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ELEMENTS OF ORDINARY DIFFERENTIAL EQUATIONS

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ELEMENTS OF ORDINARY DIFFERENTIAL EQUATIONS

VII

PREFACE

This text is intended for use in a first course in ordinary differential equations and is written for students who have had but a year's course in elementary calculus. It is designed to appeal to students majoring in engineering, science, or mathematics. The material covered is more than can be taken up in the usual one-term course, thus enabling the instructor to make a selection of topics suitable for his class. Some material not necessary for the main development is set in small type. Problems which either are more difficult or extend the theory are marked with a star (★). Answers, and some hints, are provided for practically all the problems.

Most students in American colleges and universities who major in mathematics, science, or engineering, take an introductory course in differential equations as one of their first electives in mathematics. The first course in differential equations derives a unique importance in mathematical education from this fact. Its potentialities will be realized only if the student is introduced to more advanced mathematical thought while he is learning technique and manipulation. We have tried to make the exposition appeal to the student's physical and geometric intuition and at the same time bridge the gulf between living mathematics and school mathematics.

To realize this objective we have approached the problems of existence, uniqueness, and general solutions first from an intuitive geometric point of view. These fundamentals, together with the most essential techniques for obtaining solutions, and numerous applications comprise the first part of the book. The later chapters follow a more systematic and deductive approach. They deal almost exclusively with linear equations since only for this class is a careful treatment at this level possible. Many topics are covered which are usually found only in books on specialized fields.

Some features of the book are early emphasis on geometric and numerical methods, use of the superposition principle, use of the existence theorems in development of techniques, elementary but mathematically sound development of operational calculus, systems of equations treated carefully without use of matrices, use of Green's functions, and elementary but precise treatment of power series.

A selection of sections suitable for the usual one-semester course might be the following: Chap. II, Sec. 1–10; Chap. III, Sec. 1–13 Chap. IV, Sec. 1–12, 20; Chap. V, Sec. 1–10, 12; Chap. VI, Sec. 2, 4; Chap. VII, Sec. 1, 8; Chap. VIII, Sec. 1, 2, 3.

To our wives, Dagmar and Erma, we wish to express our thanks for their generously given labor in typing the manuscript.

MICHAEL GOLOMB
MERRILL SHANKS

LAFAYETTE, IND.
April, 1950

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CHAPTER I

REVIEW AND COLLECTION OF FORMULAS

Collected in this chapter are the basic definitions and theorems from calculus most needed for differential equations. In some parts slight generalizations of familiar notions are introduced. No proofs are included.

The references cited by number refer to the bibliography on page 329.

1. Limits and Continuity of Functions. A function $f(x)$ has the "limit L as x approaches a," in symbols,

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied. For every positive number ϵ there is a positive number δ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. Intuitively, this statement asserts that if x is "close" to a, but different from a, then $f(x)$ is necessarily "close" to L. (We have tacitly assumed here that $f(x)$ is defined over an interval containing a in its interior.) Note that the existence of the limit is not dependent on the value $f(a)$ of the function at a.

An analogous definition holds for functions of several variables.

Sometimes it is desirable to let x approach a from one side only. In such a case we speak of the right-hand and left-hand limits of $f(x)$. If x approaches a from the right, that is, through values greater than a, the right-hand limit, if it exists, is indicated by

$$f(a+) = \lim_{x \to a+} f(x)$$

and the left-hand limit by

$$f(a-) = \lim_{x \to a-} f(x).$$

Thus in Fig. 1, $f(a+) = 1$, $f(a-) = 2$, while $f(a) = 3$. Note that the existence of the limits $f(a+)$ and $f(a-)$ does not depend on the value of the function at a.

A function $f(x)$ is defined to be *continuous at the point* $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

This asserts the existence of the limit and that the limit is $f(a)$. Or in other words, if x is "close" to a, then $f(x)$ is "close" to $f(a)$. Then obviously the right-hand and left-hand limits exist and $f(a-) = f(a+) = f(a)$. A function $f(x)$ is *continuous in an interval* if it is continuous at each point of the interval.

Analogous definitions hold for functions of several variables.

Throughout most of elementary calculus only continuous functions are discussed, but even in physical applications it may be necessary to consider discontinuous functions. However, the discontinuous functions which actually occur are not, so to speak, "too badly discontinuous." A class of functions large enough for our purposes and most applications is the class of sectionally (or piecewise) continuous functions.

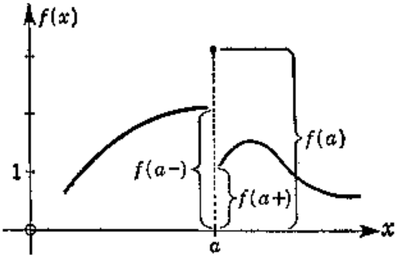


FIG. 1.

The function $f(x)$ is said to be *sectionally* (or *piecewise*) *continuous* in the interval $a < x < b$ if the interval can be subdivided into a finite number of subintervals by n suitably chosen points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

such that $f(x)$ is continuous in each of the subintervals $x_{i-1} < x < x_i$ and the right-hand and left-hand limits $f(x_i+)$ and $f(x_i-)$ exist at each x_i. [At x_0 only $f(x_0+)$, and at x_n only $f(x_n-)$ need exist.] A graph of a sectionally continuous function is shown in Fig. 2.

PROBLEMS

1. Show that arctan $1/x$ (use principal value) is sectionally continuous. Draw its graph.

2. For what values of x are the following (real valued) functions defined? Where are they continuous? Draw their graphs.

(a) $f(x) = x^2 - x - 2$.
(b)† $g(x) = 1/\sqrt{1 - x^2}$.
(c) $F(\theta) = \sqrt{\cos 2\theta}$.

3. If $P(x)$ is the postage on first-class matter (three cents per ounce or fraction thereof), draw the graph of $P(x)$ and show that it is sectionally continuous in any finite interval.

† Wherever the square-root symbol $\sqrt{}$ occurs, the *positive* root is *always* meant.

2. Derivatives and Integrals. The derivative of $f(x)$ is defined to be the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \frac{d}{dx} f(x).$$

Observe that if the derivative exists at x then certainly $f(x)$ is continuous at x.

The derivative of $y = f(x)$ is usually denoted by $y' = dy/dx$. However in cases where the independent variable is the time t, Newton's notation $\dot{y}$ is often used.

A function may be continuous and yet fail to have a derivative. For example, $y = |x|$ does not have a derivative at $x = 0$. An

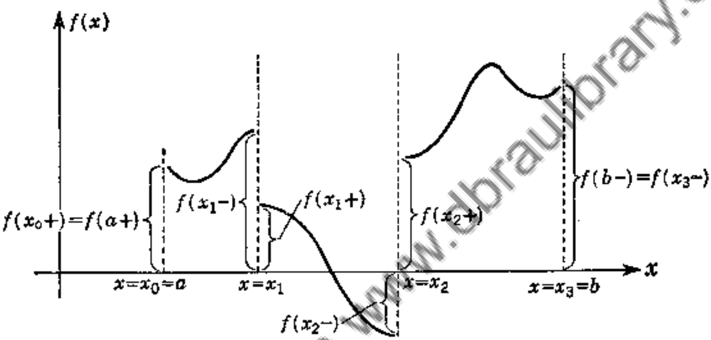


FIG. 2.

important class of functions, large enough for most physical applications, consists of the *sectionally* (or *piecewise*) *smooth* functions which are sectionally continuous and have a sectionally continuous derivative.

A function $F(x)$ is an *indefinite integral of* $f(x)$, symbolically

$$F(x) = \int f(x)\, dx$$

if

$$\frac{d}{dx} F(x) = f(x).$$

In elementary calculus it is shown that if $f(x)$ is continuous and $F(x)$ is any indefinite integral of $f(x)$, then every indefinite integral of $f(x)$ differs from $F(x)$ by a constant. In other words every indefinite integral is of the form $F(x) + C$.

We shall have occasion later to need a slightly stronger theorem which we state without proof

Theorem. If $f(x)$ is sectionally continuous in the interval $a < x < b$, there exists a function $F(x)$ continuous in the interval such that

$$\frac{d}{dx} F(x) = f(x)$$

wherever $f(x)$ is continuous. In addition every continuous indefinite integral of $f(x)$ is of the form $F(x) + C$, and

$$F(x) - F(a) = \int_a^x f(t)\, dt.$$

Briefly the point to remember is that sectionally continuous functions have continuous indefinite integrals. It should be noted that at a point of discontinuity of $f(x)$ the derivative $F'(x)$ of $F(x)$ need not exist.

The formula for integration by parts is

$$\int u\, dv = uv - \int v\, du.$$

For definite integrals this becomes

$$\int_a^b u(x)v'(x)\, dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\, dx.$$

It is often important to know that this formula is valid if $u'(x)$ and $v'(x)$ are sectionally continuous and $u(x)$ and $v(x)$ are continuous.

3. A Classification of Functions. In calculus the student learns to deal with a rather limited class of functions. We describe here (see the accompanying chart) a classification of all single-valued functions $y = f(x)$ in order to make perfectly definite the class of functions with which one normally deals. We need the following definitions.

A function $y = f(x)$ is an *algebraic function* if $y = f(x)$ satisfies an algebraic equation with coefficients which are polynomials in x. That is, if y satisfies identically

(1) $$P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_n(x) = 0$$

where n is a positive integer and $P_0(x), \ldots, P_n(x)$ are polynomials in x. If n is the least integer for which y satisfies an equation of the form (1), then n is called the degree of the function.

For example,

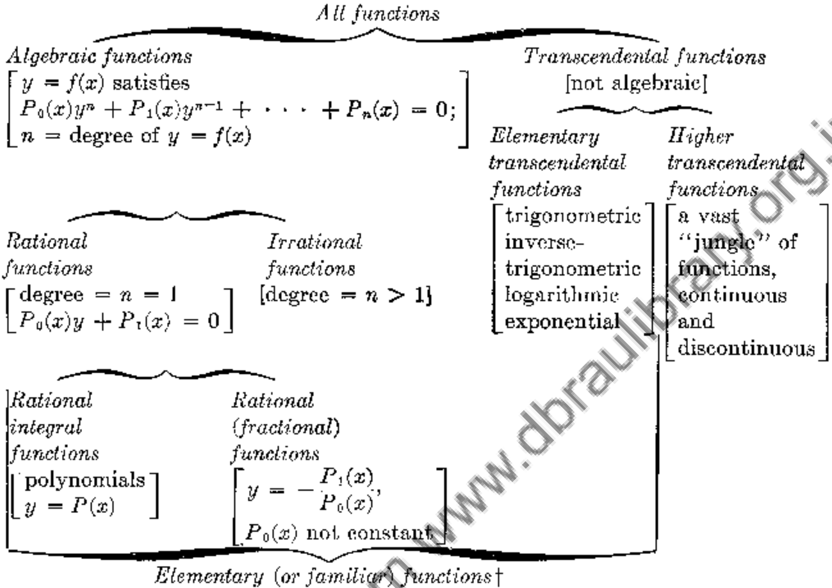
$$(1 + x^2)y^5 + (6x + x^3)y^3 + (-2 + x)y^2 + xy - 4x^2 = 0$$

defines y as an algebraic function of x. Here $n = 5$, $P_0(x) = 1 + x^2$, $P_1(x) = 0, P_2(x) = 6x + x^3, P_3(x) = -2 + x, P_4(x) = x, P_5(x) = -4x^2$.

A function which is *not algebraic* is called *transcendental.*

It is by no means evident that there are transcendental functions. Such matters as the existence of such functions are studied in the "theory of functions." We will accept the classification in the accompanying chart as being true.

CHART



† It is customary to include among the elementary functions also elementary functions of elementary functions.

Certain functions, it is seen from the chart, are called *elementary.* An equally good name would be "familiar functions," because essentially all functions encountered by the student in calculus were elementary. Now it is possible to prove, though not a trivial task, that *the derivative of an elementary function is an elementary function.* This being the case, it was possible in differential calculus to set up rules for differentiating the elementary functions. That is, one could not encounter nonelementary (or unfamiliar functions) by the process of differentiation.

