}{5}+\ldots.$$

[HARDY.]

CHAPTER VI.

INFINITE PRODUCTS.

38. Weierstrass's Inequalities.

In this article the numbers $a_1, a_2, a_3, \dots$ are supposed to be positive and less than 1; this being the case, we see that

$$(1+a_1)(1+a_2)=1+(a_1+a_2)+a_1a_2>1+(a_1+a_2).$$

Hence

$$(1+a_1)(1+a_2)(1+a_3)>[1+(a_1+a_2)](1+a_3)>1+(a_1+a_2+a_3),$$

and continuing this process we see that

(1) $$(1+a_1)(1+a_2)(1+a_3)\dots(1+a_n)>1+(a_1+a_2+a_3+\dots+a_n).$$

In like manner we have

$$(1-a_1)(1-a_2)=1-(a_1+a_2)+a_1a_2>1-(a_1+a_2).$$

Thus, since $1-a_3$ is positive, we have

$$(1-a_1)(1-a_2)(1-a_3)>[1-(a_1+a_2)](1-a_3)>1-(a_1+a_2+a_3),$$

and so we have, generally,

(2) $$(1-a_1)(1-a_2)(1-a_3)\dots(1-a_n)>1-(a_1+a_2+a_3+\dots+a_n).$$

Next, $$1+a_1=\frac{1-a_1^2}{1-a_1}<\frac{1}{1-a_1},$$

so that $$(1+a_1)(1+a_2)\dots(1+a_n)<\frac{1}{(1-a_1)(1-a_2)\dots(1-a_n)},$$

and thus, *if** $a_1+a_2+\dots+a_n$ *is less than* 1, we have, by the aid of (2), the result

(3) $$(1+a_1)(1+a_2)\dots(1+a_n)<[1-(a_1+a_2+\dots+a_n)]^{-1}.$$

---

* If $a_1+a_2+\dots+a_n$ were greater than 1, the inequality (3) would be untrue, since it would then make a positive number less than a negative number.

Similarly, we find

(4) $$(1-a_1)(1-a_2)\dots(1-a_n) < [1+(a_1+a_2+\dots+a_n)]^{-1}.$$

By combining these four inequalities, we find the results

(5) $$(1-\Sigma a)^{-1} > \Pi(1+a) > 1+\Sigma a,$$

(6) $$(1+\Sigma a)^{-1} > \Pi(1-a) > 1-\Sigma a,$$

where all the letters a denote numbers between 0 *and* 1, *such that* Σa *is less than* 1.

39. If $a_1, a_2, a_3, \dots$ are numbers between 0 and 1, the convergence of the series Σa_n is necessary and sufficient for the convergence of the products P_n, Q_n to positive limits P, Q as n increases to ∞, where

$$P_n = (1+a_1)(1+a_2)\dots(1+a_n), \quad Q_n = (1-a_1)(1-a_2)\dots(1-a_n).$$

For clearly P_n increases as n increases, and Q_n decreases.

Now, if Σa_n is convergent, we can find a number m such that

$$\sigma = a_{m+1} + a_{m+2} + a_{m+3} + \dots \text{ to } \infty < 1.$$

Then, by the inequalities (5), (6) of the last article, we have

$$\frac{1}{1-\sigma} > \frac{1}{1-(a_{m+1}+a_{m+2}+\dots+a_n)} > \frac{P_n}{P_m},$$

and $$\frac{Q_n}{Q_m} > 1-(a_{m+1}+a_{m+2}+\dots+a_n) > 1-\sigma.$$

Hence, provided that n is greater than m, we have

$$P_n < P_m/(1-\sigma),$$

and $$Q_n > Q_m(1-\sigma).$$

Thus, by Art. 2, P_n and Q_n approach definite finite limits P, Q, such that $$P \leqq P_m/(1-\sigma), \quad Q \geqq Q_m(1-\sigma).$$

But, if Σa_n is divergent, we can find m so that

$$a_1 + a_2 + \dots + a_n > N, \quad \text{if } n > m,$$

no matter how large N may be.

Hence, by the same inequalities,

$$P_n > 1+N, \quad Q_n < 1/(1+N), \quad \text{if } n > m,$$

and consequently $\lim P_n = \infty$, $\lim Q_n = 0$.

It should be observed that *if a product tends to zero as a limit, without any of its factors being zero, the product is said to*

diverge. This of course is merely a convention; but it preserves the parallelism with the theory of series.

Ex. 1. Since $\Sigma 1/n^2$ converges, the product

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\dots$$

will approach a limit *different from zero*. This is at once obvious because

$$1-\frac{1}{n^2}=\frac{(n-1)(n+1)}{n^2},$$

and so we find the product

$$Q_{n-1}=\frac{1.3}{2^2}\cdot\frac{2.4}{3^2}\cdot\frac{3.5}{4^2}\cdot\frac{4.6}{5^2}\dots\frac{(n-1)(n+1)}{n^2}=\frac{1}{2}\frac{n+1}{n},$$

so that $\lim Q_n=\frac{1}{2}$.

This value of the product is, as a matter of fact, a special case of the product formula for $\sin\theta$ (see Art. 70); the present result follows by making $\theta\to\pi$.

Similarly $\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{4^2}\right)\dots$ is convergent; and its value can be proved to be $\sinh\pi/\pi$. [See Art. 98 below.]

Ex. 2. We see similarly that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\dots$ converges; and its value is $2/\pi$. [Art. 71.]

Thus $\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\dots$ is equal to $\pi/4$.

Ex. 3. Since $\Sigma 1/n$ is divergent, the products

$$(1+\tfrac{1}{2})(1+\tfrac{1}{3})(1+\tfrac{1}{4})\dots,\quad (1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{4})\dots$$

will diverge also.

In fact $P_{n-1}=\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\dots\frac{n+1}{n}=\frac{n+1}{2},\quad Q_{n-1}=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\dots\frac{n-1}{n}=\frac{1}{n},$

so that $\lim P_n=\infty,\quad \lim Q_n=0.$

Ex. 4. *If* $a_n/a_{n+1}=1+(b_n/n)$, *where* $\lim b_n=b>0$, *then* $\lim a_n=0$.

For, under the given circumstances, we can find an index m, such that

$$b_n>\tfrac{1}{2}b>0 \quad \text{if } n\geqq m.$$

Thus we have

$$\frac{a_m}{a_{m+1}}>1+\frac{b}{2m},\quad \frac{a_{m+1}}{a_{m+2}}>1+\frac{b}{2(m+1)},\ \dots,\quad \frac{a_n}{a_{n+1}}>1+\frac{b}{2n},$$

and therefore

$$\frac{a_m}{a_{n+1}}>\left(1+\frac{b}{2m}\right)\dots\left(1+\frac{b}{2n}\right)>1+\frac{b}{2}\left(\frac{1}{m}+\frac{1}{m+1}+\dots+\frac{1}{n}\right).$$

Hence $$\lim_{n\to\infty}(a_m/a_n)=\infty,$$

so that $$\lim a_n=0.$$

It is easy to see that the argument of Art. 26 can be modified to prove that *when $a_1, a_2, a_3, \ldots$ are between 0 and 1, the values of the two infinite products*

$$P=(1+a_1)(1+a_2)(1+a_3)\ldots, \quad Q=(1-a_1)(1-a_2)(1-a_3)\ldots$$

are both independent of the order of the factors.

Since P_n increases with n, while Q_n decreases, we can choose n so that

$$P > P_n > P-\epsilon \quad \text{and} \quad Q < Q_n < Q+\epsilon.$$

Now, suppose that in any new arrangement (indicated by accents) we have to take p factors to include all the first n factors of P, Q; then we have

$$\left.\begin{array}{l} P > P_r' > P_n > P-\epsilon \\ Q < Q_r' < Q_n < Q+\epsilon \end{array}\right\} \text{ if } r \geqq p.$$

and

Thus P_r' and Q_r' converge to the limits P and Q, respectively.

The argument is at once extended to shew that if P is divergent, P' must also tend to $+\infty$; and that if Q diverges to 0, Q' has the same property.

By taking logarithms, the present results could be deduced at once from Art. 26; but the reasoning given here is quite as short and goes back to first principles.

40. Absolute convergence of an infinite product.

If the series Σu_n is absolutely convergent the infinite product $\Pi(1+u_n)$ is called absolutely convergent; the product then converges to a value independent of the order of the factors.

Let us write $a_n=|u_n|$, and

$$U_n=(1+u_1)(1+u_2)\ldots(1+u_n), \quad A_n=(1+a_1)(1+a_2)\ldots(1+a_n).$$

Then it will be seen that the quotient U_{n+m}/U_n can be written in the form

$$1+s_1+s_2+\ldots+s_m,$$

where $s_1, s_2, \ldots, s_m$ are the symmetric functions of

$$u_{n+1}, \quad u_{n+2}, \quad \ldots, \quad u_{n+m}.$$

Also, A_{n+m}/A_n can be expressed similarly in terms of

$$a_{n+1}, \quad a_{n+2}, \quad \ldots, \quad a_{n+m}.$$

Accordingly, we see that

$$\left|\frac{U_{n+m}}{U_n}-1\right| \leqq \frac{A_{n+m}}{A_n}-1,$$

because every term on the left-hand side has a numerically equal term on the right; and in the latter sum all the terms are positive.

Further, $$|U_n| = A_n,$$

and so on multiplying we see that

$$|U_{n+m} - U_n| \leqq A_{n+m} - A_n.$$

But, by Art. 39, A_n tends to a definite limit A, because Σa_n is known to converge.

We can accordingly choose n so that

$$A_{n+m} - A_n < \epsilon,$$

and then $$|U_{n+m} - U_n| < \epsilon.$$

Thus U_n tends to some definite limit U, and with the above choice for n, $$|U - U_n| \leqq \epsilon.$$

Suppose next that U' denotes a derangement of the product U, A' denoting the same derangement of the product A; then from Art. 39 we know that $A' = A$.

Now let p be chosen so that U_p' contains all the factors of U_n, and consequently A_p' contains all those of A_n; then, if $r \geqq p$,

$$|U_r' - U_n| < A_r' - A_n,$$

by repeating the reasoning used above.

Hence $$|U_r' - U_n| < A' - A_n = A - A_n < \epsilon$$

by the same choice of n as previously.

Consequently

$$|U_r' - U| \leqq |U_r' - U_n| + |U - U_n| < 2\epsilon, \quad \text{if } r \geqq p,$$

and therefore U_r' tends to U as a limit; or the value of the product U' is the same as that of the product U.

Ex. To shew that some condition such as absolute convergence is necessary to allow of derangement, we may consider

$$P = (1 - \tfrac{1}{2})(1 + \tfrac{1}{3})(1 - \tfrac{1}{4})(1 + \tfrac{1}{5})\ldots,$$

which is easily seen to converge to the value $\tfrac{1}{2}$. (Ex. 1, Art. 41.)

If we derange P so as to take two negative terms to each positive term; say

$$Q = (1 - \tfrac{1}{2})(1 - \tfrac{1}{4})(1 + \tfrac{1}{3})(1 - \tfrac{1}{6})(1 - \tfrac{1}{8})(1 + \tfrac{1}{5})\ldots,$$

then we find that

$$Q_{3n}/P_{2n} = \left(1 - \frac{1}{2n+2}\right)\left(1 - \frac{1}{2n+4}\right)\ldots\left(1 - \frac{1}{4n}\right).$$

Now, by Art. 38, this product is seen to lie between

$$\frac{1}{1+\sigma_n} \quad \text{and} \quad 1-\sigma_n,$$

where
$$\sigma_n = \frac{1}{2n+2} + \frac{1}{2n+4} + \dots \frac{1}{4n}.$$

Now
$$\tfrac{1}{4} < \sigma_n < \tfrac{1}{2},$$

and so
$$\frac{1}{1+\frac{1}{4}} > Q_{3n}/P_{2n} > (1-\tfrac{1}{2}).$$

Thus, assuming that Q has a definite value, that limit lies between $\frac{4}{5}P$ and $\frac{1}{2}P$; actually, from Art. 41. I, the value of Q is $P/\sqrt{2}$ or $\dfrac{1}{2\sqrt{2}}$.

41. Further tests for the convergence of infinite products in general.

We consider next the infinite product

$$(1+u_1)(1+u_2)(1+u_3)\dots,$$

in which the numbers $u_1, u_2, u_3, \dots$ may have both signs. Without loss of generality it may be supposed that u_n is numerically less than 1; for there can only be a finite number of terms in which $|u_n|$ is greater than 1 (otherwise the product would certainly diverge or oscillate), and the corresponding factors can be omitted without affecting the question of convergence.

Now, from Art. 62, we have

$$0 < u - \log(1+u) < \tfrac{1}{2}u^2 \qquad \text{if } u \text{ is positive,}$$
$$\text{or } < \tfrac{1}{2}u^2/(1+u) \quad \text{if } 0 > u > -1.$$

Thus, if λ is the lower limit of the numbers

$$1, \quad 1+u_1, \quad 1+u_2, \quad \dots, \quad 1+u_n, \dots,$$

we have

$$0 < (u_{m+1}+u_{m+2}+\dots+u_n) - \log\{(1+u_{m+1})(1+u_{m+2})\dots(1+u_n)\}$$
$$< \tfrac{1}{2}(u_{m+1}^2+u_{m+2}^2+\dots+u_n^2)/\lambda.$$

Consequently, if the series Σu_n^2 is convergent, the difference

$$(u_{m+1}+u_{m+2}+\dots+u_n) - \log\{(1+u_{m+1})(1+u_{m+2})\dots(1+u_n)\}$$

can be made arbitrarily small by properly choosing m, no matter how large n is. Thus we have the theorem:

If the series Σu_n^2 is convergent, the infinite product

$$(1+u_1)(1+u_2)(1+u_3)\dots$$

converges if Σu_n converges; diverges to ∞ if Σu_n diverges to $+\infty$; diverges to 0 if Σu_n diverges to $-\infty$; oscillates if Σu_n oscillates.

Again, it is known that

$$u-\log(1+u) > \tfrac{1}{2}u^2/(1+u) \quad \text{if } u \text{ is positive,}$$
$$\text{or } > \tfrac{1}{2}u^2 \quad \text{if } 0 > u > -1,$$

so that

$$(u_{m+1}+u_{m+2}+\ldots+u_n)-\log\{(1+u_{m+1})(1+u_{m+2})\ldots(1+u_n)\}$$
$$> \tfrac{1}{2}(u_{m+1}^2+u_{m+2}^2+\ldots+u_n^2)/L,$$

where L is the upper limit of $1, (1+u_1), (1+u_2), \ldots, (1+u_n), \ldots$.

Hence, *if Σu_n converges* (or oscillates so that its maximum limit is not $+\infty$) while Σu_n^2 diverges, the infinite product*

$$(1+u_1)(1+u_2)(1+u_3)\ldots$$

diverges to the value 0.

The only cases not covered by the foregoing method are those in which Σu_n^2 diverges and Σu_n either diverges to $+\infty$, or has $+\infty$ as its maximum limit (in case of oscillation).

It is, perhaps, a little perplexing at first sight that when Σu_n, Σu_n^2 both diverge to ∞, the product may nevertheless converge; but it is quite easy to construct a product of this type. For, let Σc_n be a convergent, Σd_n a divergent, series of positive terms, and form the product of which the $(2n-1)$th and $2n$th terms are given by

$$1+u_{2n-1}=1+d_n, \quad 1+u_{2n}=\frac{1+c_n}{1+d_n}.$$

Then, provided that $\lim d_n=0$, the product $\Pi(1+u_n)=\Pi(1+c_n)$ and so obviously converges (by Art. 39). Further, Σu_n will diverge if $\Sigma\left(d_n+\frac{c_n-d_n}{1+d_n}\right)=\Sigma\left(\frac{d_n^2+c_n}{1+d_n}\right)$ is divergent; and then Σu_n^2 must also diverge.† This condition can be satisfied in many ways; one simple method is to take $c_n=d_n^3$, and then to adjust d_n so as to make Σd_n, Σd_n^2 both divergent, and Σd_n^3 convergent; for instance, we may take $d_n=n^{-p}$, where $\frac{1}{2} \geqq p > \frac{1}{3}$. The product is then given by $u_{2n-1}=n^{-p}$, $u_{2n}=-n^{-p}+n^{-2p}$.

Ex. 1. Since the series

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots \quad \text{and} \quad \frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$$

are both convergent, the two infinite products

$$(1+\tfrac{1}{2})(1-\tfrac{1}{3})(1+\tfrac{1}{4})(1-\tfrac{1}{5})\ldots \quad \text{and} \quad (1-\tfrac{1}{2})(1+\tfrac{1}{3})(1-\tfrac{1}{4})(1+\tfrac{1}{5})\ldots$$

converge also. In fact the first is obviously equal to 1 and the second to $\frac{1}{2}$.

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* Of course not absolutely, for then the product converges (see Art. 40).

† If Σu_n^2 were convergent, the divergence of Σu_n would imply the divergence of the product.

Ex. 2. Since the series $\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\dots$ diverges in virtue of Cesàro's theorem (Art. 23) it follows that the infinite products

$(1+\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{4})(1+\frac{1}{5})(1+\frac{1}{6})(1-\frac{1}{7})\dots$ and $(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})(1-\frac{1}{6})(1+\frac{1}{7})\dots$

are both divergent. In fact the first diverges to ∞ and the second to 0: for they are equivalent to the products

$$(1+\tfrac{1}{2})(1+\tfrac{1}{5})(1+\tfrac{1}{8})(1+\tfrac{1}{11})\dots \quad \text{and} \quad (1-\tfrac{1}{2})(1-\tfrac{1}{3})(1-\tfrac{1}{6})(1-\tfrac{1}{9})(1-\tfrac{1}{12})\dots.$$

Ex. 3. Since the series

$$\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{5}}+\dots$$

is convergent, but $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$ is divergent, it is clear that the two products

$$\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\left(1-\frac{1}{\sqrt{5}}\right)\dots$$

and

$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{4}}\right)\left(1+\frac{1}{\sqrt{5}}\right)\dots$$

both diverge to the value 0.

In fact

$$\left(1+\frac{1}{\sqrt{n}}\right)\left[1-\frac{1}{\sqrt{(n+1)}}\right]=1-\frac{1}{\sqrt{[n(n+1)]}}\left[1-\frac{1}{\sqrt{n}+\sqrt{(n+1)}}\right],$$

so that this product is always less than 1, and can be put in the form $(1-a_n)$. Further $\lim (na_n)=1$, so that our two products diverge to 0 by Art. 39.

Ex. 4. If $u_n=(-1)^n\frac{1}{2}$, it is evident that Σu_n oscillates, while $\Sigma u_n{}^2$ diverges; thus the product

$$(1-\tfrac{1}{2})(1+\tfrac{1}{2})(1-\tfrac{1}{2})(1+\tfrac{1}{2})\dots$$

must diverge to 0, which may be verified by inspection.

41·1. Alteration in the value of a non-absolutely convergent infinite product by deranging the factors.

The argument used to establish Riemann's theorem (Art. 28) requires but little change to shew that a non-absolutely convergent infinite product may be made to converge to any value, or to diverge, or to oscillate, by altering the order of the factors.

Perhaps the case of chief interest is that afforded by the infinite product $\Pi[1+(-1)^{n-1}a_n]$, where a_n is positive and $\lim (na_n)=g$. Suppose that the value of the product is P when the positive and negative terms occur alternately; and let its value be X when the limit of the ratio of the number of positive to the number of negative terms is k.

Then $$X/P=\lim_{\nu\to\infty}(1+a_{2n+1})(1+a_{2n+3})\dots(1+a_{2\nu-1}),$$

where $$\lim (\nu/n)=k.$$

Now it is plain that $\Sigma a_n{}^2$ is convergent, and therefore, as in the last article, it follows that

$$(a_{2n+1}+a_{2n+3}+\dots+a_{2\nu-1})-\log[(1+a_{2n+1})(1+a_{2n+3})\dots(1+a_{2\nu-1})]$$

can be made arbitrarily small by taking n large enough. Further, by Pringsheim's method (Art. 28), it is clear that

$$\lim (a_{2n+1}+a_{2n+3}+\dots+a_{2\nu-1})=\tfrac{1}{2}g\log k,$$

and therefore $$\log (X/P)=\tfrac{1}{2}g\log k,$$

or $$X/P=k^{\frac{1}{2}g}.$$

42. The Gamma-product.

It is evident from the foregoing articles (39, 41) that the product

$$P_n=\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\dots\left(1+\frac{x}{n}\right),\quad (x>-1),$$

is divergent except for $x=0$. But we have

$$\frac{x}{n}-\log\frac{P_n}{P_{n-1}}=\frac{x}{n}-\log\left(1+\frac{x}{n}\right)>0, \qquad \text{(Art. 62)}$$

so that the expression

$$S_n=x\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)-\log P_n$$

increases with n. Also, as in Art. 41, we have

$$S_n<\frac{x^2}{2\lambda}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right)<\frac{x^2}{\lambda}, \qquad \text{[Art. 11 (1)]}$$

where λ is either 1, if x is positive, or $1+x$, if x is negative. Hence, by Art. 2, S_n approaches a definite limit S as n increases to ∞.

Further $$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n \qquad \text{(Art. 11)}$$

approaches a definite limit C, and therefore

$$\lim (x\log n-\log P_n)$$
$$=\lim\left\{S_n-x\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right)\right\}=S-Cx.$$

Now $$x\log n-\log P_n=\log (n^x/P_n),$$

so that n^x/P_n has also a definite limit; this limit is denoted by $\Pi(x)$ in Gauss's notation.

Thus $$\Pi(x)=\lim_{n\to\infty}\frac{n^x.\,n!}{(1+x)(2+x)\dots(n+x)},$$

which, again, can be written in Weierstrass's form,

$$1/\Pi(x)=e^{Cx-S}=e^{Cx}\lim_{n\to\infty}e^{-S_n}=e^{Cx}\prod_{r=1}^{\infty}\left(1+\frac{x}{r}\right)e^{-\frac{x}{r}}.$$

When x is a positive integer, Gauss's form gives $\Pi(x)=x!$, because

$$\frac{n^x . n!}{(1+x)(2+x)\dots(n+x)}=\frac{n^x . x!}{(1+n)(2+n)\dots(x+n)}$$

$$=x!\Big/\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{x}{n}\right).$$

Although we have found it convenient to restrict $(1+x)$ to be positive, yet this is not necessary for convergence; and it is easy to see that the products for $\Pi(x)$ still converge if x has any negative value which is not an integer.*

It is easy to verify by integration by parts that Euler's integral

$$\Gamma(1+x)=\int_0^\infty e^{-t}t^x\,dt=x\int_0^\infty e^{-t}t^{x-1}\,dt=x\Gamma(x).$$

Thus $\Gamma(1+x)$ has the property of being equal to $x!$ when x is an integer; and we may therefore anticipate the equation

$$\Gamma(1+x)=\Pi(x),$$

which will be proved to be correct in Art. 178 of the Appendix.

In future we use the notation $\Gamma(1+x)$ in place of $\Pi(x)$ as the more usual in modern books.

If we change x to $x-1$ in the definition of $\Gamma(1+x)$ by the product P_n, we find that

$$\Gamma(x)=\lim_{n\to\infty}\frac{n^{x-1}.n!}{x(1+x)(2+x)\dots(n+x-1)}.$$

Thus $$\frac{\Gamma(1+x)}{\Gamma(x)}=x\lim_{n\to\infty}\frac{n}{n+x}=x,$$

or $$\Gamma(1+x)=x\Gamma(x).$$

It follows that

$$x(1+x)(2+x)\dots(n+x-1)=\Gamma(n+x)/\Gamma(x),$$

and consequently the definition leads to the equation

$$\Gamma(x)=\lim_{n\to\infty}\frac{n^x.\Gamma(n)\Gamma(x)}{\Gamma(n+x)},$$

because $$n^{x-1}.n!=n^x\Gamma(n).$$

Hence $$\lim_{n\to\infty}\frac{n^x\Gamma(n)}{\Gamma(n+x)}=1.$$

* The convergence persists also for complex values of x (see Ex. 27, p. 273).

It is often convenient to write the last equation in the form

$$\Gamma(n+x) \sim n^x \Gamma(n),$$

using the notation explained in Art. 1·1.

By reversing the foregoing argument we see that *the function $\Gamma(x)$ is completely defined by the properties*

$$\Gamma(1+x)=(x)\Gamma(x), \quad \Gamma(n+x) \sim n^x \Gamma(n),$$

together with the condition $\Gamma(1)=1$.

EXAMPLES.

1. Discuss the convergence of the products

$$\Pi[1+f(n).x^n], \quad \Pi\left[\left(1-\frac{1}{n}\right)^x\left(1+\frac{x}{n}\right)\right], \quad \Pi\left(\frac{x+x^{2n}}{1+x^{2n}}\right),$$

where $f(n)$ is a polynomial in n.

2. Prove that $$\Pi\left[\left(1-\frac{x}{c+n}\right)e^{x/n}\right]$$

converges absolutely for any value of x, provided that c is not a negative integer; and that

$$\Pi\left[1-\left(\frac{nx}{n+1}\right)^n\right]$$

is absolutely convergent if $|x|<1$.

3. If $$w_n = \frac{1}{m}\,\frac{1+a/n+b/n^2+\dots}{1+c/n+d/n^2+\dots},$$

then Πw_n diverges to 0 if $m>1$; to ∞ if $m<1$.

If $m=1$, Πw_n diverges to 0 if $a<c$, and to ∞ if $a>c$; and converges if $a=c$. [STIRLING.]

4. If $u_1=0$, $u_2=0$, $u_{2n-1}=-n^{-p}$, $u_{2n}=n^{-p}+n^{-2p}$, where $n>1$ and $\frac{1}{3}<p \leqq \frac{1}{2}$, then Σu_n, $\Sigma {u_n}^2$ are both divergent, but $\Pi(1+u_n)$ is convergent. Verify that the same is true if $\frac{1}{4}<p \leqq \frac{1}{2}$ and

$$u_{2n-1}=-n^{-p}, \quad u_{2n}=n^{-p}+n^{-2p}+n^{-3p}, \quad (n>1).$$

[*Math. Trip.* 1906.]

5. Verify the identity

$$\left(1-\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\dots\left(1-\frac{x}{n}\right)$$
$$=1-x+\frac{x(x-1)}{2!}-\frac{x(x-1)(x-2)}{3!}+\dots+(-1)^n\frac{x(x-1)\dots(x-n+1)}{n!}.$$

Shew that as n tends to infinity, the product diverges for all values of x except 0; but the series converges, provided that $x>0$.

6. Prove that $(1+x)(1+x^2)(1+x^4)(1+x^8)\dots=1/(1-x)$, if $|x|<1$.

7. Verify that $$\cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}\dots=\frac{\sin x}{x},$$

and that $$\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\frac{1}{2^3}\tan\frac{x}{2^3}+\dots=\frac{1}{x}-\cot x.$$ [EULER.]

8. Determine the value of

$$\Pi\left(1+\frac{x}{c+n}\right)e^{-x/n}$$

in terms of the Gamma functions $\Gamma(1+c)$ and $\Gamma(1+x+c)$.

9. Shew that

$$\lim_{n\to\infty}\frac{x(x+1)(x+2)\dots(x+2n-1)}{1.3.5\dots(2n-1).2x(2x+2)\dots(2x+2n-2)}=2^{x-1}.$$

$\left[\text{This product is equal to}\quad \frac{1}{2}\frac{\Gamma(x+2n)}{\Gamma(x+n)}\frac{\Gamma(n)}{\Gamma(2n)}.\right]$

10. Prove that, if k is an integer,

$$\lim_{n\to\infty}\frac{2.4.6\dots 2kn}{1.3.5\dots(2kn-1)}\cdot\frac{1.3.5\dots(2n-1)}{2.4.6\dots 2n}=\sqrt{k}.$$

11. Prove that if $u_n=\dfrac{(n+a_1)(n+a_2)\dots(n+a_k)}{(n+b_1)(n+b_2)\dots(n+b_l)}$, the product $\prod\limits_1 u_n$ can only converge if $k=l$ and $\Sigma a=\Sigma b$. When these conditions are satisfied, express the product in the form

$$\frac{\Gamma(1+b_1)\Gamma(1+b_2)\dots\Gamma(1+b_k)}{\Gamma(1+a_1)\Gamma(1+a_2)\dots\Gamma(1+a_k)}.$$

In particular, prove that

$$\prod_1^\infty\frac{n(n+a+b)}{(n+a)(n+b)}=\frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(1+a+b)}.$$

12. Prove that

$$(1-x)(1+\tfrac{1}{2}x)(1-\tfrac{1}{3}x)(1+\tfrac{1}{4}x)\dots=\frac{\Gamma(\frac{1}{2})}{\Gamma(1+\frac{1}{2}x)\Gamma(\frac{1}{2}-\frac{1}{2}x)}.$$

[Take the terms in pairs and use the last example.]

13. If $\psi(x)$ denotes $\Gamma'(x)/\Gamma(x)$, we can write (see Art. 48)

$$\psi(x)=\lim_{n\to\infty}\left(\log n-\frac{1}{x}-\frac{1}{1+x}-\frac{1}{2+x}-\dots-\frac{1}{n+x}\right).$$

Then we find $\quad \dfrac{\Gamma(x)}{\Gamma(x+y)}e^{y\psi(x)}=\prod\limits_0^\infty\left(1+\dfrac{y}{x+n}\right)e^{-y/(x+n)}.$ [MELLIN.]

14. It is easy to deduce from the theory of infinite products Abel's result (Arts. 11, 16), that Σa_n and $\Sigma a_n/s_n$ converge or diverge together. In fact consider the product $\Pi(1-a_n/s_n)=\Pi(s_{n-1}/s_n)$, which diverges to 0 if $s_n\to\infty$; so that $\Sigma(a_n/s_n)$ must also diverge (Art. 39). [Here $a_n>0$.]

15. Let a_n, b_n, c_n denote the general terms of the three hypergeometric series

$$A=F(\alpha,\beta,\gamma,1),\quad B=F(\alpha-1,\beta,\gamma,1),\quad C=F(\alpha,\beta,\gamma+1,1),$$

in which $\gamma>\alpha+\beta$. Then prove that

$$a_n-a_{n+1}=(1-\beta/\gamma)c_n-b_{n+1},$$
$$(\gamma-\alpha)(a_n-b_n)=\beta a_{n-1}+(n-1)a_{n-1}-na_n,$$
$$\lim_{n\to\infty}(na_n)=0.$$

Deduce that $$\gamma B = (\gamma - \beta) C, \quad (\gamma - \alpha)(A - B) = \beta A,$$

and that $$A = \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma(\gamma - \alpha - \beta)} C.$$

16. From Ex. 15 prove that

$$F(\alpha, \beta, \gamma, 1) \frac{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)} = F(\alpha, \beta, \gamma + n, 1) \frac{\Gamma(\gamma + n - \alpha)\Gamma(\gamma + n - \beta)}{\Gamma(\gamma + n)\Gamma(\gamma + n - \alpha - \beta)},$$

and shew that the last expression tends to the limit 1 as $n \to \infty$. Deduce that

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}.$$ [GAUSS.]

17. If $$a_n = \left[\frac{\Gamma(n)\Gamma(t)}{\Gamma(n+t)}\right]^2, \quad b_n = \left[\frac{\Gamma(n)\Gamma(1+t)}{\Gamma(n+1+t)}\right]^2,$$

verify that $$na_n - (n+1)a_{n+1} = ta_n + (t-1)a_{n+1} - (n+t)b_n,$$

and that $$(n-1)^2 n b_{n-1} - n^2(n+1)b_n = t^2[(2t-1)a_{n+1} + nb_n].$$

Shew that $\lim na_n = 0$, $\lim n^3 b_n = 0$, if $t > \frac{1}{2}$, and deduce that

$$(2t-1)\sum_1^\infty a_n - t\sum_1^\infty b_n - \sum_1^\infty nb_n = ta_1,$$

$$(2t-1)\sum_1^\infty a_n + \sum_1^\infty nb_n = 2ta_1.$$

Hence prove that $$t\sum_1^\infty b_n = 2(2t-1)\sum_1^\infty a_n - 3/t.$$

18. If $$\left.\begin{array}{ll} q_0 = \Pi(1 - q^{2n}), & q_1 = \Pi(1 + q^{2n}), \\ q_2 = \Pi(1 + q^{2n-1}), & q_3 = \Pi(1 - q^{2n-1}), \end{array}\right\} \quad (n = 1, 2, 3, \ldots),$$

the four products are absolutely convergent if $|q| < 1$.

Also $$q_0 q_3 = \Pi(1 - q^n), \quad q_1 q_2 = \Pi(1 + q^n),$$

and $$q_1 q_2 q_3 = 1.$$

Thus $$1/[(1-q)(1-q^3)(1-q^5)\ldots] = (1+q)(1+q^2)(1+q^3)\ldots.$$ [EULER.]

19. If Σu_n is absolutely convergent, the product $\Pi(1 + xu_n)$ is absolutely convergent for any value of x; and it can be expanded in an absolutely convergent series

$$1 + U_1 x + U_2 x^2 + \ldots, \quad \text{where } U_1 = \Sigma u_n.$$

Shew also that

$$\Pi(1 + xu_n)(1 + u_n/x) = V_0 + V_1(x + 1/x) + V_2(x^2 + 1/x^2) + \ldots,$$

where $$V_n = U_n + U_1 U_{n+1} + U_2 U_{n+2} + \ldots.$$

20. If $$f(x) = (1 + qx)(1 + q^3 x)(1 + q^5 x)\ldots,$$

we have at once $(1 + qx) f(q^2 x) = f(x)$; and as in the last example

$$f(x) = 1 + U_1 x + U_2 x^2 + \ldots.$$

Thus we find

$$q + q^2 U_1 = U_1, \quad q^{2n-1} U_{n-1} + q^{2n} U_n = U_n,$$

which give

$$U_1 = \frac{q}{1 - q^2}, \quad U_2 = \frac{q^4}{(1-q^2)(1-q^4)}, \quad U_3 = \frac{q^9}{(1-q^2)(1-q^4)(1-q^6)}$$

and generally $$U_n = q^{n^2}/P_n,$$

where $$P_n = (1-q^2)(1-q^4)\ldots(1-q^{2n}).$$

21. If $$F(x) = \Pi(1 + q^{2n-1}x)(1 + q^{2n-1}/x),$$
we see, from Ex. 20, that we can write
$$F(x) = V_0 + V_1(x + 1/x) + V_2(x^2 + 1/x^2) + \dots .$$
But $qxF(q^2x) = F(x)$, and so we find
$$V_1 = V_0 q, \quad V_n = V_{n-1}q^{2n-1},$$
yielding $$V_2 = V_0 q^4, \quad V_3 = V_0 q^9, \dots V_n = V_0 q^{n^2}.$$
Thus $$F(x) = V_0\{1 + q(x + 1/x) + q^4(x^2 + 1/x^2) + q^9(x^3 + 1/x^3) + \dots\}.$$
To determine V_0 we may use the results of Exs. 19, 20, from which we find
$$V_0 q^{n^2} = V_n = U_n + U_1 U_{n+1} + U_2 U_{n+2} + \dots$$
or $$V_0 = \frac{1}{P_n} + \frac{q^{2n+2}}{P_1 P_{n+1}} + \frac{q^{4n+8}}{P_2 P_{n+2}} + \dots .$$
Thus $$P_n V_0 - 1 < \frac{q^{2n}}{q_0{}^2} + \frac{q^{4n}}{q_0{}^2} + \dots, \quad \text{where } q_0 = \Pi(1 - q^{2n}),$$
because $$|q| < 1 \text{ and } P_{n+r}/P_n > q_0, \ P_r > q_0.$$
Hence $$P_n V_0 - 1 < q^{2n}/q_0{}^2(1 - q^{2n}),$$
so that $$\lim (P_n V_0) = 1, \text{ or } V_0 = 1/q_0.$$
Thus, using the notation of Ex. 17, Ch. V., we have
$$f(x, q) = q_0 \Pi(1 + q^{2n-1}x)(1 + q^{2n-1}/x),$$
from which a number of interesting results follow. [JACOBI.]

22. From Ex. 21 we find, with the notation of Ex. 18,
$$f(1, q) = q_0 q_2{}^2, \quad f(-1, q) = q_0 q_3{}^2, \quad f(q, q) = 2q_0 q_1{}^2.$$
Or, writing these equations at length, we have
$$q_0 q_2{}^2 = 1 + 2q + 2q^4 + 2q^9 + \dots,$$
$$q_0 q_3{}^2 = 1 - 2q + 2q^4 - 2q^9 + \dots,$$
$$q_0 q_1{}^2 = 1 + q^2 + q^6 + q^{12} + q^{20} + \dots,$$
where the indices in the third series are of the type $n(n + 1)$.

Again, by taking the limit of $f(x, q)/(1 + q/x)$ as x approaches $-q$, we have
$$q_0{}^3 = 1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} - \dots,$$
the indices being the same as in the third series. [Compare Art. 46.]

23. Again, from Ex. 21, we get
$$f(-1, q^2) = \Pi(1 - q^{4n}) \,.\, \Pi(1 - q^{4n-2})^2 = q_0 q_2 q_3 = q_0/q_1,$$
so that $$q_0/q_1 = 1 - 2q^2 + 2q^8 - 2q^{18} + \dots,$$
the indices being of the form $2n^2$.

Also $$f(\sqrt{q}, \sqrt{q}) = 2\Pi(1 - q^n) \,.\, \Pi(1 + q^n)^2 = 2q_0 q_1 q_2 = 2q_0/q_3,$$
so that $$q_0/q_3 = 1 + q + q^3 + q^6 + q^{10} + \dots,$$
the indices being of the form $\frac{1}{2}n(n + 1)$. [GAUSS.]

Similarly,
$$f(-q^{\frac{1}{2}}, q^{\frac{3}{2}}) = \Pi(1 - q^{3n}) \,.\, \Pi(1 - q^{3n-1})(1 - q^{3n-2}) = \Pi(1 - q^n)$$
or $$q_0 q_3 = 1 - (q + q^2) + (q^5 + q^7) - (q^{12} + q^{15}) + \dots,$$
the indices being alternately $\frac{1}{2}n(3n \pm 1)$. [EULER.]

24. Write $y=1$, and put $\sqrt{x}$, $\sqrt{q}$ in place of x, q, in the first result of Ex. 17, Ch. V. Then we have

$$[f(\pm\sqrt{x}, \sqrt{q})]^2 = f(x, q) \cdot f(1, q) + g(x, q) \cdot g(1, q).$$

Now $$f(\sqrt{x}, \sqrt{q}) = \prod (1-q^n) \cdot \prod (1+q^{n-\frac{1}{2}}x^{\frac{1}{2}})(1+q^{n-\frac{1}{2}}x^{-\frac{1}{2}}),$$

so that $$f(\sqrt{x}, \sqrt{q}) \cdot f(-\sqrt{x}, \sqrt{q}) = (q_0 q_3)^2 \cdot \prod (1-q^{2n-1}x)(1-q^{2n-1}/x)$$
$$= q_0 q_3{}^2 f(-x, q).$$

Thus, on multiplication, we find

$$(q_0 q_3{}^2)^2 [f(-x, q)]^2 = [f(x, q) \cdot f(1, q)]^2 - [g(x, q) \cdot g(1, q)]^2.$$

But $$f(1, q) = q_0 q_2{}^2, \quad g(1, q) = q^{\frac{1}{4}} f(q, q) = 2q^{\frac{1}{4}} q_0 q_1{}^2,$$

and so we have the identity

$$q_3{}^4 [f(-x, q)]^2 = q_2{}^4 [f(x, q)]^2 - 4q^{\frac{1}{2}} q_1{}^4 [g(x, q)]^2.$$

In particular, if we write $x=1$, we find the interesting result

$$q_2{}^8 = q_3{}^8 + 16q \cdot q_1{}^8,$$

which leads again to the identity

$$(1+2q+2q^4+2q^9+\dots)^4 - (1-2q+2q^4-2q^9+\dots)^4$$
$$= 16q(1+q^2+q^6+q^{12}+q^{20}+\dots)^4,$$

where the series are those given in Ex. 22 above. [JACOBI.]

Other examples on products will be found at the ends of Chapters IX., X., XI.

CHAPTER VII.

SERIES OF VARIABLE TERMS.

43. Uniform convergence of a sequence.

It may happen that the terms of a sequence depend on some variable x in addition to the index n; and this is indicated by using the notation $S_n(x)$. We assume that the sequence is convergent for all values of x within a certain interval (a, b), and then the limit

$$\lim_{n \to \infty} S_n(x)$$

defines a certain function of x, say $F(x)$, in the interval (a, b).

The condition of convergence (Art. 1) implies that, given an arbitrarily small positive number ϵ, we can determine an integer m such that

$$|S_n(x) - F(x)| < \epsilon, \quad \text{if } n > m.$$

Obviously the definition of m is not yet precise, but we can make it precise by agreeing to select the *least* integer m which satisfies the prescribed inequality. When this is done, it is natural to expect that the value of m will depend on x, and so we are led to consider a new function $m(\epsilon, x) = m(x)$, which depends on ϵ and on the nature of the sequence.

We note incidentally that, regarded as a function of ϵ, $m(x)$ is monotonic since (for any assigned value of x) m cannot decrease as ϵ diminishes.

Ex. 1. If $S_n(x) = 1/(x+n)$, where $x \geqq 0$, we have

$$F(x) = \lim_{n \to \infty} S_n(x) = 0.$$

Then the condition of convergence gives

$$x + n > 1/\epsilon,$$

so that $\qquad m(x) =$ the integral part of $(1/\epsilon) - x$, when $x < 1/\epsilon$,

or $\qquad m(x) = 0$, when $x \geqq 1/\epsilon$.

Ex. 2. If $S_n(x)=x^n$, where $0 \leqq x \leqq 1$, we have

$$F(x)=\lim_{n\to\infty} S_n(x)=0, \text{ if } x<1; \text{ and } F(1)=1.$$

Then we are to have $(1/x)^n > 1/\epsilon$, if $x<1$,

so that $m(x)=$ the integral part of $\dfrac{\log(1/\epsilon)}{\log(1/x)}$, when $x<1$.

Also, since $S_n(1)=1$ for all values of n, we must take $m(1)=0$.

Ex. 3. If $S_n(x)=\text{arc tan}\,(nx)$, where $x \geqq 0$, we have

$$F(x)=\lim_{n\to\infty} S_n(x)=\tfrac{1}{2}\pi, \text{ if } x>0; \text{ and } F(0)=0.$$

It is easily seen that

$m(x)=$ the integral part of $(\cot\epsilon)/x$, when $x>0$,

and $m(0)=0$.

Ex. 4. If $S_n(x)=nx/(1+n^2x^2)$, x being unrestricted, we have

$$F(x)=\lim_{n\to\infty} S_n(x)=0.$$

Thus $F(x)$ is here *continuous*, in contrast to Exs. 2, 3.

The condition of convergence is

$n|x|>\eta$, where $\eta=\{1+\sqrt{(1-4\epsilon^2)}\}/2\epsilon$, if $\epsilon<\frac{1}{2}$.

Thus $m(x)=$ the integral part of $\eta/|x|$, if $|x|>0$,

although $m(0)=0$.

It will be seen that in Ex. 1 the function $m(x)$ is always less than $1/\epsilon$; but in Ex. 2, $m(x)\to\infty$ as $x\to 1$; and in Exs. 3, 4, $m(x)\to\infty$ as $x\to 0$ (assuming that $\epsilon<\frac{1}{2}$). These considerations suggest a further subdivision of convergent sequences, which will prove of great importance in subsequent applications.

We shall say that *the sequence $S_n(x)$ converges* **uniformly** *in the interval (a, b), provided that for* **all** *points of the interval we can determine $\mu=\mu(\epsilon)$, so that*

$$|S_n(x)-F(x)|<\epsilon, \qquad \textit{if } n>\mu,$$

where $\mu(\epsilon)$ is **independent** of x. Then, as x varies from a to b, $m(x)$ has the fixed upper limit $\mu(\epsilon)$, and so $m(x)$ cannot tend to infinity at any point in the interval (a, b).

Thus in Ex. 1 the convergence is uniform for all positive values of x, since we can take $\mu(\epsilon)=1/\epsilon$. But in Ex. 2, the convergence is not uniform in an interval including $x=1$; although it *is* uniform in the interval $(0, c)$, if $0<c<1$, because we can then take

$$\mu(\epsilon)=\frac{\log(1/\epsilon)}{\log(1/c)}.$$

Hence in Ex. 2, $x=1$ cannot be included in any interval of uniform convergence: such a point will be called *a point of non-uniform convergence*. Similarly in Exs. 3, 4 the point $x=0$ must be excluded to ensure uniform convergence.

This distinction may be made more tangible by means of a graphical method suggested by Osgood.* The curves $y=S_n(x)$ are drawn for a succession of values of n in the same diagram; this is done in Figs. 12–15 for the sequences of Exs. 1–4. Then, if $S_n(x) \to F(x)$ *uniformly* in the interval (a, b) the *whole* of the curves

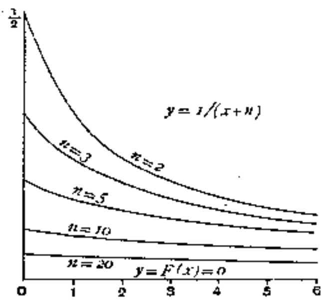

Fig. 12.

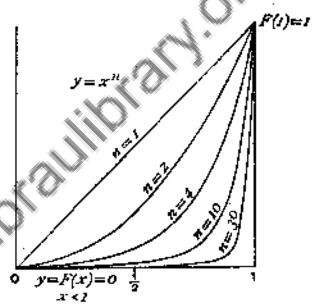

Fig. 13.

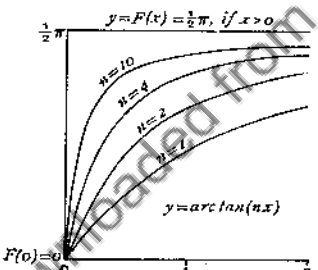

Fig. 14.

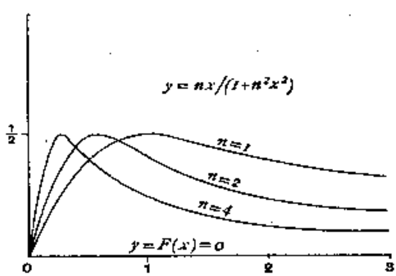

Fig. 15.

for which $n > \mu(\epsilon)$ will lie in the strip bounded by $y=F(x)\pm\epsilon$. A glance at Fig. 12 will shew that this does occur in Ex. 1. But in Ex. 2, as we see from Fig. 13, every curve $y=S_n(x)$ finally rises above $y=\epsilon$; and the larger n is taken, the nearer to $x=1$ is the point of crossing; thus $x=1$ is a point of non-uniform convergence. In the same way, Figs. 14, 15 shew that $x=0$ is a point of non-uniform convergence for each of the sequences in Exs. 3, 4.

* *Bulletin of the American Math. Society* (2), vol. 3, 1897, p. 59.

Ex. 5. The sequence $S_n(x)=x^{n-1}(1-x)$ converges *uniformly* in the interval (0, 1), because $S_n(x)<1/n$, since the maximum of $S_n(x)$ in the interval is given by $x=(n-1)/n$. The reader should contrast this result with Ex. 2, and should draw the curves $y=S_n(x)$ for a few values of n.

In order to give a definition of uniform convergence which does not involve the actual determination of $F(x)$, we introduce the following test, corresponding to that of Art. 3 for convergence.

The necessary and sufficient condition for uniform convergence in an interval is that, corresponding to any positive number ϵ, *it may be possible to find an index* m, **which is independent of x**, *and is such that*

$$|S_n(x)-S_m(x)|<\epsilon, \qquad \text{where } m=m(\epsilon),$$

for all values of n greater than m, and for all *points of the interval.*

It will be seen on comparison with Art. 3 that the only fresh condition is that m is to be independent of x, whereas the terms of the sequence are functions of x.

That the condition is *necessary* is evident, for if $S_n(x)$ tends uniformly to $F(x)$, we can write $m-1=\mu(\frac{1}{2}\epsilon)$, so that

$$|S_m(x)-F(x)|<\tfrac{1}{2}\epsilon$$

and
$$|S_n(x)-F(x)|<\tfrac{1}{2}\epsilon, \quad \text{if } n>m;$$

hence we have the inequality

$$|S_n(x)-S_m(x)|<\epsilon, \quad \text{if } n>m.$$

This condition is also *sufficient*; for if it is satisfied, $S_n(x)$ must converge to some limit, $F(x)$ say, in virtue of Art. 3; and since

$$\lim S_n(x)=F(x),$$

we have
$$|F(x)-S_m(x)|\leqq\epsilon.$$

Hence
$$|F(x)-S_n(x)|<2\epsilon, \quad \text{if } n>m,$$

and so the condition of uniform convergence to the limit $F(x)$ is satisfied.

It is useful to notice that *an interval of uniform convergence is always* closed.

Conclusions. (i) $S_n(a)$ and $S_n(b)$ tend to definite limits as n tends to infinity, and these may be called $F(a)$, $F(b)$, respectively, and $F(x)$ is now defined in the closed interval; (ii) $S_n(x)$ tends to $F(x)$ uniformly in the closed interval.

This statement has been the subject of discussion with a number of mathematicians; but it has been found in every case that their objections depended on a definition of uniform convergence which differs from the present definition. An account of other definitions will be found in Art. 49·1 below.

To avoid further misunderstandings, the following statement is made very complete:

Hypotheses. (i) $S_n(x)$ is supposed continuous in the closed interval

$$a \leqq x \leqq b, \text{ for all values of } n;$$

(ii) $S_n(x)$ tends to $F(x)$ uniformly in the open interval $a < x < b$.

Proof. In virtue of hypothesis (ii) $m = m(\epsilon)$ can be found so that

$$|S_n(x) - S_m(x)| < \epsilon, \quad \text{if } a < x < b \text{ and } n > m.$$

Also, since $S_n(x)$ is a continuous function at $x = a$, we can find values of x in the interval (a, b), such that

$$|S_n(x) - S_n(a)| < \epsilon \quad \text{and} \quad |S_m(x) - S_m(a)| < \epsilon.$$

Hence $$|S_n(a) - S_m(a)| < 3\epsilon, \quad \text{if } n > m,$$

so that $S_n(a)$ converges, and $x = a$ can be included in the interval of uniform convergence. Similarly for $x = b$.

44. Uniform convergence of a series.

If, in Art. 43, we suppose the sequence to be derived from a series of variable terms

$$f_0(x) + f_1(x) + f_2(x) + \dots \text{ to } \infty,$$

by writing $$S_n(x) = f_0(x) + f_1(x) + \dots + f_n(x),$$

we obtain the *test for uniform convergence of a series in an interval* (a, b) in the form:

It must be possible to find a number m **independent of x**, *so as to satisfy the condition*

$$|f_{m+1}(x) + f_{m+2}(x) + \dots + f_{m+p}(x)| < \epsilon, \quad \text{where } p = 1, 2, 3, \dots,$$

at all *points of the interval* (a, b).

Each of the examples given in Art. 43 can be used to construct a series by writing

$$f_n(x) = S_n(x) - S_{n-1}(x).$$

A more natural type of non-uniform convergence is given by the series:

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots.$$

Here we find $$S_n(x) = (1 + x^2) - (1 + x^2)^{-n+1},$$

so that $$F(x) = 1 + x^2 \quad (x \gtrless 0)$$

and $$F(0) = 0.$$

There is a point of non-uniform convergence at $x = 0$, as the reader will see by considering the condition

$$(1 + x^2)^{-n+1} < \epsilon, \quad \text{or} \quad (1 + x^2)^{n-1} > 1/\epsilon.$$

But, just as the general test for convergence is usually replaced by narrower tests (compare Chaps. II., III.) which are more convenient in ordinary practice, so here we usually replace the test above by one of the three following tests:

(1) Weierstrass's M-test for uniform convergence.

The majority of series met with in elementary analysis can be proved to converge uniformly by means of a test due to Weierstrass * and described briefly as the M-test:

Suppose that for all values of x in the interval (a, b), the function $f_n(x)$ has the property

$$|f_n(x)| \leqq M_n,$$

where M_n is a **positive constant, independent of x**; *and suppose that the series ΣM_n is convergent. Then the series $\Sigma f_n(x)$ is uniformly and absolutely convergent in the interval (a, b).*

The absolute convergence follows at once from Art. 18; to realise the uniform convergence, it is only necessary to remember that for any integral value of p,

$$\left|\sum_{m+1}^{m+p} f_n(x)\right| \leqq \sum_{m+1}^{m+p} M_n < \sum_{m+1}^{\infty} M_n.$$

Consequently, if we choose m so as to make the remainder in ΣM_n less than ϵ, $\left|\sum_{m+1}^{m+p} f_n(x)\right|$ is also less than ϵ; and this choice of m is obviously independent of x, so that the condition of uniform convergence is satisfied.

Series which satisfy the M-test have been called *normally convergent* by Baire. This terminology has the advantage of emphasising the fact that the M-test can be applied to nearly all series in ordinary everyday use.

Baire makes the remark that **any** uniformly convergent series can be made normally convergent by inserting brackets at suitable places. To prove this, let m_1 be taken so that

$|S_n(x) - S_{m_1}(x)| < M_1$ if $n > m_1$; then $m_2 > m_1$, so that $|S_n(x) - S_{m_2}(x)| < M_2$ and so on.

Now write $\phi_1(x) = \sum_{m_1+1}^{m_2} f_n(x)$, $\phi_2(x) = \sum_{m_2+1}^{m_3} f_n(x)$, $\phi_3(x) = \sum_{m_3+1}^{m_4} f_n(x)$, etc.

Then clearly the series $\Sigma\phi_n(x)$ satisfies the M-test; and this series is derived from $\Sigma f_n(x)$ by inserting brackets at $n = m_1, m_2, m_3, \ldots$.

* Compare also Stokes, *Math. and Phys. Papers*, vol. 1, p. 281.

(2) Abel's test for uniform convergence.

A more delicate test for uniform convergence is due, in substance, to Abel, and has been mentioned already in Art. 21 :

The series $\Sigma a_n(x)v_n(x)$ is uniformly convergent in an interval (a, b), provided that $\Sigma a_n(x)$ is uniformly convergent in the same interval; that for any particular value of x in the interval $v_n(x)$ is positive and never increases with n; and that $v_0(x)$ remains less than a fixed number K for all values of x in the interval.*

For, in virtue of the uniform convergence of $\Sigma a_n(x)$, we can find m, so that, whatever positive integer p may be,

$$a_{m+1},\quad a_{m+1}+a_{m+2},\quad \ldots,\quad a_{m+1}+a_{m+2}+\ldots+a_{m+p}$$

are all numerically less than ϵ. Then, in virtue of Abel's lemma (Art. 20), we see that

$$\left|\sum_{m+1}^{m+p} a_n(x)v_n(x)\right| < \epsilon v_m(x) < \epsilon K,$$

since, by hypothesis, $v_m(x) \leqq v_0(x) < K$.

Thus $\Sigma a_n(x)v_n(x)$ converges uniformly in the interval.

The most important special cases are (i) those in which a_n is independent of x; and (ii) those in which v_n is independent of x.

(3) Dirichlet's test for uniform convergence.

This is also more delicate than the M-test. (See Ex. 5 below.)

The series $\Sigma u_n(x)v_n(x)$ is uniformly convergent in an interval (a, b), provided that (i) *the series $\Sigma u_n(x)$ oscillates so that the absolute values of its limits of oscillation remain less than a fixed number K;* (ii) *for any particular value of x in the interval $v_n(x)$ is positive and never increases with n; and* (iii), *as n tends to ∞, $v_n(x)$ tends* **uniformly** *to zero for all values of x in the interval.*

For then, throughout the interval, the expressions

$$|u_{m+1}|,\quad |u_{m+1}+u_{m+2}|,\quad \ldots,\quad |u_{m+1}+u_{m+2}+\ldots+u_{m+p}|$$

are less than $2K$; and we can find an index m such that

$$v_m(x) < \epsilon$$

for all values of x in the interval.

* It may be useful to point out that Σa_n is not supposed to be absolutely convergent; if so, the M-test would apply, because $|a_n v_n(x)| < K|a_n|$.

Thus, using Abel's lemma as before, we see that

$$\left|\sum_{m+1}^{m+p} u_n(x)v_n(x)\right| < 2\epsilon K$$

for all points in the interval.

The important special cases should be noticed,

(i) when u_n is independent of x, and Σu_n either converges or has finite limits of oscillation;

(ii) when v_n is independent of x.

Ex. 1. *Weierstrass's M-test.*

The series $\Sigma r^n \cos n\theta$, $\Sigma r^n \sin n\theta$; $\Sigma r^n \cos n^2\theta$, $\Sigma r^n \sin n^2\theta$; $\Sigma r^n \cos (a^n\theta)$, $\Sigma r^n \sin (a^n\theta)$, $(0 < r < 1)$ converge uniformly for all real values of θ. This follows by taking $M_n = r^n$.

The series $\Sigma \dfrac{a_n x^n}{1+x^{2n}}$, $\Sigma \dfrac{a_n x^{2n}}{1+x^{2n}}$ converge uniformly for all real values of x if Σa_n is absolutely convergent.

Ex. 2. *Weierstrass's and Abel's tests.*

Fourier found the series

$$F(t) = \frac{6\theta}{\pi^2}\Sigma \frac{1}{n^2}e^{-\lambda n^2 t},$$

to represent the mean temperature at time t of a sphere originally heated to temperature θ and cooling with its surface kept at zero temperature. Here λ is a certain positive constant depending on the size, mass, and thermal properties of the sphere.

Weierstrass's test shews at once that the series converges uniformly if $t \geqq 0$; and so theorem (1) of Art. 45 gives

$$\lim_{t\to 0} F(t) = \theta.$$

The corresponding formula for the temperature at any point is

$$f(t) = \Sigma a_n e^{-n\lambda^2 t},$$

where a_n is of the form $(-1)^{n-1}(2\theta \sin n\omega)/(n\omega)$, and ω/π is equal to the quotient of the distance from the centre by the radius of the sphere. Since Σa_n converges (Art. 22), but not absolutely, we can apply Abel's test (but not Weierstrass's) to prove that $f(t)$ converges uniformly if $t \geqq 0$. Thus again we find

$$\lim_{t\to 0} f(t) = \theta.$$

Ex. 3. *Abel's test.*

Consider the case $v_n(x) = 1/n^x$, $(0 \leqq x \leqq 1)$; then $\Sigma(a_n/n^x)$ converges uniformly in the interval (0, 1) if Σa_n converges. [DIRICHLET.]

Ex. 4. *Abel's test.*

If Σa_n is convergent,

$$\Sigma a_n \frac{x^n}{1+x^n}, \quad \Sigma a_n \frac{x^n}{1+x^{2n}}, \quad \Sigma a_n \frac{nx^n(1-x)}{1-x^n}, \quad \Sigma \frac{2na_n x^n(1-x)}{1-x^{2n}},$$

converge uniformly in the interval (0, 1). [HARDY.]

Ex. 5. Consider the series

$$\frac{\cos n\theta}{n^p}, \quad \frac{\sin n\theta}{n^p}.$$

When $p > 1$, Weierstrass's M-test at once proves that both series converge uniformly for all real values of θ.

When $0 < p \leqq 1$, both series converge uniformly for any interval $(\alpha, 2\pi - \alpha)$, where α is positive; this can be proved by taking $u_n = \cos n\theta$ (or $\sin n\theta$) and $v_n = 1/n^p$ in Dirichlet's test; the value of K may be taken as $\operatorname{cosec} \frac{1}{2}\alpha$.

That the values $\theta = 0$ and 2π cannot be included in any interval of uniform convergence (if $0 < p \leqq 1$) follows, for the sine-series, from the proof of Art. 44·1 below. And the cosine-series diverges for these values of θ.

44·1. Uniform convergence of certain trigonometrical series.*

The series to be discussed are those of the types $\Sigma a_n \cos nx$, $\Sigma a_n \sin nx$, *where* (a_n) *is a sequence of positive numbers, tending steadily to zero.*

We have seen (Art. 22) that the former series converges for all values of x, other than 0 or multiples of 2π; and clearly the series cannot converge for these excepted values, unless Σa_n is convergent —in which case we can apply Weierstrass's M-test to infer uniform convergence. It follows that:

The series $\Sigma a_n \cos nx$ *can converge for all values of* x, *only if* Σa_n *converges* $(a_n > 0)$, *and then the series converges normally for all values of* x.

We pass now to the more subtle case of $\Sigma a_n \sin nx$, in regard to which the result is as follows:

The necessary and sufficient condition that the series $\Sigma a_n \sin nx$ *may converge uniformly for all values of* x *is that* $na_n \to 0$.

(i) To prove that this condition is *necessary*, consider the sum

$$R_m(x) = a_m \sin mx + a_{m+1} \sin (m+1)x + \ldots + a_p \sin px,$$

and take the special value $x = \pi/(2p+1)$.

This value makes $px < \frac{1}{2}\pi$, so that all the terms in $R_m(x)$ are positive; and so we have

$$R_m(x) > a_p\{\sin mx + \sin(m+1)x + \ldots + \sin px\},$$

because $$a_m \geqq a_{m+1} \geqq \ldots \geqq a_p.$$

The sum in { } brackets is equal to

$$\frac{\cos(m-\frac{1}{2})x - \cos(p+\frac{1}{2})x}{2\sin\frac{1}{2}x} = \frac{\cos(m-\frac{1}{2})x}{2\sin\frac{1}{2}x}, \quad \text{since } (p+\tfrac{1}{2})x = \tfrac{1}{2}\pi;$$

further, if we suppose that $p > 2m - 1$, it is easy to see that $(m-\frac{1}{2})x < \frac{1}{4}\pi$.

Thus $$\cos(m-\tfrac{1}{2})x > 1/\sqrt{2}, \quad \text{and} \quad 2\sin\tfrac{1}{2}x < x.$$

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* The following proof is taken from one published by Messrs. T. W. Chaundy and A. E. Jolliffe (*Proc. Lond. Math. Soc.* (2), vol. 15, 1916, p. 214).

Consequently the sum of sines is greater than

$$\frac{1}{x\sqrt{2}} = \frac{2p+1}{\pi\sqrt{2}} > \tfrac{2}{5}p^*.$$

It follows that

$$R_m(x) > \tfrac{2}{5}pa_p, \quad \text{if } p > 2m-1, \quad \text{and} \quad x = \pi/(2p+1),$$

and thus we cannot make $R_m(x) < \epsilon$, for this particular value of x, unless $na_n \to 0$ as $n \to \infty$.

(ii) To prove that the condition $na_n \to 0$ is *sufficient*, we note that Abel's Lemma gives

$$|R_m(x)| < a_m \operatorname{cosec} \tfrac{1}{2}x, \quad \text{if } 0 < x < \pi.$$

Now $\sin \frac{1}{2}x > x/\pi$, with the same restriction on x; so that

$$\left.\begin{array}{l}\operatorname{cosec}\frac{1}{2}x < \pi/x < m,\\ |R_m(x)| < ma_m,\end{array}\right\} \quad \text{if } \pi/m \leqq x < \pi.$$

and

If x lies between 0 and π/m, the value of π/x will be greater than m, and so may also be greater than p.

Suppose first that $\pi/x > p > m$; then all the angles $mx, (m+1)x, \ldots, px$ are less than π, and so each sine is less than the corresponding angle. Thus

$$R_m(x) < \{ma_m + (m+1)a_{m+1} + \ldots + pa_p\}x.$$

Hence, if η_m denotes the upper limit of

$$ma_m, \quad (m+1)a_{m+1}, \quad (m+2)a_{m+2}, \ \ldots \text{ to } \infty,$$

it is clear that

$$R_m(x) < px\eta_m < \pi\eta_m.$$

Again, if $p > \pi/x > m$, there is some integer r between m and p, such that $r+1 \geqq \pi/x > r$; the part of $R_m(x)$ from m to r is then covered by the last inequality, and so is less than $\pi\eta_m$. For the terms from $r+1$ to p we can use Abel's lemma, and so prove that the corresponding part of $R_m(x)$ is numerically less than $(r+1)a_{r+1} \leqq \eta_m$.

Hence, finally, we can write

$$|R_m(x)| < (\pi+1)\eta_m,$$

for any value of x, whether greater or less than π/m; and so we have

$$|R_m(x)| < \epsilon,$$

provided that m is chosen so that

$$na_n < \epsilon/(\pi+1), \quad \text{for } n \geqq m.$$

Thus the condition of uniform convergence (Art. 44) can be satisfied provided that $na_n \to 0$.

From the theorems just proved we see at once that *the series*

$$\Sigma\frac{1}{n}\cos nx, \quad \Sigma\frac{1}{n}\sin nx$$

cannot converge uniformly in any interval which includes $x=0$ (*or any multiple of* 2π).†

---

* It is easy to verify that $\pi/\sqrt{2} = 2{\cdot}2214\ldots < 2{\cdot}5 = \frac{5}{2}$.

† From Art. 65 it will be seen that the former series tends to ∞ and that the latter is discontinuous at $x=0$.

More generally, any series of the types

$$\Sigma \frac{1}{n^p} \cos nx, \quad \Sigma \frac{1}{n^p} \sin nx$$

can converge uniformly for *all* values of x, only if $p > 1$; and then, of course, the M-test is applicable.

45. Fundamental properties of uniformly convergent series. We have seen (in Arts. 25–27, 31, 34–36) how the condition of *absolute* convergence of a series enables us to perform various *algebraic* manipulations of the series; it will now appear that the condition of *uniform* convergence justifies the use of operations associated with the Calculus—such as differentiation and integration.

Cauchy and the earlier analysts (with the exception of Abel) assumed that the continuity of $F(x)$ could be deduced from that of $S_n(x)$; that this assumption is not correct follows immediately from Exs. 2, 3 of Art. 43. Further, these examples suggest that a discontinuity in $F(x)$ implies a point of non-uniform convergence; although Ex. 4, Art. 43, indicates that non-uniform convergence does not necessarily involve the discontinuity of $F(x)$.

Again, if we wish to integrate $F(x)$, the equation

$$\int_{c_1}^{c_2} \{\lim_{n\to\infty} S_n(x)\}\, dx = \lim_{n\to\infty} \int_{c_1}^{c_2} S_n(x)\, dx$$

is not necessarily true either, as will be seen from the examples given on p. 133 below.

(1) *If the series $F(x) = \Sigma f_n(x)$ is uniformly convergent in the interval (a, b), and if each of the functions $f_n(x)$ is continuous in the interval, so also is the sum $F(x)$.*

For, in virtue of the definition of uniform convergence, the number $m = m(\epsilon)$ can be chosen **independently** of x in such a way that

$$|f_m(x) + f_{m+1}(x) + \dots \text{ to } \infty| < \epsilon, \quad \text{if } a \leqq x \leqq b,$$

no matter how small ϵ may be. Now write

$$f_0(x) + f_1(x) + f_2(x) + \dots + f_{m-1}(x) = S_m(x),$$

and it is then clear that

$$|F(x) - S_m(x)| < \epsilon, \quad (a \leqq x \leqq b).$$

Thus if c is any value of x within the interval (a, b), we have

$$|F(c)-S_m(c)| < \epsilon,$$

so that $$|F(c)-F(x)| < 2\epsilon + |S_m(c)-S_m(x)|.$$

Now m being **fixed**, $S_m(x)$ is a continuous function of x, and therefore we can find a value

$$\delta = \delta(c, m, \epsilon) = \delta(c, \epsilon),$$

such that $$|S_m(c)-S_m(x)| < \epsilon, \quad \text{if } |c-x| < \delta.$$

Hence $$|F(c)-F(x)| < 3\epsilon, \quad \text{if } |c-x| < \delta.$$

Thus $F(x)$ is a continuous function in the neighbourhood of the point c.

It is not unusual for beginners to miss the point of the foregoing proof; and it is therefore advisable to show how the argument fails when applied to such a series as

$$(1-x)+(x-x^2)+(x^2-x^3)+\dots, \quad \text{(Ex. 2, Art. 43)}$$

when we take $c=1$.

Here $$f_m(x)+f_{m+1}(x)+\dots \text{ to } \infty = x^m \quad \text{if } 0 < x < 1,$$

and $$f_m(1)+f_{m+1}(1)+\dots \text{ to } \infty = 0.$$

Thus, if we wish to make both these remainders less than ϵ, we must choose m, if we can, so that

$$x^m < \epsilon, \quad \dots\dots\dots\dots(A)$$

but to make $$|S_m(1)-S_m(x)| < \epsilon$$

we must take $$1-x^m < \epsilon$$

or $$x^m > 1-\epsilon, \quad \dots\dots\dots\dots(B)$$

and the two inequalities (A) and (B) are mutually contradictory (supposing that $\epsilon < \frac{1}{2}$).

Consequently the two steps used in the general argument are incompatible here; and the reason for this difficulty lies in the fact that the inequality (A) does not lead to a determination of *m independent of x*, when x can approach as near to 1 as we please. The assumption that the series converges uniformly enables us to ensure that the condition corresponding to (A) does not contradict (B).

(2) *If the series $F(x)=\Sigma f_n(x)$ is uniformly convergent in the interval (a, b), and if each of the functions $f_n(x)$ is continuous in the interval, we may write*

$$\int_{c_1}^{c_2} F(x)\,dx = \sum \int_{c_1}^{c_2} f_n(x)\,dx, \quad \text{if } a \leqq c_1 < c_2 \leqq b.$$

For, in virtue of the uniform convergence of $\Sigma f_n(x)$, we can choose $m=m(\epsilon)$ so that

$$|F(x)-S_n(x)| < \epsilon, \quad \text{if } n > m, \quad \text{and } a \leqq x \leqq b.$$

Integrate the last inequality from c_1 to c_2, and we have the result *

$$(1)\quad \left|\int_{c_1}^{c_2} F(x)\,dx - \int_{c_1}^{c_2} S_n(x)\,dx\right| < \epsilon(c_2-c_1) \leqq \epsilon(b-a), \quad \text{if } n > m.$$

Now write

$$\phi_n = \int_{c_1}^{c_2} S_n(x)\,dx = \int_{c_1}^{c_2} f_0(x)\,dx + \int_{c_1}^{c_2} f_1(x)\,dx + \ldots + \int_{c_1}^{c_2} f_{n-1}(x)\,dx;$$

then it is clear from the previous inequality (1) that the sequence (ϕ_n) converges, and that its sum is equal to

$$\int_{c_1}^{c_2} F(x)\,dx.$$

That is,
$$\sum_0^\infty \int_{c_1}^{c_2} f_n(x)\,dx = \int_{c_1}^{c_2} F(x)\,dx.$$

By a change of notation we may write †

$$\sum \int_a^x f_n(\xi)\,d\xi = \int_a^x F(\xi)\,d\xi. \quad (a \leqq x \leqq b)$$

It should be noted further that this series of integrals converges uniformly in the interval (a, b), in virtue of inequality (1) above.

In this form, the process is known as *term-by-term integration of series*.

The reader will probably find less difficulty here in realising the importance of the condition that m should be independent of x. It is not, however, easy to give a really simple example of a non-uniformly convergent *series* in which term-by-term integration leads to erroneous results. The following method shews how a variety of *sequences* can be constructed in which the process fails.

Take $S_n(x) = nx f(nx^2)$, where $f(\xi)$ is a positive decreasing function for which the integral $\int_0^\infty f(\xi)\,d\xi$ converges to some value J. Then $\xi f(\xi) \to 0$ as $\xi \to \infty$ (Arts. 9, 11); and so $S_n(x) \to 0$ as $n \to \infty$ for any positive value of x, while $S_n(0) = 0$. Thus we have

$$F(x) = \lim_{n\to\infty} S_n(x) = 0, \quad \text{for } x \geqq 0,$$

and accordingly
$$\int_0^a F(x)\,dx = 0, \quad \text{if } a > 0.$$

* It follows from Theorem (1) that $F(x)$ is continuous in the interval (a, b), and we assume that any continuous function is integrable; so that $\int_{c_1}^{c_2} F(x)\,dx$ is determinate.

† It is not correct to write simply $\Sigma \int f_n(x)\,dx = \int F(x)\,dx$, because the constants of integration might happen to lead to a non-convergent series after integration.

But $$\int_0^a S_n(x)\,dx = \tfrac{1}{2}\int_0^{na^2} f(\xi)\,d\xi \quad \text{by writing } \xi = nx^2;$$

and so $$\int_0^a S_n(x)\,dx \to \tfrac{1}{2}J, \quad \text{as } n \to \infty.$$

That is, $$\lim_{n\to\infty}\int_0^a S_n\,dx > \int_0^a \lim_{n\to\infty} S_n(x)\,dx,$$

so that the process of term-by-term integration fails for this class of sequences.

Two simple cases are given by taking

$$f(\xi) = e^{-\xi} \quad \text{or} \quad 1/(1+\xi^2), \quad \text{for which } J = 1 \text{ or } \tfrac{1}{2}\pi.$$

The figure below (Fig. 16) shews the approximation curves for the former case, the peaks being given by $x = 1/\sqrt{(2n)}$, $y = \sqrt{(n/2e)}$; for the latter the

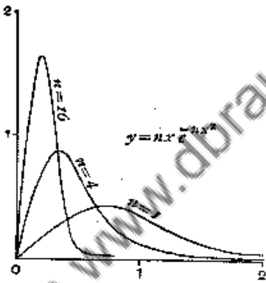

FIG. 16.

curves are similar in *shape* to those of Fig. 15, but the peaks are given by $x = 1/\sqrt{n}$, $y = \tfrac{1}{2}\sqrt{n}$, so that the general appearance does not differ much from Fig. 16.

In a general way we can see from the shape of these curves that the area below $y = S_n(x)$ is greater than that of a triangle joining the origin to the peak. And in each of these examples the area of this triangle (being $\tfrac{1}{2}xy$) has a constant value independent of n; so that we should expect the area below $S_n(x)$ *not* to tend to zero, in spite of the fact that $S_n(x)$ does so.

Two examples of *series* illustrating the failure of term-by-term integration are given in Exs. 14, 15 at the end of the chapter. But Ex. 14 uses a series which ceases to converge at the critical point $(x = 1)$; and in Ex. 15 the failure is less easy to prove in an elementary way.

Other examples of sequences are constructed in Exs. 16, 17.

Of course the argument above assumes that the range of integration is *finite*; the conditions under which an infinite series can be integrated from 0 to ∞, say, belong more properly to the Integral Calculus; but some special cases are given in Exs. 18, 19 at the end of this chapter.

46. Differentiation of an infinite series.

If we consider Ex. 4 of Art. 43, for which

$$S_n(x)=\frac{nx}{1+n^2x^2}, \quad F(x)=0,$$

we see that

$$S_n'(0)=\lim_{x\to 0}\frac{S_n(x)-S_n(0)}{x}=\lim_{x\to 0}\frac{n}{1+n^2x^2}=n.$$

Thus $\lim_{n\to\infty} S_n'(0)=\infty$, although $F'(0)=0$.

It follows that the equation

$$\frac{d}{dx}[\lim_{n\to\infty} S_n(x)]=\lim_{n\to\infty} S_n'(x)$$

is not necessarily true when non-uniform convergence presents itself. But it should be noticed that it is the non-uniform convergence of the *differential coefficient* $S_n'(x)$ which is the cause of the failure, as will be apparent from the general theorem below.

The reader may consider similarly the case

$$S_n(x)=\frac{1}{n}\sin(nx), \quad F(x)=0;$$

here $S_n(x)$ converges uniformly to zero, but $S_n'(x)$ *oscillates*.

If the series of differential coefficients $\Sigma f_n'(x)$ *is uniformly convergent within the interval* (a, b), *its sum is equal to* $F'(x)$, *the differential coefficient of* $F(x)=\Sigma f_n(x)$; *it is assumed that the differential coefficients* $f_n'(x)$ *are continuous, and that* $\Sigma f_n(x)$ *converges in the interval.**

Write $$G(x)=\Sigma f_n'(x),$$

then, by theorem (2) of Art. 45, we have

$$\int_{c_1}^{c_2} G(x)\,dx=\Sigma\{f_n(c_2)-f_n(c_1)\}$$
$$=F(c_2)-F(c_1).$$

Thus, since $G(x)$ is a continuous function (Art. 45), it follows from the fundamental property of an integral that

$$F'(c_2)=G(c_2)$$

or $$F'(x)=G(x), \quad (a\leqq x\leqq b).$$

---

* We can infer the convergence of $\Sigma f_n(x)$ from that of $\Sigma f_n'(x)$, *if the constants of integration are properly adjusted* (as in Art. 45); this amounts to the assumption that $\Sigma f_n(x)$ converges at some one point of the interval.

A *direct* proof of the foregoing theorem is not easy without some use of the Integral Calculus; but if we restrict the proof to *normally* convergent series, we can avoid the use of integration by the following method.

Write $$\phi_n(x, h) = \frac{1}{h}\{f_n(x+h) - f_n(x)\},$$

then, by the mean-value theorem of the Differential Calculus,

$$\phi_n(x, h) = f_n'(x+\theta h), \quad \text{where* } 0 < \theta < 1.$$

Thus, if x and $x+h$ both belong to the interval (a, b), we have

$$|\phi_n(x, h)| < M_n,$$

where ΣM_n is a convergent series of positive constants, such that

$$|f_n'(x)| < M_n.$$

Accordingly $\Sigma\phi_n(x, h)$ converges uniformly for all such values of h; and therefore, by theorem (1) of Art. 45,

$$\lim_{h\to 0} \Sigma\phi_n(x, h) = \Sigma \lim_{h\to 0} \phi_n(x, h),$$

or $$\lim_{h\to 0} \frac{1}{h}\{F(x+h) - F(x)\} = \Sigma f_n'(x),$$

which is the required result.

47. It is important to bear in mind that the condition of uniform convergence is merely *sufficient* for the truth of the theorems in Arts. 45, 46; but it is by no means a *necessary* condition. In other words, this condition is too narrow; but in spite of this, no other condition of equal simplicity has been discovered as yet, and we shall not go further into the subject † here.

That uniform convergence is not necessary may be seen by considering the two following examples:

(1) Ex. 4, Art. 43, shews that non-uniform convergence does not always imply discontinuity.

(2) Consider the series

$$1 - x + x^2 - x^3 + \dots = 1/(1+x), \quad (0 < x < 1).$$

* In general θ varies with n; and this is the reason why a longer investigation is apparently inevitable when the M-test does not apply.

† Reference may be made to a paper by the author (*Proc. Lond. Math. Soc.* series 2, vol. 1, 1904, p. 187) for the general question. Many wider tests for term-by-term integration have been given by various writers; some very simple ones, due to Prof. W. H. Young, will be found in Ex. 22 at the end of this chapter.

Then $$\int_0^1 \frac{dx}{1+x} = \log 2,$$

and log 2 is also equal to the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots$$

found by integrating term-by-term.

Nevertheless $x=1$ is a point of non-uniform convergence of the series in x; because the remainder is greater than $\frac{1}{2}x^n$, and the condition $\frac{1}{2}x^n < \epsilon$ leads to a determination of n, which *cannot* be independent of x (when $x=1$ is included in the interval considered).

48. Uniform convergence of an infinite product.

The definition of uniform convergence can be extended at once to an infinite product; but applications of the principle occur less frequently in elementary analysis, and for our present purpose the following theorem will be sufficient:

If for all values of x in the interval (a, b) the function $f_n(x)$ has the property $|f_n(x)| \leqq M_n$, where M_n is a positive constant (independent of x), then if the series ΣM_n is convergent, the product

$$P(x) = [1+f_0(x)][1+f_1(x)][1+f_2(x)]\ldots \text{ to } \infty$$

is a continuous function of x in the interval, provided that all the functions $f_n(x)$ are continuous in the interval.

For, write $$[1+f_0(x)][1+f_1(x)]\ldots[1+f_{m-1}(x)] = P_m(x),$$
$$[1+f_m(x)][1+f_{m+1}(x)]\ldots \quad \text{to } \infty = Q_m(x);$$

and let A_m denote the product

$$(1+M_0)(1+M_1)\ldots(1+M_{m-1}),$$

while A is the value of the corresponding product when m tends to infinity. That A is finite follows from Art. 39.

Using the inequalities of Art. 40, we see that

$$|P(x) - P_m(x)| < A - A_m,$$

and we can accordingly choose m so that

$$|P(x) - P_m(x)| < \epsilon$$

for all values of x in the interval (a, b).

Having fixed m, we can choose δ so that

$$|P_m(x) - P_m(c)| < \epsilon, \quad \text{if } |x-c| < \delta,$$

since $P_m(x)$ is the product of a fixed number of continuous functions; and then $$|P(x) - P(c)| < 3\epsilon, \quad \text{if } |x-c| < \delta.$$

Hence $P(x)$ is a continuous function of x in the interval (a, b).

If the function $f_n(x)$ *has a derivate* $f_n'(x)$, *such that* $|f_n'(x)| < M_n$, *and if*

$$1+f_n(x) \geqq \alpha > 0$$

at all points of the interval; then the infinite product has a derivate $P'(x)$ *given by*

$$\frac{P'(x)}{P(x)} = \Sigma \frac{f_n'(x)}{1+f_n(x)}.$$

For under these conditions we have

$$\frac{f_n'(x)}{1+f_n(x)} < \frac{M_n}{\alpha},$$

so that Art. 46 can be applied; and we find, accordingly,

$$\Sigma \frac{f_n'(x)}{1+f_n(x)} = \frac{d}{dx} \Sigma \log [1+f_n(x)] = \frac{d}{dx} \log P(x) = \frac{P'(x)}{P(x)}.$$

49. Closely connected with the theory of uniform convergence is the following theorem* which is of frequent use in subsequent investigations:

Suppose that we are given a sum

$$F(n) = v_0(n) + v_1(n) + v_2(n) + \ldots + v_p(n)$$

and that we want to find the limit $\lim\limits_{n\to\infty} F(n)$, *it being understood that* p *tends steadily to infinity with* n. *Then, if*

$$\lim_{n\to\infty} v_r(n) = w_r \quad (r \text{ being fixed}),$$

the limit of $F(n)$ *is given by*

$$\lim_{n\to\infty} F(n) = w_0 + w_1 + w_2 + \ldots \text{ to } \infty = W,$$

provided that $|v_r(n)| \leqq M_r$, *where* M_r *is* **independent of** n, *and the series* ΣM_r *is convergent.*

The reader will note that the test for the theorem is substantially the same † as the M-test due to Weierstrass (Art. 44). The proof, too, is almost the same.

First choose a number $q = q(\epsilon)$, such that

$$M_q + M_{q+1} + \ldots \text{ to } \infty < \epsilon,$$

and let n be taken large enough to make $p > q$; then we have

$$|v_q + v_{q+1} + \ldots + v_p| \leqq M_q + M_{q+1} + \ldots + M_p < \epsilon$$

or

$$|F(n) - (v_0 + v_1 + v_2 + \ldots + v_{q-1})| < \epsilon.$$

* Tannery, *Fonctions d'une variable*, § 183 (in the 2nd edition).

† Here of course n takes the place of x in the test of Art. 44.

Also $|w_q + w_{q+1} + w_{q+2} + \ldots \text{ to } \infty| \leqq M_q + M_{q+1} + \ldots \text{ to } \infty < \epsilon.$

Thus

$$|F(n) - W| < |(v_0 + v_1 + \ldots + v_{q-1}) - (w_0 + w_1 + \ldots + w_{q-1})| + 2\epsilon,$$

and it is to be remembered that so far n has only been restricted by the condition $p > q$.

Now, since q is *independent* of n, we can allow n to tend to infinity in the last inequality, and then we find that the right-hand side tends to the limit 2ϵ; and we can accordingly find a value $n_0 = n_0(q, \epsilon) = n_0(\epsilon)$, such that the right-hand side is less than 3ϵ, for $n < n_0$.

Hence $$|F(n) - W| < 3\epsilon, \quad \text{if } n > n_0,$$

or $$\lim_{n \to \infty} F(n) = W = w_0 + w_1 + w_2 + \ldots \text{ to } \infty.$$

The following example will serve to shew the danger of trying to use the foregoing theorem when the M-test does not apply.

Consider the sum

$$F(n) = \log\left(1 + \frac{1}{n^2}\right) + \log\left(1 + \frac{2}{n^2}\right) + \ldots + \log\left(1 + \frac{n}{n^2}\right),$$

so that $$v_r(n) = \log\left(1 + \frac{r}{n^2}\right) \quad \text{and} \quad p = n.$$

Then obviously $$w_r = \lim_{n \to \infty} v_r(n) = 0,$$

and so the sum of the series $w_0 + w_1 + w_2 + \ldots$ is 0.

But $v_r(n)$ lies between r/n^2 and $r/(n^2 + n)$, and $\sum_1^n r = \frac{1}{2}(n^2 + n)$,

so that $$\frac{1}{2}\left(1 + \frac{1}{n}\right) > F(n) > \frac{1}{2},$$

and hence $$\lim_{n \to \infty} F(n) = \tfrac{1}{2}.$$

Another theorem of importance in this connexion is the analogous result for products:

Suppose that

$$P(n) = [1 + v_0(n)][1 + v_1(n)] \ldots [1 + v_p(n)]$$

where p tends steadily to infinity with n.

Then if $\lim_{n \to \infty} v_r(n) = w_r$, and if $|v_r(n)| \leqq M_r$ where M_r is independent of n and ΣM_r is convergent, we have the equation

$$\lim_{n \to \infty} P(n) = (1 + w_0)(1 + w_1)(1 + w_2) \ldots \text{ to } \infty.$$

The reader should have little difficulty in constructing a proof of this theorem on the lines of the foregoing, employing the results of Arts. 38, 39 to find limits for the products

$$(1+v_q)(1+v_{q+1})\dots(1+v_p)$$

and

$$(1+w_q)(1+w_{q+1})\dots \text{ to } \infty$$

in terms of the remainder $M_q+M_{q+1}+\dots$ to ∞.

To shew the need of some condition such as the M-test, we may consider the example

$$P(n)=\left(1+\frac{1}{n}\right)^p$$

in which

$$v_0=v_1=v_2=\dots=1/n,$$

so that

$$w_0=w_1=w_2=\dots=0.$$

But the equation

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^p=1$$

is not necessarily true. In fact the value of the limit depends on the value of $\lim(p/n)$, because

$$\frac{1}{n+1}<\log\left(1+\frac{1}{n}\right)<\frac{1}{n},$$

so that

$$\frac{p}{n+1}<\log\left(1+\frac{1}{n}\right)^p<\frac{p}{n}.$$

Thus

$$\lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)^p=\lim_{n\to\infty}\frac{p}{n}.$$

49·1. Historical Note* on Uniform Convergence.

The discovery of the notion of uniform convergence is generally attributed to Weierstrass (1841), Stokes (1847), and Seidel (1848). The idea is no doubt implied in Abel's theorem (Arts. 50, 51) on the continuity of power-series; but its explicit formulation is due to the three mathematicians mentioned first.

But to appreciate the development of ideas on this subject, it is necessary to formulate certain definitions differing in various ways from that adopted in Art. 43 above; this first definition refers to *uniform convergence throughout an interval*, and will be quoted as A. 1 in what follows.

Consider now the closely connected definitions:

A. 2. *Uniform convergence in the neighbourhood of a point.*

The series will be said to converge uniformly in the neighbourhood of a point ξ in the interval (a, b), if we can find $\delta=\delta(\xi)$ and $m=m(\xi,\epsilon)$, so that

$$|S_n(x)-S_m(x)|<\epsilon \quad \dots\dots\dots(1)$$

for $n>m$ and† $\xi-\delta\leqq x\leqq\xi+\delta$.

* For the substance of the following note I am indebted to Mr. G. H. Hardy's recent paper, "Sir George Stokes and the Concept of Uniform Convergence," *Proc. Camb. Phil. Soc.* vol. 19 (May 1918), p. 148.

† If ξ coincides with a, this condition is to be taken as $a\leqq x\leqq a+\delta$. Similarly if ξ coincides with b.

A. 3. *Uniform convergence at a point.*

The definition differs only from A. 2 in allowing δ *to depend on* ϵ, *as well as* ξ.

Definitions A. 1, A. 2 were in use by Weierstrass as early as 1841 or 1842; and the definition A. 2 was used in Seidel's work, published in 1848. Definition A. 3 was first explicitly formulated by W. H. Young (1903), although the idea is implicitly contained in an earlier paper by W. F. Osgood (1897).

It is important to notice here W. H. Young's theorem:* *that uniform convergence at every point of an interval involves uniform convergence throughout the interval*;† although uniform convergence *at* ξ does *not* involve uniform convergence *in the neighbourhood of* ξ.

In addition to the above definitions there is a set of less stringent conditions to which the name of *quasi-uniform convergence* has been given recently.‡ In essence the distinction from uniform convergence lies in the fact that the fundamental inequalities are satisfied by *an infinite sequence of values of* n, but not necessarily by *all* values greater than m.

The following three definitions are formulated so as to be parallel to the three preceding:

B. 1. *Quasi-uniform convergence throughout an interval.* §

The series is said to converge quasi-uniformly throughout the interval (a, b), if corresponding to every value N, we can find a value $m = m(\epsilon, N)$ greater than N, such that

$$|F(x) - S_m(x)| < \epsilon \quad \text{.........(2)}$$

for all values of x in the interval.

Arzelà and Hobson have remarked that series which satisfy B. 1 can be converted into series satisfying A. 1 by inserting brackets at appropriate places; just as uniformly convergent series can be converted into normally convergent series by insertion of brackets (Art. 44 above).

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* This theorem refers to a *closed* interval $(a \leqq x \leqq b)$: for a series might converge uniformly at *every* point of an *open* interval $(a < x < b)$ without doing so in the corresponding *closed* interval.

† Proved by the aid of the Heine-Borel Theorem (see, for example, Hardy's *Pure Mathematics* (2nd edition), Art. 105) on the following lines: Choose first ϵ, and then determine δ, m (as in definition A. 3) for every point ξ in the interval (a, b). Every point of (a, b) is included in an interval $(\xi - \delta, \xi + \delta)$ by the Heine-Borel theorem, every point of (a, b) is included in one or other of a finite sub-set of these intervals. If M is the largest m corresponding to each of the intervals of this finite sub-set, then the fundamental definition A. 1 is satisfied for $n \geqq M$ and $a \leqq x \leqq b$.

This proof needs some further elaboration to be complete; but a full discussion would be out of place here.

‡ The term *simply-uniform* was adopted by earlier writers.

§ This definition was originally introduced by Dini and Darboux; and it has been used in another form by Hobson (*Proc. Lond. Math. Soc.* vol. 1, 1903, p. 373). Dini and Hobson use the term *simply-uniformly convergent*.

B. 2. *Quasi-uniform convergence in the neighbourhood of a point.*

The series is said to converge quasi-uniformly in the neighbourhood of ξ if an interval $(\xi-\delta, \xi+\delta)$ can be found throughout which the series is quasi-uniformly convergent (B. 1). Here $\delta=\delta(\xi)$ and $m=m(\xi, \epsilon, N)$.

In effect, definition B. 2 is equivalent to the distinction originally introduced by Stokes* in 1847: in Stokes's terminology series which do *not* satisfy definition B. 2 are said *to converge infinitely slowly when* $x=\xi$.

B. 3. *Quasi-uniform convergence at a point.*

The series is said to converge quasi-uniformly at ξ, if $\delta=\delta(\xi, \epsilon, N)$ can be found such that definition B. 1 is satisfied in the interval $(\xi-\delta, \xi+\delta)$, while $m=m(\xi, \epsilon, N)$.

The idea involved in definition B. 3 is due to Dini, who proved the theorem (1) on continuity established in the following article.

It should be observed that *for series of positive terms, quasi-uniform convergence is equivalent to uniform convergence*; † for if we have found a value m satisfying inequality (2), then, since

$$S_m(x) \leqq S_n(x) \leqq F(x), \quad \text{if } n > m,$$

it follows that $\quad |F(x)-S_n(x)| < \epsilon, \quad \text{if } n > m,$

and also that $\quad |S_n(x)-S_m(x)| < \epsilon, \quad \text{if } n > m.$

Hence each A-definition is satisfied if the corresponding B-definition is satisfied.

49·2. Theorems and examples of the foregoing definitions.

(1) *The necessary and sufficient condition that $F(x)$ should be continuous at $x=\xi$ is that the series should be quasi-uniformly convergent at $x=\xi$.* [DINI.]

It is evident that the proof of Art. 45 (1) will apply if a value m has been found to satisfy inequality (2) of Art. 49·1; and the fact that δ depends on ξ, ϵ, N will not affect the final conclusion. Hence the condition is *sufficient*.‡

To see that the condition is *necessary*, note that

$$|F(x)-S_m(x)| \leqq |F(x)-F(\xi)| + |F(\xi)-S_m(\xi)| + |S_m(x)-S_m(\xi)|.$$

Since $F(x)$ is continuous at $x=\xi$, we can choose $\delta=\delta(\xi, \epsilon)$ so that

$$|F(x)-F(\xi)| < \epsilon, \quad \text{for } \xi-\delta < x < \xi+\delta,$$

and m, depending on ξ, ϵ and N, so that

$$m > N \quad \text{and} \quad |F(\xi)-S_m(\xi)| < \epsilon.$$

* *Math. and Phys. Papers*, vol. 1, pp. 236-313: see in particular p. 279. For the grounds on which the identification rests, see pp. 154-156 of Hardy's paper previously quoted.

† That is, B. 1 leads to A. 1, B. 2 to A. 2, and B. 3 to A. 3.

‡ It follows *a fortiori* that conditions A. 2, A. 3, and B. 2 all give *sufficient* conditions for continuity at a point; while A. 1 and B. 1 give sufficient conditions for continuity throughout an interval.

Having fixed m, we can choose $\delta_1 < \delta$, where $\delta_1 = \delta_1(\xi, \epsilon, m) = \delta_1(\xi, \epsilon, N)$, so that

$$|S_m(x) - S_m(\xi)| < \epsilon, \quad \text{for} \quad \xi - \delta_1 < x < \xi + \delta_1.$$

Thus $$|F(x) - S_m(x)| < 3\epsilon, \quad \text{for} \quad \xi - \delta_1 < x < \xi + \delta_1,$$

and for some value of $m > N$; and thus the condition B. 3 is satisfied.

Ex. 1. Consider the sequence $S_n(x) = nx/(1 + n^2x^2)$, for which $F(x)$ is continuous at $x = 0$. (Art. 43, Ex. 4.)

The conditions B. 3 are satisfied by taking

$$m = 2N, \quad \delta = \epsilon/2N.$$

Ex. 2. On the other hand, $S_n(x) = 1/(1 + n^2x^2)$ gives $F(x) = 0$, and $F(0) = 1$; thus the conditions B. 3 ought not to be satisfied at $x = 0$; and this conclusion is easily verified on trial.

(2) *If the sum of a series of* **positive** *terms is continuous throughout* (a, b), *then the series converges uniformly throughout* (a, b). [Dini.]

For clearly the series is continuous at every point ξ of (a, b); thus by (1) above it converges quasi-uniformly at ξ.

Since the terms of the series are all positive, it follows (as at the end of Art. 49·1) that the series converges uniformly at ξ; and since this conclusion holds for every point of (a, b), the series converges uniformly throughout (a, b) in virtue of W. H. Young's theorem. (Art. 49·1.)

EXAMPLES.

1. Shew that if $S_n(x) = x^n/(1 + x^{2n})$, $x = 1$ is a point of non-uniform convergence of $S_n(x)$ to its limit. Draw graphs of $S_n(x)$ and $\lim S_n(x)$.

2. Shew that $\Sigma \dfrac{1}{n!} \dfrac{a^n}{1 + x^2a^{2n}}$, $\Sigma \dfrac{(-1)^n}{n!} \dfrac{a^n}{1 + x^2a^{2n}}$ converge uniformly for all values of x; and that if $a < 1$ and $x < 1$, they are respectively equal to the series

$$e^a - x^2e^{a^3} + x^4e^{a^5} - \ldots,$$

and $$e^{-a} - x^2e^{-a^3} + x^4e^{-a^5} - \ldots,$$

obtained by expanding each fraction in powers of x. [Pringsheim.]

3. Shew that the series $\sum_1^\infty \dfrac{x}{n(1 + nx^2)}$ is uniformly convergent for all values of x.

[The maximum value of the general term $f_n(x)$ is given by $nx^2 = 1$, and the M-test applies.]

4. Generally consider the series

$$\sum_1^\infty \frac{x}{n^p + n^qx^2} \Bigg\} \; p, \; q \geqq 0, \text{ and } one \text{ of them} > 1 \text{ (for convergence)}.$$

[If $p > 1$, uniform convergence for all values of x is obvious; if $p \leqq 1$, the maximum of $f_n(x)$ is given by $x = n^{\frac{1}{2}(p-q)}$, and the M-test applies for all values of x if $p+q>2$. If, however, $p<1$, $q>1$ and $p+q \leqq 2$, we can use Art. 11 to prove that the sum is greater than

$$\int_1^{\infty} \frac{x\,dt}{t^p + x^2 t^q} = x^s \int_{x^r}^{\infty} \frac{dv}{v^p + v^q},$$

where $$s = (q+p-2)/(q-p), \quad r = 2/(q-p).$$

Consequently when $x \to +0$, the series $\to +\infty$ if $p+q<2$; and to a finite limit if $p+q=2$. And when $x \to -0$, the sign of the whole series is simply reversed. Thus in either case there is a discontinuity at $x=0$; which must therefore be a point of non-uniform convergence if $p<1$, $q>1$ and $p+q \leqq 2$. [HARDY.]

5. (i) Shew that the series $f(x) = \sum_1^{\infty} \frac{1}{n^3 + n^4 x^2}$ is uniformly convergent for all values of x; and that $f'(x)$ is given by term-by-term differentiation.

(ii) Shew that the series $f(x) = \sum_1^{\infty} \frac{1}{n^3 + n^4 x^2}$ is uniformly convergent for all values of x, but that $f'(0)$ does not exist.

[If we form $\{f(x) - f(0)\}/x$, the results of Ex. 4 apply at once.]

6. Generally prove that

$$f(x) = \sum \frac{1}{n^p + n^q x^2} \quad (p > 1)$$

is uniformly convergent for all values of x, and can be differentiated term-by-term if $q < 3p - 2$. But if $q \geqq 3p - 2$, the value of $f'(0)$ does not exist.

[Again form $\{f(x) - f(0)\}/x$ and apply Ex. 4.]

7. If $$f_n(x) = x^n(1 - x^n);$$
then we have $$\sum f_n(x) = x/(1-x^2), \quad \text{if } |x| < 1,$$
but $$\sum f_n(1) = 0, \text{ although } \lim_{x \to 1} [\sum f_n(x)] = \infty.$$

8. Shew that $$2 \lim_{x \to 1} \sum_1^{\infty} \frac{x^n(1-x)}{n(1-x^{2n})} = \sum_1^{\infty} \frac{1}{n^2}.$$

[For $1 - x^{2n} \geqq 2nx^n(1-x)$ and the M-test can be used.]

9. If $\sum a_n$ oscillates finitely or converges, then the series $\sum (a_n/n^x)$ is a continuous function of x, if $x \geqq c > 0$. [DIRICHLET.]

10. Shew that $$\lim_{x \to 1} \sum_1^{\infty} (-1)^{n-1} n^{-x} = \log 2.$$

11. If $\sum a_n$ converges and (μ_n) is a sequence which tends steadily to ∞ with n, the series $\sum a_n \mu_n^{-x}$ converges uniformly if $x \geqq 0$. Deduce that there is, in general, some number ξ such that $\sum c_n \mu_n^{-x}$ converges if $x > \xi$, and does not converge if $x < \xi$. Of course it is possible that the latter series may converge for *all* values of x or for *no* values of x; examples are given by $c_n = 1/n!$ or $n!$ and $\mu_n = n$. [CAHEN.]

12. If $\sum u_n$ is an absolutely convergent series of constants, shew that $\Pi(1 + u_n x)$ converges absolutely and uniformly in any finite interval.

If $\sum u_n$ converges (not absolutely) and $\sum u_n^2$ converges, $\Pi(1 + u_n x)$ converges uniformly (but not absolutely) in any finite interval.

13. Shew that the products

$$\Pi\,[1+(-1)^n x/n], \quad \Pi\,[1+(-1)^n \sin(x/n)], \quad \Pi\cos(x/n)$$

converge uniformly in any finite interval, and that the third converges absolutely.

14. Prove that if $f_n(x)=x^{n-1}(1-2x^n)$, $\quad \Sigma\int_0^1 f_n(x)\,dx=0$;

and also that
$$F(x)=\Sigma f_n(x)=\frac{1}{1+x},$$

so that
$$\int_0^1 F(x)\,dx=\int_0^1\frac{dx}{1+x}=\log 2;$$

and thus shew that term-by-term integration fails.

[This is a simplified case of an example constructed by Hardy (see the paper quoted under Ex. 15); but it is to be noticed that $f_n(1)=-1$, so that the series $\Sigma f_n(x)$ *diverges to* $-\infty$ at $x=1$.]

15. If
$$f_n(x)=e^{-2nx}-e^{-\pi n^2x^2},$$

then
$$\int_0^\infty f_n(x)\,dx=0, \quad \text{but } F(x)>0$$

for all positive values of x; and in fact

$$\int_0^\infty F(x)\,dx=\tfrac{1}{2}(\log\pi-C)=0{\cdot}142\ldots.$$

[The proof of these results is more troublesome; see Hardy, *Messenger of Mathematics*, vol. 44, p. 146. It should be observed, however, that the difficulty arises from $x=0$; the upper limit ∞ is used only to produce a simple final result.]

16. Further examples of sequences which give failure for term-by-term integration can be constructed as follows:

Let $S_n(x)=n^q x^{p-1} f(n^q x^p)$, where $p, q>0$ and $f(\xi)$ has the properties

$$\text{(i) } \xi f(\xi)\to 0 \text{ as } \xi\to\infty, \quad \text{(ii) } \int_0^\infty f(\xi)\,d\xi=J>0,$$

with the further property $f(0)=0$, in case $p=1$.

Then $F(x)=\lim\limits_{n\to\infty} S_n(x)=0$ for all values of x, but

$$\lim_{n\to\infty}\int_0^a S_n(x)\,dx=J/p, \quad \text{if } a>0.$$

In all such cases the product xy remains constant at the peaks of the curves $y=S_n(x)$.

17. Special cases of Ex. 16 are given by

$$\text{(i) } p=\tfrac{3}{2}, \quad q=1, \quad f(\xi)=\frac{1}{1+\xi^2}, \quad S_n(x)=\frac{n\sqrt{x}}{1+n^2x^3}, \quad \frac{J}{p}=\frac{\pi}{3}.$$

$$\text{(ii) } p=1, \quad q=1, \quad f(\xi)=\frac{\xi}{1+\xi^3}, \quad S_n(x)=\frac{n^2x}{1+n^3x^3}, \quad \frac{J}{p}=\frac{2\pi}{3\sqrt{3}}.$$

$$\text{(iii) } p=2, \quad q=1, \quad f(\xi)=\frac{1}{1+\xi^3}, \quad S_n(x)=\frac{nx}{1+n^3x^6}, \quad \frac{J}{p}=\frac{\pi}{3\sqrt{3}}.$$

[These examples are mentioned by T. Hayashi, *Tôhoku Math. Journal*, vol. 2, p. 44.]

18. Shew that the series $\sum_{1}^{\infty} 1/(n+x)^2$ converges uniformly if $x \geqq 0$; but that it cannot be integrated from 0 to ∞. [OSGOOD.]

19. If $f_n(x)$ is positive and satisfies the M-test for all positive values of x, prove that $\sum \int_0^{\infty} f_n(x)\,dx = \int_0^{\infty} \{\sum f_n(x)\}\,dx$, provided that *either* side is convergent.

Thus the method of term-by-term integration applies for instance to

$$\sum (n+x)^{-p}, \text{ if } p > 2; \quad \text{or } \sum (n^2+x^2)^{-q}, \text{ if } q > 1.$$

[The method of proving the general theorem is to observe that theorem (2) of Art. 45 gives

$$\int_0^X \sum f_n(x)\,dx = \sum \int_0^X f_n(x)\,dx,$$

and then to argue on the lines of Art. 31 (6) to establish the legitimacy of making X tend to ∞.]

20. If $\left|\sum_{n=0}^{m} f_n'(x)\right|$ is less than a fixed number G at all points of (a, b) and for all values of m, then if $\sum_{0}^{\infty} f_n(x)$ converges at all points of the interval (a, b), it converges uniformly. [BENDIXSON.]

[For, divide the interval into ν sub-intervals each of length $l = \delta/G$, where $\delta < \frac{1}{2}\epsilon$, ϵ being any assigned small positive number. Next find m so that at the ends of each sub-interval

$$\phi(x_r) = \sum_{m+1}^{m+p} f_n(x_r), \quad (p = 1, 2, 3, \ldots)$$

is numerically less than δ. This is possible because the series converges at each of these points, *and they are finite in number* $(\nu+1)$. Now if x is any point of the interval the distance to the nearest end of a sub-interval (say x_r) is not greater than $\frac{1}{2}l$; hence

$$|\phi(x) - \phi(x_r)| < (\tfrac{1}{2}l)(2G) = \delta,$$

because

$$|\phi'(x)| < 2G.$$

Thus

$$|\phi(x)| < |\phi(x_r)| + \delta < 2\delta < \epsilon,$$

and so the test of uniform convergence is satisfied.]

21. Apply Bendixson's test to the series

$$\sum (1/n)\cos nx, \quad \sum (1/n)\sin nx. \qquad \text{[Ex. 5, Art. 44.]}$$

22. Let the sum to n terms of a series of functions of x be denoted by $S_n(x)$; and suppose that comparison series, with sums $\sigma_n(x)$, $\Sigma_n(x)$, can be found, such that

$$\sigma_n(x) \leqq S_n(x) \leqq \Sigma_n(x),$$

for all values of n and for all points x in the interval (a, b). Then, provided that the series $\sigma_n(x)$, $\Sigma_n(x)$ are capable of being integrated term-by-term in the interval (a, b), the same is true of $S_n(x)$.

In particular, if we have $|S_n(x)| < K$,
where K is independent of n and x, then term-by-term integration is admissible. [W. H. YOUNG.]

CHAPTER VIII.

POWER SERIES.

50. The power-series $\Sigma a_n x^n$ is one of the most important types of uniformly convergent series.

We recall the result proved in Art. 10, that if

$$\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} = 1/l,$$

the power-series converges absolutely when $|x| < l$; but the series cannot converge if $|x| > l$, for then $\overline{\lim} |a_n x^n| > 1$, and so there will be an infinity of terms in the series whose absolute values are greater than 1.

Thus any power-series has an interval $(-l, +l)$ within which it converges absolutely, and outside which convergence is impossible. By writing x in place of x/l, we can reduce this interval to the special one $(-1, +1)$: and we shall suppose this done in what follows (we exclude for the moment the cases $l=0$ or ∞).

Thus suppose that we have a power-series which is absolutely convergent for values of x between -1 and $+1$: so that if k is any number between 0 and 1 the series $\Sigma |a_n| k^n$ is convergent. Then, by Weierstrass's M-test, it is clear that the series $\Sigma a_n x^n$ converges uniformly in the interval $(-k, +k)$, because in that interval $|a_n x^n| \leqq |a_n| k^n$. Hence we have the result that *a power series converges uniformly in any interval which falls* **entirely within** *its interval of absolute convergence.*

It sometimes happens that further tests (such as those given in Arts. 11–12·2) shew that the series is absolutely convergent for $|x|=1$; and then we can assert that *the series converges absolutely and uniformly in the whole interval* $(-1, +1)$, because we can compare the series $\Sigma a_n x^n$ with $\Sigma|a_n|$ and apply Weierstrass's M-test again. This test gives **normal convergence** (Art. 44) throughout the interval.

But it may happen also that Σa_n is convergent although not absolutely convergent: in this case we can apply Abel's test (Art. 44), because the sequence of variable factors x^n never increases with n, and is never greater than 1 (if $0 \leqq x \leqq 1$). Consequently, since Σa_n is supposed to converge, the series $\Sigma a_n x^n$ converges uniformly in an interval which ends at and includes $x=1$ (but need not extend as far as $x=-1$). Similarly if $\Sigma(-1)^n a_n$ is convergent the interval of uniform convergence includes $x=-1$.

Ex. 1. The series $$1+2x+3x^2+4x^3+\ldots$$ converges absolutely if $-1<x<1$ and converges uniformly in an interval $(-k, +k)$, where k is any number between 0 and 1; but the points -1, $+1$ do not belong to the region of uniform convergence.

Ex. 2. The series $$1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+\ldots$$ converges absolutely and uniformly in the interval $(-1, +1)$.

Ex. 3. The series $$1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots$$ converges absolutely if $-1<x<+1$ and converges uniformly in an interval $(-k, +1)$, where k is any number between 0 and 1; but the point -1 does not belong to the region of uniform convergence.

We now return to the cases $l=0$ or ∞, which we have hitherto left on one side. If it happens that

$$\overline{\lim}\,|a_n|^{\frac{1}{n}}=0,$$

the series $\Sigma a_n x^n$ will converge absolutely for any value of x; and the series converges uniformly in any interval $(-A, +A)$, where A may be arbitrarily great.

Ex. 4. The series $$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$ converges absolutely for any value of x and is uniformly convergent in any interval $(-A, +A)$.

On the other hand, if $\overline{\lim}\,|a_n|^{\frac{1}{n}}=\infty$, the series $\Sigma a_n x^n$ cannot converge for any value of x other than zero.

Ex. 5. An example of this is afforded by the series

$$1+x+(2!)x^2+(3!)x^3+(4!)x^4+\ldots.$$

In all the above examples, and in most other power-series commonly used, the value of l can be calculated from the equation

$$l = \lim \frac{|a_n|}{|a_{n+1}|},$$

and whenever this limit exists, its value will give the value of $\lim |a_n|^{-\frac{1}{n}}$, by Art. 149. But the existence of the limit of this quotient is by no means certain even in simple cases such as

$$1 + x^2 + x^4 + x^6 + \dots, \qquad 1 + x^4 + x^9 + x^{16} + \dots,$$

for which the quotient oscillates between 0 and ∞, although $\overline{\lim}\, |a_n|^{-\frac{1}{n}} = 1$ in both these series.

The fact that, if $\Sigma a_n x^n$ is ever convergent, the series will converge absolutely within some interval, can be established by the following method.

Suppose that the series converges for $x = x_0$, and let M be the maximum of $|a_n x_0^n|$; then

$$|a_n x^n| < M (r/r_0)^n \qquad \text{if } r = |x|, \quad r_0 = |x_0|.$$

Thus the series $\Sigma |a_n x^n|$ certainly converges if $r < r_0$, or if

$$-r_0 < x < +r_0.$$

There is an important distinction between intervals of absolute and of uniform convergence; an interval of uniform convergence *must* include its end-points, but the interval of absolute convergence need not. Or, to use a convenient terminology, the former interval is *closed*; the latter may be *unclosed*.

That the interval of absolute convergence of a power-series need not be closed is evident from Ex. 1 above, in which the series is absolutely convergent for any value of x numerically less than 1, but the series diverges for $x = 1$ and oscillates for $x = -1$. On the other hand, Ex. 2 gives an illustration of a closed interval of absolute convergence.

Now, we proved (at the end of Art. 43) that an interval of uniform convergence must be closed, whenever the function $S_n(x)$ is a continuous function of x. But for a power-series $\Sigma a_n x^n$, we have

$$S_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n,$$

which is obviously continuous for all values of x. Consequently the interval of uniform convergence of a power-series is certainly closed. This fact is not deducible from Abel's theorem (see p. 146, top, or Art. 51), for it does not appear impossible *a priori* that a power-series might diverge at $x = 1$ and yet be uniformly convergent for $|x| < 1$.

51. Abel's theorem is expressed by the equation

$$\lim_{x\to 1}(\Sigma a_n x^n)=\Sigma a_n,$$

provided that Σa_n is convergent; this of course follows from the fact (pointed out in Art. 50) that $x=1$ belongs to the region of uniform convergence and from theorem (1) in Art. 45.

Abel also shewed that when Σa_n diverges, say to $+\infty$, then $\Sigma a_n x^n$ also tends to $+\infty$ as x approaches 1. This theorem cannot be proved by any appeal to uniform convergence; but the following method applies to both theorems.

Write $\quad A_0=a_0,\quad A_1=a_0+a_1,\ldots,\quad A_n=a_0+a_1+a_2+\ldots+a_n.$

Then, since $1, x, x^2, \ldots$ is a decreasing sequence, by the second form of Abel's Lemma (Art. 20), we have

$$h(1-x^m)+h_m x^m<\sum_0^p a_n x^n<H(1-x^m)+H_m x^m,\quad \text{if } p>m,$$

or
$$h_m+(h-h_m)(1-x^m)<\sum_0^p a_n x^n<H_m+(H-H_m)(1-x^m),$$

where H, h are the upper and lower limits of $A_0, A_1, \ldots, A_{m-1}$, and H_m, h_m are those of $A_m, A_{m+1}, \ldots$ to ∞.

Since these limits are independent of p, we have

(1) $$h_m+(h-h_m)(1-x^m)\leqq\sum_0^\infty a_n x^n\leqq H_m+(H-H_m)(1-x^m).$$

Suppose first that Σa_n is convergent and has the sum s, then we can choose m so that

$$h_m\geqq s-\epsilon,\quad H_m\leqq s+\epsilon,$$

however small ϵ may be.

Now we have $\quad 1-x^m=(1-x)(1+x+x^2+\ldots+x^{m-1}),$

so that if
$$0<1-x<\delta,$$
we have
$$0<1-x^m<m\delta.$$

Consequently and
$$\left.\begin{aligned}H_m+(H-H_m)(1-x^m)&<H_m+mK\delta,\\ h_m+(h-h_m)(1-x^m)&>h_m-mK\delta,\end{aligned}\right\}\quad\text{if } 0<1-x<\delta,$$

where K is the greater of $|H-H_m|$ and $|h-h_m|$.

Thus, from (1) we have

$$s-\epsilon-mK\delta<\Sigma a_n x^n<s+\epsilon+mK\delta,$$

and so
$$s-2\epsilon<\Sigma a_n x^n<s+2\epsilon,\quad \text{if } 0<1-x<\epsilon/mK.$$

That is
$$\lim_{x\to 1}\Sigma a_n x^n=s=\Sigma a_n.$$

Secondly, if Σa_n diverges, say to $+\infty$, we can choose m so that $h_m\geqq 2N$, however large N may be; and we can take K to denote $|2N-h|$, which will not exceed $2N+|h|$.

Then $\quad h(1-x^m)+h_m x^m>h(1-x^m)+2Nx^m>2N-mK\delta,$

and so from (1) $\quad \Sigma a_n x^n>N,\quad \text{if } 0<1-x<N/mK.$

Thus
$$\lim_{x\to 1}\Sigma a_n x^n=+\infty.$$

The negative case is dealt with similarly.

Thirdly, if Σa_n *oscillates*, let $l = \underline{\lim}\, A_n$, $L = \overline{\lim}\, A_n$, and then m can be chosen so that $h_m \geqq l - \epsilon$, $H_m \leqq L + \epsilon$.

Repeating the transformations of the convergent case, we find that

$$l - 2\epsilon < \Sigma a_n x^n < L + 2\epsilon, \quad \text{if } 0 < 1 - x < \epsilon/mK,$$

and so
$$l \leqq \underline{\lim}_{x \to 1} \Sigma a_n x^n, \qquad \overline{\lim}_{x \to 1} \Sigma a_n x^n \leqq L.$$

Closely connected with the foregoing results is **the theorem of comparison for two divergent series.**

Suppose that Σa_n, Σb_n are both divergent, but that

$$f(x) = \Sigma a_n x^n, \qquad g(x) = \Sigma b_n x^n$$

are absolutely convergent for $|x| < 1$.

In the first place, suppose that $a_n/b_n \to 0$ as $n \to \infty$, we can choose m so that
$$|a_n/b_n| < \epsilon, \quad \text{if } n \geqq m.$$

Consequently, if we write

$$f_m(x) = a_0 + a_1 x + \ldots + a_{m-1} x^{m-1},$$

and if we further suppose that b_n is *positive*, we have

$$|f(x) - f_m(x)| < \epsilon(b_m x^m + b_{m+1} x^{m+1} + b_{m+2} x^{m+2} + \ldots) < \epsilon g(x).$$

Hence we have
$$\left|\frac{f(x)}{g(x)}\right| < \epsilon + \left|\frac{f_m(x)}{g(x)}\right|.$$

Now, when $x \to 1$, we have seen that $g(x) \to \infty$, but $f_m(1)$ is finite; and so we can find δ such that

$$|f_m(x)/g(x)| < \epsilon, \quad \text{if } 0 < 1 - x < \delta.$$

Hence we have also

$$|f(x)/g(x)| < 2\epsilon, \quad \text{if } 0 < 1 - x < \delta,$$

or
$$\lim_{x \to 1} \{f(x)/g(x)\} = 0, \quad \textit{when } \lim_{n \to \infty} (a_n/b_n) = 0.$$

Secondly, if $a_n/b_n \to l$, we can write

$$\frac{f(x)}{g(x)} - l = \frac{\Sigma(a_n - lb_n)x^n}{\Sigma b_n x^n},$$

which will therefore tend to zero by what has been just proved; and accordingly
$$\lim_{x \to 1} \{f(x)/g(x)\} = l = \lim_{n \to \infty} (a_n/b_n).$$

If $\lim (a_n/b_n)$ does not exist, it may still happen that $A_n/B_n \to l$, where

$$A_n = a_0 + a_1 + \ldots + a_n, \qquad B_n = b_0 + b_1 + \ldots + b_n.$$

Then we can write
$$f(x)(1 + x + x^2 + \ldots) = \Sigma A_n x^n,$$
$$g(x)(1 + x + x^2 + \ldots) = \Sigma B_n x^n,$$

and so
$$\frac{f(x)}{g(x)} = \frac{\Sigma A_n x^n}{\Sigma B_n x^n} \to l \quad \text{by the former result.}$$

In the same way we can prove that if a_n/b_n or $A_n/B_n \to \infty$, then

$$\lim_{x \to 1} f(x)/g(x) = \infty.$$

Thus we have Cesàro's theorem of comparison of divergent series:
If a_n/b_n or A_n/B_n tends to a definite limit, finite or infinite, then

$$\lim_{x\to 1}\frac{\Sigma a_n x^n}{\Sigma b_n x^n}=\lim_{n\to\infty}\frac{A_n}{B_n}=\lim_{n\to\infty}\frac{a_n}{b_n},$$

where Σb_n is a divergent series of positive terms.

Ex. 1. It is possible to obtain the first form of Abel's theorem from the last theorem, by comparing the two series

$$A_0+A_1x+A_2x^2+\dots,\quad 1+x+x^2+\dots. \qquad \text{[CESÀRO.]}$$

Ex. 2. Similarly, by comparing the series

$$A_0+(A_0+A_1)x+(A_0+A_1+A_2)x^2+\dots \quad\text{and}\quad 1+2x+3x^2+\dots,$$

we see that if $\quad \lim_{n\to\infty}\frac{1}{n}(A_0+A_1+A_2+\dots+A_{n-1})=l,$

then $\quad \lim_{x\to 1}(\Sigma a_n x^n)=l. \qquad$ [FROBENIUS.]

Ex. 3. Again, if the limit in Ex. 2 is not definite, we may consider a further mean. Suppose that

$$\lim_{n\to\infty}\frac{nA_0+(n-1)A_1+(n-2)A_2+\dots+A_{n-1}}{\frac{1}{2}n(n+1)}=l,$$

then we can compare the series

$$A_0+(2A_0+A_1)x+(3A_0+2A_1+A_2)x^2+\dots \quad\text{and}\quad 1+3x+6x^2+\dots,$$

and prove that $\quad \lim_{x\to 1}(\Sigma a_n x^n)=l.$

We note that each of the examples 1, 2, 3 includes the preceding one.

Ex. 4. As other applications, the reader may shew that

(i) $\lim_{x\to 1}(1-x)^{\frac{1}{2}}(x+x^4+x^9+x^{16}+\dots)=\frac{1}{2}\sqrt{\pi},$

(ii) $\lim_{x\to 1}\{(x+x^a+x^{a^2}+x^{a^3}+\dots)/\log(1-x)\}=-1/\log a,$

(iii) $\lim_{x\to 1}(1-x)^p(1^{p-1}x+2^{p-1}x^2+3^{p-1}x^3+\dots)=\Gamma(p), \quad$ if $p>0.$

(iv) $\lim_{x\to 1}(x-x^4+x^9-x^{16}+\dots)=\frac{1}{2}.$

In case (i), the series $x+x^4+x^9+\dots$ gives $A_n\sim n^{\frac{1}{2}}\sim\Gamma(n+\frac{3}{2})/\Gamma(n+1)$, while the series for $(1-x)^{-\frac{1}{2}}$ gives $B_n=3.5.7\dots(2n+1)/2.4.6\dots 2n.$

In case (ii) we find $A_n\sim\log n/\log a$, while the series for $\log(1-x)$ gives $B_n\sim-\log n.$

In case (iii) we use the fact that $a_n\sim\Gamma(n+p)/\Gamma(n+1).$

Finally, in case (iv) we have $A_0+A_1+\dots+A_n\sim\frac{1}{2}n.$

Lasker and Pringsheim* have proved theorems of great generality on series which diverge at $x=1$. As an example we quote the following:

If $\lambda(x)$ is a function of x, steadily increasing to ∞ with x, but more slowly than x, so that $\lim\{\lambda(x)/x\}=0$, then $\Sigma\lambda'(n)x^n$ is represented approximately by $\lambda\{1/(1-x)\}$ for values of x near to 1.

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* Pringsheim, *Acta Mathematica*, vol. 28, 1904, p. 1, where full references will be found.

52. Properties of a power-series.

The general theorems proved in Arts. 45, 46 of course apply to a power-series, so that we can make the following statements:

(1) *A power-series* $\Sigma a_n x^n$ *is a continuous function of* x *in any interval contained within its region of convergence.*

(2) *If* (c_1, c_2) *is any interval within the region of convergence*

$$\int_{c_1}^{c_2} x^k (\Sigma a_n x^n)\,dx = \Sigma \frac{a_n}{n+k+1}(c_2^{\,n+k+1} - c_1^{\,n+k+1}).$$

(3) *If* x *is any point within the region of convergence*

$$\frac{d}{dx}(\Sigma a_n x^n) = \Sigma n a_n x^{n-1}.$$

We note that the interval of absolute convergence of a power-series is not altered by differentiation or integration. This follows from the fact that* $\lim n^{\frac{1}{n}} = 1$,

so that $$\overline{\lim}\,|na_n|^{\frac{1}{n}} = \overline{\lim}\left|\frac{a_n}{n+k+1}\right|^{\frac{1}{n}} = \overline{\lim}\,|a_n|^{\frac{1}{n}} = \frac{1}{l}.$$

By applying Abel's theorem (Art. 51) to the integrated series we see that *in* (2) *the point* c_2 *may be taken at the boundary of the interval of absolute convergence, provided that the* **integrated** *series converges there, no matter whether the original series does so or not.*

An example of this has occurred already in Art. 47.

(4) *If a power-series* $f(x) = \Sigma a_n x^n$ *converges within an interval* $(-l, +l)$, *there is an interval within which* $f(x) = 0$ *has no root except, perhaps,* $x = 0$.

Suppose that the series converges for $x = x_0$, and that M is the maximum of $|a_n x_0^{\,n}|$; also write for brevity $r = |x|$, $r_0 = |x_0|$. Then, if we consider values of $r < r_0$, we have

$$|a_{m+1}x^{m+1} + a_{m+2}x^{m+2} + \dots| < M\left\{\left(\frac{r}{r_0}\right)^{m+1} + \left(\frac{r}{r_0}\right)^{m+2} + \dots\right\} = \frac{Mr^{m+1}}{r_0^{\,m}(r_0 - r)}.$$

Suppose now that a_m is the first coefficient in $\Sigma a_n x^n$ which is different from zero; then clearly

$$|f(x)| \geqq |a_m| r^m - \frac{Mr^{m+1}}{r_0^{\,m}(r_0 - r)}.$$

Thus, if $$\frac{1}{\lambda} = 1 + \frac{M}{|a_m| r_0^{\,m}}, \quad \text{so that } \lambda \leqq \tfrac{1}{2},$$

we have $$|f(x)| \geqq \frac{Mr^m}{r_0^{\,m}}\left(\frac{\lambda}{1-\lambda} - \frac{r}{r_0 - r}\right),$$

and accordingly $f(x)$ has no root, other than $x = 0$, in the interval $(-\lambda r_0, +\lambda r_0)$.

---

* For $\lim (n+1)/n = 1$, so that $\lim n^{\frac{1}{n}} = 1$ by Art. 149 in Appendix I.

(5) It is an immediate deduction from (4) that: *If two power-series* $f(x)=\Sigma a_n x^n$, $g(x)=\Sigma b_n x^n$ *are both convergent in the interval* $(-l, +l)$, *and if* $f(x)=g(x)$ *at all points in an interval* $(-c, c)$, *then*

$$a_0=b_0,\ a_1=b_1,\ a_2=b_2,\ \ldots,\ a_n=b_n,\ \ldots,$$

and the two series are identical.

It will suffice to establish the identity if we can prove that $f(x)=g(x)$ for all points of an infinite sequence (x_n) *which tends to zero as a limit*; for then the conclusion of (4) would be contradicted unless $a_n-b_n=0$.

It is not, however, sufficient to prove that $f(x)=g(x)$ for an infinite sequence of values of x. For instance, the cosine-series (Art. 59) is zero for

$$x=\pm\tfrac{1}{2}\pi,\ \pm\tfrac{3}{2}\pi,\ \pm\tfrac{5}{2}\pi,\ \ldots,$$

and the sine-series for

$$x=\pm\pi,\quad \pm 2\pi,\ \pm 3\pi,\ \ldots;$$

but these series do not vanish identically.

53. We have hitherto discussed the continuity of the power-series from the point of view of the variable x; but it sometimes happens that we wish to discuss a series $\Sigma f_n(y)\,.\,x^n$ regarded as a function of the variable y. The following theorem (due to Pringsheim) throws some light on this question: *

Suppose that a positive value X can be found such that

$$|f_n(y)|\,X^n < An^p, \qquad a \leqq y \leqq b,$$

where A, p are **fixed** *and positive, and n has* **any** *value. Then* $\Sigma f_n(y)\,.\,x^n$ *is a continuous function of y in the interval* (a, b), *provided that* $f_n(y)$ *is continuous for all finite values of n, and that* $|x| < X$.

To prove the theorem, we need only compare the series with $\Sigma A n^p \left(\frac{|x|}{X}\right)^n$, which is independent of y and is convergent when $|x| < X$; thus the series $\Sigma f_n(y)\,.\,x^n$ (by Weierstrass's test) converges uniformly with regard to y in the interval (a, b), and is therefore a continuous function of y in that interval.

It was erroneously supposed by Abel that the convergence of $\Sigma f_n(y)\,.\,X^n$ in the interval (a, b) was sufficient to ensure the continuity of $\Sigma f_n(y)x^n$ for $0 < x < X$ (assuming $f_n(y)$ continuous). But Pringsheim has constructed an example shewing that this condition is not sufficient (see Example (5) below).

* Further results have been established by Hartogs (*Math. Annalen*, vol. 62, 1906, p. 9), using more elaborate analysis.

The following examples are due to Abel, with the exception of (5):

(1) The series $1^y x + 2^y x^2 + \ldots + n^y x^n + \ldots \quad (|x| < 1)$
represents a continuous function of y.

(2) The series $x \sin y + \frac{1}{2}x^2 \sin 2y + \frac{1}{3}x^3 \sin 3y + \ldots$
is continuous as regards y when $|x| < 1$; but although the series still converges if $x = 1$, it is discontinuous at $y = 0, \pm 2\pi, \pm 4\pi, \ldots$ (see Art. 65).

(3) The series $f(y) = \frac{y}{1+y^2} + \frac{y}{4+y^2}x + \frac{y}{9+y^2}x^2 + \ldots$
is a continuous function of y if $|x| < 1$; and thus $\lim_{y \to \infty} f(y) = 0$. But if $x = 1$, the series

$$\frac{y}{1+y^2} + \frac{y}{4+y^2} + \frac{y}{9+y^2} + \ldots \text{ (see Art. 11)}$$

differs from

$$\int_1^\infty \frac{y\,dx}{x^2+y^2} = \tan^{-1} y$$

by less than the first term $y/(1+y^2)$. Thus it is evident that

$$\lim_{y \to \infty}\left(\frac{y}{1+y^2} + \frac{y}{4+y^2} + \frac{y}{9+y^2} + \ldots\right) = \frac{\pi}{2}.$$

(4) The convergence of the series

$$\Sigma[\lim_{y \to 0} f_n(y)]x^n$$

does not follow from that of $\Sigma f_n(y)x^n$ for all values of $y > 0$. Thus the series

$$\frac{\sin y}{y} + \frac{\sin 2y}{y}x + \frac{\sin 2^2 y}{y}x^2 + \ldots + \frac{\sin 2^n y}{y}x^n + \ldots$$

converges if $x < 1$, when $y > 0$; but the series $1 + 2x + 2^2x^2 + \ldots + 2^n x^n + \ldots$ diverges if $x > \frac{1}{2}$.

(5) *Pringsheim's Example*:

Let M_n tend steadily to ∞ with n in such a way that $\lim M_{n+1}/M_n = \infty$, and let $M_0 = 0$. [For example, $M_0 = 0$, $M_n = n^n$.] Then write

$$f_n(y) = \frac{M_{n+1}y^2}{1 + M_{n+1}y^2} - \frac{M_n y^2}{1 + M_n y^2},$$

and it is evident that the series $\Sigma f_n(y)x^{2n}$ converges for all real values of y and for any value* of x. Further, the functions $f_0, f_1, \ldots$ are continuous for all real values of y. *But if $x \geqq 1$, the series $\Sigma f_n(y)x^{2n}$ is discontinuous at $y = 0$.*

For $$\Sigma f_n(0)x^{2n} = 0.$$

But $$f_0(y) + f_1(y) + \ldots + f_{n-1}(y) = \frac{M_n y^2}{1 + M_n y^2},$$

and so if $|y| > 0$, $$\sum_0^\infty f_n(y) = 1.$$

---

* Because $$f_n(y)x^{2n} = \left(\frac{1}{1 + M_n y^2} - \frac{1}{1 + M_{n+1}y^2}\right)x^{2n} < \frac{x^{2n}}{M_n y^2},$$

and $\Sigma x^{2n}/M_n$ converges for any value of x, since $\lim M_{n+1}/M_n = \infty$. Of course we have taken y not to be zero; if $y = 0$, all the terms of the series are zero, and $\Sigma f_n(0) \cdot x^{2n} = 0$.

Now the terms $f_n(y)$ are *positive*, so that

$$\sum_0^\infty f_n(y) \cdot x^{2n} \geqq 1, \quad \text{if } x \geqq 1 \quad (|y|>0).$$

From these facts it is clear that the series is discontinuous at $y=0$, if $x \geqq 1$.

Of course if $|x|<1$, the series is continuous at $y=0$, because $f_n(y)$ is positive but less than 1, and so

$$|f_n(y)x^{2n}| < x^{2n};$$

and thus the Weierstrass M-test applies.

54. Multiplication and division of power-series.

As regards *multiplication* of two power-series, the results of Art. 34 shew that if both series

$$a_0+a_1x+a_2x^2+\dots, \qquad b_0+b_1x+b_2x^2+\dots$$

converge absolutely in the interval* $(-l, +l)$, their product is given by

$$c_0+c_1x+c_2x^2+\dots,$$

which converges absolutely in the same interval, where

$$c_0=a_0b_0, \qquad c_1=a_0b_1+a_1b_0, \ \dots,$$

$$c_n=a_0b_n+a_1b_{n-1}+\dots+a_nb_0, \ \dots .$$

If we apply Abel's theorem (Art. 51) to the equation

$$\Sigma c_nx^n=(\Sigma a_nx^n) \cdot (\Sigma b_nx^n),$$

we can deduce at once his theorem (Art. 34) that $C=AB$, provided that all three series converge.

For *division*, we assume first that the constant term b_0 is different from zero; and for simplicity we take it as 1. Thus we consider first

$$\frac{1}{1+b_1x+b_2x^2+\dots}=\frac{1}{1+y} \text{ say,}$$

where

$$y=b_1x+b_2x^2+\dots .$$

Now

$$(1+y)^{-1}=1-y+y^2-y^3+\dots,$$

and by Art. 36 this series may be arranged in powers of x, provided that

$$|x|<\rho/(M+1),$$

ρ being any number such that $\Sigma b_n\rho^n$ is absolutely convergent, and M the upper limit of $|b_n|\rho^n$ (of course here $s=1$).

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* If the two series have different regions of convergence, this interval will be the *smaller* of the two.

We obtain

$$(1+y)^{-1}=1-b_1x+(b_1^2-b_2)x^2-(b_1^3-2b_1b_2+b_3)x^3+\dots .$$

This series may then be multiplied by any other power-series in x, and we obtain a power-series for the quotient

$$(a_0+a_1x+a_2x^2+\dots)/(1+b_1x+b_2x^2+\dots).$$

Of course if some of the initial terms in the denominator happen to be zero, the quotient may still be found as a power-series *together with a rational fraction.*

Thus, suppose that $b_0=0$, $b_1=0$, but that b_2 is not zero; then we have

$$\frac{\Sigma a_n x^n}{\Sigma b_n x^n}=\frac{\Sigma a_n x^n}{b_2x^2(1+B_1x+B_2x^2+\dots)},$$

where $B_1=b_3/b_2$, $B_2=b_4/b_2$, ...

Then, as above, we find

$$\frac{\Sigma a_n x^n}{1+B_1x+B_2x^2+\dots}=a_0+(a_1-a_0B_1)x+(a_2-a_1B_1+a_0B_1^2-a_0B_2)x^2+\dots .$$

Thus $$\frac{\Sigma a_n x^n}{\Sigma b_n x^n}=\frac{a_0}{b_2x^2}+\frac{a_1-a_0B_1}{b_2x}+\text{a power series in } x.$$

In working out special cases beginners are apt not to carry on the denominator to a sufficient number of terms; for instance, to obtain the constant term in the last series it is necessary to include B_2, or b_4; that is, the denominator must include terms in x^4.

In practice it is often better to use the method of undetermined coefficients; thus we should write

$$\frac{a_0+a_1x+a_2x^2+\dots}{b_0+b_1x+b_2x^2+\dots}=q_0+q_1x+q_2x^2+\dots,$$

then multiply up, and we obtain, in virtue of Art. 52 (5),

$$a_0=b_0q_0,\quad a_1=b_0q_1+b_1q_0,$$

$$a_2=b_0q_2+b_1q_1+b_2q_0,\ \dots,$$

from which we get successively

$$q_0,\ q_1,\ q_2,\ \dots .$$

Or, what is practically the same thing, we may follow Newton's practice and use the ordinary method of long division, to find the successive coefficients

$$q_0,\ q_1,\ q_2,\ \dots .$$

A more exact determination of the interval of convergence for $\Sigma q_n x^n$ will be found in Art. 89 below.

55. Reversion of a power-series.

Suppose that the series

(1) $$y = a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

converges absolutely in the interval $(-l, +l)$, and that it is required to express x, if possible, as a power-series in y.

Let us try to solve the equation, formally, by assuming a power-series solution

(2) $$x = b_1 y + b_2 y^2 + b_3 y^3 + \dots,$$

and substituting (2) in equation (1).

If $\Sigma b_n y^n$ is convergent for any value of y other than zero, the resulting series may certainly be re-arranged in powers of y without altering its value, at any rate for some values of y (see Art. 36); leaving the question of what these values may be for the moment, we have, in a certain interval,

$$y = (a_1 b_1) y + (a_1 b_2 + a_2 b_1^2) y^2 + (a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3) y^3 + \dots$$

or $$a_1 b_1 = 1, \quad a_1 b_2 + a_2 b_1^2 = 0, \quad a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3 = 0,$$

and so on, in virtue of Art. 52 (5).

Thus we can determine, step-by-step, the succession of coefficients, b_n, by the equations *

$$b_1 = 1/a_1, \quad b_2 = -a_2/a_1^3, \quad b_3 = 2a_2^2/a_1^5 - a_3/a_1^4, \dots.$$

It is evident from these results that a_1 must be supposed different from 0, or the assumed solution will certainly fail.† We may then take $a_1 = 1$ without loss of generality, for the given equation can be written

$$y/a_1 = x + (a_2/a_1)x^2 + (a_3/a_1)x^3 + \dots,$$

and so, with a slight change of notation, we can start from

$$y = x + a_2 x^2 + a_3 x^3 + \dots.$$

Then the equations for $b_1, b_2, b_3, \dots$ become

$$b_1 = 1, \quad b_2 = -a_2, \quad b_3 = -a_2(2b_1 b_2) - a_3 b_1^3,$$

$$b_4 = -a_2(2b_1 b_3 + b_2^2) - a_3(3b_1^2 b_2) - a_4 b_1^4, \dots.$$

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* This method gives the coefficients as far as b_3 or b_4 without much labour; to find the general term, we can use Lagrange's series (Art. 55·1), or some other special process as in Arts. 64, 95.

† For the case when $a_1 = 0$ and a_2 is not zero, the reader may refer to Exs. B, 23–26, at the end of the chapter.

Hence $\quad b_1=1, \quad |b_2|=|a_2|, \quad |b_3|\leqq 2|a_2|\,.\,|b_2|+|a_3|,$

$$|b_4|\leqq |a_2|\{2|b_3|+b_2^2\}+3|a_3|\,.\,|b_2|+|a_4|, \ldots .$$

These equations shew that $|b_n|\leqq\beta_n$, where the β's are given by

$$\beta_1=1, \quad \beta_2=\alpha_2, \quad \beta_3=2\alpha_2\beta_2+\alpha_3,$$

$$\beta_4=\alpha_2(2\beta_3+\beta_2^2)+3\alpha_3\beta_2+\alpha_4, \ldots ,$$

provided that $\qquad |a_n|\leqq\alpha_n.$

Now the equations for the β's are the same as those which would be obtained by inserting the series

$$\xi=\eta+\beta_2\eta^2+\beta_3\eta^3+\cdots$$

in the equation $\qquad \eta=\xi-\alpha_2\xi^2-\alpha_3\xi^3-\ldots .$

Now if ρ is any positive number less than l, the series $\Sigma|a_n|\rho^n$ is convergent, and, if M is the greatest term* in this series, we have

$$|a_n|\rho^n\leqq M.$$

Thus we can put $\qquad \alpha_n=M/\rho^n,$

and then $\qquad \eta=\xi-\dfrac{M\xi^2}{\rho^2}\left(1+\dfrac{\xi}{\rho}+\dfrac{\xi^2}{\rho^2}+\ldots\right)=\xi-\dfrac{M\xi^2}{\rho(\rho-\xi)}.$

Now this equation leads to the quadratic for ξ,

$$(M+\rho)\xi^2-\rho(\rho+\eta)\xi+\rho^2\eta=0;$$

thus $\qquad 2(M+\rho)\xi=\rho(\rho+\eta)-\rho\{(\rho+\eta)^2-4\eta(M+\rho)\}^{\frac{1}{2}},$

the negative sign being taken for the square-root, because ξ and η tend to zero simultaneously.