With regard to integration the situation is distinctly different. *The integral of an elementary function* (which we know exists by the theorem of Sec. 2) *need not be an elementary function.* For example,

$$F(x) = \int \frac{dx}{\sqrt{(1 - x^2)(2 - x^2)}}$$

is not an elementary function (although this is not an easy matter to prove). It is this fact which renders the process of formal integration so troublesome. As a consequence, formal integration consists of a body of formulas and rules which *may* render the integral of an elementary function in terms of elementary functions. A table of integrals (Ref. 2) lists a large number of such integrals but *not all.* The student should not be misled into thinking that indefinite integrals not found in a table are not perfectly bona fide functions. By the theorem of Sec. 2 every continuous function $f(x)$ possesses an indefinite integral, $\int f(x)\,dx$. If this integral is an elementary function, it should be possible to express it analytically by use of a table of integrals. If the integral, however, is not elementary, its properties must be deduced somehow from the properties of the given function $f(x)$.

For future reference we consider here three properties which functions may, and sometimes do, have.

A function $f(x)$ is called an *even function* if

$$f(x) = f(-x).$$

The graph of the function then is symmetric with respect to the y axis.

A function $f(x)$ is called an *odd function* if

$$f(-x) = -f(x).$$

The graph of the function then is symmetric with respect to the origin.

A function $f(x)$ (defined for all x) is said to be periodic if there is a positive number k such that for all x

$$f(x + k) = f(x).$$

For example, $\tan (x + 2\pi) = \tan x$. If p is the smallest of such numbers k, it is called the *period* of $f(x)$. For example, $\tan x$ has period π since $\tan (x + \pi) = \tan x$ and no number smaller than π has the required property.

PROBLEMS

1. Show that $\sqrt{3 - x^2}$ is an even algebraic function of degree two.

2. Show that $5x - x^3$ is an odd rational integral function.

3. Is $\sin \sqrt{x}$ elementary? Do you think $\int \sin \sqrt{x}\,dx$ is elementary?

★4. It is possible to define *algebraic* numbers in a manner analogous to that which we used for algebraic functions. Set up the correct definition. The numbers e and π then happen not to be algebraic. (This is difficult to prove.) Such numbers are called transcendental.

5. Set up an integral for the length of the ellipse $x^2 + 2y^2 = 2$. Convince yourself that the indefinite integral involved is not an elementary function.

6. Show that the functions $\log x$, $\arcsin x$, and $\arctan x$ are integrals of simple algebraic functions of degrees 1, 2, and 1, respectively.

4. Families of Curves. Consider the equation

$$g(x, y, c) = 0. \tag{2}$$

For every value of the parameter, or arbitrary constant c, the locus of (2) is a curve. Thus Equation (2) defines a family (in fact a

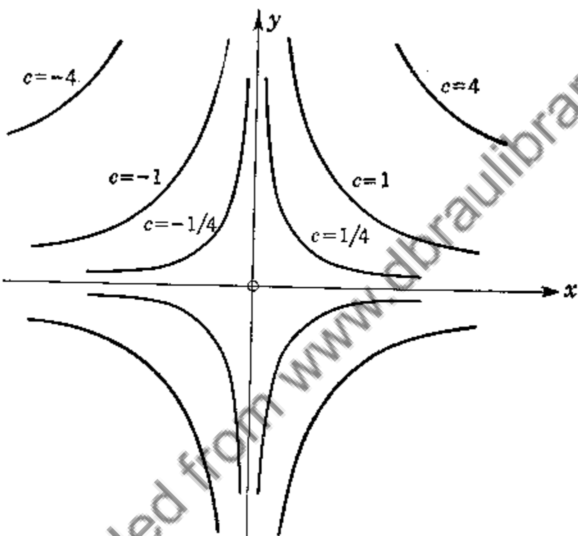


FIG. 3.

one-parameter family) of curves. It is assumed of course that the parameter c actually occurs in (2), that is, $\partial g/\partial c \neq 0$. For example, $xy - c = 0$ defines a family of equilateral hyperbolas (see Fig. 3). Since for each c Equation (2) defines y implicitly as a function of x we may also say that (2) defines a one-parameter family of functions.

The equation

$$g(x, y, c_1, \ldots, c_n) = 0 \tag{3}$$

defines an n-parameter family of curves, or functions. For example, the family of all (nonvertical) lines in the plane is given by the two-parameter family $y = mx + b$ where m and b are the parameters.

We tacitly assumed in this definition that all the parameters $c_1, \ldots, c_n$ are necessary in (3). Or in other words the number of parameters cannot be reduced. For example, the family

$$y = 2 \log c_1x + c_2$$

may be rewritten

$$y = 2 \log x + 2 \log c_1 + c_2.$$

Since $2 \log c_1 + c_2$ is but an arbitrary constant, every curve of the given family is included in the family

$$y = 2 \log x + c.$$

PROBLEMS

1. Show that $y = c_1x + c_2e^{\log c_3x} + c_4$ is a two-parameter family.

2. Sketch the families

(a) $y^2 = 2p\left(x + \frac{p}{2}\right)$. (b) $x \sin \alpha + y \cos \alpha = 2$. (c) $y = -c^2x + 2c$.

3. Show that the family $y = A \sin x + B \cos x$ can be rewritten in the form $y = c_1 \sin (x + c_2)$.

5. Envelopes. If $g(x, y, c) = 0$ is a one-parameter family, there *may* exist a curve, called the *envelope* of the family, which is tangent to each member of the given family (Fig. 4). That is, for each value of c there is a point $(x(c), y(c))$ on the envelope, and at this point both the envelope and the curve of the family have the same slope. (Note that it is not necessary that a given family have an envelope; for example, the family of parallel lines $y = x + b$ obviously has none.)

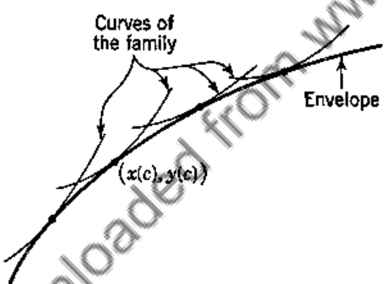


FIG. 4.

We now derive equations which define the envelope if it exists. The point $(x(c), y(c))$ is a definite point on the curve of the family associated with the parameter value c. Hence

$$g(x(c), y(c), c) = 0 \tag{4}$$

and furthermore (4) is an identity in c. Because (4) is an identity in c we get, on differentiating with respect to c,

$$\frac{\partial g}{\partial x}\frac{dx}{dc} + \frac{\partial g}{\partial y}\frac{dy}{dc} + \frac{\partial g}{\partial c} = 0. \tag{5}$$

But the slope at $(x(c), y(c))$ is obtained from

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} = 0. \tag{6}$$

Hence from (5) and (6) and using

$$\frac{dy}{dx} = \frac{dy/dc}{dx/dc}$$

we obtain

$$\frac{\partial g}{\partial c} = 0. \tag{7}$$

Thus a point (x, y) on the envelope, if one exists, must satisfy an equation obtained by eliminating c between

$$\begin{aligned} g(x, y, c) &= 0 \\ \frac{\partial}{\partial c} g(x, y, c) &= 0. \end{aligned} \tag{8}$$

The equation obtained by eliminating c in (8) is called the *eliminant* and contains the envelope. We say "contains" the envelope because other loci beside the envelope may satisfy (8). For a more detailed account of the various possibilities see Ref. 3, p. 85.

Example 1. Find the envelope of the family $y = 2cx - \frac{2}{3}c^3$. Here $g = y - 2cx + \frac{2}{3}c^3$ and we eliminate c between the equations

$$\begin{aligned} y - 2cx + \tfrac{2}{3}c^3 &= 0 \\ -x + c^2 &= 0. \end{aligned}$$

Hence

$$c = \pm\sqrt{x}$$

and

$$y = \pm 2\left(\sqrt{x}\,x - \frac{x^{\frac{3}{2}}}{3}\right) = \pm\frac{4}{3}x^{\frac{3}{2}}$$

or

$$9y^2 = 16x^3.$$

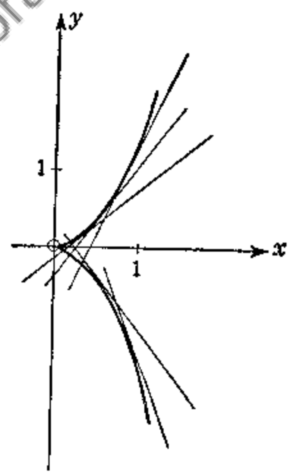


FIG. 5.

The family of curves and envelope are drawn in Fig. 5.

PROBLEM

1. Find the envelopes of the following one-parameter families and sketch the figures:

(*a*) $(x - c)^2 + y^2 = a^2$ (c is the parameter).
(*b*) $x \sin \omega + y \cos \omega = a$ (ω is the parameter).

(c) $y^2 = 2cx - c^2$.
(d) $x \tan c + y \sec c = p$ (c is the parameter).

6. Linear Independence. In the study of linear differential equations it is necessary to consider n-parameter families of functions in which the parameters occur linearly:

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \tag{9}$$

where $y_1(x), \ldots, y_n(x)$ are given functions. It is necessary to know then whether or not the family (9) can be written with fewer parameters.

Definition. The functions $y_1(x), \ldots, y_n(x)$ are said to be *linearly independent* if the identity (in x)

$$c_1 y_1(x) + \cdots + c_n y_n(x) = 0 \tag{10}$$

implies that $c_1 = c_2 = \cdots = c_n = 0$. If there exist constants $c_1, \ldots, c_n$, not all zero, for which (10) is satisfied, the functions $y_1(x), \ldots, y_n(x)$ are said to be linearly dependent.

Example 2. The functions $\sin x$, $\cos x$, $\sin (x + \delta)$ are linearly dependent. Consider

$$\begin{aligned} c_1 \sin x + c_2 \cos x + c_3 \sin (x + \delta) \\ &= c_1 \sin x + c_2 \cos x + c_3(\sin x \cos \delta + \cos x \sin \delta) \\ &= (c_1 + c_3 \cos \delta) \sin x + (c_2 + c_3 \sin \delta) \cos x. \end{aligned}$$

This last expression can vanish identically if

$$c_3 = -c \neq 0, \qquad c_1 = c \cos \delta, \qquad \text{and } c_2 = c \sin \delta.$$

PROBLEMS

1. Show that x, x^2 are linearly independent.
2. Show that $1, x, x^2, \ldots, x^n$ are linearly independent.

*3. Show that $y_1(x)$ and $y_2(x)$ are linearly independent if and only if

$$y_1(x)y_2'(x) - y_1'(x)y_2(x) \equiv 0.$$

7. Power Series. A function $f(x)$ possessing derivatives of all orders in an interval containing $x = a$ may be represented by a Taylor series with remainder

$$f(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x) \tag{11}$$

where $R_n(x)$ is the remainder after $n + 1$ terms and is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1},$$

where ξ is a number between a and x.

The (infinite) Taylor series for $f(x)$ is

$$(12) \quad f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n + \cdots .$$

Clearly the infinite series (12) converges to $f(x)$ if and only if

$$R_n(x) \to 0 \text{ as } n \to \infty.$$

It should be noted that convergence of the series (12) does not necessarily imply convergence to $f(x)$.

Perhaps the simplest way to determine the interval of convergence of the series (12) is to use the *ratio test* which asserts that the series

$$u_1 + u_2 + \cdots + u_n + \cdots$$

is convergent (absolutely) if there is an integer N and a positive number $\rho < 1$ such that for $n > N$

$$\left|\frac{u_{n+1}}{u_n}\right| \leqq \rho.$$

8. Operations on Power Series. One has a certain amount of freedom in dealing with power series. The required facts are stated (without proof) in the following theorems.

Theorem 1. *Taylor series representing $f(x)$ may be differentiated or integrated term by term. That is, if*

$$(13) \quad f(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n + \cdots \quad (|x - a| < R)$$

then

$$f'(x) = f'(a) + f''(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!} (x - a)^{n-1} + \cdots \quad (|x - a| < R)$$

and

$$\int_a^x f(t)\,dt = f(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{(n+1)!} (x - a)^{n+1} + \cdots \quad (|x - a| < R).$$

Theorem 2. *If* $f(x)$ *is given by* (13) *and* $g(x)$ *by*

$$(14)\quad g(x) = g(a) + g'(a)(x-a) + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n + \cdots \quad (|x-a| < r)$$

then $f(x) \pm g(x)$ *is given by the sum or difference of the series* (13) *and* (14) *where the resulting series converges in the smaller of the two intervals of convergence.*

Theorem 3. *The series for* $f(x)g(x)$ *is given by the product*

$$f(x)g(x) = f(a)g(a) + [f(a)g'(a) + f'(a)g(a)](x-a) + \cdots + \left[\frac{f(a)g^{(n)}(a)}{n!} + \frac{f'(a)g^{(n-1)}(a)}{(n-1)!} + \frac{f''(a)g^{(n-2)}(a)}{2!(n-2)!} + \cdots + \frac{f^{(n)}(a)g(a)}{n!}\right](x-a)^n + \cdots$$

and this series converges in the smaller of the two intervals of convergence.