But $\qquad (\rho+\eta)^2-4\eta(M+\rho)=(\lambda-\eta)(\mu-\eta),$

where $\qquad \lambda+\mu=4M+2\rho, \quad \lambda\mu=\rho^2,$

so that $\qquad \lambda=2M+\rho+2\{M(M+\rho)\}^{\frac{1}{2}},$

$$\mu=2M+\rho-2\{M(M+\rho)\}^{\frac{1}{2}}.$$

It follows that

$$2(M+\rho)\xi=\rho(\rho+\eta)-\rho^2\left(1-\frac{\eta}{\lambda}\right)^{\frac{1}{2}}\left(1-\frac{\eta}{\mu}\right)^{\frac{1}{2}},$$

and thus, since $\lambda>\mu$, the value of ξ can be expanded† in a convergent series of powers of η, provided that $0<\eta<\mu$. But this

* For an alternative determination of M, see Cauchy's inequalities in Art. 84 below.

† We anticipate here the binomial expansion of Art. 61, for the case $\nu=\frac{1}{2}$, and utilise the rule for multiplication of power-series (Art. 54).

series is clearly the same as $\Sigma\beta_n\eta^n$, which therefore converges if $0<\eta<\mu$. Now $\beta_n \geqq |b_n|$, so that finally $\Sigma b_n y^n$ is absolutely convergent in the interval $(-\mu, +\mu)$. Consequently the *formal* solution proves to be a real one, in the sense that it is certainly convergent for sufficiently small values of y.

It is perhaps advisable to point out that the interval $(-\mu, +\mu)$ has not been proved to be the extreme range of convergence of the series $\Sigma b_n y^n$; we only know that the region of convergence is not *less* than the interval $(-\mu, +\mu)$.

For instance, with the series

$$y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

we may take $\rho = 2$, $M = 2$, and then the equation for η is

$$\eta = \xi - \frac{\xi^2}{2-\xi}.$$

This is found to give

$$\lambda = 6 + 4\sqrt{2}, \quad \mu = 6 - 4\sqrt{2} = \cdot 343 \dots,$$

and so the method above gives an interval only slightly greater than $(-\frac{1}{3}, +\frac{1}{3})$. But actually (see Arts. 58, 62 below)

$$y = e^x - 1, \quad \text{so that } x = \log(1+y),$$

and the series for x converges absolutely in the interval $(-1 < y < +1)$.

55·1. Lagrange's Series.

In books on the Differential Calculus, an investigation* is commonly given for the expansion of x in powers of y, when an equation holds of the form

$$x = y\, f(x).$$

This process gives an analytical expression for the coefficients in the expansion; but it gives no information as to the conditions under which such an expansion is possible. As a matter of fact, the expansion is generally not possible unless $f(x)$ can be expanded in a convergent power-series, *the first term not being zero.*† We can then write the equation in the form

$$y = x/f(x) = \sum_1^\infty a_n x^n,$$

on carrying out the division. Thus Lagrange's problem is now seen to be, in reality, equivalent to the reversion of the power-

* See for instance Williamson, *Differential Calculus*, chap. 7; Edwards, *Differential Calculus*, chap. 18.

† Compare Exs. B, 24–26, at the end of the chapter.

series $\Sigma a_n x^n$, in the form $x = \Sigma b_n y^n$. Lagrange's investigation shews that

$$nb_n = \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[\{f(x)\}^n]_{x=0},$$

or that nb_n is equal to the coefficient of x^{n-1} in the expansion of $\{f(x)\}^n$; or, what is the same thing, to the coefficient of x^{-1} in the expansion of $(1/y)^n$.

The proof of this result is contained in that of the more general form of Lagrange's series in which $g(x)$ is expanded in powers of y, where $g(x) = \Sigma c_n x^n$ is another given power-series. We know in fact (from Arts. 36 and 55) that for sufficiently small values of $|x|$ and $|y|$, we can write

$$g(x) = p_0 + \sum_1^\infty p_n y^n,$$

where $p_0 = c_0$, $p_1 = c_1/a_1$, and the other coefficients have still to be found. The interval of convergence cannot be found, by elementary methods, until the coefficients have been determined.

Now we can differentiate this series term-by-term (Art. 52), and we find

$$g'(x) = \sum_1^\infty \left(np_n y^{n-1}\frac{dy}{dx}\right).$$

Divide now by y^r, where r is any whole number, and we get

$$\frac{g'(x)}{y^r} = \sum_1^\infty \left(np_n y^{n-r-1}\frac{dy}{dx}\right).$$

Suppose both sides of this equation to be expanded in ascending powers of x; then, on the right, there is only one term * containing x^{-1}; this one is the term $rp_r \frac{1}{y}\frac{dy}{dx}$, given by $n = r$, and there the coefficient of x^{-1} is rp_r.

---

* Except for $n = r$, we have

$$y^{n-r-1}\frac{dy}{dx} = \frac{1}{(n-r)}\frac{d}{dx}(y^{n-r}) = \frac{1}{(n-r)}\frac{d}{dx}\{x^{n-r}(A_0 + A_1x + A_2x^2 + \dots)\}$$
$$= A_0 x^{n-r-1} + \dots;$$

but in this expansion, even if n is less than r, there can be no term in x^{-1}, because x^{-1} is not the differential coefficient of any power of x.

On the other hand, if $n = r$, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_1 + 2a_2x + 3a_3x^2 + \dots}{a_1x + a_2x^2 + a_3x^3 + \dots}$$
$$= \frac{1}{x} + B_1 + 2B_2x + \dots.$$

If we now change the notation, writing n for r, it is clear that np_n is the coefficient of x^{-1} in the expansion of $g'(x)/y^n$ in ascending powers of x; this conclusion is due to Jacobi,* although the result is equivalent to Lagrange's formula

$$np_n = \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[g'(x)\{f(x)\}^n]_{x=0}.$$

It is to be observed that if the equation in x,

$$y = \sum_1^\infty a_n x^n,$$

is solved by some algebraic (or other) process, there will usually be more than one solution. The series found here gives the solution which tends to zero with y.

Ex. 1. If $y = x - ax^2$, and $g(x) = x$, we have to find the coefficient of x^{-1} in the expansion of

$$(x - ax^2)^{-n} = x^{-n}(1 - ax)^{-n}.$$

Thus we get $$np_n = \frac{n(n+1)\dots(2n-2)}{(n-1)!}a^{n-1},$$

and so $$x = y + ay^2 + 2a^2y^3 + 5a^3y^4 + \dots,$$

which converges if $|ay| < \frac{1}{4}$.

Of course this series gives only one root of the quadratic in x, namely

$$\{1 - \sqrt{(1 - 4ay)}\}/2a,$$

because this root tends to zero with y. The reader will find it instructive to verify Lagrange's series by expanding this square-root in powers of y.

Ex. 2. In like manner, if $y = x - ax^{m+1}$, we find for one root

$$x = y + ay^{m+1} + \frac{2m+2}{2!}a^2y^{2m+1} + \dots + \frac{(sm+2)(sm+3)\dots(sm+s)}{s!}a^s y^{sm+1} + \dots.$$

Ex. 3. The reader will find similarly that if $y = x(1+x)^m$, then

$$x = y - \frac{2m}{2!}y^2 + \frac{3m(3m+1)}{3!}y^3 - \frac{4m(4m+1)(4m+2)}{4!}y^4 + \dots.$$

Ex. 4. To illustrate the method of expanding $g(x)$, we take the following example: To expand e^{ax} in powers of $y = xe^{bx}$.

Here $g'(x)/y^n = ax^{-n}e^{(a-nb)x}$, and so the coefficient of x^{-1} is easily seen to be

$$a(a - nb)^{n-1}/(n-1)!.$$

Thus we find

$$e^{ax} = 1 + ay + \frac{a(a-2b)}{2!}y^2 + \frac{a(a-3b)^2}{3!}y^3 + \dots,$$

which converges if $|yb| < 1/e$.

---

* *Ges. Werke*, vol. 6, p. 37.

In particular, with $a=1$, $b=-1$, $\xi=e^x$, we obtain Eisenstein's solution of the equation $\log \xi = y\xi$ (see Ex. 11, Ch. I.), in the form of the series

$$\xi = 1 + y + 3\frac{y^2}{2!} + 4^2\frac{y^3}{3!} + 5^3\frac{y^4}{4!} + \dots, \quad \text{where } |y| < 1/e.$$

56. Applications to the theory of differential equations.

Although it is not (strictly speaking) a part of the ordinary theory of power-series, yet it seems convenient to give here three of the simplest existence-theorems from the theory of differential equations. Such theorems are available in English, only in more elaborate discussions of the theory of differential equations. But the ideas underlying these special cases involve no more difficulty than we have already encountered, for instance, in proving the existence of the reversion of a power-series (Art. 55 above).

These results relate to the existence and character of the solutions of linear differential equations of the second order, which we suppose expressed in the standard form $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = 0$.

The first theorem refers to those values of x, say $x = x_0$, for which the coefficients P, Q can be expressed in the form of power-series in $(x - x_0)$. Then it appears that the two solutions are expressible *in the same form*, with limits of convergence extending *at least as far as the smaller of those of the series for* P, Q; and such points as x_0 may be described conveniently as *ordinary points* of the differential equation.

Take now a point ($x = a$, say) in the neighbourhood of which we can express $(x-a)P$ and $(x-a)^2Q$ as power-series in $(x-a)$; one at least of P, Q being supposed to tend to infinity as $x \to a$.

Then the two solutions are usually expressible in the form $\{(x-a)^t \times \text{a power-series in } (x-a)\}$, the limits of convergence again extending *at least as far as the smaller of those of the power-series in the coefficients*; and such points as $x = a$ may be called *regular singularities* of the differential equation.

It is now clear that a study of the coefficients of a linear differential equation of the second order at once gives considerable information in reference to the character of its solutions.

To save lengthy algebra it will be supposed in the formal proofs of Arts. 56·1, 56·2, and 56·3 that the points $x = x_0$, $x = a$ are brought to the origin by making a preliminary change of variable (taking $x - x_0$ or $x - a$ respectively as the new independent variable).

56·1. Existence-theorem for an ordinary point of a differential equation of the second order.

Suppose that the ordinary point is taken as origin and that the coefficients are expressed by the two power-series

$$P=p_0+p_1x+p_2x^2+\dots, \qquad Q=q_0+q_1x+q_2x^2+\dots,$$

which converge for $|x|<R$; we shall now prove that the differential equation

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$$

has solutions of the type $y=A_0+A_1x+A_2x^2+\dots$, which converge also for $|x|<R$. Here A_0 and A_1 are arbitrary constants, while $A_2, A_3, A_4, \dots$ will be linear combinations of A_0 and A_1.

If we assume that such a solution exists, and substitute in the differential equation, we obtain the conditions

$$2A_2=-A_1p_0-A_0q_0, \qquad 2.3A_3=-2A_2p_0-A_1p_1-A_1q_0-A_0q_1,$$

and generally

$$n(n-1)A_n=-(n-1)A_{n-1}p_0-(n-2)A_{n-2}p_1-\dots-A_1p_{n-2}$$
$$-A_{n-2}q_0-\dots-A_1q_{n-3}-A_0q_{n-2}.$$

Thus $A_2, A_3, \dots$ are found successively as linear combinations of A_0 and A_1.

Also, as in Art. 55, we see that $B_n \geqq |A_n|$, if $B_1=|A_1|$, $B_0=|A_0|$, and

$$n(n-1)B_n=nB_{n-1}M+(n-1)B_{n-2}\frac{M}{r}+\dots+2B_1\frac{M}{r^{n-2}}+B_0\frac{M}{r^{n-1}},$$

where r has any value less than R and M is chosen so that

$$|p_n|<M/r^n, \qquad |q_n|<M/r^{n+1}.$$

We then see that

$$n(n-1)B_n-\frac{1}{r}(n-1)(n-2)B_{n-1}=nB_{n-1}M,$$

so that
$$\frac{B_n}{B_{n-1}}=\frac{n-2}{rn}+\frac{M}{n-1}\to\frac{1}{r},$$

or
$$\lim(B_{n-1}/B_n)=r.$$

Thus ΣB_nx^n converges if $|x|<r$; and we can now see that ΣA_nx^n converges if $|x|<R$ by taking $2r=|x|+R$. Thus the existence of the assumed solution is established.

It should be noted that the foregoing argument applies equally for solutions in descending powers of x, *provided that the expansions of P, Q take the following special forms:*

$$P=\frac{2}{x}+O\left(\frac{1}{x^2}\right),\quad Q=O\left(\frac{1}{x^4}\right)\quad \text{for } |x|>S.$$

In fact, if we substitute a trial solution

$$y=A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\ldots,$$

and introduce these special forms for P, Q, the relations for A_2, A_3, A_4, ... in terms of A_0 and A_1, are precisely similar to those already used;* and thus no fresh discussion is necessary to shew that the assumed solution converges for $|x|>S$.

Thus *the conditions for "infinity to be an ordinary point" of the differential equation are expressed by*

$$P=\frac{2}{x}+O\left(\frac{1}{x^2}\right),\quad Q=O\left(\frac{1}{x^4}\right)\quad \text{for } |x|>S.$$

56·2. Existence-theorem for a regular singular point of a linear differential equation of the second order.

Again, let the origin be taken as the regular singularity, and suppose that the coefficients are expressed in the forms

$$P=\frac{1}{x}(p_0+p_1x+p_2x^2+\ldots),\qquad Q=\frac{1}{x^2}(q_0+q_1x+q_2x^2+\ldots),$$

where the power-series converge for $|x|<R$. We shall now prove that the differential equation has solutions of the type

$$x^t(A_0+A_1x+A_2x^2+\ldots),$$

where t is *either* root of the quadratic

$$t(t-1)+tp_0+q_0=0$$

and A_0 is arbitrary, the other coefficients being multiples of A_0.

For, if we assume the existence of a solution of this type, and substitute it in the given equation, we find that

$$\{t(t-1)+tp_0+q_0\}A_0=0,$$
$$\{t(t+1)+(t+1)p_0+q_0\}A_1=-(tp_1+q_1)A_0,$$

---

* Or we may change the variable to $X=1/x$, and then the differential equation becomes

$$X^4\frac{d^2y}{dX^2}+(2X^3-PX^2)\frac{dy}{dX}+Qy=0.$$

and generally

$$\{(t+n)(t+n-1)+(t+n)p_0+q_0\}A_n$$
$$=-\{(t+n-1)p_1+q_1\}A_{n-1}-\{(t+n-2)p_2+q_2\}A_{n-2}$$
$$-\ldots-(tp_n+q_n)A_0.$$

Thus t must be a root of the *index-equation*

$$t(t-1)+tp_0+q_0=0,$$

assuming that A_0 is not zero. Then, if t' is the second root, we have

$$(t+n)(t+n-1)+(t+n)p_0+q_0=n(n+t-t').$$

Thus the conditions become

$$(1+t-t')A_1=-(tp_1+q_1)A_0, \text{ etc.,}$$

$$n(n+t-t')A_n=-\{(t+n-1)p_1+q_1\}A_{n-1}-\ldots-(tp_n+q_n)A_0,$$

from which we obtain successively A_1, A_2, A_3, ... as multiples of A_0.

To discuss the convergence of this assumed solution, we introduce an auxiliary set of coefficients B_n, such that $B_n \geqq |A_n|$. The equation corresponding to the equation for A_n will then be

$$n(n-\delta)B_n=M\left\{(n+\tau)\frac{B_{n-1}}{r}+(n-1+\tau)\frac{B_{n-2}}{r^2}+\ldots+(1+\tau)\frac{B_0}{r^n}\right\},$$

where $\delta=|t-t'|$, $\tau=|t|$, $|p_n|<M/r^n$, $|q_n|<M/r^n$, and $n>\delta$.

If we take $B_0=|A_0|$ and $B_p=|A_p|$, so long as $p\leqq\delta$, we shall have $|A_n|\leqq B_n$, when $n>\delta$. Further, the last equation gives

$$r\{n(n-\delta)B_n\}-(n-1)(n-1-\delta)B_{n-1}=M(n+\tau)B_{n-1},$$

so that

$$\frac{B_n}{B_{n-1}}=\frac{(n-1)(n-1-\delta)}{rn(n-\delta)}+\frac{M(n+\tau)}{rn(n-\delta)}\to\frac{1}{r},$$

or

$$\lim(B_{n-1}/B_n)=r.$$

Thus, as in the last article, the series $\Sigma A_n x^n$ converges if $|x|<R$; and so the assumed solution really exists.

The second solution is obtained by interchanging the parts played by the indices t, t'.

It should be noticed, however, that if $t-t'$ is equal to a positive integer m, the second solution in general breaks down on account of A_m having a zero denominator. Further, if $t'=t$, no second solution can be obtained on these lines. We shall consider these cases briefly in the following article.

It may be useful to add a special remark about the case in which the roots of the index-equation $t(t-1)+p_0t+q_0=0$ are found to be $t=0$, $t=1$ (so that $p_0=0$, $q_0=0$). Beginners are then apt to assume that the solutions should be of the character considered in Art. 56·1; but this is not the case, *unless it happens that* $q_1=0$ *as well as* $p_0=0$, $q_0=0$.

For instance the equation

$$\frac{d^2y}{dx^2}-\frac{y}{x}=0 \quad (\text{with } t=0,\ 1)$$

has a solution of the type

$$y_1=x\,(A_0+A_1x+A_2x^2+\ldots),$$

where
$$\frac{A_1}{A_0}=\frac{1}{1\,.\,2},\quad \frac{A_2}{A_1}=\frac{1}{2\,.\,3},\quad \frac{A_3}{A_2}=\frac{1}{3\,.\,4},\ \ldots.$$

But the second solution is of the form

$$y_1\log x+A_0-A_1x^2(1+\tfrac{1}{2})-A_2x^3(1+\tfrac{2}{2}+\tfrac{1}{3})-\ldots.$$

Just as in Art. 56·1 it is easy to deal with series in descending powers of x, provided that we have expansions of the type

$$P=\frac{1}{x}\Big(p_0+\frac{p_1}{x}+\frac{p_2}{x^2}+\ldots\Big),\quad Q=\frac{1}{x^2}\Big(q_0+\frac{q_1}{x}+\frac{q_2}{x^2}+\ldots\Big)\quad \text{for } |x|>S.$$

Then we assume $\quad y=\dfrac{1}{x^t}\Big(A_0+\dfrac{A_1}{x}+\dfrac{A_2}{x^2}+\ldots\Big),$

and the quadratic for t becomes now

$$t(t+1)-tp_0+q_0=0,$$

but the remainder of the discussion is effectively unaltered.

Thus *the conditions that "infinity should be a regular singularity" of the differential equation are simply*

$$P=O(1/x),\qquad Q=O(1/x^2).$$

56·3. Case of equal roots in the index-equation.

When the roots of the index-equation are equal, it is evident that

$$p_0=1-2t,\quad q_0=t^2.$$

The discussion of the first solution by means of the equations

$$n^2A_n=-\{(t+n-1)p_1+q_1\}A_{n-1}-\ldots-(tp_n+q_n)A_0$$

and
$$n^2B_n=M\left\{(n+\tau)\frac{B_{n-1}}{r}+\ldots+(1+\tau)\frac{B_0}{r^n}\right\} \qquad \text{(I.)}$$

is carried out exactly as in Art. 56·2.

To find the second solution, we begin by supposing p_0, q_0 slightly changed so as to make the second root of the index-equation $t'=t-\lambda$, where λ will be small.

The modified differential equation has two solutions, each differing slightly from the first solution; let these be denoted by

$$x^t(C_0+C_1x+C_2x^2+\ldots)$$

and

$$x^{t-\lambda}(C_0'+C_1'x+C_2'x^2+\ldots).$$

Then we have the fundamental equations

$$n(n+\lambda)C_n=-\{(t+n-1)p_1+q_1\}C_{n-1}-\ldots-(tp_n+q_n)C_0,$$

$$n(n-\lambda)C_n'=-\{(t-\lambda+n-1)p_1+q_1\}C_{n-1}'-\ldots-\{(t-\lambda)p_n+q_n\}C_0'$$

To standardise these solutions we shall assume that

$$C_0=A_0/\lambda, \qquad C_0'=-A_0/\lambda.$$

Then clearly λC_n-A_n, $\lambda C_n'+A_n$ are both of order λ (or of higher order in λ); and so we may write

$$\lambda C_n\to A_n, \quad \lambda C_n'\to -A_n, \quad C_n+C_n'\to F_n, \quad \text{as } \lambda\to 0.$$

It is our immediate object to obtain the fundamental equations for F_n.

Now, on adding together the equations for C_n, C_n' and taking the limit (as $\lambda\to 0$), we find

$$n^2F_n+2nA_n=-\{(t+n-1)p_1+q_1\}F_{n-1}-\ldots-(tp_n+q_n)F_0$$
$$-p_1A_{n-1}-p_2A_{n-2}-\ldots-p_nA_0.$$

Accordingly, we construct $G_n\geqq|F_n|$ by means of the equations

$$n^2G_n-2nB_n=M\left\{(\tau+n)\frac{G_{n-1}}{r}+(\tau+n-1)\frac{G_{n-2}}{r^2}+\ldots+(1+\tau)\frac{G_0}{r^n}\right\}$$
$$+M\left(\frac{B_{n-1}}{r}+\frac{B_{n-2}}{r^2}+\ldots+\frac{B_0}{r^n}\right).$$

Thus

$$rn^2G_n-(n-1)^2G_{n-1}$$
$$=2nrB_n-2(n-1)B_{n-1}+M(\tau+n)G_{n-1}+MB_{n-1}. \qquad \text{(II.)}$$

Since $F_0=0$, we may take $G_0=0$ also, and then it is easy* to see that

$$G_n/B_n>G_{n-1}/B_{n-1}>\ldots>G_1/B_1>0.$$

* From equation (I.) deduce that

$$rn^2B_n=\{(n-1)^2+M(\tau+n)\}B_{n-1}, \qquad \text{(III.)}$$

and then divide the coefficients of G_n and G_{n-1} in (II.) by the terms on the left and right respectively of (III.); this gives

$$\frac{G_n}{B_n}-\frac{G_{n-1}}{B_{n-1}}=\frac{2}{n}+\frac{M-2(n-1)}{(n-1)^2+M(\tau+n)}>0, \quad \text{if } 3M>2.$$

Also

$$\frac{rG_n}{G_{n-1}} = \frac{(n-1)^2}{n^2} + \frac{2r}{n} \cdot \frac{B_n}{G_n} \cdot \frac{G_n}{G_{n-1}} - \frac{2(n-1)}{n^2} \frac{B_{n-1}}{G_{n-1}} + \frac{M(\tau+n)}{n^2} + \frac{M}{n^2} \frac{B_{n-1}}{G_{n-1}}.$$

Thus we see that

$$rG_n/G_{n-1} \to 1, \quad \text{or} \quad \lim (G_{n-1}/G_n) = r.$$

Hence $\Sigma G_n x^n$ converges for $|x| < r$; and so finally $\Sigma F_n x^n$ converges for $|x| < R$, as in the other cases previously discussed.

We can now obtain the required solution as

$$\lim_{\lambda \to 0} \{x^t (C_0 + C_1 x + C_2 x^2 + \ldots) + x^{t-\lambda} (C_0' + C_1' x + C_2' x^2 + \ldots)\}.$$

Now $$x^t \{(C_0 + C_0') + (C_1 + C_1')x + (C_2 + C_2')x^2 + \ldots\} \to x^t (0 + F_1 x + F_2 x^2 + \ldots)$$

because $F_0 = 0$.

Also $$(x^{t-\lambda} - x^t)(C_0' + C_1' x + C_2' x^2 + \ldots)$$
$$= x^t \left(\frac{x^{-\lambda} - 1}{-\lambda}\right)(-\lambda C_0' - \lambda C_1' x - \lambda C_2' x^2 - \ldots)$$
$$\to x^t \log x (A_0 + A_1 x + A_2 x^2 + \ldots).$$

Hence the final solution is

$$x^t \log x (A_0 + A_1 x + A_2 x^2 + \ldots) + x^t (0 + F_1 x + F_2 x^2 + \ldots),$$

where

$$n^2 F_n + 2nA_n = -\{(t+n-1)p_1 + q_1\} F_{n-1} - \ldots - (tp_{n-1} + q_{n-1}) F_1 - p_1 A_{n-1} - p_2 A_{n-2} - \ldots - p_n A_0,$$

and in particular $$F_1 + 2A_1 = -p_1 A_0.$$

To deal with *the case when the roots of the index-equation differ by an integer m* is no more difficult, in principle, than the last investigation. But an adequate description of the steps is so lengthy that the proof must be omitted here.

We can readily see the *form* to be anticipated by taking $t=0$ in the last result and differentiating m times; the two solutions (after differentiation) have indices 0, $-m$, and are of the forms

$$\text{(i) } U, \qquad \text{(ii) } U \log x + V + x^{-m} W,$$

where U, V are power-series in x, and W is a polynomial of degree $(m-1)$.

It should be remarked further that often the most rapid method of obtaining the solutions (when the difference of indices is 0 or m)

is the method of Frobenius.* But a formal proof of convergence seems more troublesome on elementary lines.

56·4. Exercises on differential equations of the second order.

In the following examples (which are really of the nature of book-work) it is assumed that the coefficients P, Q are rational algebraic fractions in x; and accordingly there must always be at least *one* singularity. As a rule, the singularities will be supposed *regular*, except in Exs. 3, 6.

1. One regular singular point.

Suppose that the point is $x=a$, and that the indices are p, p'. Prove that

$$p=0,\ p'=-1;\quad P=2/(x-a),\ Q=0,$$

and that the solution is

$$y=A+B/(x-a).$$

If the point is taken at infinity, we find $P=0$, $Q=0$, and the solution is

$$y=A+Bx.$$

If P, Q are taken to be any polynomials, we can obtain the simplest types of a *non-regular singularity at infinity*, all other points being ordinary points.

2. Two regular singular points.

Suppose that the points are $x=a$ (indices p, p'), and $x=b$ (indices q, q'); then prove that

$$P=\frac{1-p-p'}{x-a}+\frac{1-q-q'}{x-b},\qquad Q=\frac{a-b}{(x-a)(x-b)}\left(\frac{pp'}{x-a}-\frac{qq'}{x-b}\right),$$

and deduce that $p+q=0$, $p'+q'=0$, so that actually

$$P=\frac{1-p-p'}{x-a}+\frac{1+p+p'}{x-b},\qquad Q=\frac{(a-b)^2pp'}{(x-a)^2(x-b)^2}.$$

If $y=z(x-a)^p/(x-b)^p$, verify that z has no additional singular points and has indices $(0, p'-p)$ at a; form the differential equation for z, and integrate it, and finally prove that the general solution is

$$y=A\left(\frac{x-a}{x-b}\right)^p+B\left(\frac{x-a}{x-b}\right)^{p'}.$$

If the second point (b) is at infinity, prove that

$$P=\frac{1-p-p'}{x-a},\qquad Q=\frac{pp'}{(x-a)^2},$$

which, as a matter of fact, are the values found by making $b\to\infty$ in the previous results; and shew that the solution is then

$$y=A(x-a)^p+B(x-a)^{p'}.$$

* See, for instance, Forsyth's *Differential Equations* (3rd or 4th editions), Ch. VI.

A simple example of this type of differential equation is given by Art. 68 below.

3. One regular singular point and a non-regular singularity at infinity.

Verify that these are the characteristics of the equation given by

$$P = \frac{1-p-p'}{x-a}, \quad Q = \frac{pp'}{(x-a)^2} - k^2,$$

the character at infinity being of the types $e^{\pm kx}$. (Compare Art. 116 below.)

Writing for brevity $a=0$ and $\delta = p - p'$, prove that the general solution is

$$y = Ax^p\left\{1 + \frac{k^2x^2}{2(2+\delta)} + \frac{k^4x^4}{2.4(2+\delta)(4+\delta)} + \dots\right\}$$
$$+ Bx^{p'}\left\{1 + \frac{k^2x^2}{2(2-\delta)} + \frac{k^4x^4}{2.4(2-\delta)(4-\delta)} + \dots\right\},$$

and examine the second solution when δ is an even integer.

4. Three regular singular points.

Taking the points as $x = a, b, c$ with pairs of indices (p, p'), (q, q'), (r, r'), verify that

$$P = \frac{1-p-p'}{x-a} + \frac{1-q-q'}{x-b} + \frac{1-r-r'}{x-c},$$

and

$$Q = \frac{1}{(x-a)(x-b)(x-c)}\left\{\frac{pp'(a-b)(a-c)}{x-a} + \frac{qq'(b-a)(b-c)}{x-b} + \frac{rr'(c-a)(c-b)}{x-c}\right\}$$

where $$p + p' + q + q' + r + r' = 1.$$

Also, if the third point (c) is taken at infinity, prove that

$$P = \frac{1-p-p'}{x-a} + \frac{1-q-q'}{x-b}, \quad Q = \frac{a-b}{(x-a)(x-b)}\left(\frac{pp'}{x-a} - \frac{qq'}{x-b} + \frac{rr'}{a-b}\right),$$

which, as a matter of fact, are the values found by making $c \to \infty$ in the previous results. [Papperitz.]

5. Reduction of Ex. 4 to standard form.

Write $$\xi = \frac{x-a}{x-c}\Big/\frac{b-a}{b-c}, \quad y = \frac{(x-a)^p(x-b)^q}{(x-c)^{p+q}}\eta,$$

and verify that η has indices $(0, p'-p)$ at $\xi = 0$, $(0, q'-q)$ at $\xi = 1$, and $(p+q+r, p+q+r')$ at ∞, the sum of the six indices being again unity.

Calling these indices $(0, \lambda)$, $(0, \mu)$, (ν, ν'), we find, from Ex. 4, at once

$$\frac{d^2\eta}{d\xi^2} + \left(\frac{1-\lambda}{\xi} + \frac{1-\mu}{\xi-1}\right)\frac{d\eta}{d\xi} + \frac{\nu\nu'}{\xi(\xi-1)}\eta = 0, \text{ where } \lambda + \mu + \nu + \nu' = 1,$$

which can be solved in the form (when $|\xi| < 1$)

$$\eta = AF(\nu, \nu', 1-\lambda, \xi) + B\xi^\lambda F(\nu+\lambda, \nu'+\lambda, 1+\lambda, \xi),$$

where $F(\alpha, \beta, \gamma, \xi)$ denotes Gauss's Hypergeometric series (Art. 12·2).

In this way the familiar 24 solutions of Gauss's hypergeometric differential equation can be constructed by interchanges of the points a, b, c, and of the indices p, p', etc.

6. One regular singularity and two "confluent" singularities.

Take the case of Ex. 4 with c at infinity, and let $b \to \infty$, so that two singularities tend to coincidence (or to be "confluent"); and let q, q' also tend to infinity so that $(q+q')/b$ has a finite limit l, while $(qq'-rr')/b \to s$ and $(2qq'-rr')/b^2 \to t$. Then verify that

$$P = \frac{1-p-p'}{x-a} + l, \quad Q = \frac{pp'}{(x-a)^2} + \frac{s}{x-a} + t,$$

where p, p', s, t are unrestricted.

When $l=0$ and $t=0$ this equation can be reduced to Ex. 3 by writing

$$x - a = k\xi^2,$$

which makes the new indices $2p, 2p'$ at $\xi=0$, and the new coefficients are

$$P = \frac{1-2p-2p'}{\xi}, \quad Q = \frac{4pp'}{\xi^2} + 4ks.$$

7. Special cases of Exs. 4, 5.

(i) $$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0, \qquad \text{(Arts. 67, 68)}$$

for which the indices are $(0, \frac{1}{2})$ at $x = \pm 1$ and $(n, -n)$ at ∞.

(ii) If we write in (i) $y = (1-x^2)^{\frac{1}{2}}z$, we find that the indices for z are $(0, -\frac{1}{2})$ at $x = \pm 1$, and $(n+1, -n+1)$ at ∞, so that we get

$$(1-x^2)\frac{d^2z}{dx^2} - 3x\frac{dz}{dx} + (n^2-1)z = 0,$$

which can be used to obtain the other series of Arts. 67, 68.

(iii) $$(1+t^2)\frac{d^2z}{dt^2} - 2(n-1)t\frac{dz}{dt} + n(n-1)z = 0 \qquad \text{(Art. 68·1)}$$

has indices $(0, n)$ at $t = \pm i$, and $(-n, 1-n)$ at ∞.

(iv) The functions $\{\sqrt{(1+x)} \pm 1\}^p$ give indices $(0, p)$ at $x=0$, $(0, \frac{1}{2})$ at $x=-1$, $\{-\frac{1}{2}p, -\frac{1}{2}(p-1)\}$ at ∞.

Thus $[\frac{1}{2}\{\sqrt{(1+x)}+1\}]^p = F(-\frac{1}{2}p, -\frac{1}{2}(p-1), 1-p, -x)$, (Ex. A 21)

and $\log[\frac{1}{2}\{\sqrt{(1+x)}+1\}]$ is deduced by making $p \to 0$. (Ex. A 9)

CERTAIN SPECIAL POWER SERIES.

57. The exponential limit.*

We shall prove that

$$\lim_{\nu\to\infty}(1+\xi)^\nu = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

where

$$x = \lim_{\nu\to\infty}(\nu\xi).$$

Consider first the special case when ν tends to infinity through integral values n, and write $n\xi = X$.

---

* The reader is recommended to refer to Appendix II. before proceeding further.

Then we find,* on expanding,

$$(1+\xi)^n = 1+X+\left(1-\frac{1}{n}\right)\frac{X^2}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{X^3}{3!}$$
$$+\dots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right)\frac{X^n}{n!}.$$

Now, this expression satisfies the conditions of Tannery's theorem in Art. 49; we can use as the comparison-series,

$$1+X_0+\frac{1}{2!}X_0^2+\frac{1}{3!}X_0^3+\dots,$$

where X_0 is the greatest value† of $|X|$ for any value of n. For we have

$$|v_r(n)| = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{r-1}{n}\right)\frac{|X|^r}{r!} < \frac{X_0^r}{r!},$$

where X_0 is of course independent of n. Also

$$\lim_{n\to\infty} v_r(n) = x^r/r!,$$

because $\lim\limits_{n\to\infty}\left(1-\dfrac{r}{n}\right)=1$ and $\lim\limits_{n\to\infty} X = x$. Finally the index p is equal to n, and so of course tends steadily to infinity.

Thus $\qquad \lim(1+\xi)^n = 1+x+\dfrac{1}{2!}x^2+\dfrac{1}{3!}x^3+\dots$ to ∞.

If now ν tends to infinity in any other way, ν will, at any stage, be contained between two integers n and $(n+1)$ say; and of course n will tend to infinity with ν. Thus $(1+\xi)^\nu$ will be contained between‡ $(1+\xi)^n$ and $(1+\xi)^{n+1}$; and $\nu\xi$ will be contained between $n\xi$ and $(n+1)\xi$, so that

$$\lim_{n\to\infty}(n\xi) = \lim_{n\to\infty}(n+1)\xi = x.$$

---

* For the general term in the binomial expansion of $(1+\xi)^n$ is

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}\xi^r = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{r-1}{n}\right)\frac{(n\xi)^r}{r!}.$$

† That there is a maximum is evident, because it is supposed that X approaches the limit x, as n increases to infinity.

‡ It is of course understood that the positive value of $(1+\xi)^\nu$ is taken; and then this value is obviously contained between $(1+\xi)^n$ and $(1+\xi)^{n+1}$ if ν is rational. On the other hand, if ν is irrational, the statement is a consequence of the definition of an irrational power.

Thus, from what has been proved already, we see that

$$\lim_{n\to\infty}(1+\xi)^n=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots=\lim_{n\to\infty}(1+\xi)^{n+1},$$

and since $(1+\xi)^\nu$ is contained between $(1+\xi)^n$ and $(1+\xi)^{n+1}$, it follows that

$$\lim_{\nu\to\infty}(1+\xi)^\nu=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots.$$

If we write for brevity

$$(1+\xi)^n=\sum_{r=0}^{n}f_r(x,n),$$

it will be seen that we have used the theorem

$$\lim_{n\to\infty}\sum_{r=0}^{n}f_r(x,n)=\sum_{r=0}^{\infty}\{\lim_{n\to\infty}f_r(x,n)\}.$$

That is, we have replaced a single limit by a repeated limit; and of course such a step needs justification (see the examples in Art. 49).

Special cases.

If $\xi=1/\nu$, we have the equation

$$\lim_{\nu\to\infty}\left(1+\frac{1}{\nu}\right)^\nu=1+1+\frac{1}{2!}+\frac{1}{3!}+\dots\text{ to }\infty=e.$$

The sum of this series has been calculated in Art. 7, and we found that

$$e=2{\cdot}718281828\dots.$$

If $\xi=1/(\nu-1)$, we have

$$(1+\xi)^\nu=[\nu/(\nu-1)]^\nu=(1-1/\nu)^{-\nu},$$

so that

$$\lim_{\nu\to\infty}(1-1/\nu)^{-\nu}=e.$$

These two results may be combined into the single equation

$$\lim_{\lambda\to 0}(1+\lambda)^{1/\lambda}=e,$$

where λ approaches 0 from either side.

58. The exponential function.

We may denote by the symbol $E(x)$ the exponential series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots.$$

Then (by Art. 52) we see that $E(x)$ is a continuous function of x, and that its differential coefficient is given by term-by-term differentiation, so that

$$\frac{d}{dx}E(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots=E(x),$$

because

$$\frac{d}{dx}\frac{x^r}{r!}=\frac{x^{r-1}}{(r-1)!}.$$

Further, we see from Art. 57 that

$$E(x)\times E(y)=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\cdot\lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n$$
$$=\lim_{n\to\infty}\left\{\left(1+\frac{x}{n}\right)\left(1+\frac{y}{n}\right)\right\}^n$$
$$=\lim_{n\to\infty}\left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n$$
$$=E(x+y).$$

This result can also be proved directly from the series for $E(x)$, by applying the rule (Art. 34) for multiplying two absolutely convergent series. The result leads directly to the equations (in which n, p are positive integers)

$$e^n=\{E(1)\}^n=E(n)=\{E(n/p)\}^p, \quad E(-x)=[E(x)]^{-1}.$$

Thus we see that $E(x)$ is the *positive* value of e^x for any *rational* value of x. But the equation must also be true for irrational values of x; for if $a_0, a_1, \ldots, a_n, \ldots$ represents a sequence of rational numbers whose limit is x, we have

$$e^x=\lim_{n\to\infty}e^{a_n}=\lim_{n\to\infty}E(a_n)=E(x),$$

the last step being valid because $E(x)$ is a *continuous* function (Art. 52). In future, we shall generally write e^x instead of $E(x)$; but when the index x is a complicated expression it is sometimes clearer to use $\exp x$, as in Ex. A 20, at the end of the chapter.

Ex. As a numerical example, the reader may shew that

$$e^{\frac{1}{2}\pi}=4\cdot810\ldots,$$

and hence that $\quad e^{\pi}=23\cdot14\ldots, \quad e^{2\pi}=535\cdot6\ldots.$

To obtain the first result we need only add up all the terms calculated in the example of Art. 59.

59. The sine and cosine power-series.

Write $\quad \sin x-\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}\right\}=S_n(x)$

and $\quad \cos x-\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!}\right\}=C_n(x).$

Then it is plain that $S_n(x)$, $C_n(x)$ are both continuous functions of x, and that

$$\frac{d}{dx}\{S_n(x)\}=C_n(x), \quad \frac{d}{dx}\{C_n(x)\}=-S_{n-1}(x).$$

Now $C_1(x) = \cos x - 1$ is negative, and consequently $\frac{d}{dx}\{S_1(x)\}$ is negative; but $S_1(x)$ vanishes with x, and consequently $S_1(x)$ is negative when x is positive.*

Thus $\frac{d}{dx}\{C_2(x)\} = -S_1(x)$ is positive when x is positive; but $C_2(x)$ vanishes with x, and therefore $C_2(x)$ is positive when x is positive.

Hence $\frac{d}{dx}\{S_2(x)\} = C_2(x)$ is positive when x is positive; and $S_2(x)$ vanishes with x, so that $S_2(x)$ must be positive when x is positive.

That is, $\frac{d}{dx}\{C_3(x)\} = -S_2(x)$ is negative when x is positive; and therefore, since $C_3(x)$ vanishes with x, $C_3(x)$ is negative when x is positive.

We can continue this argument, and by doing so we find that

$$\left.\begin{matrix} C_1(x),\ C_3(x),\ C_5(x),\ C_7(x),\ \ldots \\ S_1(x),\ S_3(x),\ S_5(x),\ S_7(x),\ \ldots \end{matrix}\right\} \begin{matrix}\text{are negative when} \\ x \text{ is positive,}\end{matrix}$$

while the expressions with even suffixes are positive.

This shews that $\sin x$ lies between the two expressions

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

and

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

Hence, since $\lim_{n\to\infty}\frac{x^{2n+1}}{(2n+1)!} = 0$ (see Ex. 4, p. 9),

we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ to } \infty.$$

In like manner we prove that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \text{ to } \infty.$$

These results have been established for positive values of x only; but it is evident that $\sin x$ and its series both change sign with x, while $\cos x$ and its series do not change sign, so that the results are valid also for negative values of x.

* We make use throughout this article of the fact that if y is increasing with x and is zero for $x=0$, then y (if continuous) must be positive for positive values of x; this fact is intuitive, but can be proved by arithmetic reasoning.

The figure below will serve to shew the relation between $\sin x$ and the first two or three terms in the infinite series.

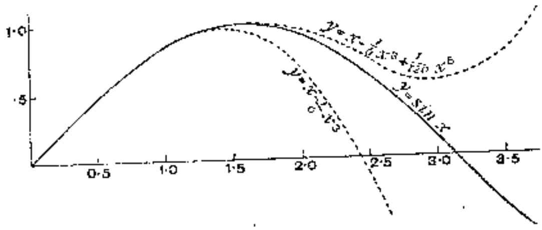

FIG. 17.

Ex. Let us calculate $\cos(\frac{1}{2}\pi)$ and $\sin(\frac{1}{2}\pi)$.

We have $x = \frac{1}{2}\pi = 1{\cdot}5708$ very nearly.

This gives

$$\begin{aligned} \tfrac{1}{2}x^2 &= 1{\cdot}2337, & \tfrac{1}{5040}x^7 &= {\cdot}0047, \\ \tfrac{1}{6}x^3 &= {\cdot}6460, & \tfrac{1}{40320}x^8 &= {\cdot}0009, \\ \tfrac{1}{24}x^4 &= {\cdot}2537, & \tfrac{1}{362880}x^9 &= {\cdot}0002, \\ \tfrac{1}{120}x^5 &= {\cdot}0797, & \tfrac{1}{3628800}x^{10} &= {\cdot}00003. \\ \tfrac{1}{720}x^6 &= {\cdot}0209, \end{aligned}$$

Hence $\cos(\frac{1}{2}\pi) = 1{\cdot}2546 - 1{\cdot}2546 = 0$, the error being less than ·00003.

Also $\sin(\frac{1}{2}\pi) = 1{\cdot}6507 - 0{\cdot}6507 = 1$, the error being less than ·00003.

60. Other methods of establishing the sine and cosine power-series.

(1) Probably the most rapid method of recalling the series to memory is to assume that $\sin x$ and $\cos x$ may be represented by power-series.

Thus if $\sin x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

we have $\cos x = \dfrac{d}{dx}(\sin x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots,$

and so $-\sin x = \dfrac{d}{dx}(\cos x) = 1 \,.\, 2a_2 + 2 \,.\, 3 \,.\, a_3 x + 3 \,.\, 4 \,.\, a_4 x^2 + \dots .$

Further, $a_0 = 0$, $a_1 = 1$, because $\sin x$ is 0 and $\cos x$ is 1, for $x = 0$.

Hence we get

$$\begin{aligned} &1 \,.\, 2 \,.\, a_2 = -a_0 = 0, \\ &2 \,.\, 3 \,.\, a_3 = -a_1 = -1, \text{ or } a_3 = -\frac{1}{3\,!}, \\ &3 \,.\, 4 \,.\, a_4 = -a_2 = 0, \\ &4 \,.\, 5 \,.\, a_5 = -a_3, \text{ or } a_5 = \frac{1}{5\,!}, \end{aligned}$$

and so on.

But of course we have no *a priori* reason for supposing that $\sin x$ and $\cos x$ can be expressed as power-series; and therefore this method is not logically complete.

(2) We may start from the series, and call them, say, $S(x)$, $C(x)$. Then multiplication of the series (by Art. 54) gives

$$S(x)C(y)+S(y)C(x)=S(x+y),$$
$$C(x)C(y)-S(x)S(y)=C(x+y).$$

Hence in particular $\{C(x)\}^2+\{S(x)\}^2=C(0)=1$

and $S(2x)=2S(x)C(x), \quad C(2x)=\{C(x)\}^2-\{S(x)\}^2.$

From these formulae we can shew that $S(x)$ and $C(x)$ satisfy the ordinary formulae of elementary trigonometry.

Further, $C(0)=1$ and $C(2)$ is negative,* so that $C(x)=0$ has at least one root between 0 and 2. But

$$\frac{d}{dx}\{C(x)\}=-S(x),$$

and $S(x)$ is always positive† for any value of x between 0 and 2. Thus $C(x)$ can have only one root between 0 and 2, because $C(x)$ steadily decreases in that interval.

Call this root α, then we have

$$C(\alpha)=0, \quad \{S(\alpha)\}^2=1,$$

and so $S(\alpha)=1$, since $S(\alpha)$ must be positive $(0<\alpha<2)$.

Hence $S(2\alpha)=2S(\alpha)C(\alpha)=0,$

$$C(2\alpha)=\{C(\alpha)\}^2-\{S(\alpha)\}^2=-1,$$

and so $S(x+2\alpha)=-S(x), \quad C(x+2\alpha)=-C(x).$

Thus $S(x+4\alpha)=+S(x), \quad C(x+4\alpha)=+C(x).$

On these formulae the whole of Analytical Trigonometry can be based; π being defined as equal to 2α.

(3) It is not difficult to prove, by induction or by the methods given in Chap. IX. below, that

$$\sin n\theta=\cos^n\theta\left\{nt-\frac{n(n-1)(n-2)}{3!}t^3+\dots\right\},$$

$$\cos n\theta=\cos^n\theta\left\{1-\frac{n(n-1)}{2!}t^2+\frac{n(n-1)(n-2)(n-3)}{4!}t^4-\dots\right\},$$

where $t=\tan\theta$, and both series terminate after $\frac{1}{2}(n+1)$ terms, when n is odd; or after $\frac{1}{2}n$ or $\frac{1}{2}(n+2)$ terms, when n is even.

Thus we have, on putting $n\theta=x$,

$$\sin x=\left(\cos^n\frac{x}{n}\right)\left\{\xi-\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{\xi^3}{3!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\left(1-\frac{4}{n}\right)\frac{\xi^5}{5!}-\dots\right\},$$

$$\cos x=\left(\cos^n\frac{x}{n}\right)\left\{1-\left(1-\frac{1}{n}\right)\frac{\xi^2}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{\xi^4}{4!}-\dots\right\},$$

where $\xi=n\tan(x/n)$.

---

* Because $C(2)=1-2+\dfrac{2}{3}-\dfrac{2^6}{6!}\left(1-\dfrac{2^2}{7\,.\,8}\right)-\dfrac{2^{10}}{10!}\left(1-\dfrac{2^2}{11\,.\,12}\right)-\dots<-\dfrac{1}{3}.$

† Because $S(x)=x\left(1-\dfrac{x^2}{2\,.\,3}\right)+\dfrac{x^5}{5!}\left(1-\dfrac{x^2}{6\,.\,7}\right)+\dots\,.$

To these expressions in brackets we can apply Tannery's theorem of Art. 49, using as the comparison-series

$$t_0 + \frac{t_0^3}{3!} + \frac{t_0^5}{5!} + \dots \text{ and } 1 + \frac{t_0^2}{2!} + \frac{t_0^4}{4!} + \cdots,$$

where $t_0 = |m \tan(x/m)| \geqq |\xi|$, m being an integer less than n, and chosen so that x lies between $-\frac{1}{2}m\pi$ and $+\frac{1}{2}m\pi$.

The argument is, in fact, almost identical with that employed in Art. 57 for the exponential limit; and we deduce that

$$\lim_{n\to\infty}\left\{\xi - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{\xi^3}{3!} + \dots\right\} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \text{ to } \infty,$$

$$\lim_{n\to\infty}\left\{1 - \left(1-\frac{1}{n}\right)\frac{\xi^2}{2!} + \dots\right\} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \text{ to } \infty.$$

Finally, $$0 < 1 - \cos\frac{x}{n} = 2\sin^2\frac{x}{2n} < \frac{x^2}{2n^2},$$

and so $$1 > \cos^n\frac{x}{n} > \left(1 - \frac{x^2}{2n^2}\right)^n > 1 - \frac{x^2}{2n}, \qquad \text{(Art. 38)}$$

and thus $$\lim_{n\to\infty}\cos^n\frac{x}{n} = 1.$$

(4) Another instructive method is to apply the process of integration by parts to the two equations

$$\sin x = \int_0^x \cos(x-t)\,dt, \quad \cos x = 1 - \int_0^x \sin(x-t)\,dt.$$

If we integrate twice by parts, we obtain

$$\sin x = \left[t\cos(x-t) - \frac{t^2}{2!}\sin(x-t)\right]_0^x - \int_0^x \frac{t^2}{2!}\cos(x-t)\,dt$$

$$= x - \int_0^x \frac{t^2}{2!}\cos(x-t)\,dt$$

and $$\cos x = 1 - \left[t\sin(x-t) + \frac{t^2}{2!}\cos(x-t)\right]_0^x + \int_0^x \frac{t^2}{2!}\sin(x-t)\,dt$$

$$= 1 - \frac{x^2}{2!} + \int_0^x \frac{t^2}{2!}\sin(x-t)\,dt,$$

and so on.

Thus we find that, in the notation of Art. 59,

$$S_n(x) = (-1)^n\int_0^x \frac{t^{2n}}{(2n)!}\cos(x-t)\,dt,$$

$$C_n(x) = (-1)^n\int_0^x \frac{t^{2n-1}}{(2n-1)!}\cos(x-t)\,dt$$

Hence, as in that article, we find

$$|S_n(x)| \leqq \int_0^{|x|} \frac{t^{2n}}{(2n)!}\,dt \leqq \frac{|x|^{2n+1}}{(2n+1)!}$$

and $$|C_n(x)| \leqq \int_0^{|x|} \frac{t^{2n-1}}{(2n-1)!}\,dt \leqq \frac{|x|^{2n}}{(2n)!}$$

61. The general binomial theorem.

We shall now discuss the general binomial series

$$f(x)=1+\nu x+\nu(\nu-1)\frac{x^2}{2!}+\nu(\nu-1)(\nu-2)\frac{x^3}{3!}+\dots \text{ to } \infty.$$

We know from elementary algebra that if ν is a positive integer this series terminates and represents $(1+x)^\nu$. We proceed now to examine the corresponding theorem for other values of ν.

By Art. 12·2, the series is absolutely convergent if $|x|<1$; and so (Art. 50) the series is uniformly convergent in any interval $(-k, +k)$, where $0<k<1$.

Now

$$f'(x)=\nu\left\{1+(\nu-1)x+(\nu-1)(\nu-2)\frac{x^2}{2!}+\dots \text{ to } \infty\right\}=\nu g(x) \text{ say,}$$

where $g(x)$ differs from $f(x)$ by having $(\nu-1)$ in place of ν.

Also $$(1+x)g(x)=1+(\nu-1)x+(\nu-1)(\nu-2)\frac{x^2}{2!}+\dots$$
$$+\quad x+\quad 2(\nu-1)\frac{x^2}{2!}+\dots$$

or $$(1+x)g(x)=1+\nu x+\nu(\nu-1)\frac{x^2}{2!}+\dots=f(x),$$

so that $$(1+x)f'(x)=\nu f(x).$$

Hence we see that $$\frac{d}{dx}\left\{\frac{f(x)}{(1+x)^\nu}\right\}=0,$$

or $$f(x)=A(1+x)^\nu,$$

where A is independent of x. But $f(0)=1$; and consequently, if we choose the positive value for $(1+x)^\nu$, we have $A=1$; that is,

$$f(x)=(1+x)^\nu.$$

This result has, of course, been proved only for an interval $(-k, +k)$; let us now see if it can be extended to include the points -1, $+1$. The quotient of the nth term in the series by the $(n+1)$th is

$$-n/(n-\nu-1)x, \qquad \text{if } n>\nu+1,$$

and so the series converges at $x=-1$ if ν is positive (Art. 12·2), and at $x=+1$ if $\nu+1$ is positive (Art. 19). Thus by Abel's theorem (Art. 51) the sum of the series at $x=-1$ is 0, if ν is positive; and at $x=+1$ the sum is 2^ν if $\nu+1$ is positive.

Other methods.

(1) The most rapid method for recalling the series to memory is to solve the differential equation

$$(1+x)f'(x) = \nu f(x),$$

by assuming a series $f(x) = 1 + a_1 x + a_2 x^2 + \dots$.

On substitution, we find that

$$a_1 = \nu, \quad 2a_2 + a_1 = \nu a_1, \quad 3a_3 + 2a_2 = \nu a_2, \quad \text{etc.}$$

Of course this investigation must be supplemented as above in order to complete the proof.

(2) We can multiply together two series with different values of ν (say ν_1 and ν_2) and verify (by Art. 54) that their product is a similar series in which the coefficient of $x^n/n!$ is a polynomial of degree n in ν_1 and ν_2, the terms of highest degree being $\nu_1{}^n$ and $\nu_2{}^n$. Now when ν_1, ν_2 are any integers greater than n, this coefficient is equal to

$$(\nu_1 + \nu_2)(\nu_1 + \nu_2 - 1) \dots (\nu_1 + \nu_2 - n + 1)$$

by the elementary binomial theorem. Hence this expression represents the form of the coefficient generally. Thus the product is equal to $f(x)$, where $\nu = \nu_1 + \nu_2$; compare Art. 96, below. Then we can apply the same argument as was used for the exponential series (Art. 58) to prove that $f(x)$ must be the νth power of its value for $\nu = 1$.

(3) The proof given in the example of Art. 36 may be regarded as one of the best of a purely algebraic type.

(4) We have
$$(1+x)^\nu = 1 + \nu \int_0^x (1 + x - t)^{\nu-1} dt$$
$$= 1 + \nu x + \nu(\nu-1) \int_0^x (1+x-t)^{\nu-2} t \, dt,$$

where we obtain the last line by integrating by parts. Continuing thus, we get

$$(1+x)^\nu - \left\{1 + \nu x + \nu(\nu-1)\frac{x^2}{2!} + \dots + \nu(\nu-1)\dots(\nu-n+1)\frac{x^n}{n!}\right\}$$
$$= \nu(\nu-1)\dots(\nu-n)\int_0^x (1+x-t)^{\nu-n-1}\frac{t^n}{n!}dt.$$

Now, if x is positive, $(1+x-t)$ lies between 1 and $(1+x)$, so that

$$\int_0^x (1+x-t)^{\nu-n-1}\frac{t^n}{n!}dt < \frac{x^{n+1}}{(n+1)!},$$

provided that $n > \nu - 1$.

On the other hand, if x is negative, we can only say that

$$\left|\int_0^x (1+x-t)^{\nu-n-1}\frac{t^n}{n!}dt\right| < |1+x|^{\nu-n-1}\frac{|x|^{n+1}}{(n+1)!}, \quad \text{if } n > \nu - 1.$$

Thus, in either case, if $|x| < 1$, the difference

$$(1+x)^\nu - \left\{1 + \nu x + \ldots + \nu(\nu-1)(\nu-2)\ldots(\nu-n+1)\frac{x^n}{n!}\right\}$$

tends to zero as n increases to infinity.

When $|x| \geqq 1$, it is sometimes useful to replace $(1+x)^\nu$ by a selected number of terms from the binomial series (compare Art. 116 below); and it is then necessary to make an estimate of the error introduced by this step. We can obtain such an estimate from the formulae just worked out by integrating by parts. The results are easily found to be: *

If $x > 1$, and $n > \nu$, the error involved in using the first n terms of the series, in place of $(1+x)^\nu$, is less than the next term of the series.

But if x is negative and numerically greater than 1, the previous estimate must be multiplied by $|1+x|^{\nu-n}$.

If $x = -1$, it is interesting to note that *we can sum the binomial series to a finite number of terms.* Thus we have

$$1-\nu+\tfrac{1}{2}\nu(\nu-1)=(1-\nu)(1-\tfrac{1}{2}\nu),$$

$$1-\nu+\tfrac{1}{2}\nu(\nu-1)-\tfrac{1}{6}\nu(\nu-1)(\nu-2)=(1-\nu)(1-\tfrac{1}{2}\nu)(1-\tfrac{1}{3}\nu),$$

and so on.

Hence the sum to $(n+1)$ terms is

$$(1-\nu)\left(1-\frac{1}{2}\nu\right)\left(1-\frac{1}{3}\nu\right)\ldots\left(1-\frac{1}{n}\nu\right).$$

It is clear from Art. 39 that as n tends to ∞ this sum tends to 0 if ν is positive, and to ∞ if ν is negative.†

62. The logarithmic series.

We take as our definition of the natural logarithm the equation

$$\log(1+x)=\int_0^x \frac{dt}{1+t}. \qquad \text{(See Appendix II.)}$$

Now when $|x| < 1$, we can write

$$(1+t)^{-1}=1-t+t^2-t^3+\ldots \text{ to } \infty,$$

the series converging uniformly from $t=0$ to $t=x$. Hence (by Art. 52 (2)) we obtain the series

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots \text{ to } \infty.$$

* It should be noted that we have here written $(n-1)$ for n in the actual formulae worked out under (4) above.

† Because the series $1+\frac{1}{2}+\frac{1}{3}+\ldots$ to ∞ is divergent.

However, it is not necessary to make use of uniform convergence in order to integrate term-by-term; for we have

$$\frac{1}{1+t}=1-t+t^2-\ldots+(-1)^{n-1}t^{n-1}+(-1)^n\frac{t^n}{1+t}.$$

Thus $$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\ldots+(-1)^{n-1}\frac{1}{n}x^n$$
$$+(-1)^n\int_0^x \frac{t^n}{1+t}\,dt.$$

If x is positive, the last integral is clearly less than

$$\int_0^x t^n dt=\frac{x^{n+1}}{n+1},$$

which tends to zero as n tends to infinity, provided that $0<x\leqq 1$. Thus *the logarithmic series is valid* even for $x=1$.*

This result follows also from Abel's theorem (Art. 51); and has been obtained previously in the form

$$\log 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots. \qquad \text{(Art. 19)}$$

On the other hand, when x is negative, we can only say that

$$\left|\int_0^x \frac{t^n}{1+t}\,dt\right|$$

is less than $$\frac{1}{1+x}\int_0^{|x|} t^n dt=\frac{|x|^{n+1}}{(n+1)(1+x)},$$

and from this expression it would be expected that $x=-1$ must be excluded from the region of convergence of the logarithmic series; and, as a matter of fact, the series

$$-(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots)$$

has been proved to be divergent (Art. 7, Ex. 2).

It should be noted that *even if $x>1$, the difference between* $\log(1+x)$ *and the first n terms of the series is less than the following term.*

The special case $n=1$ leads to the results

$$0<x-\log(1+x)<\tfrac{1}{2}x^2, \qquad \text{if } x>0.$$
$$0<x-\log(1+x)<\tfrac{1}{2}x^2/(1+x), \qquad \text{if } -1<x<0.$$

---

* That is, the operation of term-by-term integration can here be extended beyond the region of uniform convergence of the integrated series (compare Art. 47 and Art. 52).

Another method.

From the identity $$(1+x)^y = e^{y \log(1+x)},$$
we see that (Arts. 58, 61)

$$1 + yx + y(y-1)\frac{x^2}{2!} + y(y-1)(y-2)\frac{x^3}{3!} + \dots$$
$$= 1 + y\log(1+x) + \frac{1}{2!}\{y\log(1+x)\}^2 + \frac{1}{3!}\{y\log(1+x)\}^3 + \dots .$$

It is now necessary to consider whether the first of these series can be re-arranged in powers of y without changing its value. By Art. 26, this derangement will be permissible if the series

$$1 + \eta\xi + \eta(\eta+1)\frac{\xi^2}{2!} + \eta(\eta+1)(\eta+2)\frac{\xi^3}{3!} + \dots$$

is convergent, where $\xi = |x|$, $\eta = |y|$.

But the last series is the expanded form of $(1-\xi)^{-\eta}$, which is convergent if $\xi < 1$; that is, if $|x| < 1$. Thus the derangement will not alter the sum.

Hence we get (from the coefficients of y and y^2) the equations

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots \text{ to } \infty,$$

$$\tfrac{1}{2}\{\log(1+x)\}^2 = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3(1+\tfrac{1}{2}) + \tfrac{1}{4}x^4(1+\tfrac{1}{2}+\tfrac{1}{3}) - \tfrac{1}{5}x^5(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}) + \dots \text{ to } \infty.$$

Similar (but less simple) series may be deduced for higher powers of $\log(1+x)$. [Compare Chrystal's *Algebra*, Ch. XXVIII. § 9.]

63. For purposes of numerical computation of logarithms it is better to use the series

$$\log\frac{1+x}{1-x} = 2(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \dots),$$

which can be found from the previous series by writing $-x$ for x and then subtracting; or directly, by integrating $1/(1-x^2)$. In either way the remainder after n terms is seen to be less than

$$\frac{2x^{2n+1}}{(2n+1)(1-x^2)}.$$

Then, by writing $x = 1/(2p+1)$, we deduce the formula

$$\log\left(\frac{p+1}{p}\right) = 2\left\{\frac{1}{2p+1} + \frac{1}{3}\,\frac{1}{(2p+1)^3} + \frac{1}{5}\,\frac{1}{(2p+1)^5} + \dots\right\},$$

which is a very convenient form for numerical work.

Ex. **The natural logarithms from** log 2 **to** log 10.

By writing $p = 2, 3, 4$, we obtain three series, the first of which is

$$\log\frac{3}{2} = 2\left\{\frac{1}{5} + \frac{1}{3}\left(\frac{1}{5^3}\right) + \frac{1}{5}\left(\frac{1}{5^5}\right) + \dots\right\}.$$

The second and third give similar series for $\log\frac{4}{3}$ and $\log\frac{5}{4}$.

The details of calculation for $\log \frac{3}{2}$ may be arranged as follows:

$\frac{1}{5} =$	·20 00 00 00		·20 00 00 00
$\frac{1}{5^3} =$	80 00 00	÷3	26 66 67
$\frac{1}{5^5} =$	3 20 00	÷5	64 00
$\frac{1}{5^7} =$	12 80	÷7	1 83
$\frac{1}{5^9} =$	51	÷9	06
			·20 27 32 56
			2
		$\log \frac{3}{2} =$	·40 54 65

The error involved in neglecting terms beyond the fifth is less than $\frac{2}{11}\,\frac{(\frac{1}{5})^{11}}{1-(\frac{1}{5})^2} = \frac{1}{132}\left(\frac{1}{5}\right)^9$, which cannot affect the eighth decimal. Hence the result is correct to the sixth decimal.

Similarly, we get

$$\log \tfrac{4}{3} = 2\{\cdot 14285714 + \cdot 00097182 + \cdot 00001190 + \cdot 00000017\}$$
$$= \cdot 287682 \text{ to six decimals,}$$

the error involved being again less than a unit in the eighth decimal.

Also $\log \frac{5}{4} = 2\{\cdot 11111111 + \cdot 00045725 + \cdot 00000339 + \cdot 00000003\}$
$= \cdot 223144$ to six decimals.

From these results we find the natural logarithms of all integers from 2 to 10 (with the exception of 7, which can be found similarly from $\log \frac{7}{6}$).

In particular, we have

$$\log 2 = \cdot 693147, \quad \log 3 = 1\cdot 098612,$$
$$\log 5 = 1\cdot 609438, \quad \log 10 = 2\cdot 302585.$$

Of course other series than the above have been found, which converge more rapidly, and so enable the logarithms to be easily calculated to a great number of places. To illustrate, the reader may find formulae for log 2, log 3, log 5 in terms of the three series obtained by writing $p = 15$, 24 and 80 in the general formula.

64. The power-series for arc sin x and arc tan x.

If $x = \sin\theta$, we have

$$x = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots,$$

and so, by the principle of reversion of series (Art. 55), we can express the numerically least value of θ as a convergent power-series in x, the first two terms of which are seen to be

$$\theta = x + \tfrac{1}{6}x^3 + \dots .$$

However, it is not easy to obtain the general law of the coefficients in this manner; but we can overcome the difficulty by using the Calculus.

We have in fact

$$\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{(1-x^2)}} = 1 + \frac{1}{2}x^2 + \frac{1\,.\,3}{2\,.\,4}x^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6 + \dots,$$

where θ is supposed to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, so that $\cos\theta$ is positive.

The series for $\dfrac{d\theta}{dx}$ is obtained from the binomial series by writing $\nu = -\frac{1}{2}$ and $-x^2$ for $+x$; it will therefore converge uniformly in any interval $(-k, +k)$ if $0 < k < 1$.

Hence we may integrate term-by-term and so obtain

$$\text{arc}\sin x = \theta = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{x^7}{7} + \dots,$$

which converges absolutely and uniformly in the interval $(-1, +1)$, as may be seen from the test of Art. 12·2 (5). Thus, writing $x=1$, we have the formula

$$\frac{1}{2}\pi = 1 + \frac{1}{2\,.\,3} + \frac{1\,.\,3}{2\,.\,4\,.\,5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,7} + \dots,$$

but this series converges so slowly as to be quite unsuitable for numerical computation.

Although we have not found * a series for $\tan x$, we can easily find one for $\text{arc}\tan x$. For, writing $x = \tan\phi$, we have

$$\frac{d\phi}{dx} = \frac{1}{\sec^2\phi} = \frac{1}{1+x^2}.$$

* We can, of course, form such a series by dividing $\sin x$ by $\cos x$ (compare Art. 54), but there is no simple general law for the coefficients except in terms of Bernoulli's numbers (Art. 100). The first three terms are

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \dots,$$

and more coefficients are given in Art. 100 below.

The method used in Art. 54 shews that this series will certainly be convergent in any interval for which $\dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \dots < 1$; and a short calculation will shew that this is satisfied in the interval $(-1·3, +1·3)$; but by means of a theorem given in Art. 89, we can shew that the region of convergence is $(-\frac{1}{2}\pi, +\frac{1}{2}\pi)$.

Thus $\frac{d\phi}{dx}=1-x^2+x^4-\ldots+(-1)^{n-1}x^{2n-2}+(-1)^n\frac{x^{2n}}{1+x^2}$

or $$\phi=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\ldots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$$
$$+(-1)^n\int_0^x\frac{t^{2n}}{1+t^2}dt,$$

where ϕ is supposed to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

The integral last written is less, in numerical value, than
$$\int_0^{|x|}t^{2n}dt=\frac{|x|^{2n+1}}{2n+1};$$
and this tends to 0 as n tends to ∞, provided that $|x|\leqq 1$.

Hence we have * Gregory's series
$$\text{arc tan } x=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\ldots \text{ to } \infty,$$
where $-1\leqq x\leqq +1,\quad -\frac{1}{4}\pi\leqq \text{arc tan } x\leqq +\frac{1}{4}\pi.$

In particular we have
$$\tfrac{1}{4}\pi=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots.$$

The last series converges very slowly, but by the aid of Euler's method given in Art. 24, the reader will find no great difficulty in calculating $\frac{1}{4}\pi$ to five decimals, from the first 13 or 14 terms. The result is $$\tfrac{1}{4}\pi=\cdot 78540.$$

For the actual calculation of π to a large number of places, it is necessary to use special devices to increase the convergence of the series; a well-known method is to write $$\alpha=\text{arc tan }\tfrac{1}{5}.$$

Then we find $$\tan 2\alpha=\tfrac{5}{12},\quad \tan 4\alpha=\tfrac{120}{119}.$$

Hence $$\tan(4\alpha-\tfrac{1}{4}\pi)=\tfrac{1}{239}$$

or $$\tfrac{1}{4}\pi=4(\text{arc tan }\tfrac{1}{5})-(\text{arc tan }\tfrac{1}{239}).$$

For other series to calculate π see Ex. A 42, p. 196.

65. Various trigonometrical power-series.

It is clear from Art. 27 that the expansion of
$$(1-2r\cos\theta+r^2)^{-1}=1+(2r\cos\theta-r^2)+(2r\cos\theta-r^2)^2+\ldots \text{ to } \infty$$
may be arranged in powers of r without altering its value, provided that † $0<r\leqq\frac{2}{5}$.

* Of course the term-by-term integration could also have been justified by making use of the uniform convergence of the series $1-x^2+x^4-\ldots$.

† For $|2r\cos\theta|+r^2\leqq 2|r|+r^2$; and when r satisfies the condition above we have $2|r|+r^2=\frac{24}{25}<1$. Thus $|2r\cos\theta|+r^2<1$, as required by Art. 27.

The sequence of coefficients is, however, more easily determined for the fraction $(1-r\cos\theta)/(1-2r\cos\theta+r^2)$.

Write $$\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}=1+A_1r+A_2r^2+A_3r^3+\dots,$$

where A_1, A_2, A_3, ... are of course functions of θ. Then we have the identity

$$\begin{aligned}1-r\cos\theta=1&+A_1r \quad +A_2r^2 \quad +A_3r^3+\dots\\ &-2r\cos\theta-2A_1r^2\cos\theta-2A_2r^3\cos\theta-\dots\\ &\qquad +r^2 \qquad\qquad +A_1r^3+\dots,\end{aligned}$$

and hence we get, using Art. 52 (5),

$$\begin{aligned}A_1&=\cos\theta,\\ A_2&=2A_1\cos\theta-1 \quad =\cos 2\theta,\\ A_3&=2A_2\cos\theta-A_1=\cos 3\theta,\\ A_4&=2A_3\cos\theta-A_2=\cos 4\theta,\end{aligned}$$

and so on.

Thus we find the series

$$\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}=1+r\cos\theta+r^2\cos 2\theta+r^3\cos 3\theta+\dots \text{ to } \infty.$$

If we subtract 1 and divide by r, we deduce that

$$\frac{\cos\theta-r}{1-2r\cos\theta+r^2}=\cos\theta+r\cos 2\theta+r^2\cos 3\theta+r^3\cos 4\theta+\dots \text{ to } \infty.$$

Combining these two series, we get also

$$\frac{1-r^2}{1-2r\cos\theta+r^2}=1+2r\cos\theta+2r^2\cos 2\theta+2r^3\cos 3\theta+\dots \text{ to } \infty.$$

An exactly similar argument gives the formula

$$\frac{r\sin\theta}{1-2r\cos\theta+r^2}=B_1r+B_2r^2+B_3r^3+\dots \text{ to } \infty,$$

where, on multiplication, we have

$$B_1=\sin\theta, \quad B_2=2B_1\cos\theta=\sin 2\theta,$$

$$B_3=2B_2\cos\theta-B_1=\sin 3\theta, \text{ etc.}$$

Hence $$\frac{r\sin\theta}{1-2r\cos\theta+r^2}=r\sin\theta+r^2\sin 2\theta+r^3\sin 3\theta+\dots.$$

By inspection we see that all these series converge when $1+r+r^2+r^3+\dots$ converges, or when $-1<r<1$. Thus we are led to enquire whether the equations are not also true for the interval $(-1, 1)$. Now we find, identically,

$$\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}=1+r\cos\theta+r^2\cos 2\theta+\ldots+r^{n-1}\cos(n-1)\theta+R_n,$$

where
$$R_n=\frac{r^n\cos n\theta-r^{n+1}\cos(n-1)\theta}{1-2r\cos\theta+r^2},$$

so that
$$|R_n|<\rho^n(1+\rho)/(1-\rho)^2, \qquad \text{if } \rho=|r|.$$

Hence, as for the geometrical progression (Art. 6), we see that $\lim R_n=0$, if $-1<r<1$, and accordingly the first equation holds for the interval $(-1, 1)$. And the other equations can be extended similarly.

Again we have

$$\frac{d}{dr}\log(1-2r\cos\theta+r^2)=-\frac{2(\cos\theta-r)}{1-2r\cos\theta+r^2}$$
$$=-2(\cos\theta+r\cos 2\theta+r^2\cos 3\theta+\ldots)$$

by what has been proved.

Hence, integrating,* we have

$$\log(1-2r\cos\theta+r^2)=-2(r\cos\theta+\tfrac{1}{2}r^2\cos 2\theta+\tfrac{1}{3}r^3\cos 3\theta+\ldots),$$

no constant being needed because both sides tend to zero as $r\to 0$.

It may be noted that the same result is found by integrating the sine-series with respect to θ.

Also we have

$$\int_0^r\frac{\sin\theta\,dt}{1-2t\cos\theta+t^2}=\int_0^r\frac{\sin\theta\,dt}{(t-\cos\theta)^2+\sin^2\theta}=\left[\text{arc tan}\left(\frac{t-\cos\theta}{\sin\theta}\right)\right]_0^r$$
$$=\text{arc tan}\left\{\frac{(r-\cos\theta)\sin\theta-(0-\cos\theta)\sin\theta}{\sin^2\theta+(r-\cos\theta)(0-\cos\theta)}\right\}=\text{arc tan}\left(\frac{r\sin\theta}{1-r\cos\theta}\right).$$

Thus we find

$$\text{arc tan}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)=\int_0^r\frac{\sin\theta\,dt}{1-2t\cos\theta+t^2}$$
$$=\int_0^r dt\,(\sin\theta+t\sin 2\theta+t^2\sin 3\theta+\ldots)$$
$$=r\sin\theta+\tfrac{1}{2}r^2\sin 2\theta+\tfrac{1}{3}r^3\sin 3\theta+\ldots.$$

* Term-by-term integration is permissible because, if $|r|\leq k<1$, the series may be compared with $1+k+k^2+k^3+\ldots$, and Weierstrass's M-test can be applied.

We have only established the equations above on the hypothesis that $-1 < r < 1$; but we know from Art. 22 that the two series

$$\cos\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta + \dots$$
$$\sin\theta + \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta + \dots$$

are convergent, except the first for $\theta = 0$ or $2k\pi$.

Thus, by Abel's theorem (Art. 51), we have

$$\cos\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta + \dots = \lim_{r\to 1}\{-\tfrac{1}{2}\log(1-2r\cos\theta+r^2)\}$$
$$= -\tfrac{1}{2}\log(4\sin^2\tfrac{1}{2}\theta);$$

and when $0 < \theta < 2\pi$, this result can be written

$$\cos\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta + \dots = -\log(2\sin\tfrac{1}{2}\theta).$$

We find, similarly,

$$\cos\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta - \dots = \lim_{r\to 1}\tfrac{1}{2}\log(1+2r\cos\theta+r^2)$$
$$= \tfrac{1}{2}\log(4\cos^2\tfrac{1}{2}\theta);$$

which can be written as $\log(2\cos\tfrac{1}{2}\theta)$, if $-\pi < \theta < \pi$, although a fresh investigation is not necessary, because the last series can be deduced from the preceding by changing from θ to $\pi + \theta$.

In like manner we find the result

$$\sin\theta + \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta + \dots = \lim_{r\to 1}\arc\tan\left(\frac{r\sin\theta}{1-r\cos\theta}\right).$$

Now, from the figure it is evident that the angle in question is the angle ϕ, which, according to the definition of the arc tan function,

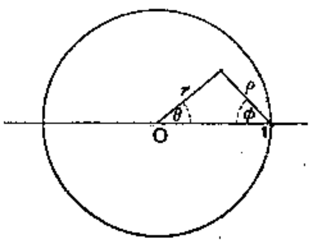

FIG. 18.

must lie between $-\tfrac{1}{2}\pi$ and $+\tfrac{1}{2}\pi$; so that $\lim\phi = \tfrac{1}{2}(\pi-\theta)$, when $0 < \theta < \pi$; and when $\pi \leqq \theta < 2\pi$, we readily find that the same formula applies, by drawing a fresh figure with θ between π and 2π.

Thus $\quad \sin\theta + \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta + \dots = \tfrac{1}{2}(\pi-\theta), \quad$ if $0 < \theta < 2\pi$.

But if $\theta=0$, or 2π, the value of the series is 0 because each term in it vanishes; thus the series is discontinuous * at $\theta=0$ and 2π.

If θ lies between $2k\pi$ and $2(k+1)\pi$, where k is an integer, positive or negative, we have

$$\sin\theta+\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta+\ldots=\tfrac{1}{2}\{\pi-(\theta-2k\pi)\}=\tfrac{1}{2}\{(2k+1)\pi-\theta\}.$$

These results have all been obtained without the use of the complex variable; although, as a matter of fact, they could be established more quickly † by assuming certain results obtained in Ch. X. below.

Thus, for instance, we have the equation

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots \text{ to } \infty,$$

provided that $|x|<1$.

If now $\qquad x=r(\cos\theta+\iota\sin\theta)$,

$$\frac{1}{1-x}=\frac{1-r\cos\theta+\iota r\sin\theta}{(1-r\cos\theta)^2+(r\sin\theta)^2}=\frac{1-r\cos\theta+\iota r\sin\theta}{1-2r\cos\theta+r^2}.$$

Thus, on taking the real and imaginary parts, we obtain the same series for

$$\frac{1-r\cos\theta}{1-2r\cos\theta+r^2} \quad\text{and}\quad \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

as were found in the previous work.

Similarly, by taking for granted the logarithmic series (Art. 95), we can obtain the series for $\log(1-2r\cos\theta+r^2)$ and $\arctan\left(\dfrac{r\sin\theta}{1-r\cos\theta}\right)$; but it will be recognised that a complete discussion on the lines of Arts. 94-96 is at the bottom more troublesome than the direct discussion given here.

EXAMPLES ‡ A.

Differentiation and Integration.

1. Justify the equation

$$\frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}-\frac{1}{a+3b}+\ldots=\int_0^1\frac{t^{a-1}}{1+t^b}\,dt \quad (a,b>0).$$

Thus the series can be found in finite terms if b/a is rational. [GAUSS.]

Deduce that

$$\left.\begin{aligned} 1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\ldots&=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}+\log 2\right),\\ \frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\ldots&=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}-\log 2\right),\end{aligned}\right\}\ \text{compare Ex. 5, Ch. IV.}$$

* Hence these are points of non-uniform convergence for the series (Art. 45); a result proved directly in Art. 44·1.

† The saving is more apparent than real, as at the bottom it depends only on rearranging the algebra. On the other hand, it is probably easier to remember the results when expressed in terms of the complex variable.

‡ In a number of these examples, the word *expansion* is used as equivalent to *power-series*; and in some cases the words *for sufficiently small values of x* are implied.

2. Prove, as in Ex. 1, that

$$1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\ldots=\frac{1}{4\sqrt{2}}\{\pi+2\log(\sqrt{2}+1)\}.$$

[*Math. Trip.* 1896.]

3. Shew that if k and n are positive integers

$$\frac{k!}{(a+nb)(a+nb+1)\ldots(a+nb+k)}=\int_0^1 t^{a+nb-1}(1-t)^k\,dt \quad (a, b>0).$$

Deduce that $$\sum_0^\infty \frac{(k!)x^n}{(a+nb)\ldots(a+nb+k)}=\int_0^1 \frac{t^{a-1}(1-t)^k}{1-xt^b}\,dt.$$

4. Prove, from Ex. 3, that

$$\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ldots=\log 2-\tfrac{1}{2},$$

$$\frac{1}{1.2.3}-\frac{1}{3.4.5}+\frac{1}{5.6.7}-\ldots=\tfrac{1}{2}(1-\log 2),$$

$$\frac{1}{2.3.4}-\frac{1}{4.5.6}+\frac{1}{6.7.8}-\ldots=\tfrac{1}{4}(\pi-3).$$

[These results are readily deducible also by rearrangement from the known series for log 2 and $\frac{1}{4}\pi$ (Arts. 62, 64). Thus the third series is

$$\frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)-\frac{1}{2}\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right)+\frac{1}{2}\left(\frac{1}{6}-\frac{2}{7}+\frac{1}{8}\right)-\ldots=\frac{1}{2}\left(\frac{\pi}{2}-\frac{3}{2}\right).]$$

5. Shew that if $\alpha>0,\ \gamma-\alpha>0$,

$$F(\alpha, \beta, \gamma, x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}\,dt,$$

and deduce that if also $\gamma-\alpha-\beta>0$,

$$F(\alpha, \beta, \gamma, 1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.$$

[The investigation of this result given in Ex. 16, Ch. VI., is better, however; because the only restriction required is $\gamma-\alpha-\beta>0$, which is necessary in order that the series $F(\alpha, \beta, \gamma, 1)$ may converge (Art. 12·2).]

6. From Ex. 16, Ch. VI., prove that

$$1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2.4}\right)^2+\left(\frac{1.3}{2.4.6}\right)^2+\left(\frac{1.3.5}{2.4.6.8}\right)^2+\ldots=\frac{\Gamma(2)}{\{\Gamma(\frac{3}{2})\}^2}=\frac{4}{\pi}.$$

[Write $\alpha=-\frac{1}{2},\ \beta=-\frac{1}{2},\ \gamma=1$. Note that Ex. 5 will not apply here.]

7. The complete elliptic integrals are

$$\left.\begin{aligned} K&=\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1.3}{2.4}\right)^2k^4+\ldots\right],\\ E&=\int_0^{\frac{1}{2}\pi}\sqrt{(1-k^2\sin^2\phi)}\,d\phi=\frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2\frac{k^2}{1}-\left(\frac{1.3}{2.4}\right)^2\frac{k^4}{3}-\ldots\right],\end{aligned}\right\}\ 0<k<1.$$

8. Prove that

$$\frac{x}{\log[1/(1-x)]}=\int_0^1(1-x)^t\,dt=1-\tfrac{1}{2}x-\tfrac{1}{12}x^2-\tfrac{1}{24}x^3-\ldots.$$

9. Shew that, if $|x| < 1$,

$$\log[\tfrac{1}{2}\{1+\sqrt{(1+x)}\}] = \frac{1}{2}\frac{x}{2} - \frac{1\,.\,3}{2\,.\,4}\frac{x^2}{4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{x^3}{6} - \dots.$$

[Differentiate, and the result follows by integrating the series for $(1+x)^{-\frac{1}{2}}$.]

10. Multiply the expansions of $(1-x)^{-1}$ and $\log(1-x)$, and deduce by integration that

$$\tfrac{1}{2}\{\log(1-x)\}^2 = \tfrac{1}{2}x^2 + \tfrac{1}{3}(1+\tfrac{1}{2})x^3 + \tfrac{1}{4}(1+\tfrac{1}{2}+\tfrac{1}{3})x^4 + \dots.$$

Prove that the result remains true for $x = -1$.

[Compare Art. 62 for another method; and also Art. 34, Ex. 2.]

11. Prove that, if $|x| < 1$,

$$\frac{1}{2}(\text{arc tan } x)^2 = \frac{x^2}{2} - \left(1+\frac{1}{3}\right)\frac{x^4}{4} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6} - \dots.$$

Shew that the result is true for $x=1$.

12. Prove that

$$-\log(1+x)\,.\,\log(1-x) = x^2 + \left(1-\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{2} + \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right)\frac{x^6}{3} + \dots.$$

[By direct multiplication, or by expanding the differential coefficient

$$(1-x)^{-1}\log(1+x) - (1+x)^{-1}\log(1-x).]$$

13. Prove that

$$\sqrt{(1+x^2)}\log\{x+\sqrt{(1+x^2)}\} = x + \frac{x^3}{3} - \frac{2}{3}\frac{x^5}{5} + \frac{2\,.\,4}{3\,.\,5}\frac{x^7}{7} - \dots.$$

Shew also that

$$\sqrt{(1-x^2)}\text{ arc sin } x = x - \frac{x^3}{3} - \frac{2}{3}\frac{x^5}{5} - \frac{2\,.\,4}{3\,.\,5}\frac{x^7}{7} - \dots.$$

Prove that both equations remain valid for $x = \pm 1$.

[For the first result, use the equation $(1+x^2)\left(\dfrac{du}{dx}-1\right) = xu$, and find a similar equation for the second.]

14. If $y = (1+x)^{-n}\log(1+x)$, shew that

$$(1+x)\frac{dy}{dx} + ny = (1+x)^{-n}.$$

Deduce the expansion

$$y = x - n(n+1)\left(\frac{1}{n}+\frac{1}{n+1}\right)\frac{x^2}{2!} + n(n+1)(n+2)\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}\right)\frac{x^3}{3!} - \dots.$$

15. Prove that

$$\tfrac{1}{2}(\text{arc tan } x)\log\frac{1+x}{1-x} = x^2 + \left(1-\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{3} + \left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\right)\frac{x^{10}}{5} + \dots.$$

[It is easy to see that

$$(1-x^4)\frac{dy}{dx} = (1-x^2)(x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\dots) + (1+x^2)(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\dots)$$

$$= 2\{x - (\tfrac{1}{3}-\tfrac{1}{5})x^5 - (\tfrac{1}{7}-\tfrac{1}{9})x^9 - \dots\}.]$$

16. Verify that

$$\tfrac{1}{2}(\text{arc tan } x)\,.\,\log(1+x^2) = S_2\frac{x^3}{3} - S_4\frac{x^5}{5} + S_6\frac{x^7}{7} - \dots,$$

where $$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n};$$

and deduce that $$\frac{\pi}{8}\log 2 = \frac{1}{3}S_2 - \frac{1}{5}S_4 + \frac{1}{7}S_6 - \dots . \qquad \text{[\textit{Math. Trip.} 1897.]}$$

[Here $$(1+x^2)\frac{dy}{dx} = x(\text{arc tan } x) + \tfrac{1}{2}\log(1+x^2)$$
$$= S_2x^2 - (S_4 - S_2)x^4 + (S_6 - S_4)x^6 - \dots .$$

For the second part, use Abel's theorem.]

Derangement of Expansions.

17. Verify the series in Exs. 8, 9, above, as far as the first four terms.

18. Determine the first three terms in the expansion of
$$x^2/\{x - \log(1+x)\}.$$

19. Expand exp (arc tan x) up to the term which contains x^5.

[*Math. Trip.* 1899.]

[We can easily expand by direct algebra; or we may note that the function satisfies the equation $(1+x^2)\dfrac{dy}{dx} = y$, and so, if $y = 1 + \Sigma a_n x^n/n!$,

we find that $a_1 = 1, \quad a_{n+1} = a_n - n(n-1)a_{n-1}$.

This gives $a_2 = 1, \quad a_3 = -1, \quad a_4 = -7, \dots$.

The possibility of the expansion follows from Art. 36.]

20. Expand $$(1+x)^{\frac{1}{x}} = \exp(1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \tfrac{1}{4}x^3 + \dots)$$
up to and including the term in x^4.

[The first three terms are $e(1 - \frac{1}{2}x + \frac{11}{24}x^2)$; the possibility of the expansion follows from Art. 36.]

21. Shew that, if $|x| < 1$,
$$[\tfrac{1}{2}\{1+\sqrt{(1+x)}\}]^p = 1 + p\left(\frac{x}{4}\right) + \frac{p(p-3)}{2!}\left(\frac{x}{4}\right)^2 + \frac{p(p-4)(p-5)}{3!}\left(\frac{x}{4}\right)^3 + \dots .$$

[*Math. Trip.* 1902.]

Verify the result for $p=3$, $p=4$ by direct expansion.

[As far as the terms in x^3, the result can be checked by noticing that
$$\frac{du}{dx} = \frac{pu}{2x}\{1 - (1+x)^{-\frac{1}{2}}\} = \frac{pu}{4}(1 - \tfrac{3}{4}x + \tfrac{5}{8}x^2 - \dots).$$

For the general term, note that this is the expansion of $(1+y)^p$, where $x = 4y(1+y)$; then use Lagrange's series, as in Ex. B. 16.]

22. If $$v = a_0 + a_1t + a_2t^2 + a_3t^3 + \dots ,$$
prove that the first and second terms in the expansions of
$$\frac{1}{v-a_0} - \frac{1}{a_1t} \quad \text{and} \quad \frac{1}{v-a_0} - \frac{1}{t}\frac{dt}{dv}$$
are respectively
$$\frac{1}{a_1}\left[-\frac{a_2}{a_1} + \left\{\left(\frac{a_2}{a_1}\right)^2 - \frac{a_3}{a_1}\right\}t\right] \quad \text{and} \quad \frac{1}{a_1}\left[\frac{a_2}{a_1} + \left\{\frac{2a_3}{a_1} - 3\left(\frac{a_2}{a_1}\right)^2\right\}t\right].$$

23. Apply the first series of Ex. 22 to prove that if

$$f(x)=\frac{1}{\phi(x)-\phi(\alpha)}-\frac{1}{(x-\alpha)\phi'(\alpha)} \quad \text{and} \quad f(\alpha)=\lim_{x\to\alpha} f(x),$$

then $$f(\alpha)=-\frac{\phi_2}{2\phi_1^2}, \quad \lim_{x\to\alpha} f'(x)=\frac{\phi_2^2}{4\phi_1^3}-\frac{\phi_3}{6\phi_1^2},$$

where $$\phi_1=\phi'(\alpha), \quad \phi_2=\phi''(\alpha), \quad \phi_3=\phi'''(\alpha).$$

Shew also that $$\frac{d}{d\alpha}\{f(\alpha)\}-\lim_{x\to\alpha} f'(x)=\frac{3\phi_2^2}{4\phi_1^3}-\frac{\phi_3}{3\phi_1^2}$$

and explain how this result follows from the second series of Ex. 22.

[Write $x=\alpha+t$, $v=\phi(x)$.]

24. Apply the last example to prove that if

$$f(x)=(\sin x-\sin\alpha)^{-1}-\{(x-\alpha)\cos\alpha\}^{-1}, \quad \text{and} \quad f(\alpha)=\lim_{x\to\alpha} f(x),$$

then $$\frac{d}{d\alpha}\{f(\alpha)\}-\lim_{x\to\alpha} f'(x)=\tfrac{3}{4}\sec^3\alpha-\tfrac{5}{12}\sec\alpha.$$

[*Math. Trip.* 1896.]

25. Prove that the coefficient of r^n in the expansion of $(1-2\mu r+r^2)^{-\frac{1}{2}}$ is

$$\frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{n!}\left\{\mu^n-\frac{n(n-1)}{2(2n-1)}\mu^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\,.\,4(2n-1)(2n-3)}\mu^{n-4}-\ldots\right\},$$

the number of terms being either $\frac{1}{2}(n+1)$ or $\frac{1}{2}n+1$.

[This is Legendre's polynomial $P_n(\mu)$.]

26. Prove that the coefficient of x^{2n} in the expansion of $(1+2px-qx^2)^{-m}$ is

$$\frac{m(m+1)\ldots(m+n-1)}{n!}q^n\left[1+\frac{n(m+n)}{2!}P+\frac{n(n-1)(m+n)(m+n+1)}{4!}P^2+\ldots\right],$$

where $P=4p^2/q$.

Shew that it is a multiple of the coefficient of t^n in the expansion of

$$\frac{1}{(1-qt)^{\frac{1}{2}}}+(n+m)\frac{p^2t}{(1-qt)^{\frac{3}{2}}}+\frac{(n+m)(n+m+1)}{2!}\frac{p^4t^2}{(1-qt)^{\frac{5}{2}}}+\ldots$$

$$=\frac{1}{\sqrt{(1-qt)}}\left(1-\frac{p^2t}{1-qt}\right)^{-(n+m)}$$

[Set as an example in differentiation. *Math. Trip.* 1898.]

27. If $[(1-xy)(1-xy^2)(1-x/y)(1-x/y^2)]^{-1}$ is expanded in powers of x, the part of the expansion which is independent of y is equal to

$$(1+x^3)/(1-x^3)(1-x^2)^2.$$ [*Math. Trip.* 1903.]

[If we expand $\{(1-xy)(1-x/y)\}^{-1}$, we obtain

$$1+x(y+1/y)+x^2(y^2+1+1/y^2)+x^3(y^3+y+1/y+1/y^3)+\ldots$$
$$=(1-x^2)^{-1}\{1+x(y+1/y)+x^2(y^2+1/y^2)+\ldots\}.$$

It is then easy to pick out the specified terms in the form

$$(1-x^2)^{-2}(1+2x^3+2x^6+\ldots).]$$

28. Prove that $$\frac{1}{(1-x)(1-y)-\lambda xy}=\Sigma\Sigma F(-m,\ -n,\ 1,\ \lambda)x^my^n,$$

where $F(-m,\ -n,\ 1,\ \lambda)$ is a (terminated) hypergeometric series. [Hardy.]

29. Prove that $\sum_0^\infty \sum_0^\infty \frac{(m+n)!}{m!\,n!}\left(\frac{x}{2}\right)^{m+n} = \frac{1}{1-x}$, if $-2 < x < 1$,

and $$\sum_0^\infty \sum_0^\infty \sum_0^\infty \frac{(m+n+p)!}{m!\,n!\,p!}\left(\frac{x}{3}\right)^{m+n+p} = \frac{1}{1-x}, \quad \text{if } -3 < x < 1,$$

and extend to any number of indices of summation. [*Math. Trip.* 1903.]

30. Shew that the following series are absolutely convergent, and by summing with respect to r first, deduce that their values are as stated:

$$\sum_{r=2}^\infty \sum_{s=2}^\infty \frac{1}{(p+s)^r} = \frac{1}{p+1}, \qquad \sum_{r=2}^\infty \sum_{s=1}^\infty \frac{1}{(2s)^r} = \log 2,$$

$$\sum_{r=1}^\infty \sum_{s=1}^\infty \frac{1}{(4s-1)^{2r+1}} = \frac{\pi}{8} - \frac{1}{2}\log 2, \qquad \sum_{r=1}^\infty \sum_{s=1}^\infty \frac{1}{(4s-1)^{2r}} = \frac{1}{4}\log 2,$$

$$\sum_{r=1}^\infty \sum_{s=1}^\infty \frac{1}{(4s-2)^{2r}} = \frac{\pi}{8}.$$ [STERN.*]

Special Series.

31. If $$f(x,y) = \frac{1}{x} - y\frac{1}{x+1} + \frac{y(y-1)}{2!}\frac{1}{x+2} - \frac{y(y-1)(y-2)}{3!}\frac{1}{x+3} + \dots,$$

shew that the series $f(x, y)$ converges if $y+1$ is positive; and if x is also positive, prove that the series is equal to $f(y+1, x-1)$.

[If $x > 0$ and $y > 0$, we can prove by Art. 45 that $f(x, y)$ is equal to $\int_0^1 t^{x-1}(1-t)^y dt$; and this can be extended to cover the case $0 > y > -1$, by Art. 175. Change the variable from t to $1-t$ to get the final result.]

32. If $$(1+bx+ax^2)^{-1} = 1 + p_1 x + p_2 x^2 + p_3 x^3 + \dots,$$

then $$1 + p_1^2 x + p_2^2 x^2 + p_3^2 x^3 + \dots = \frac{1+ax}{(1-ax)\{(1+ax)^2 - b^2 x\}}.$$

[*Math. Trip.* 1900.]

33. Shew that the sum of the squares of the coefficients in the binomial power-series is

$$\frac{\Gamma(1+2\nu)}{[\Gamma(1+\nu)]^2}, \quad \text{if } \nu > -\tfrac{1}{2}.$$ [*Math. Trip.* 1890.]

[Put $\alpha = \beta = -\nu$, $\gamma = 1$ in Ex. 16, Ch. VI.; Ex. 5 above will apply if ν is negative.]

34. Prove that

$$\log(1+x) = x(1-x) + \tfrac{1}{2}x^2(1-x^2) + \tfrac{1}{3}x^3(1-x^3) + \dots.$$

Shew that this equation fails for $x = 1$.

35. Shew that at r per cent., compound interest, a capital will increase to A times its original value in n years, approximately, where

$$n = \frac{\log A}{x}\left(1 + \frac{x}{2} - \frac{x^2}{12} + \dots\right), \quad x = \frac{r}{100}.$$

* See Dirichlet's *Vorlesungen über bestimmte Integrale* (ed. Meyer), § 117.

In particular, the capital will be doubled in the time given by the approximation

$$n = \frac{69 \cdot 3}{r} + \cdot 35,$$

and increased by half its original value when $n = \frac{40 \cdot 5}{r} + \cdot 20$.

36. If

$$\sum_0^\infty \frac{(n+a)^s}{n!} x^n = f_s(x) e^x,$$

prove that $f_s(x)$ is a polynomial of degree s in x which satisfies the equation

$$f_{s+1} = (x+a) f_s + x \frac{df_s}{dx}.$$

Shew that $f_1 = x + a$, $f_2 = (x+a)^2 + x$, $f_3 = (x+a)^3 + 3x(x+a) + x$, and that if a is positive all the roots of $f_s(x) = 0$ are real and negative, and that they are separated by the roots of $f_{s-1}(x) = 0$. [HARDY; and *Math. Trip.* 1902.]

Trigonometrical Series.

37. Prove that the series

$$\cos\theta \cdot \sin\theta + \tfrac{1}{2}\cos^2\theta \cdot \sin 2\theta + \tfrac{1}{3}\cos^3\theta \cdot \sin 3\theta + \dots$$

is convergent and is equal to $\frac{1}{2}\pi - \theta$, when θ lies between 0 and π.

[Put $r = \cos\theta$ in Art. 65.]

38. Shew that $2\sin(\frac{1}{7}\pi)$ is approximately equal to $\frac{1}{2}\sqrt{3}$, the error being about $\frac{1}{5}$th per cent. [See Ex. 6 (c), Ch. IX.]

Deduce that the side of a regular heptagon inscribed in a circle is nearly equal to the height of an equilateral triangle whose side is equal to the radius.

39. If

$$\frac{\tan y}{\tan x} = \frac{1+\lambda}{1-\lambda}, \quad \text{where } |\lambda| < 1,$$

deduce from Art. 65 that

$$y - x = \lambda \sin 2x + \tfrac{1}{2}\lambda^2 \sin 4x + \tfrac{1}{3}\lambda^3 \sin 6x + \dots .$$

40. By expansion in series (Art. 65) and term-by-term integration (Art. 45), obtain the following definite integrals:

$$\int_0^\pi \log(1 - 2r\cos\theta + r^2)\cos n\theta \, d\theta = -\pi r^n / n,$$

$$\int_0^\pi \text{arc tan}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right)\sin n\theta \, d\theta = \frac{\pi}{2}\frac{r^n}{n},$$

$$\int_0^\pi \frac{\cos n\theta}{1 - 2r\cos\theta + r^2} d\theta = \frac{\pi r^n}{1 - r^2}; \qquad \int_0^\pi \frac{\sin n\theta \sin\theta}{1 - 2r\cos\theta + r^2} d\theta = \frac{\pi}{2} r^{n-1}.$$

Here n is an integer and r lies between 0 and 1.

41. Shew that, if $|r| < 1$,

$$r\cos\theta + \tfrac{1}{3}r^3\cos 3\theta + \tfrac{1}{5}r^5\cos 5\theta + \dots = \tfrac{1}{4}\log\left(\frac{1 + 2r\cos\theta + r^2}{1 - 2r\cos\theta + r^2}\right),$$

$$r\sin\theta + \tfrac{1}{3}r^3\sin 3\theta + \tfrac{1}{5}r^5\sin 5\theta + \dots = \tfrac{1}{2}\,\text{arc tan}\left(\frac{2r\sin\theta}{1 - r^2}\right),$$

$$r\sin\theta - \tfrac{1}{3}r^3\sin 3\theta + \tfrac{1}{5}r^5\sin 5\theta - \dots = \tfrac{1}{4}\log\left(\frac{1 + 2r\sin\theta + r^2}{1 - 2r\sin\theta + r^2}\right),$$

$$r\cos\theta - \tfrac{1}{3}r^3\cos 3\theta + \tfrac{1}{5}r^5\cos 5\theta - \dots = \tfrac{1}{2}\,\text{arc tan}\left(\frac{2r\cos\theta}{1 - r^2}\right).$$

42. Series for π.

Since
$$\frac{1}{4}\pi = \operatorname{arc} \tan \frac{1}{2} + \operatorname{arc} \tan \frac{1}{3},$$

we find
$$\frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \dots\right) + \left(\frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \dots\right)$$

and
$$\frac{\pi}{4} = \frac{4}{10}\left\{1 + \frac{2}{3}\frac{2}{10} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{2}{10}\right)^2 + \dots\right\} + \frac{3}{10}\left\{1 + \frac{2}{3}\frac{1}{10} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{1}{10}\right)^2 + \dots\right\};$$

both of these results are due to Euler.

In writing these series down, we note that Ex. B. 2 below may be put in the form
$$\operatorname{arc} \tan \frac{m}{n} = \frac{mn}{m^2 + n^2}\left\{1 + \frac{2}{3}\frac{m^2}{m^2 + n^2} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{m^2}{m^2 + n^2}\right)^2 + \dots\right\}.$$

Using the fact that $\operatorname{arc} \tan \frac{1}{2} = \operatorname{arc} \tan \frac{1}{3} + \operatorname{arc} \tan \frac{1}{7}$, Clausen gave the identity $\frac{1}{4}\pi = 2 \operatorname{arc} \tan \frac{1}{3} + \operatorname{arc} \tan \frac{1}{7}$, which leads to Hutton's series
$$\frac{\pi}{4} = \frac{6}{10}\left\{1 + \frac{2}{3}\frac{1}{10} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{1}{10}\right)^2 + \dots\right\} + \frac{14}{100}\left\{1 + \frac{2}{3}\frac{2}{100} + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{2}{100}\right)^2 + \dots\right\}.$$

Again, from $\operatorname{arc} \tan \frac{1}{3} = 2 \operatorname{arc} \tan \frac{1}{7} + \operatorname{arc} \tan \frac{3}{79}$, Euler obtained the result
$$\frac{1}{4}\pi = 5 \operatorname{arc} \tan \frac{1}{7} + 2 \operatorname{arc} \tan \frac{3}{79},$$

which leads to the highly convergent series
$$\frac{\pi}{4} = \frac{7}{10}\left\{1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{2}{100}\right)^2 + \dots\right\} + \frac{7584}{10^5}\left\{1 + \frac{2}{3}\left(\frac{144}{10^5}\right) + \frac{2 \cdot 4}{3 \cdot 5}\left(\frac{144}{10^5}\right)^2 + \dots\right\}.$$

43. The various transformations of the formulae in Ex. 42 are special cases of the identity given by the late Mr. C. L. Dodgson (Lewis Carroll),
$$\operatorname{arc} \tan \frac{1}{p} = \operatorname{arc} \tan \frac{1}{p+q} + \operatorname{arc} \tan \frac{1}{p+r},$$

provided that
$$qr = 1 + p^2.$$

EXAMPLES B.

Euler's Transformation.

1. Shew by the same method as in Art. 24, that
$$a_1 x + a_2 x^3 + a_3 x^5 + a_4 x^7 + \dots = \sqrt{(1 + y^2)}\{a_1 y - (Da_1) y^3 + (D^2 a_1) y^5 - \dots\},$$

if
$$x = y/\sqrt{(1 + y^2)}, \quad \text{or} \quad y = x/\sqrt{(1 - x^2)}.$$

Similarly, prove that
$$a_1 x - a_2 x^3 + a_3 x^5 - a_4 x^7 + \dots = \sqrt{(1 - y^2)}[a_1 y + (Da_1) y^3 + (D^2 a_1) y^5 + \dots],$$

if
$$x = y/\sqrt{(1 - y^2)}, \quad \text{or} \quad y = x/\sqrt{(1 + x^2)}.$$ [EULER.]

2. By taking $a_1=1$, $a_2=\frac{1}{3}$, $a_3=\frac{1}{5}$, ... in Ex. 1, shew that if $|y|<1$,

$$\frac{1}{\sqrt{(1+y^2)}}\log\{y+\sqrt{(1+y^2)}\}=y-\frac{2}{3}y^3+\frac{2.4}{3.5}y^5-\dots,$$

$$\frac{1}{\sqrt{(1-y^2)}}\text{arc sin } y=y+\frac{2}{3}y^3+\frac{2.4}{3.5}y^5+\dots. \qquad \text{[EULER.]}$$

Prove that we may put $y=1$ in the first series.

3. Deduce from Ex. 2, that

$$\frac{2}{\sqrt{3}}\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)=1-\frac{2}{3}\frac{1}{2}+\frac{2.4}{3.5}\frac{1}{2^2}-\frac{2.4.6}{3.5.7}\frac{1}{2^3}+\dots,$$

$$\frac{\pi}{2}=1+\frac{2}{3}\frac{1}{2}+\frac{2.4}{3.5}\frac{1}{2^2}+\frac{2.4.6}{3.5.7}\frac{1}{2^3}+\dots. \qquad \text{[EULER.]}$$

4. By integrating the formulae of Ex. 2 above, prove that if $|y|<1$,

$$\tfrac{1}{2}[\log\{y+\sqrt{(1+y^2)}\}]^2=\frac{y^2}{2}-\frac{2}{3}\frac{y^4}{4}+\frac{2.4}{3.5}\frac{y^6}{6}-\dots,$$

$$\tfrac{1}{2}(\text{arc sin } y)^2=\frac{y^2}{2}+\frac{2}{3}\frac{y^4}{4}+\frac{2.4}{3.5}\frac{y^6}{6}+\dots.$$

Are these equations valid for $y=1$? [*Math. Trip.* 1897 and 1905.]

5. By a method similar to Ex. 1, shew that if

$$f(x)=b_0+b_1x+b_2x^2+\dots,$$

then $$a_0b_0+a_1b_1x+a_2b_2x^2+\dots=a_0f(x)-(Da_0)x\frac{df}{dx}+(D^2a_0)\frac{x^2}{2!}\frac{d^2f}{dx^2}-\dots.$$

[EULER.]

6. In particular, if $a_n=n^3$, we find

$$a_0=0,\quad Da_0=-1,\quad D^2a_0=6,\quad D^3a_0=-6,\quad D^4a_0=0,$$

and for $a_n=n^4$,

$$a_0=0,\quad Da_0=-1,\quad D^2a_0=14,\quad D^3a_0=-36,\quad D^4a_0=-24,\quad D^5a_0=0.$$

Thus, from Ex. 5, $$\sum\frac{n^3}{n!}x^n=(x+3x^2+x^3)e^x.$$

Similarly, $$\sum\frac{n^4}{n!}x^n=(x+7x^2+6x^3+x^4)e^x.$$

[Compare also Ex. A. 36 above.]

7. If $$S_r=\sum_1^\infty\frac{n^r}{n!},$$

S_r is an integral multiple of e, and in particular

$$S_1=e,\quad S_2=2e,\quad S_3=5e,\quad S_4=15e,\quad S_5=52e,$$
$$S_6=203e,\quad S_7=877e,\quad S_8=4140e. \qquad \text{[WOLSTENHOLME.]}$$

[These results can be deduced as in Ex. 6 or Ex. A. 36.]

8. From Ex. 5 above, prove that if

$$S_n=1^3+2^3+3^3+\dots+n^3,$$

$$\sum_1^\infty S_n\frac{x^n}{n!}=\frac{1}{4}xe^x(x^3+8x^2+14x+4),$$

and that $$\sum_1^\infty(-1)^{n-1}S_n\frac{2^n}{n!}=0. \qquad \text{[}\textit{Math. Trip.}\text{ 1904.]}$$

9. Shew, by taking $f(x)=(1-x)^{-p}$ in Ex. 5, that

$$px+\frac{p(p+1)}{2!}2^3x^2+\frac{p(p+1)(p+2)}{3!}3^3x^3+\dots$$
$$=\frac{px}{(1-x)^{p+3}}\{1+(3p+1)x+p^2x^2\}.$$

Obtain this result also by differentiating the series for $(1-x)^{-p}$.

10. Apply Euler's method to prove that

$$\frac{1}{m}-\frac{x}{m+1}+\frac{x^2}{m+2}-\frac{x^3}{m+3}+\dots$$
$$=\frac{1}{m(1+x)}\left\{1+\frac{1}{m+1}\left(\frac{x}{1+x}\right)+\frac{1.2}{(m+1)(m+2)}\left(\frac{x}{1+x}\right)^2+\dots\right\}.$$

[This also follows from the identity $\int_0^1\frac{t^{m-1}dt}{1+xt}=\int_0^1\frac{t^{m-1}dt}{1+x-x(1-t)}$.]

11. Apply Euler's method to prove that

$$F(\alpha,\beta,\gamma,x)=\frac{1}{(1-x)^\alpha}F\left(\alpha,\gamma-\beta,\gamma,\frac{x}{x-1}\right),$$

where $$F(\alpha,\beta,\gamma,x)=1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\dots.$$ [GAUSS.]

Miscellaneous.

12. If $y=2x/(1+x^2)$ and $|x|$, $|y|$ are both less than 1, shew that

$$x=\frac{y}{2}+\frac{1}{2}\frac{y^3}{4}+\frac{1.3}{2.4}\frac{y^5}{6}+\frac{1.3.5}{2.4.6}\frac{y^7}{8}+\dots.$$

[Use $x=\{1-\sqrt{(1-y^2)}\}/y$; or apply Lagrange's series.]

13. If $y=2x/(1-x^2)$ and $z=x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\dots$,

shew that $$z=\frac{1}{2}\left(y-\frac{2}{3}\frac{y^3}{3}+\frac{2.4}{3.5}\frac{y^5}{5}-\dots\right).$$

[If $x=\tanh\frac{1}{2}\theta$, then $y=\sinh\theta$; and Ex. 2 above gives

$$1-\frac{2}{3}y^2+\frac{2.4}{3.5}y^4-\dots=\frac{\theta}{y\sqrt{(1+y^2)}}=\frac{\theta}{\sinh\theta\cosh\theta}.$$

If we multiply by $dy=\cosh\theta\,d\theta$ and integrate, the result follows on noting that

$$\int\frac{\theta\,d\theta}{\sinh\theta}=\int\frac{dx}{x}\log\left(\frac{1+x}{1-x}\right).]$$

14. From the expansion of $(1-\frac{1}{50})^{-\frac{1}{2}}$ determine the value of $\sqrt{2}$ to 12 decimal places. [1·414213562373.]

Obtain in the same way the cube root of 2 from the expansion of $(1+\frac{3}{125})^{\frac{1}{3}}$. [1·25992105.] [EULER.]

Lagrange's Series.

15. If $x=y(a+y)$, prove that

$$y=\frac{x}{a}-\frac{x^2}{a^3}+\frac{4}{1\,.\,2}\frac{x^3}{a^5}-\frac{5\,.\,6}{1\,.\,2\,.\,3}\frac{x^4}{a^7}+\ldots+(-1)^{n-1}\frac{(2n-2)!}{n!(n-1)!}\frac{x^n}{a^{2n-1}}+\ldots.$$

[Use Lagrange's series; or expand $\frac{1}{2}\{-a+\sqrt{(a^2+4x)}\}$.]

16. Use Lagrange's series to establish the equation

$$(1-x)^\nu=1-\nu t+\frac{\nu(\nu-3)}{2!}t^2-\frac{\nu(\nu-4)(\nu-5)}{3!}t^3+\ldots,$$

where $\quad t=x(1-x) \quad$ and $\quad |t|<\frac{1}{4}$.

Similarly obtain the results of Exs. A. 9, 21, and B. 12.

17. Prove that

$$\text{arc tan } x=t-\frac{4}{1}\frac{t^3}{3}+\frac{6\,.\,7}{1\,.\,2}\frac{t^5}{5}-\frac{8\,.\,9\,.\,10}{1\,.\,2\,.\,3}\frac{t^7}{7}+\ldots,$$

where $\quad t=x(1+x^2) \quad$ and $\quad |t|^2<\frac{4}{27}$.

18. Prove that the coefficient of x^{n-1} in the expansion of $[x/(e^x-1)]^n$ is $(-1)^{n-1}$. [Wolstenholme; and *Math. Trip.* 1904.]

Prove that the coefficient of x^{n-1} in the expansion of

$$(1+x)^{2n-1}(2+x)^{-n}$$

is $\frac{1}{2}$. [*Math. Trip.* 1906.]

[Use Lagrange's series (1) for $y=e^x-1$; and (2) for $\log(1-y)$, where

$$y=x(2+x)/(1+x)^2=1-1/(1+x)^2.]$$

19. If $f(x)$ is a power-series in x, whose lowest term is x, shew that the coefficient of $1/x$ in the expansion of $[1/f(x)]^n$, in ascending powers of x, is n times the coefficient of x^n in the expansion of $g(x)$, the function inverse to $f(x)$.

Determine the coefficient for the following forms of $f(x)$:

(1) $\sin x$; (2) $\tan x$; (3) $\log(1+x)$; (4) $1+x-\sqrt{(1+x^2)}$;
(5) $\sinh x$; (6) $\tanh x$. [Wolstenholme.]

[The results are:

(1) 0 or $\dfrac{1\,.\,3\,.\,5\ldots(n-2)}{2\,.\,4\,.\,6\ldots(n-1)}$; (2) 0 or $(-1)^{(n-1)/2}$; (3) $1/(n-1)!$;

(4) $\frac{1}{2}n$; (5) 0 or $(-1)^{(n-1)/2}\dfrac{1\,.\,3\,.\,5\ldots(n-2)}{2\,.\,4\,.\,6\ldots(n-1)}$; (6) 0 or 1.

The values 0 occur when n is even.]

20. Shew that

$$2^{n-1}+n\,.\,2^{n-2}+\frac{n(n+1)}{2!}2^{n-3}+\ldots+\frac{n(n+1)\ldots(2n-2)}{(n-1)!}=4^{n-1}.$$

[*Math. Trip.* 1903.]

[This is the coefficient of $1/x$ in the expansion of $\{x(1-x)\}^{-n}(1-2x)^{-1}$, and is therefore equal to na_n if

$$\Sigma a_n y^n=-\tfrac{1}{2}\log(1-2x), \text{ where } y=x(1-x).$$

Hence $\quad \Sigma a_n y^n=-\frac{1}{4}\log(1-4y)$, or $a_n=4^{n-1}/n$.]

An alternative way of stating the result is to say that *the sum of the first n terms in the binomial series for $(1-\frac{1}{2})^{-n}$ is equal to the remainder.*

21. Shew that, if n is a positive integer,

$$(t+a)^n = t^n + na(t+b)^{n-1} + \dots + n_r a(a-rb)^{r-1}(t+rb)^{n-r} + \dots + a(a-nb)^{n-1},$$

where n_r is the ordinary binomial coefficient

$$n(n-1)\dots(n-r+1)/r! \qquad \text{[ABEL.]}$$

[Take the result of Ex. 4, Art. 55·1, multiply by e^{tx}, and equate coefficients of $x^n/n!$.

Several authors have considered the validity of the equation, also due to Abel,

$$\phi(t+a) = \phi(t) + a\phi'(t+b) + \frac{a(a-2b)}{2!}\phi''(t+2b) + \dots,$$

but their results cannot be given here. We may remark, however, that the theorem fails if $\phi(t)$ is $\log t$ or a *negative* power of t. The most recent results are due to Pincherle (*Acta Math.*, Bd. 28, 1904, p. 225).]

22. Expand t^m and $\log t$ in powers of x, where

$$t^{-\beta} - t^{-\alpha} = (\alpha - \beta)x,$$

and determine the interval of convergence.

[Write $t^{\alpha-\beta} = 1 + y$ and apply Lagrange's series; or otherwise.]

23. Extend the method of Art. 55 to prove that if

$$y = a_2x^2 + a_3x^3 + a_4x^4 + \dots,$$

there are two expansions for x of the form

$$x_1 = b_1y^{\frac{1}{2}} + b_2y + b_3y^{\frac{3}{2}} + \dots, \quad x_2 = -b_1y^{\frac{1}{2}} + b_2y - b_3y^{\frac{3}{2}} + \dots.$$

Shew also that if $g(x) = c_0 + c_1x + c_2x^2 + \dots$,

$$g(x_1) + g(x_2) = 2c_0 + d_1y + d_2y^2 + d_3y^3 + \dots,$$

where nd_n is the coefficient of $1/x$ in the expansion of $g'(x)/y^n$.

24. As a particular case of the last example, shew that if

$$y(1 + ax + bx^2 + cx^3 + \dots) = x^2,$$

then
$$x_1 + x_2 = ay + (ab + c)y^2 + \dots.$$

25. If $y = x^2(1+x)^{-m}$, we find $x_1 + x_2 = \Sigma a_n y^n$,

where
$$a_1 = m, \quad a_2 = m(2m-1)(2m-2)/3!,$$
$$a_3 = m(3m-1)(3m-2)(3m-3)(3m-4)/5!, \text{ etc.}$$

26. It is easy to write down the general forms for the expansion of $x_1 + x_2$ in the following cases:

$$y = x^2 + ax^m; \quad y = x^2e^{ax}.$$

Theorems of Abel and Frobenius.

27. With the notation of Ex. 9, shew that

$$\lim_{x\to 1} \frac{F(\alpha, \beta, \gamma, x)}{(1-x)^{\gamma-\alpha-\beta}} = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \quad \text{if } \gamma < \alpha+\beta,$$

$$\lim_{x\to 1} \frac{F(\alpha, \beta, \gamma, x)}{\log[1/(1-x)]} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}, \quad \text{if } \gamma = \alpha+\beta. \qquad \text{[GAUSS.]}$$

28. If $\sum na_n$ is convergent, so also is $\sum a_n$ (Ex. 2, Ch. III.), and if $f(x) = \sum a_n x^n$, then $\sum na_n = \lim_{x\to 1} \{f(1) - f(x)\}/(1-x)$. [Stolz.]

[Note that $(1-x^n)/n(1-x)$ gives a decreasing sequence of factors, and apply Abel's theorem.]

29. If $v_n(x)$ decreases as n increases and $\lim_{x\to 1} v_n(x) = 1$, extend Art. 51 to prove that if $\sum a_n$ is convergent or divergent (to $+\infty$),

$$\lim_{x\to 1} \sum a_n v_n(x) = \sum a_n \quad \text{or} \quad \infty.$$

Also shew that if A_n/B_n tends to a definite limit l,

$$\lim_{x\to 1} \{\sum a_n v_n(x)\}/\{\sum b_n v_n(x)\} = l,$$

provided that B_n is always positive and that B_n tends to ∞.

30. If the coefficients a_n, b_n satisfy the conditions of Ex. 29, and $f_n(x)$ decreases as n increases (but is always positive), prove that $\sum a_n f_n(x)$ will converge provided that $\sum b_n f_n(x)$ does, if $\lim B_n f_n = 0$. Deduce that when $f_n(1) = 0$, and $\lim_{x\to 1} \sum b_n f_n(x) = \sigma > 0$, then $\lim_{x\to 1} \sum a_n f_n(x) = \sigma l$.

[Apply the lemma of Art. 148.]

31. If $Dv_n = v_n - v_{n+1}$, $D^2 v_n = v_n - 2v_{n+1} + v_{n+2}$, and $\lim_{n\to\infty} (nv_n) = 0$, shew that

$$v_0 = D^2 v_0 + 2D^2 v_1 + 3D^2 v_2 + \ldots .$$

Writing $f_n(x) = D^2 v_n$ in Ex. 30, shew that if $D^2 v_n$ is positive, and if

$$\lim_{n\to\infty} n^2 D^2 v_n = 0,$$

then the series $D^2 v_0 + 2D^2 v_2 + 3D^2 v_4 + \ldots$, $v_0 - 2v_1 + 3v_2 - 4v_3 + \ldots$ converge and are equal. If further v_n has the limit 1 as x tends to 1, then the last series has the limit of $\frac{1}{4}$ as x tends to 1.

32. Use the method of Ex. 31 to shew that if $D^2 v_n$ is positive, and if $\lim_{n\to\infty} nDv_n = 0$ and $\lim_{x\to 1} v_n = 1$, then $\lim_{x\to 1} (v_0 - v_1 + v_2 - v_3 + \ldots) = \frac{1}{2}$.

[For another method see Ex. 3, Art. 24.]

33. From Ex. 29, prove that

$$\lim_{x\to 1} \sum_1^\infty (-1)^{n-1} \frac{x^n}{n(1+x^n)} = \frac{1}{2}\log 2 = \lim_{x\to 1} \sum_1^\infty (-1)^{n-1} \frac{x^n(1-x)}{1-x^{2n}}.$$

34. Establish the asymptotic formulae (as $x \to 1$),

$$\sum_1^\infty \frac{x^n}{n(1+x^n)} \sim \frac{1}{2}\log\left(\frac{1}{1-x}\right) \quad \text{and} \quad \sum_1^\infty \frac{n(-x)^n}{1-x^{2n}} \sim -\frac{1}{4}\frac{1}{1-x}.$$

[The difference between the two sides of the first is less than $\frac{1}{2}$; in the second, multiply by $1-x$ and use Ex. 30.]

35. On the lines of Exs. 31–34 establish the following asymptotic formulae (as $x\to 1$):

$$\left(\frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \ldots\right) \sim \frac{1}{1-x}\log\left(\frac{1}{1-x}\right),$$

$$\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \ldots\right) \sim \frac{\pi}{4}\frac{1}{1-x}.$$

[Cesàro.

CHAPTER IX.

TRIGONOMETRICAL FORMULAE.

66. Expressions for $\cos n\theta$ **and** $(\sin n\theta/\sin\theta)$ **as polynomials in** $\cos\theta$.

We have seen (Art. 65) that

$$\log(1-2r\cos\theta+r^2)=-2(r\cos\theta+\tfrac{1}{2}r^2\cos 2\theta+\tfrac{1}{3}r^3\cos 3\theta+\ldots).$$

But (Art. 62) we have also

$$\log(1-2r\cos\theta+r^2)=-\{(r\gamma-r^2)+\tfrac{1}{2}(r\gamma-r^2)^2+\tfrac{1}{3}(r\gamma-r^2)^3+\ldots\}$$

where $\gamma=2\cos\theta$; and (by Art. 27) the latter series may be rearranged in powers of r, without alteration of value, provided that $0<r\leqq\frac{2}{5}$. It is therefore evident that $\dfrac{2}{n}\cos n\theta$ is the coefficient of r^n in the expression

$$\frac{1}{n}(r\gamma-r^2)^n+\frac{1}{n-1}(r\gamma-r^2)^{n-1}+\ldots+(r\gamma-r^2),$$

because $(r\gamma-r^2)^{n+1}$, $(r\gamma-r^2)^{n+2}$, ... contain no terms in r^n.

Thus

$$\frac{2}{n}\cos n\theta=\frac{\gamma^n}{n}-\frac{(n-1)\gamma^{n-2}}{n-1}+\ldots$$
$$+(-1)^s\frac{(n-s)(n-s-1)\ldots(n-2s+1)}{(n-s)\,s!}\gamma^{n-2s}$$
$$+\ldots,$$

the number of terms being either $\frac{1}{2}(n+1)$ or $\frac{1}{2}(n+2)$.

Hence

$$2\cos n\theta=\gamma^n-n\gamma^{n-2}+\frac{n(n-3)}{2!}\gamma^{n-4}-\ldots$$
$$+(-1)^s\frac{n(n-s-1)(n-s-2)\ldots(n-2s+1)}{s!}\gamma^{n-2s}$$
$$+\ldots.$$

Similarly, we have seen that

$$\frac{r\sin\theta}{1-2r\cos\theta+r^2}=r\sin\theta+r^2\sin 2\theta+r^3\sin 3\theta+\dots.$$

Hence we deduce that $\dfrac{\sin n\theta}{\sin\theta}$ is the coefficient of r^{n-1} in the series

$$1+(r\gamma-r^2)+(r\gamma-r^2)^2+(r\gamma-r^2)^3+\dots.$$

Thus

$$\frac{\sin n\theta}{\sin\theta}=\gamma^{n-1}-(n-2)\gamma^{n-3}+\frac{(n-3)(n-4)}{2!}\gamma^{n-5}$$

$$-\dots+(-1)^s\frac{(n-s-1)(n-s-2)\dots(n-2s)}{s!}\gamma^{n-2s-1}$$

$$+\dots,$$

where the number of terms is either $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$. We note that this formula can be deduced from the last by differentiation.

It is therefore evident that both $\cos n\theta$ and $\sin n\theta/\sin\theta$ are polynomials in $\cos\theta$, of degrees n and $(n-1)$ respectively. But for some purposes it is more useful to express the functions of $n\theta$ in terms of $\sin\theta$. This we shall do in the following article.

Before leaving the formulae above, it is worth while to notice that if we write $\gamma=t+1/t$, instead of $2\cos\theta$, then $1-r\gamma+r^2=(1-rt)(1-r/t)$.

Hence $\quad \log(1-r\gamma+r^2)=\log(1-rt)+\log(1-r/t)=-\Sigma\frac{1}{n}r^n(t^n+t^{-n}),$

and so, from the foregoing argument, we get the algebraic identity

$$t^n+t^{-n}=\gamma^n-n\gamma^{n-2}+\dots \text{ as above.}$$

Similarly, $\quad \dfrac{r(t-1/t)}{1-r\gamma+r^2}=\dfrac{1}{1-rt}-\dfrac{1}{1-r/t}=\Sigma r^n(t^n-t^{-n}),$

and so we find $\quad \dfrac{t^n-t^{-n}}{t-t^{-1}}=\gamma^{n-1}-(n-2)\gamma^{n-3}+\dots$ as above.

The reader may find it instructive to contrast the former result with Ex. A. 21, Ch. VIII., writing $x=-\dfrac{4t^2}{(1+t^2)^2}=\dfrac{4}{-\gamma^2}$.

67. Forms for $\cos n\theta$ and $\sin n\theta$ in terms of $\sin\theta$.

In the formulae of the last article change θ to $(\frac{1}{2}\pi-\theta)$; then $\gamma=2\sin\theta$, and we find

$$(-1)^m 2\cos 2m\theta=\gamma^{2m}-2m\gamma^{2m-2}+\dots \text{ to } (m+1) \text{ terms},$$

$$(-1)^{m-1}\frac{\sin 2m\theta}{2\sin\theta\cos\theta}=\gamma^{2m-2}-(2m-2)\gamma^{2m-4}+\dots \text{ to } m \text{ terms},$$

if n is even and equal to $2m$.

But if n is odd and equal to $(2m+1)$, we have

$$(-1)^m 2\sin(2m+1)\theta=\gamma^{2m+1}-(2m+1)\gamma^{2m-1}+\dots \text{ to } (m+1) \text{ terms},$$

$$(-1)^m \frac{\cos(2m+1)\theta}{\cos\theta}=\gamma^{2m}\quad-(2m-1)\gamma^{2m-2}+\dots \text{ to } (m+1) \text{ terms}.$$

However, these formulae take a more elegant shape when arranged according to ascending powers of $\sin\theta$; of course it is not difficult to rearrange the expressions algebraically, but it is instructive to obtain the results in another way.

If $y=\cos n\theta$ or $\sin n\theta$, we have

$$\frac{d^2y}{d\theta^2}+n^2y=0.$$

If we write $x=\sin\theta$, this equation becomes

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0.$$

Now, if we consider the expression given above for $\cos 2m\theta$, we see that *when n is even*, $\cos n\theta$ can be expressed as a polynomial of degree n in x, containing only even powers; thus we can write

$$\cos n\theta=1+A_2x^2+A_4x^4+\dots+A_nx^n,$$

the constant term being 1, because $\theta=0$ gives $x=0$ and $\cos n\theta=1$.

If we substitute this expression in the differential equation, we find

$$\begin{aligned}0=1\,.\,2A_2+3\,.\,4A_4x^2+5\,.\,6A_6x^4+\dots+(n-1)nA_nx^{n-2}&\\ -2^2A_2x^2\quad-4^2A_4x^4\quad-\dots\qquad\qquad\qquad -n^2A_nx^n&\\ +n^2+n^2A_2x^2\quad+n^2A_4x^4\quad+\dots\qquad\qquad\qquad +n^2A_nx^n.&\end{aligned}$$

Thus $\quad 1\,.\,2\,.\,A_2+n^2=0,\qquad 3\,.\,4\,.\,A_4+(n^2-2^2)A_2=0,$

$\qquad 5\,.\,6\,.\,A_6+(n^2-4^2)A_4=0,\ \dots,$

and so $\quad A_2=-\dfrac{n^2}{2!},\ A_4=\dfrac{n^2(n^2-2^2)}{4!},\ A_6=-\dfrac{n^2(n^2-2^2)(n^2-4^2)}{6!}$, etc.

Hence $\quad \cos n\theta=1-\dfrac{n^2}{2!}x^2+\dfrac{n^2(n^2-2^2)}{4!}x^4-\dots$ to $\frac{1}{2}(n+2)$ terms when n is *even*.

Similarly, *when n is odd*, we find that $\sin n\theta$ is a polynomial of degree n, which contains only odd powers of x; thus we write

$$\sin n\theta=nx+A_3x^3+A_5x^5+\dots+A_nx^n,$$

the first coefficient being determined by considering that for $\theta=0$,

$$x=0,\quad \frac{dx}{d\theta}=1,\quad \frac{d}{d\theta}(\sin n\theta)=n.$$

Hence, on substitution, we find

$$\begin{aligned}0 = 2\,.\,3A_3x + 4\,.\,5A_5x^3 + 6\,.\,7A_7x^5 + \ldots + (n-1)nA_nx^{n-2}\\ -nx - 3^2A_3x^3 - 5^2A_5x^5 - \ldots \qquad -n^2A_nx^n\\ +n^3x + n^2A_3x^3 + n^2A_5x^5 + \ldots \qquad +n^2A_nx^n.\end{aligned}$$

Thus $\quad 2\,.\,3A_3 + (n^2-1)n = 0, \quad 4\,.\,5A_5 + (n^2-3^2)A_3 = 0, \ldots,$
giving

$$\sin n\theta = nx - \frac{n(n^2-1^2)}{3!}x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 - \ldots$$

to $\frac{1}{2}(n+1)$ terms,

n being *odd*.

To verify the algebraic identity between these results and those of Art. 66, consider in particular $n=6$. Then Art. 66 gives

$$2\cos 6\theta = \gamma^6 - 6\gamma^4 + 9\gamma^2 - 2$$

or

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

Change from θ to $(\frac{1}{2}\pi - \theta)$, and we get

$$\cos 6\theta = 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta$$

$$= 1 - \frac{6^2}{2!}\sin^2\theta + \frac{6^2(6^2-2^2)}{4!}\sin^4\theta - \frac{6^2(6^2-2^2)(6^2-4^2)}{6!}\sin^6\theta,$$

in agreement with the above formula for $\cos n\theta$.

Again, take $n=7$; from Art. 66 we have

$$2\cos 7\theta = \gamma^7 - 7\gamma^5 + 14\gamma^3 - 7\gamma$$

or

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta.$$

Hence, changing θ to $\frac{1}{2}\pi - \theta$, we have

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta,$$

and on writing $n=7$ in the above formula for $\sin n\theta$, the results are found to agree.

By differentiating the formulae just obtained for $\sin n\theta$ and $\cos n\theta$, we find

$$\frac{\cos n\theta}{\cos\theta} = 1 - \frac{n^2-1^2}{2!}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{4!}\sin^4\theta - \ldots$$

to $\frac{1}{2}(n+1)$ terms,

when n is *odd*; and

$$\frac{\sin n\theta}{\cos\theta} = n\sin\theta - \frac{n(n^2-2^2)}{3!}\sin^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{5!}\sin^5\theta - \ldots$$

to $\frac{1}{2}n$ terms,

when n is *even*.

The reader will find that these formulae lead to

$$\cos 7\theta/\cos\theta = 1 - 24\sin^2\theta + 80\sin^4\theta - 64\sin^6\theta$$

and

$$\sin 6\theta/\cos\theta = 6\sin\theta - 32\sin^3\theta + 32\sin^5\theta,$$

and that these formulae agree with those of Art. 66 on writing $\frac{1}{2}\pi - \theta$ for θ and reversing the order of the terms.

68. The expressions obtained in the last article are restricted by certain conditions on the value of n. Let us now see if these conditions can be removed in any way.

Take for example the infinite series

$$y=1-\frac{n^2}{2!}x^2+\frac{n^2(n^2-2^2)}{4!}x^4-\ldots, \qquad (x=\sin\theta)$$

which was proved to terminate and to represent $\cos n\theta$, when n is *even*.

If n is an odd integer, or is not an integer, the series does not terminate. It is natural to consider whether it is convergent, and if so, to investigate its sum.

The test (5) of Art. 12·2 shews at once that the series converges absolutely when $|x|=1$; and so, as we have proved in Art. 50, the series converges absolutely and uniformly for $|x|\leqq 1$.

Thus we can differentiate the series term-by-term, as in Art. 52; and on substituting in the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0,$$

it is easy to verify (as in Art. 67) that the series gives a solution of the equation.

It follows from the general theory of Art. 56·1 that this equation has a particular solution of the type y, which corresponds to the special values $A_0=1$, $A_1=0$. From the general theory, we can anticipate that the solution will converge if $|x|<1$, because the coefficients P, Q are multiples of $(1-x^2)^{-1}=1+x^2+x^4+\ldots$, which converges for $|x|<1$. But it happens here that the series still converges for $|x|=1$ as well as for smaller values.

Thus y is again a solution of the differential equation in θ,

$$\frac{d^2y}{d\theta^2}+n^2y=0,$$

and is accordingly of the form

$$y=A\cos n\theta+B\sin n\theta,$$

where A, B are independent of θ.

It is usual in elementary text-books to take this result for granted, on the ground that no solution of a second-order differential equation can contain more than two arbitrary constants. But it seems worth while to obtain a simple formal proof as follows: Write $y=z\cos n\theta$, then using accents to indicate differentiation with respect to θ, we find

$$y''=z''\cos n\theta-2nz'\sin n\theta-n^2z\cos n\theta.$$

Hence
$$y'' + n^2 y = 0$$
gives
$$z'' \cos n\theta - 2nz' \sin n\theta = 0,$$
which can be integrated at once on multiplying by $\cos n\theta$. The result is
$$z' \cos^2 n\theta = \text{const.} = nB, \text{ say},$$
or
$$z' = nB \sec^2 n\theta.$$

Integrating again, we have
$$z = A + B \tan n\theta,$$
or
$$y = z \cos n\theta = A \cos n\theta + B \sin n\theta,$$
where A, B are arbitrary constants.

It is perhaps worth while to note that if we assume that $y = y_0$, $\frac{dy}{d\theta} = y_1$ for $\theta = 0$, this process determines A, B in the course of the investigation: for (at $\theta = 0$) $z' = y' = y_1$, and so $B = y_1/n$, and (at $\theta = 0$) $z = y = y_0$, and so $A = y_0$.

To find the values of A, B we note that
$$y = 1, \quad \frac{dy}{dx} = 0 \quad \text{for } x = 0.$$

Thus we also have
$$y = 1, \quad \frac{dy}{d\theta} = 0 \quad \text{for } \theta = 0,$$
provided that θ lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.*

Thus we find that $A = 1$, $B = 0$, and accordingly for any value of n,
$$\cos n\theta = 1 - \frac{n^2}{2!}x^2 + \frac{n^2(n^2 - 2^2)}{4!}x^4 - \dots \text{ to } \infty,$$
where
$$x = \sin\theta, \quad \text{and} \quad -\tfrac{1}{2}\pi \leqq \theta \leqq +\tfrac{1}{2}\pi.$$

In particular we have the elegant result
$$\cos \tfrac{1}{2}n\pi = 1 - \frac{n^2}{2!} + \frac{n^2(n^2 - 2^2)}{4!} - \dots \text{ to } \infty.$$

On differentiation with respect to x we find the result
$$\frac{\sin n\theta}{\cos\theta} = nx - \frac{n(n^2 - 2^2)}{3!}x^3 + \frac{n(n^2 - 2^2)(n^2 - 4^2)}{5!}x^5 - \dots,$$
which, however, converges only for $|x| < 1$; the series diverges for $x = 1$, as might be anticipated from the fact that
$$|\sin n\theta/\cos\theta| \to \infty \quad \text{as } \theta \to \tfrac{1}{2}\pi,$$
unless n is an even integer.

* When $x = \sin\theta$ is given, there is one and only one value of θ between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, because x steadily increases in this range of values of θ. Similarly, there is only one value of θ between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, because x steadily decreases in that range for θ: and so on.

We prove on exactly similar lines that

$$z = nx - \frac{n(n^2-1^2)}{3!}x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 - \dots$$

is another solution of the above differential equation; and from the values

$$z=0, \quad \frac{dz}{dx}=n, \quad \text{at } x=0$$

we deduce the result

$$\sin n\theta = nx - \frac{n(n^2-1^2)}{3!}x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 - \dots,$$

where $x = \sin\theta$ and $-\frac{1}{2}\pi \leqq \theta \leqq +\frac{1}{2}\pi$.

In particular we have

$$\sin\tfrac{1}{2}n\pi = n - \frac{n(n^2-1^2)}{3!} + \frac{n(n^2-1^2)(n^2-3^2)}{5!} - \dots.$$

By differentiating with respect to x, we have also

$$\frac{\cos n\theta}{\cos\theta} = 1 - \frac{n^2-1^2}{2!}x^2 + \frac{(n^2-1^2)(n^2-3^2)}{4!}x^4 - \dots,$$

which converges only for $|x| < 1$; as might be anticipated since $|\cos n\theta/\cos\theta| \to \infty$ as $\theta \to \frac{1}{2}\pi$, unless n is an odd integer.

If formulae are required for values of θ between, say, $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, it is only necessary to replace θ by $\pi - \theta$ in the above results; and similarly for other ranges of values of θ.

68·1. Formulae for $\cos n\theta$, $\sin n\theta$ derived from de Moivre's theorem.

The foregoing investigations have been carried out by means of formulae which are entirely independent of the complex variable; and, as a matter of fact, these formulae are best established on the above lines.