Theorem 4. *If* $g(a) \neq 0$, *then* $f(x)/g(x)$ *can be expanded in a Taylor series and the series represents* $f(x)/g(x)$ *in some interval containing* a. (Note that here we do not state the interval of convergence.)

9. Complex Power Series. In our discussion so far the variable x in a power series was presumed to be a real number. On detailed examination of the meaning of convergence it is found that we may also consider power series when x is a complex number z. Instead of speaking then of the "interval" of convergence, we speak of the "circle" of convergence $|z - a| < R$.

Assuming then that we may consider series of complex numbers, we can use these series to define new functions of a complex variable.

In particular we wish to define the exponential function for complex values of the argument. We have the familiar expansion

$$(15)\qquad e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

which is valid for all real z. By our understanding the series in (15) is convergent also for all complex numbers z. We therefore define a function "exp z"

$$\exp z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots,$$

where z is a complex number. Then $\exp z = e^z$ if z is real and is a new function otherwise. It is customary to continue to use e^z as the notation for $\exp z$, but if this is done it is essential to remember that we mean the function defined by the power series since in elementary analysis complex numbers as exponents are not discussed. For example, e^{1+i} now means the complex number

$$e^{1+i} = 1 + (1 + i) + \frac{(1 + i)^2}{2!} + \cdots + \frac{(1 + i)^n}{n!} + \cdots .$$

It can be proved that the function $\exp z = e^z$ obeys the familiar laws of exponents; for example, $e^a e^b = e^{a+b}$.

There is an important relation connecting the exponential function and the sine and cosine which is obtained as follows. Set $z = ix$ in (15), then

(16) $$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots .$$

Remembering that $i^2 = -1$, $i^3 = -i$, etc., and rearranging the terms in (16) (a permissible operation on power series), we get

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).$$

The series in parentheses are the familiar expansions of $\sin x$ and $\cos x$, so we have

(17) $$e^{ix} = \cos x + i \sin x.$$

This last is Euler's formula.†

Replacing x by $-x$ in (17) we get

(18) $$e^{-ix} = \cos x - i \sin x.$$

For $x = 2\pi$ or π we get

$$e^{2\pi i} = 1$$
$$e^{\pi i} = -1,$$

relations which seem startling to the student when seen for the first time.

The functions $\sin x$ and $\cos x$ are now easily expressed in terms of the exponential function by simply solving (17) and (18) for $\sin x$ and $\cos x$:

(19) $$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

† Leonhard Euler, Swiss, 1707–1783, was a highly prolific mathematician. There are numerous relations in the mathematical and physical literature each of which is referred to as "Euler's equation"!

10. Hyperbolic Functions. In analogy with Equations (19) it is possible to define two functions which have many properties in common with the trigonometric functions. These new functions are called the *hyperbolic sine* and *hyperbolic cosine* and are, respectively,

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{20}$$

The name "hyperbolic" is used here since these functions may also be defined in a geometric manner using a hyperbola just as the sine and cosine are defined using a circle.

The remaining hyperbolic functions are defined from their analogy with the trigonometric functions; for example,

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \text{ etc.}$$

It is a simple matter to prove that

$$\frac{d}{dx}\sinh x = \cosh x,$$

$$\frac{d}{dx}\cosh x = \sinh x,$$

and

$$\cosh^2 x - \sinh^2 x = 1.$$

The proofs are left as an exercise for the reader.

We will include the hyperbolic functions in the class of elementary functions.

11. Partial Differentiation. If $f(u, v, w, \ldots)$ is a function of several variables $u, v, w, \ldots$, and in turn $u, v, w, \ldots$ are functions of $x, y, z, \ldots$,

$$u = u(x, y, z, \ldots)$$
$$v = v(x, y, z, \ldots)$$
$$w = w(x, y, z, \ldots), \text{ etc.}$$

then

$$F(x, y, z, \ldots) = f(u(x, y, z, \ldots), v(x, y, z, \ldots), w(x, y, z, \ldots), \ldots)$$

is a function of the variables $x, y, z, \ldots$. The partial derivatives of $F(x, y, z, \ldots)$ are obtained by the so-called "chain rule"

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} + \cdots$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} + \cdots$$

Etc.

If

$$f(x, y, z) = 0$$

defines z implicitly as a function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z},$$

$$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

12. Determinants of Second and Third Order. A determinant of the second order is a square array of four numbers bordered by vertical lines

(21)
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

and to which is assigned the value $A = a_{11}a_{22} - a_{21}a_{12}$. For example,

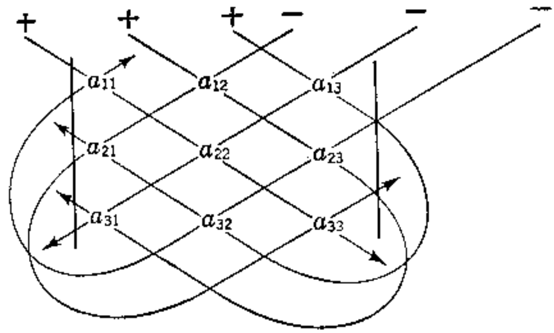
$$\begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix} = (1)(5) - (-3)(-2) = 5 - 6 = -1.$$

Note that we have used two subscripts in (21) to denote the *element* a_{ij} of the determinant. The first subscript indicates the row and the second subscript the column in which the element occurs.

In the same manner a determinant of the third order is defined to be

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}.$$

The expansion of a third-order determinant can be obtained by multiplying the elements following the arrows indicated (on p. 16) and adding the products with the signs indicated on the arrows.



For example,

$$\begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & 1 \\ -2 & 3 & 2 \end{vmatrix} = (1)(2)(2) + (-1)(1)(-2) + (2)(3)(3) - (2)(2)(-2) - (-1)(3)(2) - (1)(3)(1)$$
$$= 4 + 2 + 18 + 8 + 6 - 3 = 35.$$

An important property of third-order (and in fact of nth-order) determinants concerns the minors and cofactors of the elements. The *minor of the element* a_{ij} (in the ith row and jth column) is denoted by α_{ij} and is the determinant of the second order remaining after deleting from the array the ith row and jth column. For example,

$$\alpha_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}.$$

The *cofactor of the element* a_{ij} is denoted by A_{ij} and is equal to the minor α_{ij} taken with the sign $(-1)^{i+j}$

$$A_{ij} = (-1)^{i+j}\alpha_{ij}.$$

For example,

$$A_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = (-1)^{2+1}\alpha_{21}.$$

Theorem 1. *The determinant is equal to the sum of the products of the elements of any row by the corresponding cofactors.*

For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}.$$

***Theorem* 2.** *The sum of the products of the elements of any row by the cofactors of any other row is zero.*

For example,

$$a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} = 0.$$

The content of Theorems 1 and 2 may be given as follows:

$$(22) \qquad a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} = \begin{cases} A & (\text{if } i = j) \\ 0 & (\text{if } i \neq j) \end{cases}$$

When $i = j$ in (22), we have the "expansion of the determinant by cofactors of the ith row."

Theorem 3. Theorems 1 and 2 are valid for columns as well as rows.

For columns, Equation (22) becomes

$$(23) \qquad a_{1j}A_{1k} + a_{2j}A_{2k} + a_{3j}A_{3k} = \begin{cases} A & (\text{if } j = k) \\ 0 & (\text{if } j \neq k) \end{cases}$$

When $j = k$ we have the "expansion of the determinant by cofactors of the jth column."

From Theorem 3 it is possible to obtain *Cramer's rule* for the solution of three simultaneous linear equations in three unknowns x_1, x_2, x_3:

$$(24) \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

To solve for x_i multiply the first equation in (24) by A_{1i}, the second by A_{2i}, and the third by A_{3i}, and add the resulting equations. There is obtained

$$(25) \qquad x_1(a_{11}A_{1i} + a_{21}A_{2i} + a_{31}A_{3i}) + x_2(a_{12}A_{1i} + a_{22}A_{2i} + a_{32}A_{3i}) + x_3(a_{13}A_{1i} + a_{23}A_{2i} + a_{33}A_{3i}) = b_1A_{1i} + b_2A_{2i} + b_3A_{3i}.$$

Now it follows from (23) that only one of the parentheses on the left side of (25) is different from zero, namely, the coefficient of x_i, and that one is equal to A. Hence

$$(26) \qquad Ax_i = b_1A_{1i} + b_2A_{2i} + b_3A_{3i}.$$

But the right side of (26) is just the expansion of the determinant obtained by replacing in the determinant A the ith column by b_1, b_2, b_3. If the value of the resulting determinant is denoted by B_i, we have

$$(27) \qquad Ax_i = B_i,$$

or if $A \neq 0$

$$(28) \qquad x_i = \frac{B_i}{A},$$

which is Cramer's rule.

For more equations in more unknowns the Equations (27) and (28) are also valid. In that case A and B_i are nth-order determinants.

Several inferences may be drawn from (27) and (28). First, if $A \neq 0$ there is a unique solution of (24) given by (28). Second, if $A = 0$ the equations are consistent only if $B_i = 0$ for $i = 1, 2, 3$.

The linear system (24) is said to be *homogeneous* if $b_1 = b_2 = b_3 = 0$. Equation (24) is then obviously satisfied by the "trivial solution" $x_1 = x_2 = x_3 = 0$. Usually we are interested in nontrivial solutions. We may then assert that *a system of homogeneous linear equations has only the trivial solution if $A \neq 0$. Conversely, if $A = 0$ one may prove that nontrivial solutions necessarily exist.*

For greater detail see Ref. 8.

PROBLEMS

1. Find the cofactors of the determinant

$$\begin{vmatrix} -1 & -2 & -3 \\ 2 & 1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$$

Verify Theorems 1-3 for this determinant.

2. Expand

$$\begin{vmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix}$$

by cofactors of the third row.

3. Show, using the theorems of this section, that the system

$$\begin{aligned} x - 2y + 3z &= 5 \\ 2x - y - z &= 2 \\ x - 5y + 10z &= 0 \end{aligned}$$

is inconsistent.

4. Does the system

$$\begin{aligned} 2x - y - z &= 0 \\ x + 3y - z &= 0 \\ x - 11y + z &= 0 \end{aligned}$$

have a nontrivial solution?

CHAPTER II

GEOMETRIC FUNDAMENTALS FOR FIRST-ORDER DIFFERENTIAL EQUATIONS

1. Introduction. The study of differential equations had its origin in the investigation of physical laws. Because the basic physical laws are generally stated in a form involving derivatives (that is, as differential equations), the determination of the relations among the quantities concerned requires the solution of differential equations. With this practical stimulus the theory of differential equations has been developed by a multitude of investigators and is still growing at a rapid rate and still receiving impetus from physical problems. In spite of its intimate relation to applications the theory of differential equations has an independent existence of its own consisting of a self-coherent body of knowledge. It is a fact, perhaps curious to one unacquainted with the details, that only by the independent study of the theory of differential equations for its own sake can sufficient clarity of understanding be obtained to allow confident application to physical problems.

The present chapter states the geometric problem that is posed by a first-order differential equation and aims at giving the student sufficient insight into the problem to remove some of the mystery from the subject.

2. Differential Equations Defined. Suppose that $x, y, z, \ldots$ are independent variables and that $u, v, w, \ldots$ are functions of $x, y, z, \ldots$. A differential equation asserts that there is some functional relationship between the independent variables $x, y, z, \ldots$, the dependent variables $u, v, w, \ldots$, and some of the derivatives of $u, v, w, \ldots$ with respect to $x, y, z, \ldots$. For example,

(1) $\dfrac{d^2z}{dt^2} + t^2\dfrac{dz}{dt} - 3zt = t + 1$ (z, dependent variable; t, independent variable)

(2) $\dfrac{d^2u}{dx^2}\dfrac{du}{dx} + 3u = 5$ (u, dependent variable; x, independent variable)

(3) $F\left(x, y, \dfrac{dy}{dx}\right) = 0$ (y, dependent variable; x, independent variable)

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y)$$ (u, dependent variable; x, y, independent variables)

(5) $$\frac{d^3 u}{dt^3} + \frac{dv^4}{dt} = 0$$ (u, v, dependent variables; t, independent variable)

If there is but *one independent variable*, as in equations (1), (2), (3), and (5), the differential equation is called *ordinary*. If there are several independent variables, so that the derivatives involved are *partial derivatives*, the equation is a *partial* differential equation, as in (4). This book is concerned solely with ordinary differential equations so that differential equation will always mean ordinary differential equation unless the contrary is explicitly stated.