There are, however, certain other formulae (not independent of the previous results) which are most easily found by anticipating de Moivre's theorem (Art. 74 below).

Suppose in the first place that n is a positive integer; then the formula

$$\cos n\theta + \iota\sin n\theta = (\cos\theta + \iota\sin\theta)^n$$

can be written as $\cos^n\theta(1+\iota t)^n$, where $t = \tan\theta$.

Thus we find, on applying the binomial theorem and dividing into real and imaginary parts,*

$$\cos n\theta = \cos^n\theta\,(1 - n_2t^2 + n_4t^4 - n_6t^6 + \dots),$$
$$\sin n\theta = \cos^n\theta\,(nt - n_3t^3 + n_5t^5 - n_7t^7 + \dots),$$

both expressions terminating; when n is even, the cosine-formula has $\frac{1}{2}(n+2)$ terms and the sine-formula has $\frac{1}{2}n$ terms; when n is odd, each formula has $\frac{1}{2}(n+1)$ terms.

To see the essential equivalence with Arts. 66, 67 we may consider the same values of n as before. Thus for $n=6$, the above formulae give

$$\cos 6\theta = \cos^6\theta\,(1 - 15t^2 + 15t^4 - t^6),$$
$$\sin 6\theta = \cos^6\theta\,(6t - 20t^3 + 6t^5).$$

Now write s for $\sin\theta$ and then $\cos^6\theta = (1-s^2)^3$, $t^2 = s^2/(1-s^2)$, giving

$$\cos 6\theta = (1-s^2)^3 - 15s^2(1-s^2)^2 + 15s^4(1-s^2) - s^6$$
$$= 1 - 18s^2 + 48s^4 - 32s^6,$$

and
$$\sin 6\theta/\cos\theta = 6s(1-s^2)^2 - 20s^3(1-s^2) + 6s^5$$
$$= 6s - 32s^3 + 32s^5.$$

Both of these agree with the formulae found in Art. 67.

When n is *not* a positive integer, the same formulae will hold, provided that the conditions of Art. 96 below are satisfied. These conditions may be summed up as follows:

It is necessary that $t = \tan\theta$ should be numerically less than unity, and that $-\frac{1}{4}\pi < \theta < +\frac{1}{4}\pi$.

From the point of view of differential equations, these series are found as solutions of the equation

$$\frac{d^2y}{d\theta^2} + n^2y = 0, \quad \text{or} \quad (1+t^2)\frac{d}{dt}\left\{(1+t^2)\frac{dy}{dt}\right\} + n^2y = 0.$$

It is easily found that the indices of this equation are $(\frac{1}{2}n, -\frac{1}{2}n)$ at $t = \pm\iota$, and that infinity is an ordinary point.

If we now write $y = z\cos^n\theta = z(1+t^2)^{-n/2}$, it is clear that the indices for z become

$$n,\ 0 \text{ at } t = \pm\iota, \quad \text{and} \quad -n,\ 1-n \text{ at } \infty.$$

Thus the differential equation for z is now seen to be (Art. 56·4, Ex. 7)

$$(1+t^2)\frac{d^2z}{dt^2} - 2(n-1)t\frac{dz}{dt} + n(n-1)z = 0,$$

and on substituting the series

$$z = A_0 + A_1t + A_2t^2 + \dots,$$

we find that
$$\frac{A_{r+1}}{A_{r-1}} = -\frac{(n-r+1)(n-r)}{r(r+1)}.$$

---

* Here n_1, n_2, n_3, ... denote the binomial coefficients.

This leads to the form

$$z = A_0(1 - n_2 t^2 + \dots) + \frac{A_1}{n}(nt - n_3 t^3 + \dots)$$

in agreement with the previous results.

69. Various deductions from Art. 67.

We have seen that

$$\frac{\sin n\theta}{\sin\theta} = n + A_2 \sin^2\theta + \dots + A_{n-1}\sin^{n-1}\theta, \quad (n \text{ odd})$$

or
$$\frac{\sin n\theta}{\sin\theta\cos\theta} = n + B_2\sin^2\theta + \dots + B_{n-2}\sin^{n-2}\theta, \quad (n \text{ even})$$

where the coefficients are the same as those worked out in Art. 67, but are not needed in an explicit form at present.

Now the left-hand side vanishes for

$$\theta = \pm\alpha, \quad \pm 2\alpha, \quad \pm 3\alpha, \dots, \quad \text{where } \alpha = \pi/n,$$

so that the right-hand side (regarded as a polynomial in $\sin\theta$) must have roots

$$\sin\theta = \pm\sin\alpha, \quad \pm\sin 2\alpha, \quad \pm\sin 3\alpha, \dots.$$

When n is odd, there are $(n-1)$ of these roots which are all different; and these are given by

$$\sin^2\theta = \sin^2\alpha, \quad \sin^2 2\alpha, \dots, \sin^2\tfrac{1}{2}(n-1)\alpha.$$

But if n is even, there are $(n-2)$ different roots given by

$$\sin^2\theta = \sin^2\alpha, \quad \sin^2 2\alpha, \dots, \sin^2\tfrac{1}{2}(n-2)\alpha.$$

Thus we can factorise the formulae as follows:

$$\frac{\sin n\theta}{\sin\theta} = n\left(1 - \frac{\sin^2\theta}{\sin^2\alpha}\right)\left(1 - \frac{\sin^2\theta}{\sin^2 2\alpha}\right)\dots\left\{1 - \frac{\sin^2\theta}{\sin^2\frac{1}{2}(n-1)\alpha}\right\},$$

$$\frac{\sin n\theta}{\sin\theta\cos\theta} = n\left(1 - \frac{\sin^2\theta}{\sin^2\alpha}\right)\left(1 - \frac{\sin^2\theta}{\sin^2 2\alpha}\right)\dots\left\{1 - \frac{\sin^2\theta}{\sin^2\frac{1}{2}(n-2)\alpha}\right\},$$

where the first line refers to odd values of n and the second to even values.

If we compare these with the explicit forms given in Art. 67, we can deduce various identities, such as

$$\frac{n^2-1}{6} = \frac{1}{\sin^2\alpha} + \frac{1}{\sin^2 2\alpha} + \dots + \frac{1}{\sin^2\frac{1}{2}(n-1)\alpha}, \quad (n \text{ odd})$$

$$\frac{n^2-4}{6} = \frac{1}{\sin^2\alpha} + \frac{1}{\sin^2 2\alpha} + \dots + \frac{1}{\sin^2\frac{1}{2}(n-2)\alpha}, \quad (n \text{ even}),$$

which are deduced by considering the coefficients of $\sin^2\theta$.

In a similar way we prove the identities

$$\frac{\cos n\theta}{\cos\theta}=\left(1-\frac{\sin^2\theta}{\sin^2\beta}\right)\left(1-\frac{\sin^2\theta}{\sin^2 3\beta}\right)\cdots\left\{1-\frac{\sin^2\theta}{\sin^2(n-2)\beta}\right\},\quad (n \text{ odd})$$

$$\cos n\theta=\left(1-\frac{\sin^2\theta}{\sin^2\beta}\right)\left(1-\frac{\sin^2\theta}{\sin^2 3\beta}\right)\cdots\left\{1-\frac{\sin^2\theta}{\sin^2(n-1)\beta}\right\},\quad (n \text{ even})$$

where $\beta=\pi/2n$ and only the *odd* multiples of β appear.

On comparing these with the forms of Art. 67, from the terms in $\sin^2\theta$, we see that

$$\frac{n^2-1}{2}=\frac{1}{\sin^2\beta}+\frac{1}{\sin^2 3\beta}+\cdots+\frac{1}{\sin^2(n-2)\beta},\quad (n \text{ odd})$$

$$\frac{n^2}{2}=\frac{1}{\sin^2\beta}+\frac{1}{\sin^2 3\beta}+\cdots+\frac{1}{\sin^2(n-1)\beta},\quad (n \text{ even}).$$

Again, if we consider the formulae of Art. 66, it is evident that $(\cos n\theta-\cos n\omega)$ may be expressed as a polynomial of degree n in $\cos\theta$, the term of highest degree being $2^{n-1}\cos^n\theta$. But the expression $(\cos n\theta-\cos n\omega)$ is zero if

$$n\theta=\pm n\omega,\quad 2\pi\pm n\omega,\quad 4\pi\pm n\omega,\ \ldots.$$

Thus the factors of the polynomial in question will be n different expressions of the form

$$\cos\theta-\cos\omega,\quad \cos\theta-\cos(\omega\pm 2\alpha),\quad \cos\theta-\cos(\omega\pm 4\alpha),\ \ldots,$$

where, as before, α denotes π/n.

It is easily seen that the n different factors can be taken as

$$\cos\theta-\cos\omega,\quad \cos\theta-\cos(\omega+2\alpha),\quad \cos\theta-\cos(\omega+4\alpha),\ \ldots\quad \cos\theta-\cos(\omega+2(n-1)\alpha),$$

because $\cos(\omega-2r\alpha)=\cos(\omega+2(n-r)\alpha)$.

Hence we have the identity

$$\cos n\theta-\cos n\omega=2^{n-1}\prod_{r=0}^{n-1}\{\cos\theta-\cos(\omega+2r\alpha)\}.$$

If we write $\theta=0$ in this expression we have

$$\sin^2\tfrac{1}{2}n\omega=2^{2n-2}\prod_{r=0}^{n-1}\sin^2(\tfrac{1}{2}\omega+r\alpha),$$

or, with a change of notation,

$$\sin n\theta=\pm 2^{n-1}\prod_{r=0}^{n-1}\sin(\theta+r\alpha).$$

But the $\pm$ sign is really $+$, because, if $0<\theta<\alpha$, all the factors are positive, and it is easily seen that both sides change sign together (when θ passes through any multiple of α).

69·1. Equations with roots $\sin^2(r\pi/n)$, $\cos^2(r\pi/n)$, $\tan^2(r\pi/n)$, where r, n are positive integers.

These equations are constructed immediately by writing

$$\sin n\theta/\sin\theta = 0,$$

which gives at once $$n\theta = \pm r\pi.$$

Then, to obtain the actual equations required, we have only to express the function $\sin n\theta/\sin\theta$ in terms of

(i) $\sin\theta$, as in Art. 67,

or (ii) $\cos\theta$, as in Art. 66,

or (iii) $\tan\theta$, as in Art. 68·1,

to obtain the desired equations.

The method is illustrated by taking $n=5$, which gives

(i) $5 - 20x^2 + 16x^4 = \sin 5\theta/\sin\theta = 0$,

when $$x = \pm\sin\tfrac{1}{5}\pi, \quad \pm\sin\tfrac{2}{5}\pi.$$

(ii) $\gamma^4 - 3\gamma^2 + 1 = 0$,

when $$\gamma = \pm 2\cos\tfrac{1}{5}\pi, \quad \pm 2\cos\tfrac{2}{5}\pi.$$

(iii) $5 - 10t^2 + t^4 = 0$,

when $$t = \pm\tan\tfrac{1}{5}\pi, \quad \pm\tan\tfrac{2}{5}\pi.$$

It will be readily found that any two of these equations can be derived from the third, as we should expect. The one which happens to be easiest to solve is (ii), giving

$$\gamma^2 = \tfrac{1}{2}(3 \pm \sqrt{5}), \quad \gamma = \tfrac{1}{2}(\pm\sqrt{5} + 1).$$

It is easy to see that this leads to the ordinary elementary results

$$\cos 36° = \tfrac{1}{4}(\sqrt{5}+1), \quad \cos 72° = \tfrac{1}{4}(\sqrt{5}-1).$$

The general theory of the solution of equations for $\sin(r\pi/n)$, etc., has led to many interesting investigations by Abel and Gauss, to mention only two prominent names; a few samples of Gauss's results are given in Exs. 13–16 of Chap. X. But a more striking conclusion (though less easy to obtain by comparatively elementary means) is that for $n=17$ and 257 (and generally for $n=2^{2^m}+1$ when this is a prime number), the final equations are soluble by quadratics; and thus it is possible to construct 17-sided and 257-sided regular polygons by Euclidean constructions with ruler and compass only.*

* These are the next in order to Euclid's own construction for $n=5$. That the case $n=5$ is capable of solution by quadratics has just been proved above. The theory for $n=17$ is indicated in Ex. 16 of Ch. X. below.

It is, however, quite certain that any practical draughtsman can construct a regular figure of any number of sides by various approximate methods * with far greater accuracy than would be obtained by applying Gauss's construction for the 17-sided polygon ; but this fact has nothing to do with the theoretical beauty of Gauss's work.

70. Expressions of $\sin\theta$ and $\cos\theta$ as infinite products.

We have seen in article 69 that, if n is an odd integer,

$$\frac{\sin n\phi}{n\sin\phi}=\prod_{r=1}^{\frac{1}{2}(n-1)}\left(1-\frac{\sin^2\phi}{\sin^2 r\alpha}\right),$$

where $\alpha=\pi/n$. Thus, if we write $n\phi=\theta$ we have

$$\frac{\sin\theta}{n\sin(\theta/n)}=\prod_{r=1}^{\frac{1}{2}(n-1)}\left\{1-\frac{\sin^2(\theta/n)}{\sin^2(r\pi/n)}\right\}.$$

To this equation we can apply the second theorem of Art. 49 ; we have, in fact,†

$$\left|\frac{\sin^2(\theta/n)}{\sin^2(r\pi/n)}\right|<\frac{\theta^2}{4r^2},$$

because $r\pi/n$ is less than $\frac{1}{2}\pi$. Now this expression is independent of n, and the series $\Sigma\theta^2/4r^2$ is convergent ; consequently, the theorem applies. But we have

$$\lim_{n\to\infty} n\sin(\theta/n)=\theta,$$

and
$$\lim_{n\to\infty}\frac{\sin^2(\theta/n)}{\sin^2(r\pi/n)}=\lim_{n\to\infty}\frac{n^2\sin^2(\theta/n)}{n^2\sin^2(r\pi/n)}=\frac{\theta^2}{r^2\pi^2}.$$

Consequently,
$$\frac{\sin\theta}{\theta}=\prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right).$$

The special value $\theta=\frac{1}{2}\pi$ leads at once to *Wallis's Theorem* :

$$\frac{2}{\pi}=\prod_{1}^{\infty}\left(1-\frac{1}{4r^2}\right)=\prod_{1}^{\infty}\frac{(2r-1)(2r+1)}{2r\,.\,2r}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots$$

or
$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\text{ to }\infty.$$

The reader should find no difficulty in expressing $\cos\theta$ as an infinite product by a similar method.

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* Such as Ex. A, 38 of Ch. VIII.

† We see, by differentiation or from the graph of $\sin x$, that $\sin x/x$ decreases as x increases from 0 to π : thus

$$1>(\sin x)/x>2/\pi,\qquad\text{if } 0<x<\tfrac{1}{2}\pi.$$

Consequently, $$n\sin(r\pi/n)>2r,\qquad\text{if } r<\tfrac{1}{2}n.$$

Also $|n\sin\theta/n|<|\theta|$, for any value of θ ; and so the inequality follows.

We have in fact (Art. 69)

$$\cos\theta = \prod_{r=1}^{n-1}\left\{1-\frac{\sin^2(\theta/n)}{\sin^2(r\pi/2n)}\right\}, \quad r=1, 3, 5, \ldots, n-1,$$

where n is even.

Here the comparison-series is $\Sigma\theta^2/r^2$, and the result is

$$\cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots.$$

Alternative methods are to write

$$\cos\theta = \frac{\sin 2\theta}{2\theta} \Big/ \frac{\sin\theta}{\theta},$$

and to appeal directly to the sine-product; or to write $\frac{1}{2}\pi-\theta$ for θ in that product and then rearrange the factors.

It is perhaps worth while to refer briefly to an incomplete "proof" given in some of the older books. Since $\sin\theta$ vanishes for $\theta=0$ and for $\theta=\pm r\pi$, and since $\sin\theta/\theta \to 1$ as $\theta\to 0$, it is urged that $\sin\theta/\theta$ must be of the form given above; but exactly the same argument would apply equally to the function $a^\theta\sin\theta$, where a is any real number, so that this "proof" only suggests that $\prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right)$ is probably of the form $a^\theta\sin\theta/\theta$; we cannot prove that a is 1 on these lines. In this connexion, it may be noted that if we separate $1-\frac{\theta^2}{r^2\pi^2}$ into factors $\left(1-\frac{\theta}{r\pi}\right)$, $\left(1+\frac{\theta}{r\pi}\right)$ and then take more positive than negative factors (say p positive to every q negative factors), the value of the product is $\left(\frac{p}{q}\right)^{\frac{\theta}{\pi}}\left(\frac{\sin\theta}{\theta}\right)$. This follows from Art. 41 by writing

$$a_{2r-1}=a_{2r}=\theta/r\pi.$$

We have already pointed out the danger of applying the theorem of Art. 49 to cases when the M-test does not hold good. An additional illustration of this risk may be given here.

Since $\sin(\pi-\phi)=\sin\phi$, it follows that the values of $\sin(r\pi/n)$ when r ranges from $\frac{1}{2}(n+1)$ to $(n-1)$ are the same as those when r ranges from 1 to $\frac{1}{2}(n-1)$, but in the reverse order.

Hence
$$\frac{\sin^2\theta}{\{n\sin(\theta/n)\}^2} = \prod_{r=1}^{n-1}\left\{1-\frac{\sin^2(\theta/n)}{\sin^2(r\pi/n)}\right\},$$

and if we apply the theorem here, we appear to get

$$\frac{\sin^2\theta}{\theta^2} = \prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right),$$

which contradicts the result obtained before. Of course the explanation is that the inequality $\frac{\sin^2(\theta/n)}{\sin^2(r\pi/n)} < \frac{\theta^2}{4r^2}$ is no longer true, since $r\pi/n$ may be greater than $\frac{1}{2}\pi$; and it is, in fact, impossible to construct a convergent comparison-series ΣM_r.

71. Weierstrass's formula for the sine-product.

It is sometimes useful to express the sine-product in a form in which there is only a single factor in each term; and it is at first sight natural to write

$$\frac{\sin\theta}{\theta}=\prod_{-\infty}^{+\infty}{}'\left(1-\frac{\theta}{n\pi}\right), \text{ using } n \text{ in place of } r,$$

where the accent implies that $n=0$ is omitted from the product. This is, however, apt to lead to errors, because $\prod^{\infty}\left(1-\frac{\theta}{n\pi}\right)$ is divergent (see Art. 39); and we must either write

$$\frac{\sin\theta}{\theta}=\lim_{N\to\infty}\prod_{-N}^{+N}{}'\left(1-\frac{\theta}{n\pi}\right),$$

or else modify the factors so as to ensure the convergence of the product. The simplest modification is due to Weierstrass, and is given by the formula*

$$\frac{\sin\theta}{\theta}=\prod_{-\infty}^{\infty}{}'\left\{e^{\theta/n\pi}\left(1-\frac{\theta}{n\pi}\right)\right\}.$$

In the first place the last product is absolutely convergent.

For we have $$e^x=1+x+\frac{x^2}{2!}+\dots. \qquad \text{(Art. 58)}$$

Thus $$1+x<e^x<1+x+x^2+\dots, \quad \text{if } 0<x<1.$$

Hence $$1-x^2<e^x(1-x)<1, \quad \text{if } 0<x<1.$$

But if x is negative and numerically less than 1, we see that

$$1+x<e^x<1+x+\tfrac{1}{2}x^2<1+x+x^2. \qquad \text{(Art. 19)}$$

Thus $$1-x^2<e^x(1-x)<1-x^3.$$

Hence, if $|e^{\theta/n\pi}(1-\theta/n\pi)|=|1+u_n|$, it is clear that $|u_n|$ is less than $\theta^2/n^2\pi^2$. Thus Σu_n is absolutely convergent, and so $\Pi(1+u_n)$ is also absolutely convergent.

To evaluate the product we need only notice that, since it is convergent, its value is equal to

$$\lim_{N\to\infty}\prod_{-N}^{+N}{}'\left\{e^{\theta/n\pi}\left(1-\frac{\theta}{n\pi}\right)\right\}.$$

In this product the corresponding positive and negative terms can be taken together, and the result is

$$\lim_{N\to\infty}\prod_{1}^{N}\left(1-\frac{\theta^2}{n^2\pi^2}\right)=\frac{\sin\theta}{\theta},$$

by our previous result.

---

* Π' is used to imply that $n=0$ is omitted from the product.

The cosine-product may be expressed similarly as

$$\cos\theta = \prod_{-\infty}^{\infty}\left\{e^{2\theta/(2n+1)\pi}\left(1-\frac{2\theta}{(2n+1)\pi}\right)\right\}.$$

We have seen (Art. 42) that

$$\frac{1}{\Gamma(1+x)} = e^{Cx}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n},$$

where C is Euler's constant; it will be recognized now that this is, so to speak, *half* the Weierstrassian form of the sine-product. Changing the sign of x, we have

$$\frac{1}{\Gamma(1-x)} = e^{-Cx}\prod_{n=1}^{\infty}\left(1-\frac{x}{n}\right)e^{x/n},$$

and so $$\frac{1}{\Gamma(1-x)\Gamma(1+x)} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) = \frac{\sin\pi x}{\pi x}.$$

Thus, since $\Gamma(1+x)=x\Gamma(x)$, we have one of Euler's formulae

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}.$$

In particular, Wallis's product is equivalent to the result

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

71·1. Formulae for $\cot\theta$ and allied results.

If we take the formula (Art. 69)

$$\sin n\phi = 2^{n-1}\prod_{0}^{n-1}\sin(\phi+r\alpha), \quad (\alpha=\pi/n)$$

and differentiate logarithmically, we obtain the result

$$(A) \qquad n\cot n\phi = \sum_{0}^{n-1}\cot(\phi+r\alpha).$$

Differentiating again, we find similarly

$$(B) \qquad n^2\operatorname{cosec}^2 n\phi = \sum_{0}^{n-1}\operatorname{cosec}^2(\phi+r\alpha).$$

For our purpose it will be convenient to suppose n to be odd; and then write $n=2m+1$. We take then

$$n\cot n\phi = \cot\phi + \sum_{1}^{m}\cot(\phi+r\alpha) + \sum_{m+1}^{n-1}\cot(\phi+r\alpha).$$

Now $$\cot(\phi+r\alpha) = \cot(\phi+r\alpha-\pi) = \cot\{\phi-(n-r)\alpha\},$$

since $$n\alpha = \pi,$$

so that
$$\sum_{m+1}^{n-1}\cot(\phi+r\alpha)=\sum_{m+1}^{n-1}\cot\{\phi-(n-r)\alpha\}$$
$$=\sum_{1}^{m}\cot(\phi-r\alpha).$$

Thus, finally, we have

(A_1)
$$n\cot n\phi=\cot\phi+\sum_{1}^{m}\{\cot(\phi+r\alpha)+\cot(\phi-r\alpha)\},$$

and similarly

(B_1)
$$n^2\operatorname{cosec}^2 n\phi=\operatorname{cosec}^2\phi+\sum_{1}^{m}\{\operatorname{cosec}^2(\phi+r\alpha)+\operatorname{cosec}^2(\phi-r\alpha)\}$$

where $n=2m+1$.

We shall now obtain corresponding formulae for $\cot\theta$ and $\operatorname{cosec}^2\theta$ in the form of infinite series.

We write $n\phi=\theta$, and then make $n\to\infty$ in the previous formulae; and the first step is to obtain comparison-series in accordance with Tannery's theorem of Art. 49.

Thus we write
$$\frac{1}{n}\{\cot(\phi+r\alpha)+\cot(\phi-r\alpha)\}=\frac{-n\sin 2\phi}{n^2(\sin^2 r\alpha-\sin^2\phi)}.$$

Now we have
$$|n\sin 2\phi|<2n\phi=2\theta,\quad |n\sin\phi|<n\phi=\theta,$$
and
$$|n\sin r\alpha|>2r\quad\text{(as in Art. 70).}$$

Thus, as soon as $2r$ exceeds θ, we see that
$$\frac{1}{n}|\cot(\phi+r\alpha)+\cot(\phi-r\alpha)|<\frac{2\theta}{4r^2-\theta^2}.$$

It follows that, for values of $r>\frac{1}{2}\theta$, we can compare (A_1) (after division by n) with the convergent series
$$\Sigma 2\theta/(4r^2-\theta^2).$$

Hence Art. 49 applies; and since
$$\lim_{n\to\infty}\frac{1}{n}\cot\phi=\lim_{\phi\to 0}\frac{\phi}{\theta\tan\phi}=\frac{1}{\theta}$$
and
$$\lim_{n\to\infty}\frac{1}{n}\cot(\phi+r\alpha)=\lim_{\phi,\,\alpha\to 0}\frac{\phi+r\alpha}{(\theta+r\pi)\tan(\phi+r\alpha)}=\frac{1}{\theta+r\pi},$$
while
$$\lim_{n\to\infty}\frac{1}{n}\cot(\phi-r\alpha)=\frac{1}{\theta-r\pi},$$
we see that the result is

(A_2)
$$\cot\theta=\frac{1}{\theta}+\sum_{1}^{\infty}\left(\frac{1}{\theta+r\pi}+\frac{1}{\theta-r\pi}\right).$$

It is natural to write (A_2) in the simpler form $\sum_{-\infty}^{+\infty} 1/(\theta-r\pi)$; but (see Art. 8) this form is not precise unless written either as

$$(A_3) \qquad \cot\theta = \lim_{N\to\infty}\sum_{-N}^{N}\left(\frac{1}{\theta-r\pi}\right)$$

or as

$$(A_4) \qquad \cot\theta = \frac{1}{\theta}+\sum_{-\infty}^{\infty}{}'\left(\frac{1}{\theta-r\pi}+\frac{1}{r\pi}\right).$$

Of these (A_4) corresponds to Weierstrass's form of Art. 71 for the sine-product; and it is possible to obtain (A_2)–(A_4) by differentiating the various product-formulae directly. But the final proof of uniform convergence is no easier than the investigation just given.

To deal with the series (B_1) is now easy; for it is at once evident that our previous inequalities give

$$\frac{1}{n^2}\{\operatorname{cosec}^2(\phi+r\alpha)+\operatorname{cosec}^2(\phi-r\alpha)\} = \frac{n^2\sin^2(\phi-r\alpha)+n^2\sin^2(\phi+r\alpha)}{n^4(\sin^2\phi-\sin^2 r\alpha)^2}$$

$$< \frac{(\theta-r\pi)^2+(\theta+r\pi)^2}{(4r^2-\theta^2)^2} = \frac{2(r^2\pi^2+\theta^2)}{(4r^2-\theta^2)^2}.$$

Thus Art. 49 can again be applied, because the last series is convergent.

Then $$\lim_{n\to\infty}\left(\frac{1}{n^2}\operatorname{cosec}^2\phi\right) = \lim_{\phi\to 0}\left(\frac{\phi^2}{\theta^2\sin^2\phi}\right) = \frac{1}{\theta^2}$$

and

$$\lim_{n\to\infty}\left\{\frac{1}{n^2}\operatorname{cosec}^2(\phi+r\alpha)\right\} = \lim_{\phi,\alpha\to 0}\left\{\frac{(\phi+r\alpha)^2}{(\theta+r\pi)^2\sin^2(\phi+r\alpha)}\right\} = \frac{1}{(\theta+r\pi)^2}.$$

Thus our result is

$$(B_2) \qquad \operatorname{cosec}^2\theta = \frac{1}{\theta^2}+\sum_{r=1}^{\infty}\left\{\frac{1}{(\theta+r\pi)^2}+\frac{1}{(\theta-r\pi)^2}\right\},$$

which can be written in the form

$$(B_3) \qquad \operatorname{cosec}^2\theta = \sum_{-\infty}^{\infty}\frac{1}{(\theta-r\pi)^2},$$

without introducing any further modifications, as in (A_3) or (A_4).

These formulae can be found by differentiating (A_4); the proof of uniform convergence will be found in Art. 71·2 below.

In exactly the same way the identities (Art. 69),

$$\frac{n^2-1}{6n^2} = \sum_{1}^{\frac{1}{2}(n-1)}\frac{1}{n^2\sin^2(r\pi/n)} \quad (n \text{ odd})$$

and

$$\frac{1}{2} = \sum_{1}^{n-1}\frac{1}{n^2\sin^2(r\pi/2n)} \quad (r \text{ odd, } n \text{ even})$$

give, on applying Tannery's theorem of Art. 49, the two series

$$\frac{1}{6}=\sum_1^\infty \frac{1}{r^2\pi^2}, \quad \frac{1}{2}=\sum_1^\infty \frac{4}{(2s-1)^2\pi^2},$$

or

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$$

and

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}.$$

Of course there is no difficulty in deducing the second of these from the first, because

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\right)$$

$$=\left(1-\frac{1}{4}\right)\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right),$$

the transformations being justified by Art. 26. [Compare Exs. 2, 3, Ch. IV.]

71·2. Alternative development of the theory.

It is possible to develop the results of the previous articles by starting from the series

$$F(x)=\sum_{-\infty}^{+\infty}\frac{1}{(x-n\omega)^2}.$$

This series converges absolutely and uniformly in any closed regions from which the points $x=n\omega$ are excluded; this follows at once by using the M-test (Art. 44). Thus $F(x)$ is a continuous function, and its integral may be calculated by term-by-term integration (Art. 45).

Further

$$F(x+\omega)=\sum_{-\infty}^{\infty}\frac{1}{(x+\omega-n\omega)^2}=\sum_{-\infty}^{\infty}\frac{1}{(x-m\omega)^2}=F(x),$$

so that $F(x)$ is a periodic function, its period being ω.

When $|x|$ is less than $|\omega|$, we can write

$$F(x)=\frac{1}{x^2}+\sum_1^\infty\left\{\frac{1}{(n\omega-x)^2}+\frac{1}{(n\omega+x)^2}\right\},$$

and, since $|x|<|\omega|$, we have

$$\frac{1}{(n\omega-x)^2}+\frac{1}{(n\omega+x)^2}=\frac{2}{n^2\omega^2}\left(1+\frac{3x^2}{n^2\omega^2}+\frac{5x^4}{n^4\omega^4}+\ldots\right),$$

so that *

$$F(x)=\frac{1}{x^2}+c_0+c_2x^2+c_4x^4+\ldots,$$

where $c_0=\frac{2}{\omega^2}\sum_1^\infty\frac{1}{n^2}, \quad c_2=\frac{6}{\omega^4}\sum_1^\infty\frac{1}{n^4}, \quad c_4=\frac{10}{\omega^6}\sum_1^\infty\frac{1}{n^6}$, etc.

* This implies a reversal of order of summation in a repeated series; it is easy to verify that the condition of absolute convergence (Arts. 31, 33) is satisfied here.

Thus $F(x)$ is an even function of x and tends to infinity like $1/x^2$, as $x \to 0$.

Now consider the function $G(x)$ derived by integrating $F(x)$; to make this function definite we shall write *

$$G(x)-\frac{1}{x}=\int_0^x \left\{\frac{1}{\xi^2}-F(\xi)\right\} d\xi.$$

Then we have clearly $G'(x)=-F(x)$, so that †

$$G(x)=\frac{1}{x}+\sum_{-\infty}^{+\infty}{}' \left(\frac{1}{x-n\omega}+\frac{1}{n\omega}\right),$$

and for values of $|x| < |\omega|$, we find the power-series

$$G(x)=\frac{1}{x}-c_0 x-\frac{1}{3}c_2 x^3-\frac{1}{5}c_4 x^5-\ldots.$$

Thus $G(x)$ is an odd function of x and tends to infinity like $1/x$, as $x \to 0$.

Also $$G'(x+\omega)=-F(x+\omega)=-F(x)=G'(x),$$

so that $$G(x+\omega)-G(x)=\text{const.}$$

To evaluate this constant we can write

$$G(x)=\lim_{N\to\infty}\left\{\frac{1}{x}+\sum_{-N}^{N}{}'\left(\frac{1}{x-n\omega}+\frac{1}{n\omega}\right)\right\}$$

$$=\lim_{N\to\infty}\sum_{-N}^{+N}\frac{1}{x-n\omega},$$

and so $$G(x+\omega)-G(x)=\lim_{N\to\infty}\left\{\frac{1}{x+(N+1)\omega}-\frac{1}{x-N\omega}\right\}=0.$$

If we write $x=-\frac{1}{2}\omega$, and remember that $G(x)$ is an odd function, it follows that

$$G(\tfrac{1}{2}\omega)=0.$$

Thus $G(x)$ *is an odd function, with period* ω, *and* $G(\frac{1}{2}\omega)=0$.

We now proceed to integrate again; ‡ but to remove the

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* Observe that we *cannot* write simply

$$G(x)=-\int_0^x F(\xi)\,d\xi,$$

because $F(\xi)\to\infty$ like $1/\xi^2$ as $\xi\to 0$.

† Σ' implies that $n=0$ is to be omitted from the summation.

‡ Term-by-term integration is permissible because the series for $G(x)$ is uniformly convergent under the same conditions as the series for $F(x)$: see Art. 45 (2).

logarithms from the formulae, we shall introduce a function $H(x)$, defined by

$$\log\left\{\frac{H(x)}{x}\right\}=\int_0^x\left\{G(\xi)-\frac{1}{\xi}\right\}d\xi,$$

so that $$H'(x)/H(x)=G(x).$$

Thus we have

$$\log\left\{\frac{H(x)}{x}\right\}=\sum_{-\infty}^{+\infty}{}'\left\{\log\left(\frac{n\omega-x}{n\omega}\right)+\frac{x}{n\omega}\right\}$$

or $$H(x)=x\prod_{-\infty}^{\infty}{}'\left\{e^{x/n\omega}\left(1-\frac{x}{n\omega}\right)\right\}.$$

Also, if $|x|<|\omega|$, we have

$$\log\left\{\frac{H(x)}{x}\right\}=-\tfrac{1}{2}c_0x^2-\tfrac{1}{12}c_2x^4-\tfrac{1}{30}c_4x^6-\dots.$$

Thus $H(x)$ is also an odd function of x, and tends to zero as $x\to 0$, in such a way that $H(x)/x\to 1$.

Again we have $$\frac{H'(x+\omega)}{H(x+\omega)}=\frac{H'(x)}{H(x)}$$

in virtue of the periodic property of $G(x)$. Thus we find

$$\frac{H(x+\omega)}{H(x)}=\text{const.}$$

To determine the constant, let us write $x=-\frac{1}{2}\omega$, which gives

$$H(\tfrac{1}{2}\omega)/H(-\tfrac{1}{2}\omega)=-1,$$

because $H(x)$ is an odd function of x.

Thus, in general $$H(x+\omega)=-H(x),$$

and so $$H(x+2\omega)=-H(x+\omega)=+H(x).$$

Thus $H(x)$ *is an odd function of* x, *with period* 2ω *and such that* $H(x)/x\to 1$ *as* $x\to 0$, *while* $H(x+\omega)=-H(x)$.

Now $H(x)$ is continuous* for all values of x, such that $|x|<|\omega|$; and thus, from the periodic character of $H(x)$, it is clear that $H(x)$ is continuous for all finite values of x.

Assuming that these properties uniquely determine the sine-function,† we can write

$$H(x)=\frac{\omega}{\pi}\sin\left(\frac{\pi x}{\omega}\right),$$

* The series for $\log\{H(x)/x\}$, being derived by integrating a uniformly convergent series, is itself uniformly convergent (Art. 45); and thus the function is continuous.

† It seems impossible to *prove* this without some reference to Theory of Functions.

and then the two other functions are

$$G(x)=\frac{H'(x)}{H(x)}=\frac{\pi}{\omega}\cot\left(\frac{\pi x}{\omega}\right),$$

$$F(x)=-G'(x)=\left(\frac{\pi}{\omega}\right)^2\operatorname{cosec}^2\left(\frac{\pi x}{\omega}\right).$$

Thus, the various series of the present article all agree with those obtained in Arts. 70–71·1.

We are now able to obtain the coefficients $c_0, c_2, c_4, \ldots$ in finite terms.

Write for brevity $\omega=\pi$, so that $H(x)=\sin x$.

Then,* by Art. 59,

$$\begin{aligned}\log\left\{\frac{H(x)}{x}\right\}&=\log\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+\ldots\right)\\&=-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+\ldots\\&\quad-\frac{1}{2}\left(\quad\frac{x^4}{36}-\frac{x^6}{360}+\ldots\right)\\&\quad+\frac{1}{3}\left(\qquad-\frac{x^6}{216}+\ldots\right)\\&\quad-\ldots\\&=-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}-\ldots.\end{aligned}$$

Hence $$c_0=\frac{1}{3},\quad c_2=\frac{1}{15},\quad c_4=\frac{2}{189},\ \ldots,$$

giving $$\sum_1^\infty\frac{1}{n^2}=\frac{\pi^2}{6},\quad\sum_1^\infty\frac{1}{n^4}=\frac{\pi^4}{90},\quad\sum_1^\infty\frac{1}{n^6}=\frac{\pi^6}{945},\ \ldots$$

(compare Art. 100 below).

It is convenient to note also that these values (for c_0, c_2, c_4) lead to the expansions:

$$G(x)=\cot x=\frac{1}{x}-\frac{1}{3}x-\frac{1}{45}x^3-\frac{2}{945}x^5-\ldots,$$

$$F(x)=\operatorname{cosec}^2 x=\frac{1}{x^2}+\frac{1}{3}+\frac{1}{15}x^2+\frac{2}{189}x^4+\ldots.$$

* The justification for these steps is contained in Art. 36; the present case corresponds to $b_0=0$, in the notation of that article.

EXAMPLES.

Reference may also be made to Examples on Ch. XI.

1. Shew that, if $|x| < 1$,

$$\frac{(1-x)\cos\theta}{1-2x\cos 2\theta+x^2}=\cos\theta+x\cos 3\theta+x^2\cos 5\theta+\ldots+x^n\cos(2n+1)\theta+\ldots,$$

and deduce that, if $\gamma=2\cos 2\theta$,

$$\frac{\cos 13\theta}{\cos\theta}=\gamma^6-\gamma^5-5\gamma^4+4\gamma^3+6\gamma^2-3\gamma-1.$$

2. From the formula (of Art. 68)

$$\sin(m\theta)=mx-\frac{m(m^2-1)}{3!}x^3+\ldots \text{ to } \infty,$$

(where $\theta=\text{arc}\sin x$ and lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$), obtain the power-series for $(\text{arc}\sin x)^3$: namely

$$\frac{1}{6}(\text{arc}\sin x)^3=\frac{1}{2}\frac{x^3}{3}+\left(\frac{1}{1^2}+\frac{1}{3^2}\right)\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\ldots \text{ to } \infty.$$

Obtain similarly the formula of Ex. B. 4, Ch. VIII., for $\frac{1}{2}(\text{arc}\sin x)^2$.

3. From formula (B) of Art. 71·1, prove that

$$\Sigma\cot(\theta+r\alpha)\operatorname{cosec}^2(\theta+r\alpha)=n^3\cot n\theta\operatorname{cosec}^2 n\theta.$$

Deduce that, if $\beta=\frac{1}{4}\alpha=\pi/4n$,

$$2n^3=\cot\beta\operatorname{cosec}^2\beta-\cot 3\beta\operatorname{cosec}^2 3\beta+\cot 5\beta\operatorname{cosec}^2 5\beta-\ldots \text{ to } n \text{ terms.}$$

[*Math. Trip.* 1901.]

4. If n is odd and equal to $2m+1$, shew that

$$\sum_{r=1}^{m}\tan^4(r\pi/n)=\tfrac{1}{6}n(n-1)(n^2+n-3). \qquad [\textit{Math. Trip.}\ 1903.]$$

5. If n is odd, shew that

$$\sum_{r=1}^{n-1}\operatorname{cosec}^2(r\pi/n)=\tfrac{1}{3}(n^2-1).$$

If $n=abc\ldots k$, where $a, b, c, \ldots, k$ are primes, shew that the above sum, if extended *only to values of r which are prime to n*, is equal to

$$\tfrac{1}{3}(a^2-1)(b^2-1)(c^2-1)\ldots(k^2-1). \qquad [\textit{Math. Trip.}\ 1902.]$$

6. (a) Shew that the roots of the equation

$$x^3+x^2-2x-1=0$$

are

$$2\cos(\tfrac{2}{7}\pi),\quad 2\cos(\tfrac{4}{7}\pi),\quad 2\cos(\tfrac{6}{7}\pi).$$

[Write $x+2=\gamma^2$ in the formula of Art. 66.]

6. (b) Shew that the roots of the equation $x^3-x^2+\frac{1}{7}=0$ are given by

$$x\sqrt{7}=2\sin(\tfrac{2}{7}\pi),\quad 2\sin(\tfrac{4}{7}\pi),\quad 2\sin(\tfrac{8}{7}\pi).$$

[If $y=2\sin\theta$, the formula of Art. 67 can be written

$$y^3=\pm\sqrt{7}(y^2-1).$$

To distinguish between the two sets of three roots note that $2\sin(\frac{2}{7}\pi)$, $2\sin(\frac{4}{7}\pi)$ are both greater than 1, while $2\sin\frac{8}{7}\pi$ lies between -1 and 0.]

6. (*c*) If $\sin\alpha=\sqrt{3}/4$, shew that

$$\sin 7\alpha=\sqrt{3}/256.$$

Deduce that the side of a regular heptagon inscribed in a circle of radius r is nearly equal to the height of an equilateral triangle whose side is equal to r.
[Ex. A. 38, Ch. VIII.]

7. Prove that

$$\prod_{-\infty}^{\infty}\left\{1-\frac{x^2}{(n+c)^2}\right\}=1-\frac{\sin^2\pi x}{\sin^2\pi c} \quad\text{and}\quad \frac{x+c}{c}\prod_{-\infty}^{\infty}\left(1+\frac{x}{n+c}\right)e^{-x/n}=\frac{\sin\pi(x+c)}{\sin\pi c}.$$
[EULER.]

Determine the value of the limit of the product $\prod_{-M}^{N}\left(1+\frac{x}{n+c}\right)$ when $M, N\to\infty$ so that $N/M\to k$.

8. Prove that $$\prod_{-\infty}^{\infty}\left\{1-\frac{4x^2}{(n\pi+x)^2}\right\}=-\frac{\sin 3x}{\sin x}.$$ [EULER.]

9. Shew that

$$(1-x)(1+\tfrac{1}{3}x)(1-\tfrac{1}{5}x)(1+\tfrac{1}{7}x)\ldots=\cos(\tfrac{1}{4}\pi x)-\sin(\tfrac{1}{4}\pi x).$$

[Group the terms in pairs and apply Ex. 11, Ch. VI., obtaining

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\Big/\Gamma\left(\frac{1-x}{4}\right)\Gamma\left(\frac{3+x}{4}\right);$$

or write out the product form for $\sin\{\frac{1}{4}\pi(1-x)\}/\sin(\frac{1}{4}\pi)$.]

10. Shew directly from the products for $\sin x$ and $\cos x$ that

$$\sin(x+\tfrac{1}{2}\pi)=\cos x,\quad \cos(x+\tfrac{1}{2}\pi)=-\sin x,$$

and deduce the periodic properties of the sine and cosine.

11. Deduce the infinite product for $\sin x$ from the equation

$$\sin(\pi x)=2x\int_0^{\frac{1}{2}\pi}\cos(2xt)\,dt$$

by means of the series for $\cos(2xt)$ in powers of $\sin t$ (Art. 68).

12. If $$u_n=\frac{(n+a_1)(n+a_2)\ldots(n+a_k)}{(n+b_1)(n+b_2)\ldots(n+b_k)},$$

where $\Sigma a=\Sigma b$ and none of $b_1, b_2, \ldots, b_k$ are zero, then

$$\prod_{-\infty}^{\infty}u_n=\frac{\sin(a_1\pi)\sin(a_2\pi)\ldots\sin(a_k\pi)}{\sin(b_1\pi)\sin(b_2\pi)\ldots\sin(b_k\pi)}=P.$$ [EULER.]

If $\Sigma b-\Sigma a=\delta$, shew that

$$u_0\prod_{-\infty}^{\infty}{}'(u_n e^{\delta/n})=\lim_{\nu\to\infty}\prod_{-\nu}^{+\nu}u_n=P,$$

and that $$\lim_{M,N\to\infty}\prod_{-M}^{N}u_n=L^{-\delta}P,$$

where $L=\lim(N/M)$.

Find the value of $\prod_{-\infty}^{\infty}{}'(u_n e^{\delta/n})$ when $n=0$ is excluded and some of $b_1, b_2, \ldots, b_k$, are zero.

13. Prove that $\lim_{\nu \to \infty} \left\{ \sum_{m=-\nu}^{\nu} \sum_{n=-\nu}^{\nu} \frac{1}{(x-m)(x-n)} \right\} = -\pi^2$,

where in the double summation all values $m=n$ are excluded.

[*Math. Trip.* 1895.]

14. Shew that

$$\pi \{\cot(\pi x) - \cot(\pi a)\} = \sum_{-\infty}^{\infty} \left(\frac{1}{x-n} - \frac{1}{a-n} \right) = \sum_{-\infty}^{\infty} \frac{a-x}{(x-n)(a-n)},$$

$$\pi^3 \cot(\pi x) \operatorname{cosec}^2(\pi x) = \sum_{-\infty}^{\infty} \frac{1}{(x-n)^3},$$

$$\pi^4 \left\{ \operatorname{cosec}^4(\pi x) - \frac{2}{3} \operatorname{cosec}^2(\pi x) \right\} = \sum_{-\infty}^{\infty} \frac{1}{(x-n)^4}.$$

[EULER.]

15. Shew that

$$\operatorname{cosec} \theta = \frac{1}{\theta} - \left(\frac{1}{\theta - \pi} + \frac{1}{\theta + \pi} \right) + \left(\frac{1}{\theta - 2\pi} + \frac{1}{\theta + 2\pi} \right) - \dots,$$

$$\sec \theta = -\left(\frac{1}{\theta - \frac{1}{2}\pi} - \frac{1}{\theta + \frac{1}{2}\pi} \right) + \left(\frac{1}{\theta - \frac{3}{2}\pi} - \frac{1}{\theta + \frac{3}{2}\pi} \right) + \dots .$$

[We can get the expansion for cosec θ from the identity

$$\operatorname{cosec} \theta = \cot(\tfrac{1}{2}\theta) - \cot \theta.]$$

Deduce that

$$\frac{\pi}{4} \left(\sec \frac{\pi}{2x} - 1 \right) = \frac{1}{x^2 - 1} - \frac{1}{3^2 x^2 - 3} + \frac{1}{5^2 x^2 - 5} - \frac{1}{7^2 x^2 - 7} + \dots .$$

[EULER.]

16. Shew that

$$\frac{\pi}{2} \cot \frac{\pi z}{2} = \frac{1}{z} + \left(\frac{1}{z-2} + \frac{1}{z+2} \right) + \left(\frac{1}{z-4} + \frac{1}{z+4} \right) + \dots,$$

$$\frac{\pi}{2} \tan \frac{\pi z}{2} = -\left(\frac{1}{z-1} + \frac{1}{z+1} \right) - \left(\frac{1}{z-3} + \frac{1}{z+3} \right) - \dots .$$

Deduce that $\frac{1}{x-y} + \sum_{-\infty}^{\infty}{}' \left\{ \frac{1}{n + x + (-1)^{n-1} y} - \frac{1}{n} \right\} = \frac{\pi \cos \pi x}{\sin \pi x - \sin \pi y}$,

$n=0$ being excluded from the summation. [*Math. Trip.* 1896.]

17. Find the sum of the series

$$\sum_{-\infty}^{\infty}{}' \frac{5n-2}{6n^3 - 5n^2 + n},$$

the value $n=0$ being excluded.

$\left[\text{Note that} \quad \frac{5n-2}{6n^3 - 5n^2 + n} = \left(\frac{1}{n - \frac{1}{2}} - \frac{1}{n} \right) + \left(\frac{1}{n - \frac{1}{3}} - \frac{1}{n} \right). \right]$

18. If the general term of a series Σu_n can be divided into partial fractions in the form

$$u_n = \sum_a \frac{A}{n+a}, \quad \text{where } \Sigma A = 0,$$

then
$$\sum_{-\infty}^{\infty} u_n = \sum_a A \pi \cot(a\pi),$$

where all the numbers a are supposed different from zero.

19. If the general term u_n of a series is expressed in the form

$$\frac{A}{(2n)^3}+\frac{B}{(2n)^2}+\frac{C}{2n}+\frac{A'}{(2n-1)^3}+\frac{B'}{(2n-1)^2}+\frac{C'}{2n-1},$$

prove that $\sum_1^\infty u_n$ can be summed by means of these formulae, provided that

$$A+7A'=0 \quad \text{and} \quad C+C'=0.$$

Apply to the particular example

$$u_n=\frac{48n^2-36n+7}{8n^3(2n-1)^3},$$

proving that the sum is equal to $\frac{1}{4}\pi^2$.

20. We have seen (Ch. VI., Ex. 17) that if

$$S_t=\left\{\frac{\Gamma(1)\,\Gamma(t)}{\Gamma(1+t)}\right\}^2+\left\{\frac{\Gamma(2)\,\Gamma(t)}{\Gamma(2+t)}\right\}^2+\left\{\frac{\Gamma(3)\,\Gamma(t)}{\Gamma(3+t)}\right\}^2+\dots,$$

then $\quad S_{t+1}=\dfrac{2}{t}(2t-1)S_t-\dfrac{3}{t^2}, \quad$ if $t>\frac{1}{2}$.

Now $\quad S_1=\dfrac{1}{6}\pi^2$ by Art. 71·1 so that $S_2=\dfrac{1}{3}(\pi^2-9), \quad S_3=\pi^2-\dfrac{39}{4}$.

Thus $\quad \left(\dfrac{1}{1\,.\,2\,.\,3}\right)^2+\left(\dfrac{1}{2\,.\,3\,.\,4}\right)^2+\left(\dfrac{1}{3\,.\,4\,.\,5}\right)^2+\dots=\dfrac{1}{16}(4\pi^2-39)$.

CHAPTER X.

COMPLEX SERIES AND PRODUCTS.

72. The algebra of complex numbers.

We assume that the reader has already become acquainted with the leading features of the algebra of complex numbers. The fundamental laws of operation are as follows:

If $\qquad x=\xi+\iota\eta, \quad x'=\xi'+\iota\eta',$

then
$$x+x'=\xi+\xi'+\iota(\eta+\eta'), \qquad (addition)$$
$$x-x'=\xi-\xi'+\iota(\eta-\eta'), \qquad (subtraction)$$
$$xx'=\xi\xi'-\eta\eta'+\iota(\xi\eta'+\xi'\eta), \qquad (multiplication)$$
$$\frac{x'}{x}=\frac{\xi\xi'+\eta\eta'}{\xi^2+\eta^2}+\iota\,\frac{\xi\eta'-\xi'\eta}{\xi^2+\eta^2}, \qquad (division).$$

It is easily seen that these laws include those of real numbers as a special case; and that these four operations can be carried out without exception (excluding division by zero). Further, these laws are consistent with the relations with which we are familiar for real numbers, such as
$$x(y+z)=xy+xz,$$
$$xy=yx, \quad x(yz)=(xy)z.$$

Thus any of the ordinary algebraic identities, which are established in the first instance for real numbers, are still true if the letters are supposed to represent complex numbers.

It is natural to ask whether other assumptions might not be made which would be equally satisfactory. Thus the analogy for addition might suggest for multiplication
$$xx'=\xi\xi'+\iota\eta\eta'.$$

But this is inconsistent with the relation $x+x=2x$, since $x'=2$ would then give
$$2x=2\xi+\iota(0)=2\xi,$$
whereas
$$x+x=\xi+\xi+\iota(\eta+\eta)=2\xi+2\iota\eta.$$

Since the assumption $\iota^2 = -1$ together with the ordinary associative, commutative and distributive laws are sufficient to fix the law of multiplication, we might try to find some other law of multiplication, by assuming that $\iota^2 = \alpha + \iota\beta$, where α, β are some fixed real numbers. It can then be shewn (see Stolz, *Allgemeine Arithmetik*, Bd. II., pp. 8–12) that we are either led back to the assumptions made above, or else we are forced to admit the existence of numbers x_1, x_2 such that the product $x_1 x_2$ is zero without either x_1 or x_2 being zero. Thus the assumption $\iota^2 = -1$ corresponds to the only *simple* natural extension of the laws of algebra as formulated for real numbers.

73. Argand's diagram.

The reader is doubtless also familiar with the usual representation of the complex number* $x = \xi + \iota\eta$ by a point with rectangular coordinates (ξ, η).

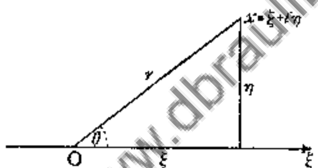

FIG. 19.

Nevertheless it may be convenient to give a brief summary of the method.

If we introduce polar coordinates r, θ, we can write

$$x = r(\cos\theta + \iota\sin\theta).$$

We shall call $r = (\xi^2 + \eta^2)^{\frac{1}{2}}$ the *absolute value* of x (it is sometimes also called the *modulus* of x); and we shall denote it by the symbol $|x|$. This, of course, is quite consistent with the notation used previously; for if x is *real*, $|x|$ will be either $+x$ or $-x$, according as x is positive or negative.

We call θ the *argument* of x: it is sometimes called the *phase* or *amplitude* of x.

From the diagram the meaning of $x + x'$, $x - x'$ is now evident.

If we draw through A, AB, CA equal and parallel to OA', then B, C are respectively $x + x'$ and $x - x'$. The fact that $x + x' = x' + x$

* Probably the reader will be more accustomed to the notation $z = x + \iota y$ for a complex number and its "real and imaginary" parts. In the present account x has been deliberately adopted to represent a complex number, so as to bring out as strongly as possible the points of similarity between power-series of the real and of the complex variable.

is represented by the geometrical theorem that $A'B$ is equal and parallel to OA (Euclid I. 33).

Since $\quad OB < OA + AB \quad$ or $\quad OB < OA + OA'$,

we have the relation $\quad |x+x'| < |x| + |x'|$,

and similarly, $\quad |x-x'| < |x| + |x'|$.

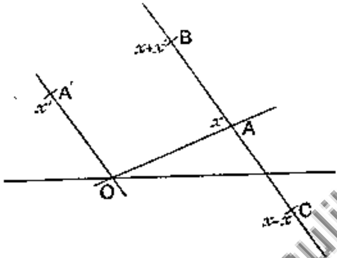

FIG. 20.

Again, supposing $OA' < OA$, we have

$$OB + AB > OA \quad \text{or} \quad OB > OA - OA'.$$

Thus, and so also

$$\left.\begin{aligned}|x+x'| &> |x| - |x'| \\ |x-x'| &> |x| - |x'|\end{aligned}\right\}, \quad \text{if } |x| > |x'|.$$

It is easy to prove similarly that

$$|x+y+z+w| < |x| + |y| + |z| + |w|,$$

and generally, that $\quad |\Sigma x| < \Sigma |x|$.

These facts can also be proved algebraically; for example, consider the first inequality and write

$$R = |x+x'|, \quad \text{so that } R^2 = (\xi + \xi')^2 + (\eta + \eta')^2.$$

Then we have $\quad R^2 = r^2 + r'^2 + 2(\xi\xi' + \eta\eta')$.

Hence $\quad (r+r')^2 - R^2 = 2(rr' - \xi\xi' - \eta\eta')$,

and this is certainly positive if $\xi\xi' + \eta\eta'$ is zero or negative. But if $\xi\xi' + \eta\eta'$ is positive, we have

$$(rr')^2 - (\xi\xi' + \eta\eta')^2 = (\xi\eta' - \xi'\eta)^2,$$

so that $\quad rr' \geqq \xi\xi' + \eta\eta'$,

the sign of equality occurring only if $\quad \xi\eta' - \xi'\eta = 0$.

Thus in all cases $\quad (r+r')^2 - R^2 \geqq 0$

or $\quad r + r' - R \geqq 0$,

and the sign of equality can appear only if $\xi\eta' - \xi'\eta = 0$ and $\xi\xi' + \eta\eta' > 0$; which is represented geometrically by supposing that OA' falls along OA.

On the other hand, when this inequality is satisfied, we have also

$$|X_n - X_m| < \epsilon, \quad |Y_n - Y_m| < \epsilon, \quad \text{if } n > m,$$

and therefore the sequences (X_n), (Y_n) are convergent.

Thus (S_n) converges; and therefore the condition is *sufficient.*

As an application of this principle we consider the interpretation of t^α, where $t = \cos\theta + \iota\sin\theta$ and α is *irrational.* We note as a preliminary that *

$$\begin{aligned}|(\cos\phi + \iota\sin\phi) - (\cos\psi + \iota\sin\psi)|\\ &= \sqrt{\{(\cos\phi - \cos\psi)^2 + (\sin\phi - \sin\psi)^2\}} = \sqrt{\{2 - 2\cos(\phi-\psi)\}}\\ &= 2\,|\sin\tfrac{1}{2}(\phi-\psi)| < |\phi - \psi|.\end{aligned}$$

Now, if (α_n) is any sequence of *rational* numbers which has α as its limit, we can find m so that

$$|\alpha_n - \alpha_m| < \epsilon, \quad \text{if } n > m.$$

Thus

$$|\{\cos(\alpha_n\theta) + \iota\sin(\alpha_n\theta)\} - \{\cos(\alpha_m\theta) + \iota\sin(\alpha_m\theta)\}| < \epsilon\,|\theta|$$

if $n > m$; and so the sequence $t^{\alpha_n} = \cos(\alpha_n\theta) + \iota\sin(\alpha_n\theta)$ is convergent. It is therefore natural to define t^α as $\lim t^{\alpha_n}$; but it is of course to be remembered that *all* the limits †

$$\lim\{\cos\alpha_n(\theta + 2k\pi) + \iota\sin\alpha_n(\theta + 2k\pi)\} \quad (k = \pm 1, \pm 2, \pm 3, \text{ to } \infty)$$

may equally well be regarded as included in the symbol t^α. Thus special care must be taken to specify the meaning to be attached to t^α; for most purposes it is sufficient to retain only the value which reduces to 1 when θ is zero (that is, the value given by $k=0$).

Convergence and divergence of a series of complex terms.

If $a_n = \xi_n + \iota\eta_n$, we have

$$a_1 + a_2 + \ldots + a_n = (\xi_1 + \xi_2 + \ldots + \xi_n) + \iota(\eta_1 + \eta_2 + \ldots + \eta_n) = X_n + \iota Y_n.$$

Then if X_n, Y_n are separately convergent to the limits s, t respectively as n tends to ∞, we say that

$$a_1 + a_2 + \ldots \text{ to } \infty$$

converges to the sum $s + \iota t$.

But if either X_n or Y_n diverges so that $|X_n + \iota Y_n| = |\Sigma a_n|$ diverges, we shall say that Σa_n diverges; and generally, we shall call a series *divergent* if $|X_n + \iota Y_n|$ diverges.

* The geometrical interpretation of this inequality is that a chord of a circle is less than the arc.

† All these values are unequal, because no integer k makes $k\alpha$ equal to an integer; of course k must not vary with n.

It is easy to see from this definition and the foregoing discussion for sequences that *the necessary and sufficient condition for convergence* is simply that, corresponding to any real positive number ϵ, we can find m such that

$$|a_{m+1}+a_{m+2}+\ldots+a_{m+p}|<\epsilon,$$

no matter how great p may be.

It is not very easy to frame a definition of *oscillatory* series of complex numbers, which shall be consistent with the definition adopted for series of real terms in Art. 6. It is clear that in the first place there are *nine* possible types for the behaviour of a complex number $X_n+\iota Y_n$, as n tends to infinity; for each part X_n, Y_n may converge (C), diverge (D), or oscillate (O). Of these nine types, *one* has been defined as *convergent* (represented by C, C); and *five* as divergent (represented by C, D; O, D; D, D; D, C; D, O). The remaining *three* are represented by C, O; O, O; O, C; and may fairly be called *oscillatory*.

On the other hand, if the limits of oscillation of either part are $-\infty$ to $+\infty$, then the absolute value $|X_n+\iota Y_n|$ *may* tend to $+\infty$ as a limit.*

However, to avoid minute discussions of so many possible types in dealing with actual series, it seems to be simplest to agree to call all series *divergent* for which $|X_n+\iota Y_n|\to+\infty$, even though this agreement will now class some series as divergent, which might be called oscillatory with more accuracy.†

We can formulate a similar definition for the convergence of an infinite product of complex factors; but as a rule we need only absolutely convergent products, which will be discussed in the next article.

76. Absolute convergence of a series of complex terms.

If $a_n=\xi_n+\iota\eta_n$ and if $\Sigma|a_n|$ is convergent, we shall say that Σa_n is *absolutely convergent*. It is evident in this case, from Art. 18,

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* It would *not* do so if, say,

$$X_n=n\left\{1+2\cos\left(\frac{n\pi}{3}\right)\right\},\quad Y_n=1.$$

For then $X_n\to+\infty$ if $n=6m$, $6m+1$, or $6m+5$,

$X_n\to-\infty$ if $n=6m+3$,

$X_n=0$ if $n=6m+2$ or $6m+4$.

Thus $|X_n+\iota Y_n|\to\infty$ except for $n=6m+2$ or $6m+4$,

but $|X_n+\iota Y_n|=1$ for these values.

† For example, $X_n=(-1)^n n,\quad Y_n=1/n$

gives $|X_n+\iota Y_n|=\sqrt{(n^2+1/n^2)}\to\infty.$

On the other hand, the series

$$X_n=(-1)^n n,\quad Y_n=0$$

(which consists purely of *real* terms) would be classed as *divergent* by this definition; and as *oscillatory* by the definition of Art. 6.

that the separate series $\Sigma\xi_n$, $\Sigma\eta_n$ are both absolutely convergent, because

$$|\xi_n| \leqq |a_n| \quad \text{and} \quad |\eta_n| \leqq |a_n|.$$

It follows therefore that Σa_n is convergent in virtue of our definition (Art. 75); and by Art. 26 the sums $\Sigma\xi_n$, $\Sigma\eta_n$ are independent of the order of the terms. Hence also, *the sum of an absolutely convergent series is independent of the order of arrangement.*

It is probably obvious from what has been said as to series of real terms, that absolute convergence is not *necessary* for convergence. An example of a series of complex terms which converges, although not absolutely, is

$$\iota + \tfrac{1}{2}(\iota)^2 + \tfrac{1}{3}(\iota)^3 + \tfrac{1}{4}(\iota)^4 + \dots$$
$$= -(\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \dots) + \iota(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots).$$

For both real and imaginary parts converge, by Art. 19, and the sum is $-\frac{1}{2}\log 2 + \frac{1}{4}\pi\iota$, by Arts. 62, 64; but the series of absolute values is

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots,$$

which diverges by Art. 7 (Ex. 2) or Art. 11.

It is evident from Arts. 25, 28 that the sum of a non-absolutely convergent series may be altered by derangement.

Absolute convergence of an infinite product of complex factors.

The infinite product $\Pi(1+a_n)$ is said to be *absolutely convergent* if the product $\Pi(1+|a_n|)$ is convergent. It follows at once that *if Σa_n is absolutely convergent, so also is $\Pi(1+a_n)$, and conversely.*

For we know that $\Sigma|a_n|$ converges: and so by Art. 39 the two products $\Pi(1+|a_n|)$ and $\Pi(1-|a_n|)$ are convergent. Similarly, if either of these two products is convergent, so also is $\Sigma|a_n|$.

However, it is not quite obvious that if $\Sigma|a_n|$ converges, then $\Pi(1+a_n)$ must also converge, in the sense that we can find m, so that

$$R_m^p = \prod_{m+1}^{m+p}(1+a_n)$$

is as close to unity as we please, however large p may be.

To establish this property, we observe that if

$$A_m^p = \prod_{m+1}^{m+p}(1+|a_n|),$$

then *every* term in $(R_m^p - 1)$ has a corresponding term in $(A_m^p - 1)$; but in the latter, *every term is positive.* Hence

$$|R_m^p - 1| \leqq A_m^p - 1.$$

But $$A_m^p < \prod_{m+1}^{\infty}\{1+|a_n|\} < 1/(1-\eta_m), \qquad \text{(by Art. 38)}$$

where $$\eta_m = \sum_{m+1}^{\infty}|a_n|.$$

Thus we have $$|R_m^p - 1| < \eta_m/(1-\eta_m),$$

and since $\eta_m \to 0$ as $m \to \infty$ (because $\Sigma|a_n|$ is convergent), we can find m to make R_m^p as close to unity as we please.

The proof of Art. 40 needs only a few verbal alterations to shew that *if Σa_n is absolutely convergent the value of $\Pi(1+a_n)$ is unaltered by changing the order of the factors.*

77. Extension of Maclaurin's integral test.

It can be proved that* if $f(x)$ tends steadily to zero, as x tends to infinity, and if $\phi(x)$ tends steadily to infinity while $\phi'(x)$ tends steadily to zero, in such a way that the integral

$$\int^{\infty} f(x)\,\phi'(x)\,dx$$

is convergent, then the two series

$$\Sigma f(n)\cos\{\phi(n)\}, \quad \Sigma f(n)\sin\{\phi(n)\}$$

converge or diverge with the integrals

$$\int^{\infty} f(x)\cos\{\phi(x)\}\,dx, \quad \int^{\infty} f(x)\sin\{\phi(x)\}\,dx$$

respectively.

Without going into the proof of this general theorem, we shall investigate the most useful special case—namely the series†

$$\Sigma n^{-\mu}, \text{ where } \mu = \alpha + \iota\beta, \quad (\alpha > 0).$$

Now let us write

$$U_n = \frac{1}{n^{\mu}} - \int_n^{n+1}\frac{dx}{x^{\mu}},$$

and then‡ $$U_n = \int_n^{n+1} dx\left(\frac{1}{n^{\mu}} - \frac{1}{x^{\mu}}\right) = \int_n^{n+1} dx \int_n^x \frac{\mu\, d\xi}{\xi^{\mu+1}}.$$

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* See Bromwich, *Proc. Lond. Math. Soc.* (2), vol. 6, 1908, p. 327; and G. H. Hardy, *ibid.*, vol. 9, 1910, p. 126.

† In the notation of the general theorem, this series is given by taking

$$f(x) = x^{-\alpha}, \quad \phi(x) = \beta \log x.$$

See Arts. 93, 96; and note that the conditions laid down for the general theorem are verified here.

‡ We assume that the ordinary rules for differentiation and integration remain formally unaltered when complex indices are used. This assumption can be fully justified quite easily as soon as the definition of complex powers has been fully cleared up: for instance, on the lines adopted in Arts. 93, 96 below.

Thus, if we write $$|\mu| = \sqrt{(\alpha^2+\beta^2)} = \rho, \text{ say},$$

we have $$|U_n| \leqq \int_n^{n+1} dx \int_n^x \frac{\rho\, d\xi}{\xi^{\alpha+1}},$$

because $|\xi^{\mu+1}| = \xi^{\alpha+1}$, or

$$|U_n| \leqq \frac{\rho}{\alpha}\int_n^{n+1} dx\left(\frac{1}{n^\alpha}-\frac{1}{x^\alpha}\right).$$

Now $\alpha > 0$, by hypothesis; and in the last integral x lies between n and $n+1$, so that

$$\frac{1}{n^\alpha}-\frac{1}{x^\alpha} \leqq \frac{1}{n^\alpha}-\frac{1}{(n+1)^\alpha}.$$

Substituting, we find that

$$|U_n| \leqq \frac{\rho}{\alpha}\left\{\frac{1}{n^\alpha}-\frac{1}{(n+1)^\alpha}\right\}.$$

Thus, summing to n terms, we find that

$$|U_1|+|U_2|+\ldots+|U_n| \leqq \frac{\rho}{\alpha}\left\{1-\frac{1}{(n+1)^\alpha}\right\} \leqq \frac{\rho}{\alpha}, \quad \text{since } \alpha > 0.$$

And thus, by Art. 2, the series $\Sigma|U_n|$ is convergent; or ΣU_n is absolutely convergent, and its sum is less than ρ/α in absolute value.

Thus,* on writing out the sum of ΣU_n to $(n-1)$ terms, we see that

$$1+\frac{1}{2^\mu}+\frac{1}{3^\mu}+\ldots+\frac{1}{n^\mu}-\int_1^n \frac{dx}{x^\mu}$$

tends to a definite limit as n tends to infinity.

But $$\int_1^n \frac{dx}{x^\mu} = \frac{1}{\mu-1}\left(1-\frac{1}{n^{\mu-1}}\right),$$

and so, if $\alpha > 1$, this integral tends to the limit $1/(\mu-1)$ as $n\to\infty$; and so the series $\Sigma n^{-\mu}$ is then convergent.

On the other hand, if $0<\alpha<1$, the integral behaves like $n^{1-\mu}/(1-\mu)$ as $n\to\infty$ (because the absolute value of this expression also tends to infinity); and accordingly the same holds for the sum of $\Sigma n^{-\mu}$. Or, in symbols,

$$1+\frac{1}{2^\mu}+\ldots+\frac{1}{n^\mu} \sim \frac{n^{1-\mu}}{1-\mu}, \quad \text{if } 0<\alpha<1.$$

If we adopt the convention of Art. 75, the series $\Sigma n^{-\mu}$ would be called divergent, if $0<\alpha<1$; because its absolute value tends

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* Note that $|n^{-\mu}| = n^{-\alpha}$, and so $n^{-\mu}$ tends to zero as n tends to infinity, because $\alpha > 0$.

to infinity, although both real and imaginary parts oscillate between $-\infty$ and $+\infty$.

The only case remaining for discussion is given by $\alpha=1$; and then $n^{1-\mu}/(1-\mu)$ oscillates between *finite* limits, so that the same is true for the series* $\Sigma n^{-\mu}$. All that we can then assert is that *the range of oscillation* is equal to $2/\beta$ for both real and imaginary parts of the sum.

Hence, finally, we sum up our results as follows:

The series $\Sigma n^{-\mu}$ (*where* $\mu=\alpha+\iota\beta$) *is convergent if* $\alpha>1$; *divergent* (*in the sense of Art.* 75) *if* $\alpha<1$, *and the sum to n terms is represented asymptotically by the formula*

$$n^{1-\mu}/(1-\mu).$$

When $\alpha=1$ (*and* β *is not zero*), *the series oscillates finitely*; *and the range of oscillation is equal to* $2/\beta$, *both for the real and for the imaginary parts of the sum.*

78. Ratio-Tests for absolute convergence.

Since the series to be tested is $\Sigma|a_n|$, which consists solely of *positive* terms, it is evident that we can apply at once any suitable test from Chap. II.; but nearly every series of importance is covered by a natural extension of the ratio-test (5) of Art. 12·2. Thus it seems worth while to formulate this extension in a form specially convenient for application to series of complex terms; similar extensions for other tests will be given in the small type below.

We suppose then that the quotient a_n/a_{n+1} can be expressed in the form

$$\frac{a_n}{a_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^\lambda}\right), \quad \begin{cases}\mu=\alpha+\iota\beta\\ \lambda>1\end{cases}$$

where in most ordinary series the index λ is 2.

Then, to test for absolute convergence, we form the quotient of absolute values

$$\frac{|a_n|}{|a_{n+1}|}=\left|1+\frac{\mu}{n}+O\left(\frac{1}{n^\lambda}\right)\right|.$$

This expression is clearly equal to

$$\sqrt{\left[\left\{1+\frac{\alpha}{n}+O\left(\frac{1}{n^\lambda}\right)\right\}^2+\left\{\frac{\beta}{n}+O\left(\frac{1}{n^\lambda}\right)\right\}^2\right]}=\sqrt{\left\{1+\frac{2\alpha}{n}+O\left(\frac{1}{n^\kappa}\right)\right\}},$$

where κ is the smaller of the indices λ and 2.

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* Note that the *range* of oscillation is the same for the series as for the integral: although the actual limits of oscillation are *not* the same. This is due to the fact that the difference tends to a definite limit.

Thus we have

$$\frac{|a_n|}{|a_{n+1}|}=1+\frac{\alpha}{n}+O\left(\frac{1}{n^\kappa}\right), \quad (\kappa>1)$$

and so the results of (5), Art. 12·2, can be immediately applied, shewing that $\Sigma|a_n|$ converges if $\alpha>1$, and diverges if $\alpha \leqq 1$.

Thus the rule now applicable to series of complex terms is:

If the quotient a_n/a_{n+1} can be expressed in the form

$$\frac{a_n}{a_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^\lambda}\right), \quad \begin{cases}\mu=\alpha+\iota\beta \\ \lambda>1\end{cases}$$

then the series Σa_n is absolutely convergent, provided that $\alpha>1$; but cannot converge absolutely, *if $\alpha \leqq 1$.**

Pringsheim's tests for absolute convergence of a complex series.†

Since $\quad |a_n|=\sqrt{(\xi_n^2+\eta_n^2)}, \quad$ if $a_n=\xi_n+\iota\eta_n$,

it is evident that the square-root complicates some of the tests given in Chap. II. for series of positive terms.

Of course the tests of Art. 9 can be at once changed to

$$\overline{\lim}\, C_n^2 \cdot |a_n|^2 < \infty, \qquad (convergence)$$

$$\underline{\lim}\, D_n^2 \cdot |a_n|^2 > 0, \qquad (divergence);$$

but the same transformation cannot, as a rule, be applied to the ratio-tests of Arts. 12·1 and 12·2.

Thus, the condition

$$\underline{\lim}\left\{D_n^2\left|\frac{a_n}{a_{n+1}}\right|^2-D_{n+1}^2\right\}>0$$

is by no means sufficient to ensure the convergence of $\Sigma|a_n|$; because whenever $\lim D_n=\infty$ (which is usually the case), the above condition does not exclude the possibility

$$\underline{\lim}\left\{D_n\left|\frac{a_n}{a_{n+1}}\right|-D_{n+1}\right\}=0,$$

and this may occur with a divergent series.

For instance, with $1/a_n=n\log n$ and $D_n=n \quad (n>2)$, we find

$$D_n^2\left|\frac{a_n}{a_{n+1}}\right|^2-D_{n+1}^2>2(n+1)/\log n.$$

Thus, $$\lim\left\{D_n^2\left|\frac{a_n}{a_{n+1}}\right|^2-D_{n+1}^2\right\}=\infty,$$

but yet $\Sigma|a_n|$ diverges, as we have proved in Art. 11 (2); and in fact it will be found that

$$\lim\left\{D_n\left|\frac{a_n}{a_{n+1}}\right|-D_{n+1}\right\}=0.$$

* It may still *converge*; but the condition of *absolute* convergence is broken when $a \leqq 1$. This rule is contained in one given by Weierstrass (see below, Art. 79).

† *Archiv für Math. u. Phys.* (3), Bd. 4, 1902, pp. 1-19.

Pringsheim therefore has introduced the conditions

$$\underline{\lim} \left\{ D_n \left| \frac{a_n}{a_{n+1}} \right|^2 - \frac{D_{n+1}^2}{D_n} \right\} > 0, \quad (convergence)$$

$$\overline{\lim} \left\{ D_n \left| \frac{a_n}{a_{n+1}} \right|^2 - \frac{D_{n+1}^2}{D_n} \right\} < 0, \quad (divergence)$$

which are substantially equivalent to the conditions of Art. 12·1.

For if the first condition is satisfied, we can find ρ and m so that

$$D_n \left| \frac{a_n}{a_{n+1}} \right|^2 - \frac{D_{n+1}^2}{D_n} \geqq \rho > 0, \quad \text{if } n > m.$$

Thus,
$$D_n \left| \frac{a_n}{a_{n+1}} \right| - D_{n+1} \geqq \rho \Big/ \left\{ \left| \frac{a_n}{a_{n+1}} \right| + \frac{D_{n+1}}{D_n} \right\}.$$

Now in all cases of practical interest,* it is possible to assign a fixed upper limit to $\left| \frac{a_n}{a_{n+1}} \right|$ and $\frac{D_{n+1}}{D_n}$, say l; then $D_n \left| \frac{a_n}{a_{n+1}} \right| - D_{n+1} \geqq \rho/2l$, and so the first condition of Art. 12·1 is justified, proving convergence.

But if the second condition holds, we can find m so that

$$\left\{ D_n \left| \frac{a_n}{a_{n+1}} \right| - D_{n+1} \right\} \left\{ \left| \frac{a_n}{a_{n+1}} \right| + \frac{D_{n+1}}{D_n} \right\} < 0, \quad \text{if } n > m$$

so that the first factor must be negative, leading to divergence as in Art. 12·1.

It is easy to transform Pringsheim's conditions by writing $D_n = f(n)$ as in Art. 12·2, and then we find

$$\underline{\lim}\, \kappa_n > 0, \textit{ convergence}; \quad \overline{\lim}\, \kappa_n < 0, \textit{ divergence},$$

where
$$\left| \frac{a_n}{a_{n+1}} \right|^2 = 1 + 2\frac{f'(n)}{f(n)} + \frac{\kappa_n}{f(n)}.$$

The only fresh condition to establish is that $\{f'(n)\}^2/f(n) \to 0$.

For
$$\lim_{x\to\infty} \frac{\{f'(x)\}^2}{f(x)} = \lim_{x\to\infty} \frac{2f'(x)f''(x)}{f'(x)} = \lim_{x\to\infty} 2f''(x),$$

in virtue of L'Hospital's rule (Appendix I., Art. 147); now we assumed that $f''(x) \to 0$, and so $\{f'(x)\}^2/f(x)$ also tends to zero.

In particular, let us take

(1) $f(n) = 1$, (2) $f(n) = n$, (3) $f(n) = n \log n$.

We obtain, after a few transformations, the following conditions: †

(1) $\overline{\underline{\lim}} \left| \frac{a_n}{a_{n+1}} \right|^2 \begin{matrix} > 1, \\ < 1, \end{matrix}$ $\begin{matrix} (convergence) \\ (divergence). \end{matrix}$

(2) $\overline{\underline{\lim}}\, n \left\{ \left| \frac{a_n}{a_{n+1}} \right|^2 - 1 \right\} \begin{matrix} > 2, \\ < 2, \end{matrix}$ $\begin{matrix} (convergence) \\ (divergence). \end{matrix}$

(3) $\overline{\underline{\lim}}\, (\log n) \left[n \left\{ \left| \frac{a_n}{a_{n+1}} \right|^2 - 1 \right\} - 2 \right] \begin{matrix} > 2, \\ < 2, \end{matrix}$ $\begin{matrix} (convergence) \\ (divergence). \end{matrix}$

* Pringsheim admits slightly more general conditions.

† The inequalities $\overline{\underline{\lim}}\, P \begin{matrix} > 1 \\ < 1 \end{matrix}$ are here used as equivalent to the two $\underline{\lim}\, P > 1$, $\overline{\lim}\, P < 1$.

79. Ratio-tests for non-absolute convergence.

Again supposing that the quotient a_n/a_{n+1} can be reduced to the special form given at the beginning of Art. 78, let us consider what information can be obtained when $\alpha \leqq 1$.

In the first place we observe that (when $n > 1$), Art. 96 below* gives

$$\frac{(n+1)^\mu}{n^\mu} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right),$$

and accordingly we have

$$\frac{n^\mu a_n}{(n+1)^\mu a_{n+1}} = 1 + O\left(\frac{1}{n^\kappa}\right),$$

where, as before, κ denotes the smaller of the indices λ and 2. Thus if $b_n = n^\mu a_n$, it is easy to see that the quotient

$$b_{n+1}/b_n = 1 + O\left(\frac{1}{n^\kappa}\right)$$

is the typical factor of an absolutely convergent product (since $\kappa > 1$) by Art. 76.

Thus b_n tends to a definite limit as $n \to \infty$; and so we can write

$$a_n = O(n^{-\mu}),$$

and

$$|a_n| = O(n^{-\alpha}),$$

because α is the real part of μ.

It follows that if $\alpha \leqq 0$, $|a_n|$ does not tend to zero as $n \to \infty$; and accordingly *the series* Σa_n *cannot converge at all unless* $\alpha > 0$.

When $0 < \alpha < 1$, it is clear that $|a_n| \to 0$, and thus a_n also tends to zero, so that convergence *may* occur. We can now appeal to Art. 77, which proves that $\Sigma n^{-\mu}$ diverges (in the sense explained in Art. 75); and so Σa_n also diverges in the same sense.

* Although it is convenient to place the present results here (for the purpose of grouping together all rules on ratio-tests for convergence), yet we make no use of the present rules in establishing the convergence of the binomial power-series for values of x such that $|x| < 1$; these rules are, however, used in the discussion of $(1+x)^\nu$ at points on the circle of convergence $|x| = 1$. Here the expansion of $(1+1/n)^\mu$ has been assumed for the special value $x = 1/n < 1$; but the same result can be deduced from the relation

$$\left(1+\frac{1}{n}\right)^\mu - \left(1+\frac{\mu}{n}\right) = \mu(\mu-1)\int_0^{1/n}\left(1+\frac{1}{n}-t\right)^{\mu-2} t\,dt.$$

It will be convenient to sum up the results of these considerations in a single rule :*

If $$\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^\lambda}\right), \quad \begin{cases}\mu = \alpha + \iota\beta \\ \lambda > 1\end{cases}$$

then

(i) *if* $\alpha > 1$, Σa_n *is absolutely convergent ;*

(ii) *if* $\alpha = 1$, *and* β *is not zero,* Σa_n *oscillates finitely ;*

(iii) *if* $0 < \alpha < 1$, Σa_n *diverges (in the sense of* Art. 75, *so that* $|\Sigma a_n| \to \infty$), *although the individual terms* a_n *still tend to zero ;*

(iv) *if* $\alpha \leqq 0$, *the terms* a_n *do not tend to zero, and convergence is accordingly impossible.*

It will be noticed that for series of the present type, *non-absolute convergence cannot occur.* But although the present series include almost all those which are commonly wanted in analysis, yet it is easy to construct non-absolutely convergent series, such as the series given in Art. 76.

It will be convenient to establish here the further result that :

When a_n/a_{n+1} *is of the form given above, the series* $\Sigma(a_n - a_{n+1})$ *is absolutely convergent, provided that* $\alpha > 0$.†

In fact we have

$$\frac{a_n - a_{n+1}}{a_n} = 1 - \left\{1 - \frac{\mu}{n} + O\left(\frac{1}{n^\kappa}\right)\right\} = \frac{\mu}{n}\left\{1 + O\left(\frac{1}{n^{\kappa-1}}\right)\right\},$$

and accordingly $$\frac{|a_n - a_{n+1}|}{|a_n|} = \frac{\rho}{n}\left\{1 + O\left(\frac{1}{n^{\kappa-1}}\right)\right\}.$$

But $$|a_n| = O(n^{-\alpha}),$$

and accordingly $$|a_n - a_{n+1}| = O\{n^{-(1+\alpha)}\}, \quad \text{since } \kappa > 1.$$

Thus $\Sigma|a_n - a_{n+1}|$ converges (provided that $\alpha > 0$) in virtue of Art. 11 (1); and so the result stated is established.

Ex. As a special example we may take the hypergeometric series $F(\alpha, \beta, \gamma, 1)$ considered in Art. 12·2, for which

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)},$$

so that $$\mu = 1 + \gamma - (\alpha + \beta).$$

* First given in its general form by Weierstrass (*Ges. Werke*, vol. 1, p. 185). This includes Gauss's rule (Art. 12·2) as a special case.

† It will be noticed that $\Sigma(a_n - a_{n+1})$ is obviously convergent, because $a_n \to 0$ when $\alpha > 0$; it is not obvious, however, that the convergence is *absolute*.

Then we see that, if the real part of $(\gamma-\alpha-\beta)$ is positive, the hypergeometric series $F(\alpha, \beta, \gamma, 1)$ converges absolutely; and that if this real part is not positive, the series cannot converge.

80. Abel's Lemma (for complex series).

The preliminary transformation of Art. 20 remains unchanged; but, if the letters denote complex numbers, we cannot make any advance beyond equation (A) without bringing in some condition with reference to the series of differences,

$$(v_1-v_2),\quad (v_2-v_3),\ \ldots,\ (v_{p-1}-v_p).$$

The simplest general hypothesis appears to be *the assumption that the series of differences* $\Sigma(v_n-v_{n+1})$ *is absolutely convergent.**

Since the sum of this series to $(n-1)$ terms is equal to (v_1-v_n), it follows that v_n has a definite limit (say l) as n tends to infinity; and if we now take the absolute value of equation (A) of Art. 20, we obtain the inequality

$$\left|\sum_1^p a_n v_n\right| \leqq H\{|v_1-v_2|+|v_2-v_3|+\ldots+|v_{p-1}-v_p|+|v_p|\},$$

where H is any number not less than the upper limit to $|s_1|$, $|s_2|, \ldots |s_p|$. If the expression in { } brackets is called V_p, it is easy to see that

$$V_{p+1}-V_p=|v_p-v_{p+1}|+|v_{p+1}|-|v_p| \geqq 0.$$

Thus the sequence (V_n) is an *increasing* sequence of positive numbers; and so tends to a limit V given by the equations

$$V-\lambda=|v_1-v_2|+|v_2-v_3|+\ldots \text{ to } \infty=\Sigma\,|v_n-v_{n+1}|$$

where $$\lambda=\lim_{n\to\infty}|v_n|.$$

Hence *the new form of Abel's lemma is* †

$$\left|\sum_1^p a_n v_n\right| \leqq HV,$$

where H *is not less than the upper limit of* $|s_1|, |s_2|, \ldots |s_p|$, *and* V *is defined by the last pair of equations.*

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* In Art. 20 we assumed these differences to be all *positive*; so that then the condition of absolute convergence follows at once.

† In Art. 20 $$V-\lambda=(v_1-v_2)+(v_2-v_3)+\ldots \text{ to } \infty$$ $$=v_1-\lambda,$$
and so $$V=v_1.$$

81. Further tests for convergence.

The reader will find no difficulty in modifying the proof of Art. 18 to establish the theorem:

(1) *If* Σa_n *is absolutely convergent, so also is* $\Sigma a_n v_n$, *provided that* $|v_n|$ *never exceeds a fixed number* K.

(2) **Abel's test for complex series.**

If the series Σa_n *is convergent, and if the series* $\Sigma(v_n - v_{n+1})$ *is absolutely convergent, then* $\Sigma a_n v_n$ *is convergent.**

Let us apply Abel's Lemma (Art. 80) to the remainder in $\Sigma a_n v_n$, namely

$$\sum_{m+1}^{m+p} a_n v_n = a_{m+1}v_{m+1} + a_{m+2}v_{m+2} + \ldots + a_{m+p}v_{m+p}.$$

We see that
$$\left|\sum_{m+1}^{m+p} a_n v_n\right| < H_m V_m,$$

where H_m is not less than the upper limit of

$$|a_{m+1}|, \quad |a_{m+1}+a_{m+2}|, \quad \ldots, \quad |a_{m+1}+a_{m+2}+\ldots+a_{m+p}|,$$

and $$V_m - \lambda = |v_{m+1} - v_{m+2}| + |v_{m+2} - v_{m+3}| + \ldots \text{ to } \infty,$$

while $$\lambda = \lim |v_n|.$$

Now clearly $V_m \leqq V_1 = V$ (in the notation of Art. 80), and since Σa_n converges, we can find m so that $H_m \leqq \epsilon$, however small ϵ may be; and then

$$\left|\sum_{m+1}^{m+p} a_n v_n\right| < \epsilon V,$$

so that $\Sigma a_n v_n$ satisfies the fundamental condition of convergence (Art. 75).

(3) **Dirichlet's test for complex series.**

If the series Σa_n *oscillates between finite limits, and the series* $\Sigma(v_n - v_{n+1})$ *is absolutely convergent, and if, further,* $\lim v_n = 0$, *then* $\Sigma a_n v_n$ *is convergent.*

In fact we need only note that here H_m remains less than a fixed number K, and that K can be taken independent of m (although it naturally depends on the limits of oscillation of Σa_n); also $\lambda = 0$, and m can accordingly be chosen so that

$$V_m < \epsilon.$$

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* As a rule Σa_n will *not* be absolutely convergent or we could apply the first test above.

Then we have $$\left|\sum_{m+1}^{m+p} a_n v_n\right| < \epsilon K,$$

and accordingly the fundamental test for convergence is again satisfied (Art. 75).

Ex. The series

$$t + \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 + \tfrac{1}{4}t^4 + \ldots, \quad \text{where} \quad t = \cos\theta + \iota\sin\theta, \text{ so that } |t| = 1,$$

converges except for $t = 1$, that is, $\theta = 0$.

For $$\sum_{m+1}^{m+p} t^n = t^{m+1}(1 - t^p)/(1 - t),$$

and so $$\left|\sum_{m+1}^{m+p} t^n\right| \leqq 2/|1 - t|;$$

and if $v_n = 1/n$, $\Sigma(v_n - v_{n+1})$ is a convergent series of positive terms.

It is evident that this series is *not* absolutely convergent; because the series of moduli is $\Sigma 1/n$, which diverges.

82. Double series of complex terms.

The reader should have little difficulty in modifying the definitions and theorems given in Chap. V., so as to apply to double series with complex terms.

In practical work we usually deal with absolutely convergent double series only; and with reference to such series we have the general theorem (Arts. 31, 33):

If a double series has been proved to be absolutely convergent for **any** *mode of summation, it will be absolutely convergent for* **all** *modes of summation, and the sum of the series is independent of the mode of summation.*

As an example we shall consider the series

$$\sum_{(m, n)} (x - \Omega)^{-\nu},$$

where $\Omega = m\omega + n\omega'$, and x, ω, ω' are complex (but such that the quotient ω'/ω is not real).

In the first place we note that we can write

$$|x - \Omega|^2 = am^2 + 2bmn + cn^2 + 2fm + 2gn + k,$$

where $\omega = \alpha + \iota\beta$, $\omega' = \alpha' + \iota\beta'$, $a = \alpha^2 + \beta^2$, $b = \alpha\alpha' + \beta\beta'$, $c = \alpha'^2 + \beta'^2$, and f, g, k depend on ξ, η, where $\xi + \iota\eta = x$.

Now $ac - b^2 = (\alpha\beta' - \alpha'\beta)^2$, which is always positive because ω'/ω is not real, so that $\alpha\beta' - \alpha'\beta$ is not zero.

Thus, as in Ex. 3, Art. 32, we can write

$$A^2(m + n)^2 > am^2 + 2bmn + cn^2 > B^2(m + n)^2,$$

where A^2 is the greatest of a, c, and we can take $B^2 = (ac - b^2)/(a + c - 2b)$.

Now the discussion of the convergence depends on large values of m and n; and consequently we can find positive values of λ, μ, such that

$$A(m+n+\lambda)^2 > am^2+2bmn+cn^2+2fm+2gn+k > B(m+n-\mu)^2,$$

provided that m and n are sufficiently great.

Thus we have

$$A(m+n+\lambda) > |x-\Omega| > B(m+n-\mu),$$

or

$$A^{-p}(m+n+\lambda)^{-p} < |x-\Omega|^{-p} < B^{-p}(m+n-\mu)^{-p},$$

assuming that p is positive.

Hence, applying the test (2) of Art. 33, we see that $\Sigma|x-\Omega|^{-p}$ converges or diverges according as $\Sigma_N N^{1-p}$ converges or diverges; thus $\Sigma|x-\Omega|^{-p}$ converges if $p>2$, and diverges if $p \leqq 2$; and *accordingly* $\Sigma(x-\Omega)^{-p}$ *is absolutely convergent only if* $p>2$.

It has been tacitly assumed that x is not equal to Ω for any value of m, n; but if x should be equal to some value of Ω, it will be necessary simply to omit one term from the series. The remaining terms of the series will still converge absolutely (if $p>2$).

It has also been assumed that in the double series m and n are restricted to be positive (as in Chap. V.); but in most applications m and n are allowed to take negative integral values as well as positive. When this is done, the conclusion is easily seen to be unaltered, although the result is proved most quickly by superposing four separate series in which m and n are restricted to be positive (the signs of ω, ω' being reversed as necessary).

We conclude that:

The series $\Sigma(x-\Omega)^{-p}$, *in which* $\Omega=m\omega+n\omega'$, *and* m, n *vary independently from* $-\infty$ *to* $+\infty$, *is absolutely convergent, if* $p>2$, *and only under this condition; so that the sum of the series is independent of the mode of summation, provided that* $p>2$.

In particular this result holds for $p=3$, which is the most interesting special case (Art. 103).

Ex. 1. Prove that $\Sigma'\Omega^{-3}=0$, $\Sigma'\Omega^{-5}=0$, $\Sigma'\Omega^{-7}=0$, etc., where the accent implies that the term $m=0$, $n=0$ is omitted.

Ex. 2. Prove that

$$\Sigma'\left\{\frac{1}{(x-\Omega)^2}-\frac{1}{\Omega^2}\right\},\quad \Sigma'\left(\frac{1}{x-\Omega}+\frac{1}{\Omega}+\frac{x}{\Omega^2}\right)$$

are absolutely convergent.

83. Uniform convergence.

After what has been explained in Chapter VII., there will be no difficulty in appreciating the idea of uniform convergence for a series $\Sigma f_n(x)$, when x is complex; the only essential point of novelty being that now any region of uniform convergence usually consists of an area in the (ξ, η) plane (if $x=\xi+\iota\eta$) instead of an interval (or segment) of the axis of x. It is also sometimes convenient to use the idea of *uniform convergence along a curve,* which should present no fresh difficulty to the reader. Just as in Chapter VII., an area of uniform convergence is always *closed,* so that the boundary must be included in the area.

(1) **Weierstrass's M-test for uniform convergence** remains almost unaltered, thus:

If for all points $(x=\xi+\iota\eta)$ within a certain area A, the function $f_n(x)$ has the property that

$$|f_n(x)| \leqq M_n,$$

where M_n is a positive **constant**, *and if the series ΣM_n converges, then $\Sigma f_n(x)$ converges absolutely and uniformly at all points within A.* For brevity we may call such series *normally convergent,* in agreement with Art. 44.

Abel's and Dirichlet's tests for uniform convergence (for a complex series) are obtained at once from the analysis of Art. 81.

(2) **Abel's test for uniform convergence.**

If the series $\Sigma a_n(x)$ is uniformly convergent, and if the sum $\Sigma|v_n-v_{n+1}|$ and $\lim |v_n|$ *are both less than a constant K for all points x within an area A, then the series $\Sigma a_n(x)v_n(x)$ is uniformly convergent for all points within the area A.*

In fact, for all such points, we have

$$V-\lambda < K, \quad \lambda < K, \quad \text{and so } V < 2K,$$

using the notation of Art. 81; thus the remainder $\sum_{m+1}^{m+p} a_n(x)v_n(x)$ is numerically less than $2\epsilon K$ for any point within the area A, proving uniform convergence.

(3) **Dirichlet's test for uniform convergence.**

If the series $\Sigma a_n(x)$ oscillates between finite limits for all points x within a certain region A, and if $\Sigma(v_n-v_{n+1})$ is normally convergent while $v_n(x)$ tends to zero uniformly at all points x within the area A, then $\Sigma a_n(x)v_n(x)$ is uniformly convergent within the area A.

In fact, here $H_m < 2K$, where K depends on the limits of oscillation of $\Sigma a_n(x)$, but is independent of x, in accordance with the hypothesis as to the limits of oscillation of $\Sigma a_n(x)$.

Further, $\lambda=0$, and by means of the comparison series of positive constants for $\Sigma|v_n-v_{n+1}|$ we can find m so that $V_m<\epsilon$, and then

$$\left|\sum_{m+1}^{m+p} a_n(x)v_n(x)\right| < 2\epsilon K$$

for all points within the region A. Thus the proof of uniform convergence is completed.

In the applications which we have in view, a_n will depend on the variable x while v_n is independent of x.

The proof of Art. 45 (1) can be modified at once to shew that *$\Sigma f_n(x)$ is a continuous function of x within any region of uniform convergence, provided that the separate functions $f_n(x)$ are continuous in the same region.*

The discussion of differentiation and integration with respect to the *complex* variable x falls outside the scope of this book; but it is not out of place to mention that (once the fundamental notions have been made clear) the results of Arts. 45-47 remain practically unaltered.

It is evident also that Art. 48 remains valid, when x is a complex variable; and that the two theorems of Art. 49 remain valid, when the functions $v_r(n)$ are complex.

It is often necessary to integrate a complex function with respect to a *real* variable; in particular it is useful to consider: **The mean value of a continuous function $f(x)$ along a circle $|x|=r$**, which is defined by the equation

$$\mathfrak{M}f(x)=\frac{1}{2\pi}\int_0^{2\pi} f(x)\,d\theta, \quad \text{where } x=r(\cos\theta+\iota\sin\theta).$$

The existence of a definite mean value is inferred at once from the continuity of $f(x)$, just as in Art. 161 of Appendix II.; and the following conclusions are immediate consequences of the definition:

(i) $\mathfrak{M}f(x)=a$, if $f(x)$ is equal to a constant a,

(ii) $|\mathfrak{M}f(x)|<M$, if $|f(x)|<M$ on the circle,

(iii) $\mathfrak{M}x^k=0$, $\mathfrak{M}x^{-k}=0$, if k is an integer (not zero),

because $x^k=r^k(\cos k\theta+\iota\sin k\theta)$ and

$$\int_0^{2\pi}\cos(k\theta)\,d\theta=0, \quad \int_0^{2\pi}\sin(k\theta)\,d\theta=0.$$

Further, from Art. 45 (2), we deduce that *if the series* $\Sigma f_n(x)$ *converges uniformly to the sum* $F(x)$ *for all points on the circle* $|x|=r$, *then*

$$\mathfrak{M}F(x)=\Sigma\{\mathfrak{M}f_n(x)\}.$$

We can define the mean value without using the Integral Calculus, by supposing the circumference divided into ν equal parts at $x_1, x_2, \ldots x_\nu$, and writing

$$\mathfrak{M}f(x)=\lim_{\nu\to\infty}\frac{1}{\nu}\{f(x_1)+f(x_2)+\ldots+f(x_\nu)\}.$$

This method leads to the results (i)–(iii) just proved; and thus Cauchy's inequalities (p. 249) can be established without the Calculus.

84. Circle of convergence of a power-series $\Sigma a_n x^n$.

From Art. 10 it is evident that the series is absolutely convergent if

$$\overline{\lim}\,|a_n x^n|^{\frac{1}{n}}<1,$$

and the series certainly cannot converge if

$$\overline{\lim}\,|a_n x^n|^{\frac{1}{n}}>1,$$

because then $a_n x^n$ cannot tend to zero as a limit.

Hence, if we write, as in Art. 50,

$$\overline{\lim}\,|a_n|^{\frac{1}{n}}=1/l$$

(where, of course, l is real and positive), the power-series converges absolutely if $|x|<l$; and cannot converge if $|x|>l$.

To interpret this geometrically, let a circle of radius l be drawn in Argand's diagram; then the series is absolutely convergent at

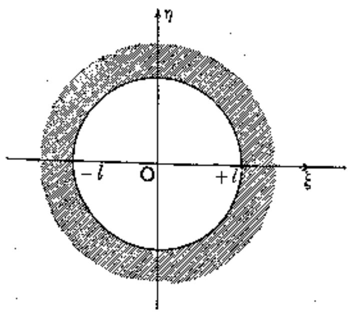

FIG. 23.

any point within the circle, and cannot converge at any point outside the circle. The circle is called *the circle of convergence*; and it will be seen that, when a_n is real, the interval of convergence $(-l, +l)$ obtained in Art. 50 is a diameter of the circle.

For most series of practical importance in analysis, the quotient $|a_n|/|a_{n+1}|$ has a definite limit l, when n tends to infinity; and then l is equal to the radius of convergence as just defined (Art. 149).

Of course we know nothing at present with respect to points on the circle of convergence; but when $l=1$ (a case to which every other can be reduced, excepting $l=0$ and ∞) we can usually obtain information by means of Weierstrass's rule given in Art. 85 below.

When Σa_n is absolutely convergent and $l=1$, the region of uniform convergence is the whole of the circle $|x|=1$, *including the circumference*; this is evident from Weierstrass's M-test.

When $\Sigma |a_n|$ diverges and $l=1$, in general we can say only that the series converges uniformly within and on any circle $|x|=k$, where k lies between 0 and 1. We shall, however, consider this point more fully in Arts. 85 and 86.

The reader will find little difficulty in seeing that the theorems of Arts. 52-56 hold for complex power-series, certain small verbal alterations being made; some extensions of these results, such as Art. 52 (4), depend on Cauchy's inequalities below.

Since a power-series converges uniformly on every circle $|x|=r$, for which * $r<l$, we can readily obtain its mean-value along the circle by integrating term-by-term.

Thus, if $f(x)=\sum_0^\infty a_n x^n$, we have

$$\mathfrak{M} f(x)=\sum_0^\infty a_n \mathfrak{M} x^n = a_0 = f(0),$$

so that *the mean-value of a power-series along a circle* $|x|=r(<l)$ *is equal to its value at the centre.*

Similarly, we see that

$$\mathfrak{M}\{f(x)/x^n\}=a_n.$$

Thus, if M is the maximum value of $|f(x)|$ on the circle $|x|=r$, we have

$$|a_0| \leqq \mathfrak{M}|f(x)| < M \quad \text{and} \quad |a_n| \leqq \mathfrak{M}|f(x)/x^n| < M/r^n,$$

from which we deduce **Cauchy's inequalities**.

$$|a_0| < M, \quad |a_n| < M/r^n.$$

---

* If the circle $|x|=l$ belongs to the region of uniform convergence, we may of course take $r=l$.

Again, since the series

$$\frac{x}{x-c}=1+\frac{c}{x}+\frac{c^2}{x^2}+\dots, \quad |c|<r,$$

converges *uniformly* on the circle $|x|=r$, we find

$$\mathfrak{M}\frac{xf(x)}{x-c}=\sum_0^\infty c^n\mathfrak{M}\frac{f(x)}{x^n}=\sum_0^\infty a_nc^n=f(c), \quad \text{if } |c|<r.$$

Similarly, we find

$$\frac{x}{x-c}=-\left(\frac{x}{c}+\frac{x^2}{c^2}+\frac{x^3}{c^3}+\dots\right), \quad \mathfrak{M}\frac{xf(x)}{x-c}=0, \quad \text{if } |c|>r.$$

We are now in a position to obtain the extension of Art. 52 (4), that *if the first term* (a_0) *of a power-series* $f(x)$ *is not zero, there is no root of* $f(x)=0$ *within a certain circle whose centre is the origin.*

For, writing $|x|=\rho<r$, we have

$$|a_1x+a_2x^2+\dots|<M\left(\frac{\rho}{r}+\frac{\rho^2}{r^2}+\frac{\rho^3}{r^3}+\dots\right)=\frac{M\rho}{r-\rho}.$$

Hence $\qquad |f(x)|>|a_0|-M\rho/(r-\rho),$

or if $\qquad |a_0|=A, \quad |f(x)|>0, \quad$ provided that $\rho<Ar/(A+M)$.

Thus there is no root within the circle

$$|x|=Ar/(A+M).$$

It follows, as in Art. 52 (5), that *if a power-series can be proved to vanish at all points within a circle whose centre is the origin, then the series is identically zero.*

85. Behaviour of a power-series on the circle of convergence.

We assume that the circle is reduced, if necessary, to the special circle $|x|=1$; and that Σa_n is not absolutely convergent.

Then we can often apply Dirichlet's tests * of Arts. 81, 83, to prove that the series converges on the circle; and further, to establish an arc of uniform convergence for points on the circle. *We assume that* a_n *is such that* $\Sigma(a_n-a_{n+1})$ *is absolutely convergent and that* a_n *tends to zero.**

Then consider the value of

$$x^{m+1}+x^{m+2}+\dots+x^{m+p}=x^{m+1}(1-x^p)/(1-x),$$

where $|x|=1$, but x is not equal to 1.

* Note that a_n now plays the part taken by v_n in the articles quoted; and that x^n corresponds to what was there called $a_n(x)$.

It is evident that

$$|x^{m+1}+x^{m+2}+\ldots+x^{m+p}|=|1-x^p|/|1-x|,$$

which oscillates between 0 and $2/|1-x|$.

It follows that, under the above conditions, $\Sigma a_n x^n$ converges at all points of the circumference $|x|=1$, except for $x=1$; and the convergence is uniform along any arc of the circle for which $|1-x| \geqq c$, leading to an arc such as PBQ in Fig. 24, where $AP=AQ=c$.

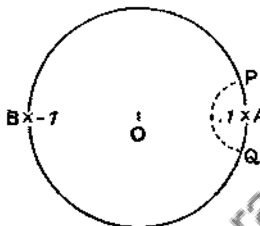

FIG. 24.

Two important special classes of such series are given by the following:

(i) *When* (a_n) *is a sequence of real numbers steadily decreasing to zero as a limit.*

For example, $a_n=1/n$, $1/n \log n$, $\log n/n$.

(ii) *When* (a_n) *is such that*

$$\frac{a_n}{a_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^\lambda}\right), \quad \begin{cases}\mu=\alpha+\iota\beta\\ \lambda>1\end{cases}$$

and $0<\alpha\leqq 1$.*

For then, as we have seen (Art. 79), $\Sigma(a_n-a_{n+1})$ is absolutely convergent, and $|a_n|$ tends to zero, as n tends to infinity. Also $|x|=1$ is the circle of convergence (Art. 84).

It should be noticed that *if* $\alpha\leqq 0$ *in case* (ii), *the series* $\Sigma a_n x^n$ *cannot converge at any point on the circle* $|x|=1$; because $|a_n x^n|=|a_n|$ does not tend to zero as n tends to infinity (Art. 79).

We see from Art. 79 that *in case* (ii) *the series cannot converge for* $x=1$ *unless* $\Sigma|a_n|$ *is convergent; and then the whole of the circumference* $|x|=1$ *is included in the region of uniform convergence.*

For examples of this type, see Exs. 38, 39 at the end of this chapter.

* Weierstrass, *Ges. Werke*, vol. 1, p. 185.

To shew that *series exist which converge, but not absolutely, at* all *points of the circle of convergence,* Pringsheim has given the following example:

Let the series Σa_n be given by $\Sigma a_n = b_1 - b_2 - b_3 + b_4 + b_5 + b_6 + b_7 - \dots$, where $b_1 = 1$, $b_2 = \frac{1}{2}$, and thereafter $b_n = 1/(n \log n)$, and the signs alternate in groups of 1, 2, 4, 8, ... terms.

Then clearly $\Sigma |a_n| = \Sigma b_n$ is divergent.

Further, it will be seen that Σa_n is convergent by noting that this series of terms can be arranged in groups of positive and negative terms; and the terms in the mth group (c_m) are numerically equal to Σb_n from $n = 2^{m-1}$ to $2^m - 1$. Thus, as in Art. 11,

$$0 < c_m - \int_{2^{m-1}}^{2^m} \frac{dx}{x \log x} < \frac{1}{2^{m-1} \log 2^{m-1}}.$$

Now this integral is

$$\int_{m-1}^{m} \frac{d\xi}{\xi} = \log\left(\frac{m}{m-1}\right), \quad \text{if } \xi = \frac{\log x}{\log 2},$$

and so we find that

$$c_m > \log\left(\frac{m}{m-1}\right); \quad c_{m+1} < \log\left(\frac{m+1}{m}\right) + \frac{1}{2^m m \log 2}.$$

Thus $$c_m - c_{m+1} > \log\left(\frac{m^2}{m^2-1}\right) - \frac{1}{2^m m \log 2} > \frac{1}{m^2} - \frac{1}{2^m m \log 2} > 0.$$

Thus c_m steadily decreases; and since

$$c_m \sim 1/m,$$

it follows that $c_m \to 0$, as $m \to \infty$.

Hence Σa_n converges by Art. 19.

Further, we have

$$\begin{aligned}\Sigma |a_n - a_{n+1}| &= (b_1 + b_2) + (b_2 - b_3) + (b_3 + b_4) + (b_4 - b_5) + \dots \\ &= b_1 + 2b_2 + 2b_4 + 2b_8 + \dots \\ &< 1 + \tfrac{2}{2} + \tfrac{2}{4} + \tfrac{2}{8} + \dots,\end{aligned}$$

and this is clearly convergent.

Thus $\Sigma a_n x^n$ converges at all points of the circle $|x| = 1$; for we have just proved that the series converges for $x = 1$ (that is, that Σa_n is convergent); and Dirichlet's test (Art. 81) enables us to assert that the series converges at all other points of the circle.

That the convergence on $|x| = 1$ is not absolute convergence follows from the fact that $\Sigma |a_n x^n| = \Sigma |a_n|$, which is a divergent series.

86. Abel's theorem and allied theorems.

Let us suppose next that the series $\Sigma a_n x^n$ converges, but *not absolutely,** for some point $x = x_0$.

* When absolute convergence can be asserted, we can appeal to the M-test and deduce at once *normal* convergence (as in Art 83); the whole discussion is then much easier.

Now we have seen (Art. 84) that any power-series converges absolutely at all points within the circle of convergence, and that it will not converge at any external point; thus $x=x_0$ must lie on the circumference of the circle of convergence.

If we write in the first place $x=kx_0$, and treat k as a real variable varying from 0 to 1, we shall be considering points on the radius joining the origin to the point x_0; and the series to be discussed is then equal to

$$\Sigma(a_n x_0^n) k^n.$$

It follows at once * from Art. 50 (the original Abel's theorem) that

$$\lim_{k \to 1} \Sigma(a_n x_0^n) k^n = \Sigma a_n x_0^n.$$

Or, returning to the original notation, we may write

$$\lim_{x \to x_0} \Sigma a_n x^n = \Sigma a_n x_0^n$$

when the point x travels up to the point x_0 along the radius of the circle of convergence, assuming that the series on the left converges.

In like manner, if $\Sigma a_n x_0^n$ diverges (in the sense of Art. 75), we see that the modulus of $\Sigma a_n x^n$ tends to infinity, when x tends to x_0, along the radius.

The results, so far, are substantially due to Abel in his classical memoir on the binomial series. The next generalisation is due to Picard,† who proved that by a slight modification in Abel's analysis it is possible to allow x to tend to x_0 *along any curve which does not touch the circle of convergence at x_0.* But instead of giving the proof of this generalisation, we shall now proceed to establish uniform convergence within an area which is attached to the circle at the point $x=x_0$.

By taking x/x_0 as a new variable, we see at once that there is no loss of generality in supposing that the special point x_0 coincides with $x=1$; and in the remainder of the article we shall assume that Σa_n is convergent (but not absolutely), so that the circle of convergence is now $|x|=1$, and we shall discuss the uniformity of convergence in the neighbourhood of $x=1$.

If we take $v_n = x^n$ in Art. 80, we find that $\lambda = \lim |v_n| = 0$, and

$$|v_n - v_{n+1}| = |1-x| \,.\, |x|^n,$$

so that $$V = \Sigma |v_n - v_{n+1}| = |1-x| / \{1-|x|\} \geqq 1.$$

---

* Strictly speaking, the real and imaginary parts of the series $\Sigma a_n x_0^n$ should be considered separately; but the result follows immediately.

† *Traité d'Analyse*, t. 2, 1893, p. 73.

We can now use Abel's test for uniform convergence (Art. 83), because the series Σa_n is convergent. Thus the series $\Sigma a_n x^n$ will converge uniformly in any area for which

$$|1-x| \leqq K\{1-|x|\},$$

where K is any assigned constant greater than 1, and of course $|x| < 1$.

To interpret this inequality we observe that it may be written

$$\rho \leqq K(1-r) \quad \text{or} \quad (K-\rho)^2 \geqq K^2 r^2,$$

where $$1-x=\rho(\cos\phi+\iota\sin\phi), \quad \text{(see fig. 25).}$$

Thus $$K^2-2K\rho+\rho^2 \geqq K^2(1-2\rho\cos\phi+\rho^2)$$

or $$(K^2-1)\rho \leqq 2(K^2\cos\phi-K).$$

In this condition, ϕ lies between $\pm\frac{1}{2}\pi$, and the equation

$$(K^2-1)\rho=2(K^2\cos\phi-K), \quad (-\tfrac{1}{2}\pi<\phi<\tfrac{1}{2}\pi),$$

gives the inner loop of a limaçon, which has a node at $\rho=0$; this curve is indicated roughly in figure 25 for the case $K=3$.*

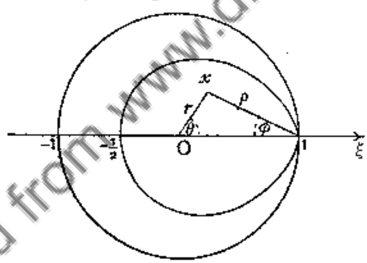

FIG. 25.

It is easy to see that the arc of the limaçon approaches the more nearly to the circle $|x|=1$, the larger K is taken.

Thus *the region of uniform convergence of* $\Sigma a_n x^n$ *may be taken as the inner loop and the contained area of any one of these limaçons.*†

If any regular curve is drawn from a point inside the circle to the point $x=1$, then, provided the curve cuts the circle at a finite angle, we can draw one of these limaçons to enclose the whole of the curve: that is, the series will converge uniformly along the curve. *Hence* $\lim \Sigma a_n x^n = \Sigma a_n$, *where* x *approaches* 1 *along any*

* At $x=1$ this argument does not hold; but here $\Sigma(v_n - v_{n+1})=0$ and $\lambda=1$, so that the point can be included in the area of uniform convergence.

† Stolz and Gmeiner (*Einleitung in die Funktionentheorie*, 1905, p. 287) use a limaçon which in the present notation would be represented by $\rho=2\cos\phi-2/K$. This limaçon lies within the loop used above.

regular curve which cuts the circle at a finite angle. This is Picard's extension of Abel's theorem to complex variables.

The theorems in Art. 51 relating to the divergence of Σa_n cannot be extended so as to hold for complex variables quite so easily, because the lemma of Art. 80 gives less precise information than the lemma of Art. 20, and it is necessary to assume that the series $\Sigma a_n x^n$ possesses some further property in addition to the divergence of Σa_n. For as a matter of fact, even if a_n is real and positive, Pringsheim has shewn that the divergence of Σa_n does not ensure

$$\lim |\Sigma a_n x^n| = \infty$$

for all paths defined as above. (See below.)

The condition introduced by Pringsheim is that of *uniform divergence*, which implies that

$$|\Sigma a_n x^n| / \Sigma a_n |x|^n \geqq \sigma > 0,$$

where $a_n > 0$ and the point x lies within the limaçon.

It then follows, as in Art. 51, that $\lim |\Sigma a_n x^n| = \infty$.

The reader will find no great difficulty in modifying the proofs given in Art. 51 so as to apply for complex variables when Pringsheim's condition is satisfied.

It is easy to verify that Pringsheim's condition is satisfied by most elementary series of analysis, such as those used in the examples of Art. 51.

To obtain Pringsheim's extension of **the comparison theorem of two divergent series**, we find that the choice of m, as in Art. 51, leads at once to the inequality

$$\left| \sum_{m+1}^{\infty} b_n x^n \right| < \epsilon \sum_{0}^{\infty} a_n |x|^n, \quad \text{if } b_n/a_n \to 0,$$

or

$$\left| \sum_{m+1}^{\infty} b_n x^n \right| < (\epsilon/\sigma) \left| \sum_{0}^{\infty} a_n x^n \right|,$$

on applying Pringsheim's condition.

The remainder of the argument proceeds exactly on the former lines. For examples, see Art. 51 and Exs. 41-45 at the end of this chapter.

To see **the necessity of the condition of uniform divergence**, Pringsheim has given the example

$$\Sigma a_n x^n = \exp\{(1-x)^{-2}\}.$$

It is easy to see that a_n is positive; and further Σa_n must diverge, because when $x \to 1$ (by *real* values) the exponential function tends to infinity. Thus Σa_n cannot converge (by using Abel's theorem).

Now

$$(1-x)^{-2} = \rho^{-2}(\cos 2\phi - \iota \sin 2\phi),$$

and so if $\frac{1}{4}\pi < \phi < \frac{1}{2}\pi$, the real part tends to $-\infty$ as $\rho \to 0$; and so the exponential function will then tend to *zero* (*not infinity*).

86·1 Converse of Abel's Theorem.

If $\lim (\Sigma a_n x^n)$ exists and is equal to a finite number A, it is not possible to infer the convergence of Σa_n without some further restriction on the coefficients. In two simple cases we can make this inference:

(i) When the coefficients a_n are all positive after a certain stage. [PRINGSHEIM.]

(ii) When $\lim na_n = 0$, and x approaches 1 by any path within the limaçon of Art. 86.* [TAUBER.]

Since in case (i) x can approach 1 by real values, we can infer from the existence of $\lim \Sigma a_n x^n$ that Σa_n cannot diverge; further, Σa_n cannot oscillate. Hence, in case (i) Σa_n converges, and is therefore equal to A, by Abel's theorem.

In case (ii), write $n|a_n| = c_n$; then we find

$$\left| \sum_0^{\nu-1} a_n(1-x^n) \right| < |1-x| \sum_0^{\nu-1} c_n,$$

because $\quad |(1-x^n)/(1-x)| = |1+x+x^2+\ldots+x^{n-1}| \leqq n;$

also, if H_ν is the upper limit to $c_\nu, c_{\nu+1}, \ldots$ to ∞, we have

$$\left| \sum_\nu^\infty a_n x^n \right| < H_\nu \frac{|x|^\nu}{\nu}(1+|x|+|x|^2+\ldots) < H_\nu \frac{|x|^\nu}{\nu\{1-|x|\}}.$$

Take then x as a point on the given path such that $|x| = 1 - 1/\nu$; we have, as in Art. 86,

$$|1-x| < K/\nu,$$

and so

$$\left| \sum_0^{\nu-1} a_n - \sum_0^\infty a_n x^n \right| < (K/\nu) \sum_0^{\nu-1} c_n + H_\nu.$$

As $\nu \to \infty$, each of the terms on the right tends to 0 (the first in virtue of Art. 149); and so

$$\lim_{\nu\to\infty} \sum_0^{\nu-1} a_n = \lim_{x\to 1} \sum_0^\infty a_n x^n = A.$$

But if na_n has no definite limit, we can infer the convergence of Σa_n from the existence of $\lim \Sigma a_n x^n$ (for some path within the limaçon), and from the condition

$$\lim \frac{1}{n}(a_1 + 2a_2 + 3a_3 + \ldots + na_n) = 0.$$

These conditions are both *necessary* for the convergence of Σa_n, and, taken together, they are *sufficient*.

Again, if $\Sigma a_n x^n$ **tends to a finite limit as** x **comes up (along the radius) to every point** t **on the circle of convergence, yet we cannot infer that** $\Sigma a_n t^n$ **converges for any single point on that circle.**

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* Tauber, *Monatshefte f. Math. u. Phys.*, vol. 8, 1897, p. 273; Pringsheim, *Münchener Sitzungsberichte*, vol. 30, 1900, p. 37, and vol. 31, 1901, p. 507. See also Landau, *Monatshefte f. Math. u. Phys.*, vol. 18, 1907, p. 19; Littlewood, *Proc. Lond. Math. Soc.* (2), vol. 9, p. 434, and Hardy and Littlewood, *ibid.* vol. 11, 1913, p. 411.

For consider Pringsheim's example

$$\Sigma a_n x^n = \exp\{(x-1)^{-1}\}$$

(which tends to 0 as $x \to 1$ along the radius).

Here, if $x = e^{\iota\phi}$,

$$\exp\{(x-1)^{-1}\} = e^{-\frac{1}{2}}\left\{\cos\left(\tfrac{1}{2}\cot\frac{\phi}{2}\right) - \iota\sin\left(\tfrac{1}{2}\cot\frac{\phi}{2}\right)\right\},$$

so that the real and imaginary parts of this function have an infinite number of maxima and minima in the neighbourhood of $\phi = 0$. But if the series $\Sigma a_n t^n$ were convergent these functions would be continuous; thus $\Sigma a_n t^n$ cannot converge.

87. Poisson's integral.

We shall now consider the question: *Can a power-series be determined so as to have given values along a definite circle, say $|x|=1$?*

Let us write the coefficients a_n in the form $\alpha_n + \iota\beta_n$ where α_n, β_n are real; and put $\Sigma\alpha_n x^n = f_1(x)$, $\Sigma\iota\beta_n x^n = f_2(x)$, so that

$$f(x) = \Sigma a_n x^n = f_1(x) + f_2(x).$$

Now suppose that when $x = \cos\theta + \iota\sin\theta$, we have

$$f_1(x) = u_1 + \iota v_1, \quad f_2(x) = u_2 + \iota v_2 \quad \text{and} \quad f(x) = u + \iota v,$$

where u_1, v_1, etc., are all real functions of θ such that

$$u = u_1 + u_2, \quad v = v_1 + v_2.$$

Then, if $|c| < 1$, we find as in Art. 84 (assuming uniform convergence of $\Sigma\alpha_n x^n$ on the circle $|x|=1$)

$$f_1(c) = \frac{1}{2\pi}\int_0^{2\pi}(u_1 + \iota v_1)\frac{x}{x-c}\,d\theta, \quad 0 = \frac{1}{2\pi}\int_0^{2\pi}(u_1+\iota v_1)\frac{x\,d\theta}{x-(1/c)}.$$

In the second integral put $1/x$ for x: this will change $u_1 + \iota v_1$ to $u_1 - \iota v_1$ (because the coefficients α_n are real), and so we have

$$0 = \frac{1}{2\pi}\int_0^{2\pi}(u_1 - \iota v_1)\frac{c}{c-x}\,d\theta.$$

If we subtract the last result from the formula for $f_1(c)$, we obtain

$$f_1(c) = \frac{1}{2\pi}\int_0^{2\pi}\frac{x+c}{x-c}u_1\,d\theta + \frac{\iota}{2\pi}\int_0^{2\pi}v_1\,d\theta.$$

Similarly, by addition we get

$$f_1(c) = \frac{1}{2\pi}\int_0^{2\pi}u_1\,d\theta + \frac{\iota}{2\pi}\int_0^{2\pi}\frac{x+c}{x-c}v_1\,d\theta.$$

In the same way we can find integrals for $f_2(c)$ in terms of u_2, v_2: the only essential change in the argument being that when x is changed to $1/x$, $u_2 + \iota v_2$ becomes $-u_2 + \iota v_2$. This, however, does not alter the final formulae; and so by addition we see that *these* *formulae remain true when suffixes are omitted throughout.* Thus $f(c)$ is completely determined (save for a constant) by a knowledge of *either* u *or* v. But, given an arbitrary continuous function for u (or v), we do not yet know that it is actually possible to determine $f(c)$ so that its real (or imaginary) part does assume the given values on the circle.

We proceed to find the limiting values of these integrals.

For example, suppose that v is an arbitrary real continuous function; then the second formula gives a value for $f(c)$ which can be expanded as a power-series in c, convergent if $|c|<1$. We shall now prove that if this function is denoted by $U+\iota V$, then V tends to v as c moves up to any point on the circle; *so that we have determined a power-series whose imaginary part has an assigned continuous value* v *along the circle* $|c|=1$.

Clearly it is sufficient to establish the result for any point on the circle; and we shall calculate the limit of V as c moves up to 1.

It will be seen that

$$\frac{x+c}{x-c}=\frac{1-r^2+2\iota r\sin(\theta-\omega)}{1-2r\cos(\theta-\omega)+r^2},$$

where now $x=\cos\omega+\iota\sin\omega$ and $c=r(\cos\theta+\iota\sin\theta)$,

so that
$$V=\frac{1}{2\pi}\int_0^{2\pi}\frac{v(1-r^2)d\omega}{1-2r\cos(\theta-\omega)+r^2}.$$

This integral V is known as **Poisson's integral**; and it is clearly a solution of Laplace's equation (in two dimensions), because it is the imaginary part of a function of the complex variable x. We shall see now that $V\to v$ as $r\to 1$; so that V solves Dirichlet's potential problem for the interior of the circle $r=1$.

From Art. 65,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r^2)d\omega}{1-2r\cos(\theta-\omega)+r^2}=\frac{1}{2\pi}\int_0^{2\pi}[1+2r\cos(\theta-\omega)+2r^2\cos 2(\theta-\omega)+\ldots]d\omega=1,$$

and, since the subject of integration is positive, the value of the integral taken over any smaller range must be less than 1.

Thus, if v_0 is the value of v for $\omega=0$, we find that

$$V-v_0=\frac{1}{2\pi}\int_0^{2\pi}\frac{(v-v_0)(1-r^2)d\omega}{1-2r\cos(\theta-\omega)+r^2};$$

and, since v is a continuous function of ω, we can determine α so that

$$|v-v_0|<\epsilon, \quad \text{if } |\omega|<2\alpha.$$

Thus
$$\frac{1}{2\pi}\left(\int_0^{2\alpha}+\int_{2\pi-2\alpha}^{2\pi}\right)\frac{|v-v_0|(1-r^2)d\omega}{1-2r\cos(\theta-\omega)+r^2}<\epsilon.$$

We have next to consider the integral from $\omega=2\alpha$ to $\omega=2\pi-2\alpha$; here, provided that $|\theta|<\alpha$, $\cos(\theta-\omega)$ is not greater than $\cos\alpha$, and so

$$1-2r\cos(\theta-\omega)+r^2 \geqq 1-2r\cos\alpha+r^2=\sin^2\alpha+(\cos\alpha-r)^2\geqq \sin^2\alpha$$

while

$$1-r^2<2(1-r).$$

Thus, if H is the upper limit to the values of $|v|$ on the circle, we have

$$\left|\frac{(v-v_0)(1-r^2)}{1-2r\cos(\theta-\omega)+r^2}\right|<\frac{4H(1-r)}{\sin^2\alpha}, \quad \text{if } |\theta|<\alpha.$$

Consequently

$$\frac{1}{2\pi}\int_{2\alpha}^{2\pi-2\alpha}\frac{|v-v_0|(1-r^2)\,d\omega}{1-2r\cos(\theta-\omega)+r^2}<\frac{4H(1-r)}{\sin^2\alpha}, \quad \text{if } |\theta|<\alpha.$$

It is therefore possible to find first α and then δ, so that

$$|V-v_0|<2\epsilon, \quad \text{if } |\theta|<\alpha, \text{ and } 1-r<\delta,$$

that is to say, *V approaches the limit v_0 as the point (r, θ) moves up towards the point 1 by any path.*

If v is continuous except at $\omega=0$ and is there discontinuous, the integral still gives a power-series for $f(c)$, and the preceding work is valid as c approaches any point on the circle except 1. To deal with the point $c=1$, suppose that v has the limit l, when $\omega\to 0$ through positive values; and the limit m when $\omega\to 0$ through negative values. Then, if we write

$$v'=v-\frac{l-m}{\pi}\sum\frac{1}{n}\sin n\omega,$$

it is evident from Art. 65 that v' becomes continuous at $\omega=0$, if we assign to v' the value $\frac{1}{2}(l+m)$ for $\omega=0$.

Further,
$$V'=V-\frac{l-m}{\pi}\sum\frac{r^n}{n}\sin n\theta=V-\frac{l-m}{\pi}\phi,$$

where ϕ represents the same angle as is indicated in Fig. 25.

Now from the theorem just established we have

$$\lim_{(r,\theta)} V'=\lim_{(\omega)} v'=\tfrac{1}{2}(l+m),$$

so that

$$\lim_{(r,\theta)} V=\tfrac{1}{2}(l+m)+(l-m)\frac{\phi_0}{\pi},$$

where ϕ_0 is the limiting value of ϕ as (r, θ) approaches 1.

In the particular case when v is given in the form $\Sigma a_n \sin n\omega$, we shall have $m=-l$, and then the result is

$$\lim_{(r,\theta)} \Sigma a_n r^n \sin n\theta=2l\phi_0/\pi.$$

It will be noted that in this case the series Σa_n cannot be convergent; for if it were convergent we should have

$$\lim_{(r,\theta)} \Sigma a_n r^n(\cos n\theta+\iota\sin n\theta)=\Sigma a_n$$

in virtue of the extension of Abel's theorem (Art. 86); that is,

$$\lim_{(r,\theta)} \Sigma a_n r^n \sin n\theta=0,$$

which is not true. Thus Σa_n cannot converge.

88. Taylor's theorem for a power-series.

We have seen that a power-series $\Sigma a_n x^n$ represents a continuous function of x, say $f(x)$, within its circle of convergence $|x|=R$;

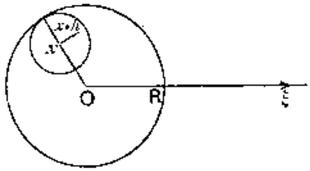

FIG. 26.

let us now attempt to express $f(x+h)$ as a power-series in h. Draw the circle of convergence, and mark a point x inside it, such that $|x|=r$; draw a second circle (of radius $R-r$), with centre x, to touch the first, and mark a point $x+h$ within the second circle.

We shall now see that $f(x+h)$ can be expressed as a power-series in h.

In fact $f(x+h)$ is the sum, by columns, of the double series

$$\begin{array}{l} a_0+a_1x+\;a_2x^2+\;\;a_3x^3\;\;+\dots \\ \quad\;\;+a_1h+2a_2xh+3a_3x^2h+\dots \\ \qquad\qquad\;+\;a_2h^2\;+3a_3xh^2+\dots \\ \qquad\qquad\qquad\qquad+\;\;a_3h^3\;\;+\dots \\ \qquad\qquad\qquad\qquad\qquad\qquad+\dots. \end{array}$$

But this series is absolutely convergent, because, if we replace each term by its absolute value, we get the series

$$|a_0|+|a_1|(r+\rho)+|a_2|(r+\rho)^2+|a_3|(r+\rho)^3+\dots,$$

where $\rho=|h|$. Now this series is convergent, because $r+\rho<R$ by the construction; and therefore the double series converges absolutely. That is, we can sum the double series by rows, without altering its value (Arts. 33 and 82).

Hence $\quad f(x+h)=f(x)+hf_1(x)+\dfrac{h^2}{2!}f_2(x)+\dfrac{h^3}{3!}f_3(x)+\dots,$

where
$$f_1(x)=a_1+2a_2x+3a_3x^2+\dots,$$
$$f_2(x)=1\,.\,2a_2+2\,.\,3a_3x+3\,.\,4a_4x^2+\dots,$$
$$f_3(x)=1\,.\,2\,.\,3a_3+2\,.\,3\,.\,4a_4x+3\,.\,4\,.\,5a_5x^2+\dots, \text{ etc.},$$

so that these series may be obtained from $f(x)$ by simply applying the formal rules for successive differentiation, without paying any attention to the meaning of the process.

The series in h may be called *Taylor's series.*

It may be useful to remark that the circle of convergence for the new series often reaches beyond the circle $|x| = R$; we know that it must reach as far as this circle, but there is no evidence that it may not extend further.

For instance, it is easy to see that if we write

$$f(x) = 1 + x + x^2 + x^3 + \dots ,$$

then
$$f(\tfrac{1}{2}\iota + h) = \frac{1}{1-\frac{1}{2}\iota} + \frac{h}{(1-\frac{1}{2}\iota)^2} + \frac{h^2}{(1-\frac{1}{2}\iota)^3} + \dots ,$$

which converges if $|h| < |1 - \frac{1}{2}\iota|$

or if $|h| < \frac{1}{2}\sqrt{5}$.

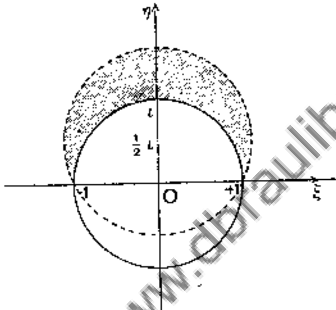

FIG. 27.

Thus the Taylor's series converges in the shaded area outside the original circle of convergence. We have thus a new power-series which *continues* the function $f(x)$ beyond the area of its original definition. Some examples of this process of continuation will be found in Exs. 30, 31, Chap. XI.

The idea of continuation is fundamental in Weierstrass's theory of Functions, but further details lie outside our province. The reader may consult Harkness and Morley's *Introduction to the Theory of Analytic Functions* for a good account of this theory.

It will be seen that in the last example that for any point P on the upper half of the circle $|x| = 1$, a further Taylor-series can be obtained within a circle, centre P, which at least reaches up to the dotted circle; and in fact the circle of convergence usually reaches further. But as a matter of fact no such Taylor-series can be found for the special point $x=1$; and accordingly *the point $x=1$ is called a singular point* for the function defined by

$$f(x) = 1 + x + x^2 + x^3 + \dots .$$

This is, of course, evident from the fact that

$$f(x) = 1/(1-x), \quad \text{when } |x| < 1.$$

But, in general, a function defined by a power-series will not be expressible in terms of known elementary functions; and then we can sometimes recognise the existence of a singular point by the following theorem:

If the coefficients of a power-series $\Sigma a_n x^n$ are all positive (at any rate after a certain stage), the series has a singular point at the point $x=R$, if R is the radius of convergence. [VIVANTI and PRINGSHEIM.]

Suppose, if possible, that, for some value of r between 0 and R, the series $\Sigma f_n(r)\frac{(x-r)^n}{n!}$ has a larger radius of convergence than $R-r$; we can then choose a real number $\rho\,(>R)$, such that the last series converges for $x=\rho$. Now this series can be arranged as a double series, *which contains here only positive terms*; it will therefore remain convergent when summed as $\Sigma a_n\{r+(\rho-r)\}^n$. That is, $\Sigma a_n x^n$ will converge for $x=\rho$, contrary to the original hypothesis; and so $x=R$ must be a singular point for the given power-series.

It must not be assumed (as might perhaps be expected from the previous example) that the power-series is *divergent* at a singular point. In fact the general theorem of Vivanti just established shews that the convergence of $\Sigma a_n R^n$ will not affect the fact that $x=R$ is a singular point for the power-series; although naturally $x=R$ will certainly be a singular point if $\Sigma a_n R^n$ diverges.

Ex. 1. The point $x=1$ is a singular point for the following series:

$$x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}+\dots$$

although the series converges to the sum $\frac{1}{6}\pi^2$ for $x=1$, and consequently Abel's theorem (Art. 86) may be applied to shew that the function is continuous within a limaçon joining up to the point $x=1$.

Ex. 2. Similarly $x=1$ is a singular point for the series

$$\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\dots.$$

This fact can be confirmed by noting that this series can be written in the form

$$\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)-\left(\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\dots\right),\quad |x|<1$$

$$=\log\left(\frac{1}{1-x}\right)-\frac{1}{x}\left\{\log\left(\frac{1}{1-x}\right)-x\right\}=1-\frac{1}{x}(1-x)\log\left(\frac{1}{1-x}\right).$$

(Art. 95.)

Thus the point $x=1$ is clearly a singular point for this series; but yet as $x\to 1$ (from the interior of the circle), the sum tends to 1, in agreement with Abel's theorem, because

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots=1.$$

In all the examples given so far, the series has only one singular point; but it is easy to construct examples with more singularities.

Ex. 3. $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$. (Art. 95.)

Write here $x = \pm \iota y$; then the series becomes

$$\pm \iota (y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{1}{7}y^7 + \dots).$$

Hence, by Vivanti's theorem, the y-series has a singular point at $y=1$; and consequently the x-series has two singular points given by $x = \pm \iota$.

Ex. 4. $x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \dots$. (Art. 95.)

It is easy to prove similarly that the points $x = \pm 1$ are both singularities of the x-series; although as a matter of fact the series converges absolutely at all points of the circle $|x| = 1$. (Art. 12·2.)

We have seen that the radius of convergence R of $\Sigma a_n x^n$ may often be determined from

$$R = \lim |a_n| / |a_{n+1}|.$$

Fabry * has proved that if $\lim (a_n/a_{n+1})$ is determinate, and equal to l (so that $R = |l|$), then the point $x = l$ is a singular point of the series.

This theorem will give at once Exs. 1, 2, but it does not give Exs. 3, 4 (because a_n/a_{n+1} oscillates between 0 and ∞); thus in some respects it is less effective than the theorem of Vivanti.

89. Extensions of Cauchy's inequalities.

If we apply the mean-value method to any circle (with centre x_0) which falls entirely within the circle of convergence $|x| = R$, it follows from Art. 88 that

$$f(x_0) = \mathfrak{M} f(x), \quad \text{for } |x - x_0| = s < R - r_0,$$

which is the analogue of Gauss's mean-value theorem in Potential-theory.

Thus we see that $|f(x_0)|$ is less than the maximum of $|f(x)|$ on any circle $|x - x_0| = s$ which falls within the circle of convergence.

It is therefore evident that if we consider the values of $|f(x)|$ corresponding to points within or on a circle $|x| = r < R$, *the greatest value of* $|f(x)|$ *must occur on the circle.* Or if, as in Art. 84, M denotes the maximum of $|f(x)|$ on the circle $|x| = r$, then $|f(x)| < M$ *at all internal points.*

Suppose next that the exact radius of convergence of $f(x)$ is not known, but that $|x| = r$ is known to be within the circle of con-

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* See Hadamard, *La Série de Taylor*, pp. 19-25; and for various extensions see Van Vleck, *Trans. Amer. Math. Soc.* vol. 1, 1900, p. 293.

vergence; and further that for all points on the circle $|x|=r$, the h-series for $f(x+h)$ converges uniformly on $|h|=s$. Then, if M is the maximum of $|f(x+h)|$ for all points such that $|x|=r$, $|h|=s$, we have by applying Cauchy's inequality to the h-series

$$\left|\frac{f_n(x)}{n!}\right| < \frac{M}{s^n} = M', \text{ say.}$$

Applying the same inequality to the x-series for $f_n(x)$ we see that

$$\frac{(m+n)(m+n-1)\ldots(m+1)}{n!}|a_{m+n}| < \frac{M'}{r^m} = \frac{M}{r^m s^n}.$$

Thus, in the expanded form of $(r+s)^{m+n}|a_{m+n}|$ *every term is less than* M; *and therefore*

$$(r+s)^{m+n}|a_{m+n}| < (m+n+1)M.$$

It follows that *the radius of convergence of* $\Sigma a_n x^n$ *is at least equal to* $(r+s)$.

This leads at once to the theorem that *there is at least one singular point on the circle of convergence of a power-series*; that is, a point in the neighbourhood of which Taylor's series cannot be applied.

In fact, if we assume that the Taylor's series is valid for all points on the circle $|x|=R$, there will be a lower limit (say S) to the radii of convergence of the Taylor's series (for points on the circle $|x|=R$).

Thus, if $0<s<S$, we can apply the foregoing argument to shew that the radius of convergence would be at least equal to $R+s$, which would be greater than R.

Thus S must be zero; and accordingly there is at least one point on the circle of convergence for which the Taylor's series does not converge.

From this inequality* we can shew that *the circle of convergence of the reciprocal of a power-series is not less than that of the primary series, unless the latter has a zero within its circle; and then the circle of the reciprocal reaches up to the zero of the primary series which is nearest to the origin.*

In fact the argument of Art. 54 shews that if L is the maximum value of $|f(x)|$ on any circle $|x|=R'<R$, then the power-series for $\{f(x)\}^{-1}$ will converge if

$$|x| < AR'/(A+L), \quad \text{if } |a_0| = A.$$

Now, if l denotes the minimum of $|f(x)|$ *within and on the circle* $|x|=R'$, it is evident that

$$l < |a_0|, \text{ or } l < A; \text{ and so } l/(l+L) < A/(A+L).$$

* H. F. Baker, *Proc. Lond. Math. Soc.* (1), vol. 34, 1902, p. 296; the discussion given there is for series in two variables, and of course can be extended to any number.

and accordingly the series for $\{f(x)\}^{-1}$ will certainly converge if

$$|x| \leqq lR'/(l+L).$$

By transferring now to a point x_1 such that $|x_1| = r_1 = lR'/(l+L)$, we can infer that $\{f(x)\}^{-1}$ expressed as a series in $(x-x_1)$ will certainly converge if

$$\frac{|x-x_1|}{R'-r_1} \leqq \frac{l}{l+L}.$$

Thus, by means of the result established above we see that the radius of convergence of $\{f(x)\}^{-1}$ is at least equal to r_2, where

$$r_2 = r_1 + (R'-r_1)l/(l+L),$$

so that $$R' - r_2 = (R'-r_1)L/(l+L) = R'\{L/(l+L)\}^2.$$

Continuing the process, the radius is seen to be not less than r_n, where

$$R' - r_n = R'\{L/(l+L)\}^n,$$

and consequently, so long as l is not zero, the radius of convergence cannot be less than R'; and so must be at least equal to R.

90. Lagrange's series.

The discussion in Arts. 55, 55·1 is not affected substantially by treating x as a complex variable; but we can now indicate a method of estimating the radius of convergence of Lagrange's series. Suppose that the series for reversion is given by

$$y = a_1x + a_2x^2 + \ldots = f(x),$$

the term a_0 being here *zero*.

Then, as proved in Art. 55·1, the expansion of $g(x)$ in powers of y is equal to $\Sigma b_n y^n$, where nb_n is the coefficient of $1/x$ in the expansion of $g'(x)/y^n$ in ascending powers of x.

Thus, applying the mean-value process,

$$nb_n = \mathfrak{M} xg'(x)/\{f(x)\}^n, \quad \text{for } |x| = r.$$

Thus, if l is the *minimum* of $|f(x)|$ and M is the maximum of $|g'(x)|$ for $|x| = r$, we have

$$n|b_n| < Mr/l^n.$$

Hence $\Sigma b_n y^n$ certainly converges if $|y| < l$.

Now l will usually depend on r, and, provided that r is less than the radii of convergence of $g(x)$ and $f(x)$, we can adjust r so as to obtain the largest value available for l.

To illustrate the method, consider the examples of Art. 55·1.

Ex. 1. Here $l = r - Ar^2$, if $A = |a|$, and the greatest value of l is given by $$Ar = \tfrac{1}{2}, \quad l = 1/4A.$$

Thus the y-series converges if $A|y| < \frac{1}{4}$ (as obtained directly in Art. 55·1).

Ex. 2. Similarly $l=r-Ar^{m+1}$, and the greatest value of l is found to be

$$\frac{m}{m+1}\{A(m+1)\}^{-\frac{1}{m}}.$$

Ex. 3. Here $l=r(1-r)^m$, $r<1$,

and the largest value is $\dfrac{m^m}{(m+1)^{m+1}}$.

Ex. 4. Here $l=re^{-Br}$, if $B=|b|$,

and the largest value of l is given by $Br=1$; and this is $1/(Be)$—in agreement with the result previously found.

It may be proved by less elementary considerations, that the best value of r is given by the root $x=x_1$ of $f'(x)=0$ which is nearest to the origin; and that the largest value of l is equal to the value of $|f(x_1)|$.

The method adopted in the present article is substantially the same as one used by Goursat.*

91. Weierstrass's double-series theorem.†

Suppose that the series

$$f_m(x)=\sum_{n=0}^{\infty} a_{m,n}x^n \quad (m=0, 1, 2, \ldots \infty)$$

are all convergent for $|x|<R$, and further that the series

$$F(x)=\sum_{m=0}^{\infty} f_m(x)$$

converges uniformly along every circle whose radius is less than R. Then

(1) *the series $\sum_{m=0}^{\infty} a_{m,n}$ converges for every value of n;*

(2) *if $A_n=\sum_{m=0}^{\infty} a_{m,n}$, then $F(x)=\sum_{n=0}^{\infty} A_n x^n$, $|x|<R$.*

For

$$\sum_{m=0}^{\infty} a_{m,n}=\sum_{m=0}^{\infty} \mathfrak{M}\{f_m(x)/x^n\},$$

the mean being taken along any circle $|x|=r_1<R$. Now on this circle $F(x)=\Sigma f_m(x)$ is uniformly convergent, and so the series $\Sigma a_{m,n}$ must be convergent and equal to $\mathfrak{M}\{F(x)/x^n\}$.

Again, if μ is any integer and

$$G(x)=\sum_{m=\mu}^{\infty} f_m(x), \quad B_n=\sum_{m=\mu}^{\infty} a_{m,n},$$

we have similarly

$$B_n=\mathfrak{M}\{G(x)/x^n\}, \text{ and so } |B_n|<M_1/r_1^n$$

if M_1 is the maximum of $G(x)$ on the circle $|x|=r_1$.

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* *Cours d'Analyse Math.* vol. 2, p. 131: for other methods see Schlömilch, *Kompendium der höheren Analysis*, vol. 2, p. 100; and H. M. Macdonald, *Proc. Lond. Math. Soc.* (1), vol. 29, p. 576.

† Weierstrass, *Ges. Werke*, vol. II. p. 205.

Hence, if $|x| = r (< r_1)$, we have

$$\left|\sum_{n=0}^{\infty} B_n x^n\right| < \sum_{n=0}^{\infty} M_1 (r/r_1)^n = M_1 r_1/(r_1 - r),$$

and by Art. 89, $|G(x)| < M_1$, so that

$$\left|G(x) - \sum_{n=0}^{\infty} B_n x^n\right| < M_1(2r_1 - r)/(r_1 - r), \quad \text{if } |x| = r.$$

Now, we have identically

$$F(x) - G(x) = \sum_{n=0}^{\infty} (A_n - B_n) x^n,$$

because this equation contains only a *finite* number (μ) of series: and so we find

$$\left|F(x) - \sum_{n=0}^{\infty} A_n x^n\right| < M_1(2r_1 - r)/(r_1 - r), \quad \text{if } |x| = r.$$

But, since $F(x)$ converges uniformly on the circle $|x| = r$, we can make M_1 as small as we please by proper choice of μ. Thus, since $F(x)$ and $\Sigma A_n x^n$ are *independent* of μ, we must have $F(x) = \sum_{n=0}^{\infty} A_n x^n$.

EXAMPLES.

Geometrical applications of Complex Numbers.

1. If the triangles AOB, BOC are directly similar, and if O bisects BK, prove that the triangle AKC is directly similar to the first pair.

[This follows at once from the algebraic identities

$$\frac{a}{b} = \frac{b}{c} = \frac{a+b}{b+c} = \frac{a-k}{c-k}.]$$

2. If x and y are complex, prove that

$$|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2),$$

and interpret this equation in Argand's diagram. Deduce that

$$|x+y| + |x-y| = |x + \sqrt{(x^2 - y^2)}| + |x - \sqrt{(x^2 - y^2)}|.$$

[HARKNESS and MORLEY.]

3. If A, B are the points in Argand's diagram which represent the roots of $ax^2 + 2bx + c = 0$, and A', B' represent the roots of $a'x^2 + 2b'x + c' = 0$, shew that the condition $ac' + a'c - 2bb' = 0$ is equivalent to the conditions

$$OA^2 = OA' \cdot OB', \quad A'\hat{O}A = A\hat{O}B',$$

where O is the mid point of AB. [*Math. Trip.* 1901.]

[Transfer to O as origin, which gives $b = 0$.]

4. Shew in a diagram the roots of the equation $32x^5 = (x+1)^5$, and prove that they are concyclic.

5. If the equation $a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = 0$

has real coefficients, and if its roots in Argand's diagram are concyclic (two being real and two complex), then

$$a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3 = 0.$$

6. If t represents a complex number such that $|t|=1$, shew that as t varies, the point

$$x=\frac{at+b}{t-c}$$

describes a circle, unless $|c|=1$, when the locus is a straight line.

[MORLEY.]

7. If t varies so that $|t|=1$, shew that the point

$$2x=at+b/t$$

in general describes an ellipse whose axes are $|a|+|b|$ and $|a|-|b|$, and whose foci are given by $x^2=ab$.

If $|a|=|b|$, prove that the point x traces out the portion of a straight line which is terminated by the two points $x^2=ab$.

8. If t varies so that $|t|=1$, prove that the point

$$x=at^2+2bt+c$$

in general describes a limaçon, whose focus is $c-b^2/a$. Find the node; and if $|a|=|b|$, shew that the limaçon reduces to a cardioid. [MORLEY.]

9. Constructions for trisecting an angle.

If $a=\cos\alpha+\iota\sin\alpha$, the determination of $\frac{1}{3}\alpha$ is equivalent to the solution of the equation in t,

$$t^3=a.$$

To effect this geometrically we use the intersections of a conic with the circle $|t|=1$; the form of the conic is largely arbitrary, but we shall give three typical constructions, the first and second of which, at any rate, were known to the later Greek geometers (*e.g.* Pappus).

(i) *A rectangular hyperbola.*

If we write our equation in the form

$$t^2=a/t,$$

and then put $t=\xi+\iota\eta$, $1/t=\xi-\iota\eta$, we find that the points trisecting the angle are given by three of the intersections with the circle $\xi^2+\eta^2=1$ of the two rectangular hyperbolas

$$\xi^2-\eta^2-(\xi\cos\alpha+\eta\sin\alpha)=0, \quad 2\xi\eta-\xi\sin\alpha+\eta\cos\alpha=0.$$

The fourth intersection of the hyperbolas is at the origin and so of course is not on the circle.

Either of these hyperbolas solves the problem, but the second is the easier to construct; its asymptotes are parallel to the axes (the one axis being an arm of the angle to be trisected), its centre is the point $(-\frac{1}{2}\cos\alpha, \frac{1}{2}\sin\alpha)$, and it passes through the centre of the circle (that is, the vertex of the angle to be trisected). Since a hyperbola is determined by its asymptotes and a point on the curve, we can now construct the hyperbola.

(ii) *A hyperbola of eccentricity* 2.

The first hyperbola in (i) cuts the circle $\xi^2+\eta^2=1$ in the same points as the hyperbola $\xi^2-3\eta^2-2(\xi\cos\alpha+\eta\sin\alpha)+1=0.$

This hyperbola has eccentricity 2, and one focus at $(\cos\alpha, \sin\alpha)$, and $\eta=0$ is the corresponding directrix; $(\cos\alpha, -\sin\alpha)$ is the vertex on the other branch of the curve. From the present point of view, this hyperbola presents itself less naturally than those given in (i); but the reverse is the case if we use geometrical properties of conics, and this was of course the method used by the Greeks.

(iii) *A parabola.*

Again, we find that the first hyperbola of (i) cuts the circle $\xi^2+\eta^2=1$ in the same points as the parabola

$$2\eta^2+\xi\cos\alpha+\eta\sin\alpha-1=0.$$

This parabola has its axis parallel to $\eta=0$, passes through the points

$$(\cos\alpha, \tfrac{1}{2}\sin\alpha), \quad (\cos\alpha, -\sin\alpha),$$

and touches the line $\xi\cos\alpha+\eta\sin\alpha-1=0$ at the point $(\sec\alpha, 0)$.

Results connected with functions of $2\pi/k$.

10. If $x=\exp(2\pi\iota/a)$ and $X=\sum_{n=0}^{a-1} x^{n^2}$, shew that

(i) $a=7$ gives $X=\iota\sqrt{7}$, (ii) $a=11$, $X=\iota\sqrt{11}$, and (iii) $a=13$, $X=\sqrt{13}$.

[Taking case (i), we find at once that $X=1+2S$, where

$$S=x+x^4+x^2, \quad S'=x^6+x^3+x^5,$$

the sequence of indices in S being given by

$$1^2, 2^2, 3^2=2+7.$$

It is easily proved that $S+S'=-1$, since $(x^7-1)/(x-1)=0$, and

$$SS'=3+S+S'=2.$$

Thus S is a root of $S^2+S+2=0$, which gives

$$X^2=-7, \quad \text{or} \quad X=\pm\iota\sqrt{7}.$$

It is easily proved that the sign must be $+$ by considering

$$\sin(2\pi/7)+\sin(8\pi/7)+\sin(4\pi/7);$$

compare Ex. 6, Ch. IX.

In like manner we deal with case (ii).

In case (iii) we write again $X=1+2S$, where now

$$S=x+x^4+x^9+x^3+x^{12}+x^{10}, \quad S'=x^2+x^8+x^5+x^6+x^{11}+x^7,$$

the sequence of indices in S being given by

$$1^2, 2^2, 3^2, \quad 4^2=3+13, \quad 5^2=12+13, \quad 6^2=10+26.$$

Here again $S+S'=-1$, but $SS'=3(S+S')=-3$.

Thus $S^2+S=3$ and $X^2=13$. That S (and therefore X) must be positive is obvious by considering that

$$\tfrac{1}{2}S=\cos(2\pi/13)+\cos(8\pi/13)+\cos(6\pi/13),$$

in which the only negative term is the second, and that term is less than the first (in numerical value).]

11. With the same notation as in the last example, shew that

(i) $a=4$ gives $X=(1+\iota)2$, (ii) $a=8$, $X=(1+\iota)2\sqrt{2}$,

(iii) $a=12$, $X=(1+\iota)2\sqrt{3}$.

[In the first case we have $x=\iota$.
In the second case, we have $x^2=\iota$, $x=(1+\iota)/\sqrt{2}$.
In the third case, $x^3=\iota$, $x=\frac{1}{2}(\sqrt{3}+\iota)$.]

12. The value of the more general sum $y=\sum_{n=0}^{a-1} x^{bn^2}$, where b is an integer prime to a, can now be inferred in these special cases. We find, in fact, that $y=X$ (in the cases of Ex. 10) if $b=k^2-Ma$, where k, M are integers. Thus, for instance,

$$\text{(i)}\ a=7,\ \begin{cases} y=+\iota\sqrt{7}, & \text{if } b=1,\ 4,\ 2, \\ \text{or} \quad =-\iota\sqrt{7}, & \text{if } b=6,\ 3,\ 5, \end{cases}$$

with similar formulae for (ii), (iii).

In Ex. 11, if

$$a=8,\ \begin{cases} y=(1+\iota)2\sqrt{2},\ \text{if } b=1, & \text{or } (-1+\iota)2\sqrt{2},\ \text{if } b=3, \\ \text{or } (-1-\iota)2\sqrt{2},\ \text{if } b=5, & \text{or } (1-\iota)2\sqrt{2},\ \text{if } b=7. \end{cases}$$

13. It will be seen from a consideration of the special cases discussed in Exs. 10, 11, that the set of values x^{-n^2} may, or may not, be equivalent to the set x^{n^2}. In the former case, a is of the form $4k+1$, where k is an integer; and the sum S consists of k pairs of terms, whose indices are complementary (that is, of the form ν, $a-\nu$). On multiplying SS' out, it is easily seen to be the same as $k(S+S')$.

Thus we find $\quad S^2+S-k=0 \quad$ or $\quad X^2=4k+1=a$.

Similarly, if a is of the form $4k+3$, we find that the terms x^{-n^2} belong to S', and then we find

$$SS'=(2k+1)+k(S+S')=k+1.$$

Thus $\quad S^2+S+k+1=0 \quad$ or $\quad X^2=-(4k+3)=-a$.

[*Math. Trip.* 1895.]

A general determination of the sign of X (and indeed a complete discussion of the distribution of indices between S and S') belongs to the problem of quadratic residues in the Theory of Numbers.*

When a is an even integer $a=2k$, where k is *odd*, we note that $x^{(n+k)^2}=-x^{n^2}$, so that X is identically zero.

When $a=4k$, the results of Ex. 11 suggest that $X=(1+\iota)\sqrt{a}$, but a complete proof of this requires some further discussion.†

14. Deduce from Ex. 10 that

$$\tan\frac{3\pi}{11}+4\sin\frac{2\pi}{11}=\sqrt{11}.$$

[*Math. Trip.* 1895.]

* For example, see Gauss, *Disq. Arithm.*, Art. 356; *Werke*, vol. 1, p. 441; *Werke*, vol. 2, p. 11; G. B. Mathews, *Theory of Numbers*, pt. 1, pp. 200–212; H. Weber, *Algebra*, vol. 1, § 179; Dirichlet, *Zahlentheorie*, §§ 111–117.

† Gauss, *Werke*, vol. 2, pp. 34–45.

[In fact,

$$\iota \tan\frac{3\pi}{11}=\frac{x^3-1}{x^3+1}=\frac{x^3-x^{33}}{1+x^3}=x^3-x^6+x^9-x+x^4-x^7+x^{10}-x^2+x^5-x^8,$$

since $x^{11}=1$. Thus, in the notation of Ex. 13,

$$\iota \tan\frac{3\pi}{11}=S-S'-2(x-x^{10}).]$$

15. The results of Ex. 10 lead to an easy geometrical construction for **the regular heptagon inscribed in a circle.** In fact, we see at once that x, x^2, x^4 are the roots of the cubic $t^3-St^2+S't-1=0$, or of

$$t^3-\tfrac{1}{2}t^2(\iota\sqrt{7}-1)-\tfrac{1}{2}t(\iota\sqrt{7}+1)-1=0,$$

so that 1, x, x^2, x^4 are roots of

$$t^2+\frac{1}{t^2}-\frac{1}{2}\left(t+\frac{1}{t}\right)-1-\frac{1}{2}\iota\sqrt{7}\left(t-\frac{1}{t}\right)=0.$$

If we write $t=\xi+\iota\eta$, $1/t=\xi-\iota\eta$, we find from the last equation

$$2(\xi^2-\eta^2)-\xi+\eta\sqrt{7}-1=0,$$

which represents a rectangular hyperbola passing through the vertices 1, x, x^2, x^4 of a regular heptagon inscribed in the circle $\xi^2+\eta^2=1$.

Another construction is given by either of the parabolas

$$4\xi^2-\xi+\eta\sqrt{7}-3=0,\quad 4\eta^2+\xi-\eta\sqrt{7}-1=0.$$

[*Oxford Sen. Schol.*, 1904.]

16. Let $x=\exp(2\pi\iota/17)$, and arrange the various powers of x according to the sequence of indices formed by taking powers of 3; thus write *

$$S=x+x^9+x^{13}+x^{15}+x^{16}+x^8+x^4+x^2=\tfrac{1}{2}(X-1),$$
$$S'=x^3+x^{10}+x^5+x^{11}+x^{14}+x^7+x^{12}+x^6.$$

Then $S+S'=-1$, $SS'=-4$, leading to $S-S'=+\sqrt{17}$, because it is easy to see that S is positive when expressed in terms of four cosines.

Next take $p=x+x^{13}+x^{16}+x^4$, $\quad q=x^3+x^5+x^{14}+x^{12}$,

$p'=x^9+x^{15}+x^8+x^2$, $\quad q'=x^{10}+x^{11}+x^7+x^6$.

Then $p+p'=S$, $\quad pp'=-1$, $\quad p-p'=\sqrt{(S^2+4)}$,

$q+q'=S'$, $\quad qq'=-1$, $\quad q-q'=\sqrt{(S'^2+4)}$,

each square-root being found to be positive as before.

Finally, put $r=x+x^{16}$, $\quad r'=x^{13}+x^4$,

and then $r+r'=p$, $\quad rr'=q$, $\quad r-r'=\sqrt{(p^2-4q)}$,

this square-root being also positive.

It follows that $\cos(2\pi/17)$ can be found from the solution of four quadratics; and accordingly **a regular 17-sided figure** can be constructed by Euclidean methods. [Gauss.]

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* We note that $3^3=10+17$, $3^4=13+68$, etc.; but in writing down the sequence of indices, multiples of 17 may be rejected. Thus the sixth index is derived from $3\times 13=5+34$, and the seventh from $3\times 5=15$, etc.

17. If p is an odd integer and q is any integer prime to p, shew that

$$\sum_{1}^{\frac{1}{2}(p-1)} \sin(2\lambda n\theta)\cot(n\theta) = \tfrac{1}{2}p - \lambda,$$

where $\theta = \pi q/p$, and λ is any integer from 1 to $p-1$ (both included). Determine the value of the sum when λ is greater than p.

[EISENSTEIN and *Math. Trip.* 1897.]

[Write $t = e^{2i\theta}$, then from the theory of partial fractions

$$\frac{px^{\lambda-1}}{x^p - 1} - \frac{1}{x-1} = \sum_{1}^{p-1} \frac{t^{n\lambda}}{x - t^n}, \quad \text{if } 1 \leqq \lambda \leqq p.$$

Take the limit of both sides as $x \to 1$, and we get

$$\tfrac{1}{2}(p+1) - \lambda = \sum t^{n\lambda}/(t^n - 1) \quad (n = 1, 2, \ldots p-1).$$

Also $\qquad -1 = \sum t^{n\lambda}$, if we now suppose $\lambda < p$,

so that $\qquad \tfrac{1}{2}p - \lambda = \tfrac{1}{2}\sum_{1}^{p-1} t^{n\lambda}\dfrac{t^n+1}{t^n-1}$.]

18. Prove similarly that, with the same notation as in the last example,

$$\sum_{0}^{p-1} \frac{\cos k(\alpha + n\theta)}{\sin(\alpha + n\theta)} = p\cot p\alpha, \quad \sum_{0}^{p-1} \frac{\sin k(\alpha + n\theta)}{\sin(\alpha + n\theta)} = p,$$

where k is odd and not greater than $2p-1$, but p need not be odd.

[*Royal Univ. of Ireland*, 1900.]

[Write $\lambda = \frac{1}{2}(k+1)$, $x = e^{-2i\alpha}$ in the partial fractions used in Ex. 17.]

Convergence of Complex Sequences.

19. If an infinite set of points is taken within a square, the set has at least one limiting point (that is, a point in whose neighbourhood there is an infinity of points of the set).

[BOLZANO and WEIERSTRASS.]

[For if the square is subdivided into four by bisecting the sides, at least one of the four contains an infinity of points of the set; repeating this argument, there is an infinity within at least one square whose side is $a/2^n$, where a is the side of the original square, and n is any integer. It is then not difficult to see that we can select a sequence of squares, each within the preceding, and each containing an infinity of points of the set; the centres of these squares then define a sequence of points which can be proved to have a limiting point. Finally, we can shew that within any square whose centre is at this limiting point, there is an infinity of points of the set.]

20. Suppose that $S_n(x) = f_0(x) + f_1(x) + f_2(x) + \ldots + f_n(x)$, and let the roots of $S_n(x) = 0$ be marked in Argand's diagram for all values of n: if these roots have $x = \alpha$ as a limiting point, the series

$$f_0(x) + f_1(x) + f_2(x) + \ldots$$

has $x = \alpha$ as a zero, provided that the series converges uniformly within an area including $x = \alpha$.

[HURWITZ.]

21. If

$$F(x, n) = \frac{n!}{x(x+1)\ldots(x+n)},$$

the series $\Sigma a_n F(x, n)$ converges absolutely, provided that $\Sigma |a_n| n^{-\xi}$ does so, where ξ is the real part of x. Thus, in particular, if $\Sigma a_n x^n$ has a radius of convergence greater than 1, $\Sigma a_n F(x, n)$ is absolutely convergent for all values of x (other than real negative integers). But if the radius of convergence is less than 1, $\Sigma a_n F(x, n)$ cannot converge.

Finally, if the radius of convergence is equal to 1, suppose that

$$\left|\frac{a_n}{a_{n+1}}\right| = 1 + \frac{\alpha}{n} + \frac{\omega_n}{n^\lambda},$$

where $\lambda > 1$ and $|\omega_n| < A$: then $\Sigma a_n F(x, n)$ is absolutely convergent if $\xi > 1 - \alpha$. [KLUYVER, see also NIELSEN, *Gammafunktion*, §§ 93, 94.]

[Note that $$F(x, n) \sim \Gamma(x) n^{-x}.]$$

22. Shew that, with the notation of the last example, the two series $\Sigma(a_n/n^x)$, $\Sigma a_n F(x, n)$ converge for the same values of x. [LANDAU.]

[Apply Abel's Lemma, taking $v_n = n^x F(x, n)$; the series $\Sigma(v_n - v_{n+1})$ and $\Sigma(1/v_n - 1/v_{n+1})$ are then easily proved to be absolutely convergent.]

23. Shew that in the notation of Ex. 21,

$$F(x-1, n) - F(x-1, n+1) = F(x, n),$$

and deduce that $\Sigma F(x, n)$ converges only when $\xi > 1$; so that $\Sigma F(x, n)$ can only converge absolutely. Shew also that $\Sigma(-1)^n F(x, n)$ converges if $\xi > 0$; and apply Ex. 22 to deduce the corresponding results for Σn^{-x}, $\Sigma(-1)^n n^{-x}$.

24. The series (see Ex. 15, Ch. I.)

$$\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \frac{x^8}{1-x^{16}} + \ldots$$

represents the function $x/(1-x)$, if $|x| < 1$, and $1/(1-x)$, if $|x| > 1$. [J. TANNERY.]

25. Shew that the series

$$\sum_0^\infty \left[\left(x + n + \frac{1}{2}\right)\log\left(1 + \frac{1}{x+n}\right) - 1\right], \quad \sum_0^\infty \left[\frac{1}{x+n} - \log\left(1 + \frac{1}{x+n}\right)\right]$$

are both convergent for all values of x, except $0, -1, -2, -3, \ldots$.

[For applications, see NIELSEN, *Gammafunktion*, §§ 33, 34.]

26. If (c_n) is a sequence of complex numbers such that $|c_n|$ tends steadily to ∞, shew that the series

$$\Sigma \frac{x^n}{c_n{}^n (x - c_n)}$$

converges absolutely for all values of x, except for $c_1, c_2, c_3, \ldots$. The series converges uniformly within the area bounded externally by the circle $|x| = R$, and internally by those circles $|x - c_n| = r$, which are contained within the circle $|x| = R$, the number r being taken small enough to prevent any overlapping of the circles.

27. The series of the last example can be simplified in case the points c_n lie along a straight line, and are such that $|c_{n+1} - c_n| \geqq k > 0$, where k is a constant; under these circumstances we can make similar statements with respect to

$$\Sigma \frac{x}{c_n(x-c_n)}, \quad \Sigma \frac{1}{(x-c_n)^2}, \quad \Pi\left(1-\frac{x}{c_n}\right)e^{x/c_n}.$$

28. Again, if the points c_n, although not distributed along a straight line, are such that no two of them are at a less distance apart than a constant k, similar statements can be made with respect to

$$\Sigma \frac{x^2}{c_n^2(x-c_n)}, \quad \Sigma\left\{\frac{1}{(x-c_n)^2}-\frac{1}{c_n^2}\right\}, \quad \Sigma \frac{1}{(x-c_n)^3}, \quad \Pi\left(1-\frac{x}{c_n}\right)e^{x/c_n+\frac{1}{2}(x/c_n)^2}.$$

A simple example of a set of this type occurs in the theory of elliptic functions, the points c_n being the vertices of a network of parallelograms.

[Here we note that not more than one point c_n can fall within a square of side $\frac{1}{2}k$; thus, if we draw squares, with centre at the origin, of sides $\frac{1}{2}k$, $\frac{3}{2}k$, $\frac{5}{2}k$, ..., not more than $8m$ points can lie between the two squares $(m-\frac{1}{2})k$, $(m+\frac{1}{2})k$. Hence $\Sigma|c_n|^{-3} < \Sigma 8m/(m-\frac{1}{2})^3 k^3$, and so $\Sigma|c_n|^{-3}$ converges.]

29. If (M_n) is a sequence of real numbers which tends steadily to ∞, and if x is a complex number whose real part is positive, the series

$$\Sigma \left|M_n^{-x} - M_{n+1}^{-x}\right|$$

is convergent.

[For, if $x = \xi + \iota\eta$ and $(M_n/M_{n+1})^{\xi} = \lambda_n$, the ratio of the general term of the given series to that of the convergent series

$$\Sigma\left(M_n^{-\xi} - M_{n+1}^{-\xi}\right)$$

is

$$R_n = \left|(1-\lambda_n^{1+\iota\eta/\xi})/(1-\lambda_n)\right|.$$

Now, $\quad \lambda_n^{1+\iota\eta/\xi} = \lambda_n\{\cos(\kappa\theta) + \iota\sin(\kappa\theta)\}, \quad$ if $\kappa = \eta/\xi, \quad \theta = \log\lambda_n$,

so that $\quad R_n^2 = \{1 - 2\lambda_n\cos(\kappa\theta) + \lambda_n^2\}/(1-\lambda_n)^2$.

Thus $\quad R_n \leqq (1+\lambda_n)/(1-\lambda_n)$,

and so if $\lambda_n \leqq \frac{1}{2}$, we see that $R_n \leqq (1+\frac{1}{2})/(1-\frac{1}{2}) \leqq 3$.

On the other hand, if $\lambda_n > \frac{1}{2}$, we can write (Art. 154)

$$(-\log\lambda_n)/(1-\lambda_n) < 1/\lambda_n < 2,$$

which leads to the result $R_n < \sqrt{(1+4\kappa^2)}$, because

$$R_n^2 = 1 + \frac{4\lambda_n}{(1-\lambda_n)^2}\{\sin^2\tfrac{1}{2}(\kappa\theta)\} < 1 + \kappa^2\left(\frac{\theta}{1-\lambda_n}\right)^2.$$

In either case there is a finite upper limit H to R_n, and so the given series converges because the comparison-series is convergent.]

30. If Σa_n is convergent, the series $\Sigma a_n M_n^{-x}$ is convergent if the real part of x is positive. Thus, *in general the region of convergence of* $\Sigma a_n M_n^{-x}$ *is bounded by a line parallel to the imaginary axis.*

Further, in case Σa_n is convergent, the series converges *uniformly* in a sector of the plane bounded by the lines $\eta = \pm\kappa\xi$, where κ is any assigned number. [CAHEN.]

[For then we can use Abel's theorem (Arts. 81, 83) in virtue of the last example.]

Power-series.

31. If R, R' are the radii of convergence of $\Sigma a_n x^n$ and $\Sigma b_n x^n$ respectively, then:

(1) The radius of convergence of $\Sigma a_n b_n x^n$ is not less than RR'.

(2) If R is less than R', R is the radius of convergence of $\Sigma(a_n + b_n) x^n$; but if $R = R'$ the radius is at least equal to R and may be greater.

[Apply the method of Art. 84.]

32. If a power-series is zero at all points of a set which has the origin as a limiting point, then the series is identically zero. [Compare Art. 52.]

33. A power-series cannot be purely real (or purely imaginary) at all points within a circle whose centre is the origin.

[Use the last example.]

34. If $\Sigma a_n x^n$ converges within the circle $|x| = R (> 0)$, shew that $\Sigma a_n \dfrac{x^n}{n!}$ converges for all values of x; and examine the relation between the regions of convergence of $\Sigma a_n x^n / n^s$ and $\Sigma a_n x^n$.

35. If $f(x) = \Sigma a_n x^n$ converges for $|x| < R$, then (see Art. 84)

$$\mathfrak{M}\,|f(x)|^2 = \Sigma\,|a_n|^2 r^{2n}, \quad \text{where } |x| = r < R.$$

Deduce Cauchy's inequalities. [GUTZMER.]

[For we have $$\mathfrak{M}\, x^{-n} f(x) = a_n,$$
and if a_n' is the conjugate to a_n,
$$\Sigma a_n' r^{2n}/x^n = f_1(r^2/x) \text{ is the conjugate to } f(x).$$
Thus $$|f(x)|^2 = f(x) f_1(r^2/x)$$
and $$\mathfrak{M}\,|f(x)|^2 = \Sigma a_n' r^{2n}\, \mathfrak{M}\, x^{-n} f(x) \quad \text{(Art. 82)}$$
$$= \Sigma a_n a_n' r^{2n} = \Sigma\,|a_n|^2 r^{2n}.]$$

36. If $f(x) = \Sigma a_n x^n$ converges for $|x| < R$, then

$$|a_1| = |f'(0)| \leqq \tfrac{1}{2} D/r,$$

where D is the maximum of $|f(x) - f(-x)|$ on the circle $|x| = r < R$.
[LANDAU and TOEPLITZ.]

[In fact, $$a_1 = \mathfrak{M}\, x^{-1} f(x), \quad -a_1 = \mathfrak{M}\, x^{-1} f(-x),$$
so that $$2a_1 = \mathfrak{M}\,[x^{-1}\{f(x) - f(-x)\}],$$
which gives the desired result.]

37. Shew that if ρ is the radius of convergence of $\Sigma a_n x^n$, the series $\Sigma a_n x^{n^2}$ will converge absolutely, provided that the argument of x is greater than $\log(1/\rho)$.

38. Examine the convergence of the power-series

$$\Sigma \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) x^{2n}, \quad \text{when } |x| = 1.$$

[Apply Weierstrass's rule, Art. 79.]

39. Discuss the convergence of the power-series

$$\Sigma \frac{x^n}{(n+a)^s}, \quad \Sigma \frac{x^n}{n \log n}, \quad \Sigma \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) \frac{x^n}{n+1}, \quad \text{for } |x| = 1.$$

[In the third, the coefficient of x^n steadily decreases; see Ex. 2, Art. 34.]

40. Discuss the convergence of

$$\left(\frac{1}{3}\right)^2 x + \left(\frac{1 \cdot 2}{3 \cdot 5}\right)^2 x^2 + \left(\frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}\right)^2 x^3 + \dots$$

and of

$$\sum_1^\infty \frac{x^n}{(n^n + n)^{\frac{1}{n}}}.$$

Abel's Theorem.

41. Deduce from Art. 86 the extensions of Exs. 2, 3, 4, Art. 51, to the complex variable; and in particular extend Frobenius's theorem so as to apply to any path of approach lying within the limaçon of Art. 86. Prove also the following result:

If $a_0 + a_1 + \dots + a_n \sim \log n$, then, as $x \to 1$,

$$\Sigma a_n x^n \sim \log\left(\frac{1}{1 - |x|}\right).$$

A further extension is quoted in Art. 51, above.

42. Consider the application of Frobenius's theorem to the series

$$1 - xt + x^4 t^2 - x^9 t^3 + \dots,$$

where t is a complex number of absolute value 1, but is not equal to -1, and x is real. It is easily proved that

$$s_0 = 1, \quad s_1 = s_2 = s_3 = \frac{1 - t^2}{1 + t}, \quad s_4 = s_5 = \dots = s_8 = \frac{1 + t^3}{1 + t}, \text{ etc.},$$

and generally $s_n = \{1 - (-t)^\nu\}/(1 + t)$, if $(\nu - 1)^2 \leqq n < \nu^2$.

Hence the arithmetic mean is found to be $1/(1 + t)$, and thus

$$\lim_{x \to 1} (1 - xt + x^4 t^2 - x^9 t^3 + \dots) = 1/(1 + t),$$

or

$$\lim_{x \to 1} (1 - x \cos\theta + x^4 \cos 2\theta - x^9 \cos 3\theta + \dots) = \tfrac{1}{2},$$

$$\lim_{x \to 1} (x \sin\theta - x^4 \sin 2\theta + x^9 \sin 3\theta - \dots) = \tfrac{1}{2} \tan(\tfrac{1}{2}\theta).$$

43. Apply Frobenius's theorem to the series

$$f_1 - (f_1 + f_2)x + (f_1 + f_2 + f_3)x^2 - (f_1 + f_2 + f_3 + f_4)x^3 + \dots,$$

where f_n is positive and decreases steadily to zero, but Σf_n diverges.

The limit is equal to

$$\begin{aligned} &\tfrac{1}{2}(f_1 - f_2 + f_3 - f_4 + \dots) \\ &= \lim_{x \to 1} \{f_1 - (f_1 + f_2)x + (f_1 + f_2 + f_3)x^2 - \dots\} \\ &= \lim_{x \to 1} \{(1 - x + x^2 - \dots)(f_1 - f_2 x + f_3 x^2 - \dots)\}. \end{aligned}$$

[Hardy.]

[We have, in fact,

$$s_1 = f_1, \quad s_2 = -f_2, \quad s_{2n-1} = f_1 + f_3 + \dots + f_{2n-1}, \quad s_{2n} = -(f_2 + f_4 + \dots + f_{2n}),$$

and so the arithmetic mean of $s_1, s_2, \dots, s_{2n}$ is

$$(t_2 + t_4 + \dots + t_{2n})/2n, \text{ where } t_{2n} = f_1 - f_2 + f_3 - \dots - f_{2n}.$$

Apply Stolz's theorem (Art. 147), and we find that the limit of the arithmetic mean is equal to $\frac{1}{2}\lim t_{2n}$. Again, from Stolz's theorem we see that $\lim (s_{2n+1}/n) = \lim f_{2n+1} = 0$, and so the arithmetic mean of $s_1, s_2, \dots, s_{2n+1}$ tends to the same limit as that of $s_1, s_2, \dots, s_{2n}$.]

44. Illustrations of the last example are given by taking

$$\lim_{x \to 1} \{1 - (1 + \tfrac{1}{2})x + (1 + \tfrac{1}{2} + \tfrac{1}{3})x^2 \dots\} = \tfrac{1}{2}\log 2,$$

$$\lim_{x \to 1} \{x \log 2 - x^2 \log 3 + x^3 \log 4 - \dots\} = \tfrac{1}{2}\log(\tfrac{1}{2}\pi).$$

[In the second we use Wallis's product (Art. 70): it is instructive to notice also that, to the base 10, $\frac{1}{2}\log(\frac{1}{2}\pi) = \cdot 098060$ to 6 decimal places, which verifies Euler's calculation given for series (5), Art. 104.]

45. It follows from Art. 11 that

$$\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - \log n$$

steadily decreases, and that its limit is Euler's constant C.

Thus the series $\quad \Sigma(-1)^{n-1}\left[\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - C - \log n\right]$

is convergent and, from the last example, its sum is seen to be

$$\tfrac{1}{2}\log 2 - \tfrac{1}{2}C + \tfrac{1}{2}\log(\tfrac{1}{2}\pi) = \tfrac{1}{2}(\log \pi - C) = \cdot 28376. \qquad \text{[HARDY.]}$$

[The value of the sum can be deduced from Wallis's product by observing that the sum to $2n$ terms is

$$\left(\log\frac{2}{1} - \frac{1}{2}\right) + \left(\log\frac{4}{3} - \frac{1}{4}\right) + \dots + \left(\log\frac{2n}{2n-1} - \frac{1}{2n}\right).]$$

CHAPTER XI.

SPECIAL COMPLEX SERIES AND FUNCTIONS.

92. The exponential power-series.

There is no difficulty in modifying the proof of Art. 57 to shew that

$$\lim_{\nu \to \infty} (1+\xi)^\nu = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = E(x),$$

where $$\lim (\nu\xi) = x,$$

and ν is real, although ξ and x are complex.

By multiplication of series, or by an argument similar to that of Art. 58, we deduce that

$$E(x) \times E(y) = E(x+y),$$

which is the fundamental equation of the exponential power-series.

As a kind of converse theorem, we shall now obtain the most general power-series,

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots,$$

which converges within some circle $|x| = R$, say, and satisfies the equation

$$f(x+y) = f(x) f(y),$$

provided that $|x|$, $|y|$, $|x+y|$ are all less than R (which certainly holds good if $|x|$ and $|y|$ are less than $\frac{1}{2}R$). Since this condition requires the equation to hold for *real* values of x, y in the interval $(-\frac{1}{2}R, +\frac{1}{2}R)$, we shall consider these values first.*

In the first place put $y=0$; then

$$f(x) \times f(0) = f(x) \quad \text{or} \quad f(0) = 1.$$

---

* We restrict x, y to be *real* so as to avoid the difficulty of differentiating with respect to a *complex* independent variable. The fact that the *coefficients* in $f(x)$ may be complex does not affect the application of Art. 52 (3), because we can differentiate the real and imaginary parts separately

Hence $a_0=1$, and so

$$f(x)=1+a_1x+a_2\frac{x^2}{2!}+\dots.$$

Again,

$$\frac{f(x+y)-f(x)}{y}=f(x)\frac{f(y)-1}{y}=f(x)\left(a_1+\frac{a_2y}{2!}+\frac{a_3y^2}{3!}+\dots\right),$$

and if we take the limit of both sides as y tends to zero, we get at once

$$f'(x)=a_1f(x).$$

Or, applying Art. 52 (3), we have

$$a_1+a_2x+a_3\frac{x^2}{2!}+\dots=a_1\left(1+a_1x+a_2\frac{x^2}{2!}+\dots\right),$$

and since this equation must hold for *all* values of x in the interval $(-\tfrac{1}{2}R,\ +\tfrac{1}{2}R)$, we must have

$$a_2=a_1^2,\quad a_3=a_1a_2,\quad a_4=a_1a_3,\ \dots\,.\qquad \text{See Art. 52 (5).}$$

That is, $\quad a_2=a_1^2,\quad a_3=a_1^3,\quad a_4=a_1^4,\ \dots,\ a_n=a_1^n,\ \dots,$

and so $\quad f(x)=1+a_1x+a_1^2\dfrac{x^2}{2!}+a_1^3\dfrac{x^3}{3!}+\dots=E(a_1x).$

We do not know from this argument that $f(x)$ satisfies *all* the conditions of the problem; but we see that *if* there is such a power-series, it can be no other than $E(a_1x)$. Now $E(a_1x)$ does satisfy the relation

$$E(a_1x)\times E(a_1y)=E\{a_1(x+y)\}$$

for any real or complex values of x, y.

Consequently our problem has been solved;* and

$$f(x)=E(a_1x),$$

where a_1 is the coefficient of x in the power-series for $f(x)$.

It is usual, and in many respects convenient, to write e^x for $E(x)$ even when x is complex. But it must be remembered that this is merely a convention; and that in an equation such as $e^{\frac{1}{2}\iota\pi}=\iota$ (see below, Art. 93) the index does not denote an ordinary power.

93. Connexion between the exponential and circular functions.

If the complex variable x in the exponential series $E(x)$ depends

* It does not follow from the foregoing that no other *function* can satisfy the relation $f(x)\times f(y)=f(x+y)$, because we have assumed $f(x)$ to be a power-series. But, if we assume that $f'(x)$ is continuous, there is no difficulty in shewing that $f(x)$ has the exponential form.

on a *real* variable t, we may differentiate term-by-term with respect to t, and obtain the same formula as if x were real:

$$\frac{d}{dt}\{E(x)\}=E(x)\frac{dx}{dt}.$$

For, suppose that corresponding to a change δt in t, x changes to $x+\delta x$; then by Art. 92

$$\frac{1}{\delta t}\{E(x+\delta x)-E(x)\}=E(x)\{E(\delta x)-1\}/\delta t.$$

And $$E(\delta x)-1=\delta x\left\{1+\frac{1}{2!}(\delta x)+\frac{1}{3!}(\delta x)^2+\dots\right\},$$

so that $$\left|\frac{1}{\delta t}\{E(x+\delta x)-E(x)\}-E(x)\frac{\delta x}{\delta t}\right|<\frac{1}{2}\left|\frac{\delta x}{\delta t}\right|\frac{|\delta x|}{1-\frac{1}{3}|\delta x|}.$$

Now $\delta x/\delta t$ approaches the limit dx/dt as $\delta t\to 0$, so that $|\delta x|\to 0$; and it follows from the last inequality that

$$\lim\frac{1}{\delta t}\{E(x+\delta x)-E(x)\}=E(x)\frac{dx}{dt}.$$

In particular, suppose that x is a pure imaginary and equal to $\iota\eta$, where η is real; then we have

$$\frac{d}{d\eta}\{E(\iota\eta)\}=\iota E(\iota\eta),$$

or, if $$E(\iota\eta)=r(\cos\theta+\iota\sin\theta),$$
we find that

$$\left(\frac{dr}{d\eta}+\iota r\frac{d\theta}{d\eta}\right)(\cos\theta+\iota\sin\theta)=\iota r(\cos\theta+\iota\sin\theta).$$

Hence $$\frac{dr}{d\eta}=0,\quad \frac{d\theta}{d\eta}=1.$$

Thus r and $\theta-\eta$ are independent of η; but for $\eta=0$, $E(\iota\eta)=1$, and so
$$r=1,\ \theta=0,\quad \text{if } \eta=0.$$

Hence, in general, $$r=1\quad\text{and}\quad\theta=\eta,$$
and so $$E(\iota\eta)=\cos\eta+\iota\sin\eta;$$
which is confirmed by the remark that

$$|E(\iota\eta)|^2=E(\iota\eta)\times E(-\iota\eta)=E(\iota\eta-\iota\eta)=E(0)=1.$$

Another method of establishing the last result is given by observing that

$$\cos\eta+\iota\sin\eta=(\cos\phi+\iota\sin\phi)^n,\qquad\text{if } \phi=\eta/n.$$

Now write $$\cos\phi+\iota\sin\phi=1+\kappa_n,$$
and we see that $$\lim_{n\to\infty}n\kappa_n=\iota\eta,$$

because $$\lim (n \sin \phi) = \eta \lim (\sin \phi / \phi) = \eta,$$

and $$\lim n(1 - \cos \phi) = \eta \lim \{(1 - \cos \phi)/\phi\} = 0.$$

Hence $$\cos \eta + \iota \sin \eta = \lim_{n \to \infty} (1 + \kappa_n)^n = E(\iota\eta)$$

by using the limit of Art. 92.

Still another method, analogous to that of Art. 59, can be used. In fact, let us write

$$\cos \eta + \iota \sin \eta - \left\{1 + \iota\eta + \frac{1}{2!}(\iota\eta)^2 + \dots + \frac{1}{n!}(\iota\eta)^n\right\} = y_n,$$

with $y_0 = \cos \eta + \iota \sin \eta - 1$.

Then $$\frac{dy_n}{d\eta} = \iota y_{n-1} \quad \text{and} \quad \frac{dy_0}{d\eta} = \iota(\cos \eta + \iota \sin \eta).$$

But, if $$y = r(\cos \theta + \iota \sin \theta) \qquad \text{for all suffixes,}$$

we have $$\frac{dy}{d\eta} = \left(\frac{dr}{d\eta} + \iota r \frac{d\theta}{d\eta}\right)(\cos \theta + \iota \sin \theta)$$

or $$\left|\frac{dy}{d\eta}\right| = \left\{\left(\frac{dr}{d\eta}\right)^2 + r^2\left(\frac{d\theta}{d\eta}\right)^2\right\}^{\frac{1}{2}} \geqq \left|\frac{dr}{d\eta}\right|.$$

Hence $$\left|\frac{dr_n}{d\eta}\right| \leqq \left|\frac{dy_n}{d\eta}\right| = |y_{n-1}| = r_{n-1}, \quad \left|\frac{dr_0}{d\eta}\right| \leqq 1,$$

and $y_0, y_1, \dots$ are all zero for $\eta = 0$. Hence we find, if η is positive, the sequence of equations

$$r_0 = |y_0| \leqq \eta, \quad r_1 = |y_1| \leqq \frac{1}{2!}\eta^2, \quad r_2 = |y_2| \leqq \frac{1}{3!}\eta^3, \dots, \quad r_{n-1} = |y_{n-1}| \leqq \frac{1}{n!}\eta^n.$$

Thus $$\lim_{n \to \infty} y_n = 0.$$

If we substitute $\iota\eta$ in the exponential series, we find

$$1 + \iota\eta - \frac{\eta^2}{2!} - \iota\frac{\eta^3}{3!} + \frac{\eta^4}{4!} + \iota\frac{\eta^5}{5!} - \dots$$

$$= \left(1 - \frac{\eta^2}{2!} + \frac{\eta^4}{4!} - \dots\right) + \iota\left(\eta - \frac{\eta^3}{3!} + \frac{\eta^5}{5!} - \dots\right),$$

and so we have now a new method of finding the sine and cosine power-series (Art. 59).

If we write $\eta = \frac{1}{2}\pi$ and π, we get the equations*

$$E(\tfrac{1}{2}\pi\iota) = \iota, \quad E(\pi\iota) = -1.$$

Using the notation explained in Art. 92, we may write

$$\cos \eta + \iota \sin \eta = e^{\iota\eta},$$

* For a discussion of the existence of π, *defined by means of these equations*, the reader should refer back to Art. 60 (2).

and changing the sign of η, we find

$$\cos\eta - \iota\sin\eta = e^{-\iota\eta};$$

thus $$\cos\eta = \frac{1}{2}(e^{\iota\eta}+e^{-\iota\eta}), \quad \sin\eta = \frac{1}{2\iota}(e^{\iota\eta}-e^{-\iota\eta}).$$

We have at present no definitions of $\cos x$ and $\sin x$ when x is complex; but it is usual and convenient to define them by the power-series already established when x is real. Then the equations

$$\cos x = \frac{1}{2}(e^{\iota x}+e^{-\iota x}), \quad \sin x = \frac{1}{2\iota}(e^{\iota x}-e^{-\iota x})$$

are true for complex values of x as well as real ones.

It follows also that any trigonometrical formulae which depend only on the addition-theorems remain unaltered for complex variables; thus in particular the formulae of Arts. 66, 67, 69 remain true.

If we write $x=\xi+\iota\eta$, it will be seen that

$$\cos x = \cos\xi\cosh\eta - \iota\sin\xi\sinh\eta,$$

$$\sin x = \sin\xi\cosh\eta + \iota\cos\xi\sinh\eta,$$

where $$\cosh\eta = \tfrac{1}{2}(e^{\eta}+e^{-\eta}), \quad \sinh\eta = \tfrac{1}{2}(e^{\eta}-e^{-\eta}).$$

We shall not elaborate the details of the analysis of the sinh and cosh functions; the results can be found in many text-books (for instance, Chrystal's *Algebra*, ch. XXIX.).

It is to be noticed that *when x is complex, the inequalities*

$$|\sin x| < |x|, \quad |\cos x| < 1$$

are no longer valid. We can, however, replace them by others, thus:

$$|\sin x| \leqq \sinh|x| = |x| + \frac{|x|^3}{3!} + \frac{|x|^5}{5!} + \dots$$

and so, if $|x| < 1$, we have

$$|\sin x| < |x|\left\{1+\frac{1}{6}+\frac{1}{6^2}+\dots\right\} < \frac{6}{5}|x|.$$

Similarly, we have

$$|\cos x| \leqq \cosh|x|;$$

and, if $|x| < 1$, we find

$$|\cos x| < \left(1+\frac{1}{2!}+\frac{1}{4!}+\dots\right) < 2.$$

94. The logarithm and its principal branch.

We have seen (Art. 93) that if η is a real angle

$$E(\iota\eta)=\cos\eta+\iota\sin\eta.$$

Hence if n is any integer (positive or negative),

$$E(2n\pi\iota)=1,$$

and since $$E(\xi+\iota\eta)=e^{\xi}(\cos\eta+\iota\sin\eta)$$

there are no solutions of the equation

$$E(\xi+\iota\eta)=1,$$

other than $$\xi=0,\quad \eta=2n\pi.$$

It follows that if we wish to solve the equation $E(y)=x$, so as to obtain the function inverse to the exponential function, the value obtained is not single-valued, but is of the form

$$y=y_0+2n\pi\iota,\quad (n=0,\ \pm1,\ \pm2,\ \ldots),$$

where y_0 is any solution of the equation.

FIG. 28.

If we represent x geometrically in Argand's diagram, we have

$$x=r(\cos\theta+\iota\sin\theta)=rE(\iota\theta).$$

But if $\log r$ is the logarithm of the real number r, defined as in Art. 154 of Appendix II., we have

$$r=E(\log r),$$

and consequently $$x=E(\log r+\iota\theta).$$

Thus we can take $y_0=\log r+\iota\theta$, and then the general solution is

$$y=\log x=\log r+\iota(\theta+2n\pi),\quad (n=0,\ \pm1,\ \pm2,\ \ldots).$$

We define the logarithmic function as consisting of all the inverses of the exponential function; and we can specify a one-valued branch of the logarithm by supposing a cut made along the negative part of the real axis, and regarding x as prevented from crossing the cut. Then we shall have

$$\log x=\log r+\iota\theta,\ \text{where}\ -\pi<\theta\leqq\pi.$$

With this determination, $\log x$ is real when x is real, which is generally the most convenient assumption. But it should be observed that such formulae as

$$\log(xx')=\log x+\log x'$$

can only be employed with caution, since it may easily happen that $(\theta+\theta')$ is greater than π, in which case we ought to write

$$\log(xx')=\log x+\log x'-2\pi\iota.$$

The reader will note that for two points such as P, Q in the diagram (Q being the reflexion of P in the negative half of the real axis),

$$\lim_{P\to Q}(\log x_P-\log x_Q)=2\pi\iota.$$

But, except at the cut, the branch selected for $\log x$ is obviously continuous over the whole plane of x; and this will be called the *principal branch* or *principal value* of the logarithmic function.

95. The logarithmic power-series.

We know from Arts. 58 and 62, that if x is real and $|x|<1$, the series

(1) $$y=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\dots$$

represents the function inverse to the exponential function

(2) $$1+x=E(y)=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\dots.$$

In other words, if we substitute the series (1) in the series (2), and then arrange according to powers of x, the result* must be $1+x$. But this transformation is merely algebraical, and, as such, is equally true whether x is real or complex.

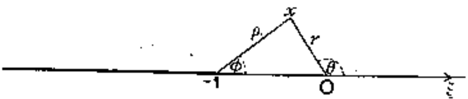

FIG. 29.

Since the series (2) converges absolutely for all values of y, the derangement implied in this transformation is legitimate (see Art. 36), provided that the series (1) is absolutely convergent. Hence, if $|x|<1$, equation (1) gives **one** value of y satisfying equation (2); and further, from (1), y is real when x is real. Thus, using the principal branch of the logarithm defined in the last article, we have

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\dots \quad (\text{if } |x|<1).$$

* It is a good exercise to verify this conclusion up to, say, x^5.

From the figure, it is evident that this equation gives

$$\log \rho + \iota\phi = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots,$$

where

$$-\pi < \phi \leqq +\pi \quad \text{(see Art. 94).}$$

This result can be confirmed by reference to Art. 65, where we proved that (if $0 < r < 1$)

$$\tfrac{1}{2}\log(1+2r\cos\theta+r^2) = r\cos\theta - \tfrac{1}{2}r^2\cos 2\theta + \tfrac{1}{3}r^3\cos 3\theta - \dots,$$

$$\text{arc tan}\frac{r\sin\theta}{1+r\cos\theta} = r\sin\theta - \tfrac{1}{2}r^2\sin 2\theta + \tfrac{1}{3}r^3\sin 3\theta - \dots.$$

If we write $x = r(\cos\theta + \iota\sin\theta)$ in the power-series (1), we get

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots = r\cos\theta - \tfrac{1}{2}r^2\cos 2\theta + \tfrac{1}{3}r^3\cos 3\theta - \dots$$
$$+\iota(r\sin\theta - \tfrac{1}{2}r^2\sin 2\theta + \tfrac{1}{3}r^3\sin 3\theta - \dots),$$

and obviously

$$\rho^2 = 1 + 2r\cos\theta + r^2,$$
$$\tan\phi = r\sin\theta/(1+r\cos\theta).$$

Thus our results are in agreement with those of Art. 65, except that we have proved that ϕ actually lies between $-\tfrac{1}{2}\pi$ and $+\tfrac{1}{2}\pi$ (because $r < 1$) instead of $-\pi$ and π.

It is easy to see (as in Art. 85) that the logarithmic series still converges for $|x| = 1$, except at the special point $x = -1$. Thus the sum of the series at any other point of the circle of convergence is found by taking the limit of the sum as $r \to 1$ (by Abel's theorem); the result obtained may be written

$$e^{\iota\theta} - \tfrac{1}{2}e^{2\iota\theta} + \tfrac{1}{3}e^{3\iota\theta} - \dots = \log(2\cos\tfrac{1}{2}\theta) + \tfrac{1}{2}\iota\theta,$$

where

$$-\pi < \theta < +\pi.$$

This again agrees with results obtained in Art. 65.

We shall obtain an independent proof of the equation

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots \quad (\text{if } |x| < 1)$$

in the course of Art. 96.

The series for arc sin x **and** arc tan x.

Again, by Art. 64, the series

(3) $$y = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5} + \dots$$

represents the function inverse to the sine-function (Art. 59).

(4) $$x = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots$$

for real values of x, y, such that $|x| \leqq 1$. Since the series (4) is absolutely convergent for all values of y, and the series (3) for

$|x| \leqq 1$, the algebraic relation between these series must persist for complex values of x, and we can accordingly write

$$\text{arc sin } x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \dots \quad (\text{if } |x| < 1),$$

since the series (4) is taken as the definition of the sine for complex values of the variable (Art. 93).

Similarly the pair of functions

(5) $$y = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \dots \quad (\text{Art. } 64),$$

(6) $$x = \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right) \Big/ \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right)$$

are inverse to one another for real values of x, such that $|x| < 1$, and we may therefore write for complex values of x

$$\text{arc tan } x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \dots \quad (\text{if } |x| < 1).$$

In these equations the values of the inverse functions are determined uniquely by the condition that the real part of each function lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; just as in Arts. 59, 64 for real variables.

To discuss the accuracy of the last statement let us consider first the equation
$$\sin(X + \iota Y) = x = \xi + \iota\eta, \text{ so that } X + \iota Y = \text{arc sin } x.$$

Then we find (as in Art. 93)

$$\xi = \sin X \cosh Y, \quad \eta = \cos X \sinh Y,$$

and so Y is given by $\dfrac{\xi^2}{\cosh^2 Y} + \dfrac{\eta^2}{\sinh^2 Y} = 1$, except as to the *sign* of Y.

But if we agree that $-\frac{1}{2}\pi < X < \frac{1}{2}\pi$, $\cos X$ is positive; and so the sign of Y is fixed by the sign of η. Having found Y, the values of $\sin X$ and $\cos X$ are known, and consequently X is fixed uniquely by the condition $-\frac{1}{2}\pi < X < +\frac{1}{2}\pi$; and so $\text{arc sin } x$ is determinate.

Similarly we find that $\text{arc cos } x$ can be uniquely determined by the condition that the real part lies between 0 and π.

Secondly, suppose that

$$\tan(X + \iota Y) = x = \xi + \iota\eta, \quad \text{so that } X + \iota Y = \text{arc tan } x.$$

Then we see that (Art. 93)

$$e^{2\iota(X+\iota Y)} = \frac{1 + \iota \tan(X + \iota Y)}{1 - \iota \tan(X + \iota Y)} = \frac{1 - \eta + \iota\xi}{1 + \eta - \iota\xi},$$

or
$$e^{-4Y} = \frac{(1-\eta)^2 + \xi^2}{(1+\eta)^2 + \xi^2},$$

which fixes Y uniquely.

Further, $$\frac{\cos 2X}{1 - \xi^2 - \eta^2} = \frac{\sin 2X}{2\xi} = \frac{1}{\sqrt{\{(1-\eta)^2 + \xi^2\}\{(1+\eta)^2 + \xi^2\}}},$$

so that $\cos 2X$ and $\sin 2X$ are known, and now X is uniquely determined by the condition $-\frac{1}{2}\pi < X < \frac{1}{2}\pi$; and so finally $\text{arc tan } x$ is determinate.

By suitable modifications of the discussion given for the arc sin function, it is easy to determine uniquely the functions inverse to the sinh and cosh functions. Of these, the function most frequently employed in Applied Mathematics is $\cosh^{-1}x$, the inverse of the cosh function; but then the definition which is generally found useful is slightly different from the above.

According to the above method, the coefficient of ι in $\cosh^{-1}x$ would be taken to lie between 0 and π, and the real part of $\cosh^{-1}x$ would then have either sign—the same sign in fact as η.

But for certain purposes it is more convenient to restrict the real part of $\cosh^{-1}x$ to be *positive*.

Thus if we write $\cosh(X+\iota Y)=x=\xi+\iota\eta,$

we have $\cosh X\cos Y=\xi, \quad \sinh X\sin Y=\eta;$

and then X is given by $\xi^2/\cosh^2 X+\eta^2/\sinh^2 X=1.$

And if X is assumed *positive*, the values of $\cos Y$, $\sin Y$ are fixed; but $\sin Y$ will have the same sign as η, which may be positive or negative. Thus Y may have any value from 0 to 2π; and the function $\cosh^{-1}x$ is then uniquely determinate.

96. The binomial power-series.

Consider the series

$$f(\nu, x)=1+\nu x+\nu(\nu-1)\frac{x^2}{2!}+\nu(\nu-1)(\nu-2)\frac{x^3}{3!}+\dots,$$

where both ν and x may be complex.

The conditions for convergence of the series readily follow from Weierstrass's rule (Arts. 79, 85); let a_n denote the coefficient of x^n. Then we have

$$\frac{a_n}{a_{n+1}}=\frac{n+1}{\nu-n}=-\left\{1+\frac{\nu+1}{n}+O\left(\frac{1}{n^2}\right)\right\}.$$

Thus *the series is always absolutely convergent for* $|x|<1$; *and* $|x|=1$ *gives the circle of convergence.*

To proceed further, write $\nu=\alpha+\iota\beta$; then Art. 79 shews that *the series is absolutely convergent on the circle* $|x|=1$, *if* α *is positive*; *and thus the series is uniformly convergent within and on the circle* $|x|=1$, *provided that* α *is positive.*

Next, *when* $-1<\alpha\leqq 0$, *the series converges (but not absolutely) on the circle* $|x|=1$, *except at* $x=-1$; *and it is uniformly convergent on any arc of that circle from which the point* $x=-1$ *is excluded* (Art. 85).

Finally, *when* $\alpha \leqq -1$, *the series does not converge at any point on the circle* $|x|=1$ (Art. 79).

In every case, the point $x=-1$ is a singular point for the power-series (by Fabry's theorem, Art. 88).*

To investigate the properties of the function $f(\nu, x)$, we form first the product $f(\nu, x) \times f(\nu', x)$ (where $|x| < 1$),

and by the ordinary rule of multiplication (Art. 54) this product can be arranged as a power-series in x; and the coefficient of x^n in the product is easily seen to be a polynomial in ν and ν', of degree n. Now the same is true of the function $f(\nu+\nu', x)$; and so we can write

$$f(\nu, x) \times f(\nu', x) - f(\nu+\nu', x) = \Sigma P_n x^n,$$

where P_n is again a polynomial of degree n in ν and ν'.

But, when ν, ν' are *any* two integers, P_n is zero, because then $f(\nu, x)=(1+x)^\nu$ and $f(\nu', x)=(1+x)^{\nu'}$. Consequently P_n must be identically zero, because, when ν' is any assigned integer, P_n vanishes for an infinity of different values of ν (namely, 1, 2, 3, ... to ∞).

Thus, we have identically,

$$f(\nu, x) \times f(\nu', x) = f(\nu+\nu', x) \quad (|x| < 1).$$

Starting from this relation we can apply the method indicated in Art. 61 (2) to prove that, when ν is a rational number,

$$f(\nu, x) = (1+x)^\nu,$$

the value of the power being uniquely determined by the fact that $f(\nu, x)$ is real when x is real.

But to deal with complex values of ν, we proceed somewhat differently. In the first place $f(\nu, x)$ can be expressed as a power-series in ν; for $f(\nu, x)$ can be regarded as the sum by columns of the double series

$$\begin{aligned} 1+\nu x-\frac{\nu}{2}x^2+\frac{\nu}{3}x^3-\frac{\nu}{4}x^4&+\ldots \\ +\frac{\nu^2}{2}x^2-\frac{\nu^2}{2}x^3+\frac{11\nu^2}{24}x^4&-\ldots \\ +\frac{\nu^3}{6}x^3-\frac{\nu^3}{4}x^4&+\ldots \\ +\frac{\nu^4}{24}x^4&-\ldots \\ &+\ldots . \end{aligned}$$

* When ν is *real*, this result follows also from Vivanti's theorem.

Now this double series is absolutely convergent because, if $|\nu| = \nu_0$ and $|x| = x_0$, the sum of the absolute values of the terms in the $(p+1)$th column is

$$\frac{\nu_0(\nu_0+1)\ldots(\nu_0+p-1)}{1 \, . \, 2 \ldots p} x_0^p,$$

which is the $(p+1)$th term of the series $f(-\nu_0, -x_0)$, which converges if $x_0 < 1$ for all values of ν_0.

The double series being absolutely convergent its sum is not altered (see Arts. 33 and 82) by changing the mode of summation to rows; this operation gives

$$f(\nu, x) = 1 + \nu X_1 + \nu^2 X_2 + \ldots ,$$

where $$X_1 = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots .$$

Now, since $f(\nu, x) \times f(\nu', x) = f(\nu+\nu', x)$, we can apply Art. 92, above, and we see that *

$$f(\nu, x) = E(\nu X_1).$$

In order to determine X_1, let us write $\nu = 1$, which gives

$$1 + x = E(X_1).$$

Thus X_1 is a value of $\log(1+x)$; and since X_1 is real when x is real, it is the principal value defined in Art. 94. We have thus a new investigation of the logarithmic series (see Art. 95).

Thus we find the equation, due to Abel,

$$f(\nu, x) = E\{\nu \log(1+x)\},$$

and for uniformity we may write conveniently

$$f(\nu, x) = (1+x)^\nu$$

on the understanding that the complex power is defined by the equation $$(1+x)^\nu = E\{\nu \log(1+x)\},$$

where the logarithm has its principal value.

In order to obtain an explicit formula for $f(\nu, x)$, we note that (as in Art. 95) $$\log(1+x) = \log \rho + \iota\phi,$$

where ρ, ϕ have the geometrical significance indicated in Fig. 29. Thus we find that

$$\begin{aligned} f(\nu, x) &= E\{\nu \log(1+x)\} \\ &= \rho^\alpha e^{-\beta\phi}\{\cos(\alpha\phi + \beta\log\rho) + \iota \sin(\alpha\phi + \beta\log\rho)\}, \end{aligned}$$

where $$\nu = \alpha + \iota\beta,$$

* Of course ν' corresponds here to x of that article; and X_1 corresponds to a_1.

a result which is also due to Abel. The above investigation is based upon the proof given by Goursat.*

The method given in the example of Art. 36 applies to complex indices. The following method is based upon a suggestion made in 1903 by Prof. A. C. Dixon:†

The relation $$f(\nu, x) \times f(\nu', x) = f(\nu + \nu', x)$$

gives at once $$f(\nu, x) = \left\{ f\left(\frac{\nu}{n}, x\right) \right\}^n = (1 + \xi)^n, \text{ say,}$$

where n is a positive integer.

Now $$n\xi = \nu\left\{x - \left(1 - \frac{\nu}{n}\right)\frac{x^2}{2} + \left(1 - \frac{\nu}{n}\right)\left(1 - \frac{\nu}{2n}\right)\frac{x^3}{3} - \left(1 - \frac{\nu}{n}\right)\left(1 - \frac{\nu}{2n}\right)\left(1 - \frac{\nu}{3n}\right)\frac{x^4}{4} + \dots\right\}.$$

But the series in { } brackets has each of its terms less, in absolute value, than the corresponding term of

$$x_0 + (1 + \nu_0)\frac{x_0^2}{2} + (1 + \nu_0)\left(1 + \frac{\nu_0}{2}\right)\frac{x_0^3}{3} + \dots,$$

which is a convergent series, independent of n; and consequently $n\xi$ converges uniformly, by the M-test; and so the limit of $n\xi$ as n tends to ∞ can be found by taking the limit of each term (Art. 49).

Hence $$\lim_{n \to \infty} (n\xi) = \nu(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots)$$
$$= \nu \log(1 + x).$$

But (Art. 92) $$\lim_{n \to \infty} (1 + \xi)^n = E(x'), \quad \text{if } x' = \lim_{n \to \infty} (n\xi).$$

Thus, $$f(\nu, x) = \lim (1 + \xi)^n = E\{\nu \log(1 + x)\}.$$

The discussion given above applies only to points within the circle $|x| = 1$.

We have seen that when $\alpha > 0$ the region of uniform convergence includes the circle $|x| = 1$; and accordingly the value of $f(\nu, x)$ on the circle is continuous with the value at internal points. Now $(1+x)^\nu$ as above defined is also continuous, and accordingly we can still write
$$f(\nu, x) = (1+x)^\nu, \quad (\text{for } |x| = 1, \text{ if } \alpha > 0).$$

Now at points on the circle we have (see Fig. 29)

$$\rho = 2\cos\tfrac{1}{2}\theta, \quad \phi = \tfrac{1}{2}\theta, \quad (-\pi < \theta < \pi),$$

* *Cours d'Analyse Mathématique*, § 275.

† Prof. Dixon has published another arrangement of this proof in the *Quarterly Journal of Mathematics*, vol. 39, 1907, p. 94.

so that the explicit formula for the last result is *

$$f(\nu, x)=(2\cos\tfrac{1}{2}\theta)^{\alpha} e^{-\frac{1}{2}\beta\theta}[\cos\{\tfrac{1}{2}\alpha\theta+\beta\log(2\cos\tfrac{1}{2}\theta)\} + \iota\sin\{\tfrac{1}{2}\alpha\theta+\beta\log(2\cos\tfrac{1}{2}\theta)\}],$$

where $\nu=\alpha+\iota\beta, \quad x=\cos\theta+\iota\sin\theta, \quad (\alpha>0).$

For the special point $x=-1$, we have the result †

$$f(\nu, -1)=0 \quad (\text{if } \alpha>0),$$

and for $x=+1$, $\theta=0$,

$$f(\nu, +1)=2^{\nu}=2^{\alpha}\{\cos(\beta\log 2)+\iota\sin(\beta\log 2)\}.$$

When $-1<\alpha\leqq 0$, the series converges uniformly on any arc of the circle from which $x=-1$ is excluded. Thus these formulae still remain valid (except at $x=-1$) when $\alpha>-1$; but it is to be remembered that the series is not absolutely convergent when α is negative.

When $\alpha\leqq -1$, the series $f(\nu, x)$ is not convergent at points on the circle $|x|=1$, and accordingly the equation

$$f(\nu, x)=(1+x)^{\nu}$$

becomes meaningless for $\alpha\leqq -1$.

This completes the analysis of the binomial series for complex values of x and ν.

The case $-1<\alpha<0$ has been discussed by Goursat as follows (*l.c. supra*).

We have seen that when $|x|<1$,

$$(1+x)f(\nu, x)=f(\nu+1, x).$$

Now this is an algebraic identity, and so, taking only the terms up to x^n, we find that

$$(1+x)S_n=S_n'+a_n x^{n+1},$$

where S_n, S_n' are the sums up to x^n, for the series $f(\nu, x)$ and $f(\nu+1, x)$ respectively.

The last result is clearly independent of the value of x; and so we may suppose $|x|=1$. Further, $|a_n|\to 0$ as $n\to\infty$ (Art. 78) and $S_n'\to(1+x)^{\nu+1}$, because the real part of $\nu+1$ is positive.

Thus $(1+x)S_n\to(1+x)^{\nu+1}$ as $n\to\infty$, if $|x|=1$. Accordingly

$$S_n\to(1+x)^{\nu}$$

as $n\to\infty$, when $|x|=1$, except for $x=-1$.

* This result is also due to Abel; and it was in this connexion that his theorem of Arts. 50, 51 first presented itself.

† This will be discussed independently in the next article.

97. The remainder in the binomial series.

In the special case $x=-1$, we have a simple formula for the sum to $n+1$ terms of the binomial series $f(\nu, -1)$; namely,

$$S_n=(1-\nu)\left(1-\frac{\nu}{2}\right)\dots\left(1-\frac{\nu}{n}\right).$$

This result has been given in Art. 61, above, for real values of ν; and it can be proved by induction without difficulty.

It follows at once from the identity found in the small type at the end of Art. 96; by changing ν to $\nu-1$, we get

$$(1+x)S_n' = S_n - (1-\nu)\left(1-\frac{\nu}{2}\right)\dots\left(1-\frac{\nu}{n}\right)(-x)^{n+1},$$

where S_n' is derived from the series for $f(\nu-1, x)$.

Now put $x=-1$, and we find

$$S_n=(1-\nu)\left(1-\frac{\nu}{2}\right)\dots\left(1-\frac{\nu}{n}\right).$$

Now applying arguments similar to those of Art. 42 and Art. 61, we see that $\qquad S_n=O(n^{-\nu}),\quad$ as $n\to\infty$.

Thus $\qquad |S_n|=O(n^{-\alpha}),\quad$ if $\nu=\alpha+\iota\beta$,

and accordingly, as $n\to\infty$,

$$S_n\to 0,\quad \text{if } \alpha>0,$$
$$|S_n|\to\infty,\quad \text{if } \alpha<0,$$

but S_n oscillates finitely, if $\alpha=0$.

For other values of x it does not appear possible to obtain any similar formula by simple algebraic methods. We can, however, obtain a simple formula by the aid of the Integral Calculus, as follows:

We have*
$$(1+x)^\nu-1=\nu x\int_0^1(1+x-xt)^{\nu-1}dt.$$

Now integrating by parts, this becomes

$$\nu x\Big[t(1+x-xt)^{\nu-1}\Big]_0^1+\nu(\nu-1)x^2\int_0^1 t(1+x-xt)^{\nu-2}dt$$
$$=\nu x+\nu(\nu-1)x^2\int_0^1 t(1+x-xt)^{\nu-2}dt.$$

Repeating the process, we obtain

$$\nu x+\tfrac{1}{2}\nu(\nu-1)x^2+\tfrac{1}{2}\nu(\nu-1)(\nu-2)x^3\int_0^1 t^2(1+x-xt)^{\nu-3}dt,$$

and so on.

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* It should be noticed that this formula is valid only for the principal value of $(1+x)^\nu$; because it is assumed that $1^\nu=1$, $1^{\nu-1}=1$, etc.

After n integrations we have the desired formula

$$(1+x)^\nu - S_n = (n+1)\, a_{n+1} x^{n+1} \int_0^1 t^n (1+x-xt)^{\nu-n-1}\, dt,$$

which gives the remainder as required.

It may be noticed that for $x=-1$ this formula gives the same result as the more elementary discussion at the beginning of this article.

One advantage of this formula is that with proper precautions *it can still be used when* $|x|>1$. The most convenient form is to note that it is usually possible to assign a simple upper limit, say H, to the absolute value of

$$(1+x-xt)^{\nu-n-1}$$

as t varies from 0 to 1. Then the absolute value of the remainder is less than

$$(n+1)|a_{n+1}x^{n+1}| \int_0^1 Ht^n\, dt = H|a_{n+1}x^{n+1}|.$$

Thus *the error in replacing* $(1+x)^\nu$ *by* S_n *is less in absolute value than* H *times the following term in the series.*

It is easy to obtain similar formulae for the logarithmic series.

Thus we have*

$$\log(1+x) = \int_0^1 \frac{x\, dt}{1+x-xt} = \left[\frac{xt}{1+x-xt}\right]_0^1 - x^2\int_0^1 \frac{t\, dt}{(1+x-xt)^2}$$

$$= x - x^2 \int_0^1 \frac{t\, dt}{(1+x-xt)^2}.$$

Repeating the process of integration by parts we find

$$\log(1+x) = x - \tfrac{1}{2}x^2 + x^3\int_0^1 \frac{t^2\, dt}{(1+x-xt)^3}.$$

After n integrations we have the formula

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots + (-1)^{n-1}\frac{1}{n}x^n$$

$$+(-1)^n x^{n+1}\int_0^1 \frac{t^n\, dt}{(1+x-xt)^{n+1}}.$$

Thus, if we can assign an upper limit H to the absolute value of

$$(1+x-xt)^{-n-1},$$

as t varies from 0 to 1, it follows that:

The error in replacing $\log(1+x)$ *by the first* n *terms in the logarithmic series is less in absolute value than* H *times the following term in the series.*

* Here again the principal value of the logarithm is used, so as to give $\log 1 = 0$.

98. The infinite products for $\sin x$ and $\cos x$.

The identities of Art. 69 remain true for complex values of x, and we deduce, as in Art. 70,

$$\frac{\sin x}{n \sin (x/n)} = \prod_{r=1}^{m} \left\{1 - \frac{\sin^2(x/n)}{\sin^2(r\pi/n)}\right\}, \quad \text{if } n = 2m+1.$$

Now, since n is to tend to ∞, we can always ensure that n is greater than $|x|$, and so Art. 93 gives

$$|\sin (x/n)| < \tfrac{6}{5}|x/n|;$$

and, since $r < \frac{1}{2}n$, $\quad \sin (r\pi/n) > 2r/n \quad$ (see Art. 70).

Hence $$\left|\frac{\sin^2(x/n)}{\sin^2(r\pi/n)}\right| < \frac{9}{25}\frac{|x|^2}{r^2}, \quad \text{if } n > |x|;$$

and consequently we can take

$$M_r = \frac{9}{25}\frac{|x|^2}{r^2}$$

in the theorem of Art. 49. Hence, as in Art. 70, we find

$$\frac{\sin x}{x} = \prod_{r=1}^{\infty}\left(1 - \frac{x^2}{r^2\pi^2}\right) = \prod_{-\infty}^{\infty}{}' \left(1 - \frac{x}{r\pi}\right) e^{\frac{x}{r\pi}}.$$

In the same way we find

$$\cos x = \prod_{r=1}^{\infty}\left\{1 - \frac{4x^2}{(2r-1)^2\pi^2}\right\}.$$

The foregoing method is the obvious extension of that used for real variables. A very elegant process has been given by Darboux.*

Thus $$\frac{\sin x}{x} = \frac{1}{2\iota x}(e^{\iota x} - e^{-\iota x}) = \lim_{n\to\infty} \frac{1}{2\iota x}\left\{\left(1 + \frac{\iota x}{n}\right)^n - \left(1 - \frac{\iota x}{n}\right)^n\right\}$$

in virtue of Art. 92.

Let us write for brevity

$$F_n(x) = \frac{1}{2\iota}\left\{\left(1 + \frac{\iota x}{n}\right)^n - \left(1 - \frac{\iota x}{n}\right)^n\right\};$$

then it is easy to obtain the factors of $F_n(x)$; for if $x = n \tan\theta$, we have

$$\left(1 + \frac{\iota x}{n}\right)^n = \sec^n\theta\, e^{\iota n\theta}, \quad \left(1 - \frac{\iota x}{n}\right)^n = \sec^n\theta\, e^{-\iota n\theta}.$$

Thus $\quad F_n(x) = 0, \quad$ if $e^{2\iota n\theta} = 1$,

or if $n\theta = \pm r\pi$, where r is an integer.

This gives $x = \pm n \tan(r\pi/n)$; and so

$$F_n(x) = Ax \prod_{r=1}^{m}\left\{1 - \frac{x^2}{n^2\tan^2(r\pi/n)}\right\}, \quad \text{if } n = 2m+1.$$

---

* Tannery, *Fonctions d'une Variable.*

Now it is easy to verify that the term of lowest degree in $F_n(x)$ is equal to x, so that $A=1$; and thus

$$F_n(x) = x\prod_{r=1}^{m}\left\{1 - \frac{x^2}{n^2\tan^2(r\pi/n)}\right\}.$$

We can again apply the theorem of Art. 49, noticing that here

$$n\tan(r\pi/n) > r\pi,$$

so that we can take $M_r = |x^2|/(r^2\pi^2)$ simply.

Ex. Prove similarly that

$$\cos x = \lim_{n\to\infty} G_n(x),$$

where
$$G_n(x) = \frac{1}{2}\left\{\left(1+\frac{\iota x}{n}\right)^n + \left(1-\frac{\iota x}{n}\right)^n\right\}$$

$$= \prod_{r=0}^{r=m}\left[1 - \frac{x^2}{n^2\tan^2\{(r+\tfrac{1}{2})\pi/n\}}\right], \quad \text{if } n=2m.$$

Deduce the cosine-product.

It is of some interest to note that Euler appears to have obtained first the products

$$\sinh x = x\prod\left(1+\frac{x^2}{r^2\pi^2}\right), \quad \cosh x = \prod\left\{1+\frac{4x^2}{(2r-1)^2\pi^2}\right\}.$$

Euler's method was to write

$$\sinh x = \lim_{n\to\infty}\frac{1}{2}\left\{\left(1+\frac{x}{n}\right)^n - \left(1-\frac{x}{n}\right)^n\right\},$$

and he then proved that this polynomial can be factorised in the form

$$x\prod_{r=1}^{m}\left\{1+\frac{x^2}{n^2\tan^2(r\pi/n)}\right\}, \quad \text{when } n=2m+1.$$

But Euler's final calculation of the limiting form needs to be supplemented by reasoning similar to that of Tannery's theorem.

99. The series of fractions for $\cot x$, $\tan x$, $\operatorname{cosec} x$.

The investigation given in Art. 71 for real angles, can be extended without difficulty to a complex variable, by making various modifications similar to those of Art. 98.

However, a method similar to that of Darboux for the sine-product leads to an easier discussion, as follows.

We have, in fact,

$$\cos x = \lim_{n\to\infty}\frac{1}{2}\left\{\left(1+\frac{\iota x}{n}\right)^{n-1} + \left(1-\frac{\iota x}{n}\right)^{n-1}\right\} = \lim_{n\to\infty}\frac{dF_n}{dx},$$

$$\sin x = \lim_{n\to\infty}\frac{1}{2\iota}\left\{\left(1+\frac{\iota x}{n}\right)^{n} - \left(1-\frac{\iota x}{n}\right)^{n}\right\} = \lim_{n\to\infty} F_n(x).$$

Thus
$$\cot x = \lim_{n\to}\left\{\frac{dF_n}{dx}\Big/F_n(x)\right\}.$$

Now $$F_n(x)=x\prod_{r=1}^{m}\left\{1-\frac{x^2}{n^2\tan^2(r\pi/n)}\right\},\quad \text{if } n=2m+1,$$
so that
$$\frac{dF_n}{dx}\Big/F_n(x)=\frac{1}{x}+\sum_{r=1}^{m}\frac{2x}{x^2-n^2\tan^2(r\pi/n)}.$$

To this it is easy to apply the theorem of Art. 49 by taking the comparison series
$$M_r=\frac{2|x|}{r^2\pi^2-|x|^2},$$
for we have $$|x^2-n^2\tan^2(r\pi/n)|>r^2\pi^2-|x|^2.$$

Thus, for all values of x, real or complex (except multiples of π), we have
$$\cot x=\frac{1}{x}+\sum_{1}^{\infty}\frac{2x}{x^2-n^2\pi^2}=\frac{1}{x}+\sum_{-\infty}^{\infty}{}'\left(\frac{1}{x-n\pi}+\frac{1}{n\pi}\right),$$
where n is taken as the variable of summation, instead of r.

Now we have the following identities:*
$$\tan x=\cot x-2\cot 2x,\quad \operatorname{cosec} x=\cot\tfrac{1}{2}x-\cot x.$$

Thus we find, on subtraction,
$$\tan x=\sum_{0}^{\infty}\frac{2x}{(n+\frac{1}{2})^2\pi^2-x^2}=-\sum_{-\infty}^{\infty}\left\{\frac{1}{x-(n+\frac{1}{2})\pi}+\frac{1}{(n+\frac{1}{2})\pi}\right\},$$
$$\operatorname{cosec} x=\frac{1}{x}+\sum_{1}^{\infty}(-1)^n\frac{2x}{x^2-n^2\pi^2}=\frac{1}{x}+\sum_{-\infty}^{\infty}{}'(-1)^n\left(\frac{1}{x-n\pi}+\frac{1}{n\pi}\right).$$

Changing from x to ιx, we find that
$$\coth x=\frac{1}{x}+\sum_{1}^{\infty}\frac{2x}{x^2+n^2\pi^2},\quad \tanh x=\sum_{0}^{\infty}\frac{2x}{(n+\frac{1}{2})^2\pi^2+x^2},$$
$$\operatorname{cosech} x=\frac{1}{x}+\sum_{1}^{\infty}(-1)^n\frac{2x}{x^2+n^2\pi^2}.$$

We note that
$$\coth\frac{x}{2}=\frac{e^{\frac{1}{2}x}+e^{-\frac{1}{2}x}}{e^{\frac{1}{2}x}-e^{-\frac{1}{2}x}}=\frac{e^x+1}{e^x-1}=1+\frac{2}{e^x-1},$$
and accordingly we have
$$\frac{1}{e^x-1}=\frac{1}{x}-\frac{1}{2}+\sum_{1}^{\infty}\frac{2x}{x^2+4n^2\pi^2}.$$

* The identities are familiar results when x is real; for other values, they follow from the formulae obtained in Art. 93.

100. The power-series for $x/(e^x-1)$.

The exponential series gives at once

$$(e^x-1)/x=1+\frac{x}{2!}+\frac{x^2}{3!}+\dots,$$

and consequently (as in Art. 54) the reciprocal function $x/(e^x-1)$ can be expanded in powers of x, provided that $|x|<\rho$, where

$$\frac{\rho}{2!}+\frac{\rho^2}{3!}+\dots\leq 1.$$

This last condition is certainly satisfied if

$$\frac{\rho}{2}\Big/\left(1-\frac{\rho}{3}\right)=1,$$

or if $$\rho=\tfrac{6}{5}=1\cdot 2.$$

Thus we can certainly write

$$\frac{x}{e^x-1}=1-\frac{x}{2}+A_2x^2+A_3x^3+A_4x^4+\dots,\quad \text{if } |x|<1\cdot 2.$$

From the last formula of Art. 99, it is clear that the function

$$\frac{x}{e^x-1}+\tfrac{1}{2}x$$

is an even function of x, so that

$$A_3=0,\ A_5=0,\ A_7=0,\ \dots.$$

Consequently we can write

$$\frac{x}{e^x-1}=1-\frac{x}{2}+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\dots,$$

where B_1, B_2, B_3, ... are called *Bernoulli's numbers.*

It is easy to verify by direct division that

$$B_1=\tfrac{1}{6},\ B_2=\tfrac{1}{30},\ B_3=\tfrac{1}{42},\ B_4=\tfrac{1}{30},\ B_5=\tfrac{5}{66},$$

but the higher numbers become very complicated.*

Again, from Art. 99 we see that

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_1^\infty\frac{2x^2}{x^2+4n^2\pi^2}.$$

* The numbers (as decimals) and their logarithms have been tabulated by Glaisher (*Trans. Camb. Phil. Soc.*, vol. 12, p. 384); and B_1 to B_{62} are given by Adams (*Scientific Papers,* vol. 1, pp. 453 and 455). (For more details, see Chrystal's *Algebra,* Ch. XXVIII. § 6.)

Now if $|x| < 2\pi$, each fraction can be expanded in powers of x, giving

$$\frac{2x^2}{x^2+4n^2\pi^2} = \frac{x^2}{2n^2\pi^2}\left(1-\frac{x^2}{4n^2\pi^2}+\frac{x^4}{16n^4\pi^4}-\dots\right).$$

Further, the resulting double series is absolutely convergent, since the series of absolute values is obtained by expanding similarly the convergent series

$$\sum_1^\infty \frac{2|x|^2}{4n^2\pi^2-|x|^2}.$$

It is therefore permissible (by Art. 82) to arrange the double series in powers of x, and then we obtain

$$\frac{x}{e^x-1} = 1-\frac{x}{2}+\frac{x^2}{2\pi^2}\left(\sum_1^\infty \frac{1}{n^2}\right)-\frac{x^4}{2^3\pi^4}\left(\sum_1^\infty \frac{1}{n^4}\right)+\frac{x^6}{2^5\pi^6}\left(\sum_1^\infty \frac{1}{n^6}\right)-\dots,$$

which is now seen to be valid for $|x|<2\pi$.*

By comparison with the former expression, we see that

$$B_1=\frac{1}{\pi^2}\sum\frac{1}{n^2},\quad B_2=\frac{3}{\pi^4}\sum\frac{1}{n^4},\quad B_3=\frac{45}{2\pi^6}\sum\frac{1}{n^6},$$

and generally
$$B_r=\frac{(2r)!}{2^{2r-1}\pi^{2r}}\sum\frac{1}{n^{2r}}.$$

We obtain thus the results (compare Arts. 71·1 and 71·2)

$$\sum\frac{1}{n^2}=\frac{\pi^2}{6},\quad \sum\frac{1}{n^4}=\frac{\pi^4}{90},\quad \sum\frac{1}{n^6}=\frac{\pi^6}{945},\quad \sum\frac{1}{n^8}=\frac{\pi^8}{9450}.$$

It is instructive to notice that (for any value of x) we have

$$\frac{2x^2}{x^2+4n^2\pi^2}=\frac{x^2}{2n^2\pi^2}\left\{1-\frac{x^2}{4n^2\pi^2}+\dots+(-1)^r\left(\frac{x^2}{4n^2\pi^2}\right)^r\right.$$
$$\left.+(-1)^{r+1}\left(\frac{x^2}{4n^2\pi^2}\right)^r\frac{x^2}{x^2+4n^2\pi^2}\right\}.$$

Thus, by addition, we see that, *when x is real, $x/(e^x-1)$ is represented by the first $(r+3)$ terms of the series with an error which is numerically less than the following term of the series;* for complex values of x, a corresponding theorem can be found, but it is necessarily more complicated.

* That 2π is the radius of convergence may be seen from one of the theorems of Art. 89; for the roots of $e^x=1$ are given by $x=2n\pi\iota$, and the least distance of any one of these from the origin is 2π.

For instance, for any real positive value of x, we have

$$0 < \frac{x}{e^x - 1} - \left(1 - \frac{x}{2}\right) < \frac{x^2}{12},$$

$$0 > \frac{x}{e^x - 1} - \left(1 - \frac{x}{2} + \frac{x^2}{12}\right) > -\frac{x^4}{720},$$

and so on.

Ex. 1. We can write

$$x \coth \frac{x}{2} = x\frac{e^x + 1}{e^x - 1} = \frac{2x}{e^x - 1} + x,$$

or again $$\frac{x}{2} \coth \frac{x}{2} = 1 + B_1 \frac{x^2}{2!} - B_2 \frac{x^4}{4!} + B_3 \frac{x^6}{6!} - \dots .$$

Thus we deduce

$$x \coth x = 1 + B_1 \frac{2^2 x^2}{2!} - B_2 \frac{2^4 x^4}{4!} + B_3 \frac{2^6 x^6}{6!} - \dots$$
$$= 1 + \tfrac{1}{3}x^2 - \tfrac{1}{45}x^4 + \tfrac{2}{945}x^6 - \dots, \quad |x| < \pi,$$

and $$x \cot x = 1 - \tfrac{1}{3}x^2 - \tfrac{1}{45}x^4 - \tfrac{2}{945}x^6 - \dots, \quad |x| < \pi,$$

the numerical coefficients being the same as in the previous series.

Ex. 2. Again

$$\tanh x = 2 \coth 2x - \coth x,$$

so from Ex. 1

$$\tanh x = \frac{B_1}{2!}(2^4 - 2^2)x - \frac{B_2}{4!}(2^8 - 2^4)x^3 + \frac{B_3}{6!}(2^{12} - 2^6)x^5 - \dots$$
$$= x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 - \tfrac{17}{315}x^7 + \dots,$$

and $$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \dots,$$

where in both series $|x| < \frac{1}{2}\pi$.

Ex. 3. Further

$$\operatorname{cosech} x = \coth \tfrac{1}{2}x - \coth x,$$

so $$x \operatorname{cosech} x = 1 - \frac{B_1}{2!}(2^2 - 2)x^2 + \frac{B_2}{4!}(2^4 - 2)x^4 - \frac{B_3}{6!}(2^6 - 2)x^6 + \dots$$
$$= 1 - \tfrac{1}{6}x^2 + \tfrac{7}{360}x^4 - \tfrac{31}{15120}x^6 + \dots$$

and $$x \operatorname{cosec} x = 1 + \tfrac{1}{6}x^2 + \tfrac{7}{360}x^4 + \tfrac{31}{15120}x^6 + \dots,$$

where in both series $|x| < \pi$.

Ex. 4. It is not possible to find similar formulae for $\operatorname{sech} x$ and $\sec x$; but it is easy to obtain the formulae

$$\operatorname{sech} x = 1 - \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 - \tfrac{61}{720}x^6 + \dots,$$
$$\sec x = 1 + \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 + \tfrac{61}{720}x^6 + \dots,$$

which are valid if $|x| < \frac{1}{2}\pi$, because $x = \pm\frac{1}{2}\pi$ give the smallest roots of the equation $\cos x = 0$ (compare Art. 89).

The numbers 1, 5, 61, 1385, ... are sometimes called *Euler's numbers*, $E_1, E_2, E_3, E_4, \dots$; but they have fewer applications than the coefficients $B_1, B_2, B_3, \dots$.

It may be noticed that

$$1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\dots=\frac{E_n\pi^{2n+1}}{2^{2n+2}(2n)!}.$$

101. Bernoullian functions.

The Bernoullian function of degree n, denoted by $\phi_n(x)$, is the coefficient of $t^n/n!$ in the expansion of

$$t\,\frac{e^{xt}-1}{e^t-1},$$

which, by the foregoing, can be expanded in powers of t if $|t|<2\pi$.

Thus we have

$$\Sigma\phi_n(x)\frac{t^n}{n!}=\left(xt+\frac{x^2t^2}{2!}+\frac{x^3t^3}{3!}+\dots\right)\left(1-\frac{1}{2}t+B_1\frac{t^2}{2!}-B_2\frac{t^4}{4!}+\dots\right),$$

so that $\phi_n(x)$ is equal to

$$x^n-\frac{n}{2}x^{n-1}+\frac{n(n-1)}{2!}B_1x^{n-2}-\frac{n(n-1)(n-2)(n-3)}{4!}B_2x^{n-4}+\dots,$$

where the polynomial terminates with either x or x^2.

From this formula, or by direct multiplication, we find that the first six Bernoullian polynomials are :

$$\begin{aligned}
\phi_1(x)&=x,\\
\phi_2(x)&=x^2-x=y,\\
\phi_3(x)&=x^3-\tfrac{3}{2}x^2+\tfrac{1}{2}x=yz,\\
\phi_4(x)&=x^4-2x^3+x^2=y^2,\\
\phi_5(x)&=x^5-\tfrac{5}{2}x^4+\tfrac{5}{3}x^3-\tfrac{1}{6}x=yz(y-\tfrac{1}{3}),\\
\phi_6(x)&=x^6-3x^5+\tfrac{5}{2}x^4-\tfrac{1}{2}x^2=y^2(y-\tfrac{1}{2}),
\end{aligned}$$

where
$$y=x(x-1)\quad\text{and}\quad z=x-\tfrac{1}{2}=\tfrac{1}{2}\frac{dy}{dx}.$$

Again, $\phi_n(x+1)-\phi_n(x)$ is the coefficient of $t^n/n!$ in the expansion of

$$\frac{t}{e^t-1}\{e^{(x+1)t}-e^{xt}\}=te^{xt},$$

so that

(A) $$\phi_n(x+1)-\phi_n(x)=nx^{n-1}.$$

Thus
$$\begin{aligned}
\phi_2(x+1)&=x^2+x,\\
\phi_3(x+1)&=x^3+\tfrac{3}{2}x^2+\tfrac{1}{2}x,\\
\phi_4(x+1)&=x^4+2x^3+x^2,
\end{aligned}$$

and generally $\phi_n(x+1)$ differs from $\phi_n(x)$ only in the sign of the coefficient of x^{n-1}.

If we write $x=1, 2, 3, \ldots$ in the difference-equation (A) and add the results, we see that, *if x is any positive integer* $(n>1)$,

$$1+2^{n-1}+3^{n-1}+\ldots+x^{n-1}=\frac{1}{n}\phi_n(x+1)=\frac{1}{n}\phi_n(x)+x^{n-1},$$

which gives *the formula of Bernoulli for the summation of the $(n-1)^{th}$ powers of positive integers.*

Generally, *if $b-a$ is any integer,*

$$a^{n-1}+(a+1)^{n-1}+(a+2)^{n-1}+\ldots+(b-1)^{n-1}=\frac{1}{n}\{\phi_n(b)-\phi_n(a)\}.$$

If we change the sign of t in the original definition, and add and subtract these two equations, we obtain, after a little reduction,

$$\text{(B)}\quad\begin{cases}\dfrac{t}{2\sinh\frac{1}{2}t}\{\cosh(x-\frac{1}{2})t-\cosh\frac{1}{2}t\}=\phi_2\dfrac{t^2}{2!}+\phi_4\dfrac{t^4}{4!}+\phi_6\dfrac{t^6}{6!}+\ldots,\\[2ex] \dfrac{t}{2\sinh\frac{1}{2}t}\{\sinh(x-\frac{1}{2})t+\sinh\frac{1}{2}t\}=\phi_1 t+\phi_3\dfrac{t^3}{3!}+\phi_5\dfrac{t^5}{5!}+\ldots.\end{cases}$$

Thus it follows that $\phi_2, \phi_4, \phi_6, \ldots$ are expressible as functions of $(x-\frac{1}{2})^2=y+\frac{1}{4}$; that is, *the even polynomials are functions only of y.*

Similarly, *the odd polynomials $\phi_3, \phi_5, \ldots$ contain $x-\frac{1}{2}=z$ as a factor, and the remaining factor is a function only of y.*

These properties are evidently verified by the polynomials $\phi_2, \phi_3, \ldots, \phi_6$, which have been tabulated above.

If we differentiate equations (B) with respect to x, we see that

$$\frac{1}{2}\frac{d\phi_2}{dx}=\phi_1-\frac{1}{2},\qquad \frac{1}{4}\frac{d\phi_4}{dx}=\phi_3,\qquad \frac{1}{6}\frac{d\phi_6}{dx}=\phi_5, \text{ etc.,}$$

and $$\frac{1}{3}\frac{d\phi_3}{dx}=\phi_2+B_1,\quad \frac{1}{5}\frac{d\phi_5}{dx}=\phi_4-B_2,\quad \frac{1}{7}\frac{d\phi_7}{dx}=\phi_6+B_3, \text{ etc.,}$$

where $B_1, B_2, \ldots$ denote Bernoulli's numbers (of Art. 100).

The general formulae will be

$$\phi'_{2m}(x)=2m\phi_{2m-1}(x),\quad (m>1)$$
$$\phi'_{2m+1}(x)=(2m+1)\{\phi_{2m}(x)+(-1)^{m-1}B_m\},\quad (m\geqq 1).$$

If we change x into $(1-x)$ in the two equations (B) containing $\cosh(x-\frac{1}{2})t$ and $\sinh(x-\frac{1}{2})t$, we see at once that

$$\phi_2(1-x)=\phi_2(x),\qquad \phi_4(1-x)=\phi_4(x), \text{ etc.,}$$
$$\phi_3(1-x)=-\phi_3(x),\qquad \phi_5(1-x)=-\phi_5(x), \text{ etc}$$

Also it is evident from the first of the two equations (B) that $y=(x-\frac{1}{2})^2-(\frac{1}{2})^2=x(x-1)$ is a factor of every even polynomial; and from the second equation we see that $\phi_3, \phi_5, \phi_7, \ldots$ contain $z=x-\frac{1}{2}$ as a factor, and since these polynomials vanish for $x=0$, $x=1$, it is clear that $\phi_3, \phi_5, \phi_7, \ldots$ are divisible by yz.

Thus we see that

$$\frac{d\phi_4}{dx}, \frac{d\phi_6}{dx}, \frac{d\phi_8}{dx}, \ldots \text{ are divisible by } y\frac{dy}{dx};$$

and so, since $\phi_4, \phi_6, \phi_8, \ldots$ have been proved to be divisible by y, these functions are now seen to be divisible by y^2.

These conclusions are confirmed for $\phi_3, \ldots, \phi_6$ by reference to the table given above.

It will be seen that the other factors of ϕ_5 and ϕ_6 are respectively $y-\frac{1}{3}$ and $y-\frac{1}{2}$; and these factors do not vanish between $x=0$ and $x=1$, because y is negative between these limits. Thus it is natural to conjecture that *the equation* $\phi_{2m}(x)=0$ *has no root between* 0 *and* 1, *while* $\phi_{2m+1}(x)$ *has only the root* $\frac{1}{2}$.

Suppose that this conjecture has been established for all values of m up to, say, $p-1$.

Accordingly $\phi_{2p-1}=\frac{1}{2p}\frac{d\phi_{2p}}{dx}$ vanishes only for $x=\frac{1}{2}$ between $x=0$ and $x=1$; and thus ϕ_{2p} either steadily increases or steadily decreases from $x=0$ to $x=\frac{1}{2}$, and varies in the opposite sense from $x=\frac{1}{2}$ to $x=1$. But $\phi_{2p}=0$ for $x=0$, $x=1$; and accordingly ϕ_{2p} cannot vanish for any value of x between 0 and 1.

Consider next the function ϕ_{2p+1}; this is known to vanish for $x=0, \frac{1}{2}, 1$; and so

$$\frac{d\phi_{2p+1}}{dx}=(2p+1)\{\phi_{2p}+(-1)^{p-1}B_p\}$$

must vanish once at least between $x=0$ and $x=\frac{1}{2}$, and also once at least between $x=\frac{1}{2}$ and $x=1$. But since ϕ_{2p} steadily increases (or steadily decreases) from $x=0$ to $x=\frac{1}{2}$, it is clear that ϕ'_{2p+1} can vanish once only in this interval; similarly, it vanishes once only between $x=\frac{1}{2}$ and $x=1$. Thus, finally, $\phi_{2p+1}=0$ can have no roots between $x=0$ and $x=1$ except $x=\frac{1}{2}$.

Thus our conjecture is now proved to be true for $m=p$. But it is known to be true for $m=2$, and therefore it is true for $m=3$; hence also for $m=4, 5, 6, \ldots$, and so on generally.

The diagram indicates the relations between $\phi_2(x)$, $\phi_3(x)$, $\phi_4(x)$, $\phi_5(x)$, and will illustrate the general argument just given.

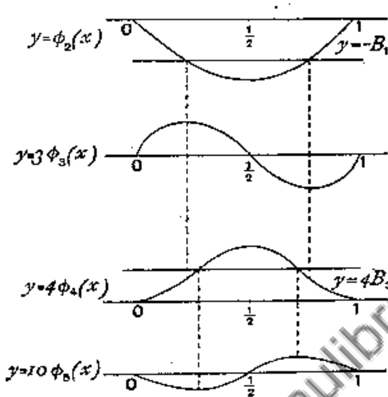

Fig. 30.

It has just been proved that ϕ_{2n} has its maximum numerical value (between $x=0$ and $x=1$) for $x=\frac{1}{2}$; to evaluate this, let us write $x=\frac{1}{2}$ in the formula (B) containing $\cosh(x-\frac{1}{2})t$. We then find

$$-\frac{t}{2}\tanh\frac{t}{4}=\frac{\phi_2 t^2}{2!}+\frac{\phi_4 t^4}{4!}+\frac{\phi_6 t^6}{6!}+\ldots \quad (\text{for } x=\tfrac{1}{2}).$$

Using Ex. 2 of Art. 100, we see that the maximum numerical values are

$$\phi_2\left(\frac{1}{2}\right)=-2B_1\left(1-\frac{1}{2^2}\right)=-\frac{1}{4},$$

$$\phi_4\left(\frac{1}{2}\right)=+2B_2\left(1-\frac{1}{2^4}\right)=+\frac{1}{16},$$

$$\phi_6\left(\frac{1}{2}\right)=-2B_3\left(1-\frac{1}{2^6}\right)=-\frac{3}{64},$$

and so on.

Thus ϕ_2, ϕ_6, ϕ_{10}, ... are all *negative* from $x=0$ to $x=1$; and ϕ_4, ϕ_8, ϕ_{12}, ... are all *positive* in the same interval.

Ex. Prove that if n is odd and k is an integer,

$$\sum_{r=0}^{k-1}\phi_n(x+r/k)=\phi_n(kx)/k^{n-1},$$

and obtain the corresponding result when n is even.

102. Euler's summation formula.

We have seen in Art. 101 that if x and n are positive integers,

$$1+2^{n-1}+3^{n-1}+\dots+x^{n-1}=\frac{1}{n}\phi_n(x)+x^{n-1}=\frac{1}{n}\phi_n(x+1)$$
$$=\frac{1}{n}x^n+\tfrac{1}{2}x^{n-1}+\frac{n-1}{2!}B_1x^{n-2}-\frac{(n-1)(n-2)(n-3)}{4!}B_2x^{n-4}+\dots,$$

this polynomial containing $\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+3)$ terms.

It is obvious that when $f(x)$ is a polynomial in x, we can obtain the value of the sum
$$f(1)+f(2)+\dots+f(x)$$
by the addition of suitable multiples of the Bernoullian functions of proper degrees. But to obtain a compact formula, we shall utilise the Calculus; and we observe that we can write the foregoing polynomial in the form

$$\int x^{n-1}dx+\tfrac{1}{2}x^{n-1}+\frac{1}{2!}B_1\frac{d}{dx}(x^{n-1})-\frac{1}{4!}B_2\frac{d^3}{dx^3}(x^{n-1})+\dots.$$

Hence *when $f(x)$ is a polynomial, we have Euler's summation formula,*

$$f(1)+f(2)+\dots+f(x)$$
$$=\int f(x)dx+\tfrac{1}{2}f(x)+\frac{1}{2!}B_1f'(x)-\frac{1}{4!}B_2f'''(x)+\dots,$$

where there is no term on the right-hand side (in its final form) which is not divisible by x.

However, the most interesting applications of this formula arise when $f(x)$ is a rational algebraic fraction, or a transcendental function, and then of course the foregoing method of proof cannot be used; and the right-hand side becomes an infinite series which may not converge.

We shall consider a number of special examples of this kind in Chapter XII. below.

As a matter of symbolic transformation it is worth noting that if
$$f(1)+f(2)+\dots+f(x)=F(x),$$
then
$$F(x+1)-F(x)=f(x+1).$$

Now assuming Taylor's theorem to be valid for these functions, we have

$$F(x+1)=F(x)+F'(x)+\frac{1}{2!}F''(x)+\dots$$
$$=e^DF(x)$$

symbolically, where D stands for d/dx.

Similarly, $$f(x+1) = e^D f(x).$$

Hence $$(e^D - 1)\,F(x) = e^D f(x),$$

or $$F(x) = \frac{e^D}{e^D - 1} f(x) = \left(1 + \frac{1}{e^D - 1}\right) f(x).$$

Now applying the expansion of Art. 100, we see that

$$1 + \frac{1}{e^D - 1} = \frac{1}{D} + \frac{1}{2} + \frac{B_1}{2!} D - \frac{B_2}{4!} D^3 + \dots,$$

which leads to the required result.

Ex. 1. If $\psi_n(x)$ is the coefficient of $t^n/n!$ in the expansion of $e^{xt}/(e^t+1)$, prove that

$$\psi_n(x) + \psi_n(x+1) = x^n,$$

and if x is a positive integer,

$$-\psi_n(0) + (-1)^x \psi_n(x) = 1^n - 2^n + 3^n - \dots + (-1)^x (x-1)^n.$$

Also $$\Psi_n(0) + (-1)^{x-1} \Psi_n(x) = 1^n - 2^n + 3^n - \dots + (-1)^{x-1} x^n,$$

where $$\Psi_n(x) = x^n - \psi_n(x).$$

Ex. 2. As particular cases of Ex. 1, we find that

$$\psi_1(x) = \tfrac{1}{2}x - \tfrac{1}{4}, \qquad \psi_2(x) = \tfrac{1}{2}x(x-1),$$
$$\psi_3(x) = \tfrac{1}{2}x^3 - \tfrac{3}{4}x^2 + \tfrac{1}{8}, \qquad \psi_4(x) = \tfrac{1}{2}x^4 - x^3 + \tfrac{1}{2}x.$$

These give, when x is a positive integer,

$$1 - 2 + 3 - 4 + \dots + (-1)^{x-1} x = -\tfrac{1}{2}x \quad \text{or} \quad +\tfrac{1}{2}(x+1),$$
$$1 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{x-1} x^2 = (-1)^{x-1} \tfrac{1}{2}(x^2 + x),$$
$$1 - 2^3 + 3^3 - 4^3 + \dots + (-1)^{x-1} x^3 = -(\tfrac{1}{2}x^3 + \tfrac{3}{4}x^2) \quad \text{or} \quad +(\tfrac{1}{2}x^3 + \tfrac{3}{4}x^2 - \tfrac{1}{4}),$$
$$1 - 2^4 + 3^4 - 4^4 + \dots + (-1)^{x-1} x^4 = (-1)^{x-1}(\tfrac{1}{2}x^4 + x^3 - \tfrac{1}{2}x),$$

and so on. In the first and third cases the alternatives are to be chosen according as x is even or odd.

Ex. 3. It is easy to see that

$$\psi_n'(x) = n\psi_{n-1}(x), \qquad \psi_n(1-x) = (-1)^n \psi_n(x),$$

$$\psi_{2n-1}(0) = -\psi_{2n-1}(1) = (-1)^n B_n \frac{2^{2n}-1}{2n}, \quad \psi_{2n}(0) = \psi_{2n}(1) = 0.$$

From the foregoing equations and from those of Ex. 2 prove that

$$\psi_5(x) = \tfrac{1}{2}x^5 - \tfrac{5}{4}x^4 + \tfrac{5}{4}x^2 - \tfrac{1}{4}, \quad \psi_6(x) = \tfrac{1}{2}x^6 - \tfrac{3}{2}x^5 + \tfrac{5}{2}x^3 - \tfrac{3}{2}x.$$

Shew also that $(x - \tfrac{1}{2})$ is a factor of the odd polynomials, and $x(x-1)$ of the even polynomials.

Ex. 4. Shew that if $f(x)$ is a polynomial in x,

$$f(1) - f(2) + f(3) - f(4) + \dots + (-1)^{x-1} f(x)$$
$$= (-1)^{x-1}\left\{\tfrac{1}{2}f(x) + \frac{2^2-1}{2!} B_1 f'(x) - \frac{2^4-1}{4!} B_2 f'''(x) + \frac{2^6-1}{6!} B_3 f^{V}(x) - \dots\right\} + \text{const.}$$

103. Development of elliptic function formulae from the algebraic side.

The following account is merely the extension to double series of what was given in Art. 71·2 above for single series.

We suppose x to be a complex variable, ω, ω' to be two periods (whose ratio is *not* real); and for brevity we write

$$\Omega = m\omega + n\omega',$$

where m, n are any two integers, positive or negative (zero included).

We have seen (Art. 82) that the series

$$F_0(x) = \sum \frac{1}{(x-\Omega)^3}$$

converges absolutely and uniformly in any region from which the points $x=\Omega$ are excluded.* Thus $F_0(x)$ is an analytic function in these regions. Also we have

$$F_0(x+\omega) = \sum \frac{1}{(x+\omega-\Omega)^3} = \sum \frac{1}{(x-\Omega)^3} = F_0(x),$$

as the only change involved is to write $(m-1)$ in place of m.

Similarly, $$F_0(x+\omega') = F_0(x).$$

Hence *$F_0(x)$ is a doubly-periodic function with periods ω, ω'.*

Further, we can write

$$F_0(x) = \frac{1}{x^3} + \sum' \frac{1}{(x-\Omega)^3},$$

where the accent implies the omission of the special term $m=0$, $n=0$. Also, if $|x|$ is less than λ, the least value of $|\Omega|$ (for any pair of integers m, n), we can write

$$\sum' \frac{1}{(x-\Omega)^3} = -\sum'\left(\frac{1}{\Omega^3} + \frac{3x}{\Omega^4} + \frac{6x^2}{\Omega^5} + \dots\right)$$
$$= -(c_2 x + 2c_4 x^3 + 3c_6 x^5 + \dots),$$

where $$c_2 = 3\sum' \frac{1}{\Omega^4}, \quad c_4 = 5\sum' \frac{1}{\Omega^6}, \quad c_6 = 7\sum' \frac{1}{\Omega^8}, \text{ etc.}$$

The reversal of the order of summation is justified by the principle of absolute convergence. Further, $\Sigma' \frac{1}{\Omega^3}$, $\Sigma' \frac{1}{\Omega^5}$, ... are zero, by another application of the same principle; for this allows us to group together pairs of terms corresponding to equal and opposite values of m, n. The pairs of terms then cancel, and these sums are accordingly zero.

* These points are supposed to be excluded by small circles of the type $|x-\Omega| = \delta$, where δ is *fixed*, although it may be arbitrarily small.

Hence, if $|x| < \lambda$, we have

$$F_0(x) = \frac{1}{x^3} - c_2 x - 2c_4 x^3 - 3c_6 x^5 - \dots .$$

We now define a function $F(x)$, such that

$$F'(x) = -2F_0(x),$$

and to make the definition precise, we write

$$F(x) - \frac{1}{x^2} = 2\int_0^x \left\{\frac{1}{\xi^3} - F_0(\xi)\right\} d\xi.$$

This gives at once

$$F(x) = \frac{1}{x^2} + \sum{}' \left\{\frac{1}{(x-\Omega)^2} - \frac{1}{\Omega^2}\right\},$$

and, if $|x| < \lambda$, $\quad F(x) = \frac{1}{x^2} + c_2 x^2 + c_4 x^4 + c_6 x^6 + \dots .$

Thus $F(x)$ is an even function and is analytic in any region from which the points $x = \Omega$ are excluded.

Since $\quad F_0(x+\omega) = F_0(x),$

we have $\quad F(x+\omega) - F(x) = \text{const.},$

and writing $x = -\frac{1}{2}\omega$, we see that the constant is zero, since $F(x)$ is an even function of x.

Thus $\quad F(x+\omega) = F(x),$

and similarly, $\quad F(x+\omega') = F(x).$

That is, $F(x)$ *is a doubly-periodic function with periods* ω, ω'.

We now define a function $G(x)$, such that

$$G'(x) = -F(x),$$

or more precisely $\quad G(x) - \frac{1}{x} = \int_0^x \left\{\frac{1}{\xi^2} - F(\xi)\right\} d\xi.$

This gives at once the formulae

$$G(x) = \frac{1}{x} + \sum{}' \left(\frac{1}{x-\Omega} + \frac{1}{\Omega} + \frac{x}{\Omega^2}\right),$$

and, if $|x| < \lambda$,

$$G(x) = \frac{1}{x} - \tfrac{1}{3} c_2 x^3 - \tfrac{1}{5} c_4 x^5 - \tfrac{1}{7} c_6 x^7 - \dots .$$

Thus $G(x)$ is an odd function, and is analytic in the same region as $F(x)$, $F_0(x)$.

We now find that

$$G(x+\omega)-G(x)=\text{const.}=\eta \text{ say,}$$

but on writing $x=-\frac{1}{2}\omega$, we only arrive at the formula

$$2G(\tfrac{1}{2}\omega)=\eta,$$

because $G(x)$ is an odd function.*

Similarly, $$G(x+\omega')-G(x)=\eta',$$

where $$2G(\tfrac{1}{2}\omega')=\eta'.$$

Thus *the function* $G(x)$ *is not doubly-periodic.*

Finally, we define a function $H(x)$ so that

$$H'(x)/H(x)=G(x),$$

or more precisely

$$\log\{H(x)/x\}=\int_0^x\left\{G(\xi)-\frac{1}{\xi}\right\}d\xi.$$

This gives at once

$$H(x)=x\Pi'\left\{\left(1-\frac{x}{\Omega}\right)e^{\phi}\right\},\quad \phi=\frac{x}{\Omega}+\frac{1}{2}\frac{x^2}{\Omega^2},$$

and, if $|x|<\lambda$,

$$\log\left\{\frac{H(x)}{x}\right\}=-\frac{1}{12}c_2x^4-\frac{1}{30}c_4x^6-\frac{1}{56}c_6x^8-\dots.$$

Thus $H(x)$ is an odd function of x; and $H(x)$ can be proved to be analytic for any value of x. This function vanishes for $x=0$, $x=\Omega$; and $H(x)/x\to 1$ as $x\to 0$.

Now $$\frac{H'(x+\omega)}{H(x+\omega)}-\frac{H'(x)}{H(x)}=\eta,$$

so that $$\log\left\{\frac{H(x+\omega)}{H(x)}\right\}=\eta x+\text{const.}$$

Taking $x=-\frac{1}{2}\omega$, we find that

$$\frac{H(x+\omega)}{H(x)}=-e^{\eta(x+\frac{1}{2}\omega)}.$$

Similarly, $$\frac{H(x+\omega')}{H(x)}=-e^{\eta'(x+\frac{1}{2}\omega')}.$$

Thus again, $H(x)$ *is not doubly-periodic.*

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* The reader may find it of interest to see that the method used for the corresponding problem in Art. 71·2 does *not* give $\eta=0$ here. It requires some theorems from the general Theory of Functions to prove that η *cannot* be zero; but there is no reason to anticipate the identity $G(\frac{1}{2}\omega)=0$.

We cannot obtain further results without making some appeal to Theory of Functions; we shall content ourselves with the assumption that a doubly periodic function, such as $F(x)$, satisfies a differential equation of the type

$$\left(\frac{dF}{dx}\right)^2 = A_0F^4 + 4A_1F^3 + 6A_2F^2 + 4A_3F + A_4,$$

where A_0, A_1, A_2, A_3, A_4 are certain constants.

Now, taking $|x| < \lambda$, we have

$$-\frac{dF}{dx} = 2F_0(x) = \frac{2}{x^3} - 2c_2x - 4c_4x^3 - 6c_6x^5 - \dots,$$

so that $$\left(\frac{dF}{dx}\right)^2 = \frac{4}{x^6} - \frac{8c_2}{x^2} - 16c_4 + (4c_2^2 - 24c_6)x^2 + \dots .$$

It follows at once that, in the differential equation for $F(x)$, we must have

$$A_0 = 0, \quad A_1 = 1.$$

Now $$\{F(x)\}^3 = \frac{1}{x^6} + \frac{3c_2}{x^2} + 3c_4 + (3c_2^2 + 3c_6)x^2 + \dots,$$

so that $$4F^3 - \left(\frac{dF}{dx}\right)^2 = \frac{20c_2}{x^2} + 28c_4 + (8c_2^2 + 36c_6)x^2 + \dots .$$

It follows now that

$$A_2 = 0, \quad 4A_3 = -20c_2.$$

Thus we form

$$4F^3 - 20c_2F - \left(\frac{dF}{dx}\right)^2 = 28c_4 + (36c_6 - 12c_2^2)x^2 + \dots .$$

As the right-hand side must be a *constant* (in order to satisfy the general theorem quoted above), it follows that *the constant* is $28c_4$, and that we have the identity

$$3c_6 = c_2^2.$$

It is usual to write the differential equation for $F(x)$ in Weierstrass's form,

$$\left(\frac{dF}{dx}\right)^2 = 4F^3 - g_2F - g_3,$$

where $$g_2 = 20c_2 = 60\sum{}'\frac{1}{\Omega^4},$$

$$g_3 = 28c_4 = 140\sum{}'\frac{1}{\Omega^6},$$

and the identity takes the form

$$7\sum{}'\frac{1}{\Omega^8}=3\left(\sum{}'\frac{1}{\Omega^4}\right)^2.$$

Written in this form, the last identity seems very remarkable; and the whole subject is full of equally striking relations.

To complete the parallelism with Art. 71·2, it may be worth while to indicate the results which could be derived for those functions on lines similar to the last. *Using the notation of that article,* we have

$$-G'(x)=\frac{\pi^2}{\omega^2}\operatorname{cosec}^2\left(\frac{\pi x}{\omega}\right)=\frac{\pi^2}{\omega^2}+\{G(x)\}^2.$$

This gives

$$\frac{1}{x^2}+c_0+c_2x^2+c_4x^4+\ldots=\frac{\pi^2}{\omega^2}+\left(\frac{1}{x}-c_0x-\frac{1}{3}c_2x^3-\ldots\right)^2,$$

and the coefficients of $1/x^2$ cancel. The following three terms yield the relations

$$c_0=(\pi/\omega)^2-2c_0,\quad \text{or}\quad c_0=\tfrac{1}{3}(\pi/\omega)^2,$$
$$c_2=c_0^2-\tfrac{2}{3}c_2,\qquad c_2=\tfrac{3}{5}c_0^2=\tfrac{1}{15}(\pi/\omega)^4,$$
$$c_4=\tfrac{2}{3}c_0c_2-\tfrac{2}{5}c_4,\qquad c_4=\tfrac{10}{21}c_0c_2=\tfrac{2}{945}(\pi/\omega)^6.$$

These values for c_0, c_2, c_4 agree with those calculated in Art. 71·2 from the series for $\log(\sin x/x)$.

The reader who is accustomed to the notation usually adopted in the theory of Weierstrass's elliptic functions, will recognise that our functions $F(x)$, $G(x)$, $H(x)$ are in reality the same as the $\wp$-, ζ-, σ-functions. The object of adopting this neutral notation here is to avoid any bias towards taking known elliptic-function properties for granted.

It will be noticed also that we have used ω, ω', η, η' to denote *twice* their usual values; so that ω is here a period (not a *half* period). The advantage of the usual notation does not show itself until a later stage; and in our group of propositions, the present notation is really easier to work with. This remark seemed necessary in order to avoid confusion on reference to the standard text-books on elliptic functions.

EXAMPLES.

General Powers of Complex Numbers.

1. If the numbers a, x are both complex, shew that when the points a^x are marked in Argand's diagram, they lie on an equiangular spiral whose angle depends only on x and not on a. [*Math. Trip.* 1899.]

Examine the special cases when x is (1) real, (2) pure imaginary; and in particular, if $x = \iota$ and a is real, prove that if $b = a^\iota$,

$$|b + 1/b| = \surd[2\{\cosh(4k\pi) + \cos 2(\log a)\}],$$

where k is an arbitrary integer.

2. If (a_n) is a sequence of complex numbers, which converges to a as a limit, and if b is another complex number, shew that values of b^{a_n} can be selected so as to form a convergent sequence, whose limit is one of the values of b^a.

3. If x is real, prove that any value of x^ι oscillates finitely both as x tends to 0 and to ∞.

4. If $|\cos x| = 1$, where $x = \xi + \iota\eta$, shew that $\sinh \eta = \pm \sin \xi$; and that if we write

$$\cos x = \cos\theta + \iota \sin\theta,$$

where θ is real, then $\sin\theta = \pm \sin^2\xi$.

Binomial Series.

5. A straight line can be drawn in the plane of the complex variable x, so that the series

$$1 - x + \frac{x(x-1)}{2!} - \frac{x(x-1)(x-2)}{3!} + \dots$$

converges to 0 on one side of the line; and the series diverges in the sense of Art. 75 on the other side of the line [*Math. Trip.* 1905.]

6. Obtain from the binomial series, or otherwise, the equation

$$(2\cos\theta)^\nu = \cos\nu\theta + \nu\cos(\nu-2)\theta + \frac{\nu(\nu-1)}{2!}\cos(\nu-4)\theta + \dots,$$

where ν is real and greater than -1. What restrictions are required as to the value of θ?

Shew that the equation ceases to be true for $\nu = \frac{1}{3}$, $\theta = \pi$, and explain why.

7. Find the sum of

$$1 + \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta + \frac{1.3.5}{2.4.6}\cos 3\theta + \dots$$

and of

$$\frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta + \frac{1.3.5}{2.4.6}\sin 3\theta + \dots.$$

[Apply Art. 96, putting $\nu = -\frac{1}{2}$, $x = -e^{\iota\theta}$.]

8. If m is positive, shew that

$$2^{\frac{m}{2}}\cos(\tfrac{1}{4}m\pi) = 1 - \frac{m(m-1)}{2!} + \frac{m(m-1)(m-2)(m-3)}{4!} - \dots,$$

and state the special form of the result when $m = 1/10$.

[Take $x = \iota$ in the expansion of $(1+x)^m$.]

9. If $t=\cos\theta+\iota\sin\theta$ and $0<r<1$, shew that

$$\frac{1}{2\pi}\int_0^{2\pi}(1+rt)^n(1+r/t)^n\,d\theta=1+r^2n_1{}^2+r^4n_2{}^2+r^6n_3{}^2+\dots$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}(1+rt)^n(1-r/t)^n\,d\theta=1-r^2n_1{}^2+r^4n_2{}^2-r^6n_3{}^2+\dots,$$

where n_1, n_2, n_3, ... are the coefficients in the binomial series.

Deduce from Abel's theorem that, if $n>0$,

$$1+n_1{}^2+n_2{}^2+\dots=\frac{2^n}{\pi}\int_0^{\pi}(1+\cos\theta)^n\,d\theta=\frac{\Gamma(2n+1)}{\{\Gamma(n+1)\}^2},$$

$$1-n_1{}^2+n_2{}^2-\dots=\frac{2^n}{\pi}\cos(\tfrac{1}{2}n\pi)\int_0^{\pi}(\sin\theta)^n\,d\theta=\frac{\Gamma(n+1)}{\{\Gamma(\frac{1}{2}n+1)\}^2}\cos(\tfrac{1}{2}n\pi).$$

[Note that the argument of $(1-r^2+2\iota r\sin\theta)^n$ approaches the limit $+\frac{1}{2}n\pi$ (as $r\to1$) when $\sin\theta$ is positive, and $-\frac{1}{2}n\pi$ when $\sin\theta$ is negative. The first summation is valid if $n>-\frac{1}{2}$, and the second if $n>-1$; but the proofs become rather more difficult when n is negative.]

Exponential Series.

10. Determine the expansion of $e^{-x\cos\theta}\cos(x\sin\theta)$ in powers of x, and deduce that

$$\int_0^x\frac{\sin t}{t}\,dt=\frac{\pi}{2}-\int_0^{\frac{1}{2}\pi}e^{-x\cos\theta}\cos(x\sin\theta)\,d\theta.$$

[Put $xe^{\iota\theta}$ for x in the exponential series.]

11. Shew that (compare Ex. 10)

$$e^{ax}\cos bx=\sum_0^{\infty}\frac{x^n}{n!}\left\{a^n-\frac{n(n-1)}{2!}a^{n-2}b^2+\frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4-\dots\right\},$$

where there are $\frac{1}{2}(n+1)$ or $\frac{1}{2}(n+2)$ terms in the brackets.

Determine a similar series for $e^{ax}\sin bx$.

12. If $x=y\sqrt{(1+ax)}$, shew that Lagrange's series for one root is

$$x=y+\tfrac{1}{2}ay^2+\sum_1^{\infty}(-1)^{n-1}\frac{1\cdot3\dots(2n-3)}{2\cdot4\dots2n}\left(\frac{a}{2}\right)^{2n}y^{2n+1},$$

and that the series converges if $|ay|<2$. [*Math. Trip.* 1902.]

13. Shew that if $|\theta|<1/e$,

$$\cos\theta=1-\theta\sin\theta+\frac{1}{2!}\theta^2\cos2\theta+\frac{2^2}{3!}\theta^3\sin3\theta-\frac{3^3}{4!}\theta^4\cos4\theta-\dots,$$

$$\sin\theta=\quad\theta\cos\theta+\frac{1}{2!}\theta^2\sin2\theta-\frac{2^2}{3!}\theta^3\cos3\theta-\frac{3^3}{4!}\theta^4\sin4\theta+\dots.$$

[*Math. Trip.* 1891.]

[Write $a=b=\iota$ in the formula of Ex. 4, Art. 55·1. The introduction of complex numbers in the place of real ones may be justified by an argument of the same type as that used in Art. 95.]

Products and Allied Series.

14. Shew that the product

$$\prod_1^\infty \{(1-e^{x/n})/\log(1-x/n)\}$$

converges absolutely except when x is a positive integer.

[If the general term is $1-a_n$, $\lim(n^2 a_n)=\frac{1}{8}x^2$.]

15. Evaluate $\quad x^2\prod_1^\infty(1+x^4/n^4) \quad$ and $\quad x^3\prod_1^\infty(1+x^6/n^6)$.

[Glaisher, *Proc. Lond. Math. Soc.* (1), vol. 7, p. 23.]

16. From Ex. 12, Ch. IX., find the values of

$$\prod_{-\infty}^\infty\left[1-\frac{x^4}{(n-c)^4}\right] \quad \text{and} \quad \prod_{-\infty}^\infty\left[1+\frac{x^4}{(n-c)^4}\right].$$

[*Math. Trip.* 1902.]

17. Prove that $\quad \dfrac{e^x-c}{1-c}=e^{\frac{1}{2}x}\lim_{\nu\to\infty}\prod_{-\nu}^{\nu}\left(1-\dfrac{x}{\log_n c}\right),$

where $\quad \log_n c=\log c+2n\pi\iota,$

any determination of $\log c$ being taken. [Hardy.]

18. Shew that $\quad \prod_2^\infty\dfrac{n^3-1}{n^3+1}=\dfrac{2}{3}.$ [Gram.]

[If $t=\frac{1}{2}(-1+\iota\sqrt{3})$, so that $t^3=1$, we have, as in Ex. 11, Ch. VI.,

$$\prod_1^\infty\frac{n^3-x^3}{n^3+x^3}=\frac{\Gamma(1+x)\,\Gamma(1+tx)\,\Gamma(1+t^2x)}{\Gamma(1-x)\,\Gamma(1-tx)\,\Gamma(1-t^2x)}.$$

Thus $\quad \dfrac{1+x+x^2}{1+x^3}\prod_2^\infty\dfrac{n^3-x^3}{n^3+x^3}=\dfrac{\Gamma(1+x)\,\Gamma(1+tx)\,\Gamma(1+t^2x)}{\Gamma(2-x)\,\Gamma(1-tx)\,\Gamma(1-t^2x)}.$

Now write $x=1$, and observe that

$$\Gamma(1-t)\,\Gamma(1-t^2)=\Gamma(2+t^2)\,\Gamma(2+t)=(1+t)(1+t^2)\,\Gamma(1+t)\,\Gamma(1+t^2)$$
$$=\Gamma(1+t)\,\Gamma(1+t^2).]$$

19. Shew that

$$\sum_{-\infty}^\infty\frac{1}{n^4+x^4}=\frac{\pi}{x^3\sqrt{2}}\,\frac{\sinh(\pi x\sqrt{2})+\sin(\pi x\sqrt{2})}{\cosh(\pi x\sqrt{2})-\cos(\pi x\sqrt{2})}.$$

[*Math. Trip.* 1888.]

[We have $\quad \dfrac{1}{n^4+x^4}=\sum_{(t)}\dfrac{t}{4x^3}\left(\dfrac{1}{n+tx}-\dfrac{1}{n}\right),$

where $t^4=-1$.

Thus, as in Ex. 18, Ch. IX., the given sum is

$$\sum_{(t)}\frac{\pi t}{4x^3}\cot(\pi t x),$$

and this gives the required result.]

20. Apply a method similar to Ex. 19 to find

$$\sum_{-\infty}^\infty\frac{1}{n^3+x^3}.$$

21. Shew that

$$\sum_{-\infty}^{\infty} (-1)^{n-1} \frac{4n^2-1}{16n^4+4n^2+1} = \pi\sqrt{2}\cosh\frac{\pi\sqrt{3}}{4}\operatorname{sech}\frac{\pi\sqrt{3}}{2}$$

[*Math. Trip.* 1898.]

[We have $\dfrac{1-4n^2}{16n^4+4n^2+1} = \dfrac{1}{2}\left(\dfrac{1}{2n-\omega} - \dfrac{1}{2n+\omega} + \dfrac{1}{2n-\omega^2} - \dfrac{1}{2n+\omega^2}\right)$,

where $\omega = \frac{1}{2}(-1+\iota\sqrt{3})$. Thus the given series is

$$-\tfrac{1}{2}\pi\{\operatorname{cosec}(\tfrac{1}{2}\pi\omega) + \operatorname{cosec}(\tfrac{1}{2}\pi\omega^2)\},$$

which can be reduced to the form given.]

22. Prove that

$$\sum_{-\infty}^{\infty} \frac{1}{(n+x)^2+y^2} = \frac{\pi}{y}\frac{\sinh(2\pi y)}{\cosh(2\pi y)-\cos(2\pi x)}.$$

Deduce that the least value of θ, when y is fixed and

$$\cot\theta = \sum_{-\infty}^{\infty} \frac{n+x}{\{(n+x)^2+y^2\}^2} \Big/ \sum_{-\infty}^{\infty} \frac{y}{\{(n+x)^2+y^2\}^2},$$

is given by $\cos\theta = 2\pi y/\sinh(2\pi y)$. [*Math. Trip.* 1892.]

[Here $$\frac{1}{(n+x)^2+y^2} = \frac{1}{2\iota y}\left\{\frac{1}{n+(x-\iota y)} - \frac{1}{n+(x+\iota y)}\right\},$$

and so the sum is $\dfrac{\pi}{2\iota y}\{\cot\pi(x-\iota y) - \cot\pi(x+\iota y)\}$.]

23. Shew that

$$\arctan\frac{y}{x} + \sum_{-\infty}^{\infty}{}' \left\{\arctan\left(\frac{y}{n+x}\right) - \frac{y}{n}\right\} = \arctan\left\{\frac{\tanh(\pi y)}{\tan(\pi x)}\right\},$$

$$\arctan\frac{y}{x} + \sum_{-\infty}^{\infty}{}' (-1)^n \left\{\arctan\left(\frac{y}{n+x}\right) - \frac{y}{n}\right\} = \arctan\left\{\frac{\sinh(\pi y)}{\sin(\pi x)}\right\}.$$

In particular we find with $y = x$,

$$\sum_{1}^{\infty} \arctan\left(\frac{2x^2}{n^2}\right) = \frac{\pi}{4} - \arctan\left\{\frac{\tanh(\pi x)}{\tan(\pi x)}\right\},$$

$$\sum_{1}^{\infty} (-1)^{n-1} \arctan\left(\frac{2x^2}{n^2}\right) = -\frac{\pi}{4} + \arctan\left\{\frac{\sinh(\pi x)}{\sin(\pi x)}\right\}.$$

[We have $$\log\sin(\pi x) = \log(\pi x) + \sum_{-\infty}^{\infty}{}' \left\{\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right\}$$

and $$\log\tan(\tfrac{1}{2}\pi x) = \log(\tfrac{1}{2}\pi x) + \sum_{-\infty}^{\infty}{}' (-1)^n \left\{\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right\}.$$

In each of these, change x to $x+\iota y$, and equate the imaginary parts on the two sides.]

24. Shew that

$$\arctan\frac{x}{1-x^2} - \arctan\frac{3x}{3^2-x^2} + \arctan\frac{5x}{5^2-x^2} - \dots = \arctan\left\{\frac{\sinh(\frac{1}{4}\pi x)}{\cos(\frac{1}{4}\pi x\sqrt{3})}\right\}.$$

[*Math. Trip.* 1891.]

[It is easy to prove that

$$\arctan\frac{ax}{a^2-x^2} = \arctan\left(\frac{x}{2a+x\sqrt{3}}\right) + \arctan\left(\frac{x}{2a-x\sqrt{3}}\right),$$

and it follows that the given series is equal to

$$\operatorname{arc\,tan}\frac{x}{2+y}+\sum_{-\infty}^{\infty}{}'(-1)^n\left\{\operatorname{arc\,tan}\left(\frac{x}{4n+2+y}\right)-\frac{x}{4n}\right\},$$

where $y=x\sqrt{3}$. Applying the result of Ex. 23, we find the formula given.]

25. The points P, Q have coordinates (p, q), $(-p, q)$ respectively; N is the point with coordinates $(na, 0)$. Shew that if

$$\theta=\sum_{-\infty}^{\infty}P\hat{N}Q,$$

then $$\tan\tfrac{1}{2}\theta=\tan(\pi p/a)\coth(\pi q/a).$$ [*Math. Trip.* 1894.]

[If we write $p+\iota q=x$, $-p+\iota q=y$, it will be found that

$$\iota\theta=\log\sin(\pi x/a)-\log\sin(\pi y/a).]$$

26. Verify that, if x is a positive integer,

$$(a+b)^{n-1}+(a+2b)^{n-1}+\dots+(a+xb)^{n-1}$$
$$=\{(a+xb+Bb)^n-(a+Bb)^n\}/nb,$$

where we are to put $B^{2s}=B_s$, $B^{2s+1}=0$ after expansion. [*Math. Trip.* 1897.]

[Compare Art. 107.]

27. Shew that

$$\frac{x}{2}\cot\frac{x}{2}=1-B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}-B_3\frac{x^6}{6!}-\dots,$$

and that $$\log\frac{x}{2\sin(\frac{1}{2}x)}=\frac{B_1}{2}\frac{x^2}{2!}+\frac{B_2}{4}\frac{x^4}{4!}+\frac{B_3}{6}\frac{x^6}{6!}+\dots.$$

28. Shew that $$\log\frac{\cosh x-\cos x}{x^2}=-\sum 2^{n+1}\cos(\tfrac{1}{2}n\pi)\frac{B_n}{2n}\frac{x^{2n}}{(2n)!}.$$

[*Math. Trip.* 1890.]

29. Assuming Stirling's formula (Art. 179), shew that

$$B_n\sim 4\sqrt{(\pi n)}(n/\pi e)^{2n},$$

when n is large.

Prove also that $$B_n/B_{n-1}\sim n^2/\pi^2.$$

[Compare Art. 106, below.]

Applications of Art. 88.