Differential equations are also classified according to the order of the highest derivative involved. Thus (3) is of the first order; (1), (2), and (4) of the second order; and (5) of the third order.

PROBLEM

1. Classify as to order.

(a) $x^3 \frac{d^3y}{dx^3} + e^y \left(\frac{d^2y}{dx^2}\right)^3 = 7.$ (b) $\frac{d^4y}{dx^4} + 5\left(\frac{dy}{dx}\right)^2 + 8y = \sin x.$

(c) $\frac{dz}{dt} = \tan z + t.$ (d) $\frac{d^2u}{dt^2}\frac{du}{dt} + \left(\frac{du}{dt}\right)^3 + u = 0.$

(e) $x\frac{dy}{dx} + y = e^x.$ (f) $\sin\left(\frac{dy}{dx}\right) + \frac{dy}{dx} = y.$

3. Solution or Integral of a Differential Equation. The general ordinary differential equation of the nth order in one dependent variable is

(6) $$f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

where the function $f(x, y, y', \ldots, y^{(n)})$ is defined for certain values of $x, y, y', \ldots, y^{(n)}$. In particular cases the range of values for which the function is defined is usually evident from its form, but in any theoretical discussion of (6) we would have to be sure of the domain of definition of the function as well as its continuity and differentiability properties. In general we will not try to be overly cautious in these matters as extra care might tend to obscure the essence of the argument. Consequently, all functions are supposed to be continuous and to have all derivatives that may be needed unless explicitly stated otherwise.

By a solution, or integral,† of (6) is meant a function $y(x)$ such that when y and its derivatives are substituted in (6) there results an identity in x.

Example 1. Show that the functions

$$y = \pm \sqrt{x^2 - cx}$$

are integrals of

$$2xy \frac{dy}{dx} = x^2 + y^2.$$

Differentiating the functions we have

$$\frac{dy}{dx} = \frac{\pm(2x - c)}{2\sqrt{x^2 - cx}}$$

and substituting in the differential equation yields

$$\frac{2x(\pm\sqrt{x^2 - cx})[\pm(2x - c)]}{2\sqrt{x^2 - cx}} = x^2 + x^2 - cx$$

or

$$2x^2 - cx = 2x^2 - cx$$

which is an identity in x.

Sometimes it is inconvenient or impossible to express y explicitly in terms of x. In this case the solution may be given in the implicit form $g(x, y) = 0$. This equation defines y implicitly in terms of x. The derivatives are then expressed in terms of x and y, and when these derivatives are substituted in the differential equation there results either an identity in x and y or an equation which is valid by virtue of the equation $g(x, y) = 0$.

Example 2. The same differential equation as in Example 1 will illustrate the implicit form for a solution. Squaring the previous solution yields

$$y^2 = x^2 - cx.$$

Differentiating implicitly yields

$$2y \frac{dy}{dx} = 2x - c$$

or

$$\frac{dy}{dx} = \frac{2x - c}{2y}.$$

† The terms "integral" and "solution" when applied to differential equations are synonymous. Integration in the sense of elementary calculus will often be referred to as *quadrature*.

Substituting in the differential equation gives

$$2xy\,\frac{2x - c}{2y} = x^2 + y^2$$

or

$$2x^2 - cx = x^2 + y^2$$

which is an identity by virtue of $y^2 = x^2 - cx$.

PROBLEM

1. Verify that the following functions are integrals of the corresponding differential equations:

(a) $y = a^2 + \frac{a}{x}$, $x^4 \left(\frac{dy}{dx}\right)^2 = y + x\,\frac{dy}{dx}$.

(b) $y = x \sin (a - x)$, $y = x\,\frac{dy}{dx} + x\sqrt{x^2 - y^2}$; $x \cos (a - x) > 0$.

(c) $y = \frac{x^2 + \log x^2 + 1}{(1 + x^2)^2}$, $x(1 + x^2)\,\frac{dy}{dx} + 4x^2 y - 2 = 0$.

(d) $x^2 = cy + c^2$, $xy'^2 = 2yy' + 4x$.

(e) $\sin y + y = x$, $\frac{dy}{dx}(y \cos y - \sin y + x) = y$.

(f) $(x - a)^2 + (y - b)^2 = r^2$, $y''^2 r^2 = (1 + y'^2)^3$.

(g) $y = c_1 e^{2x} + c_2 e^{-3x} - \frac{x}{6} - \frac{1}{36}$, $y'' + y' - 6y = x$.

(h) $y = A \cos (mt + B)$, $\frac{d^2y}{dt^2} + m^2 y = 0$.

(i) $\sin u = (A + Br)e^r + e^{-r}$, $\frac{d^2u}{dr^2} \cos u - \left(\frac{du}{dr}\right)^2 \sin u - 2\left(\frac{du}{dr}\right) \cos u + \sin u = 4e^{-r}$.

4. General Solutions. As will be seen below, a differential equation has infinitely many solutions. The student is already familiar with this fact in an especially simple case from calculus. For example,

$$\frac{dy}{dx} = f(x)$$

is a differential equation whose solution is obtained by a simple quadrature:

$$y = \int f(x)\,dx + C = F(x) + C,$$

where $F(x)$ is any indefinite integral of $f(x)$. The arbitrary constant C is called the constant of integration, and different values for C give different solutions. The solution $y = F(x) + C$ thus represents a one-parameter family of curves, and as we know from the theorem of

Sec. 2, Chap. I, *every* continuous solution of the differential equation $y' = f(x)$ is obtained from $y = F(x) + C$ by assigning some particular value to the constant C.

For more complicated first-order differential equations, however, one is unable to describe so simply all the solutions. (This will be more apparent later on.) In the case of nth-order differential equations the difficulties are even more pronounced. For these reasons we lay down the following:

Definition. A general solution of the nth-order differential equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is an n-parameter family† *of curves*

$$g(x, y, C_1, \ldots, C_n) = 0$$

such that each curve of the family is a solution of the differential equation.

We remark that we do not call such a family "the" general solution but merely "a" general solution. There is no need for the family to be composed of all solutions; and indeed as often as not there will be solutions which are not members of the family. In so far as applications are concerned, however, we will usually find that the solution we need can be found by assigning particular values to the constants occurring in a general solution.

It is by no means obvious at this point that a first-order equation possesses a general solution, but we will see that such is the case later in this chapter. The nth-order differential equation will be taken up later on. For the moment we will be concerned with a converse proposition which asserts that *an n-parameter family of curves is a general solution to some nth-order differential equation.* This may be seen as follows.

Suppose that $g(x, y, C_1, \ldots, C_n) = 0$ is the equation of the family of curves. Differentiating n times with respect to x yields n more equations, the last of which contains d^ny/dx^n. Elimination of $C_1, \ldots, C_n$ between the resulting $n + 1$ equations then gives one equation in $x, y, y', \ldots, y^{(n)}$ which is the required nth-order differential equation.

Example 3. $g(x, y, C_1, C_2) \equiv C_1y - \sin(2x + C_2) = 0$. Find a second-order differential equation which has this function as its general solution. Differentiating twice gives

$$C_1y' - 2\cos(2x + C_2) = 0$$
$$C_1y'' + 4\sin(2x + C_2) = 0.$$

† For the definition of an n-parameter family of curves see Sec. 4, Chap. I.

From the last of these and the given equation we get easily

$$y'' + 4y = 0,$$

the desired differential equation.

It would be possible of course to differentiate

$$g(x, y, C_1, \ldots, C_n) = 0$$

more than n times and so derive a differential equation of order higher than n. If this were done, however, the resulting differential equation would not have the given family as a general solution because there would be too few arbitrary constants involved.

PROBLEMS

1. In the previous set of problems (page 22) which solutions are general solutions?

2. Find differential equations for which the following are general solutions:

(a) $x^2 + (y - c)^2 = r^2$ (c is the parameter).
(b) $y = c_1x + c_2x^2$.
(c) $y^2 = x^2 + cx + 1$.
(d) $y = c_1e^{-x} + c_2e^x + x^2$.
(e) $u = A \sin 3t + B \cos 3t$.
(f) $v = c_1 \sin (3t + c_2)$.
(g) $\log y + y^2 = Ax^2 + B$.

3. Find a differential equation which has for a general solution the family of all circles with centers on the line $y = x$.

4. Find a differential equation having as a general solution the family of all ellipses with center at the origin and axes parallel to the coordinate axes.

5. First-order Equations and Direction Fields. The most general form for a first-order equation is

$$F\left(x, y, \frac{dy}{dx}\right) = 0, \tag{7}$$

where the function† $F(x, y, p)$ is defined in some region of xyp space. We will suppose that (7) can be solved for dy/dx in terms of x and y yielding

$$\frac{dy}{dx} = f(x, y), \tag{8}$$

where the function $f(x, y)$ is a continuous function of x and y in the rectangle‡ R, $a < x < b$, $c < y < d$. If there is more than one solu-

† The notation $p = dy/dx$ is standard and will henceforth be used without comment.

‡ As mentioned before, in most applications the domain where $f(x, y)$ is defined is

tion of $F(x, y, p) = 0$ for p, then each solution is of the form (8) and can be treated separately.

A solution, or integral, of (8) is a function $y(x)$ such that

$$\frac{dy}{dx} = f(x, y(x))$$

whenever the point $(x, y(x))$ lies within R. Thus if the solution $y(x)$ is represented graphically as a curve and (x, y) is a point on it, the slope of the tangent is given by (8). In other words as soon as we know that an integral curve passes through a point we also know the direction of the curve at that point. This situation is described by saying that (8) defines a *direction field.* A direction field then is given if with each point (x, y) there is associated a number $p = f(x, y)$ which is the slope of a line through that point. The direction field can be represented graphically by drawing at (x, y) a short line segment with slope $p = f(x, y)$. This line segment is called a *line element.*

Example 4. The direction field $p = x$ is indicated in Fig. 6 where some line elements are drawn for $x = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$.

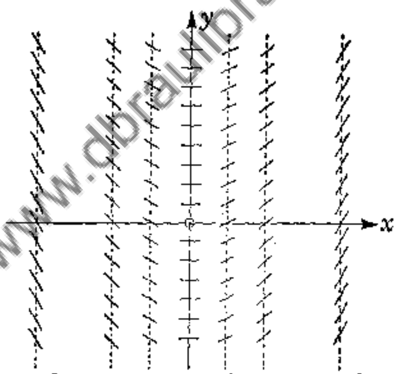


FIG. 6.

Suppose now that we have given a curve $y = y(x)$ in the rectangle R, which has at each point a tangent whose slope is given by (8). This curve then is an integral of the differential equation, and we see that finding an integral is equivalent to drawing a curve which has at each of its points the same direction as that of the direction field. This can be done roughly freehand after the direction field is drawn.

To draw the direction field for a given differential equation (8) a device may be used called the *method of isoclines.* An *isocline* is a curve through the direction field along which $p = f(x, y)$ is constant. It is itself *not* an integral curve. The family of isoclines then is the

evident from the form of the function and in many cases will be the whole plane. The situation described here contains all cases of interest to us. If we agree that a, b, c, or d may be $\pm\infty$ we have included the possibility that $f(x, y)$ is defined over the whole plane.

family of curves $f(x, y) = k$. Choosing a few values for k will indicate the family of isoclines. Now along each isocline draw several parallel line elements, that is, line segments of slope $f(x, y) = k$. We have now a sketch of the direction field. This was the procedure used in Fig. 6 where the dotted lines are the isoclines.

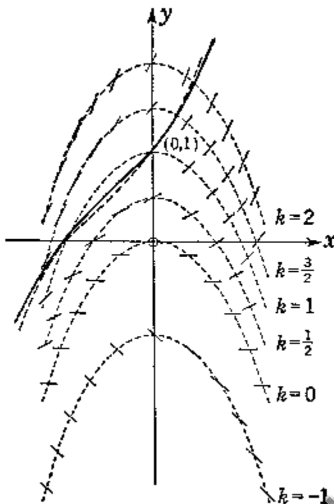


FIG. 7.