30. If $$f(x)=1+\nu x+\nu(\nu-1)\frac{x^2}{2!}+\nu(\nu-1)(\nu-2)\frac{x^3}{3!}+\dots,$$

it is easily verified that, with the notation of Art. 88,

$$f_1(x)=\nu\left[1+(\nu-1)x+(\nu-1)(\nu-2)\frac{x^2}{2!}+\dots\right]=\nu f(x)/(1+x),$$

$$f_2(x)=\nu(\nu-1)\left[1+(\nu-2)x+(\nu-2)(\nu-3)\frac{x^2}{2!}+\dots\right]=\nu(\nu-1)f(x)/(1+x)^2,$$

and so on. Thus, we obtain the transformation

$$f(x_1) = f(x) + \frac{\nu f(x)}{1+x}(x_1 - x) + \frac{\nu(\nu-1) f(x)}{(1+x)^2}\frac{(x_1-x)^2}{2!} + \dots,$$

or

$$f(x_1) = f(x) f\left(\frac{x_1 - x}{1+x}\right).$$

The two series on the right-hand are both convergent if $|x| < 1$ and $|x_1 - x| < |1+x|$, and the latter condition is satisfied for some points x_1 which are outside the circle $|x_1| = 1$; we have thus obtained a *continuation* of the binomial series. Repeating the process, we obtain

$$f(x_n) = f(x) f\left(\frac{x_1 - x}{1+x}\right) f\left(\frac{x_2 - x_1}{1+x_1}\right) \dots f\left(\frac{x_n - x_{n-1}}{1+x_{n-1}}\right),$$

where we assume that the broken line from x to x_n is drawn so that

$$|x_{r+1} - x_r| < |1 + x_r| \quad (r = 0, 1, \dots, n-1).$$

For example, by taking

$$1 + x = 1, \quad 1 + x_1 = \frac{1+i}{\sqrt{2}}, \quad 1 + x_2 = i, \quad 1 + x_3 = \frac{-1+i}{\sqrt{2}}, \quad 1 + x_4 = -1,$$

we find
$$\frac{x_{r+1} - x_r}{1 + x_r} = \frac{1+i}{\sqrt{2}} - 1, \quad \left|\frac{x_{r+1} - x_r}{1 + x_r}\right|^2 = 2 - \sqrt{2} < 1,$$

so that
$$f\left(\frac{x_{r+1} - x_r}{1 + x_r}\right) = e^{-\frac{1}{4}\beta\pi}[\cos(\tfrac{1}{4}\alpha\pi) + i\sin(\tfrac{1}{4}\alpha\pi)],$$

where
$$\nu = \alpha + i\beta.$$

Thus we are led to $f(-2) = e^{-\beta\pi}[\cos(\alpha\pi) + i\sin(\alpha\pi)]$.

But it should be noticed that if we take a broken line passing *below* the real axis, we find $f(-2) = e^{\beta\pi}[\cos(\alpha\pi) - i\sin(\alpha\pi)]$; we thus obtain two different values for $f(-2)$ by approaching -2 along different paths. This indicates (what we know to be the case) that $f(x)$ is many-valued unless $\beta = 0$ and α is an integer.

31. A method similar to the last example can be applied to

$$\phi(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots,$$

for which we find

$$\phi_1(x) = (1+x)^{-1}, \quad \phi_2(x) = -(1+x)^{-2}, \quad \phi_3(x) = (2!)(1+x)^{-3}, \dots,$$

and so
$$\phi(x_1) = \phi(x) + \phi\left(\frac{x_1 - x}{1+x}\right).$$

Using the same points x, x_1, x_2, x_3, x_4, as in the last example, we get $\phi(-2) = i\pi$. And with a broken line passing below the real axis, we get $\phi(-2) = -i\pi$.

CHAPTER XII.

ASYMPTOTIC SERIES AND TRIGONOMETRICAL SERIES.

104. Historical remarks on the use of non-convergent series.*

Before the methods of Analysis had been put on a sure footing, and in particular before the theory of convergence had been developed by Abel and Cauchy, mathematicians had little hesitation in using non-convergent series in both theoretical and numerical investigations.

In numerical work, however, they naturally used only series which are now called *asymptotic*; in such series the terms begin to decrease, and reach a minimum, afterwards increasing. If we take the sum to a stage at which the terms are sufficiently small, we may hope to obtain an approximation with a degree of accuracy represented by the last term retained; and it can be proved that this is the case with many series which are convenient for numerical calculations (see Art. 106 for examples).

An important class of such series consists of the series used by astronomers to calculate the planetary positions: it has been proved by Poincaré† that these series do not converge, but yet the results of the calculations are confirmed by observation. The explanation of this fact may be inferred from Poincaré's theory of asymptotic series (Art. 113).

But mathematicians have often been led to employ series of a

* The majority of writers on these series use the word *divergent* as including *oscillatory* series; we shall, however, except in quotations, adopt the same distinction as in the previous part of the book.

† *Acta Mathematica*, vol. 13, 1890; in particular § 13.

different character, in which the terms never decrease, and may increase to infinity. Typical examples of such series are:

(1) $1-1+1-1+1-1+\dots;$

(2) $1-2+3-4+5-6+\dots;$

(3) $1-2+2^2-2^3+2^4-2^5+\dots;$

(4) $1-2!+3!-4!+5!-6!+\dots.$

Euler considered the "sum" of a non-convergent series as the finite numerical value of the arithmetical expression from the expansion of which the series was derived. Thus he defined the "sums" of the series (1)–(3) as follows:

$$(1)=\frac{1}{1+1}=\frac{1}{2};\quad (2)=\frac{1}{(1+1)^2}=\frac{1}{4};\quad (3)=\frac{1}{1+2}=\frac{1}{3};$$

and his discussion of the series (4) will be found at the end of Art. 105.

In principle, Euler's definition depends on the inversion of two limits, which, taken in one order, give a definite value, and taken in the reverse order give a non-convergent series. Thus series (1) is

$$\lim 1-\lim x+\lim x^2-\lim x^3+\dots$$

as x tends to 1; Euler's definition replaces this by

$$\lim (1-x+x^2-x^3+\dots).$$

So, generally, if $\Sigma f_n(c)$ is not convergent, Euler would define the "sum" as $\lim_{x\to c}\Sigma f_n(x)$, when this limit is definite.

A very natural method for the numerical evaluation of non-convergent series is given by Euler's transformation (Art. 24); as an illustration we take the series used by Euler:

(5) $\log_{10}2-\log_{10}3+\log_{10}4-\dots.$

Starting at $\log_{10}10$, the differences are given in the table below (to the third order):

$\log_{10}10=$	1·0000000			
		−413927		
11	1·0413927		−36042	
		−377885		−5779
12	1·0791812		−30263	
		−347622		−4487
13	1·1139434		−25776	
		−321846		−3563
14	1·1461280		−22213	
		−299633		−2867
15	1·1760913		−19346	
		−280287		
16	1·2041200			

The next three differences (at the top) are -1292, -368, -140, as the reader can verify without much trouble.

Thus the "sum" from $\log_{10} 10$ onwards is found to be

$$\begin{aligned}&\cdot 5000000 - \{\tfrac{1}{4}(\cdot 0413927) + \tfrac{1}{8}(\cdot 0036042)\\ &\qquad + \tfrac{1}{16}(\cdot 0005779) + \tfrac{1}{32}(\cdot 0001292)\\ &\qquad + \tfrac{1}{64}(\cdot 0000368) + \tfrac{1}{128}(\cdot 0000140)\}\\ &= \cdot 5000000 - (\cdot 0108396).\end{aligned}$$

The sum of the first eight terms in the series is found to be (taking the terms in pairs)

$$\begin{aligned}&-\cdot 1760913 - \cdot 0969100 - \cdot 0669467 - \cdot 0511525\\ &= -\cdot 3911005.\end{aligned}$$

Combining these two results, the sum of the whole series appears to be

$$\begin{aligned}&\cdot 1088995 - \cdot 0108396\\ &= \cdot 098060 \text{ to six places of decimals.}\end{aligned}$$

This agrees with the evaluation given in Ex. 44, Ch. X.

The reader will find no difficulty in carrying out similar calculations for series (1), (2), (3); and the results agree with those quoted above.

Euler also attempted to evaluate (4) in this way; his result was $\cdot 4008\ldots$, agreeing to two places of decimals with those of Arts. 105, 109. But it is hard to feel convinced of the accuracy of the method here (since the corresponding power-series *cannot* converge).

To Euler's definition an objection was raised by Callet that the series (1) can also be obtained by writing $x=1$ in the series

$$(6) \qquad \frac{1+x}{1+x+x^2} = \frac{1-x^2}{1-x^3} = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + \ldots,$$

whereas the left-hand side then becomes $\frac{2}{3}$ instead of $\frac{1}{2}$.

This objection of Callet's was met by a remark of Lagrange's, who suggested that the series (6) should be written as

$$1 + 0 - x^2 + x^3 + 0 - x^5 + x^6 + 0 - x^8 + \ldots,$$

and that then the derived series would be

$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \ldots.$$

The last series gives the sums to 1, 2, 3, 4, 5, 6, ... terms as 1, 1, 0, 1, 1, 0, ...; so that the *average sum* * is $\frac{2}{3}$, which is the value

* This remark of Lagrange's has been put on a more satisfactory basis by the theorem of Frobenius (*Crelle's Journal für Math.*, vol. 89, 1880, p. 262), which was given in Art. 51, Ex. 2, above; namely, that

$$\lim_{x \to 1} (\Sigma a_n x^n) = \lim_{n \to \infty} \frac{s_0 + s_1 + \ldots + s_n}{n+1}.$$

In applying the theorem to the special series above we note that the sum $s_0 + s_1 + \ldots + s_n = (n+1) - k$, where k is the integral part of $\frac{1}{3}(n+1)$; thus

$$\lim (s_0 + s_1 + \ldots + s_n)/(n+1) = 1 - \lim k/(n+1) = \tfrac{2}{3}.$$

given by the left-hand side of (6). In the original series (1), the sums are 1, 0, 1, 0, 1, 0, ..., of which the average is $\frac{1}{2}$, agreeing with Euler's sum.

Having regard to the fact that Euler and other mathematicians made numerous discoveries by using series which do not converge, we may agree with Borel in the statement that the older mathematicians had sufficiently good *experimental* evidence that the use of such series as if they were convergent led to correct results* in the majority of cases when they presented themselves *naturally*.

A simple example of the use of a non-convergent series to obtain a correct result is afforded by a passage in Fourier's *Théorie Analytique de la Chaleur* (*Oeuvres*, t. 1, p. 206). Fourier is obtaining what we should now call a Fourier sine-series for the function $f(x) = \dfrac{\pi \sinh x}{2 \sinh \pi}$, and he finds that the coefficient of $\sin nx$ is

$$(-1)^{n-1}\left(\frac{1}{n} - \frac{1}{n^3} + \frac{1}{n^5} - \dots\right) = (-1)^{n-1} \frac{n}{1+n^2}.$$

[Compare Ex. B, 13 below.]

Thus the coefficient of $\sin x$ appears as $1-1+1-1+\dots$, and may therefore be expected to be $\frac{1}{2}$, if we adopt Euler's principle.

As a matter of fact this is correct, since

$$\int \sinh x \sin x \, dx = \tfrac{1}{2}(\cosh x \sin x - \sinh x \cos x),$$

so that

$$\frac{2}{\pi}\int_0^\pi f(x) \sin x \, dx = \tfrac{1}{2}.$$

[Compare series (4·2) of Art. 124 below.]

Abel and **Cauchy**, however, pointed out that the use of non-convergent series had sometimes led to gross errors; and, in their anxiety to place mathematical analysis on the firmest foundations, they felt obliged to banish non-convergent series from their work. But this was not done without hesitation; thus Abel writes to his former teacher Holmboë in 1826 (*Oeuvres d'Abel*, 2me. éd. t. 2, p. 256): "Les séries divergentes sont, en général, quelque chose de bien fatal, et c'est une honte qu'on ose y fonder aucune démonstration . . . la partie la plus essentielle des Mathématiques est sans fondement. Pour la plus grande partie, les résultats sont justes, il est vrai, mais c'est là une chose bien étrange. Je m'occupe à en chercher la raison, problème très intéressant."

* Borel, *Leçons sur les Séries Divergentes*, p. 9; the sketch given above is taken, with a few additions, from pp. 1–10 of this book.

And Cauchy, in the preface of his *Analyse Algébrique* (1821), writes: "J'ai été forcé d'admettre diverses propositions qui paraîtront peut-être un peu dures: par exemple, qu'une série divergente n'a pas de somme." *

Cauchy established the asymptotic property of Stirling's series (see Art. 108 below), by means of a method which can be applied to a large class of power-series. But the possibility of obtaining other useful asymptotic series was generally overlooked by later analysts; and after the time of Cauchy, workers in the regions of analysis for the most part abandoned all attempts at utilising non-convergent series. In England, however, Stokes published three remarkable papers † (dated 1850, 1857, 1868), in which Cauchy's method for dealing with Stirling's series was applied to a number of other problems, such as the calculation of Bessel's functions for large values of the variable.

But no general theory of non-convergent series was forthcoming until 1886, when papers discussing the subject were written by Stieltjes ‡ and Poincaré. § Since that time many researches have been published on the theory.

In the following articles we shall confine our exposition to the more important examples of asymptotic series, which have been found of importance in calculation—both for Pure and Applied Mathematics.

105. General considerations on non-convergent series.

In view of the results obtained in the past by the use of non-convergent series, it seems probable that we can attach a perfectly precise meaning to a non-convergent series, so that such series may be used for purposes of formal calculation, under proper restrictions. Thus we attempt to formulate rules which enable us, given a series

$$u_0+u_1+u_2+\dots$$

* Of course no one would now regard Cauchy's statement as unusual, *in the sense in which he made it.*

† See the references of Arts. 113, 116–118 below.

‡ *Annales de l'École Normale Supérieure* (3), t. 3, p. 201; we do not propose to set forth the theory of Stieltjes here. The reader may consult Van Vleck's book for a full account of this theory.

§ *Acta Mathematica*, t. 8, p. 295 (for Poincaré's theory see Art. 113 below).

(convergent or not), to associate with it a number, which is a perfectly definite function of u_0, u_1, u_2, ..., and which we call the "sum" of the series. It is of course obvious that the definition chosen is to a large extent arbitrary; but it should be such that the resulting laws of calculation agree, as far as possible, with those of convergent series.

Of course it is evident that the "sum" associated with a non-convergent series is not to be confounded with the sum of a convergent series (in the sense of Art. 6); but it will avoid confusion if the definition is such that the same operation, when applied to a convergent series, yields the sum in the ordinary sense.

It ought to be pointed out that Euler, at any rate, was perfectly aware of the distinction between his "sum" of a non-convergent series and the sum of a convergent series. Thus he says (§§ 108–111 of the *Instit. Calc. Diff.*, 1755) that the series

$$1 - 2 + 2^2 - 2^3 + 2^4 - \ldots = \frac{1}{1+2} = \frac{1}{3}$$

obviously cannot have the sum $\frac{1}{3}$ in the ordinary sense, since the sum of n terms is $S_n = \frac{1}{3}\{1 - (-2)^n\}$, and the larger n is, the more does S_n differ from $\frac{1}{3}$. And he adds, after referring to various difficulties, that contradictions can be avoided by attributing a somewhat different meaning to the word *sum*. "Let us say, therefore, that the *sum* of any infinite series is the finite expression, by the expansion of which the series is generated. In this sense the sum of the infinite series $1 - x + x^2 - x^3 + \ldots$ will be $\frac{1}{1+x}$, because the series arises from the expansion of the fraction, whatever number is put in place of x. If this is agreed, the new definition of the word *sum* coincides with the ordinary meaning when a series converges; and since divergent series have no sum, in the proper sense of the word, no inconvenience can arise from this new terminology. Finally, by means of this definition, we can preserve the utility of divergent series and defend their use from all objections."

In writing to N. Bernoulli (*L. Euleri Opera Posthuma*, t. 1, p. 536), Euler adds that he had had grave doubts as to the use of divergent series, but that he had never been led into error by using his definition of "sum." To this Bernoulli replies that the same series might arise from the expansion of more than one expression, and that if so, the "sum" would not be definite; to which Euler rejoins that he does not believe that any example of this could be given. However, Pringsheim (*Encyklopädie*, Bd. I., A. 3, 39) has given a number of examples to shew that Euler fell into error here; but in practice Euler used his definition almost exclusively in the form

$$\sum u_n = \lim_{x \to 1} (\sum u_n x^n),$$

and if restricted to this case, Euler's statement is correct.

It will be seen from these passages that Euler had views which do not differ greatly, at bottom, from those held by modern workers on this subject; although of course his attempted definition leaves something to be desired, in the light of modern analysis.

It is to be carefully borne in mind that the legitimate use of non-convergent series is always symbolic; the operations being merely convenient abbreviations of more complicated transformations in the background. Naturally this "shorthand representation" does not enable us to avoid the labour of justifying the various steps employed; but when general rules have been laid down and firmly established we may apply them with confidence in any particular case.

It may very likely be urged that we might just as well write the work in full, and so avoid all risk of misinterpretation. But experience shews that the use of the series frequently suggests suitable transformations which otherwise might never be thought of.

An example of this may be taken from Euler's correspondence with Nicholas Bernoulli (*L. Euleri Opera Posthuma*, t. 1, p. 547); where the real object of Euler's work is to shew how to attach a definite meaning to the series (4) of Art. 104.

He proves first that the series

$$x-(1!)x^2+(2!)x^3-(3!)x^4+\dots$$

satisfies *formally* the differential equation

$$x^2\frac{dy}{dx}+y=x,$$

from which he obtains the integral

$$y=\int_0^x e^{\frac{1}{x}-\frac{1}{\xi}}\frac{d\xi}{\xi}.$$

Or, if

$$\frac{1}{\xi}-\frac{1}{x}=t,$$

we find

$$y=\int_0^\infty \frac{xe^{-t}}{1+xt}\,dt,$$

in agreement with the result found in Art. 109 (2) below.

On the other hand, by using the rules which he had obtained for the transformation of convergent series into continued fractions, Euler gets

$$\frac{x}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\,\frac{3x}{1+}\,\frac{3x}{1+}\cdots,$$

and it has since been proved by Laguerre* and Stieltjes that we have actually

$$\int_0^\infty \frac{xe^{-t}}{1+xt}\,dt=\frac{x}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\cdots.$$

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* *Bulletin de la Société Math. de France*, t. 7, 1879, p. 72.

Now this relation does not suggest itself at all naturally without the use of the series; and, as already remarked, it is evident that Euler's work was entirely guided by the aim of evaluating the series (4).

Writing $x=1$, Euler obtains from the continued fraction the convergents

$$1 > \tfrac{2}{3} > \tfrac{8}{13} > \tfrac{44}{73} > \ldots > \tfrac{20}{34} > \tfrac{4}{7} > \tfrac{1}{2},$$

and by using the 13th and 14th of these, he infers the numerical value 0·5963... .

He then subtracts this decimal from 1 and infers that the value of the series (4) is 0·4036... (compare Art. 109 below).

ASYMPTOTIC SERIES.

106. Euler's use of asymptotic series.

One of the earliest and most instructive examples of the application of non-convergent series was given by Euler in applying his formula of summation (Art. 102) to calculate certain finite sums.*

Thus, taking $f(x)=1/x$, $x=n$, Euler finds

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}=\log n+\frac{1}{2n}-\frac{B_1}{2n^2}+\frac{B_2}{4n^4}-\frac{B_3}{6n^6}+\ldots+\text{const.}$$

Now this series, if continued to infinity, does not converge, because we have (Art. 99)

$$\frac{B_r}{B_{r-1}}=\frac{2r(2r-1)}{4\pi^2}\frac{\Sigma\dfrac{1}{n^{2r}}}{\Sigma\dfrac{1}{n^{2r-2}}};$$

but, if $r>3$, $\Sigma\dfrac{1}{n^{2r-2}}<1\Big/\left(1-\dfrac{1}{2^4}\right)$ (see Art. 7), and $\Sigma\dfrac{1}{n^{2r}}>1$,

so that
$$\frac{B_r}{2rn^{2r}}\Big/\frac{B_{r-1}}{2(r-1)n^{2r-2}}>\frac{15(r-1)^2}{16n^2\pi^2};$$

hence the terms in the series steadily increase in numerical value after a certain value of r (depending on n and roughly equal to the integer next greater than $1+n\pi$). It does not appear whether Euler realised that the series could never converge; but he was certainly aware of the fact that it does not converge for $n=1$. He employed

* *Inst. Calc. Diff.*, 1755 (Pars Posterior), cap. vi.

the series for $n=10$ to calculate the constant (*Euler's constant,** Art. 11),

$$C=0{\cdot}5772156649015328(6060)\ldots,$$

which he regarded as the "sum" of the series

$$\frac{1}{2}+\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}-\ldots.$$

The reason why this series can be used, although not convergent, is that *the error in the value obtained by stopping at any particular stage in the series, is less than the next term in the series.* The truth of this statement follows at once from the general theorem proved below (see Art. 107) by observing that $f^{2n}(x)$ is always positive.

To illustrate this point, consider the sums of the last series, and we find successively

$$\begin{aligned} S_2 &= {\cdot}5833, & R_2 &= -{\cdot}0061, & u_3 &= -{\cdot}0083,\\ S_3 &= {\cdot}5750, & R_3 &= +{\cdot}0022, & u_4 &= +{\cdot}0040,\\ S_4 &= {\cdot}5790, & R_4 &= -{\cdot}0018, & u_5 &= -{\cdot}0042,\\ S_5 &= {\cdot}5748, & R_5 &= +{\cdot}0024, & u_6 &= +{\cdot}0076,\\ S_6 &= {\cdot}5824, & R_6 &= -{\cdot}0052, \end{aligned}$$

after which the terms steadily increase in numerical value. Thus, from this series we cannot obtain a closer approximation than S_4, which corresponds to stopping at the numerically *least* term u_4.

This fact enables us to see at once that all of Euler's results are correct, after making a few unimportant changes.

We quote a few of Euler's results for verification:

Ex. 1. $$1+\frac{1}{2}+\ldots+\frac{1}{n}=7{\cdot}48547, \quad \text{if } n=1000,$$
$$=14{\cdot}39273, \quad \text{if } n=1000000.$$

Euler gives the values to 13 decimals.

Ex. 2. Shew that

$$1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}=\frac{1}{2}(C+\log n)+\log 2+\frac{B_1}{8n^2}-\frac{(2^3-1)B_2}{64n^4}+\ldots,$$

and that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=\log 2-\frac{1}{4n}+\frac{(2^2-1)B_1}{8n^2}-\frac{(2^4-1)B_2}{64n^4}+\ldots.$$

---

* The four additional figures in brackets were found by Gauss. By writing $n=500$ and 1000 in the series, J. C. Adams has calculated C to 260 places (*Proc. Roy. Soc.*, vol. 27, 1878; and *Math. Papers*, vol. 1, p. 459). This requires a knowledge of $B_1, B_2, \ldots, B_{62}$ which had been tabulated by Adams previously (*Math. Papers*, vol. 1, pp. 453, 455).

Ex. 3. By taking $n=5$ in the second formula of Ex. 2, we find that

$$\log 2 = \left(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots-\tfrac{1}{10}\right)+\tfrac{1}{20}-\tfrac{1}{400}+\tfrac{1}{80000},$$

with an error less than $\frac{1}{4}\times 10^{-6}$ (the next term in the series). This gives

$$\cdot 6456349 + \cdot 05 - \cdot 0025 + \cdot 0000125 - \cdot 00000025 = \cdot 693147,$$

which is correct to six places of decimals. By taking more terms we can easily calculate the value of log 2 to ten places.

Ex. 4. Find a formula for

$$\frac{1}{a+b}+\frac{1}{2a+b}+\frac{1}{3a+b}+\ldots+\frac{1}{na+b}$$

similar to Euler's formula.

Ex. 5. Taking $f(x)=1/x^2$, prove similarly that

$$\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots \text{ to } \infty = \frac{1}{n}+\frac{1}{2n^2}+\frac{B_1}{n^3}-\frac{B_2}{n^5}+\frac{B_3}{n^7}-\ldots.$$

Hence we find $\quad \dfrac{1}{10^2}+\dfrac{1}{11^2}+\dfrac{1}{12^2}+\ldots \text{ to } \infty = \cdot 1051663357,$

and we deduce that $\qquad \dfrac{\pi^2}{6} = 1\cdot 6449340668.$

Ex. 6. Shew similarly that

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\ldots = 1\cdot 2020569032.$$

Euler obtained in this manner the numerical values of $\sum 1/n^r$ from $r=2$ to 16, each calculated to 18 decimals (*l.c.* p. 456); Stieltjes has carried on the calculations to 32 decimals from $r=2$ to 70 (*Acta Math.*, vol. 10, p. 299). The values to 10 decimals (for $r=2$ to 9) are quoted in Chrystal's *Algebra*, vol. 2, p. 367.

Ex. 7. If $f(x)=1/(l^2+x^2)$, prove that

$$\frac{1}{l^2+1}+\frac{1}{l^2+2^2}+\ldots+\frac{1}{l^2+n^2}$$

$$=\frac{1}{l}\left(\frac{\pi}{2}-\theta\right)-\frac{1}{2}\left(\frac{1}{l^2}-\frac{1}{l^2+n^2}\right)+\frac{\pi}{l(e^{2l\pi}-1)}-\frac{B_1}{2}\frac{\sin^2\theta\sin 2\theta}{l^3}+\frac{B_2}{4}\frac{\sin^4\theta\sin 4\theta}{l^5}-\ldots,$$

where $\tan\theta = l/n$; the constant is determined by allowing n to tend to ∞ and using the series found at the end of Art. 99.

Ex. 8. In particular, by writing $l=n$ (in Ex. 7), we find

$$\pi = 4n\left(\frac{1}{n^2+1}+\frac{1}{n^2+2^2}+\ldots+\frac{1}{n^2+n^2}\right)+\frac{1}{n}-\frac{4\pi}{e^{2n\pi}-1}$$

$$+\frac{B_1}{1\,.\,n^2}-\frac{B_3}{3\,.\,2^2\,.\,n^6}+\frac{B_5}{5\,.\,2^4\,.\,n^{10}}-\frac{B_7}{7\,.\,2^6\,.\,n^{14}}+\ldots.$$

By writing $n=5$, Euler calculates the value of π to 15 decimals.

107. The remainder in Euler's formula.

We have seen (Art. 101) that the Bernoullian polynomials $\phi_n(t)$ satisfy the following relations:

$$\phi'_{2m}(t) = 2m\phi_{2m-1}(t) \qquad (m > 1),$$
$$\phi'_{2m+1}(t) = (2m+1)\{\phi_{2m}(t) + (-1)^{m-1}B_m\} \quad (m \geqq 1),$$

and, further, that $\phi_n(t)$ is zero both for $t=0$ and $t=1$.

It follows that

$$\int_0^1 \phi_{2n}(t)\,F''(t)\,dt = \Big[F'(t)\,\phi_{2n}(t)\Big]_0^1 - \int_0^1 2n\phi_{2n-1}(t)\,F'(t)\,dt$$
$$= -\int_0^1 2n\phi_{2n-1}(t)\,F'(t)\,dt.$$

Similarly,

$$\int_0^1 \phi_{2n-1}(t)\,F'(t)\,dt = -(2n-1)\int_0^1 \{\phi_{2n-2}(t) + (-1)^{n-2}B_{n-1}\}\,F(t)\,dt.$$

Combining these two results, we see that

$$\int_0^1 \phi_{2n}(t)\,F''(t)\,dt - 2n(2n-1)\int_0^1 \phi_{2n-2}(t)\,F(t)\,dt$$
$$= 2n(2n-1)(-1)^n B_{n-1}\int_0^1 F(t)\,dt.$$

Thus, if we write

$$X_n = \frac{1}{(2n)!}\int_0^1 \phi_{2n}(t)\,f^{2n}(x+t)\,dt,$$

we have the result

$$X_n - X_{n-1} = (-1)^n \frac{B_{n-1}}{(2n-2)!}\{f^{2n-3}(x+1) - f^{2n-3}(x)\}.$$

This relation holds for values of $n > 1$; to complete the sequence, consider the integral

$$X_1 = \tfrac{1}{2}\int_0^1 \phi_2(t)\,f''(x+t)\,dt$$
$$= \tfrac{1}{2}\int_0^1 (t^2 - t)\,f''(x+t)\,dt.$$

Transforming by a similar process, we get

$$X_1 = -\int_0^1 (t-\tfrac{1}{2})\,f'(x+t)\,dt$$
$$= -\Big[(t-\tfrac{1}{2})\,f(x+t)\Big]_0^1 + \int_0^1 f(x+t)\,dt.$$

Thus we find

$$\tfrac{1}{2}\{f(x+1) + f(x)\} = \int_0^1 f(x+t)\,dt - X_1 = \int_x^{x+1} f(\xi)\,d\xi - X_1,$$

and from the general formula we have

$$-X_1+X_2=\frac{B_1}{2!}\{f'(x+1)-f'(x)\}$$

$$-X_2+X_3=-\frac{B_2}{4!}\{f'''(x+1)-f'''(x)\}, \text{ etc.}$$

That is, we have successively

$$\tfrac{1}{2}\{f(x+1)+f(x)\}=\int_x^{x+1} f(\xi)\,d\xi+\frac{B_1}{2!}\{f'(x+1)-f'(x)\}-X_2$$

$$=\int_x^{x+1} f(\xi)\,d\xi+\frac{B_1}{2!}\{f'(x+1)-f'(x)\}$$

$$-\frac{B_2}{4!}\{f'''(x+1)-f'''(x)\}-X_3, \text{ etc.}$$

If we now write $x=a, a+1, \ldots, b-1$, where $b-a$ is any positive integer, and add the results, we obtain *Euler's summation formula* (as in Art. 102), but *with a remainder term.* Thus

$$f(a)+f(a+1)+\ldots+f(b)=\int_a^b f(\xi)\,d\xi+\tfrac{1}{2}\{f(a)+f(b)\}$$

$$+\frac{B_1}{2!}\{f'(b)-f'(a)\}-\frac{B_2}{4!}\{f'''(b)-f'''(a)\}$$

$$+\ldots+(-1)^n\frac{B_{n-1}}{(2n-2)!}\{f^{2n-3}(b)-f^{2n-3}(a)\}$$

$$-R_n,$$

where

$$R_n=\frac{1}{(2n)!}\int_0^1 \phi_{2n}(t)\{f^{2n}(a+t)+f^{2n}(a+1+t)+\ldots+f^{2n}(b-1+t)\}\,dt.$$

It is to be noticed also that

$$R_n-R_{n+1}=(-1)^n\frac{B_n}{(2n)!}\{f^{2n-1}(b)-f^{2n-1}(a)\},$$

which gives the next term in Euler's summation formula.

Now it has been proved (Art. 101) that the Bernoullian polynomials $\phi_{2n}(t)$ and $\phi_{2n+2}(t)$ are both of constant sign, but their signs are *opposite*. Thus, *if we assume that the signs of* $f^{2n}(x)$, $f^{2n+2}(x)$ *are the same and that their common sign remains constant for all values of* x *from* a *to* b, the integrals R_{n+1}, R_n have *opposite* signs.

Hence $|R_n|<|R_n-R_{n+1}|<\dfrac{B_n}{(2n)!}|f^{2n-1}(b)-f^{2n-1}(a)|.$

Thus *the error involved in omitting* R_n *from Euler's summation formula is numerically less than the next term, and has the same sign* ;

that is, the series so obtained has the same property as a convergent series of decreasing terms which have alternate signs. Theoretically, however, the convergent series can be pushed to an *arbitrary* degree of approximation, while an asymptotic series cannot; but in practice it often happens that an asymptotic series gives a better approximation for numerical work than a convergent series, as in Exs. 3, 5, 6 of the last article.

108. Application of Euler's formula to Stirling's series.

Taking $f(x)=\log x$ in the general formula, we find that

$$\log(n!)=\int_1^n \log x\,dx+\tfrac{1}{2}\log n+\frac{B_1}{2}\frac{1}{n}-\frac{B_2}{3\,.\,4}\frac{1}{n^3}$$
$$+\frac{B_3}{5\,.\,6}\frac{1}{n^5}-\ldots+\text{const.},$$

where the error at each stage is numerically less than the next term, because $f^{2n}(x)$ is negative (for all positive values of x).

This gives, on integration,

$$\log(n!)=C_1+(n+\tfrac{1}{2})\log n-n+\frac{B_1}{1\,.\,2}\frac{1}{n}-\frac{B_2}{3\,.\,4}\frac{1}{n^3}+\ldots.$$

To find the constant C_1, we use Wallis's formula (Art. 70), which gives

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\ldots\text{ to }\infty=\lim_{n\to\infty}\frac{(2^n n!)^4}{(2n!)^2(2n+1)}.$$

Thus $\log(\tfrac{1}{2}\pi)=\lim_{n\to\infty}\{4n\log 2+4\log(n!)-2\log\{(2n)!\}-\log(2n)\}.$

Now our general formula gives

$$2\log(n!)-\log\{(2n)!\}=C_1+(2n+1)\log n-2n$$
$$-(2n+\tfrac{1}{2})\log(2n)+2n+O\left(\frac{1}{n}\right)$$
$$=C_1+\tfrac{1}{2}\log n-(2n+\tfrac{1}{2})\log 2+O\left(\frac{1}{n}\right).$$

Hence

$$\log(\tfrac{1}{2}\pi)=\lim_{n\to\infty}\{4n\log 2+2C_1+\log n-(4n+1)\log 2-\log(2n)\}$$
$$=2C_1-2\log 2.$$

Thus * $$C_1=\tfrac{1}{2}\log(2\pi).$$

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* This value follows from Art. 179 in Appendix III. The reader may be warned against attempting to deduce Wallis's product from Stirling's formula; this would be an illustration of the old fallacy *ignotum per ignotius*.

Hence we have Stirling's series

$$\log(n!)=(n+\tfrac{1}{2})\log n-n+\tfrac{1}{2}\log(2\pi)+\frac{B_1}{1.2n}-\frac{B_2}{3.4n^3}+\dots$$

in which, so far, n denotes a positive integer.

To obtain the series for $\log\{\Gamma(1+x)\}$, we use the product-formula of Art. 42. Applying Euler's summation-formula from x to $x+n$, we have

$$\log\{x(1+x)\dots(n+x)\}=\int_x^{x+n}\log\xi\,d\xi+\tfrac{1}{2}\{\log x+\log(x+n)\}$$
$$-\frac{B_1}{1.2x}+\frac{B_2}{3.4x^3}-\dots+O\left(\frac{1}{n}\right).$$

Thus, subtracting this from Stirling's formula for $\log(n!)$, we find

$$-\log\left\{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\dots\left(1+\frac{x}{n}\right)\right\}=-\int_x^{x+n}\log\xi\,d\xi+\int_0^n\log\xi\,d\xi$$
$$+\tfrac{1}{2}\log x+\tfrac{1}{2}\log(2\pi)$$
$$+\frac{B_1}{1.2x}-\frac{B_2}{3.4x^3}+\dots+O\left(\frac{1}{n}\right).$$

The difference of the two integrals in the last formula is equal to

$$\int_0^x\log\xi\,d\xi-\int_n^{n+x}\log\xi\,d\xi=(x\log x-x)-x\log n-\int_0^x\log\left(1+\frac{\eta}{n}\right)d\eta,$$

putting $\xi=n+\eta$,

$$=(x\log x-x)-x\log n-\dots+O\left(\frac{1}{n}\right).$$

Thus we find that

$$x\log n-\log\left\{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\dots\left(1+\frac{x}{n}\right)\right\}$$
$$=(x+\tfrac{1}{2})\log x-x+\tfrac{1}{2}\log(2\pi)+\frac{B_1}{1.2x}-\frac{B_2}{3.4x^3}+\dots+O\left(\frac{1}{n}\right).$$

Now, when $n\to\infty$, the left-hand side tends to $\log\{\Gamma(1+x)\}$ by Art. 42, and so we have the result

$$\log\{\Gamma(1+x)\}=(x+\tfrac{1}{2})\log x-x+\tfrac{1}{2}\log(2\pi)+\frac{B_1}{1.2x}-\frac{B_2}{3.4x^3}+\dots,$$

which, as might perhaps have been anticipated, is of exactly the same form as the series originally found for $\log(n!)$. An independent discussion of this result will be found in Art. 111 below.

It is often useful to have a slightly modified form of the result which can be used when x is of the form

$$x=m+p,$$

where m is large (not necessarily an integer), and p may also be large, but is small compared with m.

For this purpose we note that

$$\log(m+p)=\log m+\frac{p}{m}-\frac{p^2}{2m^2}+\frac{p^3}{3m^3}-\dots$$

Thus if we take p to be of order $\sqrt{m}$, at most,* and reject terms of order $1/m$, we get the formulae

$$(m+p+\tfrac{1}{2})\log(m+p)=(m+p+\tfrac{1}{2})\log m+p+\tfrac{1}{2}\frac{p^2}{m}+\tfrac{1}{2}\frac{p}{m}-\tfrac{1}{6}\frac{p^3}{m^2}+O\left(\frac{1}{m}\right),$$

and $$\log\Gamma(1+m+p)=(m+p+\tfrac{1}{2})\log m-m+\tfrac{1}{2}\log(2\pi)+\tfrac{1}{2}(p^2/m)+\tfrac{1}{2}(p/m)-\tfrac{1}{6}(p^3/m^2)+O\left(\frac{1}{m}\right).$$

109. Calculation of integrals by means of asymptotic series. Various integrals of interest, both in Pure and Applied Mathematics, can be calculated most readily by means of asymptotic series. A few typical examples will be given below.

There are three methods which are usually effective in obtaining a suitable asymptotic series from a given integral:

(i) Integration by parts.

(ii) Use of symbolic operators.

(iii) Expansion of some function in the integral.

We shall consider examples of the use of each method; it should be noticed that it is usually impossible to use (iii) unless an estimate can be made as to the magnitude of the remainder in the expansion.

* This condition is usually satisfied in the majority of applications; but it is easy to modify the results in other cases. The result was published first in the *Phil. Mag.*, 1919.

(1) *The error-function integral.*

This integral is commonly expressed by the abbreviation $\operatorname{erf} x$, and is defined by the equation

$$\operatorname{erf} x=\int_0^x e^{-t^2}\,dt,$$

and a series suitable for calculation when x is small is deduced at once by expanding the exponential and integrating term-by-term. But this series is very inconvenient for numerical work when x exceeds 2.

An asymptotic series for the integral

$$u=\int_x^\infty e^{-t^2}\,dt=\sqrt{\pi}-\operatorname{erf} x$$

is readily found by writing $t^2=s$, and then

$$u=\int_{x^2}^\infty e^{-s}\frac{ds}{2\sqrt{s}}.$$

To the last integral we apply the transformation of integration by parts, which gives

$$u=\left[-\frac{e^{-s}}{2\sqrt{s}}\right]_{x^2}^\infty-\int_{x^2}^\infty\frac{e^{-s}}{2^2s^{3/2}}ds$$

$$=e^{-x^2}\left(\frac{1}{2x}\right)+\left[\frac{e^{-s}}{2^2s^{3/2}}\right]_{x^2}^\infty+\int_{x^2}^\infty\frac{1.3}{2^3}\frac{e^{-s}}{s^{5/2}}ds$$

$$=e^{-x^2}\left(\frac{1}{2x}-\frac{1}{2^2x^3}+\frac{1.3}{2^3x^5}-\frac{1.3.5}{2^4x^7}\right)+\int_{x^2}^\infty\frac{1.3.5.7}{2^5}\frac{e^{-s}}{s^{9/2}}ds,$$

where the last line is obtained from the preceding by two further integrations. Clearly this process can be continued as far as we please.

Now the remainder integral in the last formula is clearly less than

$$\frac{1.3.5.7}{2^5x^9}\int_{x^2}^\infty e^{-s}\,ds=\frac{1.3.5.7}{2^5x^9}e^{-x^2},$$

and this is the next term in the series, after those retained.

Thus *the error committed in stopping at any stage in the asymptotic series for the integral u is less than the following term in the series.**

* Of course this is the same conclusion as we arrived at in reference to the foregoing examples of Euler's method of summation.

The asymptotic series is obtained most rapidly by the symbolic method. Thus if we understand $1/D$ to denote the operation of integration from x^2 to ∞, we find that

$$\frac{1}{D}\frac{e^{-s}}{2\sqrt{s}}=\frac{e^{-s}}{2}\cdot\frac{1}{D-1}\frac{1}{\sqrt{s}}$$

$$=-\frac{e^{-s}}{2}\cdot\left(1+D+D^2+D^3+\dots\right)\frac{1}{\sqrt{s}}$$

$$=-\frac{e^{-s}}{2}\left(\frac{1}{\sqrt{s}}-\frac{1}{2}\cdot\frac{1}{s^{3/2}}+\frac{1.3}{2^2}\frac{1}{s^{5/2}}-\dots\right)$$

Remembering that $s=x^2$ is the *lower* limit of the integral, we now obtain the asymptotic series

$$u=e^{-x^2}\left(\frac{1}{2x}-\frac{1}{2^2x^3}+\frac{1.3}{2^3x^5}-\frac{1.3.5}{2^4x^7}+\dots\right)$$

as before.

If, instead of writing

$$\frac{1}{D-1}=-(1+D+D^2+\dots),$$

we use the remainder formula

$$\frac{1}{D-1}=-(1+D+D^2+\dots+D^{n-1})+\frac{D^n}{D-1},$$

it is easy to see that *we are left with the same remainder-integral as in the method of integration by parts.* From this of course we draw the same inference in regard to the magnitude of the remainder.

Finally, let us apply the method of expansions. Here we write

$$s=x^2+v$$

and
$$u=e^{-x^2}\int_0^\infty\frac{e^{-v}\,dv}{2\sqrt{(x^2+v)}}.$$

Then write
$$\frac{1}{\sqrt{(x^2+v)}}=\frac{1}{x}-\frac{1}{2}\frac{v}{x^3}+\frac{1.3}{2.4}\frac{v^2}{x^5}-\dots,$$

the remainder at any stage being less than the following term (Art. 61).

Now
$$\int_0^\infty v^n e^{-v}\,dv=n!$$

and so we obtain again the same results for u and its asymptotic series.

The methods given here fail when applied to the allied integral

$$\int_0^a e^{t^2}\,dt,$$

although, if we adopt the symbolic method and take

$$\frac{1}{D}\frac{e^s}{2\sqrt{s}}=e^s\frac{1}{1+D}\left(\frac{1}{2\sqrt{s}}\right)=e^s\left(\frac{1}{2\sqrt{s}}+\frac{1}{2^2 s^{\frac{3}{2}}}+\frac{1.3}{2^3 s^{\frac{5}{2}}}+\dots\right),$$

as a matter of fact, the result given by putting $s=a^2$ does agree with the asymptotic series deduced from more elaborate reasoning by Stieltjes (Ex. A, 8 below).

(2) *The logarithmic integral.*

The integral* $\qquad U=\int_x^\infty e^{-t}\frac{dt}{t}=-\int_0^{e^{-x}}\frac{dv}{\log v}$

has often been denoted by the symbol $-\mathrm{li}(e^{-x})$.

To obtain an asymptotic series for U, we can write†

$$\int_x^\infty\frac{e^{-t}}{t}dt=\int_1^\infty e^{-t}\frac{dt}{t}-\int_0^1(1-e^{-t})\frac{dt}{t}-\int_1^x\frac{dt}{t}+\int_0^x(1-e^{-t})\frac{dt}{t}$$

$$=-C-\log|x|+x-\frac{1}{2}\frac{x^2}{2!}+\frac{1}{3}\frac{x^3}{3!}-\frac{1}{4}\frac{x^4}{4!}+\dots,$$

where C is Euler's constant (see Appendix, Art. 178).

But this expansion, although convergent for all values of x, is unsuitable for calculation when $|x|$ is large, just as the exponential series is not convenient for calculating high powers of e. To meet this difficulty we apply methods similar to those used in (1) above.

If x is positive, we can use the method of integration by parts without difficulty; or (what is really the same method) we can use the symbolic method, writing

$$U=\frac{1}{D}\frac{e^{-t}}{t}=e^{-t}\frac{1}{D-1}\left(\frac{1}{t}\right)=-e^{-t}\left(\frac{1}{t}-\frac{1}{t^2}+\frac{1.2}{t^3}-\dots\right),$$

---

* If x is *negative*, the principal value of the integral is to be taken (see the next footnote). The symbol "li" denotes *logarithmic integral*; the meaning of this terminology is evident on writing $u=e^{-t}$, $y=e^{-x}$, and then $\mathrm{li}(y)=\int^y du/\log u$.

† When x is *negative* all these integrals are convergent except $\int_1^x\frac{dt}{t}$, of which we must take the principal value; that is

$$\lim_{\epsilon\to 0}\left(\int_1^\epsilon\frac{dt}{t}+\int_{-\epsilon}^x\frac{dt}{t}\right)=\lim_{\epsilon\to 0}\left[\log\epsilon+\log\left(-\frac{x}{\epsilon}\right)\right]$$
$$=\log(-x)=\log|x|.$$

or, finally,
$$U=e^{-x}\left(\frac{1}{x}-\frac{1}{x^2}+\frac{1\,.\,2}{x^3}-\dots\right),$$
the remainder at any stage being (as before) less than the following term in the series.

To apply the method of expansions we write $t=x+v$, and then
$$U=e^{-x}\int_0^\infty \frac{e^{-v}\,dv}{x+v},$$
while
$$\frac{1}{x+v}=\frac{1}{x}-\frac{v}{x^2}+\frac{v^2}{x^3}-\dots+(-1)^{n-1}\frac{v^{n-1}}{x^n}+(-1)^n\frac{v^n}{x^n(x+v)},$$
from which we deduce
$$U=e^{-x}\left\{\frac{1}{x}-\frac{1}{x^2}+\frac{2\,!}{x^3}-\frac{3\,!}{x^4}+\dots+(-1)^{n-1}\frac{(n-1)\,!}{x^n}+R_n\right\},$$
where
$$|R_n|=\int_0^\infty \frac{e^{-v}v^n\,dv}{x^n(x+v)}<n\,!\,x^{-(n+1)}.$$

When x is large, the terms of this series at first decrease very rapidly. Thus, *up to a certain degree of accuracy*, this series is very convenient for numerical work when x is large; but we cannot get beyond a certain approximation, because the terms finally increase beyond all limits.

For example, with $x=10$, the estimated limits for R_9, R_{10} are equal and are less than any other remainder. And the ratio of their common value to the first term in the series is about $1:2500$. To get this degree of accuracy from the first series we should need 35 terms. Again, with $x=20$, the ratio of R_{19} to the first term is less than $1:10^6$; but 80 terms of the ascending series do not suffice to obtain this degree of approximation.

When x is negative, we write
$$x=-\xi \quad\text{and}\quad t=x+v=v-\xi;$$
then we find
$$-U=e^{\xi}P\int_0^\infty \frac{e^{-v}}{\xi-v}\,dv$$
$$=e^{\xi}\left[\int_0^\infty\left(\frac{1}{\xi}+\frac{v}{\xi^2}+\frac{v^2}{\xi^3}+\dots+\frac{v^{n-1}}{\xi^n}\right)e^{-v}\,dv+P\int_0^\infty \frac{v^n e^{-v}}{\xi^n(\xi-v)}\,dv\right],$$
where P denotes the principal value of the integral. Thus
$$\mathrm{li}(e^{\xi})=e^{\xi}\left\{\frac{1}{\xi}+\frac{1}{\xi^2}+\frac{2\,!}{\xi^3}+\dots+\frac{(n-1)\,!}{\xi^n}+R_n\right\},$$
where
$$R_n=P\int_0^\infty \frac{v^n e^{-v}\,dv}{\xi^n(\xi-v)}.$$

Stieltjes * has proved by an elaborate discussion, which is too lengthy to be given here, that in this case also we get the best approximation by taking n equal to the integral part of ξ, and that the value of R_n is then of the order $e^{-\xi}(2\pi/\xi)^{\frac{1}{2}}$.

It is not without interest to note that according to Lacroix (*Calcul. Diff. et Int.*, Paris, 1819, vol. 3, p. 517) these two expansions were utilised by Mascheroni to find Euler's constant C.

Another application is to be found in the "summation" of

$$1!-2!+3!-4!+\dots,$$

taking the value of C as known (Art. 106 above).

If we write $x=1$ and equate the series of ascending powers to the series of descending powers, we find that

$$-C+\left(1-\frac{1}{2.2!}+\frac{1}{3.3!}-\dots\right)=e^{-1}(1-1!+2!-3!+\dots),$$

which gives

$$1!-2!+3!-4!+\dots=1+e\left\{C-\left(1-\frac{1}{2.2!}+\frac{1}{3.3!}-\dots\right)\right\}.$$

Lacroix gives the value 0·7965996 as the value of the series in round brackets, which yields the "sum"

$$1!-2!+3!-4!+\dots=\cdot 4036526,$$

agreeing with the result found from Euler's continued fraction in Art. 105.

Lacroix (*l.c.* p. 389) gives another calculation of this oscillatory series by using the method of approximate quadrature to evaluate the integral

$$\int_1^\infty \frac{e^{-t}}{t}\,dt=\int_0^1 \frac{e^{-1/y}}{y}\,dy=\frac{1}{e}(1-2!+3!-\dots),$$

which gives the sum ·403628... . Lacroix attributes the calculation to Euler, but without a reference; and he also suggests the application of approximate quadrature to the integral

$$\frac{1}{e}\int_0^1 \frac{dv}{1-\log v},$$

which is found by writing $v=e^{1-t}$, but he gives no numerical results.

110. Asymptotic series for integrals containing sines and cosines.

(1) *Fresnel's integrals.*

Consider the two integrals

$$U=\int_x^\infty \frac{\cos t}{\sqrt{t}}\,dt,\qquad V=\int_x^\infty \frac{\sin t}{\sqrt{t}}\,dt,\qquad (x>0),$$

* *Annales de l'École Normale Supérieure* (3), vol. 3, 1886, p. 201.

which are met with in the theory of Physical Optics, and also in the theory of deep-water waves.*

We have $$U+\iota V=\int_x^\infty \frac{e^{\iota t}}{\sqrt{t}}\,dt.$$

Thus, if we apply the symbolic process (as equivalent to integration by parts), we take

$$\frac{1}{D}\frac{e^{\iota t}}{\sqrt{t}}=e^{\iota t}\frac{1}{D+\iota}\frac{1}{\sqrt{t}}=e^{\iota t}\left(\frac{1}{\iota}-\frac{D}{\iota^2}+\frac{D^2}{\iota^3}-\dots\right)\frac{1}{\sqrt{t}}.$$

This gives, on inserting the limits,

$$U+\iota V=-e^{\iota x}\left(\frac{1}{\iota\sqrt{x}}+\frac{1}{2\iota^2 x^{\frac{3}{2}}}+\frac{1\,.\,3}{2^2\iota^3 x^{\frac{5}{2}}}+\dots\right)$$

$$=\frac{\iota e^{\iota x}}{\sqrt{x}}\left\{1+\frac{1}{2\iota x}+\frac{1\,.\,3}{(2\iota x)^2}+\frac{1\,.\,3\,.\,5}{(2\iota x)^3}+\dots\right\}$$

$$=\frac{\iota e^{\iota x}}{\sqrt{x}}(X-\iota Y),\text{ say,}$$

where $$X=1-\frac{1\,.\,3}{(2x)^2}+\frac{1\,.\,3\,.\,5\,.\,7}{(2x)^4}-\dots,$$

$$Y=\frac{1}{2x}-\frac{1\,.\,3\,.\,5}{(2x)^3}+\frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{(2x)^5}-\dots.$$

Then $U=\dfrac{1}{\sqrt{x}}(-X\sin x+Y\cos x)$, $V=\dfrac{1}{\sqrt{x}}(X\cos x+Y\sin x)$.

The remainder in the series $U+iV$ after the four terms written above is easily expressed in the form

$$\frac{1\,.\,3\,.\,5\,.\,7}{(2\iota)^4}\int_x^\infty\frac{e^{\iota t}\,dt}{t^{\frac{9}{2}}},$$

and this is numerically less than (Art. 169)

$$\frac{1\,.\,3\,.\,5\,.\,7}{2^4}\,\frac{2}{x^{\frac{9}{2}}}=\frac{2}{\sqrt{x}}\,\frac{1\,.\,3\,.\,5\,.\,7}{(2x)^4},$$

which is *twice* the modulus of the following term of the series.

It is easy to apply the method of expansion by putting $t=x+v$, and proceeding as in Art. 109 (1).

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* Historically the hydrodynamical application seems to have occurred first (see Lamb, *Proc. Lond. Math. Soc.* (2), vol. 2, 1904, p. 371); and the chief properties of the integrals were worked out by Poisson and Cauchy in connection with this problem (for references and details see Lamb's paper).

It is possible to use the asymptotic series to express X, Y in terms of another pair of integrals, which have been found useful in some calculations.

From the known integrals for $\Gamma(\frac{1}{2})$, $\Gamma(n+\frac{1}{2})$, we have

$$\int_0^\infty e^{-v}\frac{dv}{\sqrt{v}}=\sqrt{\pi},\quad \int_0^\infty e^{-v}v^n\frac{dv}{\sqrt{v}}=\frac{1\,.\,3\,\ldots\,(2n-1)}{2^n}\sqrt{\pi}$$

and so
$$X-\iota Y=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-v}}{\sqrt{v}}dv\left\{1+\frac{v}{\iota x}+\frac{v^2}{(\iota x)^2}+\ldots\right\}$$
$$=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-v}}{\sqrt{v}}dv\left(\frac{x}{x+\iota v}\right),$$

the remainder at any stage in the expanded form of the integral being numerically less than the following term.

Hence we obtain the formulae

$$X=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-v}}{\sqrt{v}}dv\frac{x^2}{x^2+v^2},\quad Y=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-v}\sqrt{v}\,dv\frac{x}{x^2+v^2}.$$

Of course we have not given a complete proof that these expressions are equal to the original integrals; but it is easy to complete the proof by differentiating with respect to x. We have, in fact,

$$\frac{d}{dx}(U+\iota V)=-\frac{e^{\iota x}}{\sqrt{x}}.$$

Thus we must have
$$\frac{d}{dx}\left\{\frac{\iota e^{\iota x}}{\sqrt{x}}(X-\iota Y)\right\}=-\frac{e^{\iota x}}{\sqrt{x}}.$$

Hence we find the condition

$$\frac{d}{dx}(X-\iota Y)+\left(\iota-\frac{1}{2x}\right)(X-\iota Y)=\iota$$

or
$$\frac{dX}{dx}-\frac{X}{2x}+Y=0,\quad \frac{dY}{dx}-\frac{Y}{2x}-X=-1.$$

It is easy to verify that these equations are satisfied by the last pair of integrals for X, Y, and that these integrals tend to 1, 0 respectively as $x\to\infty$; thus we may infer that U, V and X, Y are actually related in the manner suggested by the foregoing work. The integrals X, Y seem to be due to Cauchy, and the asymptotic expansion to Poisson (see Lamb's paper, already quoted).

It is perhaps worth while to make the additional remark that the relations between X, Y and U, V are most naturally suggested by the use of the asymptotic expansion.

(2) *The sine- and cosine-integrals.*

We shall determine next asymptotic formulae for the two integrals

$$P=\int_x^\infty\frac{\cos t}{t}dt,\quad Q=\int_x^\infty\frac{\sin t}{t}dt.$$

Then we have $$P+\iota Q=\int_x^\infty \frac{e^{\iota t}}{t}\,dt,$$

and the asymptotic formula is obtained on lines similar to those used in (1) above.

The symbolic calculation gives

$$\frac{1}{D}\frac{e^{\iota t}}{t}=e^{\iota t}\frac{1}{D+\iota}\frac{1}{t}=\frac{-\iota e^{\iota t}}{t}\left\{1+\frac{1}{\iota t}+\frac{1\,.\,2}{(\iota t)^2}+\dots\right\}.$$

Hence, on introducing the limits, we have

$$P+\iota Q=\frac{\iota e^{\iota x}}{x}\left\{1+\frac{1}{\iota x}+\frac{1\,.\,2}{(\iota x)^2}+\dots\right\}.$$

The most instructive formulae are given by dividing by $e^{\iota x}$ and taking real and imaginary parts; this gives

$$\int_x^\infty \frac{\cos(t-x)}{t}\,dt=P\cos x+Q\sin x=\frac{1}{x}\left\{\frac{1}{x}-\frac{3!}{x^3}+\frac{5!}{x^5}-\dots\right\},$$

$$\int_x^\infty \frac{\sin(t-x)}{t}\,dt=-P\sin x+Q\cos x=\frac{1}{x}\left\{1-\frac{2!}{x^2}+\frac{4!}{x^4}-\dots\right\}.$$

It is perhaps worth remarking that this *cosine*-integral is represented by a series of reciprocals of the ordinary *sine*-series, and *vice-versa*.

The second formula leads to an easy method for calculating the maxima and minima of the sine-integral, which correspond to the values $x=n\pi$; thus we find

$$I_n=\int_{n\pi}^\infty \frac{\sin t}{t}\,dt=\frac{(-1)^n}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\dots\right),\quad x=n\pi.$$

For values of n greater than 2, it is found that the calculation can be easily carried out to four decimal places; thus *

$$I_3=-\cdot 1040,\quad I_4=+\cdot 0786,$$
$$I_5=-\cdot 0631,\quad I_6=+\cdot 0528.$$

The corresponding formula for the maxima and minima of the cosine-integral is also found from the second formula, and is

$$\int_{(n+\frac{1}{2})\pi}^\infty \frac{\cos t}{t}\,dt=\frac{(-1)^{n-1}}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\dots\right),\quad x=(n+\tfrac{1}{2})\pi.$$

No investigation has been given here as to the magnitude of the remainder; but the reader should have no difficulty in seeing that the remainder is numerically less than twice the following term in each series, by applying the method used in (1) above.

* Glaisher, *Phil. Trans.*, vol. 160, 1870, p. 387.

111. Stirling's series.

An investigation of this series, which is independent of Euler's summation formula, can be given on the following lines. It is subject, however, to the drawback that the preliminary analysis is more difficult.

It can be proved that,*

$$\log \Gamma(1+x) = F(x) + 2\int_0^\infty \frac{\arc\tan(v/x)}{e^{2\pi v}-1}\,dv,$$

where $$F(x) = (x+\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log(2\pi).$$

Now (Art. 64), we have

$$\arc\tan(v/x) = (v/x) - \tfrac{1}{3}(v/x)^3 + \tfrac{1}{5}(v/x)^5 - \ldots$$
$$+(-1)^{n-1}\frac{1}{2n-1}(v/x)^{2n-1} + R_n,$$

where $$|R_n| < \frac{1}{2n+1}(v/x)^{2n+1}.$$

Hence (Art. 176, Ex. 3) we have

$$\int_0^\infty \frac{\arc\tan(v/x)}{e^{2\pi v}-1}\,dv = \frac{B_1}{1\,.\,2x} - \frac{B_2}{3\,.\,4x^3} + \frac{B_3}{5\,.\,6x^5} - \ldots$$
$$+(-1)^{n-1}\frac{B_n}{(2n-1)\,.\,2n\,.\,x^{2n-1}} + R_n',$$

where R_n' is numerically less than the first term omitted from the series.

If we take the quotient of two consecutive terms and remark that (compare Art. 106)

$$B_{n+1}/B_n = (2n+1)(2n+2)\,Q/4\pi^2,$$

where Q is a factor slightly less than 1, we see that the least value for the remainder is given by taking n equal to the integral part of πx; but the first two terms give a degree of accuracy which is ample for ordinary calculations.†

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* See, for instance, Art. 180 (App. III.); or Jordan, *Cours d'Analyse* (2nd ed.), t. 2, pp. 176–182.

† An elementary treatment of this approximation will be found (for the case when x is an integer) in a paper by the author (*Messenger of Maths.*, vol. 36, 1906, p. 81).

112. Stokes' asymptotic expression for the series

$$\Sigma \frac{\Gamma(n+a_1+1) \ldots \Gamma(n+a_r+1)}{\Gamma(n+b_1+1) \ldots \Gamma(n+b_s+1)} x^n = \Sigma X_n,$$

where x is real and $s > r$.*

Write $s-r=\mu$, $\Sigma b - \Sigma a = \lambda$, and consider the term X_{t+p}, where t is large, and p is not of higher order than $\sqrt{t}$.

Neglecting terms of order $1/\sqrt{t}$, we find from Stirling's series that

$$\log X_{t+p} = (t+p) \log x - \mu\{(t+\tfrac{1}{2}) \log t + \tfrac{1}{2} \log (2\pi) - t\} \\ -(p\mu+\lambda) \log t - \tfrac{1}{2}\mu p^2/t$$

(see the formula at the end of Art. 108).

It is convenient to suppose that x is of the form t^μ, where t is an integer (a restriction which can be removed by using more elaborate methods); and then X_t is the greatest term because $\log x = \mu \log t$, so that the terms of the first degree in p cancel. We deduce that

$$\log X_{t+p} = \mu t - \tfrac{1}{2}\mu \log (2\pi t) - \lambda \log t - \tfrac{1}{2}\mu p^2/t$$

or

$$X_{t+p} = \frac{e^{t\mu} t^{-\lambda}}{(2\pi t)^{\frac{1}{2}\mu}} \exp(-\tfrac{1}{2}\mu p^2/t).$$

This gives the asymptotic expression (combining X_{t+p} and X_{t-p})

$$\frac{e^{t\mu} t^{-\lambda}}{(2\pi t)^{\frac{1}{2}\mu}} (1+2q+2q^4+2q^9+\ldots),$$

where $q=e^{-\frac{1}{2}\mu/t}$. Making use of Art. 51, Ex. 4, we see that (since q approaches the limit 1) the series in brackets is represented approximately by $\pi^{\frac{1}{2}}(1-q)^{-\frac{1}{2}}$, or by $(2\pi t/\mu)^{\frac{1}{2}}$.

Thus the asymptotic expression is

$$\frac{e^{t\mu} t^{-\lambda}}{\mu^{\frac{1}{2}} (2\pi t)^{\frac{1}{2}(\mu-1)}}, \text{ where } t = x^{\frac{1}{\mu}}.$$

Hardy † has proved, somewhat on the same lines, that

$$f(x) = \Sigma \frac{x^{n^2}}{n^2!},$$

is represented asymptotically by $Ae^x/(2\pi x)^{\frac{1}{2}}$, where

$$A = 1 + 2(q+q^4+q^9+\ldots) \text{ and } q = e^{-2}.$$

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* *Proc. Camb. Phil. Soc.*, vol. 6, 1889; *Math. and Phys. Papers*, vol. 5, p. 221.

† *Proc. Lond. Math. Soc.* (2), vol. 2, 1904, p. 339.

113. Poincaré's theory of asymptotic series.

All the investigations of Arts. 106–111 resemble one another to the following extent:

Starting from some function $J(x)$, we develop it formally in a series
$$a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\dots .$$

This series is not convergent, but yet the sum of the first $(n+1)$ terms gives an approximation to $J(x)$ which differs from $J(x)$ by less than K_n/x^{n+1}, where K_n depends only on n and not on x.

Thus, if S_n denotes the sum of the first $(n+1)$ terms, we have
$$\lim_{x\to\infty} x^n(J-S_n)=0.$$

In all such cases, we say that *the series is asymptotic to the function*; and the relation may be denoted by the symbol
$$J(x)\sim a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots .$$

Such series were often called *semiconvergent* by older writers.

It is to be noticed, however, that the same series may be asymptotic to more than one function; for example, since
$$\lim_{x\to\infty}(x^n e^{-x})=0,$$
the same series will represent $J(x)$ and $J(x)+Ae^{-x}$.

It follows from the definition that *we can add and subtract asymptotic series as if they were convergent.*

Next, take the product of two asymptotic series, assuming that the rule of Art. 34 still applies. We then find, if
$$J(x)\sim a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots \quad\text{and}\quad K(x)\sim b_0+\frac{b_1}{x}+\frac{b_2}{x^2}+\dots ,$$
the formal product $\quad \Pi(x)=c_0+\dfrac{c_1}{x}+\dfrac{c_2}{x^2}+\dots ,$

where $\quad c_n=a_0b_n+a_1b_{n-1}+\dots+a_nb_0.$

Let S_n, T_n, Σ_n denote the sums of the first $(n+1)$ terms in these three series, so that we have, say,
$$J(x)=S_n+\rho/x^n,\quad K(x)=T_n+\sigma/x^n,$$
where ρ, σ are functions of x which tend to zero as $x\to\infty$.

Now, by definition Σ_n coincides with the product S_nT_n up to and including the terms in $1/x^n$; thus $S_nT_n-\Sigma_n$ contains terms from $1/x^{n+1}$ to $1/x^{2n}$. We can therefore write
$$S_nT_n=\Sigma_n+P_n/x^{2n},$$
where P_n is a polynomial in x, whose highest term is of degree $(n-1)$.

Thus $$\left\{J(x)-\frac{\rho}{x^n}\right\}\left\{K(x)-\frac{\sigma}{x^n}\right\}=\Sigma_n+P_n/x^{2n}$$

or $$x^n\{J(x)\,.\,K(x)-\Sigma_n\}=\rho K(x)+\sigma J(x)+(P_n-\rho\sigma)/x^n.$$

Now, as $x\to\infty$, $J(x)\to a_0$, $K(x)\to b_0$, $\rho\to 0$, $\sigma\to 0$,

and accordingly $\lim\limits_{x\to\infty} x^n\{J(x)\,.\,K(x)-\Sigma_n\}=\lim\limits_{x\to\infty} P_n/x^n=0.$

Thus the product $J(x)\,.\,K(x)$ is represented asymptotically by $\Pi(x)$; or *asymptotic series can be multiplied together as if they were convergent*; and in particular we can obtain any power of an asymptotic series by the ordinary rules.

Let us now consider the possibility of substituting an asymptotic series in a power-series. In the first place, we may evidently write $J(x)=a_0+J_1(x)$ and substitute a_0+J_1 for J in the series*

$$f(J)=c_0+c_1J+c_2J^2+c_3J^3+\dots,$$

and rearrange in powers of J_1, *provided that* $|a_0|$ *is less than the radius of convergence* (Art. 88); because $\lim J_1=0$, and we can therefore take x large enough to satisfy the restriction that $|a_0|+|J_1|$ is to be less than the radius of convergence.

This having been done, we may consider the substitution of the asymptotic series
$$\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\dots$$

for J_1 in the series
$$F(J_1)=C_0+C_1J_1+C_2J_1^2+C_3J_1^3+\dots.$$

Let us make a formal substitution, as if the series for J_1 were convergent; then we obtain some new series
$$\Sigma=D_0+\frac{D_1}{x}+\frac{D_2}{x^2}+\frac{D_3}{x^3}+\dots,$$

where
$$D_0=C_0,\ D_1=C_1a_1,\ D_2=C_1a_2+C_2a_1^2,\ D_3=C_1a_3+2C_2a_1a_2+C_3a_1^3,\text{ etc.}$$

Let us denote by S_n and Σ_n the sum of the terms up to $1/x^n$ in J_1 and Σ respectively.

Now, if $$\Sigma_n'=C_0+C_1S_n+C_2S_n^2+\dots+C_nS_n^n,$$

Σ_n' and Σ_n agree up to terms in $1/x^n$, and consequently $\Sigma_n'-\Sigma_n$ is a polynomial in $1/x$, ranging from terms in $(1/x)^{n+1}$ to $(1/x)^{n^2}$; thus

$$(1)\quad \lim_{x\to\infty} x^n(\Sigma_n'-\Sigma_n)=0.$$

* Of course c_n no longer represents $a_0b_n+\dots+a_nb_0$.

Next, if $$T_n = C_0 + C_1 J_1 + C_2 J_1^2 + \dots + C_n J_1^n,$$
we have, since S_n represents J_1 asymptotically, $\lim_{x\to\infty} x^n (J_1^r - S_n^r) = 0$, and therefore

$$(2) \quad \lim_{x\to\infty} x^n (T_n - \Sigma_n') = 0.$$

Finally, $$F - T_n = C_{n+1} J_1^{n+1} + C_{n+2} J_1^{n+2} + \dots;$$
thus, since $F(J_1)$ is convergent,

$$|F - T_n| < M J_1^{n+1},$$

where M is a constant.

Hence, we find

$$(3) \quad \lim_{x\to\infty} x^n (F - T_n) = 0,$$

because $$\lim_{x\to\infty} (x^n J_1^{n+1}) = \lim_{x\to\infty} (a_1^{n+1}/x) = 0.$$

By combining (1), (2) and (3), we see now that

$$\lim_{x\to\infty} x^n (F - \Sigma_n) = 0.$$

Thus the series Σ represents $F(J_1)$ asymptotically; and consequently *an asymptotic series may be substituted in a power-series and rearranged (just as if convergent), provided that its first term is numerically less than the radius of convergence.*

Further, a reference to the foregoing proof shews that we use the convergence of the series $f(J)$ in two places only, first in order to rearrange in powers of J_1, and secondly to establish the inequality

$$|F - T_n| < M J_1^{n+1}.$$

Now this inequality is satisfied if the series

$$C_1 J_1 + C_2 J_1^2 + C_3 J_1^3 + \dots$$

is *asymptotic* to $F(J_1)$; and then we must suppose that a_0 is zero in order to get any result at all, so that $J = J_1$ and we can entirely avoid the restriction that $f(J)$ is convergent. Thus, *an asymptotic series* **whose first term is zero** *may be substituted in another asymptotic series, and the result may be rearranged just as if both series were convergent.*

An application of the former result is to establish the *rule for division* (assuming that a_0 is not zero). For we can write

$$J(x) = a_0 (1 + K),$$

where $$K \sim \frac{a_1}{a_0 x} + \frac{a_2}{a_0 x^2} + \dots .$$

Then $$\{J(x)\}^{-1} = a_0^{-1} (1 - K + K^2 - K^3 + \dots),$$
and we can thus construct an asymptotic series for $\{J(x)\}^{-1}$ by exactly the same rule as if the series for $J(x)$ were convergent.

Thus, applying the rule for multiplication, we see that *we can divide any asymptotic series by any other asymptotic series, just as if they were convergent*.

Finally, let us consider *the integration of an asymptotic series* (in which $a_0=0$, $a_1=0$).

If
$$J(x) \sim \frac{a_2}{x^2}+\frac{a_3}{x^3}+\frac{a_4}{x^4}+\dots,$$
we have
$$|J-S_n| < \epsilon/x^n, \quad \text{if } x > x_0.$$
Thus
$$\left|\int_x^\infty J\,dx-\int_x^\infty S_n dx\right| < \frac{\epsilon}{(n-1)\,x^{n-1}}, \quad \text{if } x > x_0,$$
so that $\int_x^\infty J\,dx$ is represented asymptotically by
$$\frac{a_2}{x}+\frac{a_3}{2x^2}+\frac{a_4}{3x^3}+\dots.$$

But, on the other hand, *an asymptotic series cannot safely be differentiated without additional investigation*, for the existence of an asymptotic series for $J(x)$ does not imply the existence of one for $J'(x)$.

Thus $e^{-x}\sin(e^x)$ has an asymptotic series
$$0+\frac{0}{x}+\frac{0}{x^2}+\dots.$$
But its differential coefficient is $-e^{-x}\sin(e^x)+\cos(e^x)$, which oscillates as x tends to ∞; and consequently the differential coefficient has no asymptotic expansion.

On the other hand, if we know that $J'(x)$ has an asymptotic expansion, it must be the series obtained by the ordinary rule for term-by-term differentiation.

This follows by applying the theorem on integration to $J'(x)$; but a direct proof is quite as simple, and perhaps more instructive. We make use of the theorem that *if $\phi(x)$ has a definite finite limit as x tends to ∞, then $\phi'(x)$ either oscillates or tends to zero as a limit.*

In fact if $\phi(x)$ tends to a definite limit we can find x_0 so that
$$|\phi(x)-\phi(x_0)| < \epsilon, \quad \text{if } x > x_0.$$
Thus, since
$$\frac{\phi(x)-\phi(x_0)}{x-x_0}=\phi'(\xi), \quad \text{where } x > \xi > x_0,$$
we find
$$|\phi'(\xi)| < \epsilon/(x-x_0).$$

So $\phi'(x)$ cannot approach any *definite* limit other than zero; but the last inequality does not exclude *oscillation*, since ξ may not take *all* values greater than x_0 as x tends to ∞.

Now, if $J(x) \sim a_0 + a_1/x + a_2/x^2 + \dots$, we have

$$\lim x^{n+1}\{J(x) - S_n(x)\} = a_{n+1}.$$

Thus the differential coefficient

$$x^{n+1}\{J'(x) - S_n'(x)\} + (n+1)x^n\{J(x) - S_n(x)\},$$

if it has a definite limit, must tend to zero. But $x^n\{J(x) - S_n(x)\}$ does tend to zero, so that $\lim x^{n+1}\{J'(x) - S_n'(x)\}$, if it exists, is zero.

That is, if $J'(x)$ has an asymptotic series, it is

$$-a_1/x^2 - 2a_2/x^3 - 3a_3/x^4 - \dots .$$

It is instructive to contrast the rules for transforming and combining asymptotic series with those previously established for convergent series. Thus *any* two asymptotic series can be multiplied together: whereas the product of two convergent series need not give a convergent series (see Arts. 34, 35). Similarly *any* asymptotic series may be integrated term-by-term, although not every convergent series can be integrated (Art. 45).

On the other hand, as we have just explained, we cannot differentiate *any* asymptotic series unless we know from independent reasoning that the corresponding derivate has an asymptotic expansion; although, in dealing with a convergent series, we can apply the test for uniform convergence directly to the differentiated series, and so *infer* that the derived function has an expansion (Art. 46).

These contrasts, however, are not to be regarded as surprising. In a convergent series, the parameter with respect to which we differentiate or integrate is strictly an *auxiliary* variable, and in no way enters into the definition of the convergence of the series; but in an asymptotic series, the very definition depends on the parameter x. The contrast may be illustrated in an even more fundamental way; *any* coefficients whatever may define a perfectly good asymptotic series. Indeed an asymptotic series is not a completed whole in the same sense as a convergent series.

It is sometimes convenient to extend our definition and say that J is represented asymptotically by the series

$$\Phi + \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots\right)\Psi$$

when $\dfrac{J - \Phi}{\Psi}$ is represented by $a_0 + \dfrac{a_1}{x} + \dfrac{a_2}{x^2} + \dots$, where Φ, Ψ are two suitably chosen functions of x.

Thus, for example, we can deduce from Stirling's series the asymptotic formula

$$\Gamma(x+1) \sim e^{-x}x^x(2\pi x)^{\frac{1}{2}}\left(1+\frac{C_1}{x}+\frac{C_2}{x^2}+\dots\right),$$

where $$C_1=\frac{B_1}{2}=\frac{1}{12}, \quad C_2=-\frac{B_1{}^2}{8}=\frac{1}{288}, \text{ etc.}$$

Hitherto x has been supposed to tend to ∞ through *real, positive* values; but the theory remains unaltered if x is complex and tends to ∞ in any other definite direction. But *a non-convergent series cannot represent asymptotically the same one-valued analytic function J for all arguments of x.*

In fact, if we can determine constants M, R, such that

$$\left|J-a_0-\frac{a_1}{x}\right|<\frac{M}{|x|^2}, \quad \text{when } |x|>R,$$

it follows from elementary theorems in the theory of functions that $J(x)$ is a regular function of $1/x$, and consequently the asymptotic series must be convergent.

For different ranges of variation of the argument of x, we may have different asymptotic representations of the same function which between them give complete information as to its nature. A good illustration of this phenomenon is afforded by the Bessel functions (Arts. 116–118), which have been discussed at length by Stokes.*

114. Applications of Poincaré's theory.

An interesting and important application of Poincaré's theory is to the solution of differential equations.† The method may be summed up in the following steps:

First, a formal solution is obtained by means of a non-convergent series.

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* *Camb. Phil. Trans.*, vol. 9, 1850, vol. 10, 1857, p. 105, and vol. 11, 1868, p. 412; *Math. and Phys. Papers*, vol. 2, p. 350, vol. 4, pp. 77, 283. See also *Acta Mathematica*, vol. 26, 1902, p. 393, and *Papers*, vol. 5, p. 283. Stokes remarks that in the asymptotic series examined by him, the change in representation occurs at a value of the argument which gives the same sign to all the terms of the divergent series.

† Some interesting remarks on the sense in which an asymptotic series gives a solution of a differential equation have been made by Stokes (*Papers*, vol. 2, p. 337).

Secondly, it is shewn, by independent reasoning, that a solution exists which is capable of asymptotic representation. Thus we may either deduce a definite integral from the series first calculated; or we may find a solution as a definite integral directly, and then identify it with the series.

Thirdly, the region is determined in which the asymptotic representation is valid.

Poincaré has in fact proved* that every linear differential equation which has *polynomial* coefficients may be solved by asymptotic series; but his work is restricted to the case in which the independent variable tends to ∞ along a specified direction, and the regions are not determined. This gap has been filled by Horn in a number of special cases.†

Other applications of Poincaré's theory have been made by Barnes and Hardy in constructing the asymptotic representation of functions given by power-series. A convenient summary with very full references is given by Barnes;‡ the method adopted by Barnes is beyond the limits of this book, as it depends on the theory of contour integration; but the method of Stokes given in Art. 112 is useful in dealing with certain types of *real* series.

Before leaving the question of applications, it may be useful to point out that the ordinary Taylor's (or Maclaurin's) series of the Differential Calculus has essentially an asymptotic character ($1/x$ being changed to x), *until the remainder has been investigated.*

Even when the series

$$f(0)+xf'(0)+\tfrac{1}{2}x^2f''(0)+\dots$$

is convergent, its sum need not be equal to $f(x)$; but we can always assert that $\{f(x)-S_n(x)\}$ is of higher order than the last term in $S_n(x)$. Or, in more precise form, we can assert that

$$\lim_{x\to 0}\{f(x)-S_n(x)\}/x^n=0,$$

which has the same character as the definition adopted in Art. 113.

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* *Acta Mathematica*, vol. 8, 1886, p. 303.

† See a series of papers in the *Mathematische Annalen*, from vol. 49 onwards, and some papers in *Crelle's Journal*. A good summary of the theory with many references is given in Horn's *Gewöhnliche Differentialgleichungen*, Abschnitt VII.

‡ *Phil. Trans.*, series A, vol. 206, 1906, p. 249; see also *Quarterly Journal*, vol. 38, 1907, pp. 108, 116.

115. Differential equations.

We shall now give some examples of the way in which asymptotic series present themselves in the solution of differential equations.

Let us try first to solve the differential equation *

$$\frac{dy}{dx}=\frac{a}{x}+by, \quad (b>0),$$

by means of an asymptotic series

$$y=A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\dots .$$

On substitution, we find

$$-\frac{A_1}{x^2}-\frac{2A_2}{x^3}-\frac{3A_3}{x^4}-\dots=\frac{a}{x}+b\left(A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\dots\right).$$

This gives

$$A_0=0, \quad A_1=-\frac{a}{b}, \quad A_2=-\frac{A_1}{b}=\frac{a}{b^2},$$

$$A_3=-\frac{2A_2}{b}=-\frac{1 \, . \, 2a}{b^3}, \quad A_4=-\frac{3A_3}{b}=\frac{1 \, . \, 2 \, . \, 3 \, . \, a}{b^4}, \text{ etc.}$$

Thus we find the formal solution

$$y=-\frac{a}{bx}\left\{1-\frac{1}{bx}+2!\left(\frac{1}{bx}\right)^2-3!\left(\frac{1}{bx}\right)^3+\dots\right\},$$

and by Art. 109 this represents the integral

$$-a\int_0^\infty \frac{e^{-t}\,dt}{t+bx},$$

and it is now easy to verify directly that this integral does satisfy the given equation.

116. The modified Bessel's equation.

Following Stokes, we take the equation in the form

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{n^2}{x^2}\right)y=0,$$

and then attempt to find a solution in the form $e^{\lambda x}x^{-\frac{1}{2}}\eta$, where η proves to be an asymptotic series.

The equation for η is found to be

$$x^2\left(\frac{d^2\eta}{dx^2}+2\lambda\frac{d\eta}{dx}\right)+\{(\lambda^2-1)x^2+(\tfrac{1}{4}-n^2)\}\eta=0.$$

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* This is the simplest case of a general type of equations examined by Borel (*Annales de l'École Normale Supérieure* (3), vol. 16, 1899, p. 95).

Thus we take $\lambda^2=1$, and then, writing

$$\eta=1+\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\dots,$$

we obtain

$$\left(1.2\frac{A_1}{x}+2.3\frac{A_2}{x^2}+3.4\frac{A_3}{x^3}+\dots\right)$$
$$-2\lambda\left(A_1+2\frac{A_2}{x}+\quad 3\frac{A_3}{x^2}+\quad 4\frac{A_4}{x^3}+\dots\right)$$
$$+\left(\frac{1}{4}-n^2\right)\left(1+\frac{A_1}{x}+\quad\frac{A_2}{x^2}+\quad\frac{A_3}{x^3}+\dots\right)=0,$$

or $2\lambda A_1=\frac{1}{4}-n^2$, $4\lambda A_2=(\frac{9}{4}-n^2)A_1$, $6\lambda A_3=(\frac{25}{4}-n^2)A_2$, etc.

Thus we may take

$$A_1=\frac{1-4n^2}{8\lambda},\qquad A_2=\frac{1}{1.2}\frac{(1-4n^2)(9-4n^2)}{(8\lambda)^2},\ \text{etc.},$$

leading to the solutions

$$y=\frac{e^{\lambda x}}{\sqrt{x}}\left\{1+\frac{1-4n^2}{x}+\frac{1}{1.2}\frac{(1-4n^2)(9-4n^2)}{(8\lambda x)^2}+\dots\right\},$$

where $\lambda=\pm1$.

It is easy to see that these series cannot converge* for any value of x; they do not agree with any of the series considered up to the present, but we can write

$$\int_0^\infty e^{-t}t^{n+r-\frac{1}{2}}dt=\Gamma(n+r+\tfrac{1}{2})$$
$$=\Gamma(n+\tfrac{1}{2})\frac{1}{2^r}(1+2n)(3+2n)\dots(2r-1+2n),$$

and $$\left(1-\frac{t}{2\lambda x}\right)^{n-\frac{1}{2}}=1+\frac{(1-2n)t}{4\lambda x}+\frac{(1-2n)(3-2n)t^2}{1.2.(4\lambda x)^2}+\dots.$$

Thus the series can be written in the form

$$\frac{1}{\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-t}t^{n-\frac{1}{2}}\left(1-\frac{t}{2\lambda x}\right)^{n-\frac{1}{2}}dt.$$

When x is real and positive (n being assumed positive), this integral has a meaning only if $\lambda=-1$; and then, by Art. 61, the remainder in the binomial expansion is less than the following term (at any rate after a certain stage), and thus the same is true of the asymptotic series.

* Unless $2n$ is an odd integer; and then the series terminate.

Consequently, for $\lambda=-1$, the asymptotic series is asymptotic to the integral

$$\frac{1}{\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-t}\left\{\frac{t(t+2x)}{2x}\right\}^{n-\frac{1}{2}}dt.$$

If we write $t+x=x\cosh\theta$, and then multiply by the factor $e^{-x}x^{-\frac{1}{2}}$, we obtain the solution

$$y=\frac{x^n}{2^{n-\frac{1}{2}}\Gamma(n+\frac{1}{2})}\int_0^\infty e^{-x\cosh\theta}\sinh^{2n}\theta\,d\theta,$$

which can be proved to satisfy the original differential equation, by substituting and integrating by parts.

It may, therefore, be expected that the two original series both satisfy the differential equation; although we cannot obtain a complete proof without some assistance from the Theory of Functions.

117. Identification of the solutions of Art. 116 with known solutions.

Take first the case $n=0$; then, from the previous analysis, we have the solution

$$\int_0^\infty e^{-x\cosh\theta}\,d\theta=e^{-x}\sqrt{\left(\frac{\pi}{2x}\right)}\left\{1-\frac{1}{8x}+\frac{1^2.3^2}{2!}\frac{1}{(8x)^2}-\frac{1^2.3^2.5^2}{3!}\frac{1}{(8x)^3}+\dots\right\}.$$

To discuss the relation with known results, put $\frac{1}{2}xe^\theta=v$, and then the integral becomes

$$\int_{\frac{1}{2}x}^\infty e^{-v-\frac{1}{4}x^2/v}\frac{dv}{v},$$

and for small values of x (assumed to be real and positive) this integral (by Art. 178) approximates to the form

$$-C+\log(2/x).$$

Thus the solution has the same character near $x=0$ as the solution * denoted by $K_0(x)$; and so

$$K_0(x)=e^{-x}\sqrt{\left(\frac{\pi}{2x}\right)}\left\{1-\frac{1}{8x}+\frac{1^2.3^2}{2!}\frac{1}{(8x)^2}-\dots\right\},$$

at least for real positive values of x.

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*Gray, Mathews and MacRobert, *Bessel Functions* (1922), Ch. III. § 2. More elaborate discussions of the asymptotic series for $I_n(x)$, $K_n(x)$ will be found in Ch. V. § 3 of the same work.

Stokes has proved* that this relation persists for complex values of x, provided that the logarithm and $\sqrt{x}$ are both restricted to take their principal values (Arts. 94, 96).

To discuss the second asymptotic series (given by $\lambda=1$), we note first that the solution

$$I_0(x)=1+\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}+\frac{x^6}{2^2.4^2.6^2}+\ldots$$

can be represented, when x is large, real and positive, by the asymptotic formula†

$$e^x/\sqrt{(2\pi x)}.$$

Thus we are led naturally to the formula

$$I_0(x)=\frac{e^x}{\sqrt{(2\pi x)}}\left\{1+\frac{1}{8x}+\frac{1^2.3^2}{2!}\frac{1}{(8x)^2}+\frac{1^2.3^2.5^2}{3!}\frac{1}{(8x)^3}+\ldots\right\}.$$

As a matter of fact, this result is correct when x is real and positive; but it is clearly false if x is real and negative, because $I_0(x)$ is an even function of x and the right-hand side is not.

Stokes has shewn that, to obtain complete information, three formulae are necessary; thus suppose that $x=\xi+\iota\eta$, and that the square-root is made single-valued by means of a cut along the negative axis of ξ, so that we use always the principal value of $\sqrt{x}$. Then

$$\begin{aligned} &I_0(x)=P+\iota Q, && \text{if } \eta>0,\\ &I_0(x)=P, && \text{if } \eta=0,\quad \xi>0,\\ &I_0(x)=P-\iota Q, && \text{if } \eta<0,\end{aligned}$$

and also $\quad K_0(x)=\pi Q, \quad$ without further conditions.

Here we write

$$P=\frac{e^x}{\sqrt{(2\pi x)}}\left\{1+\frac{1}{8x}+\frac{1^2.3^2}{2!}\frac{1}{(8x)^2}+\frac{1^2.3^2.5^2}{3!}\frac{1}{(8x)^3}+\ldots\right\},$$

$$Q=\frac{e^{-x}}{\sqrt{(2\pi x)}}\left\{1-\frac{1}{8x}+\frac{1^2.3^2}{2!}\frac{1}{(8x)^2}-\frac{1^2.3^2.5^2}{3!}\frac{1}{(8x)^3}+\ldots\right\}.$$

It will be noticed that these formulae for $I_0(x)$ are actually discontinuous along the cut; and that their arithmetic mean coincides with the value given by the P-formula for $I_0(-x)$, the value of $-x$ being now real and positive.

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* *Math. and Phys. Papers*, vol. 4, p. 287.

† Write $r=0$, $s=2$, $b_1=b_2=0$, $\lambda=0$, $\mu=2$, in the general formula of Stokes (Art. 112).

A discussion for general values of n falls rather beyond our limits, but if n is an integer, we may define

$$I_n(x)=x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n I_0(x), \qquad K_n(x)=(-x)^n\left(\frac{1}{x}\frac{d}{dx}\right)^n K_0(x).$$

Then the former solution takes the form

$$I_n(x)=\frac{x^n}{2^n n!}\left\{1+\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}+\ldots;\right\}$$

and, for small values of x, the second solution approximates to

$$\frac{2^{n-1}(n-1)!}{x^n}.$$

Thus the previous results give

$$\begin{aligned} I_n(x)&=R+e^{(n+\frac{1}{2})\pi\iota}S, && \eta>0,\\ I_n(x)&=R, && \xi>0,\ \eta=0,\\ I_n(x)&=R-e^{(n+\frac{1}{2})\pi\iota}S, && \eta<0,\\ K_n(x)&=\pi S, \end{aligned}$$

where

$$R=\frac{e^x}{\sqrt{(2\pi x)}}\left\{1+\frac{1^2-4n^2}{8x}+\frac{(1^2-4n^2)(3^2-4n^2)}{2!(8x)^2}+\ldots,\right\}$$

$$S=\frac{e^{-x}}{\sqrt{(2\pi x)}}\left\{1-\frac{1^2-4n^2}{8x}+\frac{(1^2-4n^2)(3^2-4n^2)}{2!(8x)^2}-\ldots\right\}.$$

118. The ordinary Bessel function.

It is now easy to obtain the asymptotic formula for the ordinary Bessel function; in Art. 117 we replace x by ιx, and then write

$$I_n(\iota x)=\iota^n J_n(x)=e^{\frac{1}{2}n\pi\iota}J_n(x),$$

and assuming the new x to be real and positive, we find the result

$$J_n(x)=e^{-\frac{1}{2}n\pi\iota}R'+e^{\frac{1}{2}(n+1)\pi\iota}S',$$

where R', S' are obtained from R, S, respectively by the substitution of ιx for x.

Thus we get

$$R'=\frac{e^{\iota(x-\frac{1}{4}\pi)}}{\sqrt{(2\pi x)}}(U-\iota V), \quad S'=\frac{e^{-\iota(x+\frac{1}{4}\pi)}}{\sqrt{(2\pi x)}}(U+\iota V),$$

where

$$U=1-\frac{(1^2-4n^2)(3^2-4n^2)}{2!(8x)^2}+\frac{(1^2-4n^2)\ldots(7^2-4n^2)}{4!(8x)^4}-\ldots,$$

$$V=\frac{1^2-4n^2}{8x}-\frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{3!(8x)^3}+\ldots.$$

So, finally, we get the formula

$$J_n(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\{U\cos(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) + V\sin(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\},$$

where x is supposed real and positive.

It has been the practice in many books on analysis to treat the asymptotic series for $J_n(x)$ as the fundamental formula; and to deduce the formula for $I_n(x)$. In spite of the fact that the function $J_n(x)$ was investigated earlier than $I_n(x)$, it seems clearly better to follow Stokes in adopting $I_n(x)$ as the fundamental function. The function $J_n(x)$ is analogous to the sine- and cosine-functions, while $I_n(x)$ corresponds to the exponential function; and it would be found a tedious matter to derive the properties of the exponential from those of the sine-function, although the reverse process is an easy one. By analogy it is more natural to derive the asymptotic formula for $J_n(x)$ from that for $I_n(x)$; and in fact to restrict the use of $J_n(x)$ chiefly to real values of x.

It may be useful to remark that n need not be restricted to be an integer in the final formulae for $I_n(x)$ and $J_n(x)$; but further information will be found in the papers by Stokes quoted in Art. 113.

119. In addition to the asymptotic series given in the preceding articles, certain convergent series have also been found, which can be used in some cases for numerical work.

It is, however, usually impossible to obtain simple general expressions for the terms of these series; some examples are given in Exs. 9, 10 at the end of the chapter. But when the calculation can be made with sufficient accuracy from the first two or three terms, this will not be found a very serious difficulty; although the asymptotic series are almost always the easier in numerical work.

TRIGONOMETRICAL SERIES.

120. Topics included in the present section.

We shall consider first only such trigonometrical series as can be summed by immediate applications of results proved elsewhere in this book. It will be found that the majority of the Fourier series required in the ordinary applications to Mathematical Physics can be handled quite easily; and it is hoped that the results obtained will often be found useful even when more advanced methods are available.

For the sake of easy reference, the formulae will be numbered consecutively, adopting the decimal notation; so that the figure before the decimal point indicates the article and the figures after the point distinguish the particular formula. For brevity we omit the figures 12 from the references to the articles: thus, a reference 2·82 will indicate a formula in Art. 122. The figures after the point are arranged in order of magnitude according to the ordinary notation for decimal fractions; thus we should place 1·81 between 1·8 and 1·9. But we have also found it convenient to regard a group of formulae such as 1·5, 1·51, 1·52 as connected; and this occasionally prevents the strict application of the principle of numerical order of magnitude.

As a general rule, the formulae are obtained for one complete period only, the sums for other values of θ being deduced by the principle of periodicity. Thus, for instance, the series (1·1), (1·2) of Art. 121 can be summed by writing $\theta = 2k\pi + \theta'$, and choosing k so that $0 < \theta' < 2\pi$; and then we obtain the sums:

$$(1\cdot1) \quad -\log\{(-1)^k 2 \sin \tfrac{1}{2}\theta\},$$

$$(1\cdot2) \quad \tfrac{1}{2}\{(2k+1)\pi - \theta\}.$$

The interval selected for the sums is usually either given by $0 < \theta < 2\pi$, or by $-\pi < \theta < \pi$, according to the character of the series.

To facilitate easy reference, the more important series are graphed roughly; but no attempt has been made to obtain exact curves by numerical calculation and plotting on squared paper—more accurate curves will be found in books such as Carslaw's *Fourier's Series, etc.*, and Byerly's *Fourier's Series, etc.*

The methods followed are very largely those of Stokes; but we have given also Dirichlet's classical method (Arts. 126, 127) and some references to the simplest of recent results (Arts. 129–131). But fuller details as to the developments of the subject during the past twenty years would require far greater spac than we can afford.

One remark may be worth making here as to the distinction between Trigonometrical and Fourier series; it is by no means necessary that a convergent Trigonometrical series should belong to the Fourier type. An example to the contrary is given by the two series

$$\sum_{n=2}^{\infty} \frac{\cos n\theta}{\log n}, \quad \sum_{n=2}^{\infty} \frac{\sin n\theta}{\log n}.$$

In virtue of Dirichlet's test (Art. 44) these series converge uniformly in any interval $\gamma < \theta < 2\pi - \gamma$, if $0 < \gamma < \frac{1}{2}\pi$; but the coefficients are not expressible in Fourier's form. However, this distinction will not arise in the course of the work given here.

121. Series which can be summed directly.

We take as the first group of series those which are derived immediately from Art. 65. It was proved there that

$$\left.\begin{array}{ll} (1{\cdot}1) & \cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta + \ldots = -\log(2\sin\frac{1}{2}\theta) \\ (1{\cdot}2) & \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \ldots = \frac{1}{2}(\pi - \theta) \end{array}\right\} \text{ if } 0 < \theta < 2\pi.$$

Changing θ to $\pi - \theta$, we have the two results

$$\left.\begin{array}{ll} (1{\cdot}3) & \cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \ldots = \log(2\cos\frac{1}{2}\theta) \\ (1{\cdot}4) & \sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \ldots = \frac{1}{2}\theta \end{array}\right\} \text{ if } -\pi < \theta < \pi.$$

If we now combine these formulae by adding together the two cosine-series and similarly the two sine-series, we obtain the results

$$\left.\begin{array}{ll} (1{\cdot}5) & \cos\theta + \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta + \ldots = \frac{1}{2}\log(\cot\frac{1}{2}\theta) \\ (1{\cdot}6) & \sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \ldots = \frac{1}{4}\pi \end{array}\right\} \text{ if } 0 < \theta < \pi.$$

Since the last pair of series are not yet evaluated for a complete period, we note that they change sign on writing $-\pi + \theta$ for θ, and so we get the additional results

$$\left.\begin{array}{ll} (1{\cdot}5) & \cos\theta + \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta + \ldots = \frac{1}{2}\log(-\cot\frac{1}{2}\theta) \\ (1{\cdot}6) & \sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \ldots = -\frac{1}{4}\pi \end{array}\right\} \text{ if } \pi < \theta < 2\pi.$$

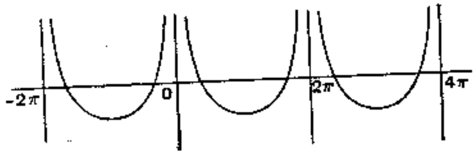
It may be noticed that if we take the difference between the cosine-series (1·1) and (1·3) and the sine-series (1·2) and (1·4), we obtain

$$\left.\begin{array}{l} \cos 2\theta + \frac{1}{2}\cos 4\theta + \frac{1}{3}\cos 6\theta + \ldots = -\log(2\sin\theta) \\ \sin 2\theta + \frac{1}{2}\sin 4\theta + \frac{1}{3}\sin 6\theta + \ldots = \frac{1}{2}(\pi - 2\theta) \end{array}\right\} \text{ if } 0 < \theta < \pi.$$

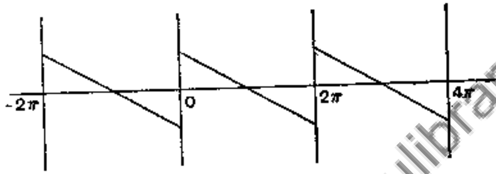
But these results agree precisely with what is given by writing 2θ for θ in the series (1·1) and (1·2), so that nothing fresh is obtained, although our algebra is verified.

The functions represented by these series are indicated roughly in the following diagrams :

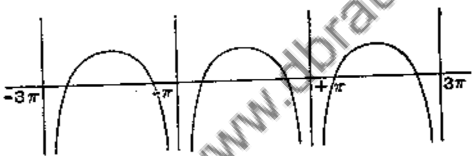
(1·1)



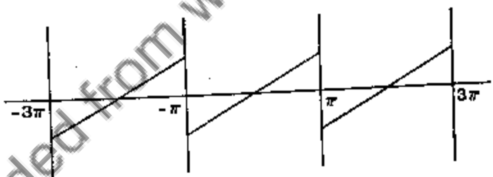
(1·2)



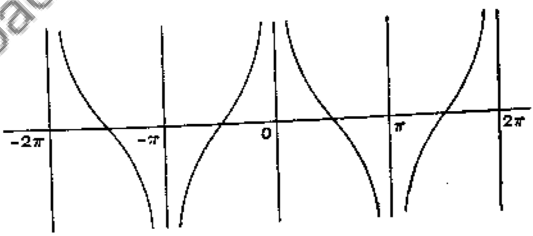
(1·3)



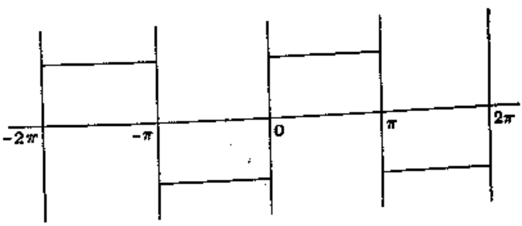
(1·4)



(1·5)



(1·6)



It is easy to derive some other interesting series by combining the foregoing series in other ways. For instance, we find that (1·1) gives

$$\cos(\theta-\alpha)+\tfrac{1}{2}\cos 2(\theta-\alpha)+\tfrac{1}{3}\cos 3(\theta-\alpha)+\ldots=-\tfrac{1}{2}\log\{4\sin^2\tfrac{1}{2}(\theta-\alpha)\}$$

and $$\cos(\theta+\alpha)+\tfrac{1}{2}\cos 2(\theta+\alpha)+\tfrac{1}{3}\cos 3(\theta+\alpha)+\ldots=-\tfrac{1}{2}\log\{4\sin^2\tfrac{1}{2}(\theta+\alpha)\}.$$

These formulae are valid for all values of θ except $\theta=\pm\alpha,\ 2\pi\pm\alpha$, etc.

Now add and subtract, and we obtain the formulae

(1·7) $$\cos\theta\cos\alpha+\tfrac{1}{2}\cos 2\theta\cos 2\alpha+\tfrac{1}{3}\cos 3\theta\cos 3\alpha+\ldots=-\tfrac{1}{4}\log\{4(\cos\theta-\cos\alpha)^2\},$$

(1·8) $$\sin\theta\sin\alpha+\tfrac{1}{2}\sin 2\theta\sin 2\alpha+\tfrac{1}{3}\sin 3\theta\sin 3\alpha+\ldots=+\tfrac{1}{4}\log\left\{\frac{\sin^2\frac{1}{2}(\theta+\alpha)}{\sin^2\frac{1}{2}(\theta-\alpha)}\right\},$$

both of which hold for all values of θ except $\theta=\pm\alpha,\ 2\pi\pm\alpha$, etc.

Similarly we can prove that

(1·71) $$\cos\theta\cos\alpha-\tfrac{1}{2}\cos 2\theta\cos 2\alpha+\tfrac{1}{3}\cos 3\theta\cos 3\alpha-\ldots=\tfrac{1}{4}\log\{4(\cos\theta+\cos\alpha)^2\},$$

(1·81) $$\sin\theta\sin\alpha-\tfrac{1}{2}\sin 2\theta\sin 2\alpha+\tfrac{1}{3}\sin 3\theta\sin 3\alpha-\ldots=\tfrac{1}{4}\log\left\{\frac{\cos^2\frac{1}{2}(\theta-\alpha)}{\cos^2\frac{1}{2}(\theta+\alpha)}\right\},$$

which are valid except for $\theta=\pi\pm\alpha,\ 3\pi\pm\alpha$, etc.

To obtain corresponding results from the sine-series (1·2), (1·4), we must first limit the angle α; say that $0<\alpha<\pi$. Then we have

$$\sin(\theta+\alpha)+\tfrac{1}{2}\sin 2(\theta+\alpha)+\tfrac{1}{3}\sin 3(\theta+\alpha)+\ldots=\tfrac{1}{2}\{\pi-(\theta+\alpha)\},\quad 0<\theta+\alpha<2\pi,$$

$$\sin(\theta-\alpha)+\tfrac{1}{2}\sin 2(\theta-\alpha)+\tfrac{1}{3}\sin 3(\theta-\alpha)+\ldots=\tfrac{1}{2}\{\pi-(\theta-\alpha)\},\quad 0<\theta-\alpha<2\pi,$$

and when θ is less than α, the sum of the second series is diminished by π.

Thus we find that the derived series are

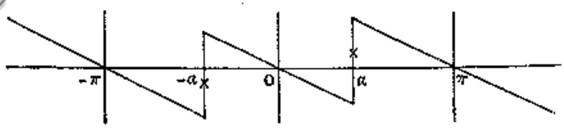
(1·9) $$\sin\theta\cos\alpha+\tfrac{1}{2}\sin 2\theta\cos 2\alpha+\tfrac{1}{3}\sin 3\theta\cos 3\alpha+\ldots=f(\theta),$$

(1·91) $$\cos\theta\sin\alpha+\tfrac{1}{2}\cos 2\theta\sin 2\alpha+\tfrac{1}{3}\cos 3\theta\sin 3\alpha+\ldots=g(\theta),$$

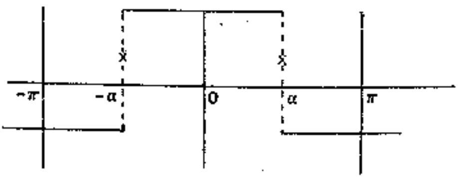
where $$\left.\begin{array}{l} f(\theta)=-\tfrac{1}{2}\theta \\ g(\theta)=\tfrac{1}{2}(\pi-\alpha)\end{array}\right\}\text{if } 0<\theta<\alpha,\qquad \left.\begin{array}{l} f(\theta)=\tfrac{1}{2}(\pi-\theta) \\ g(\theta)=-\tfrac{1}{2}\alpha\end{array}\right\}\text{if } \alpha<\theta<\pi,$$

and $$f(\alpha)=g(\alpha)=\tfrac{1}{4}(\pi-2\alpha).$$

(1·9)



(1·91)



The values outside the range $\theta=0$ to $\theta=\pi$ are given by the relations

$$f(-\theta)=-f(\theta),\quad g(-\theta)=+g(\theta).$$

Ex. 1. Prove that

$$\left.\begin{aligned}(1\cdot52)\quad &\cos\theta-\tfrac{1}{3}\cos 3\theta+\tfrac{1}{5}\cos 5\theta-\ldots=\tfrac{1}{4}\pi,\\ (1\cdot62)\quad &\sin\theta-\tfrac{1}{3}\sin 3\theta+\tfrac{1}{5}\sin 5\theta+\ldots=\tfrac{1}{2}\log(\sec\theta+\tan\theta),\end{aligned}\right\}(-\tfrac{1}{2}\pi<\theta<+\tfrac{1}{2}\pi),$$

and obtain general formulae for the sums. Draw graphs of the functions.

Ex. 2. Obtain formulae equivalent to those given for (1·9) and (1·91) above, starting from the sine-series (1·4).

Ex. 3. Sum the series $S(x)$ which is obtained by omitting all terms for which n is a multiple of k from the series

$$\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ldots,$$

proving that $S(x)=\dfrac{\pi}{2}\left(1-\dfrac{2r+1}{k}\right)$, where r is the integral part of $kx/2\pi$ and $0<x<2\pi$.

Shew that if $0<t<k$, the integral part of t is equal to

$$\frac{1}{2}(k-1)-\frac{k}{\pi}S\left(\frac{2\pi t}{k}\right).$$

[EISENSTEIN.]

Ex. 4. Obtain the series (1·5), (1·6), (1·52), (1·62) from Ex. 41, Chap. VIII.

Ex. 5. Obtain the result $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{1}{4}\pi$ from series (1·2).

122. Series which can be summed by integration.

Various series of interest can be obtained by integrating the series of Art. 121. Consider in the first place series (1·2); it follows from Ex. 5, Art. 44, that this series converges uniformly in any interval $\theta=\gamma$ to $\theta=2\pi-\gamma$, $(0<\gamma<\frac{1}{2}\pi)$. Hence the series may be integrated term-by-term (Art. 45), and we obtain the result

$$\cos\theta+\frac{1}{2^2}\cos 2\theta+\frac{1}{3^2}\cos 3\theta+\ldots=\frac{1}{4}(\theta-\pi)^2+C,$$

where $$\gamma\leqq\theta\leqq 2\pi-\gamma.$$

Now, by Weierstrass's M-test, the integrated series converges uniformly for all real values of θ; and its sum is accordingly a continuous function of θ. Thus, since $(\theta-\pi)^2$ is also continuous up to $\theta=0$ and $\theta=2\pi$, we can now write

$$\cos\theta+\frac{1}{2^2}\cos 2\theta+\frac{1}{3^2}\cos 3\theta+\ldots=\frac{1}{4}(\theta-\pi)^2+C,\quad 0\leqq\theta\leqq 2\pi.$$

To obtain the value of the constant C, write $\theta=0$, $\theta=\pi$; and then

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{1}{4}\pi^2+C,\quad 1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots=-C.$$

But on writing $p=2$ in Ex. 2, Chapter IV., the first of these series is twice the second; and so we have

$$\tfrac{1}{4}\pi^2+C=-2C,\quad\text{or } C=-\tfrac{1}{12}\pi^2.$$

This result is also readily found from the known series (Art. 71·1)

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{1}{6}\pi^2.$$

Thus, finally, we may write

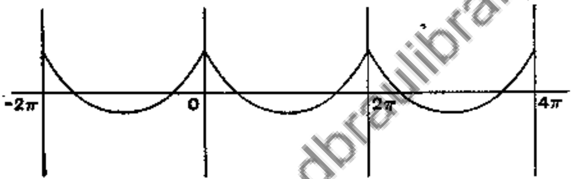
(2·1) $$\cos\theta+\frac{1}{2^2}\cos 2\theta+\frac{1}{3^2}\cos 3\theta+\ldots=\frac{1}{4}(\theta-\pi)^2-\frac{1}{12}\pi^2,\quad 0\leqq\theta\leqq 2\pi.$$

In like manner, from (1·4) we find

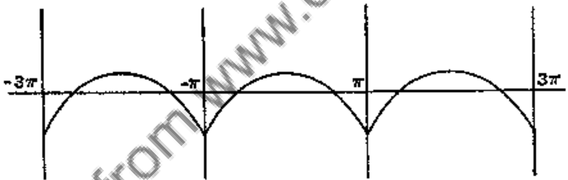
(2·2) $$\cos\theta-\frac{1}{2^2}\cos 2\theta+\frac{1}{3^2}\cos 3\theta-\ldots=-\frac{1}{4}\theta^2+\frac{1}{12}\pi^2,\quad -\pi\leqq\theta\leqq\pi.$$

The graphs are as below:

(2·1)



(2·2)



By addition of (2·1) and (2·2), we find that

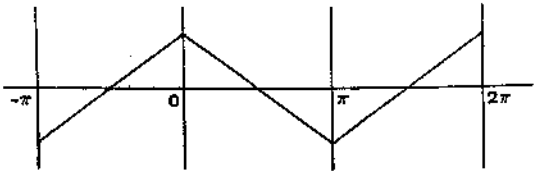
(2·21) $$\cos\theta+\frac{1}{3^2}\cos 3\theta+\frac{1}{5^2}\cos 5\theta+\ldots=\frac{1}{8}(\pi^2-2\pi\theta),\quad 0\leqq\theta\leqq\pi.$$

By changing θ to $-\theta$, we deduce that

(2·22) $$\cos\theta+\frac{1}{3^2}\cos 3\theta+\frac{1}{5^2}\cos 5\theta+\ldots=\frac{1}{8}(\pi^2+2\pi\theta),\quad -\pi\leqq\theta\leqq 0.$$

The graph is as below:

(2·21)



By changing θ to $\frac{1}{2}\pi-\theta$ in (2·21), we find that

(2·22) $$\sin\theta-\frac{1}{3^2}\sin 3\theta+\frac{1}{5^2}\sin 5\theta-\ldots=\frac{1}{4}\pi\theta,\quad -\tfrac{1}{2}\pi\leqq\theta\leqq\tfrac{1}{2}\pi,$$

$$\text{or } \tfrac{1}{4}\pi(\pi-\theta),\quad \tfrac{1}{2}\pi\leqq\theta\leqq\tfrac{3}{2}\pi.$$

Similarly, by integrating the series (1·9) of Art. 121, we have

$$\cos\theta\cos\alpha+\frac{1}{2^2}\cos 2\theta\cos 2\alpha+\frac{1}{3^2}\cos 3\theta\cos 3\alpha+\ldots=F(\theta),$$

where $F(\theta)=\frac{1}{4}\theta^2+C_1,\quad$ if $0\leqq\theta\leqq\alpha\leqq\pi$;

or $F(\theta)=\frac{1}{4}(\theta-\pi)^2+C_2,\quad$ if $0\leqq\alpha\leqq\theta\leqq\pi$.

We infer that the signs of equality may be included here by means of the fact that the integrated series converges uniformly for all real values of θ. Thus the sum is continuous at $\theta=\alpha$, and accordingly we can write

$$F(\theta)=\tfrac{1}{4}\theta^2+\tfrac{1}{4}(\alpha-\pi)^2+C,\quad\text{when } 0\leqq\theta\leqq\alpha,$$

or

$$\tfrac{1}{4}\alpha^2+\tfrac{1}{4}(\theta-\pi)^2+C,\quad\text{when } \alpha\leqq\theta\leqq\pi.$$

To obtain the value of C, write $\theta=0$, and then the series reduces to (2·1), which gives *

$$C=-\tfrac{1}{12}\pi^2.$$

Thus, finally, we can write

(2·3) $$\cos\theta\cos\alpha+\frac{1}{2^2}\cos 2\theta\cos 2\alpha+\frac{1}{3^2}\cos 3\theta\cos 3\alpha+\ldots$$

$$=\tfrac{1}{4}\theta^2+\tfrac{1}{4}(\alpha-\pi)^2-\tfrac{1}{12}\pi^2,\quad\text{if } 0\leqq\theta\leqq\alpha,$$

or

$$\tfrac{1}{4}\alpha^2+\tfrac{1}{4}(\theta-\pi)^2-\tfrac{1}{12}\pi^2,\quad\text{if } \alpha\leqq\theta\leqq\pi.$$

The reader should have no difficulty in proving that these results remain valid from $-\alpha$ to $+\alpha$, and from $+\alpha$ to $2\pi-\alpha$ respectively; and it is then easy to draw a graph to represent the function.

Similarly, we find the results

(2·31) $$\cos\theta\cos\alpha-\frac{1}{2^2}\cos 2\theta\cos 2\alpha+\frac{1}{3^2}\cos 3\theta\cos 3\alpha-\ldots$$

$$=\tfrac{1}{12}\pi^2-\tfrac{1}{4}(\alpha^2+\theta^2),\quad\text{if } -(\pi-\alpha)\leqq\theta\leqq\pi-\alpha,$$

or

$$\tfrac{1}{12}\pi^2-\tfrac{1}{4}\{(\alpha-\pi)^2+(\theta-\pi)^2\},\quad\text{if } \pi-\alpha\leqq\theta\leqq\pi+\alpha.$$

Series which are often used in Applied Mathematics* are given

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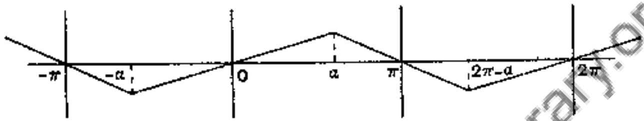
* For instance, in the problem of the *plucked string* (Rayleigh's *Theory of Sound*, Art. 127).

The formula usually given is found by multiplying by the constant $2c/\alpha(\pi-\alpha)$, (so that the sum is equal to c at $\theta=\alpha$), and then writing $\theta=\pi x/l$, $\alpha=\pi b/l$. Thus the sum is equal to cx/b from $x=0$ to b, and to $c(l-x)/(l-b)$ from $x=b$ to l.

by integrating similarly the series (1·91) of Art. 121. Repeating the process already used, we find that $(0 < \alpha < \pi)$

(2·4) $$\sin\theta\sin\alpha + \frac{1}{2^2}\sin 2\theta\sin 2\alpha + \frac{1}{3^2}\sin 3\theta\sin 3\alpha + \dots$$
$$= \tfrac{1}{2}\theta(\pi - \alpha), \qquad \text{if} \qquad -\alpha \leqq \theta \leqq \alpha.$$
or $$\tfrac{1}{2}\alpha(\pi - \theta), \qquad \text{if } \alpha \leqq \theta \leqq 2\pi - \alpha.$$

The graph is as below:

(2·4)


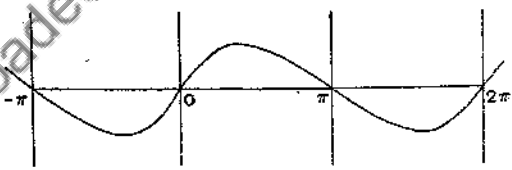
By integrating series (2·1), (2·2), (2·21), (2·22) again, we find the results

(2·5) $$\sin\theta + \frac{1}{2^3}\sin 2\theta + \frac{1}{3^3}\sin 3\theta + \dots = \frac{1}{12}\{(\theta - \pi)^3 - \pi^2\theta + \pi^3\},$$
where $0 \leqq \theta \leqq 2\pi$.

(2·6) $$\sin\theta - \frac{1}{2^3}\sin 2\theta + \frac{1}{3^3}\sin 3\theta - \dots = \frac{1}{12}(\pi^2\theta - \theta^3),$$
where $-\pi \leqq \theta \leqq \pi$.

(2·7) $$\sin\theta + \frac{1}{3^3}\sin 3\theta + \frac{1}{5^3}\sin 5\theta + \dots$$
$$= \tfrac{1}{8}(\pi^2\theta - \pi\theta^2), \text{ if } 0 \leqq \theta \leqq \pi, \quad \text{or } \tfrac{1}{8}(\pi^2\theta + \pi\theta^2), \text{ if } -\pi \leqq \theta \leqq 0.$$

The graph of (2·6) is as follows:

(2·6)


and the reader should draw a similar graph for (2·5). The graph of (2·7) is almost the same as (2·6) in a rough sketch. But in theory, the curve in (2·6) consists of the two loops of a cubic (from $-\pi$ to $+\pi$), repeated over and over again; while that in (2·7) consists of two equal parabolas.

The series

(2·71) $$\cos\theta - \frac{1}{3^3}\cos 3\theta + \frac{1}{5^3}\cos 5\theta - \dots = \frac{1}{8}\pi(\tfrac{1}{4}\pi^2 - \theta^2),$$
$$-\tfrac{1}{2}\pi \leqq \theta \leqq \tfrac{1}{2}\pi,$$

can be summed by writing $\frac{1}{2}\pi - \theta$ for θ in (2·7).

By integrating (2·5) again, we have

$$\cos\theta+\frac{1}{2^4}\cos 2\theta+\frac{1}{3^4}\cos 3\theta+\ldots=\frac{1}{48}\{2\pi^2(\theta-\pi)^2-(\theta-\pi)^4\}+C,$$
$$0\leqq\theta\leqq 2\pi.$$

Writing $\theta=0$ and $\theta=\pi$, we get

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{1}{48}\pi^4+C,\quad 1-\frac{1}{2^4}+\frac{1}{3^4}-\ldots=-C.$$

Now the second series is $\frac{7}{8}$ths of the first (by writing $p=4$ in Ex. 2 of Ch. IV.), so that $C=-\frac{7\pi^4}{720}$ (giving $\frac{1}{90}\pi^4$ for the sum of the first series, as in Art. 71·2).

Hence, finally,

(2·8) $$\cos\theta+\frac{1}{2^4}\cos 2\theta+\frac{1}{3^4}\cos 3\theta+\ldots$$
$$=\frac{1}{48}\{2\pi^2(\theta-\pi)^2-(\theta-\pi)^4-\tfrac{7}{15}\pi^4\},$$
where $0\leqq\theta\leqq 2\pi$,

and by writing $\pi+\theta$ for θ, we deduce that

(2·81) $$\cos\theta-\frac{1}{2^4}\cos 2\theta+\frac{1}{3^4}\cos 3\theta-\ldots=\frac{1}{48}(\theta^4-2\pi^2\theta^2+\tfrac{7}{15}\pi^4),$$
where $-\pi\leqq\theta\leqq+\pi$.

Also by adding (2·81) to (2·8), we find that*

(2·82) $$\cos\theta+\frac{1}{3^4}\cos 3\theta+\frac{1}{5^4}\cos 5\theta+\ldots=\frac{1}{96}\pi(4\theta^3-6\pi\theta^2+\pi^3),$$
where $0\leqq\theta\leqq\pi$,

and by changing from θ to $\frac{1}{2}\pi-\theta$, we deduce that

(2·83) $$\sin\theta-\frac{1}{3^4}\sin 3\theta+\frac{1}{5^4}\sin 5\theta-\ldots=\frac{1}{96}\pi\theta(3\pi^2-4\theta^2),$$
where $-\frac{1}{2}\pi\leqq\theta\leqq\frac{1}{2}\pi$.

Practically all these results were worked out by D. Bernoulli and Euler; we shall see (in Art. 125) the connexion between these formulae and the Bernoullian functions (Art. 101).

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* To save space we omit the completion of (2·82), (2·83); to obtain the sum of (2·82) from $\theta=-\pi$ to 0, change the sign of θ; and to deal with (2·83) change θ to $\pi-\theta$. It may be noted that the right-hand side of (2·82) can be factorised in the form $\frac{1}{96}\pi(2\theta-\pi)(2\theta^2-2\pi\theta-\pi^2)$.

Ex. 1. Prove that
$$\sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \dots$$
$$+\frac{2}{\pi}\left(\sin\theta - \frac{1}{3^2}\sin 3\theta + \frac{1}{5^2}\sin 5\theta - \dots\right) = F(\theta),$$
where
$$F(\theta) = \theta, \quad -\tfrac{1}{2}\pi \leqq \theta \leqq \tfrac{1}{2}\pi,$$
or
$$+\tfrac{1}{2}\pi, \quad \tfrac{1}{2}\pi \leqq \theta < \pi,$$
or
$$-\tfrac{1}{2}\pi, \quad -\pi < \theta \leqq -\tfrac{1}{2}\pi.$$
Draw a graph of the function.

Ex. 2. Deduce from (2·71) that
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32},$$
and that
$$1 + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \dots = \frac{3\pi^3\sqrt{2}}{128}.$$
From (2·83) shew that
$$1 - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} - \dots = \frac{11\pi^4\sqrt{2}}{1536}.$$

123. Recognition of discontinuities in the sum of a trigonometrical series.

A rapid determination of the values of θ for which a given series is likely to be discontinuous is often very useful; the method adopted here is substantially the same as one due to Stokes for the case of Fourier-series.*

In the series with which we are concerned at present the coefficients of $\cos n\theta$ and $\sin n\theta$ are either simply algebraic fractions in n, or else are the product of such fractions by terms such as $\cos n\alpha$, $\sin n\alpha$—see, for instance, series (1·9), (1·91), (2·3), (2·4) in Arts. 121, 122. The second class of series can usually be reduced to the sum or difference of two series of the first class.

See, for instance, the methods of summation adopted for (1·7)–(1·91), and (2·3), (2·4) could be divided into two series similarly. For example, (2·4) is equal to
$$\frac{1}{2}\left\{\cos(\theta-\alpha) + \frac{1}{2^2}\cos 2(\theta-\alpha) + \frac{1}{3^2}\cos 3(\theta-\alpha) + \dots\right.$$
$$\left. - \cos(\theta+\alpha) - \frac{1}{2^2}\cos 2(\theta+\alpha) - \frac{1}{3^2}\cos 3(\theta+\alpha) - \dots\right\}.$$

Thus we may restrict our work to series of the type $\Sigma a_n \cos n\theta$, $\Sigma a_n \sin n\theta$, where a_n is an algebraic fraction in n. Further, if we write $a_n = f(n)/F(n)$, where f, F are polynomials, the degree of the denominator must be less than that of the numerator, otherwise the

* *Math. and Phys. Papers*, vol. i. pp. 249 and 255.

series could not converge at all (because a_n must tend to zero, to provide for convergence).

Then, if p denotes the excess of the degree of $F(n)$ over the degree of $f(n)$, it is clear that $a_n = O(n^{-p})$, when $n \to \infty$. Hence, *if* $p \geqq 2$, *it follows from Weierstrass's M-test that the sum of the series is continuous for all real values of* θ (Arts. 44, 45).

Thus discontinuities can occur only if $p=1$; let us suppose that $na_n \to A$, then it is clear that

$$na_n - A = O(1/n) \quad \text{or} \quad a_n - \frac{A}{n} = b_n = O\left(\frac{1}{n^2}\right),$$

because a_n is a rational algebraic fraction.

Hence $$\Sigma a_n \cos n\theta - A\Sigma \frac{1}{n} \cos n\theta = \Sigma b_n \cos n\theta$$

and $$\Sigma a_n \sin n\theta - A\Sigma \frac{1}{n} \sin n\theta = \Sigma b_n \sin n\theta.$$

By what has just been proved, we see that $\Sigma b_n \cos n\theta$, $\Sigma b_n \sin n\theta$ are continuous for all real values of θ; and accordingly $\Sigma a_n \cos n\theta$, $\Sigma a_n \sin n\theta$ are discontinuous at $\theta = 0, \pm 2\pi, \pm 4\pi$, etc.

Now we know the character of these special series from the results given in (1·1) and (1·2) of Art. 121. Thus, when θ is small but positive, we can write

$$\Sigma \frac{1}{n} \cos n\theta \to \log(1/\theta), \qquad \Sigma \frac{1}{n} \sin n\theta = \tfrac{1}{2}(\pi - \theta), \qquad \theta > 0,$$

and changing the sign of θ, we obtain

$$\Sigma \frac{1}{n} \cos n\theta \to \log(-1/\theta), \quad \Sigma \frac{1}{n} \sin n\theta = \tfrac{1}{2}(-\pi - \theta), \quad \theta < 0.$$

Summing up, we can write

$$\Sigma \frac{1}{n} \cos n\theta \to \tfrac{1}{2} \log(1/\theta^2), \quad \Sigma \frac{1}{n} \sin n\theta = \tfrac{1}{2}(-\theta \pm \pi),$$

where θ is small and the ambiguous sign $\pm$ is the same as that of θ.

Consequently, if we write $B = \Sigma b_n = \Sigma(a_n - A/n)$, we find that

$$\left.\begin{aligned} &\Sigma a_n \cos n\theta \to \tfrac{1}{2} A \log(1/\theta^2) + B \\ \text{and} \quad &\Sigma a_n \sin n\theta \to \pm \tfrac{1}{2} A\pi, \end{aligned}\right\} \text{ for small values of } \theta,$$

where the ambiguous sign agrees with that of θ.

To illustrate the method, we consider the series (1·3), (1·4) of Art. 121. We write these series in the form

(1·3) $\qquad -\{\cos(\theta - \pi) + \tfrac{1}{2}\cos 2(\theta - \pi) + \tfrac{1}{3}\cos 3(\theta - \pi) + \ldots\}$,

(1·4) $\qquad -\{\sin(\theta - \pi) + \tfrac{1}{2}\sin 2(\theta - \pi) + \tfrac{1}{3}\sin 3(\theta - \pi) + \ldots\}$.

Thus (between 0 and 2π) the only discontinuity occurs at $\theta=\pi$, for both series; and since $A=1$, $B=0$, we see that the character of the cosine series is given by $-\frac{1}{2}\log\{1/(\theta-\pi)^2\}=\log\{(\theta-\pi)\}$ and that of the sine-series by $\pm\frac{1}{2}\pi$, where the ambiguous sign is the same as that of $(\pi-\theta)$. To see that the former result is correct, we note that (when θ is slightly less than π)

$$\log(2\cos\tfrac{1}{2}\theta)\rightarrow\log(\pi-\theta).$$

124. Differentiation of trigonometrical series.

Using the notation of the last article, it follows from Art. 46 that differentiation term-by-term will lead to correct results when $p\geqq 2$; except near $\theta=0,\ \pm 2\pi,\ \pm 4\pi,\ \ldots$, when $p=2$.

On the other hand, when $p=1$, we adopt the device of writing

$$a_n-A/n=b_n=O(1/n^2),\quad \text{as in the last article.}$$

Then, if $\quad \Sigma a_n\cos n\theta=F(\theta)\quad$ and $\quad \Sigma a_n \sin n\theta=G(\theta)$,

we have and
$$\left.\begin{aligned} F(\theta)+A\log(2\sin\tfrac{1}{2}\theta)&=\Sigma b_n\cos n\theta\\ G(\theta)-\tfrac{1}{2}A(\pi-\theta)&=\Sigma b_n\sin n\theta\end{aligned}\right\}\quad 0<\theta<2\pi.$$

Thus, differentiating, we find that

$$F'(\theta)+\tfrac{1}{2}A\cot\tfrac{1}{2}\theta=-\Sigma nb_n\sin n\theta$$

and

$$G'(\theta)+\tfrac{1}{2}A\qquad=+\Sigma nb_n\cos n\theta.$$

These formulae will serve to deal with nearly all series in common use in analysis; but it is often more convenient to transform them by writing

$$F(\theta)+\iota G(\theta)=P(\theta)=\Sigma a_n e^{\iota n\theta}.$$

Then combining the two equations, we find that

$$P'(\theta)+\tfrac{1}{2}A(\cot\tfrac{1}{2}\theta+\iota)=\iota\Sigma nb_n e^{\iota n\theta}$$

or

$$P'(\theta)+\frac{A\iota e^{\iota\theta}}{e^{\iota\theta}-1}=\iota\Sigma nb_n e^{\iota n\theta}$$
$$=\iota\Sigma(na_n-A)e^{\iota n\theta}.$$

Thus $\quad P'(\theta)=\iota\Sigma(na_n-A)e^{\iota n\theta}+A\iota e^{\iota\theta}/(1-e^{\iota\theta}).$

In actual work it is convenient to write *symbolically*

$$P'(\theta)=\iota\Sigma na_n e^{\iota n\theta},$$

just as if the resulting series were convergent; and then *to interpret this symbolical formula* by writing

$$A\iota\sum_1^\infty e^{\iota n\theta}=A\iota e^{\iota\theta}/(1-e^{\iota\theta}),$$

which may be regarded as a symbolic extension of the familiar equation

$$\sum_1^\infty x^n=x/(1-x),\quad |x|<1,$$

deduced by writing $x=e^{\iota\theta}$.

An incidental advantage in this way of arranging the work is that the *limits* of θ are at once suggested by the fact that the denominator vanishes for $\theta=0, \pm 2\pi$, etc.

To illustrate the method, let us take the series (1·5), (1·6) of Art. 121.

Then $$P(\theta)=e^{\iota\theta}+\tfrac{1}{3}e^{3\iota\theta}+\tfrac{1}{5}e^{5\iota\theta}+\dots,$$

so symbolically $$P'(\theta)=\iota(e^{\iota\theta}+e^{3\iota\theta}+e^{5\iota\theta}+\dots),$$

giving $$P'(\theta)=\iota e^{\iota\theta}/(1-e^{2\iota\theta}), \quad \text{if } 0<\theta<\pi,$$
$$=-\tfrac{1}{2}\operatorname{cosec}\theta.$$

Thus $$P(\theta)=-\tfrac{1}{2}\log(\tan\tfrac{1}{2}\theta)+\text{const.} \quad \text{if } 0<\theta<\pi.$$

To find the constant, let $\theta=\pi/2$, which shews that the constant is equal to
$$\iota(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\dots)=\tfrac{1}{4}\pi\iota,$$
and so we obtain again the formulae of Art. 121.

We might, of course, wish to obtain formulae applicable, say, for negative values of θ; then we find as before
$$P'(\theta)=-\tfrac{1}{2}\operatorname{cosec}\theta, \quad \text{if } -\pi<\theta<0,$$
leading to $$P(\theta)=-\tfrac{1}{2}\log(-\tan\tfrac{1}{2}\theta)+\text{const.},$$
and the constant is found to be $-\tfrac{1}{4}\pi\iota$ by writing $\theta=-\tfrac{1}{2}\pi$.

Similarly for other intervals for θ.

An example of a slightly more complicated type is given by the series
$$F(\theta)=1+2a^2\sum_1^\infty(-1)^n\frac{\cos n\theta}{n^2+a^2}.$$

This gives $$P(\theta)=1+2a^2\sum_1^\infty(-1)^n\frac{e^{\iota n\theta}}{n^2+a^2},$$

leading to the symbolical formulae
$$P''(\theta)=-2a^2\sum_1^\infty\frac{(-1)^n n^2}{n^2+a^2}e^{\iota n\theta}$$

and $$P''(\theta)-a^2P(\theta)=-a^2-2a^2\sum_1^\infty(-1)^n e^{\iota n\theta}.$$

Thus
$$P''(\theta)-a^2P(\theta)=-a^2+2a^2\frac{e^{\iota\theta}}{1+e^{\iota\theta}}=+a^2\frac{e^{\iota\theta}-1}{e^{\iota\theta}+1}=+\iota a^2\tan\tfrac{1}{2}\theta,$$
the limits for θ being given by $-\pi<\theta<+\pi$.

It follows that $$F''(\theta)-a^2F(\theta)=0,$$
supposing *a real*, and accordingly
$$F(\theta)=A\cosh a\theta, \quad -\pi<\theta<+\pi,$$
because $F(\theta)$ is an even function of θ, so that $\sinh a\theta$ is not part of the solution.

To find A, put $\theta=0$; and then

$$A=1+2a^2\sum\frac{(-1)^n}{n^2+a^2}=\frac{a\pi}{\sinh a\pi},\quad \text{by Art. 99.}$$

Thus we can write

(4·1) $$\frac{a\pi\cosh a\theta}{\sinh a\pi}=1+2a^2\Sigma(-1)^n\frac{\cos n\theta}{n^2+a^2},\quad -\pi\leqq\theta\leqq\pi$$ [Fourier],

the signs of equality being now included, since the series converges uniformly for all real values of θ.

By differentiation of (4·1) we obtain

(4·2) $$\pi\frac{\sinh a\theta}{\sinh a\pi}=2\Sigma(-1)^{n-1}\frac{n\sin n\theta}{n^2+a^2},\quad -\pi<\theta<\pi.$$ [Fourier.]

It is instructive to differentiate (4·2) by a direct application of the original method given at the beginning of this article. Thus we write

$$G(\theta)=-2\sum\frac{n\sin n(\theta+\pi)}{n^2+a^2},$$

and then $A=-2.$

Thus $$G'(\theta)-1=\sum n\left(\frac{2}{n}-\frac{2n}{n^2+a^2}\right)\cos n(\theta+\pi)=2a^2\sum\frac{(-1)^n\cos n\theta}{n^2+a^2}$$

or $G'(\theta)=F(\theta)$, which is at once verified. [*Math. Trip.* 1902.]

The reader will see without difficulty that the method adopted in the present article is substantially equivalent to a rule due to Stokes (Art. 128 below).*

Ex. 1. If $$F(\theta)=\sum_2^\infty\frac{\cos n\theta}{n^2-1},\qquad G(\theta)=\sum_2^\infty\frac{\sin n\theta}{n^2-1},$$

verify that if $0<\theta<2\pi$,

$$P'+\iota P=\iota e^{\iota\theta}\{-\lambda+\tfrac{1}{2}\iota(\pi-\theta)\},$$
$$P'-\iota P=-\iota(1+\tfrac{1}{2}e^{\iota\theta})+\iota e^{-\iota\theta}\{-\lambda+\tfrac{1}{2}\iota(\pi-\theta)\},$$

where $\lambda=\log(2\sin\tfrac{1}{2}\theta)$.

Deduce that

(4·3) $$F(\theta)=\tfrac{1}{2}+\tfrac{1}{4}\cos\theta-\tfrac{1}{2}(\pi-\theta)\sin\theta,$$

(4·31) $$G(\theta)=\tfrac{1}{4}\sin\theta-\sin\theta\log(2\sin\tfrac{1}{2}\theta).$$

* The present method was given, in a slightly condensed form, in Art. 90 of the first edition of this book. Other methods, depending on similar ideas, but more troublesome in practical work, have been suggested by Lerch (*Ann. de l'École Normale* (3), vol. 12, 1895, p. 351) and Brenke (*Annals of Mathematics* (2), vol. 8, 1907, p. 87).

Ex. 2. Verify the formula of Ex. 1 by writing

$$\frac{1}{n^2-1}=\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right),$$

and rearranging the series.

Ex. 3. Prove that if

$$F(\theta)=\tfrac{1}{3}\cos\theta+\frac{\cos 3\theta}{1.3.5}-\frac{\cos 5\theta}{3.5.7}+\frac{\cos 7\theta}{5.7.9}-\dots,$$

then $$F'''(\theta)+4F'(\theta)=0,\quad \text{if } -\tfrac{1}{2}\pi<\theta<\tfrac{1}{2}\pi.$$

Deduce that

(4·4) $$F(\theta)=\tfrac{1}{8}\pi\cos^2\theta,\quad \text{if } -\tfrac{1}{2}\pi\leqq\theta\leqq\tfrac{1}{2}\pi.$$

125. Extension of the method of Art. 124.

Suppose next that we have to deal with series in which the summation extends from $-\infty$ to $+\infty$; and that $na_n\to A$, as n tends to either $-\infty$ or $+\infty$.

We now introduce the two series*

$$\sum_{-\infty}^{\infty}{}'\frac{1}{n}\cos n\theta=0,\quad \sum_{-\infty}^{\infty}{}'\frac{1}{n}\sin n\theta=\pi-\theta,$$

where $0<\theta<2\pi$, and the accent indicates that $n=0$ is omitted.

Then, if we write

$$F(\theta)=\sum_{-\infty}^{\infty}a_n\cos n\theta,\quad G(\theta)=\sum_{-\infty}^{\infty}a_n\sin n\theta,$$

we have $$F(\theta)+\quad 0\quad =a_0+\sum_{-\infty}^{\infty}{}'b_n\cos n\theta,$$

$$G(\theta)-A(\pi-\theta)=\sum_{-\infty}^{\infty}{}'b_n\sin n\theta,$$

where, as in Art. 124, $\quad b_n=a_n-A/n.$

Thus $$F'(\theta)=-\sum_{-\infty}^{\infty}{}'nb_n\sin n\theta=-\sum_{-\infty}^{\infty}(na_n-A)\sin n\theta,$$

$$G'(\theta)+A=\sum_{-\infty}^{\infty}{}'nb_n\cos n\theta=A+\sum_{-\infty}^{\infty}(na_n-A)\cos n\theta,$$

and so we obtain the more compact formula

$$P'(\theta)=\iota\sum_{-\infty}^{\infty}(na_n-A)e^{\iota n\theta},$$

where $$P(\theta)=\sum_{-\infty}^{\infty}a_ne^{\iota n\theta}=F(\theta)+\iota G(\theta).$$

* Since the positive and negative parts of the series converge separately (Ex., Art. 22), we can group together corresponding positive and negative terms. Then the cosine-series vanishes identically, and the value of the sine-series follows from (1·2) of Art. 121.

To illustrate the method, let us take

$$F(\theta)=\sum_{-\infty}^{\infty}\frac{\cos n\theta}{n-a},\quad G(\theta)=\sum_{-\infty}^{\infty}\frac{\sin n\theta}{n-a},$$

so that $$P(\theta)=\sum_{-\infty}^{\infty}\frac{e^{\iota n\theta}}{n-a}.$$

Then $A=1$, and so we have

$$P'(\theta)=\iota\sum_{-\infty}^{\infty}\left(\frac{n}{n-a}-1\right)e^{\iota n\theta}=\iota a\sum_{-\infty}^{\infty}\frac{e^{\iota n\theta}}{n-a}=\iota aP(\theta).$$

Accordingly we have

$$P(\theta)=Ce^{\iota a\theta},\qquad (0<\theta<2\pi).$$

But, putting $\theta=\pi$, we have

$$Ce^{\iota a\pi}=-\frac{1}{a}-\frac{1}{1-a}+\frac{1}{2-a}-\frac{1}{3-a}+\dots+\frac{1}{1+a}-\frac{1}{2+a}+\frac{1}{3+a}-\dots$$

$$=-\left(\frac{1}{a}-\frac{1}{a-1}-\frac{1}{a+1}+\frac{1}{a-2}+\frac{1}{a+2}-\dots\right)=-\frac{\pi}{\sin(a\pi)}.\quad \text{(Art. 99.)}$$

Thus $$C=-\frac{2\pi\iota}{e^{2a\pi\iota}-1},$$

(5·1) and so $$P(\theta)=-\frac{2\pi\iota e^{\iota a\theta}}{e^{2a\pi\iota}-1}.$$

(5·11) Hence $$F(\theta)=-\frac{\pi\cos a(\pi-\theta)}{\sin a\pi},\quad G(\theta)=\frac{\pi\sin a(\pi-\theta)}{\sin a\pi},\quad 0<\theta<2\pi.$$

It should be noticed that a is not restricted and may be *complex*.

By writing $2a\pi\iota=t$ and $\theta=2\pi x$, the above result (5·1) may be written

(5·2) $$\frac{e^{xt}}{e^t-1}=\sum_{-\infty}^{\infty}\frac{e^{2n\pi\iota x}}{t-2n\pi\iota}$$

(5·21) or $$\frac{te^{xt}}{e^t-1}=1-\sum_{-\infty}^{\infty}{}'\frac{te^{2n\pi\iota x}}{2n\pi\iota-t},\quad \text{if } 0<x<1,$$

where $n=0$ is omitted from the summation in (5·21).