Example 5. Sketch the direction field of

$$\frac{dy}{dx} = x^2 + y \quad (R \text{ is the whole plane})$$

by the method of isoclines and draw the integral curve which passes through the point (0, 1).

The isoclines are the parabolas

$$x^2 + y = k$$

and are drawn dotted in Fig. 7 for $k = 2, \frac{3}{2}, 1, \frac{1}{2}, 0, -1$. The heavy solid curve is approximately the integral curve through (0, 1) and was drawn freehand to coincide as well as possible with the direction field. The heavy dashed line is the exact solution through (0, 1),

$$y = 3e^x - x^2 - 2x - 2$$

which was obtained by methods given in the next chapter. In many cases, however, an exact solution would not be available.

PROBLEMS

1. Draw the direction fields for the following differential equations and sketch the family of integral curves:

(a) $\frac{dy}{dx} = x^2$.

(b) $p = 1 - x$.

(c) $p = x + y$.

(d) $p = y^2$.

★(e) $p^2 = x$.

2. For what values of α and β can one solve $p^2 + \alpha p + \beta = 0$ graphically, by sketching the direction field?

3. Given the differential equation $y = px + p^2$, how many integral curves pass through the point $(-1, 2)$? Are there any points in the plane through which no integral curves pass?

6. The Basic Existence Theorem. From the preceding it is geometrically evident that there are infinitely many integral curves for Equation (8) and that they form a one-parameter family.† The parameter may be chosen in a variety of ways, one possible choice in Example 5 being the y intercept; that is, if the y intercept of the integral curve is specified, then the curve is uniquely determined.

While our considerations to date have led us to believe that Equation (8) always has a solution, we cannot as yet be assured of this fact. This assurance is provided by an *existence theorem.* Such a theorem asserts that a given problem has, under certain circumstances, a solution and possibly that with certain further restrictions the solution is unique. It is part of the business of the mathematician to provide such existence theorems. But this is by no means the only task facing the mathematician; for the knowledge that a solution exists may be of little value unless some definite technique can be given which will enable one to find that solution with at least a fair degree of accuracy.

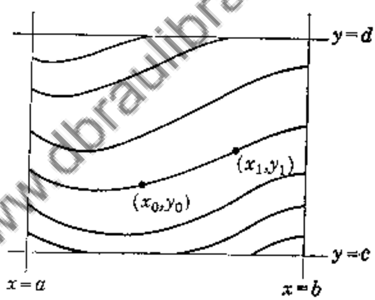


FIG. 8.

In this paragraph we will state the fundamental existence theorem for first-order equations, a proof of which may be found in Appendix A. From this result we will be able to conclude that the solutions to a first-order equation form a one-parameter family of curves.

Theorem. *If $f(x, y)$ is continuous in the rectangle R and if $\partial f(x, y)/\partial y$ is continuous in R, then through each point (x_0, y_0) of R there passes a unique integral curve $y = y(x)$ of the differential equation $dy/dx = f(x, y)$.*

The picture might appear something like that in Fig. 8. As indicated in the figure there is no necessity for any integral curve to extend from one end of the rectangle to the other.

Since through each point (x_0, y_0) in R there passes a unique integral curve, the integral curves may be written

$$y = g(x; x_0, y_0), \tag{9}$$

† For a review of the geometry of one-parameter families see Sec. 4, Chap. I, and the problems at the end of that section.

that is, to each (x_0, y_0) there is a corresponding function. It might appear from (9) that the family of integral curves is a two-parameter family because of the appearance of the two "parameters" x_0 and y_0 in (9). A moment's reflection, however, shows the fallacy of this view as infinitely many points give rise to the same solution. That is, if (x_0, y_0) and (x_1, y_1) are any two points on the same integral curve (see Fig. 8), then

$$g(x; x_0, y_0) \equiv g(x; x_1, y_1).$$

To parameterize the family of integral curves we must attach a number c to each curve. This might be done as follows. Draw a smooth curve C cutting across each integral curve† once and only once and suppose that the curve C is given in parametric form by

$$x = \varphi(c), \qquad y = \psi(c).$$

Then to each value of c we have exactly one point on each integral curve and

$$y = g(x; \varphi(c), \psi(c))$$

is the desired parameterization.

It should be noted that the existence theorem does more than assert that a solution exists through a given point (x_0, y_0) but says further that the solution is unique (that is, the only one). It is desirable sometimes to know that if only the continuity of $f(x, y)$ is assumed one is still able to make the assertion that a solution exists. The requirement of the existence of $\partial f(x, y)/\partial y$ at each point of R is sufficient to ensure that the solution through (x_0, y_0) be unique.‡ It should be observed that uniqueness applies only near (x_0, y_0), or in other words the integral curve through (x_0, y_0) might extend far enough to leave the region where $\partial f/\partial y$ exists and this extension could then fail to be unique.

Example 6. Find the portion of the plane in which the differential equation $(dy/dx) = y^2 + \arcsin x$ has a solution and determine whether the solution through any point of the region is unique.

† It is by no means obvious from the existence theorem that this is possible but a rigorous proof would take us too far afield, and as a matter of fact, it is not always possible to find a curve cutting *all* the integral curves. One can only assert that it is possible to cut across all integral curves sufficiently "near" any given one. In the case under consideration a vertical line would suffice.

‡ The requirement however is *not necessary*, it is merely sufficient. There are differential equations for which $\partial f/\partial y$ does not exist and which nevertheless have unique solutions.

Since arcsin x is defined only for $-1 \leq x \leq 1$, the integral curves cannot extend outside the strip $-1 \leq x \leq 1$. Because $y^2 + \arcsin x$ is continuous in this strip there is an integral curve through each point of the strip, and since $\partial f/\partial y$ exists and is equal to $2y$ the solution is unique.

Example 7. Determine regions of existence and uniqueness of solutions of the equation $dy/dx = 2\sqrt{y}$.

The function $\sqrt{y}$ is defined only for $y \geqq 0$, where it is continuous. The region of existence therefore is the upper half-plane $y \geqq 0$. The derivative $\partial f/\partial y = 1/\sqrt{y}$ exists only for $y > 0$, and so our existence theorem tells us only that the solution through (x_0, y_0) is unique if $y_0 > 0$ but asserts nothing if $y_0 = 0$. As a matter of fact the family of integral curves appears as in Fig. 9. The solution may be found by the methods of the next chapter, and any integral curve consists of half of a parabola extending as in the figure to the right of its vertex $(c, 0)$ on the x axis, while for $x < c$ it coincides with the x axis. The integral curve through (x_0, y_0) is drawn in **Fig. 9** as a heavy line.

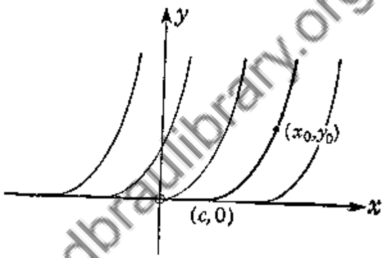


FIG. 9.

Looking at the figure it is evident that there are infinitely many integral curves passing through a point on the x axis, namely, the trivial one $y = 0$ and the others obtained by moving along the x axis to an arbitrary point $(c, 0)$ and then moving along the parabola.

PROBLEMS

1. Show that the differential equation $dy/dx = x^2 + y$ has unique integral curves passing through each point of the plane.

2. Determine regions of existence and uniqueness of solutions of the equation $dy/dx = 3y^{1/3}/2$. Sketch the family of integral curves.

3. Find those points of the plane through which integral curves of the differential equation $p = \sqrt{x}$ pass. Are these integral curves unique?

7. Orthogonal Trajectories. Suppose we have given in a rectangle R two families of curves $\varphi(x, y; \alpha) = 0$ and $\psi(x, y; \beta) = 0$, where α and β are parameters such that through each point (x_0, y_0) of R there passes a single member of each family. In general the two curves through each point could intersect at any angle. If, however, these two curves are perpendicular (orthogonal), then each curve of one family is said to be an *orthogonal trajectory* of the curves of the other family.

Example 8. Show that the circles $x^2 + y^2 - ky = 0$ are orthogonal trajectories of the family of circles $x^2 + y^2 - cx = 0$.

Solving the equations of the two circles simultaneously we find that they intersect at $(0, 0)$ and $[ck^2/(c^2 + k^2), c^2k/(c^2 + k^2)]$. The derivative for the first circle is

$$\frac{dy}{dx} = \frac{2x}{k - 2y},$$

and for the second circle,

$$\frac{dy}{dx} = \frac{c - 2x}{2y}.$$

Substituting in these derivatives the coordinates of the point of intersection $[ck^2/(c^2 + k^2), c^2k/(c^2 + k^2)]$ we get for the slopes of the tangents to the two circles

$$m_1 = \frac{2ck}{k^2 - c^2}$$

and

$$m_2 = \frac{c^2 - k^2}{2ck}.$$

Since $m_1m_2 = -1$ the circles are orthogonal. Because the circles are orthogonal regardless of k we see that the family $x^2 + y^2 - ky = 0$ is the set of orthogonal trajectories of $x^2 + y^2 - cx = 0$, and conversely. It might be objected that each curve of one family cuts each curve of the other family in two points instead of only one. However by deleting the origin from each curve this objection is removed (see Fig. 10).

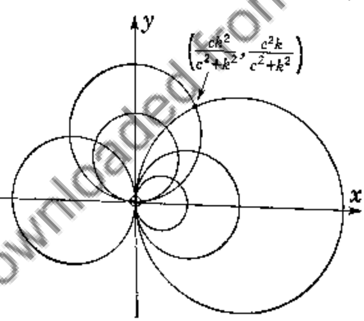


FIG. 10.

We have seen that the differential equation $p = f(x, y)$ defines a one-parameter family of curves (the integral curves) whose slope at each point is equal to $f(x, y)$. An orthogonal trajectory to this family of curves would have therefore at the point (x, y) a slope equal to $-1/f(x, y)$. Consequently the family of orthogonal trajectories satisfies the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

Example 9. Find the differential equation of the orthogonal trajectories of the family of parabolas $y^2 = 4cx$.

Differentiating, we have $2yy' = 4c$ or

$$\frac{dy}{dx} = \frac{2c}{y} = \frac{2y^2/4x}{y} = \frac{y}{2x}$$

as a differential equation satisfied by the given family; consequently the orthogonal trajectories satisfy

$$\frac{dy}{dx} = -\frac{2x}{y}.$$

PROBLEMS

1. Write down the differential equations satisfied by the family of parabolas $y^2 = 2px$ and their orthogonal trajectories.

2. Show analytically that $y = cx$ is an orthogonal trajectory of the family of circles $x^2 + y^2 = r^2$.

*3. Show that the family of curves $y^2 = 2\lambda\left(x + \frac{\lambda}{2}\right)$ is self-orthogonal. Sketch the figure.

4. Find a differential equation for the orthogonal trajectories of $y^2 = cx^3$. Show that the ellipses $2x^2 + 3y^2 = k^2$ satisfy the differential equation and are therefore orthogonal trajectories.

8. Numerical Solution of $p = f(x, y)$. We have seen that specifying one point for an integral curve to pass through will (with slight restrictions) determine a unique member of the family of integral curves. Now in applications it is usually this particular integral curve which one wishes to find. Getting a general solution to the differential equation is merely a preliminary step in picking out the particular solution required, and any method which would yield the desired particular solution would presumably be as satisfactory as any other method. Now the solution whose existence is asserted by the existence theorem may be difficult or impossible to express in terms of known functions; consequently we may find it desirable to use some graphical method (say by drawing the direction field and sketching in the solution) or some numerical method. It is such a numerical method (suggested by the geometry of the direction field) that we will describe now. The method consists in a step-by-step piecing together of an approximation to the solution. The procedure will be explained by means of an example.

Example 10. Find an approximate solution of $dy/dx = (x + y)/2$ which passes through the point (0, 1).

We will suppose that the solution is desired between $x = 0$ and $x = 2$. This limits the amount of calculation required. Any other interval over which the solution might be desired could be handled similarly.

Divide the interval from $x = 0$ to $x = 2$ into a number of subintervals, say 4. The number of subintervals to use depends on the differential equation and the accuracy required. We choose the subintervals here to be of equal length, but this is not essential. At any rate, with our choice the intervals are of length $h = 0.5$.