We can expand both sides of (5·21) in powers of t, provided that $|t|<2\pi$; and thus we find (Art. 101) the results

(5·3) $$2\sum_{1}^{\infty}\frac{\sin 2n\pi x}{2n\pi}=\tfrac{1}{2}-\phi_1(x)=\tfrac{1}{2}-x.$$

(5·4) $$2\sum_{1}^{\infty}\frac{\cos 2n\pi x}{(2n\pi)^2}=\frac{1}{2!}\{\phi_2(x)+B_1\}=\tfrac{1}{2}(x^2-x+\tfrac{1}{6}).$$

(5·5) $$2\sum_{1}^{\infty}\frac{\sin(2n\pi x)}{(2n\pi)^{2k-1}}=\frac{(-1)^k}{(2k-1)!}\phi_{2k-1}(x),\quad k>1.$$

(5·6) $$2\sum_{1}^{\infty}\frac{\cos 2n\pi x}{(2n\pi)^{2k}}=\frac{(-1)^{k-1}}{(2k)!}\{\phi_{2k}(x)+(-1)^{k-1}B_k\}.$$

These results are due (in general) to Raabe and Schlömilch; and include the series found in Art. 122 by repeated integration.

Ex. 1. Deduce (5·1) from (4·1) and (4·2) by writing ιa for a.

Ex. 2. It has been tacitly assumed that a is not an integer. If we now make $a \to m$ (a positive integer) in (5·1), prove that we obtain the results

(5·12)
$$\left.\begin{aligned} -\frac{1}{m}-\frac{\cos m\theta}{2m}+2m\sum_1^{\infty}{}'\frac{\cos n\theta}{n^2-m^2} &= -(\pi-\theta)\sin m\theta, \\ \frac{\sin m\theta}{2m}+2\sum_1^{\infty}{}' n\frac{\sin n\theta}{n^2-m^2} &= (\pi-\theta)\cos m\theta, \end{aligned}\right\} 0<\theta<2\pi,$$

where $n=m$ is omitted from the summation.

Deduce Exs. 13, 14, Chap. III.; and compare Ex. 1, Art. 124.

Ex. 3. Deduce from (5·1) that

$$\left.\begin{aligned} F(\theta)=\sum_1^{\infty}\frac{\cos n\theta}{n^2-a^2} &= \frac{\pi\sin a(\frac{1}{2}\pi-\theta)}{4a\cos\frac{1}{2}a\pi}, \\ G(\theta)=\sum_1^{\infty}\frac{n\sin n\theta}{n^2-a^2} &= \frac{\pi\cos a(\frac{1}{2}\pi-\theta)}{4\cos\frac{1}{2}a\pi}, \end{aligned}\right\} 0<\theta<\pi,$$

where $n=1, 3, 5, 7, \ldots$ to ∞.

Obtain these results also, as in Art. 124, by proving that

$$F'(\theta)=-G(\theta),\quad G'(\theta)=a^2F(\theta),$$

and, from Art. 123, that, as $\theta\to\frac{1}{2}\pi$, $F(\frac{1}{2}\pi)=0$,

while $\qquad G(\theta)\to\frac{1}{4}\pi$ as $\theta\to 0$.

Verify the conclusions also by considering $F(0)$ and $G(\frac{1}{2}\pi)$. See Chap. IX.

Ex. 4. Prove that if $2r<\lambda<2(r+1)$,

(5·7)
$$\sum_{-\infty}^{\infty}\frac{e^{\iota\lambda(x+n\pi)}}{x+n\pi}=\frac{e^{(2r+1)\iota x}}{\sin x},$$

and discuss the sum when λ is an even integer.

Ex. 5. Prove that if $0<\theta<2\pi$,

(5·8)
$$\sum_{-\infty}^{\infty}\frac{\cos(n-a)\theta}{n-a}=-\frac{\pi}{\tan a\pi},\qquad \sum_{-\infty}^{\infty}\frac{\sin(n-a)\theta}{n-a}=\pi.$$

126. Dirichlet's summation of Fourier's series.

The assumption that a function $f(x)$ can be expressed as a uniformly convergent series,

$$f(x)=a_0+\Sigma(a_n\cos nx+b_n\sin nx),\quad 0\leqq x\leqq 2\pi,$$

leads at once to the formulae for the coefficients

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx,$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx.$$

It is, however, a fact that these formulae lead to correct results in various series which do not converge uniformly; for instance, the series (1·2), (1·4), (1·6), (1·9), (1·91) of Art. 121 cannot converge uniformly for a whole period, because each of these series has at least one discontinuity.

To deal with the general question as to the representation of $f(x)$ by the above formula, take the sum to $n+1$ terms. This is easily seen to be

$$S_n(x)=\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi\left\{1+2\sum_{r=1}^{n}(\cos rx\cos r\xi+\sin rx\sin r\xi)\right\}$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi\left\{1+2\sum_{r=1}^{n}\cos r(x-\xi)\right\}$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi\left\{\frac{\sin(n+\frac{1}{2})(x-\xi)}{\sin\frac{1}{2}(x-\xi)}\right\}.$$

Divide the integral into two, from 0 to x and from x to 2π (assuming that $0<x<2\pi$); in the former write $x-\xi=2v$, and in the latter write $\xi-x=2v$. Then we have

$$S_n(x)=\frac{1}{\pi}\int_0^{\frac{1}{2}x} f(x-2v)\frac{\sin(2n+1)v}{\sin v}dv$$
$$+\frac{1}{\pi}\int_0^{\pi-\frac{1}{2}x} f(x+2v)\frac{\sin(2n+1)v}{\sin v}dv.$$

It follows from Art. 174 that

$$S_n(x)\to\tfrac{1}{2}\{f(x-0)+f(x+0)\}\text{ as } n\to\infty,$$

and that, at a point of continuity for $f(x)$,

$$S_n(x)\to f(x).$$

If, however, $x=0$ or 2π, we have

$$S_n(0)=S_n(2\pi)=\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\frac{\sin(n+\frac{1}{2})\xi}{\sin\frac{1}{2}\xi}d\xi$$
$$=\frac{1}{\pi}\int_0^{\pi} f(2v)\frac{\sin(2n+1)v}{\sin v}dv$$
$$\to\tfrac{1}{2}\{f(0)+f(2\pi)\}.$$

It has been tacitly assumed that $f(x)$ satisfies *Dirichlet's conditions*:

The function $f(x)$ is supposed to have only a limited number of maxima and minima, and a limited number of discontinuities (including infinities), between the limits 0 and 2π.*

* Provided that $\int_0^{2\pi} f(\xi)\,d\xi$ is absolutely convergent.

That these conditions are not *necessary* for the truth of the theorem is well known; but it is not easy to give more general conditions without going deeply into the subject. In any case these conditions are wider than is really necessary for the ordinary applications in Mathematical Physics.

127. Summation of sine- and cosine-series.

In many of the series specially studied here and in many applications to problems of Physics, it is found that we may consider series containing only sines or only cosines. In such cases the function $f(x)$ may be regarded as *arbitrary* only for a *half* period; for instance, if $f(x)=\Sigma b_n \sin nx$, we have $f(-x)=-f(x)$, and so the values of $f(x)$ from $x=0$ to π suffice to give the values also from $x=-\pi$ to 0, so that the function is really known over a *complete* period.

Assume then in the first place that we have a uniformly convergent series from $x=0$ to π,

$$f(x)=\Sigma b_n \sin nx.$$

It follows that $$b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\, dx.$$

Thus, on summation we find

$$S_n(x)=\frac{2}{\pi}\int_0^\pi f(\xi)\,d\xi\left(\sum_{r=1}^{n}\sin rx \sin r\xi\right)$$

$$=\frac{1}{\pi}\int_0^\pi f(\xi)\,d\xi\left[\sum_{r=1}^{n}\{\cos r(x-\xi)-\cos r(x+\xi)\}\right]$$

$$=\frac{1}{2\pi}\int_0^\pi f(\xi)\,d\xi\left[\frac{\sin(n+\frac{1}{2})(x-\xi)}{\sin\frac{1}{2}(x-\xi)}-\frac{\sin(n+\frac{1}{2})(x+\xi)}{\sin\frac{1}{2}(x+\xi)}\right].$$

Suppose first that $0<x<\pi$; then we write, as in Art. 126,

$$S_n(x)=\frac{1}{\pi}\int_0^{\frac{1}{2}x} f(x-2v)\frac{\sin(2n+1)v}{\sin v}\,dv$$

$$+\frac{1}{\pi}\int_0^{\frac{1}{2}(\pi-x)} f(x+2v)\frac{\sin(2n+1)v}{\sin v}\,dv$$

$$-\frac{1}{\pi}\int_{\frac{1}{2}x}^{\frac{1}{2}(\pi+x)} f(-x+2v)\frac{\sin(2n+1)v}{\sin v}\,dv.$$

Thus, from Art. 174, we see that, when $f(x)$ satisfies Dirichlet's conditions,

$$S_n(x) \to \tfrac{1}{2}\{f(x-0)+f(x+0)\}, \quad \text{as } n \to \infty,$$

or

$$\to f(x) \text{ at a point of continuity.}$$

It is evident that $S_n(0)=0, =S_n(\pi)$, and that

$$S_n(-x) = -S_n(x),$$

so that in general

$$S_n(x) \to -f(-x) \quad \text{when } -\pi < x < 0;$$

although this can, of course, be obtained at once from the integral.

In like manner, by assuming a cosine-series, we get the formulae

$$f(x) = a_0 + \Sigma a_n \cos nx \quad \text{from } x=0 \text{ to } \pi,$$

and

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx.$$

The sum to $n+1$ terms is then easily expressed by the integral

$$C_n(x) = \frac{1}{2\pi}\int_0^\pi f(\xi)\,d\xi\left[\frac{\sin(n+\frac{1}{2})(x-\xi)}{\sin\frac{1}{2}(x-\xi)} + \frac{\sin(n+\frac{1}{2})(x+\xi)}{\sin\frac{1}{2}(x+\xi)}\right],$$

and, as for $S_n(x)$, we obtain

$$C_n(x) \to \tfrac{1}{2}\{f(x-0)+f(x+0)\} \quad \text{as } n \to \infty,$$

or

$$C_n(x) \to f(x) \quad \text{at a point of continuity.}$$

But we find

$$C_n(0) = \frac{1}{\pi}\int_0^\pi f(\xi)\,d\xi\,\frac{\sin(n+\frac{1}{2})\xi}{\sin\frac{1}{2}\xi},$$

and so

$$C_n(0) \to f(+0) \quad \text{as } n \to \infty.$$

Also

$$C_n(\pi) = \frac{1}{\pi}\int_0^\pi f(\xi)\,d\xi\,\frac{\sin(n+\frac{1}{2})(\pi-\xi)}{\sin\frac{1}{2}(\pi-\xi)}$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} f(\pi-2v)\,dv\,\frac{\sin(2n+1)v}{\sin v},$$

and so

$$C_n(\pi) \to f(\pi-0) \quad \text{as } n \to \infty.$$

It should be noticed that in most of the physical problems to which these series are applied $f(x)$ is continuous between the limits $x=0, \pi$; but it quite often happens that $f(0)$ and $f(\pi)$ do not vanish. *Under these conditions the sine-series will usually be discontinuous (and consequently non-uniformly convergent) at $x=0$ and π; while the cosine-series remains continuous right up to the ends of the interval.*

This remark is illustrated by considering the two series for (say) the function $f(x) = x$ from $x = 0$ to π. We find from (1·4) that

$$x = 2(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \ldots), \quad \text{if } 0 \leqq x < \pi,$$

and from (2·21) that

$$x = \frac{1}{2}\pi - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right), \quad \text{if } 0 \leqq x \leqq \pi.$$

Of course series such as (1·1), (1·3) and (1·5) indicate that fairly simple cosine-series exist in which discontinuities occur at $x = 0$, or $x = \pi$; but such cases usually correspond to an *infinity* in the function, and this is unlikely to arise in the ordinary applications to Physics.

Ex. 1. Verify the two series for x by direct evaluation of the Fourier-series formulae.

Ex. 2. Obtain similarly two series for x^2 from $x = 0$ to π; and confirm the results from (1·4), (2·7) and (2·2).

Ex. 3. Confirm (1·9), (1·91), (2·4), (4·1), (4·2) by direct calculation of the Fourier-coefficients.

128. Stokes's transformation for finding discontinuities and for differentiating a Fourier series.*

Consider first a sine-series for the interval $(0, \pi)$, and suppose that there is a possible discontinuity of amount μ in $f(x)$, at say $x = \alpha$; but we assume that, in general, the differential coefficients $f'(x)$ and $f''(x)$ exist throughout the interval. Then, on integrating by parts, we find that

$$\int f(\xi)\sin n\xi\, d\xi = -\frac{1}{n}f(\xi)\cos n\xi + \frac{1}{n^2}f'(\xi)\sin n\xi - \frac{1}{n^2}\int f''(\xi)\sin n\xi\, d\xi.$$

Write b_n, b_n'' for the coefficients of $\sin nx$ in the sine-series for $f(x)$, $f''(x)$, respectively; then, on taking the last equation between the limits 0 and π, we find that

(8·1) $$\tfrac{1}{2}\pi b_n = \frac{1}{n}\{f(0) - (-1)^n f(\pi) + \mu\cos n\alpha\} - \frac{1}{n^2}\mu'\sin n\alpha' - \frac{1}{n^2}(\tfrac{1}{2}\pi b_n''),$$

where μ' represents a possible discontinuity in $f'(x)$ at $x = \alpha'$.

Thus, if the form of b_n is known as an algebraic rational fraction of n, we can determine $f(0)$, $f(\pi)$, μ and α by considering the coefficient of $1/n$; but, as a rule, we cannot find μ' and α' until b_n'' is found.

* *Math. and Phys. Papers*, vol. 1, p. 236; the paper is dated 1847 and contains Stokes's views on uniform convergence (see Art. 49·1).

In many applications to physical problems we know that $f(x)$ and $f'(x)$ will have no discontinuities between 0 and π, and the values of $f(0)$, $f(\pi)$ are known or can be determined without difficulty; then we find the result for b_n''

(8·2) $$b_n''=-n^2b_n+\frac{2n}{\pi}\{f(0)-(-1)^nf(\pi)\}.$$

If in addition we know $f''(0)$, $f''(\pi)$, we can find the coefficient in the Fourier-series for $f^{iv}(x)$ by using the formula

(8·21) $$b_n^{iv}=-n^2b_n''+\frac{2n}{\pi}\{f''(0)-(-1)^nf''(\pi)\},$$

and so on to any order; but it is usually unnecessary to go further.

Now consider the cosine-series; then we have the equation

$$\int f(\xi)\cos n\xi\, d\xi=\frac{1}{n}f(\xi)\sin n\xi+\frac{1}{n^2}f'(\xi)\cos n\xi-\frac{1}{n^2}\int f''(\xi)\cos n\xi\, d\xi,$$

leading to the formula

(8·3) $$\tfrac{1}{2}\pi a_n=\frac{1}{n}(-\mu\sin n\alpha)+\frac{1}{n^2}\{-\mu'\cos n\alpha'-f'(0)+(-1)^nf'(\pi)\}$$
$$-\frac{1}{n^2}(\tfrac{1}{2}\pi a_n''),$$

where a_n'' refers similarly to the cosine-series for $f''(x)$.

Thus we can determine μ and α (but not $f(0)$, $f(\pi)$) from the coefficient of $1/n$ in a_n. If it is known that there are no discontinuities in $f(x)$, $f'(x)$ between $x=0$ and $x=\pi$, then we have the formula for a_n''

(8·4) $$a_n''=-n^2a_n+\frac{2}{\pi}\{(-1)^nf'(\pi)-f'(0)\},$$

with the special formula

$$a_0''=\frac{1}{\pi}\{f'(\pi)-f'(0)\}.$$

It may be convenient to note here the corresponding formulae when the function $f(x)$ is given from $x=0$ to $x=l$, and the Fourier-series are expressed in the forms

$$\sum b_n\sin(n\pi x/l),\quad a_0+\sum a_n\cos(n\pi x/l).$$

Then we have*

(8·5) $$b_n=\frac{2}{l}\int_0^l f(\xi)\sin(n\pi\xi/l)\,d\xi,$$

* Stokes, *l.c.* pp. 256, 259, 287.

and in the sine-series for $f''(\xi)$,

(8·51) $$b_n'' = -\frac{n^2\pi^2}{l^2}b_n + \frac{2n\pi}{l^2}\{f(0) - (-1)^n f(l)\}.$$

Similarly for the cosine-series,

(8·6) $$a_0 = \frac{1}{l}\int_0^l f(\xi)\,d\xi, \quad a_n = \frac{2}{l}\int_0^l f(\xi)\cos(n\pi\xi/l)\,d\xi,$$

and in the cosine-series for $f''(\xi)$,

(8·61) $$a_n'' = -\frac{n^2\pi^2}{l^2}a_n - \frac{2}{l}\{f'(0) - (-1)^n f'(l)\},$$

$$a_0'' = \qquad -\frac{1}{l}\{f'(0) - f'(l)\}.$$

To illustrate Stokes's methods, let us take

$$b_n = n/(n^2 + a^2),$$

then we see from (8·1) that

$$f(0) = \tfrac{1}{2}\pi, \quad f(\pi) = 0,$$

because $\qquad b_n - 1/n = O(1/n^2).$

Further, (8·2) gives

$$b_n'' = n^2\left(\frac{1}{n} - \frac{n}{n^2 + a^2}\right) = a^2 b_n,$$

so that finally $\qquad f''(x) = a^2 f(x).$

Thus $$f(x) = \frac{\pi}{2}\frac{\sinh a(\pi - x)}{\sinh a\pi},$$

where the constants of integration have been found from the conditions $\qquad f(\pi) = 0, \quad f(0) = \tfrac{1}{2}\pi.$

Ex. 1. Discuss similarly the series (1·2), (1·4), (1·9), (1·91), (4·2).

Similarly if we assume that

$$f''(x) = a^2 f(x)$$

and $$f'(0) = 0, \quad f'(\pi) = a^2\pi,$$

we find that $$f(x) = a\pi\frac{\cosh ax}{\sinh a\pi}.$$

Then (8·4) gives $\qquad a^2(a_n) = a_n'' = -n^2 a_n + \frac{2}{\pi}\{(-1)^n a^2\pi\},$

or $$a_n = (-1)^n\frac{2a^2}{n^2 + a^2};$$

while $$a_0 = \frac{1}{a^2}\frac{1}{\pi}(a^2\pi) = 1.$$

Hence we obtain again the series (4·1).

We conclude with one of Stokes's own examples: To find the velocity-potential when a rectangular box of infinite length (containing fluid) is made to rotate with angular velocity ω about the line of centres.

We have then to solve

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,$$

subject to

$$\frac{\partial\phi}{\partial x}=-\omega(y-\tfrac{1}{2}b)\quad\text{at } x=0 \quad\text{or } a,$$

$$\frac{\partial\phi}{\partial y}=+\omega(x-\tfrac{1}{2}a)\quad\text{at } y=0 \quad\text{or } b.$$

We assume*

$$\phi=\sum Y_n\cos(n\pi x/a),$$

where Y_n is a function of y. Then using (8·61) we find that

$$\frac{d^2Y_n}{dy^2}=+\frac{n^2\pi^2}{a^2}Y_n-\frac{2\omega}{a}\{1-(-1)^n\}(y-\tfrac{1}{2}b)$$

by introducing the values of $\dfrac{\partial\phi}{\partial x}$ at $x=0$ and $x=a$.

Now, from (2·21) we have

$$\frac{x}{a}-\frac{1}{2}=-\sum\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{a}\right),\quad\text{where } n=1, 3, 5, \ldots .$$

Thus, using the values of $\dfrac{\partial\phi}{\partial y}$ at $y=0$, $y=b$, we see that

$$\frac{dY_n}{dy}=-\frac{4\omega a}{n^2\pi^2}\text{ at } y=0, b,\quad(n=1, 3, 5, \ldots).$$

Hence finally

$$Y_n=\frac{4\omega a}{n^2\pi^2}(y-\tfrac{1}{2}b)+\frac{8\omega a^2}{n^3\pi^3}\frac{\sinh\{n\pi(\frac{1}{2}b-y)/a\}}{\cosh(\frac{1}{2}n\pi b/a)},$$

and this result is equivalent to the one given by Stokes.†

Ex. 2. Solve similarly the problem of finding the velocity-potential of fluid motion inside a rotating sector.‡

Here

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial\theta^2}=0,$$

and

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta}=\omega r\text{ at }\theta=0,\ \theta=\alpha\quad\text{and}\quad\frac{\partial\phi}{\partial r}=0\text{ at } r=1.$$

Then assuming

$$\phi=\sum R_n\cos(n\pi\theta/\alpha),$$

prove that

$$R_n=\frac{8\omega\alpha^2 r^{n\pi/\alpha}}{n\pi(n^2\pi^2-4\alpha^2)}-\frac{4\omega\alpha r^2}{n^2\pi^2-4\alpha^2},\quad(n=1, 3, 5, \ldots).$$

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* It should be noted that we might equally well start from $\Sigma X_n\cos(n\pi y/b)$; to obtain the best results we suppose here $b>a$, so that the final series converges very fast.

† *L.c.* pp. 288 and 191.

‡ Stokes, *l.c.* p. 306.

Ex. 3. By assuming that $f''(x)=a^2f(x)$ and $f'(0)=-\frac{1}{2}a^2\pi=f'(\pi)$ prove from (8·4) that

$$\frac{a\pi}{2}\frac{\sinh a(\frac{1}{2}\pi-x)}{\cosh\frac{1}{2}a\pi}=2a^2\sum\frac{\cos nx}{n^2+a^2},\quad n=1,3,5,\ldots,$$

and verify this from the series (4·1).

Ex. 4. Similarly, by assuming that

$$f''(\theta)=-4f(\theta)\quad\text{from }\theta=0\text{ to }\theta=\alpha,$$

and that $$f'(0)=1=f'(\alpha),$$

prove that

$$\frac{\sin(2\theta-\alpha)}{2\cos\alpha}=-\sum\frac{4\alpha}{n^2\pi^2-4\alpha^2}\cos\frac{n\pi\theta}{\alpha},\quad(n=1,3,5,\ldots).$$

Verify by writing $a=2\iota\alpha/\pi$, $x=\pi\theta/\alpha$ in Ex. 3.

Hence sum the coefficient of r^2 in the formula for ϕ of Ex. 2.

129. Fejér's theorem on Fourier's series.

It will be recalled that (in Ex. 2, Art. 51) we obtained the theorem (due to Frobenius) that if σ_n is the arithmetic mean of $s_0, s_1, \ldots, s_{n-1}$, and if σ_n has a definite limit, then

$$\lim_{r\to 1}(a_0+a_1r+a_2r^2+\ldots)=\lim_{n\to\infty}\sigma_n.$$

A specially interesting example of this mean-value is due to Fejér, who applied the process to the Fourier-series for a function $f(x)$ which does not satisfy Dirichlet's conditions (Art. 126), so that the Fourier-series may not converge.

In fact, if

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\cos n\theta\,d\theta,\quad b_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\sin n\theta\,d\theta,$$

and $$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta,$$

and if we write $u_0=a_0,\quad u_n=a_n\cos nx+b_n\sin nx,$

we find that $$s_n=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\frac{\sin(n+\frac{1}{2})(\theta-x)}{\sin\frac{1}{2}(\theta-x)}\,d\theta,$$

and that the arithmetic mean of $s_0, s_1, \ldots, s_{n-1}$ is

$$\sigma_n=\frac{1}{2n\pi}\int_0^{2\pi}f(\theta)\left\{\frac{\sin\frac{1}{2}n(\theta-x)}{\sin\frac{1}{2}(\theta-x)}\right\}^2d\theta.$$

By dividing this integral into two, as in Art. 126, we find from Ex. 7, Art. 173 (App. III.), that σ_n tends to the limit $f(x)$ if the function $f(x)$ is continuous. The extension to cases when $f(x)$ has a finite number of ordinary discontinuities presents no fresh difficulty; but the proof under the single restriction that $f(x)$ must be integrable is beyond our range.*

It is easily seen that if $f(x)$ is continuous the convergence of σ_n to its limit $f(x)$ is *uniform* for all values of x from 0 to 2π. Thus we have

$$\lim_{n \to \infty} \int_0^{2\pi} \{f(x) - \sigma_n\}^2 \, dx = 0.$$

Since $$\sigma_n = a_0 + \sum_{r=1}^{n-1} \frac{n-r}{n} (a_r \cos rx + b_r \sin rx),$$

we find that (paying attention to the definitions of a_r, b_r)

$$J_n = \int_0^{2\pi} \{f(x) - \sigma_n\}^2 \, dx$$
$$= \int_0^{2\pi} \{f(x)\}^2 \, dx - 2\pi \left\{ a_0^2 + \tfrac{1}{2} \sum_{r=1}^{n} \left(1 - \frac{r^2}{n^2}\right)(a_r^2 + b_r^2) \right\}.$$

Thus $$a_0^2 + \tfrac{1}{2} \sum_{r=1}^{m} (a_r^2 + b_r^2)\left(1 - \frac{r^2}{n^2}\right) < \frac{1}{2\pi} \left\{ \int_0^{2\pi} \{f(x)\}^2 \, dx - J_n \right\},$$

where m is any number less than n; and, taking the limit as n tends to ∞, we find

$$a_0^2 + \tfrac{1}{2} \sum_{r=1}^{m} (a_r^2 + b_r^2) \leqq \frac{1}{2\pi} \int_0^{2\pi} \{f(x)\}^2 \, dx, \quad \text{because } \lim J_n = 0.$$

Thus the series $\sum_{r=1}^{\infty} (a_r^2 + b_r^2)$ is convergent (Art. 7), and so we may apply Tannery's theorem (Art. 49) to J_n, which gives†

$$\frac{1}{2\pi} \int_0^{2\pi} \{f(x)\}^2 \, dx = a_0^2 + \tfrac{1}{2} \sum_{r=1}^{\infty} (a_r^2 + b_r^2).$$

It follows that Σa_n^2, Σb_n^2 are convergent; thus (Ex. 15, Ch. II.), we see that the series

$$\frac{1}{n} \Sigma |a_n| \text{ and } \frac{1}{n} \Sigma |b_n|$$

are convergent.

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* L. Fejér, *Math. Annalen*, vol. 58, 1904, p. 51; Lebesgue, *Séries Trigonométriques*, Paris, 1906, pp. 92–104.

† This result is due to de la Vallée Poussin; see also a paper by Hurwitz (*Math. Annalen*, vol. 57, 1903, p. 425).

Hence the series

$$a_0 x + \sum \frac{1}{n} \{a_n \sin nx + b_n (1 - \cos nx)\}$$

is normally convergent; and its sum is therefore equal to the sum found by taking the arithmetic mean. But this is equal to

$$\lim_{n \to \infty} \int_0^x \sigma_n \, dx = \int_0^x f(x) \, dx,$$

because σ_n converges uniformly to the value $f(x)$.*

130. Poisson's Integral.

By applying Frobenius's theorem (Ex. 2, Art. 51) to Fejér's result (Art. 129), we now see that if a_n, b_n are the usual Fourier-constants of $f(\theta)$, we have the result

(10·1) $$\lim_{r \to 1} \{a_0 + \Sigma(a_n \cos n\theta + b_n \sin n\theta) r^n\} = f(\theta),$$

provided that $f(\theta)$ is a continuous function from $\theta = 0$ to 2π. This result is easily seen to be the same as the conclusion obtained in Art. 87 above.

There have been "proofs" of Fourier's series published which amount to proving the last equation, more or less correctly; and then assuming that the limit of the left-hand in (10·1) is equal to Fourier's series.

Naturally, if the Fourier's series can be proved to converge, its sum is equal to the limit in (10·1) by virtue of Abel's theorem (Art. 51); but the only simple conditions under which we can infer the convergence of the Fourier-series from the existence of the limit (10·1) are derived from Tauber's (Art. 86·1). These conditions are $\lim (na_n) = 0$, $\lim (nb_n) = 0$; but, as we have seen in Arts. 121–124, there are many interesting series for which these conditions are not verified.

Thus, in general, it is easier to follow Dirichlet's method, as given in Art. 126, rather than to attempt to build up a proof on these lines. On the other hand, in many physical problems, it is the existence of the limit (10·1), rather than the convergence of the Fourier's series, which proves to be of importance.†

* This result is also due to de la Vallée Poussin.

† Reference should be made to the paper by Stokes (frequently quoted in the foregoing Articles), *Math. and Phys. Papers*, vol. 1, p. 236—see, in particular, Section I., and the remarks on p. 237.

131. Character of the approximation curves near a discontinuity in a Fourier-series.

Assuming that the discontinuity is finite and occurs at $\theta=\alpha$, we can apply the process of Arts. 123, 128 to express the discontinuous part of the series by means of a suitable series of the type

$$A\Sigma\frac{1}{n}\sin n(\theta-\alpha),$$

and accordingly the behaviour of any Fourier-series near a discontinuity can be determined by the study of the special series $A\Sigma(1/n)\sin n\theta$ in the neighbourhood of $\theta=0$.

We have seen (Art. 121) that the discontinuity in this series is equal to $A\pi$; and it is natural to conjecture that the approximation-curves tend to a limiting form which includes a straight line of length $A\pi$ joining the separate parts of the curve representing $f(\theta)$, as in the sketch below.

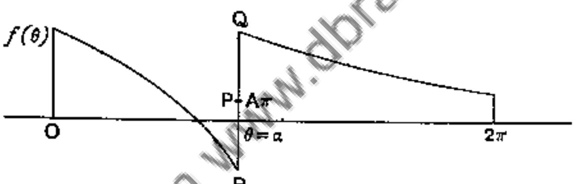

FIG. 31.

This conclusion, however, is not quite correct. It appears from the analysis given in Art. 132 below, that the first maximum to the right on the approximation curve $S_n(\theta)$ tends to a limiting height $A(1{\cdot}85194)$ above the point P representing the sum of the series at the point $\theta=\alpha$; and similarly, by changing from $\theta-\alpha$ to $\alpha-\theta$, we see that the first minimum to the left tends to a limiting depth $A(1{\cdot}85194)$ below the point P.

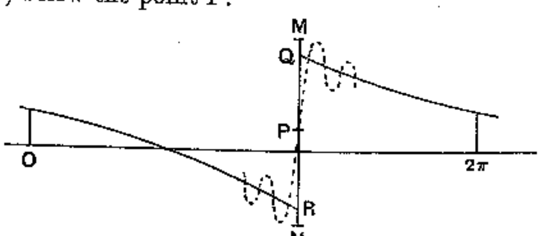

FIG. 32.

It will be noted that P is midway between the points Q, R in virtue of Dirichlet's analysis (Art. 126); and so the maximum and minimum limiting points M, N are at distances $A(\cdot 28114)$ above and below the points Q, R, as in the sketch below.

The phenomenon just described was first definitely pointed out by Willard Gibbs,* although once the remark has been made, the phenomenon is almost obvious on glancing at any carefully drawn set of approximation-curves. The most elaborate set of such diagrams was drawn by Michelson and Stratton,† who went up to $n=80$; but the phenomenon is clearly indicated in much less elaborate curves, such as those given by Byerly ‡ and Carslaw.§

132. Fejér's lemma.‖

Although it is not absolutely necessary for other applications in this book, it will be convenient now to investigate certain additional properties of the sum

$$S_n(x) = \sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \dots + \frac{1}{n}\sin nx$$

for values of x between 0 and π; the range from π to 2π is then given by the relation

$$S_n(2\pi - x) = -S_n(x).$$

To investigate the maxima and minima of $S_n(x)$ we note first that

$$S_n'(x) = \cos x + \cos 2x + \dots + \cos nx = \cos \tfrac{1}{2}(n+1)x \sin \tfrac{1}{2}nx / \sin \tfrac{1}{2}x.$$

It is readily seen that the turning-points are as follows:

$$\textit{Maxima } x = \frac{\pi}{n+1},\quad \frac{3\pi}{n+1},\quad \frac{5\pi}{n+1}, \dots$$

$$\textit{Minima } x = \quad \frac{2\pi}{n},\quad \frac{4\pi}{n},\quad \frac{6\pi}{n}, \dots.$$

When n is odd the last terms are respectively $n\pi/(n+1)$ and $(n-1)\pi/n$; so that the last maximum is the nearest turning-point to $x=\pi$, which is an inflexion.

When n is even, $x=\pi$ belongs to both sequences, and this is accordingly not a turning-point in the strict sense, but is again an inflexion, and is also the point of contact of the curve with $y=0$. The immediately preceding turning-points are $(n-1)\pi/(n+1)$ and $(n-2)\pi/n$, of which the maximum is again the nearer to $x=\pi$.

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* *Nature*, vol. 59, 1899, p. 606.

† *Phil. Mag.* (5), vol. 45, 1898, Plate XII.

‡ *Fourier Series*, etc., p. 63.

§ *Fourier's Series*, etc., p. 49 (1st edition).

‖ For references to the earlier literature see Dunham Jackson, *Rendiconti del Palermo*, vol. 32, 1911; and Carslaw, *American Journal of Mathematics*, vol. 39, 1917, p. 185.

The investigation given here is more graphical than any of those which have been published previously; but it is hoped that it will be found correct.

It follows that the character of $S_n(x)$ near to $x=\pi$ is represented by the two rough graphs

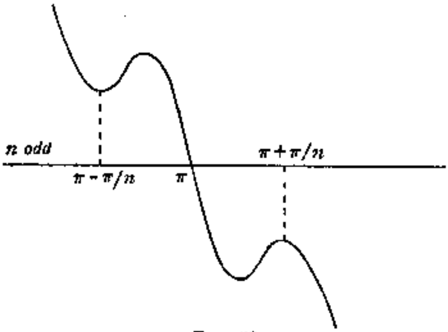


FIG. 33.

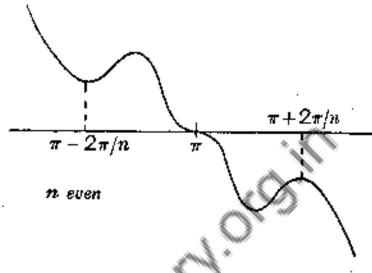


FIG. 34.

The last minimum before $x=\pi$ always gives a positive value for $S_n(x)$ because, for example,

$$S_n\left(\pi-\frac{\pi}{n}\right)=\sin\frac{\pi}{n}-\frac{1}{2}\sin\frac{2\pi}{n}+\frac{1}{3}\sin\left(\frac{3\pi}{n}\right)-\dots+(-1)^n\frac{1}{n-1}\sin\left\{\frac{(n-1)\pi}{n}\right\},$$

and in this series the terms are positive and steadily decrease, so that Art. 19 may be applied.

We consider next the relative positions of the two curves

$$y=S_n(x), \quad y=S_{n-1}(x).$$

These curves clearly intersect at the points given by $\sin nx=0$, or by $x=m\pi/n$; these points are alternately maxima for $S_{n-1}(x)$ and minima for $S_n(x)$. Further $S_n(x)>S_{n-1}(x)$ from $x=(2r-2)\pi/n$ to $(2r-1)\pi/n$, and this inequality is reversed from $x=(2r-1)\pi/n$ to $2r\pi/n$.

Thus the relative positions of the curves are as shewn in the rough graph below for part of the diagram between $x=0$ and $x=\pi$.

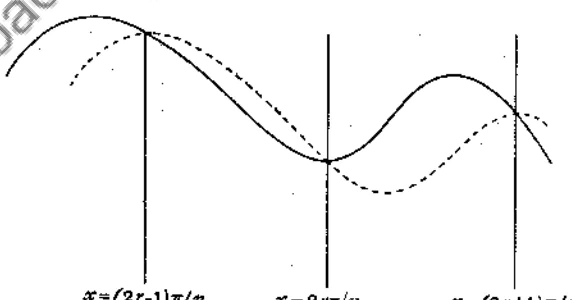


FIG. 35.

Plain curve $y=S_n(x)$. Dotted curve $y=S_{n-1}(x)$.

From this figure it is evident that *the r^{th} maximum and r^{th} minimum of $y=S_n(x)$ are in each case above the r^{th} maximum and r^{th} minimum of $y=S_{n-1}(x)$.*

Now the r^{th} minimum occurs first on the curve $n=2r+1$—on the curve $n=2r$ this point comes at $x=\pi$, and so is an inflexion.

On the curve $n=2r+1$, the r^{th} minimum is given by $x=2r\pi/n=(n-1)\pi/n$, and so is the minimum nearest to $x=\pi$.

Hence this minimum is *positive* by what has been proved above. Now, for $n=2r+2$ the r^{th} minimum is greater than for $n=2r+1$, and so is again positive; similarly for $n=2r+3$, the r^{th} minimum is greater than for $n=2r+2$; and so this minimum is again positive. Hence, generally, if $n<2r$, the r^{th} minimum is positive.

It follows that *on the curve $y=S_n(x)$, all the minima between $x=0$ and π are positive; and so all the values of y between these limits for x must be positive.*

Finally, we shall prove that the greatest maximum is the first. For we can write $S_n'(x)$ in the form

$$S_n'(x)=\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}-\tfrac{1}{2}.$$

So $S_n(x)$ is the area between the curve $y=\frac{1}{2}\sin(n+\frac{1}{2})x/\sin\frac{1}{2}x$ and the line $y=\frac{1}{2}$; now this curve gives a sequence of decreasing loops alternately positive and negative (Art. 174). The effect of including the line $y=\frac{1}{2}x$ is to decrease the positive loops and to increase the negative loops; thus the first maximum of $S_n(x)$ is still the greatest, although it does not follow that the first minimum is the least.

The general character of the graph of the curve $y=\frac{1}{2}\sin(n+\frac{1}{2})x/\sin\frac{1}{2}x$ is indicated by the sketch below which may assist in following the previous argument.

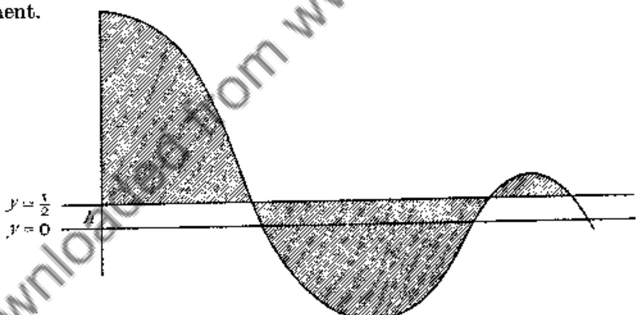


FIG. 36.

Now we have seen that the height of the first maximum increases as n increases; and so this maximum can never exceed its limit when n tends to infinity. This limit is

$$\lim_{n\to\infty}\int_0^{\pi/(n+1)}\left\{\frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}-\frac{1}{2}\right\}dx$$

$$=\lim_{n\to\infty}\int_0^{\pi(n+\frac{1}{2})/(n+1)}\left[\frac{\sin\xi}{(2n+1)\sin\{\xi/(2n+1)\}}-\frac{1}{2n+1}\right]d\xi$$

$$=\int_0^{\pi}\frac{\sin\xi}{\xi}\,d\xi=1\cdot85194.$$

Thus, *for all values of n and for all values of x between 0 and π the value of* $S_n(x)$ *lies between* 0 *and* $\int_0^\pi \frac{\sin \xi}{\xi} d\xi = 1{\cdot}85194$.

It is not difficult to discuss the series

$$f(\theta) = \sin \theta + \tfrac{1}{2} \sin 2\theta + \tfrac{1}{3} \sin 3\theta + \dots$$

by a direct method. In fact, if $S_n(\theta)$ is the sum of the first n terms in $f(\theta)$, we have, by differentiating,

$$S_n'(\theta) = \cos \theta + \cos 2\theta + \cos 3\theta + \dots + \cos n\theta = \tfrac{1}{2}\left\{\frac{\sin(n+\frac{1}{2})\theta}{\sin \frac{1}{2}\theta} - 1\right\}.$$

Thus $$S_n(\theta) = \int_0^{\frac{1}{2}\theta} \frac{\sin(2n+1)t}{\sin t}\, dt - \tfrac{1}{2}\theta.$$

Now, by Art. 174, Ex. 2 (Appendix), the limit of this integral is $\frac{1}{2}\pi$, provided that $\frac{1}{2}\theta$ lies between 0 and π; and consequently

$$f(\theta) = \tfrac{1}{2}(\pi - \theta), \qquad \text{if } 0 < \theta < 2\pi.$$

But $$f(0) = 0 = f(2\pi).$$

Thus the curve $y = f(\theta)$ consists of a line making an angle arc tan $\frac{1}{2}$ with the horizontal and two points on the horizontal axis.

A glance at Figs. 12 and 13 of Art. 43 suggests the conjecture that the limiting form of the curve $y = S_n(\theta)$ consists of the slanting line and two vertical lines, joining the slanting line to the axis (see the figure below). But as a matter of fact this is not quite correct, and the vertical lines really project above and below the slanting line.

For clearly the point

$$\theta = \lambda/n, \quad y = S_n(\lambda/n),$$

belongs to the curve $y = S_n(\theta)$, whatever the positive number λ may be. Now, as $n \to \infty$, this point approaches the limiting position

$$\theta = 0, \quad y = \int_0^\lambda \frac{\sin t}{t}\, dt,$$

in virtue of Ex. 3, Art. 174. Similarly the point

$$\theta = 2\pi, \quad y = -\int_0^\lambda \frac{\sin t}{t}\, dt,$$

belongs to the limiting form of the curve.

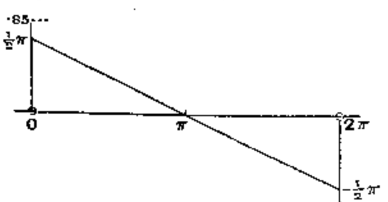


FIG. 37.

Now the integral $\int_0^\lambda (\sin t/t)\, dt$ can have any value from 0 to its maximum $1{\cdot}85194 = \frac{1}{2}\pi \times (1{\cdot}1790)$, which occurs for $\lambda = \pi$; so that in the limiting form of the curve $y = S_n(\theta)$, the two vertical lines have lengths 1·85194, instead of $\frac{1}{2}\pi = 1{\cdot}57080$, as conjectured.

Some of the approximation curves $y=S_n(\theta)$ are drawn for various values of n in Byerly's *Fourier Series*, etc., p. 63 (No. II.), and in Carslaw's *Fourier's Series*, etc., p. 49; and the curve $n=80$ is given by Michelson and Stratton, *Phil. Mag.* (5), vol. 45, 1898, Pl. XII., Fig. 5.

EXAMPLES A.

Asymptotic Series.

1. Apply Art. 107 to the function $f(x)=x^{-(1+\lambda)}$, $\lambda>0$, and deduce in particular the formula

$$\frac{1}{10^{1+\lambda}}+\frac{1}{11^{1+\lambda}}+\dots \text{ to } \infty \sim \frac{1}{10^\lambda}\left\{\frac{1}{\lambda}+\frac{1}{20}+\frac{\lambda+1}{1200}-\frac{(\lambda+1)(\lambda+2)(\lambda+3)}{72\times 10^5}+\dots\right\}.$$

Hence evaluate the sum $\qquad 1+\dfrac{1}{2^{\frac{3}{2}}}+\dfrac{1}{3^{\frac{3}{2}}}+\dots$

to five decimal places.

2. Shew from Art. 108 that

$$\log\left\{\frac{1\,.\,3\,\dots\,(2n-1)}{2\,.\,4\,\dots\,2n}\right\}=\tfrac{1}{2}\log\left(\frac{1}{n\pi}\right)-\frac{1}{8n}+O\left(\frac{1}{n^3}\right).$$

3. In certain problems on the Theory of Probability it is of interest to evaluate the quotient

$$q=n!/\{2^n(\tfrac{1}{2}n-\tfrac{1}{2}p)!\,(\tfrac{1}{2}n+\tfrac{1}{2}p)!\},$$

where n, p are large, but p is not of higher order than $\sqrt{n}$. Prove that (to order $1/n^2$)

$$\log q=\tfrac{1}{2}\log\left(\frac{2}{n\pi}\right)-\frac{p^2}{2n}-\left(\frac{1}{4n}-\frac{p^2}{2n^2}+\frac{p^4}{12n^3}\right)-\left(\frac{p^2}{3n^3}-\frac{p^4}{4n^4}+\frac{p^6}{30n^5}\right).$$

[Compare Lord Rayleigh, *Phil. Mag.* (6), vol. 37, 1919, p. 327.]

4. Obtain from Art. 107 the asymptotic expansion

$$\frac{1}{e^t-1}+\frac{1}{e^{2t}-1}+\frac{1}{e^{3t}-1}+\dots$$

$$\sim\frac{1}{t}(C-\log t)+\frac{1}{4}+\frac{(B_1)^2}{2!}\frac{t}{2}-\frac{(B_2)^2}{4!}\frac{t^3}{4}-\frac{(B_3)^2}{6!}\frac{t^5}{6}-\dots,$$

and deduce the behaviour of the series

$$\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^3}+\dots$$

as x approaches 1, by writing $t=\log(1/x)$.

[SCHLÖMILCH, *Compendium*, II., p. 238.]

5. $$\int_0^\infty\frac{e^{-kx}}{1+x^2}\,dx=\int_k^\infty\frac{dt}{t}\sin(t-k)\sim\frac{1}{k}-\frac{2!}{k^3}+\frac{4!}{k^5}-\dots,$$

the error obtained by stopping at any stage being less than the following term in the series. [CAUCHY and DIRICHLET.]

[For the first integral use the identity

$$\frac{1}{1+x^2}=1-x^2+x^4-\dots\pm\frac{x^{2n}}{1+x^2}.$$

The equality between the integrals is suggested by the series and can be established directly.]

6. $$\int_0^\infty \frac{e^{-kx^2}}{1+x^2}dx = \sqrt{\pi}e^k\int_{\sqrt{k}}^\infty e^{-t^2}dt \sim \frac{\sqrt{\pi}}{2\sqrt{k}}\left[1-\frac{1}{2k}+\frac{1\,.\,3}{(2k)^2}-\frac{1\,.\,3\,.\,5}{(2k)^3}+\dots\right],$$

the error being again less than the following term.

[Apply the same methods again.]

By writing $$\int_{\sqrt{k}}^\infty e^{-t^2}dt = \int_0^\infty e^{-t^2}dt - \int_0^{\sqrt{k}} e^{-t^2}dt$$

and integrating the latter by parts, we find that the first integral is equal to

$$\frac{\pi}{2}\left[e^k - \sum_0^\infty \frac{k^{n+\frac{1}{2}}}{\Gamma(n+\frac{3}{2})}\right].$$

7. Generally, if $0 < s < 1$, we find

$$\int_0^\infty \frac{e^{-kx}x^{-s}}{1+x}dx = \Gamma(1-s)e^k\int_k^\infty e^{-t}t^{s-1}\,dt = \frac{\pi}{\sin(s\pi)}\left[e^k - \sum_0^\infty \frac{k^{n+s}}{\Gamma(n+s+1)}\right]$$

$$\sim k^{s-1}\Gamma(1-s)\left[1-\frac{1-s}{k}+\frac{(1-s)(2-s)}{k^2}-\frac{(1-s)(2-s)(3-s)}{k^3}+\dots\right].$$

And similar expressions can be given for

$$\int_0^\infty \frac{e^{-kx}x^{-s}}{1+x^t}dx.$$

8. If $$u_n = P\int_0^\infty \frac{x^{2n}e^{a^2(1-x^2)}}{1-x^2}dx,$$

shew that $$u_n - u_{n+1} = \sqrt{\pi}e^{a^2}\frac{1\,.\,3\,\dots\,(2n-1)}{2^{n+1}a^{2n+1}},$$

and that $$u_0 = \int_0^\infty \frac{dx}{1-x^2}[e^{a^2(1-x^2)}-1] = \sqrt{\pi}\int_0^a e^{t^2}dt.$$

Hence prove that

$$\int_0^a e^{t^2}dt = e^{a^2}\left[\frac{1}{2a}+\frac{1}{4a^3}+\dots+\frac{1\,.\,3\,\dots\,(2n-3)}{2^n a^{2n-1}}\right]+\frac{u_n}{\sqrt{\pi}},$$

and that the value of the remainder is approximately $(a^2-n+\frac{1}{6})/\sqrt{(2n)}$.

In particular, for $a=4$, by taking 16 terms we can infer that the value of the integral $\int_0^4 e^{t^2}dt$ lies between 1149400·6 and 1149400·8.

[STIELTJES, *Acta Math.*, vol. 9, p 167.]

9. From the equation

$$\frac{1}{n^2} = \int_0^\infty te^{-nt}dt = \int_0^1 \log\left(\frac{1}{1-x}\right)(1-x)^{n-1}dx,$$

we obtain

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2} = \int_0^1 \frac{1-(1-x)^n}{x}\log\left(\frac{1}{(1-x)}\right)dx$$

$$= \int_0^1\left[\sum_{r=1}^\infty \frac{1}{r}x^{r-1}\{1-(1-x)^n\}\right]dx$$

$$= \sum_{r=1}^\infty\left[\frac{1}{r^2} - \frac{1\,.\,2\,\dots\,(r-1)}{r(n+1)(n+2)\,\dots\,(n+r)}\right].$$

Thus

$$1+\frac{1}{2^2}+\dots+\frac{1}{n^2}=\frac{\pi^2}{6}-\frac{1}{n+1}-\frac{\frac{1}{2}}{(n+1)(n+2)}-\frac{\frac{2}{3}}{(n+1)(n+2)(n+3)}-\dots.$$

[SCHLÖMILCH.]

10. Similarly prove that

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}=C+\log n+\frac{1}{2n}-\frac{1/12}{n(n+1)}-\frac{1/12}{n(n+1)(n+2)}-\dots,$$

$$\log(n!)=\tfrac{1}{2}\log(2\pi)+(n+\tfrac{1}{2})\log n-n-\frac{1/12}{n}+\frac{1/360}{n(n+1)(n+2)}-\dots.$$

11. Prove also that

$$\int_x^\infty \frac{1}{t}e^{-t}\,dt=\frac{1}{x}e^{-x}\left\{1-\frac{1}{x+1}+\frac{1}{(x+1)(x+2)}-\dots\right\},$$

the following numerators being 2, 4, 14, 38, ...

And $$\int_x^\infty \frac{1}{\sqrt{t}}e^{-t}\,dt=\frac{1}{\sqrt{x}}e^{-x}\left\{1-\frac{\frac{1}{2}}{x+1}+\frac{\frac{1}{4}}{(x+1)(x+2)}-\dots\right\},$$

the following numerators being $\frac{5}{8}$, $\frac{9}{16}$, $\frac{129}{32}$, ...

[SCHLÖMILCH, *Compendium*, II., p. 270.]

12. It must not be assumed that if $\lim(a_n/b_n)=1$ and $\sum a_n x^n$ is convergent for all values of x, that the two functions $\sum a_n x^n$, $\sum b_n x^n$ have the same asymptotic representations.

For example, consider the two series

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots,$$

$$\cos x+\frac{\sinh x}{x}=\left(1+\frac{1}{1}\right)-\frac{x^2}{2!}\left(1-\frac{1}{3}\right)+\frac{x^4}{4!}\left(1+\frac{1}{5}\right)-\frac{x^6}{6!}\left(1-\frac{1}{7}\right)+\dots.$$

13. Use the integral of Art. 180 to shew that

$$\log\frac{\Gamma(x+a)}{\Gamma(x)}=\int_0^\infty\left[ae^{-t}-e^{-xt}\left(\frac{1-e^{-at}}{1-e^{-t}}\right)\right]\frac{dt}{t},$$

and deduce the asymptotic expansion

$$a\log x+\sum_1^\infty\frac{(-1)^{n-1}}{x^n}\frac{\phi_{n+1}(a)}{n(n+1)},$$

[SONINE.]

where $\phi_n(a)$ is the Bernoullian function of Art. 101.

14. Determine a formal solution of the equation

$$x\frac{d^2y}{dx^2}+(p+q+x)\frac{dy}{dx}+py=0$$

in the form $$e^{-x}x^{-q}\left(1+\frac{A_1}{x}+\frac{A_2}{x^2}+\dots\right),$$

and express the result as a definite integral.

$\left[\text{The integral is } \frac{e^{-x}}{x^q\Gamma(q)}\int_0^\infty t^{q-1}\left(1+\frac{t}{x}\right)^{p-1}e^{-t}\,dt,\quad \text{if } q>0.\right]$

15. Obtain a formal solution of the differential equation

$$\frac{d^2y}{dx^2}+(2n+1-x^2)y=0$$

in the form $$e^{-\frac{1}{2}x^2}x^n\left[1-\frac{n(n-1)}{4x^2}+\frac{n(n-1)(n-2)(n-3)}{4\,.\,8x^4}-\dots\right],$$

and express this series by means of a definite integral.

16. Obtain the asymptotic solution of the differential equation

$$\frac{d^2u}{dy^2}-\tfrac{1}{3}yu=0,$$

by writing $y=3z^{\frac{2}{3}}$, $u=vz^{-\frac{1}{6}}$; and prove that the equation reduces to

$$\frac{d^2v}{dz^2}-\left(4-\frac{5}{36z^2}\right)v=0,$$

which gives the solution

$$v\sim e^{\pm 2z}\left(1+\frac{1\,.\,5}{1}\zeta+\frac{1\,.\,5\,.\,7\,.\,11}{1\,.\,2}\zeta^2+\frac{1\,.\,5\,.\,7\,.\,11\,.\,13\,.\,17}{1\,.\,2\,.\,3}\zeta^3+\dots\right),$$

where $\zeta=\pm 1/(144z)$.

[STOKES, *Math. and Phys. Papers*, vol. 2, p. 329; vol. 4, pp. 77, 283.]

17. Apply Euler's formula (Art. 107) to obtain the asymptotic formulae (as $x\to 0$),

$$\frac{1}{1+x}+\frac{1}{2(1+2x)}+\frac{1}{3(1+3x)}+\dots\sim C+\log\frac{1}{x}+\frac{x}{2}-B_1\frac{x^2}{2}+B_2\frac{x^4}{4}-\dots.$$

Use Ex. 26, p. 519, to prove that

$$\operatorname{sech} x+\operatorname{sech} 2x+\operatorname{sech} 3x+\dots\sim\tfrac{1}{2}\{(\pi/x)-1\}.$$

If we attempt to continue the last asymptotic formula as a power-series in x, all the coefficients will be found to be zero: and as a matter of fact, the next term in the approximation is $(2\pi/x)e^{-\pi^2/x}$.

18. Apply the method of Art. 112 to the series

$$F(q)=\sum x^n q^{n^2}\quad(x>1,\ q<1),$$

and prove that as $q\to 1$,

$$F(q)\sim\exp\left\{\frac{1}{4}\frac{(\log x)^2}{\log(1/q)}\right\}.$$

Discuss in the same way the series $\sum x^n q^{n^3}$.

EXAMPLES B.

Trigonometrical Series.

1. Prove from (2·4) that if $0<\beta<\alpha<\frac{1}{2}\pi$,

$$\sum\frac{1}{n^2}\sin n\theta\sin n\alpha\cos n\beta=\tfrac{1}{2}\theta(\pi-\alpha),\quad\text{if } 0<\theta<\alpha-\beta,$$

$$\text{or}\quad\tfrac{1}{4}\pi(\theta+\alpha-\beta)-\tfrac{1}{2}\alpha\theta,\quad\text{if } \alpha-\beta<\theta<\alpha+\beta,$$

$$\text{or}\quad\tfrac{1}{2}\alpha(\pi-\theta),\quad\text{if } \alpha+\beta<\theta<\pi.$$

Write $\theta=\pi x/l$, $\alpha=\pi b/l$, $\beta=\pi at/l$; multiply by $\log 2c/\alpha(\pi-\alpha)$; and so obtain some of the formulae for the vibrations of a plucked string. Similarly discuss the remaining formulae. [RAYLEIGH, *Theory of Sound*, vol. 1, Art. 129.]

2. Deduce from Ex. 1 formulae for the sum of the series

$$\Sigma \frac{1}{n} \sin n\theta \sin n\alpha \sin n\beta.$$

And interpret the results similarly in terms of the string.

[RAYLEIGH, *Theory of Sound*, vol. 1, Art. 129.]

3. Show that

$$\Sigma \frac{1}{n} \sin 2n\theta \sin^2 n\phi = \tfrac{1}{4}\pi, \quad \begin{Bmatrix} 0 < 2\theta < \pi \\ \theta < \phi < \pi - \theta \end{Bmatrix}.$$

Deduce that

$$\Sigma \frac{1}{n^2} \sin^2 n\theta \sin^2 n\phi = \tfrac{1}{4}\pi\theta, \quad \Sigma \frac{1}{n^2} \sin^4 n\theta \sin^2 n\phi = \tfrac{1}{8}\pi\theta,$$

$$\Sigma \frac{1}{n^4} \sin^4 n\theta \sin^2 n\phi = \tfrac{1}{8}\pi\theta^3,$$

with certain restrictions on θ and ϕ. [H. N. DAVIS.]

4. Show that, if $0 \leqq y < x \leqq \frac{1}{2}$,

$$\sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \frac{\cos(2m\pi x)\cos(2n\pi y)}{m^2 - n^2} \right\} = \pi^2[\tfrac{1}{2}\{\phi_2(y) - \phi_2(x)\} + \tfrac{1}{4}(y - \tfrac{1}{2})],$$

where $m = n$ is omitted from the series; but that the sum is zero if $x = y$.

Show further that the order of summation is immaterial, except when $x = 0$, $y = 0$. (See Ex. 12, p. 101.)

5. Deduce from series (1·7) the results

$$\int_0^{\pi} \log(\cos x - \cos \alpha)^2 \, dx = -2\pi \log 2,$$

$$\int_0^{\pi} \log(\cos x - \cos \alpha)^2 \cos nx \, dx = -(2\pi/n) \cos n\alpha.$$

6. Another form of the first integral in Ex. 5 is

$$\int_0^{\frac{1}{2}\pi} \log(a^2 \cos^2 \theta - b^2 \sin^2 \theta)^2 \, d\theta = \pi \log\{\tfrac{1}{4}|a^2 + b^2|\};$$

hence show that if $r^2 < 1$,

$$\int_0^{\pi} \frac{\log 4(\cos x - \cos \alpha)^2}{1 - 2r\cos x + r^2} \, dx = \frac{2\pi}{1 - r^2} \log(1 - 2r\cos\alpha + r^2).$$

7. Prove from series (1·8) that

$$\int_0^{\pi} \sin nx \log\left\{\frac{\sin^2 \frac{1}{2}(x + \alpha)}{\sin^2 \frac{1}{2}(x - \alpha)}\right\} dx = \frac{2\pi}{n} \sin n\alpha,$$

and deduce that

$$\mathrm{P}\int_0^{\pi} \frac{\cos nx \,.\, dx}{\cos x - \cos \alpha} = \pi \frac{\sin n\alpha}{\sin \alpha} \quad (0 < \alpha < \pi).$$

The last result is easily verified by using the trigonometrical identity

$$\frac{\cos nx - \cos n\alpha}{\cos x - \cos \alpha} = \frac{1}{\sin \alpha}\{\sin n\alpha + 2\cos x \sin(n-1)\alpha + \ldots + 2\cos(n-1)x \sin \alpha\}.$$

8. Differentiate the series

$$F(\theta)=\sum_1^\infty \frac{\cos(x+n)\theta}{x+n}, \quad G(\theta)=\sum_1^\infty \frac{\sin(x+n)\theta}{x+n}.$$

By integrating with respect to θ, deduce that

$$\sum_0^\infty \frac{\cos(x+n)\theta}{x+n}=\frac{1}{2}\int_\theta^\pi \frac{\cos(x-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}d\theta+\cos(\pi x)\sum_0^\infty \frac{(-1)^n}{x+n},$$

$$\sum_0^\infty \frac{\sin(x+n)\theta}{x+n}=\frac{1}{2}\int_\theta^\pi \frac{\sin(x-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}d\theta+\sin(\pi x)\sum_0^\infty \frac{(-1)^n}{x+n}.$$

The series $\sum_0^\infty(-1)^n/(x+n)$ is sometimes denoted by $\beta(x)$, and can be expressed by means of the ψ-function (see Ex. 13, p. 115), since

$$\frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right)\right\}=\sum_0^\infty \frac{(-1)^n}{x+n}=\beta(x).$$

Thus, *if x is rational*, the series can be summed in finite terms; the case $x=1$ has occurred in Arts. 65. Let us take $x=\frac{1}{2}$ as a further example.

We get
$$\sum_0^\infty \frac{\cos(n+\frac{1}{2})\theta}{n+\frac{1}{2}}=\frac{1}{2}\int_\theta^\pi \operatorname{cosec}\tfrac{1}{2}\theta\,d\theta=\log\cot\tfrac{1}{4}\theta,$$

$$\sum_0^\infty \frac{\sin(n+\frac{1}{2})\theta}{n+\frac{1}{2}}=\beta(\tfrac{1}{2})=2(1-\tfrac{1}{3}+\tfrac{1}{5}-\dots)=\tfrac{1}{2}\pi,$$

where $0<\theta<\pi$.

If θ/π is rational the series may be expressed by means of ψ-functions; and so the integrals are then expressible in the same form.

If we allow θ to tend to 0 in the sine-series, we get the result (Art. 125)

$$\frac{1}{2}\int_0^\pi \frac{\sin(x-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}d\theta=\tfrac{1}{2}\pi-\sin(\pi x)\beta(x).$$ [HARDY.]

9. Shew that if we attempt to find a Fourier sine-series for $\cot x$ from $x=0$ to $x=\pi$, we obtain the series

$$2(\sin 2x+\sin 4x+\sin 6x+\dots),$$

and verify Fejér's general theorem (Art. 129) in this special case.

10. From 5·1, p. 370, deduce that if λ lies between the two even integers $2r$, $2(r+1)$,

$$\sum_{-\infty}^{\infty}\frac{e^{i\lambda(\xi+n\pi)}}{\xi+n\pi}=\frac{e^{(2r+1)i\xi}}{\sin\xi},$$

and examine the case $\lambda=2r$.

[Write $\theta=(\lambda-2r)\pi$, $a=-\xi/\pi$.]

11. Shew that

$$\sum_0^\infty(-1)^n\frac{e^{inx}}{(n+1)(n+2)}=(e^{-ix}+e^{-2ix})\log(1+e^{ix})-e^{-ix},$$

and divide this equation into real and imaginary parts.

12. Shew that
$$\sum_{-\infty}^{\infty}(-1)^n\frac{e^{2inv\pi}}{t-n}=\frac{\pi e^{2i\theta t\pi}}{\sin t\pi},$$
where θ is the difference between v and the integer nearest to v.

[Write $\pi(1+2\theta)$ for θ in the series 5·1, p. 370, and observe that $-\frac{1}{2}<\theta<+\frac{1}{2}$.]

13. By writing ia and $-ia$ for a in 5·1, p. 370, or otherwise, shew that
$$\pi\frac{\cosh a(\pi-x)}{\sinh a\pi}=\frac{1}{a}+2a\sum_{1}^{\infty}\frac{\cos nx}{n^2+a^2},\quad 0\leqq x\leqq 2\pi,$$
$$\pi\frac{\sinh a(\pi-x)}{\sinh a\pi}=\quad 2\sum_{1}^{\infty}\frac{n\sin nx}{n^2+a^2},\quad 0<x<2\pi.$$
Deduce each of these from the other by differentiating [*Math. Trip.* 1902.]

14. Prove that, if $0\leqq\theta\leqq\pi$,
$$\frac{\cos 4\theta}{1\,.\,2}+\frac{\cos 6\theta}{2\,.\,3}+\frac{\cos 8\theta}{3\,.\,4}+\dots=\cos 2\theta-\left(\frac{\pi}{2}-\theta\right)\sin 2\theta+\sin^2\theta\log(4\sin^2\theta),$$
$$\frac{\sin 4\theta}{1\,.\,2}+\frac{\sin 6\theta}{2\,.\,3}+\frac{\sin 8\theta}{3\,.\,4}+\dots=\sin 2\theta-(\pi-2\theta)\sin^2\theta-\sin\theta\cos\theta\log(4\sin^2\theta);$$
and find the sums of
$$\sum_{2}^{\infty}\frac{\cos n\theta}{n^2-1},\quad \sum_{2}^{\infty}\frac{\sin n\theta}{n^2-1}.$$

APPENDIX I.

ARITHMETIC THEORY OF IRRATIONAL NUMBERS AND LIMITS.

133. Infinite decimals.

If we apply the ordinary process of division to convert a rational fraction a/b to a decimal, it is evident that either the process must terminate or else the quotients must recur after $(b-1)$ divisions at most; for in dividing by b, there are not more than $(b-1)$ different remainders possible (namely 1, 2, 3, ... $b-1$).

For instance, $\frac{1}{8} = \cdot 125$, terminating after three divisions.

Again $\frac{5}{7} = \cdot\dot{7}1428\dot{5}$, recurring after six $(=7-1)$ divisions.

Also $\frac{5}{14} = \cdot 3\dot{5}71428\dot{8}$, recurring after seven divisions.

And $\frac{2}{13} = \cdot\dot{1}5384\dot{6}$, recurring after six $(=\frac{1}{2}(13-1))$ divisions.

If the decimal part is purely periodic and contains p figures, the decimal can be expressed in the form $P/(10^p-1)$, by means of the formula for summing a Geometrical Progression (Art. 6). Thus b must be a factor of, or equal to, 10^p-1; and so b is not divisible by either 2 or 5. Conversely, it follows from Euler's extension of Fermat's theorem that when b is not divisible by either 2 or 5, an index p can be found so that 10^p-1 is divisible by b; thus a/b is of the form $P/(10^p-1)$ and so can be expanded as a periodic decimal with p figures in the period.

But if the decimal part is mixed, containing n non-periodic and p periodic figures (as for $\frac{5}{14}$), b must contain either 2^n or 5^n as a factor; because the decimal part of $(a\times 10^n)/b$ will be purely periodic. The relation between p and the other prime factors of b cannot be discussed so simply. But it is proved in the Theory of Numbers (see, for instance, Gauss, *Disquisitiones Arithmeticae*, §§ 83–92, 308–318) that if $b=2^\alpha 5^\beta r^\rho s^\sigma t^\tau \ldots$, where $r, s, t, \ldots$ are primes (not 2 or 5), then n is the greater of α, β, while p is a factor of the L.C.M. of $r^{\rho-1}(r-1)$, $s^{\sigma-1}(s-1)$, $t^{\tau-1}(t-1)$, ...

If now we agree to replace a terminated (say $\cdot 125$) by an infinite decimal $\cdot 125000\ldots$, it will be evident that *any rational fraction can be expressed as an infinite decimal.**

* According to the rules of arithmetic, we could also replace $\cdot 125$ by $\cdot 124\dot{9}$, but it is more convenient to have a unique form, and we shall adhere to $\cdot 125000\ldots$.

But we can easily see that the rational numbers do not exhaust *all* infinite decimals.

Thus consider the decimal

$$\cdot 1010010001000010\ldots,$$

which consists of unities separated by zeros, the number of zeros increasing by one at each stage. Clearly this decimal neither terminates nor recurs: and it is therefore *not rational*.

Similarly, we may take a decimal

$$\cdot 1110101000101000 1\ldots$$

formed by writing unity when the order of the decimal place is prime (1, 2, 3, 5, 7, ...), and zero when it is composite (4, 6, 8, 9, 10, ...). This cannot be rational, since the primes do not form a sequence which recurs (in rank), and their number is infinite, as appears from the Theory of Numbers.

If the primes recurred in rank after a certain stage, it would be possible to find the integers a, b, such that all the numbers

$$a, a+b, a+2b, \ldots$$

would be prime. Now this is impossible, since $a+ab$ is divisible by a; and therefore the primes do not recur.

If the primes were finite in number, we could denote them by p_1, p_2, p_3, ..., p_n; and then the number

$$(p_1 p_2 p_3 \ldots p_n)+1$$

would not be divisible by any prime, which is absurd. Thus the number of primes is infinite. This theorem and proof are due to Euclid (Bk. ix, Prop. 20).

As an example of a different type, consider the infinite decimal obtained by applying the regular arithmetic process for extracting the square-root to a non-square such as 2; this process gives a sequence of digits *

$$1\cdot 414213562373\ldots.$$

This decimal has the property that, if A_n denotes the value of the first n digits after the point,

$$\left(A_n+\frac{1}{10^n}\right)^2 > 2 > A_n^2,$$

which gives $\quad 2-A_n^2 < 3/10^n, \quad$ since $\quad 2A_n+1/10^n < 3.$

* A rapid way of finding the decimal is to use the series of Euler, Ex. B. 14, Ch. VIII.

To see that this decimal cannot terminate or recur, we have only to prove that *there is no rational fraction whose square is equal to* 2.

This is nearly obvious, but we can give a formal proof thus: Suppose, if possible, that $(a/b)^2=2$; we may assume that a, b are positive integers which are mutually prime, and therefore at least one of them is *odd*. Now since $a^2=2b^2$, a cannot be *odd*; so that b must be odd. But if we write $a=2c$, we get $2c^2=b^2$, so that b cannot be odd; we thus arrive at a contradiction.

134. The order of the system of infinite decimals.

It is possible, and in many ways it is distinctly best, to build up the whole theory of rational numbers on the basis of *order*, regarding the numbers as marks distinguishing certain objects arranged in a definite order. If, as usual, we place the larger numbers to the right of the smaller, along a straight line, we shall then regard the inequalities $A>B$, $B>C$ simply as meaning that the mark A is to the right of B and the mark C to the left of B, so that B falls between A and C.

We shall now prove that *we can obtain the same arrangement by reference to the corresponding infinite decimals, without comparing the rational numbers directly.*

Suppose that we find

$$A=a_0+\frac{a_1}{10}+\frac{a_2}{10^2}+\dots+\frac{a_n}{10^n}+\dots,$$

$$B=b_0+\frac{b_1}{10}+\frac{b_2}{10^2}+\dots+\frac{b_n}{10^n}+\dots,$$

and that the integers a_r, b_r are the same up to a certain stage;* say that we find

$$a_0=b_0,\quad a_1=b_1,\ \dots,\ a_{n-1}=b_{n-1},\quad \text{but } a_n>b_n.$$

Write
$$A_r=a_0+\frac{a_1}{10}+\frac{a_2}{10^2}+\dots+\frac{a_r}{10^r},$$

with a corresponding interpretation for B_r. Then we have

$$A_n-B_n=(a_n-b_n)/10^n\geqq 1/10^n.$$

* They cannot be always the same, or A would be equal to B. Note that a_0, b_0 may be negative, but that $a_1, a_2, \dots, b_1, b_2, \dots$ are all positive and less than 10.

Also $10^n(B-B_n)$ is a rational number, and is less than 1, in virtue of the method of finding B_n from B.

Thus $$B < B_n + 1/10^n,$$
while $$A \geqq A_n$$
and $$A_n \geqq B_n + 1/10^n.$$
Hence $$A > B.$$

Thus, *in order to determine the relative position of two infinite decimals (derived from rational fractions), we need only compare their digits, until we arrive at a stage where the corresponding digits are different; the relative value of these digits determines the relative position of the two decimals.*

By extending this rule to *all* infinite decimals (whether derived from rational numbers or not) we can assign a perfectly definite order to the whole system: for example, the decimal ·1010010001... given in Art. 133 would be placed between the two decimals

·1010000... (zeros) and ·1011000... (zeros),

and also between

·101001000... (zeros) and ·101001100... (zeros),

and so on.

Similarly, we may shew that the infinite decimal derived from extracting the square root of 2 must be placed between $\frac{24}{17}$ and $\frac{58}{41}$. For, by division, we find
$$\tfrac{24}{17} = 1{\cdot}411\ldots, \quad \tfrac{58}{41} = 1{\cdot}4146\ldots,$$
so that, in agreement with the rule,
$$\tfrac{24}{17} < 1{\cdot}41421\ldots < \tfrac{58}{41}.$$

It must not be forgotten that *at present the new infinite decimals are purely formal expressions*, although, as we have explained, they fall in perfectly definite order into the scheme of infinite decimals derived from rational fractions.

135. Additional arithmetical examples of infinite decimals which are not rational.

Consider first the sequence of fractions (a_n), where
$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}.$$

If m is any integer, and $n > m$, we find
$$a_n - a_m = \frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \ldots + \frac{1}{n!},$$

which is less than

$$\frac{1}{m!}\left\{\frac{1}{m+1}+\frac{1}{(m+1)^2}+\dots+\frac{1}{(m+1)^{n-m}}\right\}<\frac{1}{m!}\left(\frac{1}{m+1}\right)\Big/\left(1-\frac{1}{m+1}\right).$$

Thus, if $n > m+1$, the decimal for a_n lies between the decimals given by

$$a_{m+1}=a_m+\frac{1}{(m+1)!} \quad\text{and}\quad a_m+\frac{1}{m(m!)}.$$

In this way we find successively the limits

1·66	and 1·75,	$m=2$
1·708	and 1·720,	$m=3$
1·7166	and 1·7188,	$m=4$
1·71805	and 1·71834,	$m=5$
1·71825	and 1·71829,	$m=6$

and so on.

As m increases, these two decimals become more and more nearly equal; and we are thus led to construct an infinite decimal (1·718281828 ...), which we regard as equivalent to the expression

(1) $$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots \text{ to } \infty.$$

It will now be proved that this infinite decimal cannot agree with the one which corresponds to any rational number.

For the decimal corresponding to

$$\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots+\frac{1}{n!}$$

must be less than the decimal derived from

$$\frac{1}{2}+\frac{1}{2\,.\,3}+\frac{1}{2\,.\,3^2}+\dots+\frac{1}{2\,.\,3^{n-2}}.$$

And the last expression is

$$\frac{1}{2}\left(1-\frac{1}{3^{n-1}}\right)\Big/\left(1-\frac{1}{3}\right)=\frac{3}{4}\left(1-\frac{1}{3^{n-1}}\right).$$

Hence, no matter how many terms we take from $\frac{1}{2!}+\frac{1}{3!}+\dots$, the decimal derived from their sum will be less than the decimal ·75.

But if c is any integer greater than 1,

$$\frac{1}{2!}>\frac{1}{c+1},\quad \frac{1}{3!}>\frac{1}{(c+1)(c+2)},\quad \frac{1}{4!}>\frac{1}{(c+1)(c+2)(c+3)},$$

and so on. Thus the decimal representing any number of terms from

$$\frac{1}{c+1}+\frac{1}{(c+1)(c+2)}+\frac{1}{(c+1)(c+2)(c+3)}+\dots$$

must be less than ·75.

Suppose now, if possible, that (1) could lead to an infinite decimal agreeing with the decimal derived from a/c, where a and c are positive integers. Multiply by $c!$, and (1) becomes

$$\{(2\,.\,3\,\dots\,c)+(3\,.\,4\,\dots\,c)+(4\,.\,5\,\dots\,c)+\dots+c+1\}$$
$$+\frac{1}{c+1}+\frac{1}{(c+1)(c+2)}+\dots.$$

The terms in {} brackets give some integer I, say, and so we find that

$$a(c-1)!-I=\frac{1}{c+1}+\frac{1}{(c+1)(c+2)}+\dots,$$

that is, an integer equal to a decimal which is less than ·75, which is absurd. Thus no fraction such as a/c can give the same infinite decimal as (1) does.

Consider next the continued fractions

$$b_n=1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\dots \text{ to } n \text{ terms.}$$

Here, we recall the facts that if

$$b_m=p/q,\quad b_{m+1}=r/s,$$

then
$$|ps-qr|=1,$$

while b_n lies between b_m and b_{m+1} if $n > m+1$.

Thus we find for the successive values of b_n,

$$1,\ 2,\ \tfrac{3}{2},\ \tfrac{5}{3},\ \tfrac{8}{5},\ \tfrac{13}{8},\ \dots,$$

and so, converting to decimals, we see that b_n lies between the two sequences

$$1,\ 1{\cdot}5,\ 1{\cdot}6,\ 1{\cdot}615,\ \dots,\quad\text{and}\quad 2,\ 1{\cdot}67,\ 1{\cdot}625,\ 1{\cdot}6191,\ \dots.$$

As m increases, b_m and b_{m+1} become more and more nearly equal, and lead to an infinite decimal 1·6180340..., which can be considered as equivalent to the infinite continued fraction

$$(2)\qquad 1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\dots \text{ to } \infty.$$

This infinite decimal cannot be derived from a rational fraction a/c; for if it were we should have the inequality

$$\left|\frac{p}{q}-\frac{a}{c}\right|<\left|\frac{p}{q}-\frac{r}{s}\right|=\frac{1}{qs},$$

so that
$$|pc-aq| > c/s.$$

Since $(pc-aq)$ is an integer, the last condition gives $c > s$; but this is obviously absurd, because the denominator of the cth convergent is greater than c (if $c > 5$).

Similarly, the continued fraction

(3) $$1+\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots$$

can be proved to lead to an infinite decimal which is not rational.

The reader who is familiar with the theory of continued fractions will see that the square of (3) converges to the value 2; while (2) can be interpreted in connexion with the first geometrical example of Art. 136 below.

As a somewhat different example, it is easy to see that the infinite decimal

(4) $$\log_{10}2 = \cdot 301029995663981\ldots$$

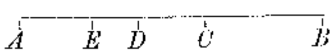
cannot be rational. For if it were equal to a/c, we should have $10^a = 2^c$; but 10^a must end with 0, whereas 2^c ends with 2, 4, 6 or 8. Thus $10^a = 2^c$ is impossible.

Similarly, we can see that 3, 5, 6, 7, 11, ... cannot have rational logarithms.

136. Geometrical examples.

From the examples given in Arts. 133, 135 it is evident that the system of rational numbers is by no means sufficient to fulfil all the needs of algebra. We shall now give an example to shew that it does not suffice for geometry.

Let a straight line AB be divided at C in "golden section" (as in Euclid, Book II., prop. 11), so that $AC : CB = AB : AC$. It is



then easy to see that AC must be greater than CB, but less than twice CB.* Cut off AD equal to CB; it follows at once that AC is divided at D in the same ratio as AB is divided at C. For we have

$$AC : AD = AC : CB = AB : AC,$$

and consequently

$$AD : DC = AD : (AC-AD) = AC : (AB-AC) = AC : CB.$$

Also AD is less than half of AB.*

* The first follows from the definition; and so we see that $AB = AC + CB$ is less than twice AC. Now, since $AC : CB = AB : AC$, it follows that AC is less than twice CB.

Thus if we repeat this process $2n$ times, we arrive at a line AN, which is less than the 2^nth part of AB.

Now suppose, if possible, that AC/AB can be expressed as a rational fraction r/s; then $AD/AB = CB/AB$ is $(s-r)/s$, and DC/AB is $(2r-s)/s$. Hence AE/AB is $(2r-s)/s$ and ED/AB is $(2s-3r)/s$. Continuing this argument, we see that AN/AB must be some multiple of $1/s$; and so cannot be less than $1/s$. But we have seen that AN/AB is less than $1/2^n$, so that we are led to a contradiction, because we can choose n so that 2^n exceeds s. Thus the ratio $AC : AB$ cannot be rational.

It is not difficult to prove similarly that the ratio of the side to the diagonal of a square is not expressible as a rational fraction. In fact, let ABC in the figure represent half a square of which AB is a diagonal; it is at once evident that AB is greater than AC and less than $2AC$. Cut off $BD = BC$, and erect DE perpendicular to

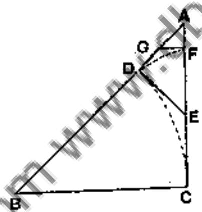


FIG. 38.

AB at D; then we have $ED = DA$, and $EC = ED$, because BE is a line of symmetry for the quadrilateral $BCED$. Thus $EC = DA$. If we repeat the same construction on the triangle ADE, we see in the same way that

$$AF = FG = GD.$$

Thus $AD\ (=\frac{1}{2}FC)$ is less than half AC; and similarly, AF is less than half AD. Thus, by continuing the construction, we arrive at an isosceles triangle ANP, such that AN is less than the 2^nth part of AC.

But if AC/AB is a rational fraction r/s, then AD/AB is $(s-r)/s$, so that AF/AB is $(3r-2s)/s$; and continuing the process, we see that AN/AB is not less than $1/s$, or that AN/AC is not less than $1/r$, which leads to a contradiction, as before.

Ex. The reader may shew geometrically that the continued fraction (2) of the last article converges to the ratio $AB : AC$; while (3) converges to the ratio of the diagonal to the side of a square.

137. A special classification of rational numbers.

The examples of Arts. 133–136 indicate the need for developing some theory of *irrational numbers.* But before proceeding to a formal definition, which will be found in the next article, we shall give some considerations which shew how infinite decimals which do not recur lead up to Dedekind's definition.

The infinite decimal 1·41421... discussed in Art. 133 enables us to divide *all* rational numbers into two classes :

(A) *The lower class*, which contains all rational fractions (such as $\frac{7}{5}, \frac{1}{7}$) less than or equal to some term of the sequence of terminated decimals

$$1{\cdot}4,\quad 1{\cdot}41,\quad 1{\cdot}414,\quad 1{\cdot}4142, \text{ etc.}$$

(B) *The upper class*, which contains all rational fractions (such as $\frac{7}{4}, \frac{8}{1}$) greater than *every* term of the sequence.

It is then clear that

(i) Any number in the upper class is greater than every number in the lower class.

(ii) There is no greatest number in the lower class; and no least number in the upper class.

To see the truth of the second statement, we may observe that, if

$$l = (4 + 3k)/(3 + 2k),$$

we have $\quad l - k = 2(2 - k^2)/(3 + 2k), \quad 2 - l^2 = (2 - k^2)/(3 + 2k)^2.$

Hence, if k is any rational number of the lower class, we have $l > k$, because $k^2 < 2$; and, for the same reason, $l^2 < 2$, so that l will also belong to the lower class. There is therefore no greatest number in the lower class.

If now we suppose k to be a rational number of the upper class, we prove by a similar argument that l is also a number of the upper class, but is less than k.

Ex. 1. Prove similarly that if $k^2 < N$, then $l > k$, $l^2 < N$, where

$$l = (Na + bk)/(b + ak) \quad \text{and} \quad b^2 > Na^2.$$

Ex. 2. Establish inequalities similar to those of Ex. 1, taking

$$l = k(3N + k^2)/(N + 3k^2).$$ [DEDEKIND.]

Ex. 3. The formula, corresponding to Ex. 2, for the nth root of N is

$$l = k\{(n+1)N + (n-1)k^n\}/\{(n-1)N + (n+1)k^n\}.$$

Ex. 4. Utilise the last example to find approximations to $2^{\frac{1}{3}}$; the first two may be taken as 1, $\frac{5}{4}$.

The classification of rational numbers which has been just described can, however, be obtained by a different process. From the arithmetical process of extracting the square-root of 2, it is evident that

$$(1{\cdot}4)^2, \quad (1{\cdot}41)^2, \quad (1{\cdot}414)^2, \quad (1{\cdot}4142)^2, \ldots$$

are all less than 2; but the sequence contains numbers which are as close to 2 as we please. Thus the lower class contains every positive rational number whose square is less than 2; and it also contains all negative rational numbers. Since the two classes together contain *all* rational numbers, it follows that the upper class must contain every positive rational number whose square is greater than 2.

Thus the same classification is made by putting,

(A) In the *lower class*, all negative numbers and all positive numbers whose square is less than 2.

(B) In the *upper class*, all positive numbers whose square is greater than 2.

138. Dedekind's definition of irrational numbers.

Suppose that some rule has been chosen which separates *all* rational numbers into two classes, such that any number in the upper class is greater than every number in the lower class. Thus, if a number k belongs to the upper class, so also does every rational number greater than k.

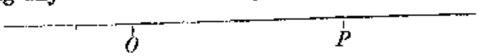
There are then three mutually exclusive possibilities :

(1) There may be a number g in the lower class which is greater than every other number in that class.

(2) There may be a number l in the upper class which is less than every other number in that class.

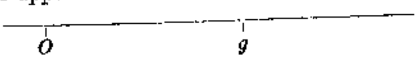
(3) Neither g nor l may exist.

The cases (1), (2) lend themselves very readily to geometrical interpretation, by representing any rational number by a point on a line. Thus OP will



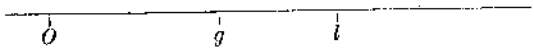
represent the fraction m/n, if the length OP is m times the nth part of the unit of length.

In case (1), the upper class consists of all rational points to the right of g



on the line; and the lower class consists of g and all rational points to the left of g.

Similarly, we can illustrate case (2). It might be thought at first sight that g and l might exist simultaneously; but this is excluded by the hypothesis



that *all* rational numbers are to be classed. Now $\frac{1}{2}(g+l)$ is rational and falls between g and l; and this would escape classification.

That there are cases in which neither g nor l can exist is clear from the example given in the last article, where it was proved that there could be no greatest number in the lower class, and no least number in the upper class.

For example, let us illustrate on a straight line the approximations to $\sqrt{2}$, which are derived from the convergents to the continued fraction

$$1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots$$

The convergents of the lower class are seen to be

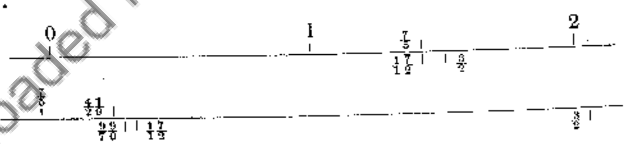
$$1,\quad \tfrac{7}{5},\ \tfrac{41}{29},\ \ldots,$$

while $\qquad \tfrac{3}{2},\ \tfrac{17}{12},\ \tfrac{99}{70},\ \ldots$

are those of the upper class.

It may be observed that if p/q is a convergent of either class, the next convergent of the same class is $(3p+4q)/(2p+3q)$, while the intermediate convergent of the other class is $(p+2q)/(p+q)$.

The representative points are as shewn in the diagrams, the second figure being a large-scale reproduction of the segment of the first which falls between $\frac{7}{5}$ and $\frac{3}{2}$.



It is clear that in case (3) the rule gives a cleavage or *section* in the rational numbers; and to fill up the gap so caused in our number-system, we agree to *regard every such section as defining a new number*, and in particular we may *regard this new (irrational) number as being equivalent to the lower class of rational numbers.* This constitutes *Dedekind's definition of irrational numbers.** For it is clear from what has been said that these new numbers cannot be rational.

On the other hand, in cases (1), (2) there is no section, and so no new number is introduced.

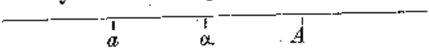
* Other definitions have been framed by Méray, Weierstrass and G. Cantor.

139. Definitions of equal, greater, less; deductions.

For the present we use the following notation:

An irrational number is denoted by a Greek letter, such as α, β; the numbers of the corresponding lower class by small italics, as a, b; those of the upper class by capital italics, as A, B. The classes themselves may be denoted by adding brackets, as (a), (A).

These definitions may be indicated graphically thus



It is an obvious extension of the ordinary use of the symbols $<$, $>$, to write $a<\alpha<A$, $b<\beta<B$.

In particular, we say that α *is positive* when 0 belongs to (a); α *is negative* when 0 belongs to (A).

Two irrational numbers are equal, if their classes are the same; in symbols we write $\alpha=\beta$ if $(a)=(b)$ and $(A)=(B)$.

The reader who is acquainted with Euclid's theory of ratio will recognise that this definition of equality is exactly the same as that which he adopts in Book V. of the *Elements*. Euclid in fact says that $A:B=C:D$, provided that the inequalities $nA \gtreqless mB$ are accompanied by $nC \gtreqless mD$, for any values of m, n whatever. In Dedekind's theory, the inequality $nA>mB$ implies that m/n is in the lower class defining $A:B$; thus Euclid's definition implies that the two ratios $A:B$ and $C:D$ have the same lower class and the same upper class.

On the other hand, *the number α is less than the number β*, when part of the upper class (A) belongs to the lower class (b), so that at least one rational number r belongs both to (A) and to (b).

This definition of inequality also coincides with Euclid's.

It follows at once from the definition that if $\alpha<\beta$ and $\beta<\gamma$, then $\alpha<\gamma$.

Again, *if $\alpha<\beta$, an infinite number of rational numbers fall between α and β*. For at least one rational number r exists such that $\alpha<r<\beta$. Now there is no greatest number in the class (b), so that we can find another rational number s which belongs to (b) and is greater than r. Thus

$$\alpha<r<s<\beta.$$

Then if x, y are any two positive integers, we have

$$r<\frac{rx+sy}{x+y}<s,$$

so that all these rational numbers lie between α and β.

140. Deductions from the definitions.

Any irrational number (α) *can be expressed as an infinite decimal.*

For there will be some integer n_0 (positive or negative) such that n_0 belongs to the lower class (a) and n_0+1 belongs to the upper class (A). Then consider the rational fractions

$$n_0, \quad n_0+\tfrac{1}{10}, \quad n_0+\tfrac{2}{10}, \quad n_0+\tfrac{3}{10}, \quad \dots, \quad n_0+\tfrac{9}{10}, \quad n_0+1;$$

some of these belong to class (a), the rest to class (A). Suppose that $n_0+n_1/10$ is the greatest in class (a), so that we have

$$n_0+\frac{n_1}{10}<\alpha<n_0+\frac{n_1+1}{10}.$$

Continuing this process, we arrive at the result

$$n_0+\frac{n_1}{10}+\frac{n_2}{10^2}+\dots+\frac{n_r}{10^r}<\alpha<n_0+\frac{n_1}{10}+\frac{n_2}{10^2}+\dots+\frac{n_r+1}{10^r}.$$

If we call these two decimal fractions a_r, A_r, it is plain that A_r-a_r $(=1/10^r)$ can be made less than any prescribed rational fraction merely by taking r to be sufficiently great; and if we continue the process indefinitely we see that α is the number defined by the infinite decimal $n_0\cdot n_1n_2n_3\dots$.

The argument just given shews also that *we can always determine numbers A, a belonging to the two classes such that $A-a$ is less than an arbitrarily small rational fraction.*

It is useful to note further that a_r can be chosen so as to exceed any prescribed number a' of the lower class. For let a'' be another number of the lower class which is greater than a'; and then choose r so that $10^r>1/(a''-a')$. Then

$$A_r-a_r<a''-a', \text{ or } a_r-a'>A_r-a''.$$

But $A_r>a''$, so that $a_r>a'$.

141. Modified form of Dedekind's definition.

Suppose that a classification of the rational numbers has the following properties:

(1) if a belongs to the lower class, so does every rational number less than a;

(2) if A belongs to the upper class, so does every rational number greater than A;

(3) every number a is less than any number A;

(4) numbers A, a can be found in the two classes such that $A-a$ is less than an arbitrary rational fraction.

Such a classification defines a single number, rational or irrational. For any rational number r which does not belong to either class must lie between the two classes, since any number less than a number of the lower class must also belong to the lower class; and therefore r must exceed every number of the lower class: similarly, r must be less than every number of the upper class. Hence, if a, A are any two members of the two classes $a < r < A$.

Suppose now that s is a second rational number which belongs to neither class; then $a < s < A$. Hence $|r-s|$ must be less than $A-a$; but this is impossible, since by hypothesis A, a can be chosen so that $A-a$ is less than any assigned rational fraction.

Consequently, not more than one rational number can escape classification; if there is one such number, the classification may be regarded as defining that number; but if there is no rational number which escapes classification, we have obtained a Dedekind section, and have therefore defined an irrational number.

142. Algebraic operations with irrational numbers.

The negative of an irrational number α is defined by means of the lower class $-A$ and the upper class $-a$; it is denoted naturally by $-\alpha$.

The reciprocal of an irrational number α is defined most easily by restricting the classes at first to contain only terms of *one* sign; and then the reciprocal $1/\alpha$ is defined by the lower class $1/A$ and the upper class $1/a$. Thus if the number α is positive, the complete specification of the classes for $1/\alpha$ will be given by putting the whole of $1/A$ in the lower class, together with all negative numbers, while the upper class will contain only the positive part of $1/a$; and a corresponding definition is easily framed for $1/\alpha$ when α is negative.

The absolute value of an irrational number α is always positive and is equal to α or $-\alpha$, according as α is positive or negative; it is denoted by $|\alpha|$.

Addition of two irrationals.

Suppose α, β to be the two given irrationals, so that $a < \alpha < A$, $b < \beta < B$. Then we classify the rational numbers by making $a+b$ a typical member of the lower class and $A+B$ of the upper class. This rule obviously satisfies conditions (1)–(3) of Art. 141, and so defines a *single* number. To prove that it satisfies condition (4) we note that

$$(A+B)-(a+b)=(A-a)+(B-b),$$

and, as explained in Art. 140, we can find a, A and b, B so as to make $A-a$ and $B-b$ each less than $\frac{1}{2}\epsilon$; and then $(A+B)-(a+b)$ is less than ϵ. Hence our classification defines a number which may be rational or irrational; this number is called the sum $\alpha+\beta$.

It follows at once that $\alpha+(-\alpha)=0$; for here the lower class is represented by the type $a-A$, and the upper class by $A-a$. That is, the lower class consists of all negative rational numbers and the upper class of all positive rational numbers; hence, zero is the only rational number not classed, and therefore is the number defined by the classification.

Subtraction of irrationals.

In virtue of the relation $\beta+(-\beta)=0$, we may define $\alpha-\beta$ as equal to the sum $\alpha+(-\beta)$.

Multiplication of positive irrational numbers.

For simplicity of statement, we omit the *negative* numbers from the lower classes; and then we define the product $\alpha\beta$ by using the type ab for the lower class and AB for the corresponding upper class. To prove that this classification defines a single number, let ϵ denote an arbitrary positive rational fraction less than 1; and choose any rational number R which is greater than $\alpha+\beta+1$. Next find numbers A, a and B, b such that $A-a<\epsilon_1$, $B-b<\epsilon_1$, where

$$\epsilon_1=\epsilon/R.$$

The determination of A, a and B, b is possible in virtue of Art. 140. Then we have

$$AB-ab<(a+\epsilon_1)(b+\epsilon_1)-ab$$

or $$AB-ab<\epsilon_1(a+b+\epsilon_1)<\epsilon_1(a+b+1)<\epsilon_1 R.$$

That is, $$AB-ab<\epsilon.$$

Thus, as in Art. 141, the classification by means of ab and AB defines a single number which may be rational or irrational; and this number is called $\alpha\beta$.

In particular, if $\beta=1/\alpha$, the product is equal to 1; for the lower class is represented by a/A and the upper class by A/a. That is, the lower class contains all rational numbers less than 1 and the upper class all rational numbers greater than 1. Consequently the product is equal to 1, the single rational number which escapes classification.

Multiplication of negative irrational numbers is reduced at once to that of positive numbers by agreeing to accept the ordinary "rule of signs" as established for rational numbers.

Division of irrationals.

In consequence of the relation $(1/\beta)\times\beta=1$, we may define the quotient α/β as equal to the product $\alpha\times(1/\beta)$.

It is now evident that any of the fundamental laws of algebra which have been established for rational numbers remain true for irrational numbers.

Thus, we have the following laws:

$$\alpha+0=\alpha,\quad \alpha+\beta=\beta+\alpha,\quad \alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma,$$
$$\alpha\times 1=\alpha,\quad \alpha\beta=\beta\alpha,\quad \alpha(\beta\gamma)=(\alpha\beta)\gamma,$$
$$\alpha(\beta+\gamma)=\alpha\beta+\alpha\gamma,$$
$$|\alpha|+|\beta|\geqq|\alpha+\beta|\geqq|\alpha|-|\beta|.$$

For example, let us prove the theorem $\alpha+\beta=\beta+\alpha$.

By definition we have

$$(a+b)<\alpha+\beta<(A+B)$$

and

$$(b+a)<\beta+\alpha<(B+A).$$

But $a+b=b+a$ and $A+B=B+A$, so that $\alpha+\beta$ and $\beta+\alpha$ are defined by the same two classes and are accordingly equal.

The reader will find it a good exercise to write out proofs of the other laws in a similar way. After this he may attempt to construct a theory of irrational indices and of logarithms, on the foundation of Dedekind's theory. It is necessary to define α^λ first and then to prove that $\alpha^\lambda\cdot\alpha^\mu=\alpha^{\lambda+\mu}$, and so on; finally shewing that the equation in λ, $\alpha^\lambda=\beta$, has a root; here α, β are positive and λ, μ may be either positive or negative.

143. The principle of convergence for monotonic sequences whose terms may be either rational or irrational.

A monotonic sequence (a_n) leads to a section in the system of rational numbers as follows:

Suppose for definiteness that the sequence is an increasing one, in which the terms remain less than a fixed number A, so that

$$a_1\leqq a_2\leqq a_3\leqq\ldots\leqq a_n\leqq\ldots<A.$$

Now if k is any rational number, one of two alternatives must occur; either some term in the sequence (a_n) will be equal to or greater than k, or else every term of the sequence will be less than k. We define the class (b) as the class of all rational numbers k which satisfy the first condition, the class (B) as the class of all which satisfy the second condition. Typical numbers of the class (b) are the rational numbers which belong to the sequence; while (B) contains every rational number greater than A, and possibly some rational numbers less than A.

It is clear that the classes (b), (B) together contain *all* rational numbers, and therefore give a section which defines some number β, rational or irrational. We may call (B), (b) the upper and lower classes respectively, defined by the sequence (a_n).

Now every rational number greater than β belongs to the upper class (B), and is therefore greater than any term a_n. And the same is true of every irrational number γ greater than β; for there will be rational numbers between γ and β, and these rational numbers are greater than any term a_n: thus γ is also greater than any term a_n. Consequently *no term in the sequence* (a_n), *whether rational or irrational, can exceed* β.

On the other hand, every rational number less than β must belong to the lower class (b). Now if ϵ is any positive number, there will be rational numbers between β and $\beta-\epsilon$; and, since these numbers are less than β, they must belong to the lower class (b). That is, *there must be some term of the sequence, say* a_m, *which is greater than or equal to* $\beta-\epsilon$.

Hence, since $\quad a_n \geqq a_m, \qquad$ if $n > m$,

we have $\quad \beta \geqq a_n > \beta-\epsilon, \quad$ if $n > m$.

That is, $$\lim a_n = \beta. \qquad \text{[By def. Art. 1.]}$$

A good example of such a sequence is afforded by the terminated decimals derived from an infinite decimal; and it will be seen at once that the section described here is an obvious extension of the method used in Art. 137 above.

Suppose next that the terms of the sequence, while still increasing, do not remain less than any fixed number A. It is then evident that if $a_m > A$, we have $a_n > A$ if $n > m$.

Thus $$\lim_{n\to\infty} a_n = \infty. \qquad \text{[By def. Art. 1.]}$$

Exactly similar arguments can be applied to a decreasing sequence.

As an example we shall give a proof of the theorem that *any continuous monotonic function attains just once every value between its greatest and least values.* Suppose that $f(x)$ steadily increases from $x=a$ to $x=c$, so that

$$b < d, \quad \text{if } f(a)=b \text{ and } f(c)=d.$$

Then if l is any number between b and d, we consider $f\{\frac{1}{2}(a+c)\}$, which is also between b and d; suppose that this is found to be less than l, write then

$$a_1=\tfrac{1}{2}(a+c), \quad b_1=f(a_1)<l,$$
$$c_1=c, \qquad d_1=f(c_1)>l.$$

On the other hand, when $f\{\frac{1}{2}(a+c)\}$ is greater than l, we write

$$a_1=a, \qquad b_1=f(a_1)<l,$$
$$c_1=\tfrac{1}{2}(a+c), \quad d_1=f(c_1)>l.$$

Continuing the process we construct two sequences (a_n), (c_n), the first never decreasing and the second never increasing; and $c_n-a_n=(c-a)/2^n$, so that (a_n), (c_n) have a common limit k. Also by the method of construction it is evident that $f(a_n)<l<f(c_n)$; unless it happens that at some stage we find $f(a_n)=l$, in which case the theorem requires no further discussion.

Now since $f(x)$ is *continuous* we can find an integer ν such that $f(c_n)-f(a_n)<\epsilon$, if $n>\nu$; and both $f(k)$ and l are contained between $f(a_n)$ and $f(c_n)$. Thus we can find ν so that

$$|f(k)-l|<\epsilon, \quad \text{if } n>\nu,$$

and therefore, as in Art. 1·2 (6), $f(k)=l$. From the method of construction it is clear that there is only one value such as k; and this is also evident from the monotonic nature of $f(x)$.

144. Maximum and minimum limiting values of a sequence of rational or irrational terms.

Suppose first that all the terms of the sequence are less than some fixed rational number R, and let r be a smaller rational number, such that an infinity of terms a_n are greater than r. Then if we bisect the interval (r, R) by $\frac{1}{2}(r+R)$, it is evident that either an infinity or a finite number of terms a_n fall between $\frac{1}{2}(r+R)$ and R; in the former case we write $r_1=\frac{1}{2}(r+R)$, $R_1=R$; in the latter we write $r_1=r$, $R_1=\frac{1}{2}(r+R)$. We have thus constructed a smaller interval (r_1, R_1) which contains an infinity of terms a_n; and we can repeat the process as often as we wish. A few stages are indicated in the diagram.

Fig. 39.

Then the sequence $r, r_1, r_2, r_3, \dots$ never decreases, and remains less than R; and so the sequence (r_n) determines a number G (which may of course be either rational or irrational, as in the last article). Again the sequence (R_n) has the same limit G, because

$$R_n-r_n=(R-r)/2^n \to 0$$

as $n \to \infty$. Thus, if ϵ is an arbitrarily small positive number, we can find n so that $r_n \geqq G-\epsilon$, $R_n \leqq G+\epsilon$.

Consequently *an infinite number of terms* a_n *lie between* $G-\epsilon$ *and* $G+\epsilon$, *but only a finite number are greater than* $G+\epsilon$.

Thus we can determine a sub-sequence from (a_n) which has G as its limit; and we can find a certain stage after which all the terms of the sequence are less than $G+\epsilon$; thus no convergent sub-sequence can have a limit greater than G. *These properties shew that* G *is the maximum limit of the sequence* (a_n). (See Art. 5·2.)

If no such number as R can be found in the foregoing argument, there are numbers of the sequence (a_n) greater than any assignable number, *so that the maximum limit is then* ∞. On the other hand, if no such number as r can be found, there will be only a *finite* number of terms greater than $-N$, however large N may be, and consequently

$$\lim a_n = -\infty.$$

For the sake of uniformity we may say even then that the sequence has a maximum limit, which is, of course, $-\infty$.

All the foregoing discussion can be at once modified to establish the existence of a *minimum limit* (g or $-\infty$).

145. The general principle of convergence is both necessary and sufficient.

The principle is stated explicitly in Art. 3; and it is understood that the terms of the sequence may be rational or irrational.

In the first place, the condition is obviously *necessary*; for if $\lim a_n = l$, we know that an index m can be found to correspond to ϵ, in such a way that

$$|l-a_n| < \epsilon, \quad \text{if } n \geqq m.$$

Thus $\quad |a_n-a_m| \leqq |l-a_n|+|l-a_m| < 2\epsilon, \quad \text{if } n > m.$

In the second place, the condition is *sufficient*; for let m be fixed so that

$$|a_n-a_m| < \epsilon, \quad \text{if } n > m,$$

or

$$a_m-\epsilon < a_n < a_m+\epsilon, \quad \text{if } n > m.$$

Then it follows from the last article that the sequence (a_n) has a finite maximum limit G; and then an infinity of terms fall between $G-\epsilon$ and $G+\epsilon$. Choose one of these, say a_p, whose index p is greater than m. Thus we have

$$G-\epsilon < a_p < G+\epsilon.$$

Also $a_p - \epsilon < a_m < a_p + \epsilon$, since $p > m$.

Thus $G - 2\epsilon < a_m < G + 2\epsilon$,

and since $a_m - \epsilon < a_n < a_m + \epsilon$, if $n > m$,

it follows that $G - 3\epsilon < a_n < G + 3\epsilon$, if $n > m$.

Thus $a_n \to G$; and consequently the sequence is convergent. Of course in this case $g = G$, the two extreme limits being equal in a convergent sequence.

Various proofs of this general theorem have been published, some being apparently much shorter than the foregoing series of articles. But on examining the foundations of the shorter investigations it will be seen that in all cases the apparent brevity is obtained by avoiding the definition of an irrational number. This virtually implies a shirking of the whole difficulty; for this difficulty consists essentially * in proving that (under the condition of Art. 3) a sequence may be used to define a "number."

146. First theorem on limits of quotients.

If $\lim a_n = 0$ *and* $\lim b_n = 0$; *and if, in addition, the sequence* (b_n) **steadily decreases**, then

$$\lim \frac{a_n}{b_n} = \lim \frac{a_n - a_{n+1}}{b_n - b_{n+1}},$$

provided that the **second** *quotient has a definite limit, finite or infinite.*

Suppose first that the limit is finite and equal to l; then if ϵ is an arbitrarily small positive fraction, m can be found so that

$$l - \epsilon < \frac{a_n - a_{n+1}}{b_n - b_{n+1}} < l + \epsilon, \quad \text{if } n > m;$$

or, since $(b_n - b_{n+1})$ is *positive*, we have

$$(l - \epsilon)(b_n - b_{n+1}) < a_n - a_{n+1} < (l + \epsilon)(b_n - b_{n+1}).$$

Change n to $n+1, n+2, \dots n+p-1$ and add the results; then we find

$$(l - \epsilon)(b_n - b_{n+p}) < a_n - a_{n+p} < (l + \epsilon)(b_n - b_{n+p}).$$

Now take the limit of this result as $p \to \infty$; we obtain

$$(l - \epsilon) b_n \leqq a_n \leqq (l + \epsilon) b_n,$$

because by hypothesis $a_{n+p} \to 0$ and $b_{n+p} \to 0$.

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* Pringsheim (*Encyklopädie*, I. A. 3, 14) says: "As the truth of this theorem rests essentially and exclusively *on an exact definition of irrational numbers*, naturally the first accurate proofs are connected with the *arithmetical* theories of irrational numbers, and with the associated revision and improvement of the older *geometrical* views."

Hence, since b_n is positive, we have

$$|(a_n/b_n)-l| \leqq \epsilon, \quad \text{if } n > m,$$

or

$$\lim(a_n/b_n)=l.$$

On the other hand, if the given limit is ∞, we can find m so that

$$\frac{a_n-a_{n+1}}{b_n-b_{n+1}} > N, \quad \text{if } n > m,$$

however large N may be. By exactly the same argument as before, we obtain

$$a_n-a_{n+p} > N(b_n-b_{n+p}),$$

which leads to

$$a_n \geqq Nb_n,$$

or

$$a_n/b_n \geqq N, \quad \text{if } n > m.$$

Thus

$$\lim(a_n/b_n)=\infty.$$

There is no difficulty in extending the argument to prove that, with the same restriction on the sequence (b_n),

$$\underline{\lim}\frac{a_n-a_{n+1}}{b_n-b_{n+1}} \leqq \underline{\lim}\frac{a_n}{b_n} \leqq \overline{\lim}\frac{a_n-a_{n+1}}{b_n-b_{n+1}}.$$

This theorem should be compared with the theorem (L'Hospital's) of the Differential Calculus:

If

$$\lim_{x\to\infty}\phi(x)=0, \quad \lim_{x\to\infty}\psi(x)=0,$$

then

$$\lim_{x\to\infty}\phi(x)/\psi(x)=\lim_{x\to\infty}\phi'(x)/\psi'(x),$$

provided that the **second** *limit exists and that $\psi'(x)$ has a constant sign for values of x greater than some fixed value.*

147. Second theorem on limits of quotients.

If b_n **steadily increases** *to ∞, then*

$$\lim\frac{a_n}{b_n}=\lim\frac{a_{n+1}-a_n}{b_{n+1}-b_n},$$

provided that the **second** *limit exists.**

For if the second limit is finite and equal to l, as in Art. 146 above, we see in the same way that we can choose m so that

$$(l-\epsilon)(b_n-b_m) < a_n-a_m < (l+\epsilon)(b_n-b_m), \quad \text{if } n > m.$$

Thus, since b_n is positive,

$$X_n=(l-\epsilon)\left(1-\frac{b_m}{b_n}\right)+\frac{a_m}{b_n} < \frac{a_n}{b_n} < (l+\epsilon)\left(1-\frac{b_m}{b_n}\right)+\frac{a_m}{b_n}=Y_n.$$

* Extended by Stolz from a theorem given by Cauchy for the case $b_n=n$.

Now, since $b_n \to \infty$, we have

$$\lim X_n = \lim \left\{(l-\epsilon)\left(1-\frac{b_m}{b_n}\right)+\frac{a_m}{b_n}\right\} = l-\epsilon$$

and

$$\lim Y_n = \lim \left\{(l+\epsilon)\left(1-\frac{b_m}{b_n}\right)+\frac{a_m}{b_n}\right\} = l+\epsilon.$$

And so we can find n_0 such that

$$X_n > l-2\epsilon \quad \text{and} \quad Y_n < l+2\epsilon, \quad \text{if } n > n_0.$$

Hence $$l-2\epsilon < a_n/b_n < l+2\epsilon, \quad \text{if } n > n_0,$$

or $$\lim (a_n/b_n) = l.$$

Similarly, if the given limit is ∞, we can find m, so that

$$a_n - a_m > N(b_n - b_m), \quad \text{if } n > m,$$

however great N may be.

Thus $$\frac{a_n}{b_n} > \frac{a_m}{b_n} + N\left(1-\frac{b_m}{b_n}\right) = X_n, \text{ say,}$$

and the limit of X_n is N, as $n \to \infty$, so that we can find n_0, such that $X_n > \frac{1}{2}N$, if $n > n_0$.

Then $\quad a_n/b_n > \frac{1}{2}N, \quad$ if $n > n_0$, $\quad$ or $\lim (a_n/b_n) = \infty$.

There is no difficulty in proving similarly that with the same restriction on the sequence (b_n)

$$\underline{\lim}\, \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \leq \underline{\lim}\, \frac{a_n}{b_n} \leq \overline{\lim}\, \frac{a_n}{b_n} \leq \overline{\lim}\, \frac{a_{n+1}-a_n}{b_{n+1}-b_n}.$$

The present theorem should be compared with the following theorem (L'Hospital's) of the Differential Calculus:

If $\psi(x)$ **increases steadily** *to* ∞ *with* x, *then*

$$\lim \frac{\phi(x)}{\psi(x)} = \lim \frac{\phi'(x)}{\psi'(x)},$$

provided that the **second** *limit exists.*

It is probably not out of place to say a word on this important theorem, since few of the commoner English text-books contain a correct proof. By the extended form of the mean-value theorem we have

$$\frac{\phi(x)-\phi(a)}{\psi(x)-\psi(a)} = \frac{\phi'(\xi)}{\psi'(\xi)}, \text{ where } x > \xi > a,$$

and $\psi(x)-\psi(a)$ is positive by hypothesis. Thus, if $\phi'(x)/\psi'(x)$ tends to limit l, we can choose a so that

$$(l-\epsilon)\{\psi(x)-\psi(a)\} < \phi(x)-\phi(a) < (l+\epsilon)\{\psi(x)-\psi(a)\}.$$

And from here onwards the argument proceeds as for sequences.

Similarly, if $\phi'(x)/\psi'(x) \to \infty$, we can find a so that

$$\phi(x) - \phi(a) > N\{\psi(x) - \psi(a)\},$$

and again the same argument can be used.

Ex. 1. If

$$\left.\begin{array}{l} a_n = 1^p + 2^p + \dots + n^p \\ b_n = n^{p+1} \end{array}\right\} \qquad p+1>0,$$

we have
$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}} = \frac{1/n}{(1+1/n)^{p+1}-1}\left(1+\frac{1}{n}\right)^p.$$

Now
$$\lim \frac{(1+h)^{p+1}-1}{h} = p+1, \quad \text{as } h \to 0,$$

by the fundamental limit of the Differential Calculus.

Hence,
$$\lim \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{1}{p+1},$$

and so
$$\lim \frac{a_n}{b_n} = \frac{1}{p+1}.$$

Ex. 2. If
$$a_n = \log n, \quad b_n = n,$$

we have
$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \log\left(1+\frac{1}{n}\right),$$

so that
$$\lim\left(\frac{\log n}{n}\right) = 0. \quad \text{(Compare Art. 160.)}$$

Similarly, if $a_n = (\log n)^2$, $b_n = n$, we find

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \{\log n + \log(n+1)\}\log\left(1+\frac{1}{n}\right) < \frac{2}{n}\log(n+1),$$

which tends to 0 by the previous result.

Thus
$$\lim\{(\log n)^2/n\} = 0.$$

Similarly we can prove that $\lim\{(\log n)^k/n\} = 0$.

The reader may also verify this result by using L'Hospital's theorem.

Ex. 3. If
$$a_n = p^n, \quad b_n = n,$$

we have
$$a_{n+1} - a_n = p^n(p-1), \quad b_{n+1} - b_n = 1,$$

and hence
$$\lim(a_{n+1} - a_n) = 0, \quad \text{if } p \leqq 1,$$
$$\text{or} \quad \infty, \quad \text{if } p > 1.$$

Thus
$$\lim(p^n/n) = 0, \quad \text{if } p \leqq 1,$$
$$\text{or} \quad \infty, \quad \text{if } p > 1. \quad \text{(Art. 160.)}$$

Of course Ex. 3 is only another form of Ex. 2.

Ex. 4. *Even when* (a_n) *and* (b_n) *are both monotonic,* $\lim a_n/b_n$ *need not exist.*

In this case, the theorem shews that $\lim(a_{n+1}-a_n)/(b_{n+1}-b_n)$ does not exist. An example is given by

$$a_n = p^n\{q + (-1)^n\}, \quad b_n = p^n \quad (p > 1).$$

Here $$a_{n+1}-a_n=p^n\{pq+p(-1)^{n+1}-q-(-1)^n\}$$
$$=p^n\{(p-1)q-(p+1)(-1)^n\},$$

and so a_n steadily increases if $q>(p+1)/(p-1)$.

Then we have

$$\underline{\lim}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=q-\frac{p+1}{p-1},\quad \overline{\lim}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=q+\frac{p+1}{p-1},$$

while $$\underline{\lim}\,(a_n/b_n)=q-1,\qquad \overline{\lim}\,(a_n/b_n)=q+1.$$

Since $$(p+1)/(p-1)>1,$$

these results agree with the extended form of the theorem.

Ex. 5. *Even when* $(a_{n+1}-a_n)/(b_{n+1}-b_n)$ *oscillates infinitely,* a_n/b_n *may have a definite limit.*

For if $$a_n=3n+(-1)^n,\quad b_n=3n+(-1)^{n+1},$$

we see that $$a_{n+1}-a_n=3+2(-1)^{n+1},\quad b_{n+1}-b_n=3+2(-1)^n.$$

Thus $(a_{n+1}-a_n)/(b_{n+1}-b_n)$ oscillates between $\frac{1}{5}$ and 5, although

$$\lim\,(a_n/b_n)=1.$$

Again, if $$a_n=(n+1)^2+(-1)^n n,\quad b_n=(n+1)^2+(-1)^{n+1}n,$$

we find

$$a_{n+1}-a_n=2n+3+(-1)^{n+1}(2n+1),\quad b_{n+1}-b_n=2n+3+(-1)^n(2n+1),$$

so that $(a_{n+1}-a_n)/(b_{n+1}-b_n)$ is alternately $2(n+1)$ and $1/2(n+1)$; and so

$$\underline{\lim}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=0,\quad \overline{\lim}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\infty.$$

But $$\lim\,(a_n/b_n)=1.$$

Ex. 6. *If b_n does not steadily increase, the theorem is not necessarily true.*

For example take $$a_n=n+1,\quad b_n=\{2+(-1)^n\}n,$$

so that $$a_{n+1}-a_n=1,\quad b_{n+1}-b_n=2+(-1)^{n+1}(2n+1).$$

Consequently $$\lim\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=0,$$

but yet $$\underline{\lim}\,(a_n/b_n)=\tfrac{1}{3},\quad \overline{\lim}\,(a_n/b_n)=1.$$

Similarly L'Hospital's theorem may fail when $\psi'(x)$ changes sign infinitely often.

Thus consider $\phi(x)=x+1+\sin x\cos x,\quad \psi(x)=e^{\sin x}(x+\sin x\cos x),$ for which we find that, as $x\to\infty$, $\phi'(x)/\psi'(x)\to 0$, while $\phi(x)/\psi(x)$ oscillates between $1/e$ and e.

Ex. 7. Consider the case,

$$\phi(x)=x+a\sin x,\quad \psi(x)=x,\quad (a>0),$$

and prove that $$\lim_{x\to\infty}\phi(x)/\psi(x)=1,$$

while $$\underline{\lim}\,\phi'(x)/\psi'(x)=1-a,\quad \overline{\lim}\,\phi'(x)/\psi'(x)=1+a.$$

148. An extension of Abel's Lemma.

To determine limits for the fraction

$$X_n = \frac{b_0v_0 + b_1v_1 + \dots + b_nv_n}{a_0v_0 + a_1v_1 + \dots + a_nv_n},$$

where the terms a_r, v_r are all positive and the sequence (v_r) is monotonic.

Write $A_0 = a_0$, $A_1 = a_0 + a_1$, ..., $A_n = a_0 + a_1 + \dots + a_n$;

and $B_0 = b_0$, $B_1 = b_0 + b_1$, ..., $B_n = b_0 + b_1 + \dots + b_n$.

Then, as in Art. 20,

$$X_n = \frac{B_0(v_0 - v_1) + B_1(v_1 - v_2) + \dots + B_{n-1}(v_{n-1} - v_n) + B_nv_n}{A_0(v_0 - v_1) + A_1(v_1 - v_2) + \dots + A_{n-1}(v_{n-1} - v_n) + A_nv_n}.$$

First, if the sequence (v_n) steadily decreases, we can obtain an upper limit to X_n by writing

H_mA_r in place of B_r (for $r = m, m+1, \dots, n$)

and HA_r in place of B_r (for $r = 0, 1, \dots, m-1$),

where H, H_m are the upper limits of

$$\left(\frac{B_0}{A_0}, \frac{B_1}{A_1}, \dots, \frac{B_n}{A_n}\right), \text{ and of } \left(\frac{B_m}{A_m}, \frac{B_{m+1}}{A_{m+1}}, \dots, \frac{B_n}{A_n}\right),$$

respectively. To see that this step is justified, note that

$A_0, A_1, \dots, A_n$ and $(v_0 - v_1), (v_1 - v_2), \dots, (v_{n-1} - v_n), v_n$

are all positive.

Thus, on subtracting H_m we find

$$X_n - H_m < (H - H_m)\frac{A_0(v_0 - v_1) + A_1(v_1 - v_2) + \dots + A_{m-1}(v_{m-1} - v_m)}{A_0(v_0 - v_1) + A_1(v_1 - v_2) + \dots + A_nv_n}.$$

If we replace A_r by its value $a_0 + a_1 + \dots + a_r$, we obtain

$$X_n - H_m < (H - H_m)\frac{(a_0v_0 + a_1v_1 + \dots + a_mv_m) - A_mv_m}{a_0v_0 + a_1v_1 + \dots + a_nv_n}.$$

Or, since $H - H_m$ is positive by definition,

$$X_n < H_m + (H - H_m)\frac{a_0v_0 + a_1v_1 + \dots + a_mv_m}{a_0v_0 + a_1v_1 + \dots + a_nv_n}.$$

In like manner we prove that

$$X_n > h_m - (h_m - h)\frac{a_0v_0 + a_1v_1 + \dots + a_mv_m}{a_0v_0 + a_1v_1 + \dots + a_nv_n},$$

where h, h_m are the corresponding lower limits for B_r/A_r.

Secondly, suppose that the sequence (v_n) *steadily increases.* In the numerator of X_n the factors (v_0-v_1), (v_1-v_2), ... , $(v_{n-1}-v_n)$ are now all negative, while v_n is positive; thus the value of the numerator is increased by writing

$$hA_r \text{ in place of } B_r \quad (\text{for } r=0, 1, \ldots, m-1)$$

and $$h_m A_r \text{ in place of } B_r \quad (\text{for } r=m, m+1, \ldots, n-1),$$

while in the last term we put $H_m A_n$ in place of B_n. This changes the numerator to

$$\begin{aligned} h\{A_0(v_0-v_1)+\ldots+A_{m-1}(v_{m-1}-v_m)\} \\ +h_m\{A_m(v_m-v_{m+1})+\ldots+A_{n-1}(v_{n-1}-v_n)\}+H_m A_n v_n \\ =h_m(a_0v_0+\ldots+a_nv_n)+(H_m-h_m)A_nv_n \\ +(h_m-h)(A_mv_m-a_0v_0-\ldots-a_mv_m), \end{aligned}$$

and, since $h_m \geqq h$, this will not be decreased if we omit the negative terms in the last bracket. Hence, as the denominator is essentially positive, we obtain the result

$$X_n < h_m + \frac{(H_m-h_m)A_nv_n+(h_m-h)A_mv_m}{a_0v_0+a_1v_1+\ldots+a_nv_n}.$$

Similarly, we find

$$X_n > H_m - \frac{(H_m-h_m)A_nv_n+(H-H_m)A_mv_m}{a_0v_0+a_1v_1+\ldots+a_nv_n}.$$

Thirdly, if the sequence (v_n) *first increases and afterwards decreases.** Suppose that the term v_p is the greatest in the sequence, and let m be less than p. Then the factors (v_0-v_1), ... , $(v_{p-1}-v_p)$ are negative, while (v_p-v_{p+1}), ... , $(v_{n-1}-v_n)$, v_n are positive. Thus the numerator of X_n is not greater than

$$\begin{aligned} h\{A_0(v_0-v_1)+\ldots+A_{m-1}(v_{m-1}-v_m)\} \\ +h_m\{A_m(v_m-v_{m+1})+\ldots+A_{p-1}(v_{p-1}-v_p)\} \\ +H_m\{A_p(v_p-v_{p+1})+\ldots+A_{n-1}(v_{n-1}-v_n)+A_nv_n\} \\ =H_m(a_0v_0+\ldots+a_nv_n) \\ +(H_m-h_m)(A_pv_p-a_0v_0-\ldots-a_pv_p) \\ +(h_m-h)(A_mv_m-a_0v_0-\ldots-a_mv_m). \end{aligned}$$

Hence, by an argument similar to the last, we deduce

$$X_n < H_m + \frac{(H_m-h_m)A_pv_p+(h_m-h)A_mv_m}{a_0v_0+a_1v_1+\ldots+a_nv_n},$$

* In this case the sequence (v_n) is not, strictly speaking, monotonic; but it will save repetition to discuss the corresponding result here.

and similarly

$$X_n > h_m - \frac{(H_m - h_m)A_p v_p + (H - H_m)A_m v_m}{a_0 v_0 + a_1 v_1 + \dots + a_n v_n}.$$

Ex. Prove that if $a_n \to a$ and $\sum b_n$ is divergent, *although b_n need not be positive,*

$$\lim \frac{a_0 b_0 + a_1 b_1 + \dots + a_n b_n}{b_0 + b_1 + \dots + b_n} = a,$$

provided that

$$\frac{|b_0| + |b_1| + \dots + |b_n|}{|b_0 + b_1 + \dots + b_n|} < K. \qquad \text{[JENSEN.]}$$

149. Cauchy's theorems.

It follows at once from Art. 147, that if $(s_{n+1} - s_n)$ has a definite limit, s_n/n tends to the same limit. Thus, by writing

$$s_n = a_1 + a_2 + \dots + a_n,$$

we obtain **Cauchy's first theorem**, *if a sequence (a_n) has a limit, this limit is also equal to*

$$\lim \frac{1}{n}(a_1 + a_2 + \dots + a_n).$$

A direct proof is very simple, however; if $a_n \to l$, we can find m so that

$$l - \epsilon < a_n < l + \epsilon, \quad \text{if } n > m.$$

Hence, by addition, we have

$$(n-m)(l-\epsilon) < a_{m+1} + a_{m+2} + \dots + a_n < (n-m)(l+\epsilon).$$

Thus, writing $\qquad (a_1 + a_2 + \dots + a_n)/n = A_n,$

we see that $\left(1 - \frac{m}{n}\right)(l-\epsilon) + \frac{m}{n}A_m < A_n < \left(1 - \frac{m}{n}\right)(l+\epsilon) + \frac{m}{n}A_m$

or

$$l - \epsilon - \frac{m}{n}(l - \epsilon - A_m) < A_n < l + \epsilon + \frac{m}{n}(A_m - l - \epsilon).$$

If then n_0 is found so that

$$n_0\epsilon > m\{|l| + \epsilon + |A_m|\}, \quad \text{and} \quad n_0 \geqq m,$$

we have

$$l - 2\epsilon < A_n < l + 2\epsilon, \quad \text{if } n > n_0.$$

Hence

$$\lim A_n = l.$$

Similarly, if $a_n \to \infty$, we can find m so that $a_n > N$, if $n > m$.

Proceeding similarly, we get

$$A_n > \left(1 - \frac{m}{n}\right)N + \frac{m}{n}A_m.$$

Then choose n_0 so that $\quad n_0 > 2m\{N + |A_m|\}/N,$

and we have

$$A_n > N - \tfrac{1}{2}N = \tfrac{1}{2}N, \quad \text{if } n > n_0.$$

Thus

$$\lim A_n = \infty.$$

When $a_n \to -\infty$, we need only change all the signs to deduce the final result.

Of course the second limit may exist, when the first does not; thus, with $a_{2n-1}=1$, $a_{2n}=0$, the second limit becomes $\frac{1}{2}$, although the sequence (a_n) oscillates.

Cauchy's second theorem. *If all the terms of a sequence* (a_n) *are positive, and if* $\lim(a_{n+1}/a_n)$ *exists, so also does* $\lim a_n^{\frac{1}{n}}$; *and the two limits are equal.*

For, if $\lim(a_{n+1}/a_n)$ is finite and equal to l, we can find m, so that

$$l(1-\epsilon)<a_{n+1}/a_n<l(1+\epsilon), \quad \text{if } n \geqq m.$$

By multiplication, we obtain

$$l^{n-m}(1-\epsilon)^{n-m}<a_n/a_m<l^{n-m}(1+\epsilon)^{n-m},$$

so that $$\frac{a_m}{l^m}(1-\epsilon)^n<\frac{a_n}{l^n}<\frac{a_m}{l^m}(1+\epsilon)^n.$$

Hence $$\left(\frac{a_m}{l^m}\right)^{\frac{1}{n}}(1-\epsilon)<\frac{a_n^{\frac{1}{n}}}{l}<\left(\frac{a_m}{l^m}\right)^{\frac{1}{n}}(1+\epsilon).$$

Now, as $n\to\infty$, $(a_m/l^m)^{\frac{1}{n}}\to 1$ (Ex. 6, Ch. I.), and so we can find $n_0 \geqq m$, such that

$$\frac{1-2\epsilon}{1-\epsilon}<\left(\frac{a_m}{l^m}\right)^{\frac{1}{n}}<\frac{1+2\epsilon}{1+\epsilon}, \quad \text{if } n>n_0.$$

Hence $$1-2\epsilon<a_n^{\frac{1}{n}}/l<1+2\epsilon, \quad \text{if } n>n_0,$$

or $$\lim a_n^{\frac{1}{n}}/l=1.$$

Thus $$\lim a_n^{\frac{1}{n}}=l=\lim(a_{n+1}/a_n).$$

Again, if $\lim(a_{n+1}/a_n)=\infty$, we can find m, so that

$$a_{n+1}/a_n>N, \quad \text{if } n\geqq m,$$

however great N may be.

Hence, as above, $$a_n/a_m>N^{n-m}$$

or $$a_n^{\frac{1}{n}}>N(a_m/N^m)^{\frac{1}{n}}.$$

But, as $n\to\infty$, $$\lim(a_m/N^m)^{\frac{1}{n}}=1 \qquad \text{(Ex. 6, p. 22)};$$

thus we can find n_0, such that

$$(a_m/N^m)^{\frac{1}{n}}>\tfrac{1}{2}, \quad \text{if } n>n_0.$$

Then $$a_n^{\frac{1}{n}}>\tfrac{1}{2}N, \quad \text{if } n>n_0,$$

or $$\lim a_n^{\frac{1}{n}}=\infty.$$

The case when $\lim(a_{n+1}/a_n)=0$ can be reduced to the last by writing $a_n=1/b_n$, because a_n is positive.

It is not difficult to extend the previous argument to prove that in general

$$\underline{\lim}\,(a_{n+1}/a_n) \leqq \underline{\lim}\, a_n^{\frac{1}{n}} \leqq \overline{\lim}\,(a_{n+1}/a_n).$$

For if $\underline{\lim}\, a_{n+1}/a_n = g$, and $\overline{\lim}\, a_{n+1}/a_n = G$, we can find m so that

$$g(1-\epsilon) < a_{n+1}/a_n < G(1+\epsilon), \quad \text{if } n > m.$$

Repeating the previous arguments, we find that

$$\frac{a_n^{\frac{1}{n}}}{g} > \left(\frac{a_m}{g^m}\right)^{\frac{1}{n}}(1-\epsilon) > 1-2\epsilon, \quad \text{if } n > n_0,$$

and

$$\frac{a_n^{\frac{1}{n}}}{G} < \left(\frac{a_m}{g^m}\right)^{\frac{1}{n}}(1+\epsilon) < 1+2\epsilon, \quad \text{if } n > n_0.$$

Hence $$\underline{\lim}\, a_n^{\frac{1}{n}} \geqq g, \text{ and } \overline{\lim}\, a_n^{\frac{1}{n}} \leqq G.$$

Similarly it may be proved that

$$\underline{\lim}\,(a_{n+1}/a_n)^{\frac{1}{c_n}} \leqq \underline{\lim}\, a_n^{\frac{1}{b_n}} \leqq \overline{\lim}\,(a_{n+1}/a_n)^{\frac{1}{c_n}},$$

if $c_n = b_{n+1} - b_n$ and b_n steadily increases to ∞.

Ex. 1. To find $$\lim \frac{1}{n}(n\,!)^{\frac{1}{n}},$$

we write $$a_n = (n\,!)/n^n,$$

so that $$a_{n+1}/a_n = n^n/(n+1)^n = (1+1/n)^{-n}.$$

Thus $$\lim \frac{1}{n}(n\,!)^{\frac{1}{n}} = \lim a_n^{\frac{1}{n}} = \lim \frac{a_{n+1}}{a_n} = \frac{1}{e},$$

by Art. 155. This result can be verified at once by reference to Stirling's formula for n ! (see Art. 179).

Ex. 2. To find $\lim \frac{1}{n}\{(m+1)(m+2)\ldots(m+n)\}^{\frac{1}{n}}$, where m is fixed, we write

$$a_n = (m+1)(m+2)\ldots(m+n)/n^n,$$

and then $$\frac{a_{n+1}}{a_n} = \frac{n+m+1}{n+1}\left(1+\frac{1}{n}\right)^{-n};$$

so that the limit again is $1/e$.

Ex. 3. Similarly we find that

$$\lim \frac{1}{n}\{(n+1)(n+2)\ldots 2n\}^{\frac{1}{n}} = \frac{4}{e},$$

because $$\frac{a_{n+1}}{a_n} = \frac{2(2n+1)}{n+1}\left(1+\frac{1}{n}\right)^{-n}.$$

150. Cesàro's theorem.

If the sequences (a_n), (b_n) converge to the limits a, b, then

$$\lim \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1) = ab.$$

Write $$a_n = a + p_n, \quad \text{and} \quad |p_n| = P_n.$$

Then $p_n \to 0$, and consequently $P_n \to 0$ also.

Hence, by Cauchy's first theorem,

$$(P_1+P_2+\dots+P_n)/n \to 0.$$

Now, on substituting for $(a_1, a_2, \dots, a_n)$, the given expression becomes

$$\frac{a}{n}(b_1+b_2+\dots+b_n)+R_n,$$

where
$$|R_n| < \frac{C}{n}(P_1+P_2+\dots+P_n),$$

and C is the upper limit to $|b_1|, |b_2|, \dots |b_n|$.

Now as $n \to \infty$, C remains finite, because $b_n \to b$; and so $|R_n| \to 0$.

Further, from Cauchy's theorem,

$$(b_1+b_2+\dots+b_n)/n \to b,$$

and so the given expression tends to the limit ab.

151. The Hardy-Landau converse of Cauchy's first theorem.

If (i) a_n *is real*, (ii) either $n(a_n-a_{n+1}) < K$ or $n(a_{n+1}-a_n) < K$, (iii) $b_n=(a_1+a_2+\dots+a_n)/n$ *tends to the limit* l, *then* a_n *also tends to the limit* l.

Without loss of generality we can suppose that $l=0$, $K=1$, and that the *first* form of the two conditions (ii) is given.*

Now $a_{n+1}=(n+1)b_{n+1}-nb_n$, so that $(b_{n+1}-b_n)=(a_{n+1}-b_{n+1})/n$, and thus, on taking the sum from $n=m$ to $n=m+p-1$, we find that

(1) $$b_{m+p}-b_m=\frac{a_{m+1}-b_{m+1}}{m}+\frac{a_{m+2}-b_{m+2}}{m+1}+\dots+\frac{a_{m+p}-b_{m+p}}{m+p-1}.$$

Since $b_n \to 0$, we can find a value of ν such that $|b_n| < \epsilon$, if $n > \nu$, however small ϵ may be.

Further, we have seen (in Art. 147) that

$$\underline{\lim}\, a_n \leqq \underline{\lim}\, b_n \leqq \overline{\lim}\, b_n \leqq \overline{\lim}\, a_n,$$

and so here
$$\underline{\lim}\, a_n \leqq 0 \leqq \overline{\lim}\, a_n.$$

Thus there are three possibilities only:

(α) $\underline{\lim}\, a_n = 0 = \overline{\lim}\, a_n$, so that $a_n \to 0$,

(β) $\underline{\lim}\, a_n \leqq 0$, $\overline{\lim}\, a_n > 0$,

(γ) $\underline{\lim}\, a_n < 0$, $\overline{\lim}\, a_n \geqq 0$.

* Write $c_n=(a_n-l)/K$, then it is easy to see that $(c_1+c_2+\dots+c_n)/n \to 0$, and that $n(c_n-c_{n+1}) < 1$ if the *first* form of condition (ii) is given; when the *second* form is given, we use $d_n=(l-a_n)/K$ and then again $n(d_n-d_{n+1}) < 1$.

It will be noticed that (β) and (γ) are not exclusive, since they include the common possibility

$$\underline{\lim}\, a_n < 0, \quad \overline{\lim}\, a_n > 0;$$

but one at least of (β) and (γ) must be true, unless $a_n \to 0$. Let us suppose that (β) is true; then a positive number λ exists such that * $a_n > 2\lambda$ for one infinite set of values of n, and for another infinite set of values $a_n < \lambda$.

Choose any value $m+1 > \nu+1$ from the first set of values of n, and let $m+p+1$ be the *next* value of n belonging to the second set; then we have

$$(2) \qquad a_{m+1} > 2\lambda, \quad a_{m+p+1} < \lambda, \quad (m+p > m > \nu)$$

and also $a_{m+1}, a_{m+2}, a_{m+3}, \dots a_{m+p}$ are all greater than, or equal to, λ.

Consequently in (1) we have

$$a_{m+1} - b_{m+1} \geqq \lambda - \epsilon, \quad a_{m+2} - b_{m+2} \geqq \lambda - \epsilon, \dots a_{m+p} - b_{m+p} \geqq \lambda - \epsilon,$$

and thus we find

$$(3) \quad b_{m+p} - b_m \geqq (\lambda - \epsilon)\left(\frac{1}{m} + \frac{1}{m+1} + \dots + \frac{1}{m+p-1}\right) > \frac{p(\lambda - \epsilon)}{m+p}.$$

Further, condition (ii) gives, on summation,

$$a_{m+1} - a_{m+p+1} < \left(\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{m+p}\right) < \frac{p}{m},$$

and thus (2) yields the inequalities

$$\lambda < \frac{p}{m}, \quad p > m\lambda, \quad \frac{p}{m+p} > \frac{\lambda}{1+\lambda}.$$

Hence, substituting in (3), we have

$$(4) \qquad b_{m+p} - b_m > \frac{\lambda(\lambda - \epsilon)}{1+\lambda}.$$

But, since $m > \nu$, $|b_m| < \epsilon$ and $|b_{m+p}| < \epsilon$,

so that
$$2\epsilon > |b_{m+p} - b_m|,$$

and thus (4) yields the conclusion

$$2\epsilon(1+\lambda) > \lambda(\lambda - \epsilon)$$

or
$$\epsilon(2+3\lambda) > \lambda^2,$$

* If $\overline{\lim}\, a_n = G > 0$, there is an infinite set of values of n for which $a_n > G - \eta$, where η is any positive number; and thus (putting $\eta = \frac{1}{3}G$) an infinite set for which $a_n > \frac{2}{3}G$. Now $\underline{\lim}\, a_n \leqq 0$, and so there is another infinite set of values of n, for which $a_n < 0 + \eta$ or $a_n < \frac{1}{3}G$. So the conditions are satisfied by taking $\lambda = \frac{1}{3}G$.

But if $\overline{\lim}\, a_n = \infty$, λ may be taken as *any* positive number.

which contradicts the hypothesis that ϵ is arbitrarily small. Accordingly hypothesis (β) is inconsistent with condition (ii).

If hypothesis (γ) is true we can choose similarly * a positive number λ, and a value of $m > \nu$, such that

$$(2') \qquad -a_m < \lambda, \quad -a_{m+p} > 2\lambda,$$

while $-a_{m+1}, -a_{m+2}, \ldots, -a_{m+p}$ are all greater than, or equal to, λ.

Thus we can now write

$$b_{m+1} - a_{m+1} \geqq -\epsilon + \lambda, \ldots b_{m+p} - a_{m+p} \geqq -\epsilon + \lambda,$$

and so, reversing all the signs in (1), we find that

$$(3') \qquad b_m - b_{m+p} > \frac{p(\lambda - \epsilon)}{m+p}.$$

Also $\quad a_m - a_{m+p} < \left(\frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{m+p-1}\right) < \frac{p}{m},$

and so (2′) gives the inequalities

$$\lambda < \frac{p}{m}, \quad p > m\lambda, \quad \frac{p}{p+m} > \frac{\lambda}{1+\lambda}.$$

From here onwards the argument proceeds as before; and we conclude that hypothesis (γ) is also inconsistent with condition (ii).

Consequently the hypothesis (α) must be true, and so the theorem is proved.

The most interesting cases of the theorem arise (a) when $a_n - a_{n+1}$ is never negative (or never positive) † and (b) when

$$n \, | a_n - a_{n+1} | < K.$$

---

* If $\underline{\lim}\, a_n = g < 0$, while $\overline{\lim}\, a_n \geqq 0$, we can find an infinite set of values of n, for which $a_n < g + \eta$ and a second set for which $a_n > 0 - \eta$, where η is any positive number. We take now $\eta = -\frac{1}{3}g$, and $\lambda = -\frac{1}{3}g$; then for the first set $a_n < -2\lambda$ or $-a_n > 2\lambda$ and for the second set $a_n > -\lambda$ or $-a_n < \lambda$.

Now choose a value $m_1 > \nu$ from the second set, and the next following value m_2 from the first set; then if no values between m_1, m_2 belong to the second set, we can write $\qquad m = m_1, \quad m + p = m_2.$

But if some further values (between m_1, m_2) belong to the second set, we take m to be the greatest of these values, and again put $m + p = m_2$.

In either case there are no values between m and $m + p$ which belong to the second set: thus

$$a_{m+1} \leqq -\lambda, \quad a_{m+2} \leqq -\lambda, \ldots a_{m+p} \leqq -\lambda.$$

If $\underline{\lim}\, a_n = -\infty$, λ may be any positive number, the first set of values of n being defined by $a_n < -2\lambda$ or $-a_n > 2\lambda$; and the second set by $a_n > -\lambda$ or $-a_n < \lambda$.

† In this form of condition the proof may be simplified by observing that the sequence (a_n) is monotonic and so has a limit; and in virtue of Art. 140, the limit of (a_n) can have no other value than l.

In case (*b*) the theorem may be extended at once to complex sequences, by considering the real and imaginary parts of a_n separately.

The above proof was suggested by Mr. A. E. Jolliffe; other proofs have been given by Hardy (*Proc. Lond. Math. Soc.* (2), vol. 8), Landau (*Prace Matematyczno-Fizyczne*, vol. 21), de la Vallée Poussin (*Cours d'Analyse*, vol. 2, ed. 2, p. 157), and Cipolla (*Mem. Acad. Napoli*).

It is easy to modify the above proof to give a theorem for continuous variation, analogous to Hardy's theorem for sequences.

If $f(x)/x \to l$ *as* $x \to \infty$, *then also* $f'(x) \to l$, *provided that* either $-xf''(x) < K$ or $xf''(x) < K$.

As before, we can take $l = 0$, $K = 1$ (and use the *first* condition), without loss of generality; and let us write for brevity

$$\phi(x) = f(x)/x.$$

Then we can find ξ so that $|\phi| < \epsilon$, if $x > \xi$: and as in the foregoing, if we suppose that

$$\underline{\lim} f'(x) \leqq 0 \quad \text{and} \quad \overline{\lim} f'(x) > 0,$$

we can choose an infinite set of values such that $f'(x) > 2\lambda$, and another set for which $f'(x) < \lambda$; so choose $X\ (> \xi)$ from the first set, and let $X + h$ be the next greatest value belonging to the second set.

Then $$f'(x) > \lambda \text{ from } X \text{ to } X + h.$$

Also $$f'(x) = x\phi'(x) + \phi(x),$$

so that $$x\phi'(x) = f'(x) - \phi(x) \geqq \lambda - \epsilon \quad \text{from } X \text{ to } X + h.$$

Thus $$\phi(X+h) - \phi(X) \geqq \int_X^{X+h} \frac{\lambda - \epsilon}{x}\,dx > (\lambda - \epsilon)\frac{h}{X+h}.$$

Also $$f'(X) - f'(X+h) = \int_X^{X+h} \{-f''(x)\}\,dx < \int_X^{X+h} \frac{dx}{x} < \frac{h}{X},$$

and so $\lambda < h/X$, giving $\quad h/(X+h) > \lambda/(1+\lambda)$.

Hence finally

$$2\epsilon > |\phi(X+h) - \phi(X)| > \lambda(\lambda - \epsilon)/(1+\lambda),$$

leading to $\epsilon(2 + 3\lambda) > \lambda^2$, which is contrary to the hypothesis that ϵ is arbitrarily small.

Similarly we deal with the other alternatives.

Ex. 1. If $(a_1 + a_2 + \ldots + a_n)/n \to l$ and the product $n(a_{n+1} - a_n)$ steadily increases, prove that $a_n \to l$.

Ex. 2. State and prove the theorem corresponding to Ex. 1, for a continuous variable x.

Ex. 3. If both b_n and a_n/b_n tend steadily to infinity with n, prove that $(a_{n+1} - a_n)/(b_{n+1} - b_n)$ also tends to infinity, with a rapidity not less than that of a_n/b_n.

Ex. 4. If a_n and b_n both tend steadily to infinity with n, and if a_n/b_n tends steadily to zero, prove that $(a_{n+1}-a_n)/(b_{n+1}-b_n)$ tends also to zero, with a rapidity not less than that of a_n/b_n.

[For a discussion of Exs. 3, 4 and allied theorems, see Bortolotti, *Annali di Matematica* (3), vol. 11, 1905, p. 29.]

152. THEOREM. *If Σb_n, Σc_n are two divergent series of positive terms, then*

$$\lim \frac{c_0 s_0 + c_1 s_1 + \dots + c_n s_n}{c_0 + c_1 + \dots + c_n} = \lim \frac{b_0 s_0 + b_1 s_1 + \dots + b_n s_n}{b_0 + b_1 + \dots + b_n};$$

provided that the second limit exists, and that either (i) c_n/b_n *steadily decreases, or* (ii) c_n/b_n *steadily increases subject to the condition* *

$$\frac{c_n}{c_0 + c_1 + \dots + c_n} < K \frac{b_n}{b_0 + b_1 + \dots b_n},$$

where K is fixed.

Let us write for brevity

$$B_n = b_0 + b_1 + \dots + b_n, \qquad C_n = c_0 + c_1 + \dots + c_n,$$
$$P_n = b_0 s_0 + b_1 s_1 + \dots + b_n s_n, \qquad Q_n = c_0 s_0 + c_1 s_1 + \dots + c_n s_n.$$

Let us also write $c_n/b_n = v_n$; then

$$\frac{Q_n}{C_n} = \frac{(b_0 s_0) v_0 + \dots + (b_n s_n) v_n}{b_0 v_0 + \dots + b_n v_n}.$$

To this fraction we can apply the result of Art. 148 above, and we find, in the first case, *when* (v_n) *decreases,*

$$(1) \qquad h_m - (h_m - h) \frac{C_m}{C_n} < \frac{Q_n}{C_n} < H_m + (H - H_m) \frac{C_m}{C_n},$$

where H, h are the upper and lower limits of P_0/B_0, P_1/B_1, ..., to ∞, and H_m, h_m are those of P_m/B_m, P_{m+1}/B_{m+1}, ..., to ∞.

If $\lim P_n/B_n = l$, we can find m so that $h_m \geqq l - \epsilon$ and $H_m \leqq l + \epsilon$, and then we find from (1)

$$(l-\epsilon) - (l-h) \frac{C_m}{C_n} < \frac{Q_n}{C_n} < (l+\epsilon) + (H+l) \frac{C_m}{C_n}.$$

Now $C_n \to \infty$, so that we can choose n_0, such that

$$l - 2\epsilon < Q_n/C_n < l + 2\epsilon, \quad \text{if } n > n_0.$$

Hence,
$$\lim (Q_n/C_n) = l.$$

* The theorem is due to Cesàro, *Atti d. R. Accad. d. Lincei*, Rendiconti (IV.), 4, 1888, p. 452: it was rediscovered by Hardy, *Quarterly Journal*, vol. 38, 1907, p. 269. Of course $K > 1$ in case (2), in virtue of the fact that c_n/b_n increases.

In like manner, if $P_n/B_n \to \infty$, we can find m so that $h_m \geqq N$, and then
$$Q_n/C_n > N-(N-h)\,C_m/C_n.$$

Since $C_n \to \infty$, we can find n_0, so that
$$(N-h)C_m/C_n < \tfrac{1}{2}N, \quad \text{if } n > n_0,$$
or
$$Q_n/C_n > \tfrac{1}{2}N, \quad \text{if } n > n_0,$$
and so
$$\lim\,(Q_n/C_n) = \infty.$$

In the second case, *when $v_n = c_n/b_n$ increases,* we see that Q_n/C_n lies between
$$H_m-(H_m-h_m)\frac{B_nc_n}{C_nb_n}-(H-H_m)\frac{B_mc_m}{C_nb_m}$$
and
$$h_m+(H_m-h_m)\frac{B_nc_n}{C_nb_n}+(h_m-h)\frac{B_mc_m}{C_nb_m}.$$

Now, by hypothesis, $\dfrac{c_n}{C_n} < K\dfrac{b_n}{B_n}$,

where K is constant and greater than unity. Hence Q_n/C_n lies between
$$H_m-K(H_m-h_m)-K(H-H_m)\frac{C_m}{C_n}$$
and
$$h_m+K(H_m-h_m)+K(h_m-h)\frac{C_m}{C_n}.$$

Then proceeding as before, we see that n_0 can be found so that
$$l-2K\epsilon < Q_n/C_n < l+2K\epsilon, \quad \text{if } n > n_0,$$
so that
$$\lim\,(Q_n/C_n) = l.$$

Similarly we prove that when $P_n/B_n \to \infty$, so also does Q_n/C_n.

It is instructive to note that *in the first case the series Σc_n diverges more slowly than Σb_n, while in the second case Σc_n diverges more rapidly than Σb_n, but the final condition excludes series which diverge too fast.*

It should be noticed that if s_n tends to a definite limit, this theorem is an immediate corollary from Art. 147; for then both fractions have the same limit as s_n.

The applications of most interest arise when
$$b_0 = b_1 = \ldots = b_n = 1,$$
and then we have the result:

If Σc_n is a divergent series of positive terms, then
$$\lim\frac{c_0s_0+c_1s_1+\ldots+c_ns_n}{c_0+c_1+\ldots+c_n} = \lim\frac{s_0+s_1+\ldots+s_n}{n+1},$$

provided that the second limit exists and that either (i) c_n *steadily decreases, or* (ii) c_n *steadily increases, subject to the restriction*

$$nc_n < K(c_0 + c_1 + \dots + c_n),$$

where K *is a fixed number.*

Ex. 1. A specially interesting application arises from applying the theorem of Frobenius (Art. 51) to the series

$$a_0 + a_1 x^{c_0} + a_2 x^{c_0+c_1} + a_3 x^{c_0+c_1+c_2} + \dots,$$

where $c_0, c_1, c_2, \dots$ form an increasing sequence of positive integers, satisfying the condition last given.

Here it is evident that the series should be written in the form

$$a_0 + (0)\,x + (0)\,x^2 + \dots + a_1 x^{c_0} + (0)\,x^{c_0+1} + \dots + a_2 x^{c_0+c_1} + \dots,$$

so that $$A_\nu = a_0 + a_1 + \dots + a_n,$$

if $$c_0 + c_1 + \dots + c_{n-1} \leqq \nu < c_0 + c_1 + \dots + c_n.$$

Thus, if $s_n = a_0 + a_1 + \dots + a_n$, we have

$$A_0 + A_1 + \dots + A_\nu = s_0 c_0 + s_1 c_1 + \dots + s_{n-1} c_{n-1} + s_n(\nu - c_0 - c_1 - \dots - c_{n-1}),$$

and therefore Frobenius's mean, if it exists, is given by

$$\lim \frac{s_0 c_0 + s_1 c_1 + \dots + s_n c_n}{c_0 + c_1 + \dots + c_n},$$

which we have proved to be the same as

$$\lim (s_0 + s_1 + \dots + s_n)/(n+1),$$

provided that the last limit exists.

Ex. 2. Interesting special cases of Ex. 1 are given by taking

$$c_0 + c_1 + \dots + c_n = (n+1)^2, \quad (n+1)^3, \quad \text{etc.},$$

for which K may be taken as 2, 3 respectively.

Thus we have the results

$$\lim_{x\to 1} (a_0 + a_1 x + a_2 x^4 + a_3 x^9 + \dots) = \lim_{n\to\infty} \frac{s_0 + s_1 + \dots + s_n}{n+1}$$

and $$\lim_{x\to 1} (a_0 + a_1 x + a_2 x^8 + a_3 x^{27} + \dots) = \lim_{n\to\infty} \frac{s_0 + s_1 + \dots + s_n}{n+1}.$$

Ex. 3. But if we write* $c_0 + c_1 + \dots + c_n = 2^{n+1}$, we have $c_0 = 2$, $c_n = 2^n$, and so

$$nc_n/(c_0 + c_1 + \dots + c_n) = \tfrac{1}{2}n.$$

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* It is perhaps worth while to call special attention to the fact that the sequence

$$2,\ 4,\ 8,\ 16,\ \dots$$

does actually increase faster than

$$1,\ 8,\ 27,\ 64,\ \dots.$$

In fact the 10th, 11th and 12th terms are 1024, 2048, 4096 in the first sequence and 1000, 1331, 1728 in the second.

That is, our condition is broken, so that *we have no right to anticipate the existence of the limit,*

$$\lim_{x \to 1} (a_0 + a_1 x^2 + a_2 x^4 + a_3 x^8 + a_4 x^{16} + \ldots)$$

when the limit of $(s_0 + s_1 + \ldots + s_n)/(n+1)$ *exists*; and as a matter of fact the particular series $1 - x^2 + x^4 - x^8 + x^{16} - \ldots$ can be proved to oscillate as x tends to 1. [HARDY.]

Ex. 4. We can use this theorem to establish Cesàro's theorem of Art. 23, by taking $s_n = \pm 1$. Thus, in the notation of that article,

$$s_1 + s_2 + \ldots + s_n = p_n - q_n$$

and
$$\lim \left\{ \left(\sum_1^n c_r s_r \right) \Big/ \left(\sum_1^n c_r \right) \right\} = 0,$$

because $\sum_1^\infty c_r s_r$ converges, while $\sum_1^\infty c_r$ diverges.

Thus $(p_n - q_n)/n$ cannot approach any limit other than zero. [CESÀRO.]

Ex. 5. Write
$$B_n \left(\frac{c_n}{b_n} - \frac{c_{n+1}}{b_{n+1}} \right) = f_n$$

and
$$F_n = f_0 + f_1 + \ldots + f_n = C_n - \frac{c_{n+1}}{b_{n+1}} B_n.$$

Also let $\lambda_n = P_n / B_n$; then prove that

$$Q_n = \sum_0^n \lambda_r f_r + \lambda_n (C_n - F_n)$$

or
$$\frac{Q_n}{C_n} = \lambda_n + \frac{F_n}{C_n} \left(\frac{\Sigma \lambda_r f_r}{F_n} - \lambda_n \right).$$

Prove that, in case (i) of the theorem,* $F_n \to \infty$, but that $F_n/C_n < 1$; and by applying Art. 147 deduce that

$$(Q_n/C_n - \lambda_n) \to 0.$$ [CESÀRO.]

EXAMPLES.

Irrational Numbers.

1. (1) If A is a rational number lying between a^2 and $(a+1)^2$, prove that

$$a + \frac{A - a^2}{2a+1} < \sqrt{A} < a + \frac{A - a^2 + \frac{1}{4}}{2a+1}.$$

(2) If
$$(a + \sqrt{A})^n = p_n + q_n \sqrt{A},$$

where a, p_n, q_n are rational numbers, prove that

$$p_{n+1} = a p_n + A q_n, \quad q_{n+1} = a q_n + p_n,$$

and that
$$p_n^2 - A q_n^2 = (a^2 - A)^n.$$

Thus if a is an approximation to $\sqrt{A}$, p_n/q_n is a closer approximation.

[The approximation p_3/q_3 is the same as that used by Dedekind (see Art. 138).]

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* This is the only point in the proof at which care is necessary.

2. (1) If a, b, x, y are rational numbers such that

$$(bx-ay)^2+4(x-a)(y-b)=0,$$

prove that either (i) $x=a$ and $y=b$, or (ii) $\sqrt{(1-ab)}$ and $\sqrt{(1-xy)}$ are rational numbers. [*Math. Trip.* 1903.]

(2) If the equations

$$ax^2+2bxy+cy^2=1, \quad lx^2+2mxy+ny^2=1,$$

have only rational solutions, then

$$\sqrt{\{(b-m)^2-(a-l)(c-n)\}} \quad \text{and} \quad \sqrt{\{(an-cl)^2+4(am-bl)(cm-nb)\}}$$

are both rational. [*Math. Trip.* 1899.]

3. If α is irrational and a, b, c, d are rational, then $a\alpha+b$ is irrational unless $a=0$; and

$$(a\alpha+b)/(c\alpha+d)$$

is irrational unless $ad=bc$.

4. Any irrational number α can be expressed in the form

$$\alpha=c_0+\frac{c_1}{a}+\frac{c_2}{a^2}+\frac{c_3}{a^3}+\dots,$$

where a is an assigned positive integer and $c_1, c_2, c_3, \dots$ are positive integers less than a. Thus, in the scale of notation to base a, we may write α as a decimal,

$$c_0\cdot c_1c_2c_3\dots.$$

For example, with $a=2$, that is, in the binary scale, we find

$$\sqrt{2}=1\cdot 0110101000001\dots.$$

5. If $a_1, a_2, a_3, \dots$ is an infinite sequence of positive integers such that n can be found to make $(a_1a_2a_3\dots a_n)$ divisible by N, whatever the integer N may be, then any number α can be expressed in the form

$$\alpha=c_0+\frac{c_1}{a_1}+\frac{c_2}{a_1a_2}+\frac{c_3}{a_1a_2a_3}+\dots, \quad 0\leqq c_n<a_n.$$

When $c_n=a_n-1$ for all values of n, the fractional part of the series reduces to unity; and in order that α may be rational, c_n must be equal to a_n-1 after a certain value of n. [CANTOR.]

For instance, $\sqrt{2}=1+\frac{0}{2!}+\frac{2}{3!}+\frac{1}{4!}+\frac{4}{5!}+\frac{4+\theta}{6!}, \quad 0<\theta<\frac{1}{2}.$

The restriction that $(a_1a_2\dots a_n)$ must be divisible by N is essential; thus, if $c_n=n$, $a_n=2n+1$, we find

$$\frac{1}{3}+\frac{2}{3.5}+\frac{3}{3.5.7}+\dots \text{ to } n \text{ terms} =\frac{1}{2}\left\{1-\frac{1}{3.5.7\dots(2n+1)}\right\},$$

and so the sum to infinity is $\frac{1}{2}$, which is *rational*, although c_n is not equal to a_n-1.

6. If we can determine a divergent sequence of integers (q_n) such that

$$\lim(p_n-q_n\alpha)=0,$$

where p_n is the integer nearest to $q_n\alpha$, then α must be irrational. Apply this (a special case of Ex. 8) to the series in Ex. 7.

Establish also the converse theorem, and deduce that when α is irrational we can find an integer N such that $N\alpha - M$ is as near to any assigned number $\beta\,(0<\beta<1)$ as we please, where M is the integral part of $N\alpha$.

[For the first part, note that if α were equal to r/s,

$$|p_n - q_n\alpha| \geqq 1/s.$$

Compare Hardy and Littlewood, *Acta Mathematica*, t. 37.]

7. The sums of the series

$$\sum_1^\infty \left(\frac{1}{p}\right)^{n^2}, \quad \sum_1^\infty \left(-\frac{1}{p}\right)^{n^2}, \quad \sum_1^\infty \frac{q^n}{p^{n^2}},$$

where p, q are any positive integers (such that $q < p^2$), are irrational numbers. The same holds for the product $\Pi(1-p^{-n})$. [EISENSTEIN.]

[For simple proofs and extensions, see Glaisher, *Phil. Mag.* (4), vol. 45, 1873, p. 191, and F. Bernstein and O. Szász, *Math. Annalen*, Bd. 76.]

8. If α is the root of an algebraic equation of degree k (with integral coefficients), we can find a constant K such that

$$\left|\frac{q}{p} - \alpha\right| > \frac{1}{Kq^k},$$

where p, q are any two positive integers. Thus if we can find a divergent sequence of integers (q_n) such that

$$|p_n - q_n\alpha| < q_n^{-k},$$

where p_n is the nearest integer to αq_n, *then α is not an algebraic number* of degree k.

Consequently, if $\quad \alpha = c_0 + \dfrac{c_1}{10} + \dfrac{c_2}{10^{1.2}} + \dots + \dfrac{c_n}{10^{n!}} + \dots,$

where $c_1, c_2, c_3, \dots$ are less than 10, by taking the sequence $q_n = 10^{n!}$, we can prove that α *is transcendental*. [LIOUVILLE.]

[See Borel, *Leçons sur la Théorie des Fonctions*, ch. 2.]

9. Suppose that α is an irrational number which is converted into a continued fraction

$$\frac{1}{a_0+}\,\frac{1}{a_1+}\,\frac{1}{a_2+\dots},$$

and p_n/q_n is the convergent which precedes the quotient a_n; write further

$$Q_{n+1} = q_n A_n + q_{n-1},$$

where $\quad A_n = a_n + \dfrac{1}{a_{n+1}} + \dfrac{1}{a_{n+2}} + \dots .$

Then shew that $\quad |\sin m\alpha\pi| > K/Q_{n+1},$

if $\quad q_n < m < q_{n+1};$

and also that $\quad |\sin q_n\alpha\pi| = \sin(\pi/Q_{n+1}) = (1+\epsilon_n)\,\pi/Q_{n+1},$

where ϵ_n tends to zero as n tends to ∞.

[HARDY, *Proc. Lond. Math. Soc.* (2), vol. 3, p. 444.]

Monotonic Sequences.

10. (1) If in a sequence (a_n) each term lies between the two preceding terms, shew that it is compounded of two monotonic sequences.

(2) If a sequence of positive numbers (a_n) is monotonic, prove that the sequence (b_n) of its geometric means is also monotonic, where

$$b_n{}^n = a_1 a_2 \dots a_n.$$

(3) If $c_1, c_2, \dots, c_p$ are real positive numbers, and if

$$\mu_n = (c_1{}^n + c_2{}^n + \dots + c_p{}^n)/p,$$

prove that the sequence (μ_{n+1}/μ_n) steadily increases; and deduce that the same is true of $\mu_n^{\frac{1}{n}}$.

11. If
$$S_n = \sum_{r=1}^{n} \left\{1 - \left(\frac{r}{n}\right)^k\right\} a_r,$$

where a_r is positive and independent of n, shew that if Σa_r is convergent, its sum gives the value of $\lim S_n$ (see Art. 49).

Conversely, if $\lim S_n$ exists, shew that Σa_r converges, and that its sum is equal to the limit of S_n.

Apply to Ex. 12, taking $k=1$.

[The converse theorem is no longer true if a_n is not positive: it then forms a case of Riesz's definition for the "sum" of Σa_r. See *Cambridge Mathematical Tracts*, No. 18.]

12. Apply Cauchy's theorem (Art. 147) to prove that

$$\lim \frac{1}{n}\left\{\frac{n}{1} + \frac{n-1}{2} + \frac{n-2}{3} + \dots + \frac{1}{n} - \log(n!)\right\} = C,$$

where C is Euler's constant (Art. 11).

Prove also that for all values of n, the expression lies between 0 and 1.

[*Math. Trip.* 1907.]

13. Prove that if

$$\lim \left\{\lambda (a_{n+1} - a_n) + (1-\lambda)\frac{a_n}{n}\right\} = l, \quad \text{where } \lambda \text{ is positive,}$$

then
$$\lim (a_{n+1} - a_n) = \lim \frac{a_n}{n} = l. \qquad \text{[MERCER.]}$$

[It follows at once from Cauchy's theorem (Art. 147) that if $(a_{n+1} - a_n)$ tends to a limit, so also does $b_n = a_n/n$, and that these limits are equal; thus their common value must also be equal to l. We have now to prove that the sequence $(a_{n+1} - a_n)$ cannot oscillate; and it is clearly sufficient to shew, that b_n cannot oscillate. Now the given expression is equal to

$$c_n = \lambda\{(n+1)b_{n+1} - nb_n\} + (1-\lambda)b_n,$$

and so
$$(n+1)\lambda b_{n+1} = \{(n+1)\lambda - 1\}b_n + c_n.$$

Hence b_{n+1} falls between b_n and c_n, provided that $n+1 > 1/\lambda$.

Suppose next that we choose m so that

$$l - \epsilon < c_n < l + \epsilon, \quad \text{if } n > m \geqq 1/\lambda.$$

Then b_m may be (i) greater than $l+\epsilon$, (ii) less than $l-\epsilon$, or (iii) in the interval $(l-\epsilon, l+\epsilon)$.

(i) When $b_m > l+\epsilon$, it follows that $b_m > c_m$, and so $b_m > b_{m+1} > c_m > l-\epsilon$; thus *either* $b_{m+1} > l+\epsilon$, *or* b_{m+1} falls between $l-\epsilon$ and $l+\epsilon$. Continuing the argument, we see that *either* $b_m > b_{m+1} > b_{m+2} > \ldots > l+\epsilon$ *or* at some stage b_p falls between $l-\epsilon$ and $l+\epsilon$. Under the former conditions, the sequence (b_n) steadily decreases and so tends to a limit l' not less than $l+\epsilon$; but this contradicts the previous result that if (b_n) has a limit l', the value of l' must be equal to l. Hence at some stage b_p must fall between $l-\epsilon$ and $l+\epsilon$.

(ii) If $b_m < l-\epsilon$, an argument similar to that used in (i) will show that the sequence increases until at some stage b_p falls between $l-\epsilon$ and $l+\epsilon$.

(iii) We have now to consider the consequences of having $l-\epsilon < b_p < l+\epsilon$ for some value of $p \geqq m$.

Since c_p falls in the same interval, and b_{p+1} lies between b_p and c_p, it follows that b_{p+1} also lies between $l-\epsilon$ and $l+\epsilon$; thus the same is true for $n=p+2$, and so for $n=p+3$ and for all values of $n \geqq p$. Hence

$$l-\epsilon < b_n < l+\epsilon, \quad \text{if } n \geqq p,$$

and so (b_n) must tend to the limit l in all cases.

This theorem has proved of great interest in establishing the equivalence of the means of Cesàro and Hölder; see I. Schur, *Math. Annalen*, vol. 74, p. 447. Other proofs of the theorem have been given by G. H. Hardy, *Quarterly Journal*, vol. 43, p. 143; and by K. Knopp in a paper immediately following Schur's. The proof by Schur does not differ substantially from that given in the first edition of this book.]

Infinite Sets of Numbers.

14. For some purposes of analysis we need to use *infinite sets of numbers which cannot be arranged as a sequence*; when a set can be arranged as a sequence, it is often called *countable* or *enumerable*.

The set of all real numbers lying between 0 and 1 is not countable. [CANTOR.]

[A proof will be found in Hobson's *Theory of Functions of a real variable*, § 56.]

15. Given any infinite set of numbers (k) we can construct a Dedekind section by placing in the upper class all rational numbers greater than *any* number k, and in the lower class all rational numbers less than *some* number k.

This section defines *the upper limit of the set*; prove that this upper limit has the properties stated on p. 16 for the upper limit of a sequence. Frame also a corresponding definition for the lower limit of the set k; and define both upper and lower limits by using the method of continued bisection (as in Art. 144).

16. The *limiting values of an infinite set of numbers* consist of numbers λ such that an infinity of terms of the set fall between $\lambda-\epsilon$ and $\lambda+\epsilon$, however small ϵ may be.

Given an infinite set of numbers (k) we can construct a Dedekind section by placing in the upper class all rational numbers which are greater than all but a finite number of the terms k, and in the lower class all rational numbers less than an *infinite* number of terms k.

This section defines *the maximum limit of the set*; prove that the maximum limit is a limiting value of the set, in accordance with the definition given above; and further that no limiting value of the set can exceed the maximum limit (compare Art. 5·2). Frame a corresponding definition for *the minimum limit* and state the analogous properties.

Continuous Functions.

17. If $f(x)$ is continuous in the interval (a, b), prove that it assumes, at least once in the interval,

(i) every value between $f(a)$ and $f(b)$,

(ii) the upper and lower limits (H and h) of $f(x)$ in the interval.

[Apply the method of continued bisection.

In case (ii) we get an infinite sequence of intervals (a_n, b_n) such that H is the upper limit of $f(x)$ in the interval (a_n, b_n); let (a_n), (b_n) tend to the common limit l. Then if $f(l) < H$, choose δ so that

$$f(x) - f(l) < \tfrac{1}{2}\{H - f(l)\}, \quad \text{if } |x - l| < \delta.$$

Then choose n so that $b_n - a_n \leqq \delta$; and we find that $f(x) < \tfrac{1}{2}\{H + f(l)\}$ at all points of (a_n, b_n), contrary to hypothesis.]

18. Prove that if $f(x)$ is continuous in an interval (a, b), then the interval can be divided into a finite number of parts (the number depending on ϵ) such that

$$|f(x_2) - f(x_1)| < \epsilon,$$

where x_1, x_2 are any two points in the same sub-division. [HEINE.]

[See G. H. Hardy, *A Course of Pure Mathematics* (2nd edit.), §§ 105-106.]

19. A function is said to be *finite in an interval* if its absolute value has a finite upper limit in the interval.

Deduce from Ex. 18 that if $f(x)$ is continuous in an interval, it is also finite in the interval; and also that δ can be found so that

$$|f(\xi_2) - f(\xi_1)| < \epsilon,$$

where ξ_1, ξ_2 are *any* two points of the interval satisfying $|\xi_2 - \xi_1| < \delta$.

APPENDIX II.

DEFINITIONS OF THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

153. In the text it has been assumed in Chapter II. that

$$\frac{d}{dx}(\log x)=\frac{1}{x},$$

and a number of allied properties of the logarithm have been used from this point onwards. It is customary in English books on the Calculus to deduce the differential coefficient of $\log x$ from the exponential limit (Art. 57) or else from the exponential series (Art. 58). It would, therefore, seem illogical to assume these properties of logarithms in the earlier part of the theory; although, no doubt, we could have obtained these limits at the beginning of the book. But from the point of view adopted here it seemed more natural to place all special limits after the general theorems on convergence. It is, therefore, desirable to indicate an independent treatment of the logarithmic function; and it is often convenient to adopt this way of introducing the function in a first course on the Calculus.*

154. Definition of the logarithmic function.

There appears to be no real need for the logarithm at the beginning of the Differential Calculus, but we require the function in the Integral Calculus as soon as fractions have to be integrated. At first it is probably best to denote $\int dx/x$ by $L(x)$, and to postpone the discussion of the nature of the function $L(x)$ until after the

* Historically, this is effectively the way in which Napier originally defined the logarithm (see Art. 156); more recently the same method has been advocated by Bradshaw (*Annals of Mathematics* (2), vol. 4, 1903, p. 51) and by Osgood, *Lehrbuch der Funktionentheorie*, Bd. 1, pp. 487-500.

definite integral has been introduced. We shall assume, for the present, the theorem that $\int_a^b y\,dx$ represents the area between a curve, the axis of x and the two ordinates $x=a$, $x=b$; an arithmetic treatment of the theorem will be given below (Art. 161).

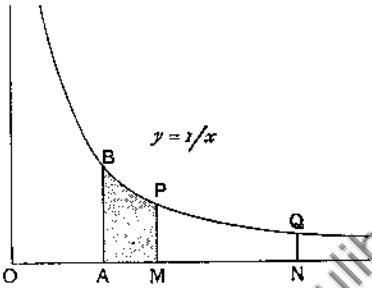


FIG. 40.

Let the rectangular hyperbola $y=1/x$ be drawn, then we shall denote by $L(x)$ the area $AMPB$ bounded by the curve, the fixed ordinate AB $(x=1)$, the axis of x and the variable ordinate MP; or, in the notation of the Calculus, we write

(1) $$L(x)=\int_1^x d\xi/\xi,$$

where, as will be evident from the figure, x is supposed ***positive.***

It is obvious from the definition that

(2) $$L(1)=0.$$

Further, if parallels are drawn through B and P to the axis of x, we obtain two rectangles, one enclosing the area $AMPB$ and the other entirely within $AMPB$.

Thus we have

(3) $$x-1 > L(x) > (x-1)/x,$$

or, with a slight change of notation,

(3a) $$x > L(1+x) > x/(1+x).$$

Although (3) has only been proved when $x > 1$, yet it is easy to shew similarly that the inequalities (3) hold good *algebraically*, when $x < 1$. But care must be taken to notice that when x is less than 1, in (3), or negative, in (3a), all the members of the inequalities are *negative*; thus, for the numerical values the inequalities would have to be reversed.

For instance, we get from (3),

$$-\tfrac{1}{2} > L(\tfrac{1}{2}) > -1.$$

But in numerical value $$\tfrac{1}{2} < |L(\tfrac{1}{2})| < 1.$$

Again, if we take an ordinate NQ, such that $ON=2x$, it is clear that the area $PMNQ=L(2x)-L(x)$ and lies between the rectangles $MN.NQ$ and $MN.MP$. The areas of these rectangles are respectively

$$x\left(\frac{1}{2x}\right)=\frac{1}{2} \quad \text{and} \quad x\left(\frac{1}{x}\right)=1,$$

so that

(4) $$1 > L(2x)-L(x) > \tfrac{1}{2}.$$

Thus, writing $x=1, 2, 4, \ldots$, we get

$$1 > L(2) > \tfrac{1}{2}, \quad \text{since } L(1)=0,$$
$$1 > L(4)-L(2) > \tfrac{1}{2},$$
$$1 > L(8)-L(4) > \tfrac{1}{2},$$

and so on.

It follows by addition that

$$n > L(2^n) > \tfrac{1}{2}n.$$

Now, if $x > x_0$, it is evident from the figure that

$$L(x) > L(x_0).$$

Hence, if $$2^{n+1} > x > 2^n,$$
we have $$L(2^{n+1}) > L(x) > L(2^n),$$
and so $$n+1 > L(x) > \tfrac{1}{2}n;$$
thus it is clear that $L(x)$ *tends to infinity with* x (Art. 1·2, Note 2), or

(5) $$\lim_{x\to\infty} L(x)=\infty.$$

Again, if we write $x=1/t$, we have

$$\frac{dx}{x}=-\frac{dt}{t},$$

so that $$L(x)=\int_1^x\frac{dx}{x}=-\int_1^t\frac{dt}{t}=-L(t), \text{ or}$$

(6) $$L(x)=-L(1/x).$$

Hence, since $1/x$ tends to infinity as x approaches zero, $L(x)$ will tend to negative infinity, or in symbols

(7) $$\lim_{x\to 0} L(x)=-\infty.$$

Again, *the function* $L(x)$ *is continuous for all positive values of* x. For as above we see that $L(x+h)-L(x)$ lies between two rectangles, one of which is numerically equal to h/x and the other to $h/(x+h)$. Thus, if $|h|<\delta$,

$$|L(x+h)-L(x)|<\delta/(x-\delta),$$

and so $$|L(x+h)-L(x)|<\epsilon, \quad \text{if } |h|<\epsilon x/(1+\epsilon),$$

which proves the continuity of $L(x)$.

It follows from the fact that integration is the reverse operation to differentiation that

(8) $$\frac{d}{dx}\{L(x)\}=\frac{1}{x};$$

but without appealing to this general fact we can obtain this result (8) by noticing that $\{L(x+h)-L(x)\}/h$ is contained between $1/x$ and $1/(x+h)$. Thus

$$\frac{d}{dx}\{L(x)\}=\lim_{h\to 0}\frac{1}{h}[L(x+h)-L(x)]=\frac{1}{x}.$$

If $x=a$, b are two ordinates such that $b>a>1$, we find that

$$L(b)-L(a)<(b-a)/a$$

by using the same argument as we used to establish (4). Further, from (3), we have $L(a)>(a-1)/a$.

Hence $$\frac{L(a)}{a-1}>\frac{1}{a}>\frac{L(b)-L(a)}{b-a},$$

or, by adding numerators and denominators,* we find that

$$\frac{L(a)}{a-1}>\frac{L(b)}{b-1}>\frac{L(b)-L(a)}{b-a},\quad \text{where } b>a>1.$$

Consequently, *the function $L(x)/(x-1)$ decreases as x increases*; which corresponds to the nearly obvious geometrical fact that *the mean ordinate between AB and MP decreases as x increases.*

As an exercise, the reader may prove this result by differentiation.

The figure below gives a general idea of the course of the logarithmic function.

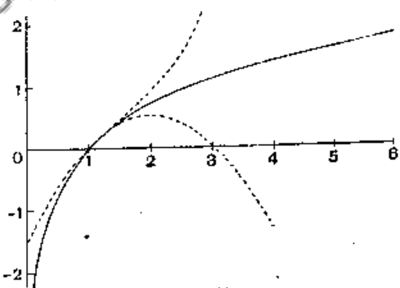


FIG. 41.

The dotted lines represent the curves

$$y=(x-1)-\tfrac{1}{2}(x-1)^2,$$
$$y=(x-1)-\tfrac{1}{2}(x-1)^2+\tfrac{1}{3}(x-1)^3.$$

* If $\frac{p}{q}>\frac{r}{s}$, then $\frac{p}{q}>\frac{p+r}{q+s}>\frac{r}{s}$, provided that q and s are both positive.

155. Fundamental properties of the logarithmic function.

In the formula
$$L(u)=\int_1^u \frac{dx}{x},$$
change the independent variable from x to ξ by writing $x=\xi/v$; we find then
$$L(u)=\int_v^{uv}\frac{d\xi}{\xi}=\int_1^{uv}\frac{d\xi}{\xi}-\int_1^v\frac{d\xi}{\xi},$$
or, going back to the definition,
$$L(u)=L(uv)-L(v).$$

Thus

(1) $$L(uv)=L(u)+L(v).$$

From equation (1) it follows at once that

(2) $$L(u^n)=nL(u),$$

where n is any rational number.*

Now we have proved (Art. 154), that $L(x)$ is a continuous function, which steadily increases from $-\infty$ to $+\infty$ as x varies from 0 to ∞. Thus there is one and only one real root of the equation $L(x)=1$ (see Art. 143); let this root be denoted by e, as usual, so that $L(e)=1$.

Then equation (2) gives, for rational values of n,

(3) $$L(e^n)=n,$$

which proves that $L(x)$ must agree with the logarithm to base e, as ordinarily defined; we shall therefore write $\log x$ in place of $L(x)$ in future.

We can obtain approximations to the numerical value of e by observing that equation (3) of Art. 154 gives, on writing $x=1+1/n$,
$$\frac{1}{n}>\log\left(1+\frac{1}{n}\right)>\frac{1}{n+1}, \quad \text{if } n>0,$$
or

(4) $$1>\log\left(1+\frac{1}{n}\right)^n>\frac{n}{n+1}.$$

Thus, as n increases, $\log(1+1/n)^n$ tends to 1 as its limit; and, since the logarithmic function is continuous and monotonic, $(1+1/n)^n$ must tend to e.

* Equation (2) may be used to establish the existence of roots which are not evident on geometrical grounds; for example, the fifth root. Of course, from the point of view adopted in this book, it is more natural to establish the existence of such roots by using Dedekind's section (see Art. 138).

Similarly, we prove that

(5) $$1 < \log\left(1-\frac{1}{n}\right)^{-n} < \frac{n}{n-1}, \quad \text{if } n > 1.$$

Thus, we find two formulae

(6) $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}.$$

It is easy to give a direct proof that the two expressions in (6) have a definite limit. For we have proved (Art. 154, end) that $(\log x)/(x-1)$ decreases as x increases.

Thus, if $x = 1+1/n$, we see that $\log(1+1/n)^n$ increases with n; and therefore $(1+1/n)^n$ does so. In the same way we prove that $(1-1/n)^{-n}$ decreases as n increases.

But from (4) and (5) we see that $(1+1/n)^n$ is less than $(1-1/n)^{-n}$, and is therefore less than $(1-1/2)^{-2}$ if $n > 2$.

Thus $(1+1/n)^n < 4$, and consequently $(1+1/n)^n$ converges to a definite limit e (by Art. 144).

As a matter of fact, however, these limits for e are not very convenient for numerical computation; and their geometric mean gives a better approximation.

For it will be seen that

$$\log\frac{n+1}{n-1} = \int_0^{\frac{1}{n}} \frac{2dt}{1-t^2} = \int_0^{\frac{1}{n}} 2\left(1+\frac{t^2}{1-t^2}\right)dt.$$

Hence

$$\int_0^{\frac{1}{n}} 2(1+t^2)dt < \log\left(\frac{n+1}{n-1}\right) < \int_0^{\frac{1}{n}} 2\left(1+\frac{t^2}{1-1/n^2}\right)dt,$$

or $$\frac{2}{n}\left(1+\frac{1}{3n^2}\right) < \log\left(\frac{n+1}{n-1}\right) < \frac{2}{n}\left\{1+\frac{1}{3(n^2-1)}\right\}.$$

Thus we have

(7) $$1+\frac{1}{3n^2} < \log\left(\frac{n+1}{n-1}\right)^{\frac{n}{2}} < 1+\frac{1}{3(n^2-1)},$$

so that $\left(\frac{n+1}{n-1}\right)^{\frac{n}{2}}$ differs from e only by terms of order $1/3n^2$.

With $n=100$ it will be found that

$$\left(1+\frac{1}{n}\right)^n = 2\cdot7048, \quad \left(1-\frac{1}{n}\right)^{-n} = 2\cdot7320, \quad \left(\frac{n+1}{n-1}\right)^{\frac{n}{2}} = 2\cdot7184,$$

the third of which is only wrong by a unit in the last place.

156. Napier's Logarithms.

The recent tercentenary * of the publication of Napier's logarithms has revived interest in the details of Napier's methods.

Napier's definition of Logarithms.

Suppose that a point P moves on a straight line CA, starting from a point A with velocity u, and moving so that the velocity of P is always proportional to CP; and that simultaneously a second point Q moves on a second straight line, starting from a point B with constant velocity v. Then in Napier's original definition BQ was called the logarithm of CP, when the constant velocity v was equal to the initial velocity u.

Napier's tables were constructed for trigonometrical purposes; and before Napier's time the current trigonometrical tables gave the various functions in the form of whole numbers, taking the radius CA (Fig. 42) as a suitable power of 10, and tabulating the values of PT and CP as the sine and cosine of PCT, respectively. Thus, to obtain an accuracy corresponding to modern 7-figure tables, the radius was taken as 10^7; and this was the value chosen by Napier.

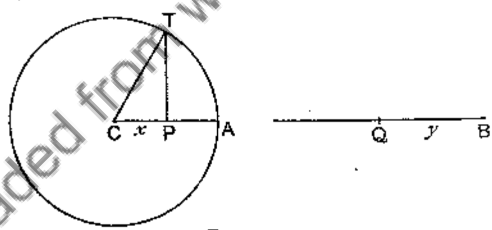


FIG. 42.

To compare the present definition with Art. 154 above, let us write in the figure $c = CA$, $x = CP$, $y = BQ$; then we have

$$\frac{dx}{dt} = -u\frac{x}{c}, \quad \frac{dy}{dt} = v.$$

Hence $$\frac{dy}{dx} = -\frac{v}{u}\frac{c}{x} = -\frac{c}{x}, \quad \text{if } v = u,$$

and so $$y = -\int_c^x \frac{c}{\xi}\, d\xi = c \log\left(\frac{c}{x}\right).$$

* John Napier (1550-1617) was Baron of Merchiston, near Edinburgh. The *Mirifici Logarithmorum Canonis Descriptio* was published in 1614: tercentenary celebrations were held in Edinburgh in July 1914. For additional details the reader may consult the Tercentenary Memorial Volume published by the Royal Society of Edinburgh in 1915.

Thus, if $\log_N$ denotes Napier's logarithms, we have the fundamental formula

$$\log_N x = c \log\left(\frac{c}{x}\right),$$

where, actually, $c=10^7$, and the range of values of x in Napier's table is from c to 0.

It is easy to prove, directly from Napier's definition, that to numbers in geometrical progression correspond logarithms in arithmetical progression, and similarly

$$\log_N x_2 - \log_N x_1 = \log_N x_4 - \log_N x_3$$

if $$x_2 : x_1 = x_4 : x_3.$$

This formula was the basis of Napier's rules for applications of logarithms, and was used also in his fundamental calculations described in the following Article.

Subsequently Briggs * remarked that the rules of calculation would be simplified by choosing v/u so that the value $x=c/10$ should correspond to $y=c$; thus in general

$$y = c\frac{\log(c/x)}{\log 10} = c\log_{10}\left(\frac{c}{x}\right).$$

But Napier pointed out that the rules of calculation would be still further simplified by supposing $y=0$ to correspond to $x=1$, so leading to the relation

$$\log(x_1 x_2) = \log x_1 + \log x_2.$$

Thus finally Briggs adjusted the constants so as to make $y=0$, c correspond to $x=1$, c, respectively. This gives in general

$$y = c\frac{\log x}{\log c} = 10^9 \log_{10} x,$$

if $c=10^{10}$; and this forms the basis of Briggs's *Arithmetica Logarithmica* of 1624.

157. Napier's method of calculating logarithms.

In order to calculate his fundamental table of logarithms, Napier adopted an approximation for the difference between the logarithms of two nearly equal numbers. Thus suppose that P_1, P_2 represent two positions of the point P in Fig. 42, with $CP_1 > CP_2$; then Napier pointed out that the time taken in passing from P_1 to P_2 must be longer than it would be for a constant velocity equal to the actual velocity at P_1, and, on the other hand, it must be less than

* Henry Briggs, sometime Fellow of St. John's College, Cambridge; Gresham Professor of Geometry, London, at the date of his correspondence with Napier; subsequently Savilian Professor, Oxford.

if the constant velocity were supposed equal to the actual velocity at P_2. Thus

$$\frac{P_1P_2}{CP_1} < \frac{Q_1Q_2}{CA} < \frac{P_1P_2}{CP_2};$$

and a closer approximation is to be anticipated by taking

$$\frac{Q_1Q_2}{CA} = \frac{1}{2}\left(\frac{P_1P_2}{CP_1} + \frac{P_1P_2}{CP_2}\right),$$

the arithmetic mean of the two limits assigned previously.

In symbols this approximation may be written

$$\log_N b - \log_N a = \frac{c}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right), \qquad \text{(A)}$$

or

$$\log\left(\frac{a}{b}\right) = \frac{1}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right),$$

which is practically the same as (7) of Art 155.

To estimate the error in the approximation we may use the equation

$$\tfrac{1}{2}(a-b)\left(\frac{1}{a}+\frac{1}{b}\right) - \log\left(\frac{a}{b}\right) = \int_b^a (a-x)(x-b)\frac{dx}{x^3}$$

$$= \int_0^1 \frac{p^3 t(1-t)\,dt}{(1+pt)^3},$$

where $t=(x-b)/(a-b)$, $p=(a-b)/b$.

Thus when p is small and positive, the error is less than

$$p^3\int_0^1 t(1-t)\,dt = \tfrac{1}{6}p^3,$$

but is of this order of magnitude.

The error in Napier's approximation (A) is therefore of the order

$$\tfrac{1}{6}c\{(a-b)/b\}^3;$$

and it seems certain that Napier was aware that his method of approximation was very close, although probably he had to depend only on arithmetical tests of its accuracy.

Another approximation was used by Napier in calculations of a less fundamental type; this was obtained by using a constant velocity equal to that at some point between P_1 and P_2.

Thus in symbols $$\log_N b - \log_N a = \frac{c}{k}(a-b), \qquad \text{(B)}$$

where $$b < k < a.$$

If $k=\frac{1}{2}(a+b)$, the approximation (B) corresponds to (7) of Art. 155, and the error is easily found to be of the order

$$\tfrac{1}{12}c\{(a-b)/b\}^3.$$

But in general k will differ from $\frac{1}{2}(a+b)$ by an amount less than $\frac{1}{2}(a-b)$; and then the error in (B) is seen to be of the order

$$\tfrac{1}{2}c\{(a-b)/b\}^2.$$

Thus with $c=10^7$ and $(a-b)/b<\frac{1}{4}\times10^{-3}$, the errors involved in using (B) can never be as much as $\frac{5}{16}$ (or say $\frac{1}{3}$); that is, the approximation (B) can be applied with safety to interpolate from Napier's Third Table (for which the ratio of consecutive entries is $1-\dfrac{1}{2\times10^3}:1$) by selecting the number given in the table which is nearest to the number whose logarithm is required.

The approximation (A) is used in calculating the logarithms in Napier's First Table: this table contains the values of

$$X_r=c\left(1-\frac{1}{c}\right)^r=10^7\left(1-\frac{1}{10^7}\right)^r,\quad r=1,\,2,\,\ldots\,10,$$

and then $$\log_N X_r=r\log_N X_1.$$

Applying the approximation (A) to the values c, and $X_1=c-1$, it follows that

$$\log_N X_1=\tfrac{1}{2}\left(1+\frac{c}{c-1}\right)=1+\frac{1}{2c},$$

the final error being actually of the order $\frac{1}{3}c^{-2}$; Napier then took

$$\log_N X_1\;=1{\cdot}00000005,$$

and so $$\log_N X_{100}=100{\cdot}000005,$$

the error in the second being of order $\frac{1}{3}\times10^{-12}$. The effect of this error in the Third Table is at most multiplied by

$$50\times20\times70=7\times10^4,$$

and so is at most 2 in the eighth decimal.

Napier's Second Table contains the values of

$$Y_r=c\left(1-\frac{100}{c}\right)^r=10^7\left(1-\frac{1}{10^5}\right)^r,\quad r=1,\,2,\,\ldots\,50,$$

and then $$\log_N Y_r=r\log_N Y_1.$$

To approximate to Y_1, Napier used the fact that

$$X_{100}-Y_1={\cdot}0004950,\quad Y_1=9999900,$$

and applied approximation (A) again, giving here

$$\log_N Y_1=\log_N X_{100}+{\cdot}0004950$$
$$=100{\cdot}0005000,$$

the error in which is of order $\frac{1}{3}\times 10^{-8}$, so that the ultimate effect in the Third Table is at most $\frac{5}{3}\times 10^{-4}$, or say 2 in the fourth decimal.

The value of X_{100} can be verified at once from the binomial series; this gives

$$\frac{X_{100}}{c}=1-\frac{100}{c}+\frac{100}{c}\cdot\frac{99}{2c}-\frac{100}{c}\cdot\frac{99}{2c}\cdot\frac{98}{3c}+\dots,$$

or $$X_{100}-Y_1=\frac{9900}{2c}\text{, neglecting } \tfrac{1}{3}\times 10^{-9}.$$

The error in $\log_N Y_1$ is estimated most readily from the fact that

$$\log_N Y_1=-c\log\left(1-\frac{100}{c}\right)=100\left\{1+\frac{1}{2}\left(\frac{100}{c}\right)+\frac{1}{3}\left(\frac{100}{c}\right)^2+\dots\right\},$$

using the logarithmic series.

Then $$\log_N Y_{50}=5000{\cdot}025000,$$

the error in which is of order $\frac{1}{3}\times 10^{-7}$; but the value given in the Second Table for Y_{50} is in error owing to some arithmetical slip. The table gives

9995001·222927

instead of the value 9995001·224804,

which is found by using the binomial expansion

$$Y_{50}=10^7\left(1-\frac{50}{10^5}+\frac{50}{10^5}\,\frac{49}{2\times 10^5}-\frac{50}{10^5}\,\frac{49}{2\times 10^5}\,\frac{48}{3\times 10^5}+\dots\right)$$
$$=10^7-5000+1{\cdot}225-{\cdot}000196+\dots,$$

the terms omitted being of order 3×10^{-8}.

This arithmetical slip affected the whole of the logarithms in Napier's Third Table, which contains the values of the products

$$Z_r W_s/c,\quad\text{for } r=0, 1, 2, \dots, 20, \text{ and } s=0, 1, 2, \dots, 68,$$

where $$Z_r=c\left(1-\frac{1}{2\times 10^3}\right)^r,\quad W_s=c\left(1-\frac{1}{100}\right)^s;$$

and the corresponding logarithms are given by

$$(r\log_N Z_1+s\log_N W_1).$$

To calculate $\log Z_1$, Napier took a fourth proportional V, such that *

$$c:V=Y_{50}:Z_1,$$

* Most rapidly calculated, as Napier remarked, in the form of
$$c-V:c=(Y_{50}-Z_1):Y_{50}.$$

and then the value of $\log_N V$ can be found from the first table. Napier found that

$$c - V = 1\cdot2235387, \quad \text{instead of } 1\cdot225417,$$

which would follow from the value of Y_{50} given above.

Then approximation (A) gives

$$\log_N Z_1 - \log_N Y_{50} = c - V,$$

neglecting terms of order 10^{-7} here; and so Napier gave

$$\log_N Z_1 = 5001\cdot2485387$$

in place of $\log_N Z_1 = 5001\cdot250417$,

which follows by using the correct value of Y_{50}.

That this value is correct to the last figure given follows from the logarithmic series, which shews that

$$\log_N Z_1 = -c \log\left(1 - \frac{1}{2000}\right) = 5000\left(1 + \frac{1}{4\times10^3} + \frac{1}{12\times10^6} + \dots\right)$$
$$= 5000 + 1\cdot25 + \cdot0004166\dots,$$

the terms omitted being of order 2×10^{-7}.

The last of Napier's fundamental logarithms is $\log_N W_1$; now we find from the binomial theorem that

$$Z_{20} = 10^7\left\{1 - \frac{20}{2\times10^3} + \frac{20}{2\times10^3}\cdot\frac{19}{4\times10^3} - \frac{20}{2\times10^3}\cdot\frac{19}{4\times10^3}\cdot\frac{18}{6\times10^3} + \dots\right\}$$

or $\quad Z_{20} - W_1 = 475 - 1\cdot425 + \cdot003028125 - \cdot00000484\dots$

$$= 473\cdot57802$$

with an error of order 10^{-6}; this was found by Napier with an 8 in the last place instead of 2.

From this Napier calculated that

$$\log_N W_1 - \log_N Z_{20} = 478\cdot3502551,$$

which is in error by about $\cdot00006$, corresponding to the error in Napier's value for Z_{20}.

It is possible to calculate this result from the approximation (A) directly; but Napier derived it by finding first U, such that

$$c : U = Z_{20} : W_1.$$

Then U is most nearly equal to Y_5, in the Second Table; and U' is formed such that

$$c : U' = U : Y_5.$$

It is found that U' is nearly equal to X_{22} in the First Table; and then the approximation (A) is applied in the form

$$\log_N U' - \log_N X_{22} = -(U' - X_{22}).$$

Thus finally the result is

$$\log_N W_1 - \log_N Z_{20} = \log_N U = \log_N Y_5 - \log_N U'$$
$$= 5 \log_N Y_1 - 22 \log_N X_1 + (U' - X_{22})$$
$$= 478{\cdot}0025 + {\cdot}34770.$$

Actually U need not be calculated out completely, U' being given by

$$U' : Y_5 = Z_{20} : W_1.$$

This leads to the value

$$U' = 10^7 - 22 + {\cdot}34772,$$

while

$$X_{22} = 10^7 - 22 + {\cdot}00002.$$

It follows that the errors in Napier's Third Table, and in the derived logarithms, are all due to the errors in Y_{50} and Z_{20}; and thus the resultant errors can be estimated as a defect of

$${\cdot}001878 - {\cdot}00006 \text{ in } 5000,$$

or say

$${\cdot}364 \text{ in } 10^6.$$

For instance Napier gave the result

$$\log_N 10^6 = 23025842{\cdot}34,$$

while modern calculations lead to

$$\log 10 = 2{\cdot}3025850930$$

or

$$\log_N 10^6 = 23025850{\cdot}93.$$

There is thus an error in defect equal to 8·59; while the error estimated at the above rate would be equal to 8·37.

Similarly Napier gives for 9°

$$\text{sine} = 1564345 \; (= 10^7 \times {\cdot}1564345),$$

and the corresponding value of $\log_N$ is given as

$$18551174.$$

The actual value should be

$$10^7 \log \operatorname{cosec} 9^\circ = 23025850{\cdot}93 \times \log_{10} \operatorname{cosec} 9^\circ$$
$$= 23025850{\cdot}93 \times 0{\cdot}8056676$$
$$= 18551182,$$

so that the error in defect is about 8; and by the above rule the error would be 6·73.

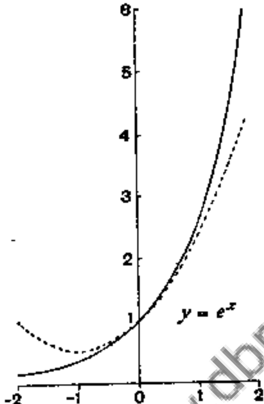
On the other hand, the value given by Napier for his logarithm of the cosine of 9° is 123881, which agrees exactly with the value derived from the product $23025850{\cdot}93 \times \log_{10} \sec 9^\circ$.

158. The exponential function.

Since the logarithmic function $\log y$ steadily increases as y increases from 0 to $+\infty$, it follows from Art. 143 that, corresponding to any assigned real value of x, there is a real positive solution of the equation

$$\log y = x.$$

We call y the *exponential function* when x is the independent variable and write $y = \exp x$; the graph of the function can be obtained by interchanging x and y in Fig. 41, p. 439, and then reversing the direction of the axis of x. The figure obtained is shewn below:



Dotted curve, $y = 1 + x + \frac{1}{2}x^2$.

FIG. 43.

It is evident that *the exponential function is single-valued*,* because $\log y_2 > \log y_1$, if $y_2 > y_1$. Thus two different values of y cannot correspond to the same value of x in the equation $\log y = x$, so that y is a single-valued function of x.

Suppose now that

$$y = \exp x, \quad y + k = \exp(x + h),$$

so that $$x = \log y, \quad x + h = \log(y + k),$$

then $$h = \log(y + k) - \log y = \log(1 + k/y).$$

Thus, from equation (3) of Art. 154 we have

(1) $$k/(y+k) < h < k/y,$$

or, since y and $y + k$ are positive,

(2) $$hy < k < h(y + k).$$

Accordingly k has the same sign as h, or *the function* $\exp x$ *increases with* x, and consequently *the exponential function is continuous.*

* Generally, the function inverse to a given function is single-valued in any interval for which the given function steadily increases (or steadily decreases).

We have proved in equation (3) of Art. 155 that, when x is *rational*, the exponential function is the *positive* value of e^x. If now x is irrational, defined by upper and lower classes (A), (a), we have

$$e^a = \exp a < \exp x < \exp A = e^A,$$

because the exponential function increases with x. Also since $\exp x$ is continuous, $e^A - e^a$ can be made as small as we please; and consequently (compare p. 407) $\exp x$ is the single number defined by the classes (e^a), (e^A). Thus $\exp x$ coincides with Dedekind's definition of e^x, when x is irrational; and so *for all values of x, the exponential function is the positive value of e^x.*

Since $\log 1 = 0$, it follows that $e^0 = 1$; thus from the continuity of the exponential function we see that

(3) $$\lim_{x \to 0} e^x = e^0 = 1.$$

Again, because $\log y + \log y' = \log (yy')$, we have

(4) $$e^{x+x'} = e^x \cdot e^{x'}.$$

Of course (3) and (4) agree with the ordinary laws of indices, as established for rational numbers and rational indices in books on algebra.

From the definitions of the logarithmic and exponential functions it follows at once that

$$\frac{1}{y}\frac{dy}{dx} = 1, \quad \text{if } y = \exp x, \quad \text{so that } \frac{dy}{dx} = y.$$

Thus *the exponential function has a derivate equal to itself*, that is,

(5) $$\frac{d}{dx}(e^x) = e^x.$$

This result can also be deduced at once from (2) above.

Again (3a) of Art. 154 gives

$$x > \log (1 + x),$$

or (6) $$e^x > 1 + x, \quad \text{for any value of } x.$$

If we now change the sign of x, we get $e^{-x} > 1 - x$; from which we deduce

(7) $$e^x < 1/(1 - x), \quad \text{if } 0 < x < 1.$$

These simple inequalities are often sufficient to obtain useful properties of the exponential function.*

* The geometrical meaning of (6) is simply that the exponential curve lies entirely above any of its tangents.

159. Some miscellaneous inequalities.

When $x > 1$, (3) of Art. 154 gives

(1) $$\log x < x - 1 < x;$$

thus if n is any positive index,

(2) $$\log x^n < x^n \quad \text{or} \quad \log x < x^n/n.$$

Again, from the same article, we see that if x and n are positive,

$$x/(n+x) < \log(1 + x/n) < x/n.$$

Thus, we find

(3) $$e^{\xi} < \left(1 + \frac{x}{n}\right)^n < e^x, \quad \text{if } \xi = \frac{xn}{n+x}.$$

Now, as $n \to \infty$, $\xi \to x$; and the exponential is a continuous function; thus it follows from (3) that

(4) $$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$

Similarly, we can prove that if $n > x > 0$,

(3a) $$e^{x_1} > \left(1 - \frac{x}{n}\right)^{-n} > e^x, \quad \text{where } x_1 = \frac{xn}{n-x}.$$

Here $x_1 > x$ as $n \to \infty$, so that

$$e^x = \lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^{-n}.$$

When n is a positive integer, we have

$$\left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2}\left(1 - \frac{1}{n}\right) + \dots + \left(\frac{x}{n}\right)^n,$$

and since all the terms in this expression are positive, (3) gives

$$e^x > 1 + x + \frac{x^2}{2}\left(1 - \frac{1}{n}\right),$$

and consequently, by taking the limit as $n \to \infty$,

(5) $$e^x \geqq 1 + x + \tfrac{1}{2}x^2, \quad \text{if } x > 0.$$

Similarly, we can prove that, if $x > 0$, e^x is greater than any finite number of terms taken from the exponential series.

160. Some limits; the logarithmic scale of infinity.

From the definition of Art. 154 it is clear that

$$\frac{\log x - \log a}{x - a} < \frac{1}{a}, \quad \text{if } x > a.$$

Hence also

$$\log x < \log a + \epsilon(x-a), \quad \text{if } x > a > 1/\epsilon.$$

or

$$\frac{\log x}{x} < \frac{\log a}{x} + \epsilon\left(1-\frac{a}{x}\right), \quad \text{if } x > a > 1/\epsilon.$$

Now when $x \to \infty$, the expression on the right tends to ϵ, and we can accordingly find * $\xi \geqq 1/\epsilon$, so that

$$\log x/x < 2\epsilon, \quad \text{if } x > \xi.$$

Thus

(1) $$\log x/x \to 0, \quad \text{as } x \to \infty.$$

By changing x first to x^n and secondly to x^{-n}, and noting that $\log x^n = n \log x$, we now see that

(2) $$\lim_{x\to\infty}(\log x/x^n) = 0, \quad \text{if } n > 0,$$

and

(3) $$\lim_{x\to 0}(x^n \log x) = 0, \quad \text{if } n > 0.$$

Again, from (3) of the last article we see that

$$e^x > (x/n)^n, \quad \text{if } x > 0,\ n > 0.$$

From this it is clear that e^x tends to infinity with x, and with great rapidity.

Again, changing n to $n+1$ in the last inequality, we find that

$$\frac{e^x}{x^n} > \frac{x}{(n+1)^{n+1}},$$

which tends to infinity with x, when n is any fixed positive index. Thus we can write

(4) $$\lim_{x\to\infty}(e^x/x^n) = \infty, \quad \text{if } n > 0.$$

We can also obtain (1) and (4) by appealing to L'Hospital's rule, Art. 146 above.

Thus $$\lim_{x\to\infty}\frac{\log x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0,$$

but as a matter of fact the foregoing proof is actually the same as the proof of the general theorem given in Art. 146.

Similarly, $$\lim_{x\to\infty}\frac{e^{ax}}{x} = \lim_{x\to\infty}\frac{ae^{ax}}{1} = \infty, \quad a > 0.$$

If we write $a = 1/n$ and raise the last to the nth power, we get (4).

---

* For instance we might take

$$\frac{\log a}{\xi} - \frac{\epsilon a}{\xi} = \epsilon \quad \text{or} \quad \xi = \frac{\log a}{\epsilon} - a,$$

assuming this greater than $1/\epsilon$, which is always true when ϵ is small enough.

The limits (1)–(4) form the basis of *the logarithmic scale of infinity.* It follows from (2) that $\log x$ tends to ∞ more slowly than any positive power of x, however small its index may be; hence, *a fortiori*, $\log(\log x)$ tends to ∞ still more slowly, and so on. On the other hand, we see from (4) that e^x tends to ∞ faster than any power of x, however large its index may be; and hence, *a fortiori*, e^{e^x} tends to ∞ still faster, and so on. Thus we can construct a succession of functions, all tending to ∞, say,

$$\ldots < \log(\log x) < \log x < x < e^x < e^{e^x} < \ldots,$$

and *each function tends to ∞ faster than any power of the preceding function, but more slowly than any power of the following function.*

It is easy to see, however, that this scale by no means exhausts all types of increase to infinity. Thus, for instance, the function

$$e^{(\log x)^2} = x^{\log x}$$

tends to ∞ more slowly than e^x, but more rapidly than any (fixed) power of x.

Similarly,

$$x^x = e^{x \log x}$$

tends to infinity more rapidly than e^x, but more slowly than e^{x^2} or than e^{e^x}.

Other examples will be found at the end of this Appendix (Exs. 11, 14, p. 459). See also G. H. Hardy's Tract, "Orders of Infinity," No. 12 of the Cambridge Mathematical Tracts.

161. The existence of an area for the rectangular hyperbola.

We give here a proof that the rectangular hyperbola has an area which can be found by a definite limiting process; this seems essential, since comparatively few English books give an adequate arithmetic discussion of the area of a curve. The method applies at once to any continuous curve, although the diagram refers only

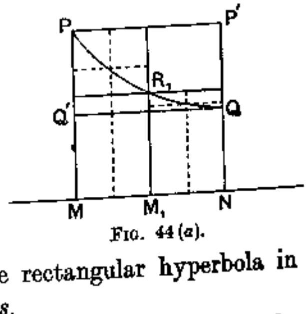


FIG. 44 (*a*).

to a curve like the rectangular hyperbola in which the ordinate constantly *decreases*.

Consider any strip of the figure, bounded by the curve (PQ), the axis of x (MN), and two ordinates MP and NQ. We can associate with this strip an *outer* rectangle $PMNP'$ and an *inner* rectangle $Q'MNQ$.

Here of course the outer rectangle corresponds to the first ordinate MP and the inner rectangle to the last ordinate NQ of the strip; but if there are several maxima and minima on the curve between P and Q, the outer rectangle will correspond to the upper limit (or to the greatest maximum), and the inner rectangle to the lower limit (or to the least minimum).*

Now bisect MN at M_1 and draw the ordinate M_1R_1. This divides the original strip into two, and each strip has an outer and an inner rectangle; namely, in the figure, PM_1, R_1N outside the curve, R_1M, QM_1 inside. But when the strip is subdivided into two, the upper limit (or the greatest maximum) in the whole strip is also the upper limit in one of the two subdivisions; and (in general) is greater than the upper limit in the other subdivision. Hence the sum of the two new outer rectangles must be less than the original outer rectangle; thus in our particular diagram PM_1+R_1N is obviously less than the whole rectangle PN.

By similar reasoning the sum of the two inner rectangles is greater than the original inner rectangle; in particular R_1M+QM_1 is greater than the rectangle QM.

If we bisect MM_1 and M_1N again, we obtain four outer and four inner rectangles; and again the sum of the outer rectangles has been diminished while the sum of the inner has been increased by the further bisection.

When MN is divided into 2^n equal parts, let us denote the sum of the outer rectangles by S_n and the sum of the inner rectangles by s_n. Then

$$S_0 > S_1 > S_2 > \dots > S_n > \dots,$$

and

$$s_0 < s_1 < s_2 < \dots < s_n < \dots .$$

Also $$S_n > s_n, \quad (n=0, 1, 2, 3, \dots).$$

Now, in our special diagram, we see that the difference S_1-s_1 is the sum of the two rectangles PR_1, R_1Q, which is equal to

$$\tfrac{1}{2}MN(MP-NQ)=\tfrac{1}{2}(S_0-s_0).$$

* To follow the reasoning here, and in what follows, the reader will find it useful to sketch several curves with a number of maxima and minima of various relative magnitudes: and other curves with (finite) discontinuities.

Similarly, $$S_2 - s_2 = \tfrac{1}{2}(S_1 - s_1) = \frac{1}{2^2}(S_0 - s_0),$$

and generally $$S_n - s_n = \tfrac{1}{2}(S_{n-1} - s_{n-1}) = \dots = \frac{1}{2^n}(S_0 - s_0).$$

It is therefore clear from Art. 143 that S_n and s_n approach a common limit as n increases; this limit, say A, is called the area of the figure $PMNQ$.

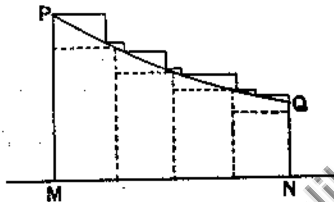


FIG. 44 (*b*).

But it is essential to prove that *we find the same area A in whatever way the base MN is supposed divided to form the rectangles.**

Let Σ denote the sum of the outer rectangles when MN is divided up in any manner, regular or irregular; and let σ denote the sum of the corresponding inner rectangles. Then a glance at Fig. 44 (*b*) will shew that for any value of n we have †

$$\Sigma > s_n, \quad \sigma < S_n,$$

where of course Σ, σ are quite independent of n.

Thus, since $$\lim S_n = A = \lim s_n,$$

we have $$\Sigma \geqq A, \quad \sigma \leqq A.$$

But $$\Sigma - \sigma \leqq \beta(MP - NQ),$$

where β is the breadth of the widest rectangle contained in the sums Σ, σ.

Hence we can choose a value δ such that

$$\Sigma - \sigma < \epsilon, \quad \text{if } \beta < \delta,$$

* This theorem was originally proved by Newton (see his *Principia*, Lemmas 2, 3). The discussion given by Newton is a condensed form of the following proof: the additions made here are intended to lead up to Riemann's condition, as Newton's proof cannot be applied unless the curve steadily rises or steadily falls (as in the case of the rectangular hyperbola sketched).

† To avoid confusion we have only indicated the rectangles Σ and s_n, the latter being dotted; the reader will have no difficulty in constructing a similar figure for σ and S.

and therefore, since $\Sigma \geqq A \geqq \sigma$, we have

$$\Sigma - A < \epsilon, \quad A - \sigma < \epsilon, \quad \text{if } \beta < \delta.$$

Thus,
$$\lim_{\beta \to 0} \Sigma = A = \lim_{\beta \to 0} \sigma.$$

That is, we obtain the *same* area A, in whatever way the base MN is divided, *provided that the largest sub-division tends to zero.*

162. Extension of the discussion to curves which are not monotonic.

When the function to be integrated is not monotonic, but is finite in the interval (a, b) (for definition, see Ex. 19, p. 435), we construct S_n, s_n as above; and we can see that S_n and s_n are each monotonic, and so have definite limits as n tends to infinity.

But we cannot make the inference that $S_n - s_n \to 0$, without introducing some further condition. The most natural condition is due to Riemann.

Suppose that in *any* sub-division of (a, b) into sub-intervals $\eta_1, \eta_2, \ldots, \eta_\nu$, we denote by ω_r the difference between the upper and lower limits (or between the greatest maximum and the least minimum) of the function in the interval η_r, then Riemann assumes that it is possible to find a value for δ so as to make

$$\sum_{r=1}^{\nu} \eta_r \omega_r < \epsilon,$$

for all modes of division of the interval such that $\eta_1, \eta_2, \ldots, \eta_\nu$ are *all* less than δ.

Now $S_n - s_n = \Sigma \eta_r \omega_r$, if η_r refers to the subdivisions constructed by successive bisection. Thus under Riemann's condition

$$\lim (S_n - s_n) = 0,$$

and so
$$\lim S_n = \lim s_n = A, \text{ say.}$$

Then, just as before, we prove that for *any* mode of division

$$\Sigma \geqq A, \quad \sigma \leqq A,$$

and by Riemann's condition, we have also

$$\Sigma - \sigma < \epsilon, \quad \text{if } \eta_r < \delta.$$

Thus
$$\lim \Sigma = A = \lim \sigma.$$

It is easy to shew that *any continuous function satisfies Riemann's condition*; for (see Ex. 18, App. I.) we can find δ so that $\omega_r < \epsilon/(b-a)$, if $\eta_r < \delta$. Thus we find $\Sigma \eta_r \omega_r < \epsilon$, because $\Sigma \eta_r = b - a$.

163. Extension of definition to double integrals.

It is also easy to extend the definition of integration to a function of two variables, say x, y; let us consider the meaning of

$$\iint f(x, y)\, dx\, dy,$$

where x ranges from a to b, and y from a' to b'.

If we divide (a, b) into 2^m equal parts and (a', b') into 2^n equal parts, we obtain two sums

$$S_{m,n} = \Sigma H_{\mu,\nu} \gamma_{\mu,\nu}, \quad s_{m,n} = \Sigma h_{\mu,\nu} \gamma_{\mu,\nu},$$

where $H_{\mu,\nu}$ and $h_{\mu,\nu}$ are the upper limit (or greatest maximum) and lower limit (or least minimum) in a sub-rectangle $\gamma_{\mu,\nu}$. Then, just as above, we see that $S_{m,n}$ decreases if either m or n is increased, while $s_{m,n}$ increases; thus $S_{m,n}$ and $s_{m,n}$ have each a limit when m, n tend to infinity *in any manner* (see Ch. V., Art. 31).

Further, if $f(x, y)$ is *continuous* we prove as above that

$$\lim S_{m,n} = \lim s_{m,n} = V, \text{ say.}$$

Now we have, from the definition of single integrals,

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} S_{m,n}\right) = \int_a^b dx \int_{a'}^{b'} f(x, y)\,dy$$

and

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} S_{m,n}\right) = \int_{a'}^{b'} dy \int_a^b f(x, y)\,dx,$$

so that these *two repeated integrals are each equal to V, and therefore to each other.*

EXAMPLES.

1. Prove directly from the integral for $\log x$ that $2\frac{1}{2} < e < 3$.

[For we have $\qquad \log(2\frac{1}{2}) = \int_1^{2\frac{1}{2}} \frac{dt}{t}, \quad \log 3 = \int_1^3 \frac{dt}{t}.$

If we take these integrals from 1 to $1\frac{1}{4}$, from $1\frac{1}{4}$ to $1\frac{1}{2}$, etc., we find that

$$\log(2\tfrac{1}{2}) < \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \dots + \tfrac{1}{9} < 1.$$

$$\log 3 > \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \dots + \tfrac{1}{12} > 1.]$$

2. Determine which of the two expressions

$$(\tfrac{1}{2}e)^{\sqrt{3}}, \quad \sqrt{(2)}^{\frac{1}{2}\pi}$$

is the greater. [*Oxford Sen. Schol.*]

[Take logarithms and note that

$$\sqrt{3}/(\sqrt{3} + \tfrac{1}{4}\pi) < \tfrac{2}{5}\sqrt{3} < \cdot 6929 < \log 2,$$

since (Art. 63) $\log 2 > \cdot 6931$.]

3. When x is positive, shew that the functions

$$\frac{\log(1+x)}{x} \quad \text{and} \quad (1+x)\frac{\log(1+x)}{x}$$

are both monotonic; and sketch their graphs.

4. If $1 + x > 0$, prove that $x^2 > (1+x)\{\log(1+x)\}^2$. [*Math. Trip.* 1906.]

[Write $\log(1+x) = 2\xi$, and use the fact that $e^{\xi} - e^{-\xi} > 2\xi$ if ξ is positive.]

5. Prove that as x ranges from -1 to ∞, the function

$$\frac{1}{\log(1+x)} - \frac{1}{x}$$

remains continuous and steadily decreases from 1 to 0. [*Math. Trip.* 1894.]

[From the last example, we see that the derivate is negative; discontinuity is only possible at $x=0$, and when $|x|$ is small we find that

$$\frac{1}{\log(1+x)}-\frac{1}{x}=\frac{1}{2}-\frac{x}{12}+\dots,$$

the series converging if $|x|<1$.]

6. Prove that to the base $(1+1/p)^p$, where p is large, the logarithm of any number N is equal to

$$\log_e N \times \left(1+\frac{1}{2p}-\frac{1}{12p^2}+\frac{1}{24p^3}-\dots\right).$$

7. In J. Burgi's tables the value of $\log 10$ is given as $2{\cdot}30270022$ (in modern notation). Verify that this is consistent with the value $p=10^4$, assuming $\log_e 10=2{\cdot}3025850939$.

Shew also that Napier's value (quoted in Art. 157) is not consistent with taking $p=-10^7$.

[This disproves the statement, made by some German writers, that Napier's logarithms were calculated on the same lines as J. Burgi's.]

8. If $a_n \to l$ as n tends to ∞, prove that

$$\lim n(a_n^{\frac{1}{n}}-1)=\log l, \quad \lim(1+a_n/n)^n=e^l.$$

Deduce that

$$\lim\left\{\frac{1}{p}(a_1^{\frac{1}{n}}+a_2^{\frac{1}{n}}+\dots+a_p^{\frac{1}{n}})\right\}^n=(a_1a_2\dots a_p)^{\frac{1}{p}}.$$

9. If ρ_n is numerically less than a fixed number A, independent of n, and if

$$\log\left(1+\frac{\rho_n}{n}\right)=\frac{\sigma_n}{n},$$

then $$\overline{\lim}\,\sigma_n=\overline{\lim}\,\rho_n.$$

Also if $$\log\left(1+\frac{1}{n}+\frac{\rho_n}{n\log n}\right)=\frac{1}{n}+\frac{\sigma_n}{n\log n},$$

then $$\overline{\lim}\,\sigma_n=\overline{\lim}\,\rho_n.$$

[Compare Art. 12·2 (4).]

10. Use the last example to shew that if

$$\log\frac{a_n}{a_{n+1}}=\frac{\mu}{n}+O\left(\frac{1}{\mu^\lambda}\right), \quad (\lambda>1),$$

the series of positive terms Σa_n converges if $\mu>1$, and otherwise diverges.

Deduce that the series

$$\Sigma(n!)^2 n^{n^2-1}e^n(n-1)^{-(n+1)^2}$$

is divergent. Compare (4) and (5) of Art. 12·2.

11. Shew how to determine X so that

$$e^x>Mx^N, \quad \text{if } x>X,$$

where M, N are any assigned large numbers.

[We have to make $$x > \log M + N \log x,$$
which (since $\log x < 2\sqrt{x}$) can be satisfied by taking $x > 2 \log M$ and $16N^2$. But as a rule these determinations of X are unnecessarily large.]

12. What is the largest number which can be expressed algebraically by means of three digits? Estimate the number of digits in this number when written in the ordinary system of numeration.

[The number of digits in 9^{9^9} is found to be 369,693,100. If they could be written 16 to the inch, this number would extend over 360 miles.]

13. The logarithmic function $\log x$ is not a rational function of x.

[Apply Art. 160.]

14. Arrange the following functions in the order of the rapidity with which they tend to infinity with x:
$$x^x, \quad x^{\log x}, \quad (\log x)^x, \quad (\log x)^{(\log x)^2}, \quad (\log x)^{\log \log x}, \quad (\log \log x)^{\sqrt{\log x}}.$$
Indicate the position of each of these functions amongst the members of the standard logarithmic scale.

15. If we assume the binomial series for any integral exponent, and suppose n to be an integer greater than $|x|$, we find
$$\left(1+\frac{x}{n}\right)^n = 1 + x + \left(1-\frac{1}{n}\right)\frac{x^2}{2!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!} + \dots \text{ to } (n+1) \text{ terms},$$
$$\left(1-\frac{x}{n}\right)^{-n} = 1 + x + \left(1+\frac{1}{n}\right)\frac{x^2}{2!} + \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\frac{x^3}{3!} + \dots \text{ to } \infty.$$
Deduce that, if x is positive,
$$\left(1+\frac{x}{n}\right)^n < 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \text{ to } \infty < \left(1-\frac{x}{n}\right)^{-n},$$
and so obtain the exponential series.

16. Shew that
$$e^x \sum_1^\infty \frac{(-1)^{n-1} x^n}{n(n!)} = \sum_1^\infty \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)\frac{x^n}{n!}.$$

[If the product on the left is called v, we get
$$\frac{dv}{dx} - v = \frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots.$$
Taking $v = \Sigma a_n x^n / n!$, we get at once, since $a_0 = 0$,
$$a_1 = 1, \quad a_n - a_{n-1} = 1/n.$$
If we obtain the series for v by means of the rule for multiplication, we find the identity
$$n - \frac{1}{2}\,\frac{n(n-1)}{2!} + \frac{1}{3}\,\frac{n(n-1)(n-2)}{3!} - \dots \text{ to } n \text{ terms} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n},$$
which is easily verified directly.]

17. Shew that, as $x \to 0$, [Euler.]
$$\left(\frac{e^x}{e^x - 1} - \frac{1}{x}\right) \to \frac{1}{2}, \quad \frac{1}{x^2}\{e^x - 1 - \log(1+x)\} \to 1.$$

18. If $\chi_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \frac{1}{2}\log\{n(n+1)\} - C$, where C is Euler's constant (Art. 11), prove that

$$0 < \chi_n < \tfrac{1}{6}\{n(n+1)\}^{-1}. \qquad \text{[CESÀRO.]}$$

[It will be seen that

$$\chi_{n-1} - \chi_n = \int_0^1 \frac{x^2 dx}{n(n^2 - x^2)} < \frac{1}{3}\frac{1}{n(n^2-1)},$$

which gives the result.]

19. A good approximation to the function of Ex. 18 is given by taking

$$\chi_n = \frac{1}{6\{n(n+1) + \frac{1}{5}\}},$$

the error in which is of the order $1/(150n^6)$. [A. LODGE.]

[Apply Euler's series (Art. 106).]

20. Prove that

$$e^{-x}\left\{\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \frac{x^{n+3}}{(n+3)!} + \ldots\right\} = \frac{x^n}{n!}\left(\frac{x}{n+1} - \frac{x^2}{n+2} + \frac{1}{2!}\frac{x^3}{n+3} - \ldots\right).$$

Deduce that the expansion of

$$\log\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}\right) = x - \frac{x^n}{n!}\left\{\frac{x}{n+1} - \frac{x^2}{n+2} + \frac{x^3}{2!(n+3)} - \ldots - \frac{x^{n+1}}{n!(2n+1)}\right\}$$

as far as the term in x^{2n+1}.

[Differentiate the first equation, and both sides reduce to $e^{-x}(x^n/n!)$.]

21. Shew that the sequence

$$a_1 = 1, \quad a_2 = e^2, \quad a_3 = e^{e^3}, \quad a_4 = e^{e^{e^4}}, \ldots$$

tends to infinity more rapidly than any member of the exponential scale.

22. Prove that the series

$$\Sigma(\log n)^p n^{-q}$$

converges if $q > 1$ or if $q = 1$ and $p < -1$; and otherwise diverges.

APPENDIX III.

SOME THEOREMS ON INFINITE INTEGRALS AND GAMMA-FUNCTIONS.

164. Infinite integrals: definitions.

If either the range is infinite or the subject of integration tends to infinity at some point of the range, an integral may be conveniently called *infinite*,* as differing from an ordinary integral very much in the same way as an infinite series differs from a finite series.

In the case of an infinite integral, the method commonly used to establish the existence of a finite integral will not apply, as will be seen if we attempt to modify the proof of Art. 161. We must accordingly frame a new definition:

First, *if the range is infinite, we define the integral* $\int_a^\infty f(x)dx$ *as equal to the limit* $\lim_{\lambda\to\infty}\int_a^\lambda f(x)dx$ *when this limit exists.*

Secondly, *if the integrand tends to infinity at either limit (say that* $f(x)\to\infty$ *as* $x\to a$*), we define the integral* $\int_a^b f(x)dx$ *as equal to the limit*

$$\lim_{\delta\to0}\int_{a+\delta}^b f(x)dx \quad (\delta>0)$$

when this limit exists.

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* Following German writers (who use *uneigentlich*), some English authors have used the adjective *improper* to distinguish such integrals as we propose to call *infinite*. The term used here was introduced by Hardy (*Proc. Lond. Math. Soc.* (ser. 1), vol. 34, p. 16, footnote), and has several advantages, not the least of which is the implied analogy with the theory of *infinite* series.

If the integrand tends to infinity at a point c within the range of integration, it is usually best to divide the integral into two, and then we should define the integral by the equation

$$\int_a^b f(x)\,dx = \lim_{\delta\to 0}\int_a^{c-\delta} f(x)\,dx + \lim_{\delta_1\to 0}\int_{c+\delta_1}^b f(x)\,dx.$$

But in certain problems the two limits in the last equation are both infinite, while the *sum* of the two integrals tends to a finite limit if δ_1/δ tends to a finite limit; we then define *the principal value of the integral* by the equation

$$P\int_a^b f(x)\,dx = \lim_{\delta\to 0}\left[\int_a^{c-\delta} f(x)\,dx + \int_{c+\delta}^b f(x)\,dx\right].$$

It is at once evident that we can extend the use of the terms *converge*, *diverge*, and *oscillate* * so as to apply to these definitions.

Exs. (of convergence).

1. $$\int_0^\infty \frac{dx}{1+x^2} = \lim_{\lambda\to\infty}\int_0^\lambda \frac{dx}{1+x^2} = \lim_{\lambda\to\infty}(\arctan\lambda) = \frac{\pi}{2},$$

2. $$\int_0^\infty e^{-ax}\,dx = \lim_{\lambda\to\infty}\int_0^\lambda e^{-ax}\,dx = \lim_{\lambda\to\infty}\frac{1}{a}(1-e^{-a\lambda}) = \frac{1}{a}, \quad \text{if } a>0,$$

3. $$\int_0^1 \frac{dx}{x^k} = \lim_{\delta\to 0}\int_\delta^1 \frac{dx}{x^k} = \lim \frac{1-\delta^{1-k}}{1-k} = \frac{1}{1-k}, \quad \text{if } 0<k<1,$$

4. $$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \lim_{\delta\to 0}\int_0^{1-\delta}\frac{dx}{\sqrt{(1-x^2)}} = \lim_{\delta\to 0}[\arcsin(1-\delta)] = \frac{\pi}{2}$$

5. $$\int_{-a}^b \frac{dx}{x^{\frac{1}{3}}} = \lim_{\delta\to 0}\int_{-a}^{-\delta}\frac{dx}{x^{\frac{1}{3}}} + \lim_{\delta_1\to 0}\int_{\delta_1}^b \frac{dx}{x^{\frac{1}{3}}}$$
$$= \lim_{\delta\to 0}\frac{3}{2}(\delta^{\frac{2}{3}} - a^{\frac{2}{3}}) + \lim_{\delta_1\to 0}\frac{3}{2}(b^{\frac{2}{3}} - \delta_1^{\frac{2}{3}})$$
$$= \frac{3}{2}(b^{\frac{2}{3}} - a^{\frac{2}{3}}),$$

6. $$P\int_{-a}^b \frac{dx}{x} = \lim_{\delta\to 0}\left(\int_{-a}^{-\delta}\frac{dx}{x} + \int_\delta^b \frac{dx}{x}\right)$$
$$= \lim_{\delta\to 0}\left(-\log\frac{a}{\delta} + \log\frac{b}{\delta}\right) = \log\left(\frac{b}{a}\right),$$

where in the last two integrals we suppose a and b to be positive. It should be remarked that in the last case we should have

$$\int_{-a}^{-\delta}\frac{dx}{x} + \int_{\delta_1}^b \frac{dx}{x} = \log\frac{b}{a} + \log\frac{\delta}{\delta_1},$$

which, of course, does not tend to a definite limit unless δ/δ_1 does so.

* Stokes, *Math. and Phys. Papers*, vol. 1, p. 241

Exs. (of divergence and oscillation).

$$\int_0^\infty \frac{dx}{1+x} \text{ diverges and } \int_0^\infty \sin x\,dx \text{ oscillates,}$$

$$\int_0^1 \frac{dx}{x} \text{ diverges and } \int_0^1 \frac{dx}{x^2}\sin\left(\frac{1}{x}\right) \text{ oscillates.}$$

It must not be supposed that the two types of infinite integrals are fundamentally different. An infinite integral of one type can always be transformed so as to belong to the other type; thus, if $f(x)\to\infty$ as $x\to b$, but is continuous elsewhere in the interval (a, b), we can write

$$\xi=\frac{x-a}{b-x} \quad\text{or}\quad x=\frac{a+b\xi}{1+\xi}.$$

Then $$\int_a^b f(x)\,dx=\int_0^\infty f\left(\frac{a+b\xi}{1+\xi}\right)\frac{(b-a)\,d\xi}{(1+\xi)^2}$$

and the integrand in ξ is everywhere finite.*

Ex. $$\int_0^1 \frac{dx}{(1-x^2)^{\frac{1}{2}}}=\int_0^\infty \frac{d\xi}{(1+\xi)(1+2\xi)^{\frac{1}{2}}}.$$

By reversing this transformation it may happen that an integral to ∞ can be expressed as a *finite* integral.

Ex. When $x=1/\xi$, $\int_1^\infty x^{-s}dx$ becomes $\int_0^1 \xi^{s-2}d\xi$, which is a finite integral if $s \geqq 2$ (both integrals still converge if $2>s>1$).

It is also possible in many cases to express a convergent infinite integral of the second type as a finite integral by a change of variable. Thus we have

$$\int_0^1 \frac{f(x)}{(1-x^2)^{\frac{1}{2}}}\,dx=\int_0^{\frac{1}{2}\pi} f(\sin\theta)\,d\theta$$

by writing $x=\sin\theta$, and the latter integral is finite if $f(x)$ is finite in the interval (0, 1) (for definition see **Ex. 19, p. 435**). Kronecker in his lectures on definite integrals states that such a transformation is always possible, but although this is theoretically true, it is not effectively practicable† in all cases.

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* Care must be taken in applying this kind of transformation when the infinity of $f(x)$ is *inside* the range of integration. Here it is usually safer to divide the integral into two, as already explained.

† If $f(x)\to\infty$ as $x\to a$, we can write $\int_x^b f(x)dx=\xi$, and introduce ξ as a new variable. Similarly in other cases; and in the same sense we can always express a divergent integral in the form $\int^\infty d\xi$.

165. Special case of monotonic functions.

Although, as we have pointed out in the last article, the definition of a definite integral requires in general a modification for the case of an infinite integral, yet we can obtain a direct definition of the integral as the limit of a sum, when the integrand steadily increases or steadily decreases.

Suppose first that in the integral $\int_a^\infty f(x)dx$ the function $f(x)$ steadily decreases to 0 for values of x greater than c; we may then consider only the integral $\int_c^\infty f(x)dx$, because the integral from a to c falls under the ordinary rules. Then let $x_0(=c)$, $x_1, x_2, x_3, \ldots$ be a sequence of values increasing to ∞; we have, as in Art. 11,

$$(x_{n+1}-x_n)f(x_n) > \int_{x_n}^{x_{n+1}} f(x)dx > (x_{n+1}-x_n)f(x_{n+1})$$

Thus, if the integral $\int_c^\infty f(x)dx$ converges to the value I,

$$(1) \qquad \sum_0^\infty (x_{n+1}-x_n)f(x_n) \geqq I \geqq \sum_0^\infty (x_{n+1}-x_n)f(x_{n+1}).$$

Of the two series in (1), the second certainly converges, in virtue of the convergence of the integral and the fact that the series contains only positive terms. The first need not converge, if the rate of increase of (x_n) is sufficiently rapid; for instance, with $x_n = 2^{2^n}$ and $f(x)=1/x^2$, it will be found that every term in the series is greater than $\frac{1}{2}$.

However, by taking x_n to be a properly chosen function of some parameter h (as well as of n), we can easily ensure the convergence of both series in (1); and we can also prove that the two series have a common limit as $x_{n+1}-x_n$ is made to tend to zero by varying h; this common limit must be equal to I, in virtue of the inequalities (1).

For example, suppose that $x_{n+1}-x_n$ is independent of n and equal to h, say; then $x_n = c+nh$ and we have

$$\sum_0^\infty (x_{n+1}-x_n)f(x_n) \quad = h[f(c)+f(c+h)+f(c+2h)+\ldots],$$

$$\sum_0^\infty (x_{n+1}-x_n)f(x_{n+1}) = h[f(c+h)+f(c+2h)+f(c+3h)+\ldots].$$

It follows that the difference between the two sums is $hf(c)$, so that both are convergent, and their difference tends to 0 with h; hence

$$I=\lim_{h\to 0} h[f(c)+f(c+h)+f(c+2h)+\ldots].$$

In like manner, if x_{n+1}/x_n is independent of n and equal to q, say, so that $x_n=cq^n$, we have

$$\sum_0^\infty (x_{n+1}-x_n)f(x_n) \quad =c(q-1)[f(c)+qf(cq)+q^2f(cq^2)+\ldots]$$

and $$\sum_0^\infty (x_{n+1}-x_n)f(x_{n+1})=c[(q-1)/q][qf(cq)+q^2f(cq^2)+\ldots].$$

Thus we can again infer the convergence of the first series from that of the second, and we see that

$$I=\lim_{q\to 1} c(q-1)[f(c)+qf(cq)+q^2f(cq^2)+\ldots].$$

Ex. 1. Consider $\int_0^\infty xe^{-x}dx$, with $x_n=nh$.

We have then
$$\begin{aligned} I&=\lim_{h\to 0} h^2[e^{-h}+2e^{-2h}+3e^{-3h}+\ldots]\\ &=\lim_{h\to 0} h^2e^{-h}(1-e^{-h})^{-2}\\ &=\lim_{h\to 0} e^h\left(\frac{h}{e^h-1}\right)^2=1,\end{aligned}$$
a value which can be verified by integration by parts.

Ex. 2. Consider $\int_c^\infty x^{-s}dx$, (where $s>1$).

Here write $x_n=cq^n$, and we get
$$\begin{aligned} I&=\lim_{q\to 1} \frac{c(q-1)}{c^s}\left(1+\frac{q}{q^s}+\frac{q^2}{q^{2s}}+\ldots\right)\\ &=\lim_{q\to 1} \frac{q-1}{c^{s-1}}\Big/\left(1-\frac{1}{q^{s-1}}\right)\\ &=\lim_{q\to 1} \frac{q^{s-1}}{c^{s-1}}\left(\frac{q-1}{q^{s-1}-1}\right)=\frac{1}{(s-1)c^{s-1}}\end{aligned}$$
by applying one of the fundamental limits of the differential calculus.

Ex. 3. It can be proved by rather more elaborate reasoning that *if $f(x)$ steadily decreases to 0 as x tends to ∞, then*

$$\int_0^\infty \sin x\, f(x)\,dx=\lim_{h\to 0} h\sum_0^\infty f(nh)\sin nh, \quad \int_0^\infty \cos x\, f(x)dx=\lim_{h\to 0} h\sum_0^\infty f(nh)\cos nh.$$

Let us consider the simple example

$$\int_0^\infty \frac{\sin x}{x}dx,$$

the sum is then
$$h+(\sin h+\tfrac{1}{2}\sin 2h+\ldots).$$

Since h is positive (and less than 2π), the sum of the series in brackets is $\frac{1}{2}(\pi-h)$, by Art. 65, and so the whole sum is

$$\tfrac{1}{2}(\pi+h),$$

which gives the limit $\frac{1}{2}\pi$; that this gives the correct value for the integral can be verified by other methods (see Ex. 1, Art. 173).

Ex. 4. The reader may verify in the same way that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \tfrac{1}{2}\pi.$$

Ex. 5. By means of the integral $\int_0^\infty x^{k-1}e^{-x}dx$, we can prove that

$$\lim_{h\to 0} h^k(e^{-h}+2^{k-1}e^{-2h}+3^{k-1}e^{-3h}+\ldots) = \int_0^\infty x^{k-1}e^{-x}dx$$
$$= \Gamma(k),$$

a result which has already been found in Art. 51 by another method.

In like manner, if $f(x)\to\infty$ as $x\to 0$, but steadily decreases as x varies from 0 to b, we can prove that when $\int_0^b f(x)dx$ converges, we have

$$\int_0^b f(x)dx = \lim_{q\to 1} b(1-q)[f(b)+qf(bq)+q^2 f(bq^2)+\ldots].$$

Ex. 6. Take $\int_0^b \log x\,dx$; we have to find

$$\lim_{q\to 1} b(1-q)[\log b + q\log(bq) + q^2\log(bq^2)+\ldots]$$
$$= \lim_{q\to 1} b(1-q)\left[\frac{\log b}{1-q}+\frac{q\log q}{(1-q)^2}\right]$$
$$= \lim_{q\to 1}[b(\log b)+bq(\log q)/(1-q)]$$
$$= b(\log b) - b,$$

as we may verify by direct integration.

In the previous work we have seen how to evaluate an infinite integral by calculating the limit of an *infinite* series; when the range is finite we can also obtain the result as the limit of a *finite* series; that is, we can replace a double limit by a single limit.

Thus, suppose that in the convergent integral $\int_a^b f(x)dx$ the integrand $f(x)\to\infty$ as $x\to a$, and that $f(x)$ steadily decreases from a to b. Then write $b-a=nh$, and an argument similar to that of Art. 11 will shew that $\int_{a+h}^b f(x)dx$ lies between the two sums

$$h[f(a+h)+f(a+2h)+\ldots+f(b-h)]$$

and

$$h[f(a+2h)+f(a+3h)+\ldots+f(b)].$$

Now, as $h \to 0$, the integral tends to a definite limit; and the difference between the two sums is $h[f(a+h)-f(b)]$, which tends to zero with h in virtue of the monotonic property of $f(x)$ (see pp. 470, 471 below). That is,

$$\int_a^b f(x)dx = \lim_{h \to 0} h\,[f(a+h)+f(a+2h)+\ldots+f(b)],$$

which gives the value of the integral as a *single* limit.

Ex. 7. Consider $\int_0^b x^{-s}dx$, where $0<s<1$.

Write $h=\frac{b}{n}$, and we have to find

$$\lim_{n\to\infty} \frac{b^{1-s}}{n^{1-s}}\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots+\frac{1}{n^s}\right)=\frac{b^{1-s}}{1-s}$$

(by Ex. 1, Art. 147, above).

Ex. 8. In the same way $\int_0^1 \log x\,dx$ is found as

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)=\lim_{n\to\infty}\log\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}=-1 \text{ (Ex. 1, Art. 149).}$$

Ex. 9. If we divide the last equation of Art. 69 by $\sin\theta$, and let θ tend to zero, we find, if $\alpha=\pi/n$,

$$n=2^{n-1}\sin\alpha\sin 2\alpha\ldots\sin(n-1)\alpha.$$

Now change from n to $2n$ and write h for α; we get, pairing the terms,

$$2n=2^{2n-2}\sin^2 h\sin^2 2h\ldots\sin^2(n-1)h.$$

Thus, extracting the square root,

$$\sin h\sin 2h\ldots\sin(n-1)h=n^{\frac{1}{2}}2^{1-n}, \quad (\text{if } h=\pi/2n),$$

and from this we can find $\int_0^{\frac{1}{2}\pi}\log\sin x$.

For this integral is equal to

$$\lim_{h\to 0} h[\log\sin h+\log\sin(2h)+\ldots+\log\sin(nh)]$$

$$=\lim_{n\to\infty}\frac{\pi}{2n}[\tfrac{1}{2}\log n-(n-1)\log 2]$$

$$=-\tfrac{1}{2}\pi\log 2.$$

166. Tests of convergence for infinite integrals with a positive integrand.

If the function $f(x)$ is positive, at least for sufficiently large values of x, it is clear that the integral $\int_a^\lambda f(x)\,dx$ steadily increases with λ; thus in virtue of the monotonic test for convergence, *the integral to* ∞ *cannot oscillate, and will converge if we can prove that*

$$\int_a^\lambda f(x)\,dx < A,$$

where A is independent of λ.

In practice the usual method of applying this test is to appeal to the *principle of comparison*, as in the case of series of positive terms; in fact, if $g(x)$ is a positive function for which $\int^\infty g(x)\,dx$ converges, then $\int^\infty f(x)\,dx$ also converges if $f(x) < g(x)$, at any rate for values of x greater than some fixed number c.

For then $$\int_c^\lambda f(x)\,dx < \int_c^\lambda g(x)\,dx < \int_c^\infty g(x)\,dx,$$

and this last expression is independent of λ.

Thus, suppose we consider $f(x)=x^\beta e^{-\alpha x}$, where α is positive and β is either positive or negative; from Art. 160 above, we find that $x^\beta e^{-\frac{1}{2}\alpha x}\to 0$ as $x\to\infty$, so that we can determine c to satisfy

$$x^\beta e^{-\frac{1}{2}\alpha x} < 1, \qquad \text{if } x > c,$$

and then $$f(x) = x^\beta e^{-\alpha x} < e^{-\frac{1}{2}\alpha x}, \qquad \text{if } x > c.$$

Now (see Ex. 2, p. **462**, Art. 164) $\int^\infty e^{-\frac{1}{2}\alpha x}\,dx$ is convergent, and consequently $\int^\infty x^\beta e^{-\alpha x}\,dx$ is also convergent.

If we write $X=e^x$, we find that

$$\int x^\beta e^{-\alpha x}\,dx = \int (\log X)^\beta X^{-(1+\alpha)}\,dX,$$

so that $\int^\infty (\log x)^\beta x^{-(\alpha+1)}\,dx$ is convergent.

Examples of this type can be multiplied to any extent by the aid of the logarithmic scale of infinity (Art. 160).

Thus *if we can find a positive index* α *and a constant B, such that* **one** *of the conditions*

$$\left.\begin{array}{l}\text{(i) } f(x) < B(\log x)^\beta x^{-(1+\alpha)},\\ \text{(ii) } f(x) < Bx^\beta e^{-\alpha x},\end{array}\right\} \qquad \alpha > 0, \quad x > c,$$

is satisfied, the integral $\int_c^\infty f(x)\,dx$ *converges.*

The comparison test for divergence runs as follows:

If $G(x)$ is always positive and $\int^\infty G(x)\,dx$ is divergent, then so also is $\int^\infty f(x)\,dx$, if $f(x) > G(x)$, at any rate after a certain value of x.

We have proved (see the small type above) that if a is positive, c can be found so that

$$x^{\beta} e^{ax} > e^{\frac{1}{2}ax}, \quad \text{if } x > c,$$

whatever the index β may be.

Now
$$\int_c^{\lambda} e^{\frac{1}{2}ax}\,dx = \frac{2}{a}\left(e^{\frac{1}{2}a\lambda} - e^{\frac{1}{2}ac}\right)$$
and this expression tends to ∞ with λ, so that the integral to ∞ is divergent.

Thus $\int^{\infty} x^{\beta} e^{ax}\,dx$ also diverges.

If $a=0$, it is easily seen that this integral diverges if $\beta \geqq -1$.

By changing the variable, we deduce as before that

$$\int^{\infty} (\log x)^{\beta} x^{-(1-a)}\,dx$$

diverges under the same conditions.

Accordingly *the integral* $\int_a^{\infty} f(x)\,dx$ *diverges if we can find an index* $a \geqq 0$, *such that one of the conditions*

$$\left.\begin{aligned}&\text{(i) } f(x) > B(\log x)^{\beta} x^{-(1-a)},\\ &\text{(ii) } f(x) > Bx^{\beta} e^{ax},\end{aligned}\right\} \quad \begin{aligned}&a > 0\\ &\text{or } a=0,\ \beta \geqq -1,\end{aligned}$$

is satisfied.

These conditions are analogous to those of Art. 11 for testing the convergence of a series of positive terms; and, as there remarked, closer tests can be obtained by making use of other terms in the logarithmic scale (although such conditions are not of importance for our present purpose). But one striking feature presents itself in the theory of infinite integrals which has no counterpart in the theory of series. *An integral* $\int^{\infty} f(x)\,dx$ *may converge even though* $f(x)$ *does not tend to the limit zero.* Naturally, we must then have an oscillatory function, for $\underline{\lim}\, f(x)=0$ is obviously necessary in all cases of convergence; but we may even have $\overline{\lim}\, f(x)=\infty$. To see, in a general way, that this is possible, we may use a graphical method.

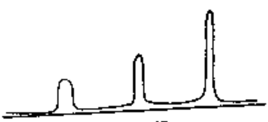

FIG. 45.

Consider a curve which has an infinite series of *peaks*, of steadily increasing height; then, it is quite possible to suppose that their widths are correspondingly decreased in such a way that the areas of the peaks form a convergent series; and consequently $\int^{\infty} f(x)\,dx$ may converge.

Let us consider in particular the function

$$f(x) = x^{\beta}/(1 + x^{\alpha}\sin^2 x), \qquad (\alpha > \beta > 0).$$

Here in general $f(x)$ is comparable with $x^{\beta-\alpha}$, but its graph comes up to the curve $y = x^{\beta}$, at every point for which x is a multiple of π.

In the interval from $n\pi$ to $(n+1)\pi$, we have

$$\frac{(n\pi)^{\beta}}{1+[(n+1)\pi]^{\alpha}\sin^2 x} < f(x) < \frac{[(n+1)\pi]^{\beta}}{1+(n\pi)^{\alpha}\sin^2 x}.$$

Now $$\int_{n\pi}^{(n+1)\pi} \frac{dx}{1+A\sin^2 x} = \frac{\pi}{(1+A)^{\frac{1}{2}}},$$

so that $$\frac{n^{\beta}\pi^{\beta+1}}{[1+(n+1)^{\alpha}\pi^{\alpha}]^{\frac{1}{2}}} < \int_{n\pi}^{(n+1)\pi} f(x)\,dx < \frac{(n+1)^{\beta}\pi^{\beta+1}}{(1+n^{\alpha}\pi^{\alpha})^{\frac{1}{2}}}.$$

From this it is evident that $\int^{\infty} f(x)\,dx$ converges or diverges with the series $\Sigma n^{\beta-\frac{1}{2}\alpha}$; that is, according as $\alpha > 2(\beta+1)$ or $\alpha \leqq 2(\beta+1)$.

And generally, if $\phi(x)$, $\psi(x)$ steadily increase to ∞ with x, the integral

$$\int^{\infty} \frac{\phi(x)\,dx}{1+\psi(x)\sin^2(x\pi)}$$

converges if $\Sigma\phi(n+1)/[\psi(n)]^{\frac{1}{2}}$ converges and diverges if $\Sigma\phi(n)/[\psi(n+1)]^{\frac{1}{2}}$ diverges.

Ex. 1. $\int_0^{\infty} \dfrac{x^{\beta}\,dx}{1+x^{\alpha}|\sin x|}$ converges or diverges according as

$\alpha > \beta+1$ or $\alpha \leqq \beta+1$. [HARDY.*]

Ex. 2. $\int_0^{\infty} \phi(x)e^{-\psi(x)|\sin x|}\,dx$ converges with $\Sigma\dfrac{\phi(n\pi+\epsilon)}{\psi(n\pi-\epsilon)}$ and diverges with $\Sigma\dfrac{\phi(n\pi-\epsilon)}{\psi(n\pi+\epsilon)}$. [HARDY.]

Ex. 3. $\int_0^{\infty} \phi(x)e^{-\psi(x)\sin^2 x}\,dx$ converges with $\Sigma\dfrac{\phi(n\pi+\epsilon)}{\sqrt{\psi(n\pi-\epsilon)}}$ and diverges with $\Sigma\dfrac{\phi(n\pi-\epsilon)}{\sqrt{\psi(n\pi+\epsilon)}}$. [DU BOIS REYMOND.]

In spite of the last result, we can prove (as in Art. 9 for series) that *if $f(x)$ steadily decreases, the condition* $\lim xf(x) = 0$ *is necessary for the convergence of* $\int^{\infty} f(x)\,dx$.

---

* *Messenger of Mathematics*, April 1902, Note VIII.

For here we have

$$\int_\lambda^\mu f(x)\,dx > (\mu-\lambda)f(\mu);$$

thus for convergence it is necessary to be able to find λ so that $(\mu-\lambda)f(\mu)$ is less than ϵ for *any* value of μ greater than λ.

Hence $\lim xf(x)=0$ is necessary for convergence. But even so, no such condition as $\lim(x\log x)f(x)=0$ is necessary in general (compare Art. 9); but it is easy to shew that if (for instance) $xf(x)$ is monotonic, then $x\log x f(x)$ must tend to 0. More generally, if $\phi(x)$ tends steadily to ∞ and $f(x)/\phi'(x)$ is monotonic, then $f(x)\phi(x)/\phi'(x)$ must tend to zero; this may be proved by changing the variable from x to $\phi(x)$. [PRINGSHEIM.]

It is perfectly easy to modify all the foregoing work* so as to apply to integrals in which the integrand tends to infinity, say at $x=0$.

The results are: *The integral* $\int_0^b f(x)dx$ *converges (if* b *is less than* 1), *provided that we can satisfy one of the conditions*

$$f(x) < Bx^{a-1}\left(\log\frac{1}{x}\right)^\beta,$$

where either (i) $a>0$ *or* (ii) $a=0,\ \beta<-1$.

On the other hand, the integral diverges when

$$f(x) > Bx^{a-1}\left(\log\frac{1}{x}\right)^\beta,$$

where (i) $a<0$ *or* (ii) $a=0,\ \beta \geqq -1$.

167. Examples.

To illustrate the last article, we consider two simple cases.

1. $$\int_0^\infty (e^{-x}-e^{-tx})\frac{dx}{x} \qquad (t>0).$$

It is easy to see that the integral converges, so far as concerns the upper limit, by applying the tests of the last article. There is an apparent difficulty at the lower limit, because of the factor $1/x$; but since

$$\frac{1}{x}(e^{-x}-e^{-tx}) = \frac{1}{x}\left[\left(1-x+\frac{x^2}{2!}-\dots\right)-\left(1-xt+\frac{x^2t^2}{2!}-\dots\right)\right]$$

$$=(t-1)-\frac{x}{2!}(t^2-1)+\dots$$

the difficulty is apparent only.

* Or we may obtain the results directly by writing $1/x$ for x in the integral.

Now the integral is $\lim_{\delta\to 0}\int_\delta^\infty (e^{-x}-e^{-tx})\frac{dx}{x}$, and

$$\int_\delta^\infty e^{-tx}\frac{dx}{x}=\int_{t\delta}^\infty e^{-x}\frac{dx}{x},$$

by changing the variable of integration.

Hence our integral is

$$\lim_{\delta\to 0}\int_\delta^{t\delta} e^{-x}\frac{dx}{x}.$$

But $\int_\delta^{t\delta} e^{-x}\frac{dx}{x}$ lies between the values found by replacing e^{-x} by $e^{-\delta}$ and by $e^{-t\delta}$; these values are respectively

$$e^{-\delta}\log t \text{ and } e^{-t\delta}\log t,$$

both of which tend to $\log t$ as δ tends to 0.

Hence $$\lim_{\delta\to 0}\int_\delta^{t\delta} e^{-x}\frac{dx}{x}=\log t,$$

and accordingly $$\int_0^\infty (e^{-x}-e^{-tx})\frac{dx}{x}=\log t.$$

2. Consider $$\int_0^\infty (Ae^{-ax}+Be^{-bx}+Ce^{-cx})\frac{dx}{x^2},$$

where a, b, c are positive and

$$A+B+C=0,\quad Aa+Bb+Cc=0.$$

It may be shewn as above that the integral converges when these conditions are satisfied.

Now consider

$$\int_\delta^\infty Ae^{-ax}\frac{dx}{x^2}=\int_{a\delta}^\infty Aae^{-x}\frac{dx}{x^2}=Aa\left(\int_\delta^\infty e^{-x}\frac{dx}{x^2}-\int_\delta^{a\delta} e^{-x}\frac{dx}{x^2}\right).$$

In virtue of the condition $\Sigma Aa=0$, it is now evident that

$$\int_\delta^\infty (\Sigma Ae^{-ax})\frac{dx}{x^2}=-\Sigma Aa\int_\delta^{a\delta} e^{-x}\frac{dx}{x^2}.$$

But

$$\int_\delta^{a\delta} e^{-x}\frac{dx}{x^2}=\int_\delta^{a\delta}\left(1-x+\frac{x^2}{2!}-\dots\right)\frac{dx}{x^2}=\frac{1}{\delta}\left(1-\frac{1}{a}\right)-\log a+\frac{1}{2!}\delta(a-1)+\dots.$$

Thus $$\int_\delta^\infty (\Sigma Ae^{-ax})\frac{dx}{x^2}=\Sigma(Aa\log a)-\frac{\delta}{2!}(\Sigma Aa^2)+\dots,$$

and so $$\int_0^\infty (\Sigma Ae^{-ax})\frac{dx}{x^2}=\Sigma(Aa\log a).$$

3. The reader can prove similarly that, if $p > n$,

$$\int_0^\infty \left(\sum_0^p A_r e^{-a_r x}\right)\frac{dx}{x^n} = -\sum_0^p A_r \frac{(-a_r)^{n-1}}{(n-1)!}\log a_r,$$

where $$\sum_0^p A_r a_r^k = 0. \quad (k=0, 1, 2, \ldots, n-1)$$

168. Analogue of Abel's Lemma.

If the function $f(x)$ steadily decreases, but is always positive, in an interval (a, b), and if $|\phi(x)|$ is less than a fixed number A in the interval, then *

$$hf(a) < \int_a^b f(x)\phi(x)\,dx < Hf(a),$$

where H, h are the upper and lower limits of the integral

$$\chi(\xi) = \int_a^\xi \phi(x)\,dx,$$

as ξ ranges from a to b.

For, assuming that $f(x)$ is differentiable, we have

$$J = \int_a^b f(x)\phi(x)\,dx = f(b)\chi(b) - \int_a^b f'(x)\chi(x)\,dx.$$

Now, since $f(b)$ is *positive* and $f'(x)$ is everywhere *negative*, we obtain a value greater than J by replacing $\chi(b)$ and $\chi(x)$ in the last expression by H, and a value less than J by replacing them by h.

Thus we find

$$hf(b) - h\int_a^b f'(x)\,dx < J < Hf(b) - H\int_a^b f'(x)\,dx$$

or $$hf(a) < J < Hf(a).$$

Similarly, if H_1, h_1 are the limits of the integral $\chi(\xi)$ in the interval (a, c) and H_2, h_2 in the interval (c, b), we find

$$h_2 f(b) - h_1\int_a^c f'(x)\,dx - h_2\int_c^b f'(x)\,dx < J$$

$$< H_2 f(b) - H_1\int_a^c f'(x)\,dx - H_2\int_c^b f'(x)\,dx$$

or $$h_1[f(a) - f(c)] + h_2 f(c) < J < H_1[f(a) - f(c)] + H_2 f(c).$$

* If $f(x)$ should be discontinuous at $x=a$, $f(a)$ denotes the limit of $f(x)$ as x tends to a through larger values.

When $\phi(x)$ is a complex function of a real variable, it is easily seen that if u is any number, real or complex, and if η_1, η_2 are the upper limits of

$$\left|\int_a^\xi \phi(x)dx - u\right|,$$

as ξ ranges from a to c and from c to b, respectively, then

$$|J - uf(a)| < \eta_1[f(a) - f(c)] + \eta_2 f(c).$$

When $f(x)$ is complex, formulae corresponding to the lemma of Art. 81 can be obtained (see *Proc. Lond. Math. Soc.*, vol. 6, 1907, p. 65); but these results are not needed for our present purpose.

The first inequality on p. 473 is equivalent to the **Second Theorem of Mean Value**. To see this, note first that $\chi(\xi)$ is continuous, and so (Ex. 17, p. 435) assumes every value between h, H at least once in the interval (a, b). Thus the inequality leads to **Bonnet's theorem**

$$J = f(a)\chi(\xi_0), \text{ where } a \leqq \xi_0 \leqq b.$$

From this **du Bois Reymond's theorem**, which is true for any monotonic function $g(x)$, follows by writing $|g(x) - g(b)|$ for $f(x)$; thus we find the form commonly quoted

$$\int_a^b g(x)\phi(x)dx = g(a)\int_a^{\xi_0}\phi(x)dx + g(b)\int_{\xi_0}^b \phi(x)dx.$$

But, since the precise value of ξ_0 cannot be determined, *these equations contain no more information than the original inequality and not so much as the inequality at the foot of p.* 473

Although the restriction that $f(x)$ is to be differentiable is of little importance here, yet it is theoretically desirable to establish such results as the foregoing with the greatest generality possible. We shall therefore give a second proof, based on one due to Pringsheim*, in which we assume nothing about the existence of $f'(x)$.

Divide the interval into n equal parts by inserting points $x_1, x_2, \ldots, x_{n-1}$, and write $x_0 = a$, $x_n = b$; then we have

$$J = \int_a^b f(x)\phi(x)dx = \sum_{r=0}^{n-1} J_r,$$

where

$$J_r = \int_{x_r}^{x_{r+1}} f(x)\phi(x)dx.$$

---

* *Münchener Sitzungsberichte*, Bd. 30, 1900, p. 209; this paper contains a more general form of the theorem, which is also deducible from the first inequality on p. 473. Another proof has been given by Hardy, *Messenger of Maths.*, vol. 36, 1906, p. 10.

Hence,* if $f_{r+1}=f(x_{r+1})$,

(1) $$J_r - f_{r+1}\int_{x_r}^{x_{r+1}}\phi(x)dx=\int_{x_r}^{x_{r+1}}[f(x)-f_{r+1}]\phi(x)dx;$$

but in virtue of the decreasing property of $f(x)$, $[f(x)-f_{r+1}]$ is positive in the last integral and is less than (f_r-f_{r+1}), so that

(2) $$\left|\int_{x_r}^{x_{r+1}}[f(x)-f_{r+1}]\phi(x)dx\right|<(f_r-f_{r+1})A(b-a)/n,$$

because $|\phi(x)|<A$ and $x_{r+1}-x_r=(b-a)/n$.

By adding up the equations (1), bearing in mind the inequality (2), we see that

(3) $$J-\sum_{r=0}^{n-1}f_{r+1}\int_{x_r}^{x_{r+1}}\phi(x)dx=R_n,$$

where $$|R_n|<\frac{A}{n}(b-a)(f_0-f_n)<\frac{A}{n}(b-a)f_0,$$

because $f_n=f(b)$ is positive.

If now we apply Abel's Lemma (Art. 20) to the sum

$$\sum_{r=0}^{n-1}f_{r+1}\int_{x_r}^{x_{r+1}}\phi(x)dx,$$

we obtain the limits hf_1 and Hf_1 for it, because

$$\sum_{r=0}^{m-1}\int_{x_r}^{x_{r+1}}\phi(x)dx=\int_a^{x_m}\phi(x)dx,$$

and the sequence $f_1, f_2, \ldots, f_n$ is decreasing.

Thus, from (3), we find

(4) $$hf_1-\frac{A}{n}(b-a)f_0<J<Hf_1+\frac{A}{n}(b-a)f_0,$$

where $$f_1=f[a+(b-a)/n].$$

If now we take the limit of (4) as n tends to infinity, we obtain the desired result.†

In exactly the same way we can make the further inference that *if c lies between a, b, and if H_1, h_1 are the upper and lower limits of* $\int_a^{\xi}\phi(x)dx$ *as ξ ranges from a to c, while H_2, h_2 are those as ξ ranges from c to b, then*

$$h_1[f(a)-f(c)]+h_2f(c)<\int_a^b f(x)\phi(x)dx<H_1[f(a)-f(c)]+H_2f(c).$$

---

* In case $f(x)$ should be discontinuous at x_{r+1}, we define f_{r+1} as the limit of $f(x)$ when x approaches x_{r+1} through *smaller* values of x; this limit will exist in virtue of the monotonic property of $f(x)$.

† It will be seen that the condition $|\phi(x)|<A$ is by no means essential, and that it may be broken at an infinity of points, provided that $\int_a^b|\phi(x)|dx$ converges; for we can then make a division into sub-intervals, for each of which $\int_{x_r}^{x_{r+1}}|\phi(x)|dx$ is less than any assigned number. But Pringsheim has proved that it is only necessary to assume that $\phi(x)$ and $f(x)\times\phi(x)$ are integrable in the interval (a, b); compare *Proc. Lond. Math. Soc.*, vol. 6, 1907, p. 62.

169. Tests of convergence in general.

Applying the general test for convergence (Art. 3), we see that *the necessary and sufficient condition for the convergence of the integral* $\int_a^\infty f(x)dx$ *is that we can find* ξ *such that*

$$\left|\int_\xi^{\xi'} f(x)dx\right| < \epsilon,$$

where ξ' *may have* **any** *value greater than* ξ *and* ϵ *is arbitrarily small.*

However, just as for infinite series, the general test for convergence is usually replaced in practice by some narrower test which can be applied more quickly. The three chief tests are the following:

1. Absolute convergence.*

The integral $\int_a^\infty f(x)dx$ will certainly converge if $\int_a^\infty |f(x)|dx$ converges, because

$$\left|\int_\xi^{\xi'} f(x)dx\right| \leqq \int_\xi^{\xi'} |f(x)|dx.$$

But naturally the analogy between such integrals and absolutely convergent series is not quite complete, since there is no *order* in the values of a function.

In particular, if $|f(x)| < -g'(x)$, where $g(x)$ steadily decreases to zero as x increases, the integral $\int_a^\infty f(x)dx$ will converge.

2. Abel's test.

An infinite integral which converges (although not **absolutely***) will remain convergent after the insertion of a factor which is monotonic and less than a fixed number (in numerical value).*

Suppose that $\int_a^\infty \phi(x)dx$ converges, and that $\psi(x)$ is a monotonic function, such that $|\psi(x)| < A$: it is then evident that $\psi(x)$ tends to some limit l as $x \to \infty$.

Thus, if we write $f(x) = l - \psi(x)$ when $\psi(x)$ increases, or $\psi(x) - l$ if $\psi(x)$ decreases, we see that $f(x)$ is positive and

* The distinction between absolute and non-absolute convergence is clearly pointed out by Stokes (*Math. and Phys. Papers*, vol. 1, p. 241).

decreases to 0; and it is obviously sufficient to prove the convergence of $\int_a^\infty f(x)\phi(x)dx$.

Now, by the analogue of Abel's Lemma (Art. 168), we see that

$$\left|\int_\xi^{\xi'} f(x)\phi(x)dx\right| < Hf(\xi) < Hf(a),$$

where H is the upper limit to

$$\left|\int_\xi^X \phi(x)\,dx\right|$$

when X ranges from ξ to ξ'. Now, in virtue of the convergence of $\int_a^\infty \phi(x)dx$, we can determine ξ so as to make $H < \epsilon/f(a)$, and then

$$\left|\int_\xi^{\xi'} f(x)\phi(x)dx\right| < \epsilon,$$

so that the integral $\int_a^\infty f(x)\phi(x)\,dx$ converges.

Hence also $\int_a^\infty \psi(x)\phi(x)dx$ converges.

3. Dirichlet's test.

An infinite integral which oscillates finitely becomes convergent after the insertion of a monotonic factor which tends to zero as a limit.

Here again we have

$$\left|\int_\xi^{\xi'} f(x)\phi(x)\,dx\right| < Hf(\xi),$$

and H will be less than some fixed constant independent of ξ;* thus, since $f(x)\to 0$, we can find ξ so that $Hf(\xi) < \epsilon$, and consequently the integral $\int_a^\infty f(x)\phi(x)dx$ is convergent.

Although the tests (2), (3) are almost immediately suggested by the tests of Arts. 21, 22, yet it is not clear that they were ever given, in a complete form, until recently. Stokes (in 1847) was certainly aware of the theorem (3) in the case $\phi(x)=\sin x$ (*Math. and Phys. Papers*, vol. 1, p. 275), but he makes no reference to any extension, nor does he indicate his method of proof. The first general statements and proofs seem to be due to Hardy (*Messenger of Maths.*, vol. 30, 1901, p. 187); his argument is somewhat different from the foregoing, and is on the lines of the following treatment of the special case $\phi(x)=\sin x$.

* Since the integral $\int_a^\xi \phi(x)dx$ oscillates *finitely* it remains less than some fixed number C for all values of ξ; thus we have $H \leqq 2C$.

In this case, the curve $y=f(x)\sin x$ oscillates between the two curves $y=f(x)$, $y=-f(x)$, as indicated roughly in the figure.

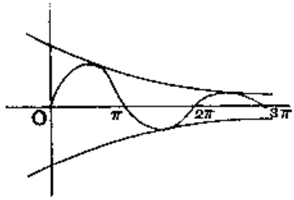

FIG. 46.

It is almost intuitively evident that the areas of the waves steadily decrease in value, and have alternate signs. In fact

$$\int_{2n\pi}^{(2n+1)\pi} f(x)\sin x\,dx = \int_0^{\pi} f(x+2n\pi)\sin x\,dx,$$

and since $\sin x$ is positive in the integral, this lies between $2f(2n\pi)$ and $2f(\overline{2n+1}\,\pi)$; so that it tends to zero as n increases to ∞. Further,

$$\int_{(2n+1)\pi}^{(2n+2)\pi} f(x)\sin x\,dx = -\int_0^{\pi} f(x+\overline{2n+1}\,\pi)\sin x\,dx,$$

which is obviously negative and numerically less than the area of the previous wave. It follows that $\int_a^{\infty}\sin x f(x)dx$ is convergent, by applying the theorem of Art. 19.

In general, if $\phi(x)$ changes sign infinitely often we can apply a similar method, using Dirichlet's test (Art. 22) to establish the convergence of the series.

If the integrand tends to ∞, say as $x\to a$, the general test for convergence and the test of absolute convergence run as follows:

The necessary and sufficient condition for the convergence of the integral $\int_a^b f(x)dx$ *is that we can find* δ *such that*

$$\left|\int_{a+\delta'}^{a+\delta} f(x)dx\right| < \epsilon,$$

where δ' *has* **any** *positive value less than* δ.

This condition is certainly satisfied if the integral $\int_a^b |f(x)|\,dx$ *converges; and the original integral is then said to converge absolutely.*

It is possible to write out corresponding modifications of Abel's and Dirichlet's tests; but such tests are not often needed in practice and are better left to the ingenuity of the reader.

Ex. 1. $\int_1^\infty \frac{\sin x}{x^p}dx$, $\int_1^\infty \frac{\cos x}{x^p}dx$ converge absolutely if $p>1$, and so does

$$\int_0^1 \frac{\cos x}{x^q}dx \text{ if } 0<q<1\text{; because } |\sin x| \leqq 1,\ |\cos x| \leqq 1.$$

Ex. 2. $\int_1^\infty \frac{\sin x}{x^p}dx$, $\int_1^\infty \frac{\cos x}{x^p}dx$ converge if $0<p<1$; and generally

$$\int^\infty \phi(x)\sin x\, dx,\ \int^\infty \phi(x)\cos x\, dx$$

converge in virtue of Dirichlet's test, if $\phi(x)$ tends steadily to zero.

For $\left|\int_a^b \sin x\, dx\right| = |\cos a - \cos b| \leqq 2$, $\left|\int_a^b \cos x\, dx\right| = |\sin b - \sin a| \leqq 2$.

Ex. 3. Further examples of Dirichlet's test are given by

$f(x)=e^{-px}$, $(\log x)^{-p}$; $\phi(x)=(\sin x)^{-\frac{1}{3}}$, $\log(4\cos^2 x)$. [HARDY, *l.c.*]

170. Frullani's integrals.

As a simple and interesting example of the tests of the last article, let us consider the value of

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx,$$

where $\phi(x)$ is such that $\int^\infty \phi(x)dx$ oscillates between finite limits (or converges).

Then, by applying Dirichlet's or Abel's test, we see that

$$\int_\delta^\infty \frac{\phi(ax)}{x}dx$$

is convergent and is equal to

$$\int_{a\delta}^\infty \frac{\phi(x)}{x}dx, \quad \text{if } a, \delta > 0.$$

Thus $$\int_\delta^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = \int_{a\delta}^{b\delta} \frac{\phi(x)}{x}dx = \int_a^b \frac{\phi(x\delta)}{x}dx.$$

and if $\phi(x)$ tends to a definite finite limit ϕ_0, as x tends to 0, the last integral has a finite range and a finite integrand: thus we have

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = \phi_0 \log\left(\frac{b}{a}\right).$$

In the same way we can prove that if

$$A+B+C=0, \quad Aa+Bb+Cc=0,$$

then $$\int_0^\infty \frac{dx}{x^2}[A\phi(ax)+B\phi(bx)+C\phi(cx)] = -(\Sigma Aa\log a)\phi'(0).$$

For examples, take $\phi(x)=\cos x$ or $\sin x$.

The former integral can also be evaluated if $\phi(x)$ tends to a definite value ϕ_1 as x tends to ∞. For we have the identity

$$\int_\delta^\lambda \frac{\phi(ax)-\phi(bx)}{x}\,dx=\int_a^b \frac{\phi(x\delta)-\phi(x\lambda)}{x}\,dx,$$

by means of which the value of Frullani's integral may be proved to be
$$(\phi_0-\phi_1)\log(b/a).$$
The integral found in Art. 167 (1) is a particular case of this formula and also of that on p. 479.

Extensions of these integrals have been considered by Lerch and by Hardy.*

171. Uniform convergence of an infinite integral.†

If we consider the integral (supposed convergent)

$$\int_a^\infty f(x, y)dx,$$

the *least* value of ξ, for which the inequality

$$\left|\int_\lambda^\infty f(x, y)\,dx\right| < \epsilon, \quad (\lambda > \xi),$$

holds, is a function of y as well as of ϵ. In correspondence with Art. 43, we say that the integral **converges uniformly** in an interval (α, β), if for **all** values of y in the interval ξ remains less than a function $X(\epsilon)$, which depends on ϵ but is **independent of** y. But, if this condition cannot be satisfied for any interval which contains a particular value y_0, then y_0 is said to be a **point of non-uniform convergence** of the integral.

Ex. 1. If $f(x, y)=1/(x+y)^2$, where $y \geqq 0$, we find that
$$\int_\lambda^\infty f(x, y)dx=1/(\lambda+y).$$
Thus $\xi=(1/\epsilon)-y$ (if $y<1/\epsilon$), or $\xi=0$ (if $y>1/\epsilon$), and so the integral is uniformly convergent for all *positive* values of y, since we may take $X(\epsilon)=1/\epsilon$.

Ex. 2. If $f(x, y)=y/(1+x^2y^2)$, we find that
$$\int_\lambda^\infty f(x, y)\,dx=\cot^{-1}(\lambda y), \quad \text{if } y \gtrless 0, \quad \text{or} =0, \quad \text{if } y=0.$$
Thus $\xi=\cot\epsilon/|y|$, if $y\gtrless 0$, or $\xi=0$, if $y=0$.
Hence $y=0$ is a point of non-uniform convergence for the integral.

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* Lerch, *Sitzungsberichte d. k. Böhmischen Gesellschaft der Wiss.*, June 2, 1893; Hardy, *Messenger of Maths.*, vol. 34, 1904, pp. 11, 102, and *Quarterly Journal*, vol. 33, 1901, p. 113. See also Art. 173, Exs. 1–4.

† Stokes, *Math. and Phys. Papers*, vol. 1, p. 283.

But, just as in the case of series, we have usually in practice to introduce a test for uniform convergence which is similar to the general test for convergence (Art. 169): *The necessary and sufficient condition for the* **uniform** *convergence of the integral* $\int_a^\infty f(x, y)dx$, *in an interval of values of y, is that we can find a value of* ξ, **independent of** y, *such that*

$$\left|\int_\xi^{\xi'} f(x, y)dx\right| < \epsilon,$$

where ξ' *has* **any** *value greater than* ξ, *and* ϵ *is arbitrarily small.*

The only fresh point introduced is seen to be the fact that ξ must be *independent of y*. The proof that this condition is both necessary and sufficient follows precisely on the lines of Art. 43, with mere verbal alterations.

But in practical work we need more special tests which can be applied more quickly; the three most useful of these tests are:

1. **Weierstrass's test.**

Suppose that for all values of y in the interval (α, β), *the function* $f(x, y)$ *satisfies the condition*

$$|f(x, y)| < M(x),$$

where $M(x)$ *is a positive function,* **independent of** y. *Then, if the integral* $\int_a^\infty M(x)dx$ *converges, the integral* $\int_a^\infty f(x, y)dx$ *is absolutely and uniformly convergent for all values of y in the interval* (α, β).

For then we can choose ξ independently of y, so that $\int_\xi^\infty M(x)dx$ is less than ϵ; and therefore

$$\left|\int_\xi^{\xi'} f(x, y)dx\right| < \int_\xi^{\xi'} M(x)dx < \int_\xi^\infty M(x)dx < \epsilon.$$

Thus the integral converges uniformly; and it converges absolutely in virtue of Art. 169, (1).

2. **Abel's test.***

The integral $\int_a^\infty f(x, y)\phi(x)dx$ *is uniformly convergent in an interval* (α, β), *provided that* $\int_a^\infty \phi(x)dx$ *converges, and that*

* Bromwich, *Proc. Lond. Math. Soc.* (2), vol. 1, 1903, p. 201.

for every fixed value of y in the interval (α, β), *the function* $f(x, y)$ *is positive and steadily decreases as x increases, while* $f(a, y)$ *is less than a constant K (independent of y).*

For then, in virtue of the analogue to Abel's Lemma, we have

$$\left|\int_{\xi}^{\xi'} f(x, y)\phi(x)dx\right| < Hf(\xi, y) < Hf(a, y) < HK,$$

where H is the upper limit to the expression $\left|\int_{\xi}^{\xi_1}\phi(x)dx\right|$, when ξ_1 ranges from ξ to ξ'. Now, since $\int_a^\infty \phi(x)dx$ is convergent, we can find ξ independently of y, so that $H < \epsilon/K$; and consequently the given integral converges uniformly.

It is evident that $\phi(x)$ may be replaced by $\phi(x, y)$, provided that $\int_a^\infty \phi(x, y)dx$ is *uniformly* convergent in the interval (α, β).

3. **Dirichlet's test.**

The integral $\int_a^\infty f(x, y)\phi(x)dx$ *is uniformly convergent in an interval* (α, β) *if* $\int_a^\infty \phi(x)dx$ *oscillates between finite limits, and the function* $f(x, y)$ *is positive and steadily decreases as x increases (y being kept constant), provided that* $f(x, y)$ *tends to zero uniformly with respect to y in the interval* (α, β).

For then $$\left|\int_{\xi}^{\xi'} f(x, y)\phi(x)dx\right| < Hf(\xi, y),$$

where H is less than some constant independent of y; we can then fix ξ, independently of y, to satisfy $f(\xi, y) < \epsilon/H$.

Again, $\phi(x)$ may contain y, provided that the extreme limits of $\int_a^\infty \phi(x)dx$ remain *finite* throughout the interval (α, β).

Ex. 3. *Weierstrass's test.*

$$\int_1^\infty \frac{\cos(xy)}{x^{1+\alpha}}dx,\ \int_1^\infty \frac{\sin(xy)}{x^{1+\alpha}}dx,\ \int_0^\infty \frac{\cos(xy)}{1+x^2}dx,\ \int_0^\infty \frac{\sin(xy)}{1+x^2}dx, \quad (\alpha > 0),$$

converge uniformly throughout any interval of variation of y.

Ex. 4. *Abel's test.*

$$\int_a^\infty e^{-xy}\frac{\cos x}{x}dx,\ \int_a^\infty e^{-xy}\frac{\sin x}{x}dx, \quad (a > 0),$$

converge uniformly in any interval $(0 \leqq y \leqq A)$, because the integrals

$$\int_a^\infty \frac{\cos x}{x}dx,\ \int_a^\infty \frac{\sin x}{x}dx$$

converge in virtue of Art. 169, Ex. 2. [STOKES, *l.c.*, p. 284.]

Generally $\int_a^\infty e^{-xy}\phi(x)dx$ converges uniformly in any similar interval, provided that $\int_a^\infty \phi(x)dx$ converges.

Ex. 5. *Dirichlet's test.*

$$\int_1^\infty \frac{\cos x}{(x^2+y^2)^{\frac{1}{2}}}\,dx,\ \int_1^\infty \frac{\sin x}{(x^2+y^2)^{\frac{1}{2}}}\,dx$$

converge uniformly throughout any interval of variation of y.

And
$$\int_1^\infty \frac{x\cos(xy)}{1+x^2}\,dx,\ \int_1^\infty \frac{x\sin(xy)}{1+x^2}\,dx,\ \int_1^\infty \frac{\sin(xy)}{x}\,dx$$

converge uniformly in any interval which does not include $y=0$.

Of course the definition of, and the tests for, uniform convergence can be modified at once so as to refer to the second type of infinite integrals.

172. Applications of uniform convergence.

An integral $\int_a^\infty f(x,y)dx$ which converges uniformly in an interval (α, β) has properties strictly analogous to those of uniformly convergent series (Arts. 45, 46); and the proofs can be carried out on exactly the same lines. Thus we find:

1. *If $f(x, y)$ is a continuous function of y in the interval (α, β), the integral is also a continuous function of y, provided that it converges uniformly in the interval (α, β).**

Only verbal alterations are needed in the proof of the corresponding theorem for series (see Art. 45).

Ex. 1. Thus (see Ex. 3, Art. 171)

$$\int_1^\infty \frac{\cos(xy)}{x^{1+a}}\,dx,\ \int_1^\infty \frac{\sin(xy)}{x^{1+a}}\,dx,\ \int_0^\infty \frac{\cos(xy)}{1+x^2}\,dx,\ \int_0^\infty \frac{\sin(xy)}{1+x^2}\,dx, \quad (a>0),$$

are continuous functions of y in any interval.

Ex. 2. If $\int_a^\infty \phi(x)dx$ is convergent, then (see Ex. 4, Art. 171)

$$\lim_{y\to 0}\int_a^\infty e^{-xy}\phi(x)\,dx=\int_a^\infty \phi(x)dx. \qquad \text{[Dirichlet.]}$$

Ex. 3. But we must not anticipate the continuity at $y=0$ of

$$\int_0^\infty \frac{x\sin(xy)}{1+x^2}\,dx,\ \int_0^\infty \frac{\sin(xy)}{x}\,dx,$$

and it is not hard to see that they are actually discontinuous.

(See Ex. 5, Art. 171, and Ex. 6, Art. 173.)

* Stokes, *l.c.*, p. 283.

2. *Under the same conditions as in* (1), *we may integrate with respect to* y *under the sign of integration, provided that the range falls within the interval* (α, β).

Again, the proof for series needs only verbal changes.

Ex. 4. If $u=\int_0^\infty \frac{\cos(xy)}{1+x^2}dx$, $\int_0^y u\,dy=\int_0^\infty \frac{\sin xy}{x(1+x^2)}dx$,

and if $v=\int_0^\infty \frac{\sin(xy)}{1+x^2}dx$, $\int_0^y v\,dy=\int_0^\infty \frac{1-\cos(xy)}{x(1+x^2)}dx$.