We substitute $x = 0$, $y = 1$ in the differential equation and find that at the initial point (0, 1) the integral curve through that point has slope equal to 0.5. As x increases from 0 to h, y will change by an amount Δy which possibly may be approximated with sufficient accuracy by the differential $dy = (dy/dx)h$. We have then

$$dy = (0.5)(0.5) = 0.25,$$

and the point $(0 + 0.5, 1 + 0.25)$ should lie approximately on the curve. The method consists merely in assuming that this new point A, namely, (0.5, 1.25), actually is on the integral curve through (0, 1) and then repeating the procedure. Thus the slope at (0.5, 1.25) is found from the differential equation to be 0.875; whence

$$dy = (0.875)(0.5) = 0.44$$

and the next approximate point B is $(0.5 + 0.5, 1.25 + 0.44)$ or (1, 1.69). Continuation of this procedure leads to the polygonal path shown in Fig. 11 as a thin solid line.

Clearly this approximate method can lead to considerable numerical labor for complicated equations, and it is therefore desirable always to choose h as large as possible without destroying the accuracy. In order to indicate the effect of choosing a smaller value for h, the approximate solution for $h = 0.25$ is drawn in Fig. 11 as a dashed polygonal line. The exact solution obtained by the methods of Chap. III is also indicated. As may be seen from the figure, the error for the curve $h = 0.25$ is about 10 per cent at $x = 2$.

It is natural to suppose that the rather crude method we have sketched is capable of some refinement, and such actually is the case. A more complete discussion of numerical methods for solving differential equations can be found in several books on the subject,† and

† See, for example, J. B. Scarborough, "Numerical Methods in Analysis," pp. 220–230, Johns Hopkins Press, 1930.

we will content ourselves here with describing a very simple refinement which is quite effective although it involves somewhat more labor than some of the other methods.

Our refinement is based on the simple observation that dy/dx is not constant in the interval of length h, and therefore, in approximating the actual change in y, Δy, we should use an average value of dy/dx rather than its value at only one end of the interval. An average value for dy/dx can be easily obtained by an iteration process as follows.

Consider our example. We found the slope at the point A to be 0.875. Since the slope at (0, 1) was 0.5, an average slope would be $(0.5 + 0.875)/2 = 0.69$. This will give us a new

$$dy = (0.69)(0.5) = 0.34$$

and a new approximate point A', namely, (0.5, 1.34). We now compute the slope at A' from the differential equation and obtain 0.92, whence also a new average slope equal to $(0.5 + 0.92)/2 = 0.71$ and another dy approximating Δy, $dy = (0.71)(0.5) = 0.36$ and therefore a third approximate point A'', namely, (0.5, 1.36). Repeating the procedure once more gives slope at A'' equal to 0.93 and average slope equal to 0.72. Whence our new $dy = 0.36$. Since this coincides with the previous value our iteration process can go no farther. We take then the point A'' as being on the desired integral curve.

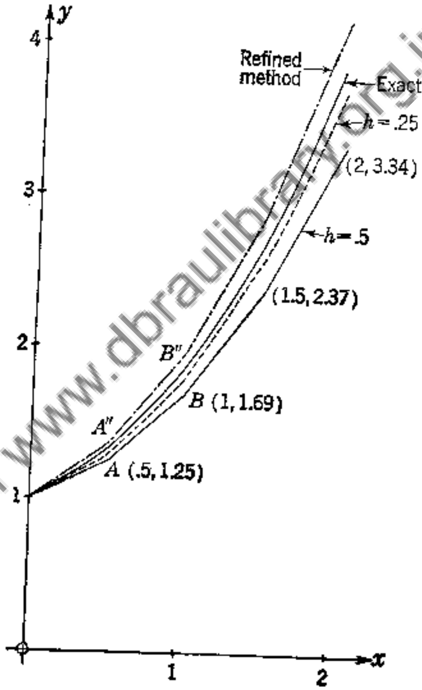


FIG. 11.

Repetition of the process yields the approximate solution drawn in Fig. 11 as a dotted line. As will be observed the accuracy is quite good.

We have illustrated a technique for numerical solution of differential equations by means of a particularly simple equation and in fact one which is readily solved exactly. It should, perhaps, then be emphasized that the method described is applicable to the most general equation of the first order $p = f(x, y)$, so that even if the special devices of the next chapter are inapplicable an approximate solution may be found.

PROBLEMS

1. Find by our approximate method the solution to $dy/dx = \frac{1}{2}$ which passes through $(0, 1)$. Why is the approximate solution exact in this case?

2. Find, between $x = 0$ and $x = 1$, an approximate solution of $dy/dx = y^2$ which passes through $(0, \frac{1}{2})$.

3. Find, between $x = 1$ and $x = 2$, an approximate solution of $dy/dx = x$ passing through $(1, 0)$. Compare your answer with the exact solution.

4. Find, between $x = 0$ and $x = 1$, an approximate solution of $dy/dx = \sin y$ passing through $(0, \pi/2)$.

9. Clairaut's Equation.† Because a general solution of a first-order differential equation consists of a one-parameter family of curves it is perhaps not unnatural to inquire as to the details if the family is particularly simple. Suppose then that the family consists of straight lines, one would conjecture that the differential equation would exhibit some interesting special features.

We therefore consider the set of all lines in the plane

$$Ax + By + C = 0, \tag{10}$$

where A, B, C are parameters. If a line is not parallel to the y axis, then $B \neq 0$ and we may solve for y in (10), getting

$$y = mx + b \tag{11}$$

where $m = -(A/B)$ and $b = -(C/B)$. Thus the collection of all (nonvertical) lines in the plane forms a two-parameter family. In order to obtain from this collection a one-parameter family we must specify some functional relationship between m and b. Let this relationship be written as $b = f(m)$, where $f(m)$ is a given function. We have therefore to consider the one-parameter family

$$y = mx + f(m) \tag{12}$$

and seek a first-order differential equation with this family as a general solution.

Differentiating (12) we get $y' = m$ and

$$y = y'x + f(y') \tag{13}$$

† Alexis C. Clairaut, French, 1713–1765, first made use of differentiation to solve differential equations in a paper devoted to the equation now named after him.

as a differential equation of which (12) is a general solution. Equation (13) is known as *Clairaut's equation* and has the virtue of extreme simplicity. Its solution can be written down by inspection: $y = mx + f(m)$.

Example 11. Find the family of tangent lines to the parabola $y = x^2$ and show that this family is a general solution of a Clairaut equation.

We find the equation of the tangent at an arbitrary point P_1, namely, (x_1, y_1), on the parabola. The slope at P_1 is $2x_1$; hence the tangent is given by

$$y - y_1 = 2x_1(x - x_1) = 2x_1x - 2x_1^2.$$

Because P_1 is on the parabola, $y_1 = x_1^2$, and we obtain for the tangent line

$$y = 2x_1x - x_1^2.$$

Since the slope $m = 2x_1$ this last equation may be written as

(14) $$y = mx - \frac{m^2}{4}.$$

Regarding m as a parameter we see from (14) that the tangents to $y = x^2$ form a one-parameter family. Clearly (14) is a general solution of the Clairaut equation

(15) $$y = px - \frac{p^2}{4}.$$

The parabola $y = x^2$ and family of tangents (14) are drawn in Fig. 12.

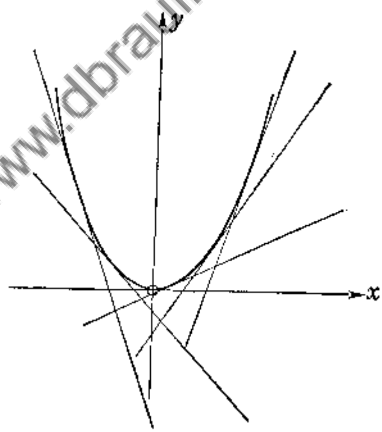


FIG. 12.

It will be observed that Clairaut's equation (13) is not solved explicitly for p. In other words it is a simple example of the implicit form for a first-order equation,

(7) $$F(x, y, p) = 0.$$

Now solving (7) for p gives p as function of x and y, which may be multivalued. In the above example, (15) is a quadratic in p so that on solving (15) for p we obtain the two equations

(16) $$p = 2x \pm 2\sqrt{x^2 - y}.$$

Hence we have a direction field with not one but two line elements at each point, and on inspecting Fig. 12 we see that at each point of the plane below the parabola $y = x^2$ there pass two integral curves of (15). If we choose, say, the positive sign for the square root in (16), we obtain a true direction field given by

$$p = 2x + 2\sqrt{x^2 - y}, \tag{17}$$

and this equation leads to the integral curves having the greater slope. The family of integral curves of (17) therefore is a family of half-lines tangent to $y = x^2$. These half-lines are drawn in Fig. 13.

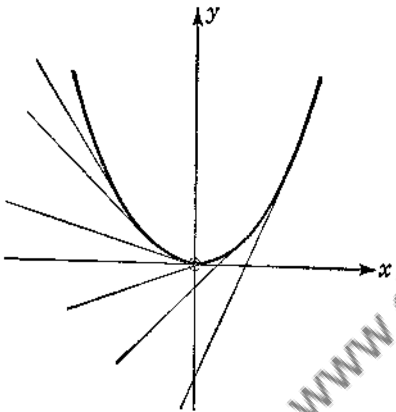


FIG. 13.

We conclude our discussion of Clairaut's equation

$$y = px + f(p) \tag{18}$$

by noting that we can solve the equation by differentiation. Differentiating (18) we have

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx},$$

or

$$\frac{dp}{dx}(x + f'(p)) = 0.$$

This last equation will certainly be satisfied if $dp/dx = 0$, or

$$p = m = \text{constant},$$

which leads to the general solution $y = mx + f(m)$. The other factor $x + f'(p)$ we disregard and merely mention that sometimes the elimination of p between $y = px + f(p)$ and $x + f'(p) = 0$ will lead to the singular solution discussed in the next section.†

PROBLEMS

1. Find a solution of $y = px - p^3$. Verify that it is a solution.

2. Show that $y = mx + 1/2m$ is the family of tangent lines to the parabola $y^2 = 2x$. What is the Clairaut equation satisfied by the given family? Is the parabola also an integral curve of the Clairaut equation? Draw the figure.

3. Show that the family of tangent lines to the circle $x^2 + y^2 = r^2$ is $y = mx \pm r\sqrt{1 + m^2}$. Write down the Clairaut equation which has this family as its solution. How many lines of the family pass through a point in the plane outside the circle? Sketch the figure.

† For greater detail see, for example, Ref. 3, pp. 85–89.

10. General, Particular, and Singular Solutions. A general solution of the first-order equation

$$F(x, y, p) = 0 \tag{7}$$

is defined to be a one-parameter family of integral curves

$$g(x, y, c) = 0. \tag{19}$$

We have seen that such a family exists when (7) is solved explicitly for p.

The integral curve obtained from (19) by assigning a particular value to c is called a *particular solution*. A particular solution therefore is associated with some general solution. Since there is no necessity for the family (19) to include all solutions, there may be integral curves not included in the collection of all particular solutions obtained from (19).†

Now it would be highly desirable if a general solution were to live up to its name and contain *all* solutions. Unfortunately this is not always the case, and we have purposely phrased our definition in such a way that no mention is made of all the solutions. From this point of view the term "general solution" loses much of its intuitive appeal, and it may be argued that some other terminology should be invented to describe the situation. A general solution is also called a "primitive." Nevertheless the term general solution seems to be so rooted in the literature on differential equations that we continue to use it and content ourselves with the definition given above. The difficulty is essentially this: the theory of differential equations is sufficiently complicated that it is hopeless to try and make statements as to the family of all solutions except in special cases. For example, if $y' = f(x)$, then every solution is of the form $y = \int f(x)\,dx + c$.

As a matter of fact we are usually interested in solving some special equation, and it is only with regard to this special equation that we should make any effort to find all the solutions, and even in special cases this may be no trivial task.

† Of course one of these solutions not included in (19) might be associated with some other general solution, say $h(x, y, k) = 0$.

It is important to note that if (x_0, y_0) is a point of the region of uniqueness of the differential equation it cannot lie on a singular solution. For if (x_0, y_0) lies in the region where integral curves are unique, then the integral curves "near" (x_0, y_0) can, as we have seen, be parameterized and the one through (x_0, y_0) becomes a particular solution. Thus the points on a singular solution are necessarily *not* in the region of uniqueness of the differential equation.

In order to have a name for integral curves which are not particular solutions we lay down the following definition: *a singular solution is an integral which is not a particular solution.*

The above definition asserts nothing about the relation of a singular solution to the members of a general solution, and little may be said simply in general. We merely remark that what we have in mind is the situation as it exists in rather elementary examples. We will see (at least in our examples) that singular solutions arise as *envelopes* to the family of curves in a general solution.