3. *The equation*

$$\frac{d}{dy}\int_a^\infty f(x, y)dx=\int_a^\infty \frac{\partial f}{\partial y}dx$$

*is valid, provided that the integral on the right converges uniformly and that the integral on the left is convergent.**

Write $\quad \phi(x, h)=\int_a^x \left[\frac{f(x, y+h)-f(x, y)}{h}-\frac{\partial f}{\partial y}\right]dx,$

and let us find ξ so that

$$\left|\int_\xi^X \frac{\partial f}{\partial y}dx\right|<\epsilon, \quad \text{if } X>\xi,$$

where ξ will be independent of y, and the inequality is correct for all values of y in the interval (α, β). Then, if $X>\xi$,

$$\int_\xi^X \frac{1}{h}[f(x, y+h)-f(x, y)]dx=\int_\xi^X dx\int_y^{y+h}\frac{1}{h}\frac{\partial f}{\partial y}dy$$
$$=\frac{1}{h}\int_y^{y+h}dy\int_\xi^X \frac{\partial f}{\partial y}dx,$$

because the value of the double integral of a continuous function, taken over a *finite* area, is independent of the order of integration (see Art. 163, p. 457).

Thus, $\quad \left|\int_\xi^X \frac{1}{h}[f(x, y+h)-f(x, y)]dx\right|<\epsilon,$

in virtue of our choice of ξ.

Now $\phi(X, h)-\phi(\xi, h)=\int_\xi^X \left[\frac{f(x, y+h)-f(x, y)}{h}-\frac{\partial f}{\partial y}\right]dx,$

so that $\quad |\phi(X, h)-\phi(\xi, h)|<2\epsilon.$

---

* The last condition is partly superfluous; compare the note on p. 133 for the case of series.

The last inequality holds for all values of X greater than ξ, and all values of $|h|$ under a certain limit. If we make X tend to ∞, we obtain

$$|\phi(\infty, h)| \leqq |\phi(\xi, h)| + 2\epsilon.$$

Since our choice of ξ is independent of h, we can now allow h to tend to zero without changing ξ; and, by definition,

$$\lim_{h\to 0} \phi(\xi, h) = 0;$$

thus we have $$\overline{\lim_{h\to 0}} |\phi(\infty, h)| \leqq 2\epsilon.$$

Since ϵ is arbitrarily small, and $\phi(\infty, h)$ is independent of ϵ, this inequality can only be true if

$$\lim_{h\to 0} \phi(\infty, h) = 0,$$

or $$\lim_{h\to 0} \int_a^\infty \frac{f(x, y+h) - f(x, y)}{h} dx = \int_a^\infty \frac{\partial f}{\partial y} dx.$$

Another theorem may be mentioned here, although the ideas involved are a little beyond our scope.

If, in the integral $F(z) = \int_a^\infty f(x, z)dx$, the function $f(x, z)$ is an analytic function of the complex variable z at all points of a certain region T of the z-plane, then $F(z)$ is analytic within T, provided that a real positive function $M(x)$ can be found which makes the integral $\int_a^\infty M(x)dx$ convergent and satisfies the condition $\left|\frac{\partial f}{\partial z}\right| < M(x)$ at all points of T.

Ex. 5. To shew the need for some condition such as that of uniform convergence, we may consider the integral $\int_0^\infty \frac{\sin(xy)}{x} dx$; if this is differentiated with respect to y under the integral sign, we find $\int_0^\infty \cos(xy)\, dx$, which does not converge.

Ex. 6. On the other hand, the equations

$$\frac{d}{dy}\int_0^\infty \frac{\cos(xy)}{1+x^2} dx = -\int_0^\infty \frac{x\sin(xy)}{1+x^2} dx,$$

$$\frac{d}{dy}\int_0^\infty e^{-xy}\frac{\sin x}{x} dx = -\int_0^\infty e^{-xy}\sin x\, dx$$

are quite correct. (See Exs. 1, 6, Art. 173.)

4. The analogue of Tannery's theorem (Art. 49).

If $$\lim_{n\to\infty} f(x, n) = g(x), \quad \lim_{n\to\infty} \lambda_n = \infty,$$

then $$\lim_{n\to\infty} \int_a^{\lambda_n} f(x, n)dx = \int_a^\infty g(x)dx,$$

provided that $f(x, n)$ tends to its limit $g(x)$ uniformly in any fixed interval, and that we can determine a positive function $M(x)$ to satisfy $|f(x, n)| \leqq M(x)$, for all values of n, while $\int_a^\infty M(x)dx$ converges.

For, let ξ be chosen so that $\int_\xi^\infty M(x)dx$ is less than ϵ, then, if n is large enough to make $\lambda_n > \xi$, we have (as in Art. 49)

$$\left|\int_a^{\lambda_n} f dx - \int_a^\infty g dx\right| < \int_a^\xi |f-g|\, dx + 2\epsilon.$$

Since ξ is fixed, $|f-g|$ will tend to zero uniformly (as n tends to ∞) in the integral on the right; and so this integral tends to zero (Art. 45 (2)). The proof can now be completed in exactly the same way as in Art. 49.

Ex. 7. To see that some test such as Tannery's is necessary, consider the integral

$$\int_0^n \left(\frac{1}{1+x^2} - \frac{n}{n^2+x^2}\right) dx = \text{arc tan}\,\frac{n-1}{n+1}.$$

Since $n/(n^2+x^2) \leqq 1/n$, we have $\lim n/(n^2+x^2) = 0$, and so if we apply the rule, without reference to the existence of $M(x)$, we find the limit

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

But $(n-1)/(n+1)$ tends to 1, so that the integral approaches the limit $\frac{1}{4}\pi$ and not $\frac{1}{2}\pi$.

The second type of infinite integrals.

The reader should find little difficulty in stating and proving results, corresponding to (1)–(4) above, for the second type of integrals.

There is only one case of practical interest which may be found to offer some difficulty; this is the problem of differentiating an integral of the type*

$$F(y) = \int_a^b f(x, y)dx, \quad b = b(y) > a,$$

in which the upper limit varies with y, and is a point of discontinuity for $\frac{\partial f}{\partial y}$, although f is continuous there.

We assume that the integral $\int_a^b \frac{\partial f}{\partial y} dx$ is uniformly convergent for all values of y belonging to the interval with which we are concerned, and that $|b'(y)|$ remains less than a constant B for these values of y.

Then we can find a constant δ such that, if $0 < \xi < \delta$, we have

$$\left|\int_{b-\delta}^{b-\xi} \frac{\partial f}{\partial y} dx\right| < \frac{\epsilon}{1+B}, \quad \text{and} \quad |f(b-\delta, y) - f(b-\xi, y)| < \frac{\epsilon}{1+B}.$$

Now, if we write

$$\phi(\xi, y) = \int_a^{b-\xi} f dx, \quad Y = \int_a^b \frac{\partial f}{\partial y} dx + b'(y) f(b, y)$$

* A very simple example is given by taking, say,

$$F(y) = \int_0^y \sqrt{\{x(y-x)\}}\, dx, \quad y > 0.$$

we have, by the ordinary theorem for differentiating an integral,

$$\frac{d\phi}{dy}=\int_a^{b-\xi}\frac{\partial f}{\partial y}dx+b'(y)f(b-\xi, y),$$

and so, using the inequalities which define δ, we find

$$\left|\frac{d\phi}{dy}-Y\right|<\epsilon, \quad \text{if } 0<\xi\leqq\delta.$$

Also,

$$\frac{d}{dy}\{\phi(\xi, y)-\phi(\delta, y)\}=\int_{b-\delta}^{b-\xi}\frac{\partial f}{\partial y}dx+b'(y)\{f(b-\xi, y)-f(b-\delta, y)\},$$

so that

$$\left|\frac{d}{dy}\{\phi(\xi, y)-\phi(\delta, y)\}\right|<\epsilon;$$

and so, using a double integral as on p. 484, we see that

$$\left|\frac{1}{h}\{\phi(\xi, y+h)-\phi(\xi, y)\}-\frac{1}{h}\{\phi(\delta, y+h)-\phi(\delta, y)\}\right|<\epsilon.$$

In the last inequality, let ξ tend to 0, and $\phi(\xi, y)$ then tends to $F(y)$, so that

$$\left|\frac{1}{h}\{F(y+h)-F(y)\}-\frac{1}{h}\{\phi(\delta, y+h)-\phi(\delta, y)\}\right|\leqq\epsilon.$$

Thus we see that

$$\left|\frac{1}{h}\{F(y+h)-F(y)\}-Y\right|<2\epsilon+\left|\frac{1}{h}\{\phi(\delta, y+h)-\phi(\delta, y)\}-\frac{d\phi}{dy}\right|.$$

Take the limit of the last inequality as h tends to zero; then the right-hand tends to 2ϵ, because δ is independent of h; thus we find

$$\overline{\lim_{h\to 0}}\left|\frac{1}{h}\{F(y+h)-F(y)\}-Y\right|\leqq 2\epsilon,$$

and so by the same argument as before, we have

$$F'(y)=Y.$$

173. Applications of Art. 172.

Ex. 1. Consider first the integral

$$J=\int_0^\infty e^{-xy}(e^{-ax}-e^{-bx})\frac{dx}{x},$$

where a, b may be complex, provided that they have their real parts positive or zero.

Then J is uniformly convergent for all positive or zero values of y.*
Now differentiate with respect to y. We obtain

$$-\int_0^\infty e^{-xy}(e^{-ax}-e^{-bx})dx=-\frac{1}{a+y}+\frac{1}{b+y},$$

and this integral converges uniformly so long as $y\geqq l>0$. Its value is therefore equal to dJ/dy, in virtue of Art. 172 (3).

* It is understood that neither a nor b is zero.

Now, $\lim_{y\to\infty} J=0$, by Art. 172 (1), so that

$$J=\int_y^\infty\left(\frac{1}{a+y}-\frac{1}{b+y}\right)dy.$$

Thus, using Ex. 2, Art. 172, we find

$$\int_0^\infty (e^{-ax}-e^{-bx})\frac{dx}{x}=\lim_{y\to 0} J=\int_0^\infty\left(\frac{1}{a+y}-\frac{1}{b+y}\right)dy.$$

The last integral can be written as $\log(b/a)$, which has the advantage of being of the same form as if a and b were real; but owing to the many-valued nature of the logarithm of a complex number, it is often safer to appeal directly to the integral.

In particular, if we write $a=1$, $b=i$, we have

$$\int_0^\infty (e^{-x}-e^{-ix})\frac{dx}{x}=\int_0^\infty\left(\frac{1}{1+y}-\frac{1}{y+i}\right)dy=\left[\log\frac{1+y}{\sqrt{(1+y^2)}}+i\tan^{-1}y\right]_0^\infty=\tfrac{1}{2}\pi i$$

or

$$\int_0^\infty (e^{-x}-\cos x)\frac{dx}{x}=0,\quad \int_0^\infty \sin x\frac{dx}{x}=\tfrac{1}{2}\pi.$$

Ex. 2. Generally, we can prove in the same way that

$$\int_0^\infty (\Sigma A e^{-ax})\frac{dx}{x}=\Sigma\int_0^\infty \frac{A}{a+y}dy=-\Sigma A\log a,$$

where $\Sigma A=0$ and the real parts of a, b, c, ... are positive or zero.

Ex. 3. By direct integration combined with (1) it will be found that

$$\int_0^\infty [e^{-ax}\{1+(a+c)x\}-e^{-bx}\{1+(b+c)x\}]\frac{dx}{x^2}=b-a+c\log\left(\frac{b}{a}\right),$$

where the logarithm is determined as before, and the real parts of a, b are not negative.

For example, if we take

$$a=1,\quad b=-i,\quad c=i,$$

we get

$$\int_0^\infty [e^{-x}\{1+(1+i)x\}-e^{ix}]\frac{dx}{x^2}=-i-1+\frac{\pi}{2},$$

or

$$\int_0^\infty [e^{-x}(1+x)-\cos x]\frac{dx}{x^2}=\frac{\pi}{2}-1,\quad \int_0^\infty (xe^{-x}-\sin x)\frac{dx}{x^2}=-1.$$

As another illustration take

$$b=1,\quad c=-(a+\tfrac{1}{2}).$$

Then we find

$$\int_0^\infty \left[\left(a-1\right)e^{-x}+\left(\frac{1}{x}-\frac{1}{2}\right)(e^{-ax}-e^{-x})\right]\frac{dx}{x}=\left(a+\frac{1}{2}\right)\log a-(a-1).$$

Ex. 4. It is easy to prove similarly that

$$\int_0^\infty [\Sigma\{A_1(1+ax)+A_2x\}e^{-ax}]\frac{dx}{x^2}=-\Sigma A_2\log a-\Sigma A_1 a,$$

where $\Sigma A_1=0$, $\Sigma A_2=0$, and the real parts of a, b, c, ... are positive or zero.

Ex. 5. By differentiating $J=\int_0^\infty \frac{\sin(xy)}{x(1+x^2)}dx$ twice, we find that

$$\frac{d^2J}{dy^2}-J=\int_0^\infty \sin(xy)\frac{dx}{x}=\frac{\pi}{2}, \quad \text{if } y>0;$$

the last result following from (1) above.

Hence, since $\lim_{y\to 0} J=0$, and J remains finite as y tends to ∞ (see Ex. 1, Art. 172), we find $J=\frac{1}{2}\pi(1-e^{-y})$.

Thus, on differentiating, we find, if y is positive,

$$\int_0^\infty \frac{\cos(xy)}{1+x^2}dx=\tfrac{1}{2}\pi e^{-y}=\int_0^\infty \frac{x\sin(xy)}{1+x^2}.$$

When y is negative we find $J=\frac{1}{2}\pi(e^y-1)$, and so the other integrals become $\frac{1}{2}\pi e^y$, $-\frac{1}{2}\pi e^y$, respectively.

Thus J and the cosine integral are continuous at $y=0$. But the third integral is discontinuous there (see Exs. 1, 3, Art. 172).

In like manner, the integral $\int_0^\infty \sin(xy)\frac{dx}{x}$ has the value $\pm\frac{1}{2}\pi$, according to the sign of y, and vanishes for $y=0$.

Ex. 6. As an example of Tannery's theorem, we take the integral

$$I_n=\frac{1}{n}\int_0^b f(x)\left(\frac{\sin nx}{x}\right)^2dx, \quad (b>0)$$

in which $|f(x)|$ is supposed less than the constant H in the interval $(0, b)$.

Here $$I_n=\int_0^{nb} f\left(\frac{x}{n}\right)\frac{\sin^2 x}{x^2}\,dx,$$

so that $$\lim_{n\to\infty} I_n=f(0)\int_0^\infty \frac{\sin^2 x}{x^2}\,dx=\frac{\pi}{2}f(0).$$

For $$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx=\left[-\frac{\sin^2 x}{x}\right]_0^\infty+\int_0^\infty \frac{\sin 2x}{x}\,dx=\frac{\pi}{2},$$

this result following from (1) above. In applying Tannery's theorem we can take $M(x)=H(\sin^2 x)/x^2$; and $f(x/n)$ tends to the limit $f(0)$ uniformly in any *fixed* interval for x. It is understood here that $f(0)$ denotes the limit of $f(x)$ as x tends to 0 through *positive* values.

Ex. 7. It follows at once from (6) that if

$$J_n=\frac{1}{n}\int_0^b f(x)\left(\frac{\sin nx}{\sin x}\right)^2dx, \quad (0<b<\pi)$$

then $$\lim_{n\to\infty} J_n=\tfrac{1}{2}\pi f(0).$$

The reader should prove that if $b=\pi$, this result must be replaced by

$$\lim J_n=\tfrac{1}{2}\pi\{f(0)+f(\pi)\}.$$

The integral J_n is interesting on account of an application to Fourier Series given by Fejér (see Art. 129).

174. Some further theorems on integrals containing another variable.

Just as Tannery's theorem (Art. 172) resembles Weierstrass's test for uniform convergence, so there is a theorem related in a similar way to Abel's test (Art. 171).

If $f(x, n)$ is positive and steadily decreases (as x increases, n being kept constant) and if $\int_a^\infty \phi(x)dx$ is convergent, then

$$\lim_{n\to\infty}\int_a^{\lambda_n} f(x, n)\phi(x)dx = \int_a^\infty g(x)\phi(x)dx,$$

provided that $\lim \lambda_n = \infty$, *that $f(x, n)$ tends to the limit $g(x)$ uniformly in any fixed interval, and that $f(a, n)$ is less than a constant A for all values of n.*

For then we can write, by Art. 168,

$$\left|\int_\xi^{\lambda_n} f(x, n)\phi(x)dx\right| < f(\xi, n)H < f(a, n)H < AH,$$

where H is the upper limit to $\left|\int_\xi^{\xi'} \phi(x)dx\right|$ as ξ' ranges from ξ to ∞. Since the last integral converges when extended to infinity, we can find ξ so as to make $AH < \epsilon$; and then also

$$\left|\int_\xi^\infty g(x)\phi(x)dx\right| < g(\xi)H < AH < \epsilon,$$

because, as x increases, $g(x)$ decreases and does not exceed A.

Consequently we find

$$\left|\int_a^{\lambda_n} f(x, n)\phi(x)dx - \int_a^\infty g(x)\phi(x)dx\right|$$
$$< 2\epsilon + \left|\int_a^\xi \{f(x, n) - g(x)\}\phi(x)dx\right|.$$

Since ξ is fixed the limit of the last integral as n tends to ∞, is zero by Art. 45 (2); and so we have

$$\overline{\lim_{n\to\infty}}\left|\int_a^{\lambda_n} f(x, n)\phi(x)dx - \int_a^\infty g(x)\phi(x)dx\right| \leqq 2\epsilon.$$

It follows that this maximum limit is zero, or

$$\lim_{n\to\infty}\int_a^{\lambda_n} f(x, n)\phi(x)dx = \int_a^\infty g(x)\phi(x)dx.$$

Dirichlet's first integral.

As an application of this theorem, consider the integral

$$J_n = \int_0^b \frac{\sin nx}{x} f(x)\,dx, \quad (b > 0).$$

If we change the variable of integration to x/n, we have

$$J_n = \int_0^{nb} f\left(\frac{x}{n}\right) \frac{\sin x}{x}\,dx.$$

Hence, if $f(x)$ is positive and never increases, our theorem can be applied, because*

$$f(0) \geqq f(x/n) > 0,$$

and $f(x/n)$ tends to the limit $f(0)$ uniformly in any fixed interval.

Hence $$\lim_{n\to\infty} J_n = f(0) \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} f(0),$$

in virtue of Art. 173, Ex. 1, above.

It is, however, easy to remove the conditions from the function $f(x)$ of being positive and never increasing. Suppose, for example, that $f(x)$ first decreases in the interval $(0, c)$, and afterwards increases in the interval (c, b). Now consider the functions $F(x)$, $G(x)$ defined by

$$\left.\begin{aligned} F(x) &= f(x) + A, \\ G(x) &= \qquad\; A \end{aligned}\right\} (0 \leqq x \leqq c)$$

and $$\left.\begin{aligned} F(x) &= f(c) + A, \\ G(x) &= f(c) + A - f(x), \end{aligned}\right\} (c \leqq x \leqq b)$$

where A is a constant such that $f(c) + A$ and $f(c) + A - f(b)$ are both positive. Then the conditions of being positive and never increasing are satisfied by both $F(x)$ and $G(x)$, so that

$$\lim_{n\to\infty} \int_0^b F(x) \frac{\sin nx}{x}\,dx = \frac{\pi}{2} F(0),$$

with a similar equation for $G(x)$. But $F(x) - G(x) = f(x)$, so that

$$\lim_{n\to\infty} \int_0^b f(x) \frac{\sin nx}{x}\,dx = \frac{\pi}{2} f(0).$$

It is easy to see that *this result can be at once extended to any case where $f(x)$ has a limited number of maxima and minima and no infinities between 0 and b.*

---

* Here we use $f(0)$ to denote the limit of $f(x)$ as x approaches 0 through *positive* values; this limit exists in virtue of the monotonic property of $f(x)$.

Dirichlet's second integral.

Consider now

$$K_n = \int_0^b \frac{\sin(2n+1)x}{\sin x} f(x)dx, \qquad (0 < b < \pi)$$

where n is an integer. We can write

$$K_n = \int_0^b \frac{\sin \nu x}{x} \phi(x)dx,$$

where $$\nu = 2n+1, \quad \phi(x) = \frac{x}{\sin x} f(x).$$

Since $x/\sin x$ steadily increases and has no infinity in the interval $(0, b)$, it follows that $\phi(x)$ will satisfy the conditions set forth in dealing with Dirichlet's first integral, provided that $f(x)$ satisfies them.*

Hence $$\lim_{n\to\infty} K_n = \tfrac{1}{2}\pi\phi(0) = \tfrac{1}{2}\pi f(0).$$

If, however, the range of integration extends up to π, we may write the integral in the form

$$\left(\int_0^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\pi}\right)\frac{\sin \nu x}{\sin x} f(x)dx,$$

and then change the variable in the second part to $\pi - x$; this gives

$$\int_0^{\frac{1}{2}\pi} \frac{\sin \nu x}{\sin x}[f(x) + f(\pi - x)]\,dx.$$

Hence $$\lim_{n\to\infty}\int_0^{\pi} \frac{\sin(2n+1)x}{\sin x} f(x)dx = \frac{\pi}{2}[f(0) + f(\pi)].$$

Ex. 1. As a verification we note that

$$\frac{\sin(2n+1)x}{\sin x}dx = 1 + 2\cos 2x + 2\cos 4x + \ldots + 2\cos 2nx,$$

so that $$\int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)x}{\sin x}dx = \tfrac{1}{2}\pi$$

and $$\int_0^{\pi} \frac{\sin(2n+1)x}{\sin x}dx = \pi,$$

which agree with the general theorems on writing $f(x) = 1$.

---

* For then we can write

$$x/\sin x = B - \chi(x), \quad f(x) = F(x) - G(x),$$

where $\chi(x)$, $F(x)$, $G(x)$ are positive and never increase in the interval $(0, b)$, while B is a positive constant. Then

$$\phi(x) = \{BF(x) + \chi(x)G(x)\} - \{BG(x) + \chi(x)F(x)\}.$$

Ex. 2. It is instructive to investigate the value of the integral

$$K_n = \int_0^b \frac{\sin(2n+1)x}{\sin x} dx, \quad (0 < b \leqq \tfrac{1}{2}\pi),$$

by means of the curve $y = \{\sin(2n+1)x\}/\sin x$. This curve is of the same general type as the one given in Fig. 46, Art. 169; except that the initial ordinate is $y = 2n+1$ and that the points of crossing the axis are

$$\pi/(2n+1), \quad 2\pi/(2n+1), \quad \ldots, \quad n\pi/(2n+1).$$

Then, using the argument given there, we see that the value of the integral K_n is expressed by a finite series of the form

$$v_0 - v_1 + v_2 - \ldots + (-)^k v_k,$$

where k is an integer such that $(2n+1)b$ lies between $(k-1)\pi$ and $k\pi$, and

$$v_0 > v_1 > v_2 > \ldots > v_k > 0.$$

Hence (if r is any integer less than k), the value of the integral K_n differs from

$$v_0 - v_1 + v_2 - \ldots + (-1)^{r-1} v_{r-1}$$

by less than v_r. Thus, changing the variable to x/ν, K_n lies between

$$\int_0^{r\pi} \frac{\sin x}{\nu \sin(x/\nu)} dx \quad \text{and} \quad \int_0^{(r+1)\pi} \frac{\sin x}{\nu \sin(x/\nu)} dx,$$

where $\nu = 2n+1$. If we make n tend to ∞, we find that the limit of the integral K_n lies between*

$$\int_0^{r\pi} \frac{\sin x}{x} dx \quad \text{and} \quad \int_0^{(r+1)\pi} \frac{\sin x}{x} dx,$$

where r is any positive integer. Thus

$$\lim_{n\to\infty} K_n = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}. \qquad \text{[Dirichlet.]}$$

Ex. 3. It is easy to see (as in Ex. 2 or otherwise) that

$$\lim_{n\to\infty} \int_0^{b_n} \frac{\sin(2n+1)x}{\sin x} dx = \int_0^\lambda \frac{\sin x}{x} dx,$$

where $\lambda = \lim(2nb_n)$. The maxima of this integral are given by $\lambda = \pi, 3\pi, 5\pi, \ldots$ and the minima by $\lambda = 2\pi, 4\pi, 6\pi, \ldots$.

Glaisher (*Phil. Trans.*, vol. 160, 1870, p. 387) has given the following numerical values for the maxima and minima, where

$$I_r = \int_{r\pi}^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} - \int_0^{r\pi} \frac{\sin x}{x} dx,$$

$$I_1 = -0{\cdot}28114, \quad I_3 = -0{\cdot}10397, \quad I_5 = -0{\cdot}06317,$$
$$I_2 = +0{\cdot}15264, \quad I_4 = +0{\cdot}07864, \quad I_6 = +0{\cdot}05276.$$

Thus the greatest value of the integral is $\tfrac{1}{2}\pi - I_1 = 1{\cdot}85194$, and the least (if $\lambda > \pi$) is $\tfrac{1}{2}\pi - I_2 = 1{\cdot}41816$.

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* From the inequalities proved in Art. 59 and in the footnote p. 213, we see that (since $x/\nu < \tfrac{1}{2}\pi$),

$$\left|\frac{x}{\nu} - \sin\frac{x}{\nu}\right| < \frac{1}{6}\frac{x^3}{\nu^3}, \quad \operatorname{cosec}\frac{x}{\nu} < \frac{2\nu}{x}, \quad \left|\frac{\sin x}{x}\right| < 1.$$

Thus $\left|\dfrac{\sin x}{x} - \dfrac{\sin x}{\nu\sin(x/\nu)}\right| = \left|\dfrac{\sin x}{x}\right| \cdot \dfrac{1}{\sin(x/\nu)} \cdot \left|\dfrac{x}{\nu} - \sin\dfrac{x}{\nu}\right| < \dfrac{1}{3}\dfrac{x^2}{\nu^2} \leqq \dfrac{\pi^2}{3}\dfrac{(r+1)^2}{\nu^2}$, in the interval $0 \leqq x \leqq (r+1)\pi$ and so this difference tends to zero uniformly

Ex. 4. If $f(x)$ is positive and steadily decreases in such a way that $\Sigma f(n\pi)$ is convergent, we can prove that

$$\lim_{n\to\infty}\int_0^\infty \frac{\sin(2n+1)x}{\sin x} f(x)dx = \frac{\pi}{2}[f(0)+2f(\pi)+2f(2\pi)+\ldots];$$

and again, if $f(x)$ tends steadily to zero,

$$\lim_{n\to\infty}\int_0^\infty \frac{\sin(2n+1)x}{\sin x}\cos x\, f(x)dx = \frac{\pi}{2}[f(0)-2f(\pi)+2f(2\pi)-\ldots].$$

In particular, if $f(x)=e^{-cx}(c>0)$, the first limit is $\frac{1}{2}\pi\coth(\frac{1}{2}c\pi)$ and the second is $\frac{1}{2}\pi\tanh(\frac{1}{2}c\pi)$.

Ex. 5. Shew that if $f(x)$ satisfies the conditions of Dirichlet's integral, and $0<s<1$, then

$$\lim_{n\to\infty} n^s\int_0^b f(x)\frac{\sin nx}{x^{1-s}}dx = f(0)\int_0^\infty x^{s-1}\sin x\,dx = f(0)\Gamma(s)\sin(\tfrac{1}{2}s\pi),$$

$$\lim_{n\to\infty} n^s\int_0^b f(x)\frac{\cos nx}{x^{1-s}}dx = f(0)\int_0^\infty x^{s-1}\cos x\,dx = f(0)\Gamma(s)\cos(\tfrac{1}{2}s\pi).$$

For the values of the integrals, see Ex. 36, p. 521.

Jordan's extension of Dirichlet's integral.

Suppose that $I_n = \int_0^b \phi(x, n) f(x)dx$, where $f(x)$ is positive and never decreases in the interval $(0, b)$, while $\phi(x, n)$ has the properties

$$\text{(i)}\quad \left|\int_0^\xi \phi(x, n)\,dx\right| < A, \quad \text{if } 0 \leqq \xi \leqq b;$$

$$\text{(ii)}\quad \lim_{n\to\infty}\int_0^\xi \phi(x, n)dx = l, \quad \text{if } 0 < c \leqq \xi \leqq b,$$

where A is a constant and c is arbitrary, but must be regarded as fixed in taking the limit (ii). Under these circumstances

$$\lim I_n = l f(0),$$

where $f(0)$ denotes the limit of $f(x)$, as on p. 491.

For we have from Art. 168, if $0<c<b$,

$$(h-h')[f(0)-f(c)]+h'f(0) < I_n < (H-H')[f(0)-f(c)]+H'f(0),$$

where H, h are the upper and lower limits of $\int_0^\xi \phi(x, n)dx$ as ξ varies from 0 to c and H', h' as ξ varies from c to b. Now, from (i) $\quad |h-h'|<2A, \quad |H-H'|<2A,$

so that $\quad |I_n - lf(0)| < 2A[f(0)-f(c)]+\eta f(0),$

if η is the greater of $H'-l$ and $l-h'$.

Now choose c so that $2A[f(0)-f(c)]<\epsilon$; and having fixed c, make n tend to infinity. Since $\lim_{n\to\infty} \eta=0$ by condition (ii), it follows that

$$\overline{\lim_{n\to\infty}} |I_n - lf(0)| \leqq \epsilon.$$

Hence $$\lim I_n = lf(0).$$

This result can be at once extended to any function $f(x)$ of the type considered in dealing with Dirichlet's integral (see p. 491).

175. Integration of series, when infinities of the integrand occur in the range.

It is obvious that infinite integrals are excluded from the discussion of Art. 45: one case of practical importance presents itself when the terms of the series are of the form $\phi(x) f_n(x)$, where $\phi(x)\to\infty$ at, say, the upper limit b. Then we can easily establish the following result:

A. *If $\Sigma f_n(x)$ converges uniformly in the interval (a, b) and $\int_a^b |\phi(x)|\,dx$ is convergent (and has the value J), then*

$$\int_a^b \phi(x)[\Sigma f_n(x)]\,dx = \Sigma \int_a^b \phi(x) f_n(x)\,dx.$$

For then we can find m, independently of x, so that

$$\left|\sum_m^p f_n(x)\right| < \epsilon, \quad \text{if } p>m.$$

Thus $$\left|\sum_m^p \int_a^b \phi(x) f_n(x)\,dx\right| < \epsilon \int_a^b |\phi(x)|\,dx = \epsilon J.$$

It follows that $\sum_0^\infty \int_a^b \phi(x) f_n(x)\,dx$ converges, and that

$$\left|\sum_m^\infty \int_a^b \phi(x) f_n(x)\,dx\right| \leqq \epsilon J.$$

At the same time we have

$$\left|\sum_0^\infty f_n(x) - \sum_0^{m-1} f_n(x)\right| \leqq \epsilon,$$

so that $$\left|\int_a^b \phi(x)\left[\sum_0^\infty f_n(x)\right]dx - \sum_0^{m-1}\int_a^b \phi(x) f_n(x)\,dx\right| \leqq \epsilon J.$$

Thus we find

$$\left|\int_a^b \phi(x)\left[\sum_0^\infty f_n(x)\right]dx - \sum_0^\infty \int_a^b \phi(x) f_n(x)\,dx\right| \leqq 2\epsilon J,$$

and, since J is fixed, we can determine m to make $2\epsilon J$ as small as we please; but in the last inequality the left-hand side is *independent of m*, and therefore must be zero.

Thus $$\int_a^b \phi(x)\left[\sum_0^\infty f_n(x)\right]dx = \sum_0^\infty \int_a^b \phi(x) f_n(x)\,dx.$$

Ex. 1. This case is illustrated by

$$\int_0^1 \log x \log(1+x)\,dx = \Sigma(-1)^{n-1}\int_0^1 \frac{x^n}{n}\log x\,dx = \Sigma\frac{(-1)^n}{n(n+1)^2}$$

$$= \Sigma(-1)^n\left[\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}\right] = 2 - 2\log 2 - \frac{1}{12}\pi^2.$$

Here the series for $\log(1+x)$ converges uniformly from 0 to 1; but $\log x \to \infty$ as $x \to 0$.

On the other hand, the series may also tend to ∞ (or it may cease to be uniformly convergent) as $x \to b$; when this happens, we can often justify term-by-term integration by means of the theorem:

B. *Suppose that $\phi(x)$ is positive in the interval (a, b), and that the terms $f_n(x)$ are all positive, then the convergence of either the integral*

$$\int_a^b \phi(x)[\Sigma f_n(x)]\,dx,$$

or the series $$\Sigma\int_a^b \phi(x) f_n(x)\,dx,$$

is necessary and sufficient to allow of term-by-term integration.

It is obvious that both conditions are necessary: the only point to be proved is that *either* of them is sufficient.

Write $$F(\delta, m) = \int_a^{b-\delta} \phi(x)\left\{\sum_0^m f_n(x)\right\}dx, \quad (\delta > 0);$$

then, since $\phi(x)$ and $f_n(x)$ are never negative, the function $F(\delta, m)$ *never decreases* as δ tends to zero and m to infinity. Thus, as in Art. 31(5), we see that if either of the repeated limits

$$\lim_{\delta\to 0}\left\{\lim_{m\to\infty} F(\delta, m)\right\},\ \lim_{m\to\infty}\left\{\lim_{\delta\to 0} F(\delta, m)\right\},$$

is convergent, so also is the other, and their values are equal.

Now Art. 45 applies to the interval $(a, b-\delta)$, so that

$$\lim_{m\to\infty} F(\delta, m) = \int_a^{b-\delta}\phi(x)\left\{\sum_0^\infty f_n(x)\right\}dx,$$

and so $$\lim_{\delta\to 0}\left\{\lim_{m\to\infty} F(\delta, m)\right\} = \int_a^b \phi(x)\left\{\sum_0^\infty f_n(x)\right\}dx. \qquad (1)$$

Similarly the other repeated limit is seen to be equal to the series

$$\sum_0^\infty \int_a^b \phi(x) f_n(x)\,dx. \quad \text{.................................(2)}$$

Hence the theorem is established; for if either the integral (1) or the series (2) is convergent, so also is the other, and their values are equal.

Ex. 2. An application of Theorem B is given by the equation

$$\int_0^1 \frac{\log(1-x)}{\sqrt{(1-x)}}dx = -\sum_1^\infty \int_0^1 \frac{x^n\,dx}{n\sqrt{(1-x)}} = -\frac{4}{3}\left(1+\frac{2}{5}+\frac{2.4}{5.7}+\frac{2.4.6}{5.7.9}+\ldots\right).$$

This result is easily verified directly; by integration by parts, the integral is found to be -4, and the sum of the series in brackets is 3 (see Ex. 2, p. 48).

We get another illustration by expanding $1/\sqrt{(1-x)}$ instead of $\log(1-x)$.

Ex. 3. Another example is given by

$$\int_0^1 \frac{x^p}{1-x}\log x\,dx = \sum_1^\infty \int_0^1 x^{n+p-1}\log x\,dx = -\sum_1^\infty \frac{1}{(n+p)^2},$$

where $p+1$ is *positive*. Here we use Theorem A to include $x=0$ in the interval and Theorem B to include $x=1$.

The special case $p=0$ gives

$$\int_0^1 \frac{\log x}{1-x}dx = -\sum_1^\infty \frac{1}{n^2} = -\frac{\pi^2}{6};$$

and if $p=-\frac{1}{2}$ the integral can also be evaluated in finite terms.

C. *When the terms $f_n(x)$ are not all positive, we can apply a similar argument in case either the integral*

$$\int_a^b |\phi(x)|\{\Sigma|f_n(x)|\}\,dx \text{ or the series } \Sigma\int_a^b |\phi|\cdot|f_n|\,dx$$

*converges.** Here we write

$$\phi f_n = \{\phi+|\phi|\}\{f+|f_n|\} - |\phi|\{f+|f_n|\} - |f_n|\{\phi+|\phi|\} + |\phi|\cdot|f_n|,$$

and then, under either of the given conditions, Theorem B can be applied to each term on the right-hand side. Of course these conditions are easily seen to be *sufficient* only and not *necessary* here. Thus, for example, if

$$f_n(x) = (-1)^{n-1}x^n, \quad a=0, \quad b=1,$$

* Hardy, *Messenger of Mathematics*, vol. 35, 1905, p. 126; Bromwich, *ibid.*, vol. 36, 1906, p. 1. It should be noticed that the argument fails if we only know that $\int_a^b \phi(x)[\Sigma f_n(x)]\,dx$ is *absolutely* convergent.

we have

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots=\int_0^1\frac{dx}{1+x}, \quad \text{(see Arts. 47, 52)}$$

although here $\Sigma|f_n(x)|=1/(1-x)$ and $\int_0^1 dx/(1-x)$ diverges.

Ex. 4. To illustrate Theorem C, consider

$$\int_0^1 \frac{x^p}{1+x}\log x\,dx=\sum_1^\infty(-1)^{n-1}\int_0^1 x^{n+p-1}\log x\,dx=\sum_1^\infty\frac{(-1)^n}{(n+p)^2}, \quad p+1>0.$$

Here we take $\phi(x)=x^p\log x$ and $\Sigma|f_n(x)|=1/(1-x)$, and then the conditions of Theorem C are satisfied (compare Ex. 3). In particular, $p=0$ gives

$$\int^1\frac{\log x}{1+x}dx=\sum_1^\infty\frac{(-1)^n}{n^2}=-\frac{\pi^2}{12}.$$

But Theorems B, C do not suffice in a number of comparatively simple cases which present themselves in practice and do not come under any really general theorem. In dealing with power-series, the remark made on p. 151, lines 14-18, is often useful; and in some cases we can apply Theorem C to $\Sigma|a_{n-1}-\lambda a_n|x^n$, taking λ to be $\lim(a_n/a_{n+1})$, and then proceed as in the following example:

Ex. 5. Consider the integral $\int_0^1\frac{x^p\log x}{(1+x)^2}dx$, $\quad p+1>0$.

If $1/(1+x)^2=\Sigma a_nx^n=\Sigma f_n(x)$, $\Sigma|f_n(x)|=1/(1-x)^2$, and Theorem C fails because the integral diverges if $1/(1-x)^2$ is put in place of $1/(1+x)^2$.

Now $a_n=(-1)^n(n+1)$ and $\lambda=-1$. Also $1/(1+x)=\Sigma(a_{n-1}+a_n)x^n$; and by Theorem C,

$$\int_0^1\frac{x^p\log x}{1+x}\Sigma(a_{n-1}+a_n)x^n=\Sigma(a_{n-1}+a_n)\int_0^1\frac{x^{n+p}\log x}{1+x}dx.$$

The coefficient of a_n on the right is

$$\int_0^1 x^{n+p}\log x\,dx=-1/(n+p+1)^2;$$

and so we find

$$\int_0^1\frac{x^p\log x}{(1+x)^2}dx=\Sigma(-1)^{n-1}\frac{n+1}{(n+p+1)^2}.$$

In particular, if $p=0$, the series reduces to $-\log 2$, and it is easily verified that this is then the value of the integral; thus our work is confirmed.

Ex. 6. To illustrate the result given on p. 151, consider the equation

$$\int_0^1\frac{x^p}{1+x}dx=\frac{1}{p+1}-\frac{1}{p+2}+\frac{1}{p+3}-\ldots, \quad (p+1>0);$$

this is valid, because the resulting series converges.

Ex. 7. Further applications of Theorems B, C are given by the equations

$$\left.\begin{aligned}\int_0^1 \frac{(\log x)^{2r-1}}{1-x}\,dx &= -(2r-1)!\left(1+\frac{1}{2^{2r}}+\frac{1}{3^{2r}}+\dots\right) = -\frac{2^{2r-2}}{r}B_r\pi^{2r},\\ \int_0^1 \frac{(\log x)^{2r-1}}{1+x}\,dx &= -(2r-1)!\left(1-\frac{1}{2^{2r}}+\frac{1}{3^{2r}}-\dots\right) = -\frac{2^{2r-1}-1}{2r}B_r\pi^{2r},\end{aligned}\right\}\text{Art. 100,}$$

$$\int_0^1 \frac{dx}{x}\log\frac{1+x}{1-x} = 2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right) = \frac{\pi^2}{4}.$$

From the last integral we can obtain the results

$$\int_0^\infty \frac{dx}{x}\log\left(\frac{x+y}{x-y}\right)^2 = \pi^2, \qquad \text{if } y>0,$$

$$\text{or} = 0, \qquad \text{if } y = 0,$$

$$\text{or} = -\pi^2, \quad \text{if } y<0.$$

Ex. 8. By changing the variable in Ex. 6 from x to t, where $x=e^{-2at}$, we find

$$\int_0^\infty \frac{\cosh(bt)}{\cosh(at)}dt = \frac{\pi}{2a}\sec\left(\frac{\pi b}{2a}\right), \quad 0<b<a.$$

Similarly, starting from the equation

$$\int_0^1 \frac{1-x^{p-1}}{1-x}\,dx = 1-\frac{1}{p}+\frac{1}{2}-\frac{1}{p+1}+\dots, \qquad (p>0)$$

which can be established by Theorem B, we find

$$\int_0^\infty \frac{\sinh(bt)}{\sinh(at)}dt = \frac{\pi}{2a}\tan\left(\frac{\pi b}{2a}\right), \quad 0<b<a.$$

176. Integration of an infinite series over an infinite interval.

The method of proof employed in Art. 45 does not justify the deduction of the equation

$$\int_a^\infty \phi(x)\left[\sum_0^\infty f_n(x)\right]dx = \sum_0^\infty \int_a^\infty \phi(x) f_n(x)\,dx,$$

from the knowledge that $\Sigma f_n(x)$ is uniformly convergent for all values of x greater than a.

A. However, if in addition we know that $\int_a^\infty |\phi(x)|\,dx$ is convergent, the method used to prove Theorem A of Art. 175 can be at once modified to establish the desired result.

But it is often necessary to justify the equation when either $\int_a^\infty |\phi(x)|\,dx$ is divergent or $\Sigma f_n(x)$ can only be proved to converge uniformly over a fixed interval;* and then some new test must be introduced.

Thus, for example, if

$$f_0(x)+f_1(x)+\dots+f_n(x)=S_n(x)=(2x/n^2)e^{-x^2/n^2}, \lim_{n\to\infty} S_n(x)=0;$$

and the maximum of $S_n(x)$ is $\sqrt{2}/n\sqrt{e}$, so that $S_n(x)$ converges uniformly to its limit in any interval for x. But yet we find, taking $\phi(x)=1$,

$$\int_0^\infty S_n(x)\,dx=1\,; \text{ so that } \lim_{n\to\infty}\int_0^\infty S_n(x)\,dx=1,$$

and this is not the same as $\int_0^\infty [\lim_{n\to\infty} S_n(x)]\,dx$.

This illustrates the case when $\int_a^\infty |\phi(x)|\,dx$ diverges (because $\phi(x)=1$); the other difficulty arises in the integration of series such as the exponential series $\Sigma x^n/n!$, which converges uniformly in any fixed interval (which may be arbitrarily great) but does not converge uniformly in an infinite interval.

B. Many cases of practical importance are covered by the following test:

If $\Sigma f_n(x)$ converges uniformly in any fixed interval $a \leqq x \leqq b$, where b is arbitrary, and if $\phi(x)$ is continuous for all finite values of x, then

$$\int_a^\infty \phi(x)[\Sigma f_n(x)]\,dx=\Sigma\int_a^\infty \phi(x)f_n(x)dx\,;$$

provided that either the integral $\int_a^\infty |\phi(x)|\{\Sigma|f_n(x)|\}\,dx$ or the series $\Sigma\int_a^\infty |\phi(x)|\,.\,|f_n(x)|\,dx$ is convergent.

For, by means of the identity

$$\phi f_n=\{\phi+|\phi|\}\{f_n+|f_n|\}-|\phi|\,.\{f_n+|f_n|\}-|f_n|\,.\{\phi+|\phi|\}+|f_n|\,.\,|\phi|,$$

we can at once reduce this theorem to the case in which ϕ and f_n are never negative.

In this case the function

$$F(\lambda, \mu)=\int_a^\lambda \phi(x)[\sum_0^\mu f_n(x)]\,dx$$

never decreases as λ, μ increase; and consequently we can repeat the

* The distinction between uniform convergence over a *fixed* and over an *infinite* interval may be illustrated by the two examples $S_n(x)=x/n$ and $S_n(x)=1/(x+n)$. The former converges uniformly to zero in any interval $(0, b)$, where b is *fixed*, but may be taken arbitrarily great; the latter converges uniformly to zero for *all* positive values of x.

arguments given in Art. 31 (5) to prove that if $\lim_{\lambda\to\infty}(\lim_{\mu\to\infty} F(\lambda, \mu))$ exists, so also does the other repeated limit, and the two limits are equal.

But, in virtue of the uniform convergence of $\Sigma f_n(x)$, we have

$$\lim_{\mu\to\infty} F(\lambda, \mu) = \int_a^\lambda \phi(x)[\sum_0^\infty f_n(x)]dx,$$

so that $$\lim_{\lambda\to\infty}\{\lim_{\mu\to\infty} F(\lambda, \mu)\} = \int_a^\infty \phi(x)[\sum_0^\infty f_n(x)]\,dx.$$

The other repeated limit is seen in the same way to be

$$\sum_0^\infty \int_a^\infty \phi(x) f_n(x) dx,$$

and so the test is established.

Ex. 1. Consider $$\int_0^\infty \frac{\sin(bx)}{e^{ax}-1}dx,$$

where a is positive, and $b = p + iq$, where $|q| = s < a$; since

$$|\sin(bx)| = [\sinh^2(qx) + \sin^2(px)]^{\frac{1}{2}} < \cosh sx < e^{sx}$$

and the integral $$\int_0^\infty [e^{sx}/(e^{ax}-1)]\,dx$$

is convergent, it follows from Theorem **B** that term-by-term integration is permissible,* because the terms in the series

$$1/(e^{ax}-1) = e^{-ax} + e^{-2ax} + e^{-3ax} + \dots$$

are all *positive*. Thus we have

$$\int_0^\infty \frac{\sin(bx)}{e^{ax}-1}dx = \frac{b}{a^2+b^2} + \frac{b}{(2a)^2+b^2} + \frac{b}{(3a)^2+b^2} + \dots.$$

In the case when $a = 2\pi$, this expression is equal (by Art. 100) to

$$\frac{1}{2}\left(\frac{1}{e^b-1} - \frac{1}{b} + \frac{1}{2}\right),$$

and so in general it is equal to

$$\frac{\pi}{a}\left(\frac{1}{e^{2\pi b/a}-1} - \frac{a}{2\pi b} + \frac{1}{2}\right).$$

Ex. 2. In like manner we prove that

$$\int_0^\infty \frac{\sin(bx)}{e^{ax}+1}dx = \frac{b}{a^2+b^2} - \frac{b}{(2a)^2+b^2} + \frac{b}{(3a)^2+b^2} - \dots$$

$$= \frac{1}{2}\left[\frac{1}{b} - \frac{\pi}{a\sinh(\pi b/a)}\right],$$

by Art. 99.

---

* Note that exactly the same argument enables us to include 0 in the range of integration, although the series diverges there.

Ex. 3. Taking the case $a=2\pi$, expand both sides of Ex. 1 in powers of b. In this case the application of the theorem depends upon the integral

$$\int_0^\infty \frac{\sinh\{|b|x\}}{e^{2\pi x}-1}dx,$$

which converges if $|b|<2\pi$. Thus we find

$$\int_0^\infty \frac{x^{2r-1}dx}{e^{2\pi x}-1}=\frac{B_r}{4r};$$

see Art. 100 and compare Art. 175, Ex. 7.

Ex. 4. Similarly, by expanding $\sin(bx)$ in powers of x, we find that if $0<b<a$,

$$\int_0^\infty e^{-ax}\sin(bx)dx=\frac{b}{a^2}\left(1-\frac{b^2}{a^2}+\frac{b^4}{a^4}+\dots\right)=\frac{b}{a^2+b^2}.$$

And without restriction on b, we have from the values of $\Gamma(\frac{1}{2})$, $\Gamma(\frac{3}{2})$, ...,

$$\int_0^\infty e^{-ax^2}\cos(2bx)dx=\frac{\sqrt{\pi}}{2\sqrt{a}}\left(1-\frac{b^2}{a}+\frac{1}{2!}\frac{b^4}{a^2}-\dots\right)=\frac{\sqrt{\pi}}{2\sqrt{a}}\exp\left(-\frac{b^2}{a}\right).$$

C. However, Theorem B does not cover all cases which are required. For example, it is not hard to see that the series

$$\sum_1^\infty \frac{\sin(nx)}{n^p x}, \quad (p>1)$$

can be integrated term-by-term between the limits 1 and ∞, although the test given above fails.* This case and others are covered by the following test:

Write $\int_a^x f_n(x)dx=g_n(x)$ *and suppose that the series* $\Sigma f_n(x)$ *converges uniformly in any fixed interval* (a, b), *while the series* $\Sigma g_n(x)$ *converges uniformly in an infinite interval* $(x \geqq a)$; *then*

(1) $\Sigma\left[\int_a^\infty f_n(x)\,dx\right]$ *converges,*

(2) $\int_a^\infty [\Sigma f_n(x)]\,dx$ *converges,*

(3) *the values of* (1) *and* (2) *are equal.*

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For, by Art. 45, we have

$$\int_a^x [\Sigma f_n(x)]\,dx=\Sigma g_n(x).$$

And, since $\Sigma g_n(x)$ is uniformly convergent, we have (Art. 45)

$$\int_a^\infty [\Sigma f_n(x)]\,dx=\lim_{x\to\infty}[\Sigma g_n(x)]=\Sigma \lim_{x\to\infty} g_n(x)=\Sigma\left[\int_a^\infty f_n(x)dx\right].$$

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* See the second paper quoted on p. 497.

177. The inversion of a repeated infinite integral.

It is by no means easy to determine fairly general conditions under which the equation*

$$(1) \qquad \int_a^\infty dx \int_b^\infty f(x, y)\,dy = \int_b^\infty dy \int_a^\infty f(x, y)\,dx$$

is correct.

Here we shall simply consider the easiest case, when either $f(x, y)$ is *positive* or else the integrals still converge when $f(x, y)$ is replaced by $|f(x, y)|$.

Let us write

$$F(\lambda, \mu) = \int_a^\lambda dx \int_b^\mu f(x, y)\,dy = \int_b^\mu dy \int_a^\lambda f(x, y)\,dx,$$

this equation being valid (see p. 457) if, as we suppose, $f(x, y)$ is continuous for all finite values of x, y (or at least for all such as come under consideration). Further, write

$$\phi(x, \mu) = \int_b^\mu f(x, y)\,dy, \quad \psi(x) = \lim_{\mu \to \infty} \phi(x, \mu) = \int_b^\infty f(x, y)\,dy,$$

assuming the convergence of the last integral. Let the interval (a, λ) be subdivided by continued bisection into n sub-intervals, each of length l, and let $h_r(\mu)$ denote the minimum of $\phi(x, \mu)$ in the rth interval; then, as in Art. 162, we have

$$F(\lambda, \mu) = \lim_{n \to \infty} \sum_{r=1}^{n} l h_r(\mu).$$

Now this sum cannot decrease as n and μ tend to infinity;† and so we may use theorem (5), Art. 31, which gives

$$(2) \qquad \lim_{\mu \to \infty} \left\{ \lim_{n \to \infty} \sum_1^n l h_r(\mu) \right\} = \lim_{n \to \infty} \left\{ \lim_{\mu \to \infty} \sum_1^n l h_r(\mu) \right\},$$

provided that *one* of these limits converges. Thus

$$(3) \qquad \lim_{\mu \to \infty} F(\lambda, \mu) = \lim_{n \to \infty} \sum_1^n l k_r, \quad \text{if } k_r = \lim_{\mu \to \infty} h_r(\mu).$$

Now we shall prove below (see the small type, p. 504) that

* For wider conditions, see a paper in the *Proc. Lond. Math. Soc.* (2), vol. 1, 1903, p. 187, and other papers quoted there. Reference may also be made to Gibson's *Calculus*, Ch. XXI. (2nd ed.), and Jordan's *Cours d'Analyse*, t. 2, §§ 71, 72.

† As regards n, see the argument of Art. 162; and $\phi(x, \mu)$ increases with μ (because $f(x, y)$ is not negative), so that the same is true of $h_r(\mu)$.

k_r is the minimum of $\psi(x)=\lim\phi(x, \mu)$ in the rth interval; and so, using Art. 162 again, we have

$$(4) \qquad \lim_{n\to\infty}\sum_1^n lk_r=\int_a^\lambda \psi(x)dx.$$

Since the integral in (4) is supposed convergent (otherwise equation (1) would be obviously meaningless), the equation (4) shews that the right-hand limit in (2) exists; and so the assumption made above is justified. From the equations (3) and (4) we see that*

$$\int_b^\infty dy\int_a^\lambda f(x, y)dx=\int_a^\lambda dx\int_b^\infty f(x, y)dy.$$

From (3) and (4) it is also clear that

$$\int_a^\infty dx\int_b^\infty f(x, y)dy=\lim_{\lambda\to\infty}\{\lim_{\mu\to\infty}F(\lambda, \mu)\};$$

and similarly, we find that the second integral in (1) is equal to the repeated limit of $F(\lambda, \mu)$ taken in the reverse order.

Now $F(\lambda, \mu)$ cannot decrease, as λ and μ increase, so that we can again apply theorem (5) of Art. 31; and we obtain de la Vallée Poussin's theorem:

Equation (1) *above is correct, provided that* **both** *the integrals*

$$\int_a^\infty f(x, y)dx, \quad \int_b^\infty f(x, y)dy$$

are convergent, and that **either** *of the repeated integrals converges.*

It will be seen that (by using $f+|f|$ in place of f) we can extend the theorem to cases when f changes sign, provided that the integrals all remain convergent when $|f|$ is put in place of f.

We have still to prove that *if $h(\mu)$ is the minimum of $\phi(x, \mu)$ in any interval $(p\leqq x\leqq q)$, then $h(\mu)$ tends to a limit k, which is the minimum of $\psi(x)$ in the same interval.*

From the definition of $h(\mu)$, we have

$$\phi(x, \mu)\geqq h(\mu),$$

and so, on making μ tend to infinity, we find

$$(5) \qquad \psi(x)\geqq k.$$

If it happens that $\phi(p, \mu)\leqq k$, it is evident that $\psi(p)\leqq k$, also; and so we see from (5) that $\psi(p)=k$, and consequently k is the minimum of $\psi(x)$ in the interval (p, q).

* Note that we do *not* use any condition of uniform convergence, as in Art. 172 (2); instead, we have the condition that f is nowhere negative.

But if $\phi(p, \mu) > k \geqq h(\mu)$, it follows from Ex. 17, p. 435, that the equation $\phi(x, \mu) = k$ has at least one root in the interval (p, q); let ξ_μ denote the *least* root.* Then, if $\nu > \mu$, we have†

$$\phi(\xi_\mu, \nu) \geqq \phi(\xi_\mu, \mu) = k,$$

so that $\xi_\nu \geqq \xi_\mu$, and ξ_ν therefore tends to a limit ξ as ν tends to infinity. Again

$$\phi(\xi_\nu, \mu) \leqq \phi(\xi_\nu, \nu) = k, \quad (\nu > \mu),$$

so that, on making ν tend to infinity, we have

$$\phi(\xi, \mu) \leqq k.$$

Thus $\psi(\xi) \leqq k$, and so from (5) we find that $\psi(\xi) = k$, which is therefore again the minimum of $\psi(x)$.

Ex. As an application we shall establish the equations

$$\int_0^\infty e^{-yt}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)dt = 2\int_0^\infty \frac{x\,dx}{(x^2+y^2)(e^{2\pi x}-1)},$$

$$\int_0^\infty \frac{e^{-yt}}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)dt = 2\int_0^\infty \frac{\arctan(x/y)}{e^{2\pi x}-1}dx,$$

where the real part of y is positive and the arc tan function is determined so as to vanish with x.

We have seen (Ex. 1, Art. 176) that

$$\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} = 2\int_0^\infty \frac{\sin(xt)}{e^{2\pi x}-1}dx,$$

and therefore

$$\int_0^\infty e^{-yt}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)dt = 2\int_0^\infty e^{-yt}\,dt \int_0^\infty \frac{\sin(xt)}{e^{2\pi x}-1}dx.$$

Now the last integral is absolutely convergent, since

$$\int_0^\infty \frac{|\sin(xt)|}{e^{2\pi x}-1}dx < \int_0^\infty \frac{xt\,dx}{e^{2\pi x}-1} = \frac{t}{24}, \quad \text{(Ex. 3, Art. 176)},$$

and

$$|e^{-yt}| = e^{-\xi t}, \quad \text{if } y = \xi + i\eta.$$

Thus

$$2\int_0^\infty |e^{-yt}|\,dt \int_0^\infty \frac{|\sin(xt)|}{e^{2\pi x}-1}dx < \frac{1}{12\xi^2},$$

which proves the absolute convergence; we can therefore invert the order of integration without altering the value of the integral, and we then find

$$\int_0^\infty e^{-yt}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)dt = 2\int_0^\infty \frac{x\,dx}{(x^2+y^2)(e^{2\pi x}-1)}.$$

Now, if we write $y = \xi + i\eta$ in the last equation, we can integrate with respect to ξ under the integral sign, between ξ_0 and ∞; for

$$\left|e^{-yt}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\right| < \frac{1}{12}te^{-\xi t},$$

and so

$$\int_{\xi_0}^\infty d\xi \int_0^\infty \left|e^{-yt}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\right| dt < \frac{1}{12\xi_0}, \quad (\xi_0 > 0)$$

* If the equation has an infinite set of roots, the limiting values of the set are also roots (because ϕ is continuous); and so the set *attains* its lower limit, which is therefore the least root.

† The reader is advised to use a figure in following the argument here.

so that this double integral is absolutely convergent. Similarly we find that the right-hand side is absolutely convergent, since $|x^2+y^2|\geqq \xi^2$, so that

$$\int_{\xi_0}^{\infty} d\xi \int_0^{\infty} \frac{x\,dx}{|x^2+y^2|\,.\,(e^{2\pi x}-1)} \leqq \int_{\xi_0}^{\infty} \frac{d\xi}{\xi^3} \int_0^{\infty} \frac{x\,dx}{e^{2\pi x}-1} = \frac{1}{24\xi_0}.$$

Thus we find the further equation

$$\int_0^{\infty}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt \int_{\xi_0}^{\infty} e^{-yt}\,d\xi = 2\int_0^{\infty}\frac{dx}{e^{2\pi x}-1}\int_{\xi_0}^{\infty}\frac{x\,d\xi}{(x^2+y^2)},$$

which gives $$\int_0^{\infty}\frac{e^{-y_0 t}}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt = 2\int_0^{\infty}\frac{\arc\tan (x/y_0)}{e^{2\pi x}-1}\,dx,$$

where $y_0=\xi_0+i\eta$.

178. The Gamma-integral.

In Art. 42 we have seen that

$$\Gamma(1+x) = \lim_{n\to\infty} \frac{n^x\,.\,n!}{(1+x)(2+x)\ldots(n+x)}.$$

We shall now express this function by means of an infinite integral when α, the real part of $1+x$, is positive.

Write $$I_s=\int_0^1 y^{x+s}(1-y)^{n-s}\,dy,$$

then, using the method of integration by parts, we find that

$$I_s/I_{s+1}=(n-s)/(1+s+x),$$

and so $$I_0/I_n=n!/\{(1+x)(2+x)\ldots(n+x)\}.$$

But $$I_n=1/(n+1+x),$$

so that $$\Gamma(1+x)=\lim n^{x+1} I_0,$$

or, changing the variable by writing $t=ny$, we have

$$\Gamma(1+x)=\lim_{n\to\infty}\int_0^n\left(1-\frac{t}{n}\right)^n t^x\,dt.$$

We can apply Tannery's theorem (Art. 172 (4)) to the last integral; for we have*

$$|e^{-t}t^x-(1-t/n)^n t^x| < t^{1+\alpha}/(2n),$$

and so (since α is positive) the integrand converges to the limit $e^{-t}t^x$ uniformly in any *fixed* interval for t. Further, we have

$$|(1-t/n)^n t^x| < e^{-t}t^{\alpha-1},$$

and the integral $\int_0^{\infty} e^{-t}t^{\alpha-1}\,dt$ is convergent, because α is positive.

---

* Actually $1-e^t\left(1-\frac{t}{n}\right)^n=\int_0^t e^v\left(1-\frac{v}{n}\right)^{n-1}\frac{v}{n}\,dv$, so that, when t is positive, $e^{-t}-(1-t/n)^n$ is positive and less than $t^2/(2n)$.

Thus all the conditions are satisfied, and so we find

$$\Gamma(1+x)=\int_0^\infty e^{-t}t^x\,dt.$$

A somewhat similar integral can be found for Euler's constant; we have seen (Art. 11) that

$$C=\lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right).$$

But $$1+\frac{1}{2}+\dots+\frac{1}{n}=\int_0^1(1+x+x^2+\dots+x^{n-1})\,dx=\int_0^1\frac{1-x^n}{1-x}\,dx.$$

Thus we find, on writing $x=1-t/n$,

$$1+\frac{1}{2}+\dots+\frac{1}{n}=\int_0^n\left[1-\left(1-\frac{t}{n}\right)^n\right]\frac{dt}{t}.$$

And $$\log n=\int_1^n\frac{dt}{t};$$

hence $$C=\lim_{n\to\infty}\left[\int_0^1\left\{1-\left(1-\frac{t}{n}\right)^n\right\}\frac{dt}{t}-\int_1^n\left(1-\frac{t}{n}\right)^n\frac{dt}{t}\right],$$

and, by the same method as before, we obtain as the limit

$$C=\int_0^1(1-e^{-t})\frac{dt}{t}-\int_1^\infty e^{-t}\frac{dt}{t}=-\lim_{\delta\to0}\left[\log\delta+\int_\delta^\infty e^{-t}\frac{dt}{t}\right].$$

Since $$\int_\delta^\infty\frac{dt}{t(1+t)}=\log\frac{1+\delta}{\delta},$$

we see that $$C=\lim_{\delta\to0}\left[\int_\delta^\infty\left(\frac{1}{1+t}-e^{-t}\right)\frac{dt}{t}-\log(1+\delta)\right]$$
$$=\int_0^\infty\left(\frac{1}{1+t}-e^{-t}\right)\frac{dt}{t}.$$

Another form is easily obtained by changing the variable from t to $1/t$ in the integral $\int_1^\infty e^{-t}\,dt/t$; this gives

$$C=\int_0^1(1-e^{-t}-e^{-1/t})\frac{dt}{t}.$$

A number of definite integrals for C can be obtained from the expression

$$-\lim_{\delta\to0}\left[\log\delta+\int_\delta^\infty e^{-t}\frac{dt}{t}\right].$$

Amongst them are the following:

$$C=\int_0^\infty\left(\frac{1}{1+t^2}-e^{-t}\right)\frac{dt}{t}=\int_0^\infty\left(\frac{1}{e^t-1}-\frac{e^{-t}}{t}\right)dt=\int_0^\infty e^{-t}\log\frac{1}{t}\,dt.$$

It is easy to see from Art. 180 below that

$$\Gamma'(1)=\int_0^\infty\frac{dt}{t}\left(e^{-t}-\frac{t}{e^t-1}\right)=-C.$$

Useful properties of the Gamma-function.

1. When x is a positive integer, we can write

$$\Gamma(1+x)=\lim_{n\to\infty}\frac{n!\,.\,n^x}{(1+x)(2+x)\dots(n+x)}=\lim_{n\to\infty}\frac{x!\,.\,n^x}{(1+n)(2+n)\dots(x+n)}=x!;$$

a result which is also easily obtained from the definite integral, using the method of integration by parts.

2. $$\Gamma(x)\Gamma(1-x) = \lim \frac{n!\,n^{x-1}}{x(1+x)\ldots(n-1+x)} \cdot \frac{n!\,n^{-x}}{(1-x)(2-x)\ldots(n-x)}$$

$$= \lim \frac{n+x}{n} \frac{(n!)^2}{x(1-x^2)(2^2-x^2)\ldots(n^2-x^2)}$$

$$= \lim \left[x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots\left(1-\frac{x^2}{n^2}\right)\right]^{-1}$$

or $$\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x). \quad \text{(Art. 98.)}$$

3. Writing $x=\frac{1}{2}$ in the last result, we find, since $\Gamma(\frac{1}{2})$ is positive,

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

4. $$\Gamma(x+\tfrac{1}{2})\Gamma(x+1) = \lim \frac{(n!)^2\,n^{2x-\frac{1}{2}}}{(\frac{1}{2}+x)(1+x)(\frac{3}{2}+x)\ldots(n+x)}$$

$$= \lim \frac{(n!)^2\,n^{2x-\frac{1}{2}}\,2^{2n}}{(1+2x)(2+2x)\ldots(2n+2x)}.$$

Also $$\Gamma(2x+1) = \lim \frac{(2n)!\,(2n)^{2x}}{(1+2x)(2+2x)\ldots(2n+2x)}.$$

Thus $$2^{2x}\frac{\Gamma(x+\frac{1}{2})\Gamma(x+1)}{\Gamma(2x+1)} = \lim \frac{(n!)^2\,2^{2n}}{(2n)!\,n^{\frac{1}{2}}}.$$

Since this last expression does not contain x, we can find its value by putting $x=0$; this gives $\Gamma(\frac{1}{2})$ or $\sqrt{\pi}$, a result which can also be obtained by appealing directly to the definition.

179. Stirling's asymptotic formula for the Gamma-function when x is real, large and positive.

In the integral

$$\Gamma(1+x) = \int_0^\infty e^{-t}t^x\,dt,$$

the maximum of the integrand is $e^{-x}x^x$ and occurs for $t=x$, so if we write

$$e^{-t}t^x = (e^{-x}x^x)\,e^{-y^2},$$

the range of values $(-\infty, 0, +\infty)$ for y will correspond precisely to the range $(0, x, \infty)$ for t.

Thus $$\Gamma(1+x) = e^{-x}x^x\int_{-\infty}^{\infty} e^{-y^2}\frac{dt}{dy}\,dy.$$

Now, taking logarithms, we have

$$y^2 = (t-x) - x\log(t/x)$$

so that $$2y\frac{dy}{dt} = 1 - \frac{x}{t}.$$

But the properties of the logarithmic function shew that y^2 lies between*

$$\frac{x}{2}\left(\frac{t-x}{x}\right)^2 \quad\text{and}\quad \frac{x}{2}\left(\frac{t-x}{t}\right)^2.$$

Thus, since y has the same sign as $t-x$, we see that y lies between

$$\left(\frac{x}{2}\right)^{\frac{1}{2}}\frac{t-x}{x} \quad\text{and}\quad \left(\frac{x}{2}\right)^{\frac{1}{2}}\frac{t-x}{t}.$$

Thus, $(x/2)^{\frac{1}{2}}(1/y)$ lies between

$$x/(t-x) \quad\text{and}\quad t/(t-x).$$

And therefore, since $t/(t-x)=1+x/(t-x)$, we see that $t/(t-x)$ must lie between

$$\frac{1}{y}\left(\frac{x}{2}\right)^{\frac{1}{2}} \quad\text{and}\quad 1+\frac{1}{y}\left(\frac{x}{2}\right)^{\frac{1}{2}}.$$

Hence $\dfrac{dt}{dy}=\dfrac{2ty}{t-x}$ lies between

$$(2x)^{\frac{1}{2}} \quad\text{and}\quad 2y+(2x)^{\frac{1}{2}}.$$

Accordingly, we have

$$\Gamma(1+x)=e^{-x}x^x\int_{-\infty}^{\infty}e^{-y^2}[(2x)^{\frac{1}{2}}+\xi]\,dy,$$

where $|\xi|<2|y|$.

Now† $$\int_{-\infty}^{+\infty}e^{-y^2}dy=\pi^{\frac{1}{2}},\quad \int_{-\infty}^{\infty}|y|\,e^{-y^2}dy=1,$$

and accordingly

$$\left|\frac{\Gamma(1+x)}{e^{-x}x^x(2\pi x)^{\frac{1}{2}}}-1\right|<\frac{2}{(2\pi x)^{\frac{1}{2}}}.$$

Hence $$\lim_{x\to\infty}\frac{\Gamma(1+x)}{e^{-x}x^x(2\pi x)^{\frac{1}{2}}}=1,$$

or, as we may write it,

$$\Gamma(1+x)\sim e^{-x}x^x(2\pi x)^{\frac{1}{2}}.$$

---

* We have, if $(t-x)/x=\tau,\quad y^2=x\int_0^{\tau}\theta\,d\theta/(1+\theta),$
which obviously lies between $\frac{1}{2}x\tau^2$ and $\frac{1}{2}x\tau^2/(1+\tau)^2$.

† $\int_{-\infty}^{\infty}e^{-y^2}dy=2\int_0^{\infty}e^{-y^2}dy=\int_0^{\infty}e^{-x}x^{-\frac{1}{2}}dx=\Gamma(\frac{1}{2})=\pi^{\frac{1}{2}}$. (Art. 178 (3).)

Again, we see that

$$|\log \Gamma(1+x) - \log\{e^{-x}x^x(2\pi x)^{\frac{1}{2}}\}| < 2/[(2\pi x)^{\frac{1}{2}} - 2],$$

so that $\log \Gamma(1+x) \sim (x+\frac{1}{2}) \log x - x + \frac{1}{2} \log (2\pi),$

using the symbol $\sim$ in the extended sense explained in Art. 113.

If we subtract $\log x$, we obtain

$$\log \Gamma(x) \sim (x-\tfrac{1}{2}) \log x - x + \tfrac{1}{2} \log (2\pi).$$

The foregoing method is due to Liouville, who gave it in his *Journal de Mathématiques* (t. 11, 1846, p. 464).

Ex. Consider the value of

$$\phi(x) = n^{nx}\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\dots\Gamma\left(x+\frac{n-1}{n}\right)\Big/\Gamma(nx),$$

where n is a positive integer.

If we change x to $x+1$, we see that

$$\phi(x+1)/\phi(x) = n^n\left[x\left(x+\frac{1}{n}\right)\dots\left(x+\frac{n-1}{n}\right)\right]\Big/nx(nx+1)\dots(nx+n-1) = 1.$$

Hence $\phi(x) = \phi(x+1) = \phi(x+2) = \dots = \phi(x+s).$

But when y is large, $\Gamma(y+a) \sim \Gamma(y) \cdot y^a$ (Art. 42), so that

$$\phi(y) \sim n^{ny}y^{\frac{1}{2}(n-1)}[\Gamma(y)]^n/\Gamma(ny),$$

or, using the asymptotic formula above,

$$\phi(y) \sim n^{ny}y^{\frac{1}{2}(n-1)}[e^{-ny}y^{ny}(2\pi/y)^{\frac{1}{2}n}][e^{ny}(ny)^{-ny}(2\pi/ny)^{-\frac{1}{2}}].$$

Hence, as y tends to infinity, $\phi(y)$ tends to the limit $(2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}}$, and we have already proved that $\phi(x) = \phi(x+s)$, where s is an arbitrarily great positive integer, so that we must have $\phi(x) = (2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}}$.

The special case $n=2$ has been discussed in Art. 178 (4).

180. Integrals for $\log \Gamma(1+x)$.

We have proved (Ex. 1, Art. 173) that if the real parts of a, b are positive,

$$\log\frac{b}{a} = \int_0^\infty \frac{dt}{t}(e^{-at} - e^{-bt}).$$

Hence, *if the real part of* $1+x$ *is positive*, we have

$$\log\left(\frac{r}{r+x}\right) = \int_0^\infty \frac{dt}{t}(1-e^{-xt})e^{-rt}. \qquad (r = 1, 2, 3, \dots)$$

$$\text{Now } \log \Gamma(1+x) = \lim_{n\to\infty} \log \frac{n^x n!}{(1+x)(2+x)\dots(n+x)}$$

$$= \lim_{n\to\infty}\left[x \log n + \sum_1^n \log\left(\frac{r}{r+x}\right)\right].$$

Thus we are led to consider the function

$$S(x, n) = x \log n + \sum_1^n \log\left(\frac{r}{r+x}\right)$$

$$= x\int_0^\infty (e^{-t} - e^{-nt})\frac{dt}{t} + \sum_1^n \int_0^\infty (1-e^{-xt})e^{-rt}\frac{dt}{t}.$$

Now $\displaystyle\sum_1^n e^{-rt} = e^{-t}(1-e^{-nt})/(1-e^{-t}) = (1-e^{-nt})/(e^t-1),$

so that $\displaystyle S(x, n) = \int_0^\infty \left(xe^{-t} - \frac{1-e^{-xt}}{e^t-1}\right)\frac{dt}{t} - \int_0^\infty e^{-nt}\left(x - \frac{1-e^{-xt}}{e^t-1}\right)\frac{dt}{t}$

$$= F(x) + G(x, n), \text{ say.}$$

It is to be observed that both in $F(x)$ and in $G(x, n)$ the integrands are finite at $t=0$.

For if $t < 2\pi$, we can write (Art. 100)

$$\frac{1-e^{-xt}}{e^t-1} = x - \tfrac{1}{2}(x+x^2)t + \dots,$$

so that $\displaystyle \frac{1}{t}\left(x - \frac{1-e^{-xt}}{e^t-1}\right) = \tfrac{1}{2}(x+x^2) + X_1 t + X_2 t^2 + \dots,$

and similarly for the other integrand.

Thus, when $t<1$, $\displaystyle \frac{1}{t}\left|x - \frac{1-e^{-xt}}{e^t-1}\right|$ cannot exceed some fixed value, independent of t; but if $t>1$, this expression is less than $|x| + \dfrac{e+1}{e-1}$, because $|e^{-xt}| < e^t$ (since the real part of $1+x$ is positive). Thus we can determine a value X, independent of t, such that

$$\left|\frac{1}{t}\left(x - \frac{1-e^{-xt}}{e^t-1}\right)\right| < X.$$

Then $\displaystyle |G(x, n)| < \int_0^\infty X e^{-nt}\,dt$

$$< X/n,$$

or $\displaystyle \lim_{n\to\infty} G(x, n) = 0.$

so that

Hence $\displaystyle \log\Gamma(1+x) = \lim_{n\to\infty} S(x, n) = F(x)$

$$= \int_0^\infty \frac{dt}{t}\left(xe^{-t} - \frac{1-e^{-xt}}{e^t-1}\right).$$

This integral can be divided into two parts, and we find

$$\log \Gamma(1+x) = \phi(x) + \psi(x),$$

where $$\phi(x) = \int_0^\infty \left[xe^{-t} - \frac{1}{e^t - 1} + \left(\frac{1}{t} - \frac{1}{2}\right)e^{-xt}\right]\frac{dt}{t}$$

and $$\psi(x) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)e^{-xt}\frac{dt}{t} = 2\int_0^\infty \frac{\arctan(y/x)}{e^{2\pi y} - 1}\,dy,$$

the last expression following from the example of Art. 177. The advantage of this transformation is due to two facts, first that the value of $\phi(x)$ can be found in terms of elementary functions; and secondly that $\psi(x)$ tends to zero if $|x|$ tends to ∞ in such a way that the real part of x also tends to ∞.

For, in the course of the example of Art. 177, we proved that

$$|\psi(x)| < 1/12\xi,$$

where ξ is the real part of x. Thus when ξ tends to ∞, we have

$$\lim \psi(x) = 0.$$

The limit is also 0, when η tends to ∞, ξ being kept positive (see Ex. 56, p. 525).

As regards $\phi(x)$, we have

$$\phi(x) - \phi(1) = \int_0^\infty \left[(x-1)e^{-t} + \left(\frac{1}{t} - \frac{1}{2}\right)(e^{-xt} - e^{-t})\right]\frac{dt}{t}$$

$$= (x + \tfrac{1}{2})\log x - (x - 1)$$

by Ex. 3, Art. 173. Thus we see that, if $A = 1 + \phi(1)$,

$$\phi(x) = (x + \tfrac{1}{2})\log x - x + A.$$

To determine A, we make use of (4), Art. 178, which gives

$$\log \Gamma(x + \tfrac{1}{2}) + \log \Gamma(x+1) + 2x \log 2 - \log \Gamma(2x+1) = \tfrac{1}{2}\log \pi.$$

Thus we have, since $\lim \psi(x) = 0$,

$$\lim\left[\phi(x - \tfrac{1}{2}) + \phi(x) + 2x \log 2 - \phi(2x)\right] = \tfrac{1}{2}\log \pi,$$

which gives, on inserting the value of $\phi(x)$,

$$\lim\left[A + x\log\left(1 - \frac{1}{2x}\right) + \frac{1}{2} - \frac{1}{2}\log 2\right] = \frac{1}{2}\log \pi$$

or* $$A = \tfrac{1}{2}\log(2\pi)$$ (compare Ex. 55, p. 525).

* The value of A can also be found from Stirling's asymptotic formula (Art. 179); or by a device due to Pringsheim (*Math. Annalen*, Bd. 31, p. 473).

Thus we can write

$$\log \Gamma(1+x) = (x+\tfrac{1}{2}) \log x - x + \tfrac{1}{2} \log 2\pi + \psi(x).$$

where
$$\psi(x) = 2\int_0^\infty \frac{\arc\tan(y/x)}{e^{2\pi y}-1}\,dy$$

and
$$|\psi(x)| < 1/12\xi.$$

It is often convenient to have a formula for $\Gamma(1+x)$, when x is real and varies in the neighbourhood of a fixed large value. Thus, if we write
$$x = \nu + \alpha,$$
where ν is large and α may be large, but is of the order $\sqrt{\nu}$ at most, we obtain the asymptotic expression*

$$\log \Gamma(1+x) \backsim (\nu+\alpha+\tfrac{1}{2}) \log \nu - \nu + \tfrac{1}{2} \log 2\pi + \tfrac{1}{2}\alpha^2/\nu,$$

where the error is of order α/ν.

Hence
$$\Gamma(1+\nu+\alpha) \backsim (2\pi\nu)^{\frac{1}{2}} \nu^{\nu+\alpha} e^{-\nu+\frac{1}{2}\alpha^2/\nu}.$$

In several books on analysis, the integrals for $\log \Gamma(1+x)$ are found by a somewhat different method due to Dirichlet.

In outline, this proof is as follows:

(1) Differentiate the Gamma-integral, and we find

$$\Gamma'(1+x) = \int_0^\infty e^{-t} t^x \log t\,dt = \int_0^\infty e^{-t} t^x\,dt \int_0^\infty (e^{-v} - e^{-vt}) \frac{dv}{v}.$$

(2) Invert the order of integration, and we obtain

$$\frac{\Gamma'(1+x)}{\Gamma(1+x)} = \int_0^\infty [e^{-v} - (1+v)^{-(1+x)}] \frac{dv}{v} = \lim_{\delta\to 0} \left[\int_\delta^\infty e^{-v} \frac{dv}{v} - \int_\delta^\infty (1+v)^{-(1+x)} \frac{dv}{v}\right].$$

Also
$$\int_\delta^\infty \frac{dv}{v}(1+v)^{-(1+x)} = \int_{\log(1+\delta)}^\infty \frac{e^{-xy}}{e^y - 1}\,dy, \quad \text{if } 1+v = e^y.$$

(3) We must next prove that

$$\lim_{\delta\to 0} \int_{\log(1+\delta)}^\delta \frac{e^{-xy}}{e^y-1}\,dy = 0,$$

and then we have
$$\frac{\Gamma'(1+x)}{\Gamma(1+x)} = \int_0^\infty \left(\frac{e^{-y}}{y} - \frac{e^{-xy}}{e^y-1}\right) dy.$$

(4) Finally, if we integrate the last equation, we arrive at the same integral as before for $\log \Gamma(1+x)$.

The reader will find it a good exercise in the use of Arts. 166, 172, 177, to shew that the steps (1)–(4) are legitimate. Proofs will be found in Jordan's *Cours d'Analyse* (t. 2, 2me éd., pp. 176–182).

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* We have
$$\phi(\nu+\alpha) = \phi(\nu) + \alpha\phi'(\nu) + \tfrac{1}{2}\alpha^2\phi''(\nu+\theta\alpha), \quad (0<\theta<1),$$
so that here we get
$$\left(\nu+\frac{1}{2}\right)\log\nu - \nu + \frac{1}{2}\log(2\pi) + \alpha\left(\log\nu + \frac{1}{2\nu}\right) + \frac{1}{2}\alpha^2\left[\frac{1}{\nu+\theta\alpha} - \frac{1}{2}\frac{1}{(\nu+\theta\alpha)^2}\right].$$

EXAMPLES.

Tests of Convergence.

1. Determine the values of a, b for which the integrals

(1) $\int_0^\infty x^{a-1}\cos x\,dx$, (2) $\int_0^\infty x^{a-1}\sin x\,dx$, (3) $\int_0^\infty \frac{x^{a-1}dx}{1+x}$, (4) $\int_0^\infty \frac{x^{a-1}-x^{b-1}}{1-x}dx$

are convergent.

2. Discuss the continuity of the integral

$$\int_0^\infty \frac{\sin y\,dx}{1-2x\cos y+x^2}$$

regarded as a function of y. Sketch its graph. [*Math. Trip.* 1904.]

3. Discuss the convergence of the integrals

$$\int_0^\infty \tanh x\left(\frac{x+1}{x^3+x+1}\right)^{\frac{1}{2}}dx, \quad \int_0^\infty \frac{\sin(xy)\,dx}{\sqrt{(x^2-x+1)}}.$$ [*Math. Trip.* 1893.]

4. If $0<\kappa\leqq\frac{1}{2}$, both the series and the integral

$$\sum^\infty \frac{\sin n\theta}{n^\kappa+a\sin n\theta}, \quad \int^\infty \frac{\sin x\,dx}{x^\kappa+a\sin x}$$

are divergent if $a>0$, although both converge if $a=0$. When $\kappa>\frac{1}{2}$, the series and integral are both convergent.

Reconcile these results with Dirichlet's tests (Arts. 22 and 171). [Hardy.]

5. If $f(x)$ tends *steadily* to zero as $x\to\infty$, prove from Art. 166 that we can infer the convergence of $\int^\infty xf'(x)dx$ from that of $\int^\infty f(x)dx$, provided that $f(x)$ is monotonic.

Similarly, shew that if (a_n) is a monotonic sequence, the convergence of $\Sigma n(a_n-a_{n+1})$ can be deduced from that of Σa_n.

6. Apply the method of Art. 166 to prove that, if α, β, γ are positive, the integral

$$\int^\infty \frac{e^{\alpha x}dx}{e^{\beta x}\sin^2 x+e^{\gamma x}\cos^2 x}$$

converges if $\beta+\gamma>2\alpha$, and diverges if $\beta+\gamma\leqq 2\alpha$.

Deduce that, if $\beta>1>\gamma>0$ and $\beta+\gamma>2$, the integral

$$\int^\infty \frac{dt}{t\{(l_1t)^\beta\sin^2(l_2t)+(l_1t)^\gamma\cos^2(l_2t)\}}$$

is convergent, where $l_1t=\log t$, $l_2t=\log(\log t)$.

Shew that in the last integral the integrand tends *steadily* to zero, but that no test of the logarithmic scale suffices to establish the convergence of the integral.

State and prove corresponding results for series. [Hardy.]

7. Shew that the series $\sum \frac{n^{\alpha-1}}{a+n^{\beta}\sin^2(n\pi\lambda)}$ diverges if a is positive and λ is *rational* (in contrast to the corresponding integral in Art. 166). But if λ is the root of an algebraic equation of degree $m>1$, the series converges if $\beta>\alpha+2m$. However, irrational values of λ can be constructed for which the series will diverge, whatever β may be.

[Compare a paper by Hardy, *Proc. Lond. Math. Soc.* (2), vol. 3, pp. 444–9.]

8. Shew that the integrals

$$\int^{\infty}\cos\{f(x)\}dx,\quad \int^{\infty}\sin\{f(x)\}dx$$

are convergent, provided that $f'(x)$ tends steadily to infinity with x.

Prove also that $\int^{\infty} f'(x)\sin\{e^{f(x)}\}dx$ is convergent no matter how rapidly $f'(x)$ tends to infinity.

[In the first case it is *not* sufficient that $f(x)$ tends steadily to infinity, as we may see by taking $f(x)=x$.]

9. Although (see Ex. 8) $\int^{\infty}\cos(x^2)dx$ and $\int^{\infty}\sin(x^2)dx$ are convergent, prove that $\Sigma\cos(n^2\theta)$ and $\Sigma\sin(n^2\theta)$ cannot converge if θ/π is rational.

Change of Variables.

10. If $g(\xi)$ is an odd function of ξ, prove by dividing the range into intervals $(0, \frac{1}{2}\pi)$, $(\frac{1}{2}\pi, \pi)$, $(\pi, \frac{3}{2}\pi)$, ... and introducing the new variables x, $\pi-x$, $x-\pi$, $2\pi-x$, ... respectively, that

$$\int_0^{\infty} g(\sin x)\frac{dx}{x}=\int_0^{\frac{1}{2}\pi} g(\sin x)\frac{dx}{\sin x},$$

provided that both integrals converge.

Deduce that

$$\int_0^{\infty}\sin^{2n+1}x\frac{dx}{x}=\tfrac{1}{2}\pi\frac{1.3\ldots(2n-1)}{2.4\ldots 2n},$$

$$\int_0^{\infty}\tan^{-1}(a\sin x)\frac{dx}{x}=\tfrac{1}{2}\pi\sinh^{-1}a,\quad \text{(Ex. 15.)}$$

$$\int_0^{\infty}(\log\cos^2 x)\frac{\sin x}{x}dx=-\pi\log 2.\qquad \text{[Wolstenholme.]}$$

11. Apply the same method to prove that, if $f(\xi)$ is an even function of ξ, $a>0$ and $0\leqq\kappa\leqq 2$

(1) $\int_0^{\infty} f(\sin x)\frac{dx}{x^2}=\int_0^{\frac{1}{2}\pi} f(\sin x)\frac{dx}{\sin^2 x}$;

(2) $\int_0^{\infty} f(\sin x)\frac{2a\,dx}{a^2+x^2}=\sinh 2a\int_0^{\frac{1}{2}\pi}\frac{f(\sin x)dx}{\sinh^2 a+\sin^2 x}$

$$=2\int_0^{\frac{1}{2}\pi} f(\sin x)\left(1+2\sum_1^{\infty}e^{-2na}\cos 2nx\right)dx;$$

(3) $\int_0^\infty f(\sin x)\frac{2a\cos(\kappa x)}{a^2+x^2}dx$

$$=\int_0^{\frac{1}{2}\pi} f(\sin x)\left[\cosh(\kappa a)\frac{\sinh 2a}{\sinh^2 a+\sin^2 x}-\sinh(\kappa a)\right]dx$$

$$=2\int_0^{\frac{1}{2}\pi} f(\sin x)\left[e^{-\kappa a}+2\cosh(\kappa a)\sum_1^\infty e^{-2na}\cos 2nx\right]dx;$$

(4) $\int_0^\infty f(\sin x)\cos(\kappa x)\frac{dx}{x^2}=\int_0^{\frac{1}{2}\pi} f(\sin x)(\operatorname{cosec}^2 x-\kappa)dx.$

[It is understood that all the integrals converge; the series used are given in Ex. 14, p. 225, and Ex. 22, p. 314.]

12. Illustrations of the last example are:

(1) $\int_0^\infty (\log\cos^2 x)\frac{dx}{x^2}=-\pi,\quad \int_0^\infty (\log\cos^2 x)^2\frac{dx}{x^2}=4\pi\log 2,$

$$\int_0^\infty (\log\cos^2 x)(\log\sin^2 x)\frac{dx}{x^2}=2\pi(2\log 2-1).$$

(2) $\int_0^\infty \cos^2 x\frac{dx}{a^2+x^2}=\frac{\pi}{4a}(1+e^{-2a}),\quad \int_0^\infty (\log\cos^2 x)\frac{dx}{a^2+x^2}=\frac{\pi}{a}\log\tfrac{1}{2}(1+e^{-2a}),$

and a similar formula containing $\log\sin^2 x$ and $\log\frac{1}{2}(1-e^{-2a})$.

(3) $\int_0^\infty \cos(\kappa x)(\log\cos^2 x)\frac{dx}{a^2+x^2}=\frac{\pi}{a}[\cosh(\kappa a)\log(1+e^{-2a})-e^{-\kappa a}\log 2],$

and a similar formula containing $\log\sin^2 x$ and $\log(1-e^{-2a})$.

(4) $\int_0^\infty \cos(\kappa x)(\log\cos^2 x)\frac{dx}{x^2}=\pi(\kappa\log 2-1);$

but in this case there is no corresponding formula with $\log\sin^2 x$.

[DE LA VALLÉE POUSSIN and HARDY.]

Differentiation and Integration.

13. Calculate the integrals

$$u=\int_0^1\log(x^2+y^2)dx,\quad u_0=\int_0^1\log x^2 dx,$$

and prove that $$\lim_{y\to 0}(u-u_0)/y=\pm\pi,$$

the ambiguous sign being the same as the sign of y. Explain why this limit is not the same as the integral

$$\int_0^1 \lim_{y\to 0}[\{\log(x^2+y^2)-\log x^2\}/y]\,dx.$$ [STOLZ.]

14. Prove by differentiation, or by expanding in powers of α, that

$$\int_0^\infty \log(1+\alpha\operatorname{sech} x)dx=\tfrac{1}{2}[\pi\sin^{-1}\alpha-(\sin^{-1}\alpha)^2].$$

Obtain two other integrals by writing $\alpha=i\beta$, where β is real; and verify these results by differentiation and expansion. [HARDY.]

15. Prove similarly that

$$\int_0^{\frac{1}{2}\pi} \log(1+a\sin x)\frac{dx}{\sin x} = \tfrac{1}{2}[\pi\sin^{-1}a - (\sin^{-1}a)^2]$$

and
$$\int_0^{\frac{1}{2}\pi} \log(1+a\sin^2 x)\frac{dx}{\sin^2 x} = \pi[\sqrt{(1+a)}-1].$$

Obtain four other integrals by writing $a = i\beta$. [WOLSTENHOLME.]

16. By integrating the equation

$$\int_0^\infty \frac{\cos(xy)}{a^2+x^2}dx = \frac{\pi}{2a}e^{-ay}, \text{ where } a>0,\ y>0,$$

with respect to y, prove that

$$\int_0^\infty \frac{\sin(xy)}{x(a^2+x^2)}dx = \frac{\pi}{2a^2}(1-e^{-ay}), \quad \int_0^\infty \frac{1-\cos(xy)}{x^2(a^2+x^2)}dx = \frac{\pi}{2a^3}(e^{-ay}+ay-1),$$

$$\int_0^\infty \frac{xy-\sin(xy)}{x^3(a^2+x^2)}dx = \frac{\pi}{2a^4}[1-ay+\tfrac{1}{2}(ay)^2-e^{-ay}],$$

and so on, the terms introduced on the left being those of the sine and cosine power-series and the terms on the right being those of the exponential series. [*Math. Trip.* 1902.]

17. Justify differentiating the integral

$$\int_0^{\frac{1}{2}\pi} \tan^{-1}(a^2\tan^2 x)dx, \quad (a>0)$$

under the integral sign, and so prove that its value is $\pi\tan^{-1}\{a/(a+\sqrt{2})\}$.

Change the variable to θ, where $a^2\tan^2 x = \cot\theta$, and deduce that if we put $\sqrt{\kappa} = 2a/(a^2-1)$,

$$\int_0^{\frac{1}{2}\pi} \tan^{-1}\left\{\frac{2\sqrt{(\tan\theta)}}{\sqrt{\kappa}(1+\tan\theta)}\right\}d\theta = \pi\tan^{-1}\left\{\frac{1}{\sqrt{(2\kappa)}+\sqrt{(1+\kappa)}}\right\}.$$

Examine the special cases $\kappa=2$ (Wolstenholme) and $\kappa=8$ (*Oxford Senior Scholarship*).

18. By differentiation or otherwise, prove that if a, b are positive,

$$\int_0^\infty \log\left(1+\frac{a^2}{x^2}\right)dx = \pi a, \quad \int_0^\infty \frac{\log(a^2+x^2)}{b^2+x^2}dx = \frac{\pi}{b}\log(a+b),$$

$$\int_0^\infty \tan^{-1}(ax)\tan^{-1}(bx)\frac{dx}{x^2} = \tfrac{1}{2}\pi\left[a\log\left(1+\frac{b}{a}\right)+b\log\left(1+\frac{a}{b}\right)\right],$$

$$\int_0^\infty \log\left(1+\frac{a^2}{x^2}\right)\log\left(1+\frac{b^2}{x^2}\right)dx = 2\pi\left[a\log\left(1+\frac{b}{a}\right)+b\log\left(1+\frac{a}{b}\right)\right].$$

19. By differentiation or otherwise, prove that, if a is positive,

$$\int_0^{\frac{1}{2}\pi} \tan^{-1}(\sinh a\sin x)dx = \int_0^a \frac{t\,dt}{\sinh t}$$

$$= \frac{1}{4}\pi^2 - 2\left[(1+a)e^{-a} + \frac{1}{3^2}(1+3a)e^{-3a} + \frac{1}{5^2}(1+5a)e^{-5a}+\dots\right].$$

[*Math. Trip.* 1892.]

20. Shew that if a, b, c are positive, a being the greatest of the three,

$$\int_0^\infty \sin ax \cos bx \cos cx \frac{dx}{x} = \tfrac{1}{2}\pi, \qquad \text{if } a > b+c,$$
$$\text{or } \tfrac{1}{4}\pi, \qquad \text{if } a < b+c.$$

Deduce by integration that

$$\int_0^\infty \sin ax \sin bx \cos cx \frac{dx}{x^2} = \tfrac{1}{2}\pi b, \qquad \text{if } a > b+c,$$
$$\text{or } \tfrac{1}{4}\pi(a+b-c), \qquad \text{if } a < b+c,$$

and
$$\int_0^\infty \sin ax \sin bx \sin cx \frac{dx}{x^3} = \tfrac{1}{2}\pi bc, \qquad \text{if } a > b+c,$$
$$\text{or } \tfrac{1}{8}\pi(2bc+2ca+2ab-a^2-b^2-c^2), \qquad \text{if } a < b+c.$$

In particular, $\displaystyle\int_0^\infty \sin tx \sin^2 x \frac{dx}{x^3} = \tfrac{1}{2}\pi t(1-\tfrac{1}{4}t)$, if $0 < t < 2$,

or $\tfrac{1}{2}\pi$, if $t > 2$.

21. Prove that, if $t > |a_1| + |a_2| + \ldots + |a_n|$,

$$\int_0^\infty \sin tx \,.\, \prod_1^n \sin a_r x \,.\, \prod_1^p \cos b_r x \,.\, \frac{dx}{x^{n+1}} = \tfrac{1}{2}\pi(a_1 a_2 \ldots a_n). \quad \text{[STÖRMER.]}$$

22. The results of Exs. 20, 21 can also be found by integration by parts; this method gives at once

$$\int_0^\infty (\Sigma A \cos ax) \frac{dx}{x^{2n+2}} = (-1)^{n+1} \tfrac{1}{2}\pi \,\Sigma A \frac{a^{2n+1}}{(2n+1)!},$$
$$\int_0^\infty (\Sigma A \cos ax) \frac{dx}{x^{2n+1}} = (-1)^n \,\Sigma A \log a \frac{a^{2n}}{(2n)!},$$

where $\Sigma A = 0, \ \Sigma A a^2 = 0, \ \Sigma A a^4 = 0, \ \ldots, \ \Sigma A a^{2n} = 0.$

Establish similar formulae for integrals which contain sums of sines; and prove that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n dx = \frac{1}{(n-1)!} \frac{\pi}{2^n} \left[n^{n-1} - n(n-2)^{n-1} + \frac{n(n-1)}{2!}(n-4)^{n-1} - \ldots \right],$$

the number of terms in the bracket being $\tfrac{1}{2}n$ or $\tfrac{1}{2}(n+1)$. [WOLSTENHOLME.]

Dirichlet's Integrals.

23. Apply the theorem of Art. 172 (4) to justify the equation

$$\lim_{c \to 0} \int_0^\infty f(x+ct) e^{-t^2} dt = \tfrac{1}{2}\sqrt{\pi} f(x),$$

and deduce that

$$\lim_{c \to 0} \frac{1}{c} \int_{-\infty}^\infty f(y) e^{-(y-x)^2/c^2} dy = \tfrac{1}{2}\sqrt{\pi} \lim_{\delta \to 0} [f(x+\delta) + f(x-\delta)].$$

[WEIERSTRASS.]

24. Apply Abel's test of uniform convergence to prove that if $f(t)$ is monotonic (at least after a certain stage) and continuous, then

$$\lim_{n \to \infty} \int_a^\infty f(t) \sin(nt) \frac{dt}{t} = \pi f(0), \ \tfrac{1}{2}\pi f(0), \ \text{or } 0,$$

according as a is negative, zero, or positive.

Deduce that if x is positive and $\int^\infty f(t)dt$ is convergent, then

$$\int_0^\infty \cos(xv)dv\int_0^\infty f(t)\cos(vt)dt=\frac{\pi}{2}f(x),$$

and the same result is true if the cosines are *both* replaced by sines. [FOURIER.]

25. By taking $f(x)=e^{-ax}$ $(a>0)$, deduce from the last example that

$$\int_0^\infty \frac{v\sin(xv)}{a^2+v^2}dv=\frac{\pi}{2}e^{-ax}=\int_0^\infty \frac{a\cos(xv)}{a^2+v^2}dv.$$

Consider similarly the integrals given by taking $f(x)=1$ from $x=0$ to 1, and $f(x)=0$ from 1 to ∞. [FOURIER.]

26. From the integrals

$$\operatorname{sech} x=2\int_0^\infty \frac{\cos 2xt}{\cosh \pi t}dt,\quad \operatorname{sech}^2 x=4\int_0^\infty \frac{t\cos 2xt}{\sinh \pi t}dt,$$

$$e^{-x^2}=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\cos 2xt\,dt,$$

prove by the method of Ex. 4, Art. 174, that

$$\sum_{-\infty}^{\infty}\operatorname{sech}(x+n\omega)=\frac{\pi}{\omega}\sum_{-\infty}^{\infty}\frac{\cos(2n\pi x/\omega)}{\cosh(n\pi^2/\omega)},$$

$$\sum_{-\infty}^{\infty}\operatorname{sech}^2(x+n\omega)=\frac{2\pi^2}{\omega^2}\sum_{-\infty}^{\infty}\frac{n\cos(2n\pi x/\omega)}{\sinh(n\pi^2/\omega)},$$

$$\sum_{-\infty}^{\infty}e^{-(x+n\omega)^2}=\frac{\sqrt{\pi}}{\omega}\sum_{-\infty}^{\infty}e^{-n^2\pi^2/\omega^2}\cos(2n\pi x/\omega).$$ [SCHLÖMILCH.]

Integration of Series.

27. Prove that (see Ex. 40, p. 195), if a, b are positive,

$$\int_0^{\frac12\pi}\log(a^2\cos^2x+b^2\sin^2x)dx=\pi\log\left(\frac{a+b}{2}\right),$$

$$\int_0^{\frac12\pi}\log(a^2\cos^2x+b^2\sin^2x)\cos 2nx\,dx=-\frac{\pi}{n}\left(\frac{b-a}{b+a}\right)^n,$$

and verify that these results remain correct when $b=0$ and when $a=0$.

Deduce that, if $r^2<1$ and p, q are positive,

$$(1-r^2)\int_0^{\pi}\frac{\log(a^2\cos^2x+b^2\sin^2x)}{1-2r\cos x+r^2}dx=2\pi\log[\tfrac12\{a(1+r^2)+b(1-r^2)\}],$$

$$\int_0^\pi \frac{\log(a^2\cos^2x+b^2\sin^2x)}{p^2\cos^2x+q^2\sin^2x}dx=\frac{2\pi}{pq}\log\left(\frac{aq+bp}{p+q}\right).$$

28. Using the series of Ex. 1, Ch. IX., prove that

$$\int_0^\pi \frac{\cos\frac12\phi\,d\phi}{1+2t^2\cos\phi+t^4}=\frac{2\tanh^{-1}t}{t(1+t^2)},\quad \int_0^\pi\frac{\cos\frac12\phi\,d\phi}{1-2t^2\cos\phi+t^4}=\frac{2\tan^{-1}t}{t(1-t^2)}.$$

Deduce that $\displaystyle\int_0^\pi \tan^{-1}\left(\frac{2t^2\sin\phi}{1-t^4}\right)\frac{d\phi}{\sin\frac12\phi}=8\tan^{-1}t\,.\,\tanh^{-1}t,$

and verify this result by expanding in powers of t. [HARDY.]

29. From Art. 65, shew that (if $r^2<1$)

$$\frac{(1-r^2)\sin x}{(1-2r\cos x+r^2)^2}=\sin x+2r\sin 2x+3r^2\sin 3x+\dots.$$

Shew also that

$$\int_0^\pi \frac{\sin^2 x\cos x\,dx}{(1-2r\cos x+r^2)^2}=\frac{\pi r}{1-r^2},\quad \int_0^\pi \frac{x\sin x\,dx}{1-2r\cos x+r^2}=\frac{\pi}{r}\log(1+r).$$

30. (1) Prove that $\displaystyle\int_0^\infty \frac{t^{2n-1}dt}{\sinh(\pi t)}=\frac{2^{2n}-1}{2n}B_n$ (Ex. 7, Art. 175)

and $\displaystyle\int_0^\infty \frac{t^{2n}dt}{\cosh(\pi t)}=\frac{E_n}{2^{2n+1}}$, (Ex. 8, Art. 175)

where E_n is Euler's number (**Ex. 4, p. 299**).

(2) By expanding in powers of α, shew that

$$\int_0^\infty e^{-x}(1-e^{-\alpha x})\frac{dx}{x}=\log(1+\alpha).$$

31. From the series for $\log(4\sin^2 x)$, $\log(4\cos^2 x)$ (see Art. 65), prove that

$$\int_0^{\frac{1}{2}\pi}\cos 2nx\log(4\sin^2 x)dx=-\frac{\pi}{2n},\quad \int_0^{\frac{1}{2}\pi}\cos 2nx\log(4\cos^2 x)dx=(-1)^{n-1}\frac{\pi}{2n},$$

$$\int_0^{\frac{1}{4}\pi}\log(\cot^2 x)dx=2\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\right),$$

$$\int_0^{\frac{1}{12}\pi}\log(\cot^2 x)dx=\frac{4}{3}\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\right).$$

Deduce that

$$\int_0^{\frac{1}{2}\pi}\{\log(4\sin^2 x)\}^2dx=\frac{1}{6}\pi^3=\int_0^{\frac{1}{2}\pi}\{\log(4\cos^2 x)\}^2dx,$$

$$\int_0^{\frac{1}{2}\pi}\log(4\sin^2 x).\log(4\cos^2 x)dx=-\frac{1}{12}\pi^3.$$ [WOLSTENHOLME.]

[Compare Exs. 46, 47; and note that the only difficulties arise in extending the rule for term-by-term integration up to the limits.]

32. Use Art. 175 to justify the following transformations;

$$\sum_1^\infty \frac{1}{n^2}\frac{1}{2^n}=\int_0^1\log\left(\frac{1}{x}\right)\frac{dx}{2-x}$$
$$=\int_0^1\log\left(\frac{1}{1-x}\right)(1-x+x^2-\dots)dx$$
$$=1-\tfrac{1}{2}(1+\tfrac{1}{2})+\tfrac{1}{3}(1+\tfrac{1}{2}+\tfrac{1}{3})-\dots$$
$$=\tfrac{1}{12}\pi^2-\tfrac{1}{2}(\log 2)^2.$$ [LEGENDRE.]

33. Shew that

$$\int_0^\infty e^{-t^2}\sin(2tx)\frac{dt}{t}=\sqrt{\pi}\int_0^x e^{-v^2}dv$$ (Ex. 4, Art. 176),

and verify the equation by making x tend to ∞.

34. If $\sinh x.\sinh y=1$, prove that $\displaystyle\int_0^\infty y\,dx=\tfrac{1}{4}\pi^2$.

[Write $t=e^{-x}$ and use **Ex. 7, Art.** 175.] [*Math. Trip.* 1902.]

Gamma Functions, etc.

35. By writing $x+y=\xi$, $y=\xi\eta$, shew that if the real parts of r, s are positive,

$$\Gamma(r)\Gamma(s)=\int_0^\infty e^{-x}x^{r-1}dx\int_0^\infty e^{-y}y^{s-1}dy=\int_0^\infty e^{-\xi}\xi^{r+s-1}d\xi\int_0^1 \eta^{r-1}(1-\eta)^{s-1}d\eta,$$

and deduce that

$$\Gamma(r)\Gamma(s)/\Gamma(r+s)=\int_0^1 \eta^{r-1}(1-\eta)^{s-1}d\eta.$$

36. If $U=\int_0^\infty e^{-xt}t^{n-1}dt$, where $x=\xi+i\eta$ and $\xi>0$, shew that

$$\frac{\partial U}{\partial \xi}=-\frac{n}{x}U, \quad \frac{\partial U}{\partial \eta}=-\frac{in}{x}U,$$

and hence prove that $U=\Gamma(n)/x^n$, if $n>0$.

By using Ex. 2, Art. 172, deduce that if $0<n<1$,

$$\int_0^\infty \cos t \,.\, t^{n-1}dt=\Gamma(n)\cos(\tfrac{1}{2}n\pi), \quad \int_0^\infty \sin t\,.\,t^{n-1}dt=\Gamma(n)\sin(\tfrac{1}{2}n\pi),$$

and verify that the last result is correct if $-1<n<1$.

Obtain the corresponding formulae for

$$\int_0^\infty \cos(x^p)dx \quad \text{and} \quad \int_0^\infty \sin(x^p)dx, \quad p>1. \qquad \text{[Cauchy.]}$$

37. If the real part of x lies between $-k$ and $-(k+1)$, where k is a positive integer, prove that

$$\Gamma(x)=\int_0^\infty t^{x-1}\left[e^{-t}-1+t-\ldots+(-1)^{k+1}\frac{t^k}{k!}\right]dt. \qquad \text{[Cauchy.]}$$

[Apply the process of integration by parts to the integral for $\Gamma(x+k)$.]

38. Shew that if α, β are real,

$$\left\{\frac{\Gamma(\alpha)}{|\Gamma(\alpha+i\beta)|}\right\}^2=\prod_0^\infty\left[1+\frac{\beta^2}{(\alpha+n)^2}\right]. \qquad \text{[Mellin.]}$$

If $x=i\eta$, shew that

$$|\Gamma(1+x)|=\sqrt{\{(\pi\eta)/\sinh(\pi\eta)\}}.$$

39. If

$$A=\int_0^1\frac{dx}{\sqrt{(1-x^4)}}, \quad B=\int_0^1\frac{x^2dx}{\sqrt{(1-x^4)}},$$

express A, B in terms of Gamma-functions, and prove that

$$\Gamma(\tfrac{1}{4})=(\tfrac{1}{2}\pi)^{\frac{1}{4}}A^{\frac{1}{2}}.$$

Assuming the value of $\Gamma(\tfrac{1}{4})$ given in Ex. 41, deduce that [Gauss.]

$$A=1\cdot311029, \quad B=0\cdot599070.$$

40. Similarly express the integrals

$$\int_0^1\frac{dx}{\sqrt{(1-x^3)}}, \quad \int_0^1\frac{x\,dx}{\sqrt{(1-x^3)}}$$

in terms of $\Gamma(\tfrac{1}{3})$, and so obtain numerical values for them. [Gauss.]

41. Deduce from the product formula for $\Gamma(1+x)$ that if $|x|<2$,

$$\log \Gamma(1+x)=\tfrac{1}{2}\log\left\{\frac{\pi x}{\sin(\pi x)}\right\}-\tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right)+C_1x-C_3x^3-C_5x^5-\dots,$$

where $C_1=1-C \quad =0{\cdot}422783, \qquad C_7=\frac{1}{7}\sum_2^\infty n^{-7}=0{\cdot}0011928,$

$$C_3=\tfrac{1}{3}\sum_2^\infty n^{-3}=\quad 673530, \qquad C_9=\tfrac{1}{9}\sum_2^\infty n^{-9}=\qquad 2232,$$

$$C_5=\tfrac{1}{5}\sum_2^\infty n^{-5}=\quad 73856, \qquad C_{11}=\tfrac{1}{11}\sum_2^\infty n^{-11}=\qquad 449.$$

As a numerical exercise, prove that

$$\log_{10}\Gamma(\tfrac{5}{4})=\bar{1}{\cdot}957321, \quad \log_{10}\Gamma(\tfrac{4}{3})=\bar{1}{\cdot}950841.$$

It will also be found from this series that if $\Gamma(1+i)=re^{i\theta}$, then

$$\log_{10}r=\bar{1}{\cdot}71731 \quad \text{and} \quad \theta=-{\cdot}30163.$$

These give $\qquad \Gamma(1+i)=0{\cdot}49802-(0{\cdot}15495)i;$

a result calculated to 7 decimals by Gauss, from Stirling's series (Art. 111), writing $x=10+i$.

42. If $\qquad \psi(x)=\Gamma'(x)/\Gamma(x)=\dfrac{d}{dx}\{\log\Gamma(x)\},$

prove that $\qquad \psi(x)=\lim\limits_{n\to\infty}\left[\log n-\left(\dfrac{1}{x}+\dfrac{1}{1+x}+\dots+\dfrac{1}{n+x}\right)\right]$

$$=-C-\frac{1}{x}+\sum_1^\infty\left(\frac{1}{n}-\frac{1}{n+x}\right),$$

where C is Euler's constant.

Shew that $\qquad \psi(x)-\psi(y)=\sum\limits_0^\infty\left(\dfrac{1}{n+y}-\dfrac{1}{n+x}\right).$

Deduce from Arts. 178, 179 that

$$\psi(1+x)-\psi(x)=1/x, \quad \psi(1-x)-\psi(x)=\pi\cot(\pi x),$$

$$\psi(2x)=\tfrac{1}{2}[\psi(x)+\psi(x+\tfrac{1}{2})]+\log 2,$$

$$\psi(rx)=\frac{1}{r}\left[\psi(x)+\psi\left(x+\frac{1}{r}\right)+\dots+\psi\left(x+\frac{r-1}{r}\right)\right]+\log r,$$

$$\psi(x)+C=\int_0^1\frac{1-t^{x-1}}{1-t}\,dt \quad \text{(if the real part of } x \text{ is positive).}$$

Obtain the particular results,

$$\psi(1)=-C, \quad \psi(2)=1-C, \quad \psi(3)=\tfrac{3}{2}-C, \;\dots,$$

$$\psi(\tfrac{1}{2})=-C-2\log 2, \quad \psi(\tfrac{3}{2})=2-C-2\log 2.$$

Shew also that

$$\psi'(1)=\sum_1^\infty(1/n^2)=\tfrac{1}{6}\pi^2, \quad \psi'(\tfrac{1}{2})=\sum_0^\infty 4/(2n+1)^2=\tfrac{1}{2}\pi^2.$$

43. Shew that, if p, q are positive integers,

$$\psi(p/q)+C=\lim_{t\to 1}f(t),$$

where $\qquad f(t)=-t^p\log(1-t^q)-q\sum\limits_0^\infty\dfrac{t^{p+nq}}{p+nq}$

$$=-t^p\log\{(1-t^q)/(1-t)\}+(1-t^p)\log(1-t)+\sum_{r=1}^{q-1}\omega^{-pr}\log(1-\omega^r t),$$

if $\qquad \omega=\cos(2\pi/q)+i\sin(2\pi/q).$

Deduce from this and the corresponding formula with $q-p$ in place of p, that

$$\psi\left(\frac{p}{q}\right)+C=-\log q-\tfrac{1}{2}\pi\cot\left(\frac{p\pi}{q}\right)+\sum_{r=1}^{q-1}\cos\left(\frac{2\pi rp}{q}\right)\log\left\{2\sin\left(\frac{r\pi}{q}\right)\right\}.$$

Obtain the particular results,

$$\psi(\tfrac{1}{3})+C=-\tfrac{3}{2}\log 3-\tfrac{1}{6}\pi\sqrt{3},\quad \psi(\tfrac{1}{4})+C=-3\log 2-\tfrac{1}{2}\pi,$$
$$\psi(\tfrac{2}{3})+C=-\tfrac{3}{2}\log 3+\tfrac{1}{6}\pi\sqrt{3},\quad \psi(\tfrac{3}{4})+C=-3\log 2+\tfrac{1}{2}\pi.\qquad \text{[Gauss.]}$$

44. Similar results can be obtained for the function $\beta(x)=\sum_{0}^{\infty}\frac{(-1)^n}{x+n}$; thus, shew that

$$\beta(x)+\beta(1+x)=1/x,\quad \beta(x)+\beta(1-x)=\pi\operatorname{cosec}(\pi x),$$
$$\beta(x)=\frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right)\right]=\psi(x)-\psi\left(\frac{x}{2}\right)-\log 2,$$
$$\lim_{x\to\infty}[\psi(x)-\log x]=0,\quad \lim_{x\to\infty}\beta(x)=0,$$
$$\beta(x)=\int_0^1\frac{t^{x-1}}{1+t}dt\quad\text{(if the real part of } x \text{ is positive).}$$

In particular, prove that

$$\beta(1)=\log 2,\qquad \beta(\tfrac{1}{2})=\tfrac{1}{2}\pi,$$
$$\beta(\tfrac{1}{3})=\log 2+\tfrac{1}{3}\pi\sqrt{3},\quad \beta(\tfrac{2}{3})=-\log 2+\tfrac{1}{3}\pi\sqrt{3}.$$

45. If
$$f(a)=\int_0^{\frac{1}{2}\pi}\sin^{2a-1}x\,dx=\frac{\sqrt{\pi}}{2}\frac{\Gamma(a)}{\Gamma(a+\frac{1}{2})},$$

prove from Art. 172 (3) that we may differentiate under the integral sign, provided that a is positive.

Hence

$$f'(a)=2\int_0^{\frac{1}{2}\pi}\sin^{2a-1}x\,.\log\sin x\,.\,dx=\frac{\sqrt{\pi}}{2}\frac{\Gamma(a)}{\Gamma(a+\frac{1}{2})}[\psi(a)-\psi(a+\tfrac{1}{2})]$$

and
$$f''(a)=4\int_0^{\frac{1}{2}\pi}\sin^{2a-1}x\,.(\log\sin x)^2\,.\,dx$$
$$=\frac{\sqrt{\pi}}{2}\frac{\Gamma(a)}{\Gamma(a+\frac{1}{2})}[\{\psi(a)-\psi(a+\tfrac{1}{2})\}^2+\psi'(a)-\psi'(a+\tfrac{1}{2})].$$

46. Shew from Ex. 45 that

$$\int_0^{\frac{1}{2}\pi}\sin x\,.\log\sin x\,.\,dx=\log 2-1,$$
$$\int_0^{\frac{1}{2}\pi}\sin x\,.(\log\sin x)^2\,.\,dx=(\log 2-1)^2+1-\tfrac{1}{12}\pi^2,$$
$$\int_0^{\frac{1}{2}\pi}\frac{\log\sin x}{\sqrt{(\sin x)}}dx=-\frac{\sqrt{\pi}}{4\sqrt{2}}\{\Gamma(\tfrac{1}{4})\}^2,$$
$$\int_0^{\frac{1}{2}\pi}\sqrt{(\sin x)}\,.\log\sin x\,.\,dx=\sqrt{2\pi^3}(\pi-4)/\{\Gamma(\tfrac{1}{4})\}^2,$$
$$\int_0^{\frac{1}{2}\pi}\log\sin x\,.\,dx=-\tfrac{1}{2}\pi\log 2,$$
$$\int^{\frac{1}{2}\pi}(\log\sin x)^2\,.\,dx=\tfrac{1}{2}\pi[(\log 2)^2+\tfrac{1}{12}\pi^2].$$

47. Justify the differentiation of the equation (Ex. 35)

$$\int_0^{\frac{1}{2}\pi} \sin^{2\alpha-1} x \cos^{2\beta-1} x\, dx = \tfrac{1}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Deduce that

$$\int_0^{\frac{1}{2}\pi} \log\sin x \,.\, \log\cos x \,.\, dx = \tfrac{1}{2}\pi[(\log 2)^2 - \tfrac{1}{24}\pi^2],$$

$$\int_0^{\frac{1}{2}\pi} \sin x \,.\, \log\sin x \,.\, \log\cos x \,.\, dx = 2 - \log 2 - \tfrac{1}{8}\pi^2.$$

Miscellaneous.

48. From the series

$$\operatorname{sech} x = 2(e^{-x} - e^{-3x} + e^{-5x} - \dots),$$

prove that if the real part of c is greater than -1,

$$\int_0^\infty \frac{e^{-cx}}{\cosh x}\, dx = \tfrac{1}{2}\left[\psi\left(\frac{c+3}{4}\right) - \psi\left(\frac{c+1}{4}\right)\right].$$

[See Ex. 42 and use Art. 52 (3).]

49. From the last example deduce that, if the real part of a is positive and not greater than 1,

$$\int_0^\infty \frac{\sinh ax}{\cosh x}\frac{dx}{x} = \log\cot\tfrac{1}{4}(1-a)\pi,$$

and hence, if λ is real, prove that

$$\int_0^\infty \cos\lambda x \tanh x \frac{dx}{x} = \log\coth\tfrac{1}{4}\lambda\pi, \qquad \text{[*Math. Trip.* 1889.]}$$

$$\int_0^\infty \frac{\sin\lambda x}{\cosh x}\frac{dx}{x} = 2\tan^{-1}(\tanh\tfrac{1}{4}\lambda\pi). \qquad \text{[HARDY.]}$$

50. From Ex. 8, Art. 175, prove that if the real part of a is positive and not greater than $\frac{1}{2}$,

$$\int_0^\infty \frac{\sinh^2 ax}{\sinh x}\frac{dx}{x} = \tfrac{1}{2}\log\sec a\pi. \qquad \text{[*Math. Trip.* 1895.]}$$

51. Deduce from Ex. 35 and Art. 175 B, that if x and a are positive

$$\frac{\Gamma(x)\Gamma(a)}{\Gamma(x+a)} = \frac{1}{x} - \frac{a-1}{x+1} + \frac{(a-1)(a-2)}{2!(x+2)} - \frac{(a-1)(a-2)(a-3)}{3!(x+3)} + \dots.$$

Shew also that

$$\left[\frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})}\right]^2 = \frac{1}{x} + \frac{1^2}{4}\frac{1}{x(x+1)} + \frac{1^2.3^2}{4.8}\frac{1}{x(x+1)(x+2)} + \dots$$

[To obtain the latter series, expand $\eta^{-\frac{1}{2}}(1-\eta)^{x-1}$ in the form

$$\eta^{-\frac{1}{2}}(1-\eta)^{x-\frac{1}{2}}\left(1 + \tfrac{1}{2}\eta + \frac{1.3}{2.4}\eta^2 + \dots\right).]$$

52. Obtain the first integral of Ex. 49 from the series

$$\operatorname{sech} x = 2(e^{-x} - e^{-3x} + e^{-5x} - \ldots)$$

by applying Frullani's integral to the separate terms.

Obtain similarly the following integrals:

$$\int_0^\infty e^{-ax} \tanh x \frac{dx}{x} = \log \frac{a}{4} + 2 \log \left\{ \Gamma\left(\frac{a}{4}\right) \Big/ \Gamma\left(\frac{a+2}{4}\right) \right\},$$

$$\int_0^\infty e^{-ax}(1 - \operatorname{sech} x) \frac{dx}{x} = -\log \frac{a}{4} + 2 \log \left\{ \Gamma\left(\frac{a+3}{4}\right) \Big/ \Gamma\left(\frac{a+1}{4}\right) \right\},$$

where the real part of a is positive. [HARDY.]

53. Write down the form of Frullani's integral when $\phi(x) = 1/(1+e^{-x})$; and deduce that when p is positive,

$$\int_0^\infty \left(\frac{\sinh px}{\cosh px + \cos qx} - \frac{\sinh px}{\cosh px + \cos rx} \right) \frac{dx}{x} = \tfrac{1}{2} \log \left(\frac{p^2+q^2}{p^2+r^2} \right).$$

[*Math. Trip.* 1890.]

54. The following integrals are allied to Frullani's integral

$$\left.\begin{aligned} &\int_0^\infty (\sin mx - \sin nx)^2 \frac{dx}{x^2} = \tfrac{1}{2}\pi \,|m-n|, \\ &\int_0^\infty (e^{-mx} - e^{-nx})^2 \frac{dx}{x^2} = 2m \log \frac{m+n}{2m} + 2n \log \frac{m+n}{2n} \end{aligned}\right\} \quad m, n \geqq 0.$$

$$\int_{-\infty}^\infty [\phi(x-a) - \phi(x-b)]\, dx = (b-a)[\phi(\infty) - \phi(-\infty)].$$

Evaluate the first of these integrals when m, n have opposite signs.

55. By changing the variable from t to $2t$ in Art. 180, prove that

$$\phi(1) = A - 1 = \int_0^\infty \frac{dt}{t} \left[\left(\frac{1}{t} + 1 \right) e^{-2t} - \frac{1}{t} e^{-t} \right] + \frac{1}{2} \int_0^\infty \frac{1 - e^{-t}}{e^t + 1} \frac{dt}{t}.$$

Shew from Ex. 3, Art. 173, that of these integrals the first is equal to $\log 2 - 1$ and the second to $\frac{1}{2} \log (\frac{1}{2}\pi)$. (See Ex. 52.)

56. Prove that if ξ is positive, the function

$$\frac{e^{-\xi t}}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right)$$

steadily decreases as t increases from 0 to ∞ (see p. 297). By applying Art. 168, deduce that, if $x = \xi + i\eta$ in the formulae of p. 512,

$$|\psi(x)| < \tfrac{1}{8}/|\eta|.$$

57. Deduce from Ex. 56 that if $x = \xi + i\eta$, where ξ is positive and fixed, but η tends to infinity,

$$|\Gamma(1+x)| \sim \sqrt{(2\pi)}\, r^{\xi + \frac{1}{2}} e^{-\frac{1}{2}\pi\eta},$$

[PINCHERLE.]

where $r = |x|$. (Compare Ex. 38, p. 521.)

MISCELLANEOUS EXAMPLES.

1. Shew that $\sum_1^\infty \left[n \log\left(\frac{2n+1}{2n-1}\right) - 1\right] = \frac{1}{2}(1 - \log 2).$

[The series can be summed to n terms; or we may express the general term in the form $\int_0^1 \{x^2/(4n^2 - x^2)\}\, dx.$]

2. Discuss the convergence of the series

$$\Sigma n^k[\sqrt{(n+1)} - 2\sqrt{n} + \sqrt{(n-1)}].$$ [*Math. Trip.* 1890.]

3. If
$$C_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n,$$
prove that
$$C_n - C_{2n} = \int_0^1 \frac{t^{2n}}{1+t}\, dt,$$
and deduce that Euler's constant is equal to
$$1 - \int_0^1 \frac{dt}{1+t}(t^2 + t^4 + t^8 + t^{16} + \dots).$$ [CATALAN.]

4. Prove that, if $\mu > \nu$, the sum
$$\sum_{n=0}^{\mu}{}' \frac{2\nu}{\nu^2 - n^2}, \quad (n = \nu \text{ excluded}),$$
tends to the limit $\log\{(1+k)/(1-k)\}$, when μ, ν tend to infinity in such a way that ν/μ tends to k. [*Math. Trip.* 1894.]

5. Apply Euler's method (Art. 24) to shew that
$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots = \cdot 915965\dots, \quad 1 - \frac{1}{3^4} + \frac{1}{5^4} - \frac{1}{7^4} + \dots = \cdot 988944\dots.$$

[For other methods of transforming and evaluating these series see GLAISHER, *Messenger of Maths.*, vol. 33, 1903, pp. 1, 20.]

6. If s_n denotes $2^{-n} + 3^{-n} + 4^{-n} + \dots$ to ∞, prove by conversion into double series that
$$s_2 + s_3 + s_4 + \dots = 1, \qquad s_2 + s_4 + s_6 + \dots = \tfrac{3}{4},$$
$$\tfrac{1}{2}s_2 + \tfrac{1}{3}s_3 + \tfrac{1}{4}s_4 + \dots = 1 - C, \quad s_2 + \tfrac{1}{2}s_4 + \tfrac{1}{3}s_6 + \dots = \log 2.$$
[See WOOLSEY JOHNSON, *Bull. Am. Math. Soc.*, vol. 12, 1906, p. 477.]

7. If θ is positive, prove that

$$\sum_1^\infty (3\coth^4 n\theta - 4\coth^2 n\theta + 1) = 4\sum_1^\infty n^3(\coth n\theta - 1).$$

[Write $x = e^{-2\theta}$ and convert into a double series.] [*Math. Trip.* 1894.]

8. Shew that the series

$$\Sigma \frac{x^n(x^{2n+1}-1)}{(x^{2n}+1)(x^{2n+2}+1)}$$

does not converge uniformly in any interval including $x=1$.

[*Math. Trip.* 1901.]

9. (1) If $f_n(x) = nx^{n-1} - (n+1)x^n$, prove that

$$\sum_1^\infty f_n(x) = 1, \quad 0 \leqq |x| < 1,$$

and deduce that $\int_0^1 \Sigma f_n(x)\,dx = 1$, while $\Sigma \int_0^1 f_n(x)\,dx = 0$.

(2) Prove that the series obtained by differentiating

$$\Sigma \frac{1}{n^2}\log(1+nx^2)$$

is uniformly convergent for all real values of x, including $x=0$. Is the same true of the given series?

10. Prove that

$$\left(\frac{1}{a} + \frac{1}{2}\frac{x}{a+2} + \frac{1.3}{2.4}\frac{x^2}{a+4} + \dots\right)\left(1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \dots\right)$$
$$= \frac{1}{a}\left[1 + \frac{a+1}{a+2}x + \frac{(a+1)(a+3)}{(a+2)(a+4)}x^2 + \dots\right],$$

$$\log(1+x^2).\tan^{-1}x = 2[\tfrac{1}{3}(1+\tfrac{1}{2})x^3 - \tfrac{1}{5}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})x^5 + \dots],$$

$$[F(\tfrac{1}{2}, 1, 2, x)]^2 = F(1, \tfrac{3}{2}, 3, x), \quad [F(1, \tfrac{3}{2}, 3, x)]^2 = F(2, \tfrac{5}{2}, 5, x).$$

[All these can be obtained by direct multiplication; but the law of the coefficients is more quickly determined by differentiation or some other special device.]

11. Shew how to calculate log 2, log 3, log 5, log 7 from the five series a, b, c, d, e given by writing

$$x = \tfrac{1}{31}, \tfrac{1}{49}, \tfrac{1}{161}, \tfrac{1}{251}, \tfrac{1}{4801}$$

respectively in the series for $\log\{(1+x)/(1-x)\}$; and prove that

$$a - 2b + c = d + 2e.$$

[For results to 260 decimals, see ADAMS, *Math. Papers*, vol. 1, p. 459.]

12. Prove that as x tends to 1, [CESÀRO.]

$$x + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^9 + \tfrac{1}{4}x^{16} + \dots \sim -\tfrac{1}{2}\log(1-x).$$

13. Sketch the graphs from 0 to 2π of the functions

$$\sin 5x + \frac{1}{2}\sin 10x + \frac{1}{3}\sin 15x + \dots,$$

$$\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \dots,$$

$$\sin 2x + \frac{1}{3}\sin 6x + \frac{1}{5}\sin 10x + \dots.$$

14. If $$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

prove that $$\frac{1}{s}[f(x)+f(x+a)+f(x+2a)+\ldots+f\{x+(s-1)a\}],$$

where $a=2\pi/s$, contains only those terms of the original series in which n is a multiple of s.

[This result is of some importance in the numerical applications of Fourier's series; see WEDMORE, *Journal Instit. Elect. Engineers*, vol. 25, 1896, p. 224, and LYLE, *Phil. Mag.* (6), vol. 11, 1906, p. 25.]

15. If $\nu=4n^2-1$, so that ν takes the values (1.3), (3.5), (5.7), ... for $n=1, 2, 3, \ldots$, prove that

$$\Sigma\frac{1}{\nu}=\frac{1}{2},\quad \Sigma\frac{1}{\nu^2}=\frac{1}{16}(\pi^2-8),\quad \Sigma\frac{1}{\nu^3}=\frac{1}{64}(32-3\pi^2),\quad \Sigma\frac{1}{\nu^4}=\frac{1}{768}(\pi^4+30\pi^2-384).$$

[Take $x=\frac{1}{2}$ in the series of Ex. 14, p. 225.]

16. Shew that $$1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\ldots=\frac{\pi}{2\sqrt{3}},$$

$$\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{17^2}+\frac{1}{19^2}+\frac{1}{29^2}+\frac{1}{31^2}+\ldots=\frac{\pi^2}{36}(2-\sqrt{3}),$$

by giving x special values in the series of Ex. 14, p. 225.

17. Prove that if n is even

$$\sum_{r=1}^{n}\tan^{-1}\left(\sec\frac{2r\pi}{2n+1}\sinh x\right)=\tan^{-1}\left\{\frac{\sinh nx}{\cosh(n+1)x}\right\}.$$

[*Math. Trip.* 1907.]

18. Shew that the remainder after n terms in the first series of Ex. 27, p. 315, is

$$(-1)^{n-1}\frac{x^{2n+1}}{(2n)!\sin x}\int_0^1\cos(xt)\,\phi_{2n}(t)\,dt,$$

where $\phi_n(t)$ denotes the Bernoullian function defined in Art. 101.

[*Math. Trip.* 1905.]

19. Shew that

$$\lim_{n\to\infty}\left[\left(\frac{1}{n}\right)^n+\left(\frac{2}{n}\right)^n+\ldots+\left(\frac{n-1}{n}\right)^n\right]=\frac{1}{e-1}.$$

[WOLSTENHOLME.]

[Use Arts. 102, 49 and the series for $1/(e^x-1)$ in Art. 100.]

20. Discuss the convergence of the series

$$\sum_{-\infty}^{\infty}\frac{\cos(x+n\alpha)}{\cos(y+n\beta)},$$

where x, y, α, β are complex.

[*Math. Trip.* 1892.]

Discuss the convergence of the products

$$\prod_1^{\infty}\left(1+\frac{1}{n^2x}\right),\quad \prod_1^{\infty}\left[1+\frac{1}{n^2(x^n-1)}\right],\quad \prod_1^{\infty}\frac{1+e^{2nx}}{1+e^{(2n-1)x}},$$

for all values of the complex variable x.

[*Math. Trip.* 1893.]

21. Prove that if

$$S_0 = 1! - 2! + 3! - 4! + \dots, \qquad \text{(Arts. 105, 109)}$$

then

$$1(1!) - 2(2!) + 3(3!) - 4(4!) + \dots = 1 - 2S_0,$$

$$1^2(1!) - 2^2(2!) + 3^2(3!) - 4^2(4!) + \dots = 5S_0 - 2,$$

and generally $\Sigma(-1)^{n-1} n^k (n!)$ is of the form $\alpha S_0 + \beta$, where α, β are integers (positive or negative).

22. Shew that

$$\int_0^x \frac{e^t}{\sqrt{t}}\,dt \sim \frac{e^x}{\sqrt{x}}\left(1 + \frac{1}{2x} + \frac{1\,.\,3}{2^2 x^2} + \frac{1\,.\,3\,.\,5}{2^3 x^3} + \frac{1\,.\,3\,.\,5\,.\,7}{2^4 x^4} + \dots\right).$$

[For an application to an important physical problem, see LOVE, *Phil. Trans.*, A, vol. 207, 1907, pp. 195–197.]

23. If

$$X_n = \int_0^x e^{-x} \frac{x^n}{n!}\,dx,$$

prove that

$$\lim_{x\to\infty} (\lim_{n\to\infty} X_n) = 0, \quad \lim_{n\to\infty} (\lim_{x\to\infty} X_n) = 1.$$

24. From the power-series for $\frac{1}{2}[\log(1+x)]^2$, shew that if $-\pi < \theta < \pi$,

$$[\log(4\cos^2 \tfrac{1}{2}\theta)]^2 - \theta^2 = 8[\tfrac{1}{2}\cos 2\theta - \tfrac{1}{3}(1+\tfrac{1}{2})\cos 3\theta + \tfrac{1}{4}(1+\tfrac{1}{2}+\tfrac{1}{3})\cos 4\theta - \dots],$$

and obtain a corresponding formula for $[\log(4\sin^2 \frac{1}{2}\theta)]^2$.

Shew also that

$$\theta^2 = \tfrac{1}{3}\pi^2 - 4\left(\cos\theta - \frac{1}{2^2}\cos 2\theta + \frac{1}{3^2}\cos 3\theta - \dots\right).$$

Deduce the integrals of Ex. 31, p. 520.

25. If

$$u_n = -2n \int_0^{\frac{1}{2}\pi} \cos 2nx \log(2\cos \tfrac{1}{2}x)\,dx,$$

prove that

$$u_n - u_{n+1} = (-1)^n/(2n+1),$$

and that

$$u_n = \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \dots \pm \frac{1}{2n-1}\right). \qquad \text{[CATALAN.]}$$

26. Prove that if k is an integer and $|r| < 1$,

$$\int_0^\pi \frac{(1-r^2)\cos nx}{1 - 2r\cos kx + r^2}\,dx = 0, \text{ if } n \text{ is not divisible by } k,$$

$$= \pi r^\nu, \text{ if } n = \nu k.$$

Deduce that if $k = \alpha\kappa$ and $l = \alpha\lambda$, where κ, λ are co-prime integers,

$$\int_0^\pi \frac{(1-r^2)(1-s^2)\,dx}{(1-2r\cos kx + r^2)(1-2s\cos lx + s^2)} = \pi\frac{1 + r^\lambda s^\kappa}{1 - r^\lambda s^\kappa}. \qquad \text{[HARDY.]}$$

27. Prove that if a is positive and $y^2 = ax^2 + 2bx + c$,

$$\int_{x_0}^{x_1} \frac{dx}{y} = \frac{2}{\sqrt{a}} \tanh^{-1}\left\{\frac{(x_1 - x_0)\sqrt{a}}{y_1 + y_0}\right\}.$$

If also $ac - b^2$ is positive and equal to p^2, prove that

$$\int_{x_0}^{x_1} \frac{dx}{y^2} = \frac{1}{p}\tan^{-1}\left\{\frac{p(x_1 - x_0)}{ax_1x_0 + b(x_1 + x_0) + c}\right\},$$

where the angle lies between 0 and π.

[For a discussion of these and other similar cases, see BROMWICH, *Messenger of Maths.*, vol. 35, 1906, p. 131.]

28. Prove that if λ, μ are real but not necessarily positive,

$$\int_0^{\frac{1}{2}\pi} \log(\lambda\cos^2\theta+\mu\sin^2\theta)^2\,d\theta = 2\pi\log\{\tfrac{1}{2}|(\sqrt{\lambda}+\sqrt{\mu})|\}.$$

Deduce that if $u=ax^2+2bx+c$, $u'=a'x^2+2b'x+c'$, where u is positive for all real values of x (so that $p=ac-b^2>0$),

$$\int_{-\infty}^{\infty}\log u^2\frac{dx}{u}=\frac{2\pi}{\sqrt{p}}\log\frac{4p}{a},\quad \int_{-\infty}^{\infty}\log\left(\frac{u'}{u}\right)^2\frac{dx}{u}=\frac{2\pi}{\sqrt{p}}\log\left\{\frac{|q+\sqrt{pp'}|}{2p}\right\},$$

where $q=\frac{1}{2}(ac'+a'c)-bb'$, $p'=a'c'-b'^2$. Express the second result in a real form, when p' is negative.

29. Prove that if the integral $\int_0^{\frac{1}{2}\pi} f(\sin^2 x)\,dx$ is convergent and equal to $\frac{1}{2}A\pi$, then the integral

$$\int^{\infty}[A-f(\sin^2 x)]\phi(x)\,dx$$

converges, provided that $\phi(x)$ tends steadily to zero.

Deduce the convergence of

$$\int^{\infty}\log(4\cos^2 x)\phi(x)\,dx,\quad \int^{\infty}\log(4\sin^2 x)\phi(x)\,dx. \qquad \text{[Hardy.]}$$

30. Prove that in the sense defined by Pringsheim (Ch. V.),

$$\lim_{\lambda,\,\mu\to\infty}\int_0^{\lambda}\int_0^{\mu}\sin(ax+by)x^{r-1}y^{s-1}\,dx\,dy \qquad (r,\,s>0)$$
$$=a^{-r}b^{-s}\Gamma(r)\Gamma(s)\sin\tfrac{1}{2}(r+s)\pi.$$

Prove also that if $0<\alpha<\pi$,

$$\lim_{\lambda,\,\mu\to\infty}\int_0^{\lambda}\int_0^{\mu}e^{i(x^2+2xy\cos\alpha+y^2)}\,dx\,dy=\frac{i\alpha}{2\sin\alpha}.$$

[See Hardy, *Messenger of Maths.*, vol. 32, 1903, pp. 92, 159.]

31. Prove that if the real part of x is positive,

$$(1)\quad \int_0^1 e^{-t}t^{x-1}\,dt=\frac{1}{e}\sum_0^{\infty}\frac{1}{x(x+1)\dots(x+n)},$$

$$(2)\quad \int_0^1\frac{t^{x-1}\,dt}{1+t}=\frac{1}{2}\sum_0^{\infty}\frac{n!}{x(x+1)\dots(x+n)}\,\frac{1}{2^n}.$$

32. Prove that if in the interval $(a,\,b)$ the function $|f(x,\,n)|$ is less than 1 for all values of n, and if the function $\phi(x)$ is positive and has a convergent integral from a to b, then

$$\lim_{n\to\infty}\int_a^b f(x,\,n)\phi(x)\,dx=\int_a^b\{\lim_{n\to\infty}f(x,\,n)\}\phi(x)\,dx,$$

provided that $f(x,\,n)$ tends to its limit uniformly in any interval which does not contain $x=a$.

Deduce that

$$\lim_{n\to\infty}\int_0^{\frac{1}{2}\pi}e^{-n\sin x}\phi(x)\,dx=0,$$

$$\lim_{n\to\infty}\int_0^{\frac{1}{2}\pi}[p+(1-p)\sin^2 x]^{\frac{1}{2}n}\phi(x)\,dx=0, \qquad (0<p<1).$$

INDEX OF SPECIAL INTEGRALS, PRODUCTS, AND SERIES.

(The numbers refer to pages, not to articles.)

Numerical values.

Power-series.

Products.

Trigonometrical series.

GENERAL INDEX.*

(The numbers refer to pages, not to articles.)

* No attempt has been made to index references to authors; but where a theorem is usually quoted in the text under an author's name, it appears under that name in the index.

* Commonly attributed to Cauchy; it occurs in Maclaurin's *Fluxions*, 1742, Art. 350.

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