It is geometrically evident that an envelope to the curves of a general solution must itself be a solution because it has at each of its points the same slope as a member of the family and thus a slope coinciding with that of the direction field. (Exception must be made to this statement in the case of vertical tangents.)

The technique for finding the envelope (if one exists) is to eliminate the parameter c between the two equations (see Sec. 5, Chap. I)

$$g(x, y, c) = 0 \tag{20}$$
$$\frac{\partial g}{\partial c}(x, y, c) = 0.$$

The resulting equation, called the *eliminant* of Equations (20), then will contain the envelope. We say "contains the envelope" because the eliminant may be factorable and its graph may contain other loci besides the envelope.

A word of caution is necessary regarding the use of Equations (20) in finding the envelope. In the first place, Equations (20) may be inconsistent, in which case there may be no envelope. In the second place, even if Equations (20) are consistent and possess an eliminant, the curve of the eliminant may not be an envelope.

We have seen that a singular solution of a first-order differential equation may be obtained by finding the envelope of the family of curves of a general solution, $g(x, y, c) = 0$. Because the eliminant of (20) may contain other loci besides the envelope, *the eliminant must always be checked in the differential equation to determine whether or not it is a solution.* If it is a solution, it will be a singular solution. A curve whose equation is contained in the eliminant, and which is not a singular solution, is called an *extraneous locus.*†

Now let us see how these considerations apply to Clairaut's equation. For this purpose we return to Example 11 of the previous section.

† For greater detail see, for example, Ref. 3, pp. 85–89.

Example 11 (*continued*). From our mode of construction of the family $y = mx - (m^2/4)$ it is clear that the parabola $y = x^2$ is an envelope. Let us see if we get the parabola by the technique for finding envelopes described above. The Equations (20) now are

$$g(x, y, m) \equiv y - mx + \frac{m^2}{4} = 0$$
$$\frac{\partial g}{\partial m}(x, y, m) \equiv \quad -x + \frac{m}{2} \quad = 0.$$

Eliminating m between these equations we get $y = x^2$ as the eliminant. As we have mentioned it is necessary to check to see whether the parabola actually is a solution. Differentiating $y = x^2$ we get $y' = 2x$, and substituting in the differential equation $y = px - \dfrac{p^2}{4}$ we get

$$x^2 = 2xx - \frac{(2x)^2}{4} = x^2,$$

which is an identity in x. Thus $y = x^2$ is a singular solution.

It is interesting to examine our differential equation $y = px - (p^2/4)$ in the light of the existence theorem of Sec. 6. We solved this equation for p in Example 11, getting

$$(16) \qquad p = 2x \pm 2\sqrt{x^2 - y},$$

and by choosing only the positive square root obtained

$$(17) \qquad p = 2x + 2\sqrt{x^2 - y},$$

which gave us a single-valued direction field. We now examine (17) for regions of existence and uniqueness of solutions. Clearly (17) is defined (real) and continuous if $x^2 - y \geqq 0$, and this region coincides with the points of the plane on or below the parabola $y = x^2$. (Above the parabola we have $y > x^2$.) Thus the region of existence of a solution is given by $x^2 \geqq y$.

Taking the partial derivative with respect to y of $2x + 2\sqrt{x^2 - y}$ we get $-1/\sqrt{x^2 - y}$, and this derivative exists and is real if $x^2 - y > 0$. The region where this inequality is satisfied consists of the points *below* the parabola $y = x^2$, and this therefore certainly is a region of uniqueness of the differential equation. Examination of Fig. 13 will make this clear. We need to discuss the points on the parabola to determine whether they too should be included in the region of uniqueness. But it is quite clear that there are at least two solutions through a point on the parabola, namely, the straight-line solution and the singular solution (the parabola itself). Thus the region of uniqueness is precisely the portion of the plane below the parabola.

PROBLEMS

1. The equation $(dy/dx)^2 = (1 - y^2)/y^2$ has a general solution

$$y^2 + (x - c)^2 = 1.$$

Find the singular solution, if any. Sketch the family of integral curves.

2. Find the eliminant of the family of curves $(y - k)^2 = x(x - 1)^2$. Is it an envelope? Sketch the figure.

3. Find a differential equation which has the family $1 + ky + k^2x^2 = 0$ as a general solution. Determine any singular solution and sketch the figure.

4. Verify that $y = \sin (x + c)$ is a general solution of $p^2 = 1 - y^2$. Find the singular solution, if any.

5. Since the points on a singular solution have more than one integral curve through them, they are not points of uniqueness for the solutions. Apply this fact to the differential equations of Probs. 1 and 4 in order to determine the singular solution.

MISCELLANEOUS PROBLEMS

1. Draw the direction field for the equation

$$\frac{dy}{dx} = x^2 + y^2.$$

2. Draw the direction field and sketch the family of integral curves of the equation

$$p^2 + px - y = 0.$$

3. Find by the method of isoclines an approximate solution of

$$\frac{dy}{dx} = 2x - y$$

passing through (1, 1).

4. For the direction field of Prob. 3 find the locus of all points (x, y) in the plane such that the line element at (x, y) points toward the origin.

***5.** Show that for the direction field

$$\frac{dy}{dx} = ax + by + c$$

the locus of points (x, y) in the plane whose line elements are directed toward an arbitrary, but fixed, point (x_0, y_0) is a conic.

6. Find a differential equation with $y^2 = 2cx - c^2$ as a general solution. Sketch the family of integral curves and find the singular solution.

7. A function $f(x)$ satisfies the equation

$$\frac{d}{dx} f(x) = x^2 + [f(x)]^2$$

and $f(1) = 0$. Find approximately $f(1.5)$ to two decimal places.

8. Find a differential equation having $ky = k^2x + 1$ as a general solution. Does it have a singular solution?

9. Find the envelope of the family of lines $x \sin \omega + y \cos \omega = 1$. Sketch the figure.

10. Determine regions of existence and uniqueness for the differential equation $p^2 = \sin y$. Sketch the direction field and the integral curves.

11. Find a differential equation having the family of all nonvertical lines in the plane as a general solution.

12. Find a differential equation having the family of all circles as a general solution.

★13. Find a first-order equation having among its solutions the family of all straight lines through the origin and all circles concentric with the origin. (Hint: Look up the answer.)

14. Find the solution of $dy/dx = x$ which passes through (0, 0) (*a*) by integration; (*b*) by using simplest numerical method; (*c*) by using refined numerical method. Compare the results. If one keeps repeating the refined method is the solution exact?

CHAPTER III

TECHNIQUES FOR SOLVING FIRST-ORDER EQUATIONS. APPLICATIONS

1. Introduction. The geometry of the direction field defined by $p = f(x, y)$ has made the existence of a solution plausible, and by the existence theorem of Sec. 6, Chap. II, we have verified this conjecture. Furthermore, approximate methods are available to find solutions satisfying initial conditions. Nevertheless we are little better able to find integrals of first-order equations than before. The student has met this situation in integral calculus. Suppose that one is required to find an indefinite integral of some continuous function $f(x)$. We write down $\int f(x)\, dx$ and can be assured that it exists. That is, there is a function $F(x)$ whose derivative is $f(x)$. Nevertheless expressing this function in terms of elementary functions may be either difficult or impossible. Consequently much time is spent in calculus learning certain tricks which enable one to manipulate complicated functions until they can be easily integrated. Looking at things in this way it would be surprising if anything less involved were to occur in differential equations. The situation then that we are confronted with is as follows: To consider those types of first-order equations which are simple enough to be solved with little difficulty and occur often enough to justify their consideration.

The most important types of first-order equations are, first, exact equations; second, equations with variables separable; and third, linear equations. These are discussed in this chapter. If an equation does not fall under one of these types, an effort is usually made to transform it into one of these types by some substitution. Such substitutions are considered briefly. The special type $p = (a_1x + b_1y + c_1)/(a_2x + b_2y + c_2)$ is considered because of its importance in nonlinear equations and also because it introduces in a simple way the notion of singular point in a direction field.

2. Exact Equations. We suppose the first-order equations to be solved for dy/dx.

(1)
$$\frac{dy}{dx} = f(x, y).$$

It will be convenient to write (1) in an altered form. The function $f(x, y)$ may be expressed as the quotient of two functions

(2) $$f(x, y) = \frac{-M(x, y)}{N(x, y)}.$$

This is possible in many ways, a trivial one being obtained with $N(x, y) = 1$. The reason for the minus sign will appear shortly. Equation (1) now may be written

$$\frac{dy}{dx} N(x, y) + M(x, y) = 0.$$

Multiplying this last equation by dx and observing that $(dy/dx)\, dx$ is the differential of y we get

(3) $$M(x, y)\, dx + N(x, y)\, dy = 0,$$

which may be called the differential form of Equation (1).

Now suppose that we have an integral curve of (1) or (3) given in the implicit form

(4) $$u(x, y) = c = \text{constant}.$$

The total differential of u along the curve is zero,

(5) $$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy = 0$$

and, therefore,

(6) $$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y}.$$

Since (4) is an integral curve, (1) and (6) are identical, and the right member of (6) has the form of the right member of (2). Comparing (3) and (5) we inquire as to the circumstances under which the left member of (3) is the total differential of some function u, that is, when

(7) $$M(x, y) = \frac{\partial u}{\partial x} \qquad \text{and} \qquad N(x, y) = \frac{\partial u}{\partial y}.$$

If the left member of (3) is the total differential of a function $u(x, y)$, or what is the same, if Equations (7) are satisfied, then the differential equation (3) is called exact.

We can obtain a necessary condition for (7) to be valid as follows. Since $\partial^2 u/\partial x\, \partial y = \partial^2 u/\partial y\, \partial x$, Equation (7) implies that

(8) $$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}.$$

The importance of (8) lies in the fact that it is also sufficient. This may be seen by actually constructing the function $u(x, y)$. Set

(9) $$u(x, y) = \int^x M(x, y)\, dx + C(y),$$

where $C(y)$, the constant of integration, is a function of y which will be determined shortly and $\int^x$ signifies that we are integrating with respect to x holding y constant. Clearly $\partial u/\partial x = M(x, y)$.

Now from (7) we want $\partial u/\partial y$ to equal $N(x, y)$; so we set

(10) $$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int^x M(x, y)\, dx + \frac{dC}{dy} = N(x, y)$$

and investigate whether $C(y)$ may be determined to fit our requirements. Solving for dC/dy we get

(11) $$\frac{dC}{dy} = N(x, y) - \frac{\partial}{\partial y}\int^x M(x, y)\, dx,$$

and C may be found by a quadrature.

Now this "constant" of integration $C(y)$ must be a function of y alone; consequently the right member in (11) must be independent of x. This will be the case if

(12) $$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\int^x M(x, y)\, dx\right] = 0.$$

Carrying out the differentiation in (12) we get $\partial N/\partial x - \partial M/\partial y$, which is zero by virtue of (8).

We therefore have the following:

Theorem. If M and N are functions of x and y for which

$$\partial M/\partial y = \partial N/\partial x,$$

there is a function $u(x, y)$ whose total differential is $du = M\,dx + N\,dy$, and $u(x, y) =$ constant is an integral curve of $M\,dx + N\,dy = 0$.

Example 1. Show that $(x + y^2)\, dy + (y - x^2)\, dx = 0$ is exact and integrate.

We have $N = x + y^2$ and $M = y - x^2$ so that $\partial M/\partial y = 1 = \partial N/\partial x$ and (7) is satisfied.

To integrate the equation it is desirable to carry out the operations of (9) and (10) rather than to use (9) and (11) as "formulas" into which we substitute M and N. Thus,

$$u(x, y) = \int^x (y - x^2)\, dx + C(y) = xy - \frac{x^3}{3} + C(y),$$

whence

$$\frac{\partial u}{\partial y} = x + \frac{dC}{dy} = N = x + y^2.$$

Therefore,

$$\frac{dC}{dy} = y^2$$

and

$$C = \frac{y^3}{3}.$$

It is unnecessary at this point to add the constant of integration; for $C(y)$ is *any* function satisfying (10). Hence

$$u(x, y) = xy - \frac{x^3}{3} + \frac{y^3}{3}$$

and the integral of the differential equation is

$$xy - \frac{x^3}{3} + \frac{y^3}{3} = k = \text{constant}.$$

In some cases it may be possible to see by inspection that Equation (3) is the total differential of a function u. If such is the case the solution may be written down at once. For instance:

Example 2. Solve

$$\frac{dy}{x} - \frac{y\,dx}{x^2} = 0.$$

Writing this as

$$\frac{x\,dy - y\,dx}{x^2} = 0$$

we see at once that the left member is the differential of y/x so that the general solution is y/x = constant.

In other cases proper grouping of the terms in the left member of (3) will allow us to integrate by inspection. For instance:

Example 3. Solve

$$\frac{(y^2 - xy)\,dx}{xy^2} + \frac{x\,dy}{y^2} = 0.$$

Writing this as

$$\frac{dx}{x} - \frac{y\,dx - x\,dy}{y^2} = 0$$

we see that the left member is the total differential of $\log x - x/y$ so that the solution is

$$\log x - \frac{x}{y} = \text{constant}.$$

This technique of integration by inspection is greatly facilitated by detailed familiarity with the formulas of calculus. Naturally, since the equations of Examples 2 and 3 are exact, the earlier and longer technique will work as well.

PROBLEMS

1. Determine which of the following equations are exact:

(a) $\dfrac{dy}{dx} + y \cos x = e^{2x}$.

(b) $(1 + x^2)p + 2xy = 1$.

(c) $(x^2 + 2xy - y^2)\,dx - (x^2 - 2xy - y^2)\,dy = 0$.

(d) $\dfrac{x\,dy}{x^2 + y^2} - \dfrac{y\,dx}{x^2 + y^2} = 0$.

2. Test the following for exactness and, if exact, solve:

(a) $(y + 2x)\,dx + x\,dy = 0$.

(b) $\dfrac{(xy + 1)}{y}\,dx + \dfrac{2y - x}{y^2}\,dy = 0$.

(c) $(x^2 - y^2)\,dx = 2xy\,dy$.

(d) $\dfrac{dy}{y^2 + y} = \dfrac{dx}{x^2 - x}$.

(e) $(ax + by)\,dx + (bx + cy)\,dy = 0$.

(f) $(2x^2 + 4xy)\,dx - (2x^2 + y^2)\,dy = 0$.

(g) $\dfrac{x + y}{y - 1}\,dx - \dfrac{1}{2}\left(\dfrac{x + 1}{y - 1}\right)^2 dy = 0$.

(h) $(\sin x + x \sin y)\,dy - (\cos y - y \cos x)\,dx = \sin y\,dy$.

(i) $[\sin x \tan y + \cos (x + y)]\,dx - [\cos x \sec^2 y - \cos (x + y)]\,dy = dx$.

(j) $(2y + e^y \sin x)\,dy + e^y \cos x\,dx = 0$.

(k) $\left[\log (x^2 + y^2) + \dfrac{2x^2}{x^2 + y^2} + y\right] dx + \left(\dfrac{2xy}{x^2 + y^2} + x\right) dy = 0$.

(l) $\dfrac{y}{1 - x^2y^2}\,dx + \dfrac{x}{1 - x^2y^2}\,dy + dy = 0$.

(m) $\cos 2x \tan y\,dx + \sin x \cos x \sec^2 y\,dy + \sec^2 y\,dy = 0$.

(n) $(\cos 2x \tan x \cos y + \tan x \cos y)\,dx - \sin^2 x \sin y\,dy = 0$.

(o) $y^{\frac{1}{3}}[(xy)^{\frac{1}{2}} + (xy)^{\frac{4}{3}}]\,dx + x^{\frac{1}{3}}[(xy)^{-\frac{1}{2}} + (xy)^{-\frac{4}{3}}]\,dy = 0$.

(p) $\dfrac{-y + 1}{(x - y + 1)^2}\,dx + \dfrac{x}{(x - y + 1)^2}\,dy = 0$.

(q) $\dfrac{(x + y)^2}{1 + (x + y)^2}\,dx + \left[\dfrac{1}{1 + (x + y)^2} + 1\right] dy = 0$.

3. Integrate by inspection

(a) $x\,dy + y\,dx = 0.$

(b) $x\,dx + \dfrac{y\,dx - x\,dy}{y^2} = 0.$

(c) $x^2dx + \dfrac{dy}{1 + y^2} = 0.$

3. Variables Separable. The simplest exact equation is one in which M is a function of x alone and N a function of y alone:

(13) $$M(x)\,dx + N(y)\,dy = 0.$$

Then $\partial M/\partial y = 0 = \partial N/\partial x$, which verifies that (13) is exact. Using the technique for exact equations we have

$$u(x, y) = \int^x M(x)\,dx + C(y) = F(x) + C(y),$$

where $F(x)$ is any indefinite integral of $M(x)$. Then

$$\frac{\partial u}{\partial y} = \frac{dC}{dy} = N(y),$$

so

$$C(y) = \int^y N(y)\,dy = G(y),$$

where $G(y)$ is any indefinite integral of $N(y)$. The general solution of (13) then is

(14) $$F(x) + G(y) = k = \text{constant}.$$

Clearly, setting the differential of the left member of (14) equal to zero gives $M\,dx + N\,dy = 0$. In other words, the solution could have been obtained by integrating (13) by inspection. This is possible because of the fact that x and y are separated in (13). Any equation which can be reduced to the form (13) is said to have its *variables separable*. This type is much the simplest to handle so that the mode of attack on more complicated equations is often motivated by a desire to reduce the given equation by means of some change of variable to a form where the variables are separable.

Example 4. Solve $x\,dx + y\,dy = 0$ and discuss the family of integral curves.

Integrating the equation as it stands gives

$$\frac{x^2}{2} + \frac{y^2}{2} = C = \text{constant}$$

or $x^2 + y^2 = C$, where we have replaced $2C$ by C which is certainly permissible since both $2C$ and C are arbitrary constants. The integral curves then comprise a family of concentric circles with center at the origin for $C \geqq 0$ and imaginary loci for $C < 0$.

It may be noted that no more generality is obtained by adding constants of integration to the integrals $\int x\,dx$ and $\int y\,dy$. For this gives $x^2/2 + k_1 + y^2/2 + k_2 = C$ or $x^2 + y^2 = 2C - k_1 - k_2$, and $2C - k_1 - k_2$ is but an arbitrary constant.

Example 5. Solve $x\,dy + y\,dx = 0$.

Here the variables are not separated but obviously may be separated by dividing both sides by xy. (Naturally we must assume $x \neq 0 \neq y$.) Then

$$\frac{dy}{y} + \frac{dx}{x} = 0,$$

and the general solution is

$$\log y + \log x = C = \log xy$$

or

$$xy = e^C = k.$$

PROBLEMS

1. Find general solutions of the following:

(a) $\dfrac{dy}{dx} = \dfrac{y - 1}{xy}$.

(b) $x^3 + y^3 \dfrac{dx}{dy} = 0$.

(c) $xy\,dy = (y - 1)(x + 1)\,dx$.

(d) $\sqrt{1 - x^2}\,dy = \sqrt{1 - y^2}\,dx$.

(e) $(y^2 - y)\,dx = (x^2 + x)\,dy$.

(f) $x\,dy + (2x^2 - 1)\tan y\,dx = 0$.

(g) $(x^2 + x - 1)y\,dx + x(x^2 - 1)\,dy = 0$.

(h) $L\dfrac{di}{dt} + Ri = 0$.

(i) $2\sqrt{y}\,e^{2x}\,dx + \dfrac{dy}{1 + y} = 0$.

(j) $dx + (1 - x^2)\cot y\,dy = 0$.

(k) $(a^2 + y^2)\,dx + (a^2 + x^2)\,dy = 0$.

(l) $e^{x-y}\,dy + dx = 0$.

(m) $(x^2 + 1)\sin y\,dy + 2x\cos y\,dx = 0$.

(n) $y' = 1 - y^2$.

(o) $\dfrac{dy}{dx} = \dfrac{y^2 - 2y + 5}{-x^2 - 2x - 2}$.

2. Determine the integral curves of the following differential equations passing through the point indicated:

(a) $y^2(1 + x)\,dx - x^3\,dy = 0$; $(1, 2)$.
(b) $\sin x \sec^2 x \cos^2 y\,dx + dy = 0$; $(\pi, 0)$.
(c) $\rho \dfrac{d\theta}{d\rho} = k$; $(\rho = 1, \theta = 0)$.
(d) $e^x \sin y\,dx + (1 + e^x) \cos y\,dy = 0$; $\left(0, \dfrac{\pi}{4}\right)$.
(e) $\dfrac{dy}{dx} = \dfrac{x}{y}$; $(2, 1)$.
(f) $x^2 y\,dy - dx = x^2\,dx$; $(2, 2)$.
(g) $ye^{3x}dx - (1 + e^x)\,dy = 0$; $(\log 2, 3)$.
(h) $y(x\,dy - y\,dx) = x\,dy$; $(2, 1)$.
(i) $(xy - 1)\left(1 + \dfrac{dy}{dx}\right) = (y - x)\left(\dfrac{dy}{dx} - 1\right)$; $(e + 1, 2)$.
(j) $y' = \cos y\,[\cos (x + y) + \cos (x - y)]$; $\left(\dfrac{\pi}{6}, \dfrac{\pi}{4}\right)$.
(k) $yy' = xe^{x^2+y^2}$; $(\sqrt{\log 2}, 0)$.
(l) $x(y^2 + 1)\,dx + ye^{2x}dy = 0$; $(0, 0)$.

3. Find a function equal to its derivative.

***4.** Find all differentiable functions $f(x)$ which satisfy the functional equation

$$f(x' + x'') = f(x')f(x'') \text{ for all } x', x''.$$

Hint: Show that $f(x)$ satisfies the differential equation

$$\frac{d}{dx} f(x) = kf(x),$$

where k is a constant.

4. Integrating Factors. It may happen that Equation (3) is not exact as it stands but that multiplying both sides of (3) by a function $u(x, y)$ will render it exact. Such a function is called an *integrating factor*. The situation is exemplified by the following:

Example 6. Solve $x\,dy - y\,dx = (x^2 + y^2)\,dx$.

Dividing through by $x^2 + y^2$ (the integrating factor is $1/(x^2 + y^2)$) we have

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = dx,$$

and dividing numerator and denominator of the left side by x^2 yields

$$\frac{(x\,dy - y\,dx)/x^2}{1 + (y^2/x^2)} = dx.$$

The left member is now recognizable as the differential of arctan y/x; so the general solution may be written down by inspection, namely,

$$\arctan \frac{y}{x} = x + C.$$

From the above example it is clear that the discovery of an integrating factor may require considerable ingenuity and familiarity with the formulas of calculus. Consequently, as a practical technique it is severely limited by the skill of the user. Although it is not a simple matter to write down an integrating factor, it can easily be shown that the general equation of the first order (3) always possesses an integrating factor. This may be seen as follows:

Equation (3) is equivalent to

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}. \tag{2}$$

From the existence theorem of Chap. II it follows that the integral curves of (2) can be parameterized in any region where $\partial(-M/N)/\partial y$ exists. If these integral curves are written in the form $y = f(x, C)$, we may suppose that the solution is solved for the parameter C, giving

$$u(x, y) = C, \tag{4}$$

which is merely an implicit form for the family of integral curves. Therefore

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy = 0 \tag{5}$$

and

$$M\, dx + N\, dy = 0 \tag{3}$$

both hold along an integral curve. Consequently

$$\frac{dy}{dx} = \frac{-M}{N} = \frac{-\partial u/\partial x}{\partial u/\partial y}$$

or

$$\frac{\partial u/\partial x}{M} = \frac{\partial u/\partial y}{N}. \tag{15}$$

If we denote the common ratio in (15) by $\mu(x, y)$ we have

$$\frac{\partial u}{\partial x} = \mu M, \qquad \frac{\partial u}{\partial y} = \mu N,$$

and (5) becomes

(16) $$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = \mu(M\,dx + N\,dy) = 0.$$

In other words μ is an integrating factor.

As a matter of fact (3) has infinitely many integrating factors. Consider $\mu\varphi(u)$ where μ and u are as above and φ is an arbitrary continuous function of u. Then

$$\begin{aligned}\mu\varphi(u)(M\,dx + N\,dy) &= \varphi(u)(\mu M\,dx + \mu N\,dy)\\ &= \varphi(u)\,du\\ &= dv,\end{aligned}$$

where $v = \int\varphi(u)\,du$. Hence, $\mu\varphi(u)$ is also an integrating factor.

As has been mentioned, to find an integrating factor is, in general, as difficult as solving the differential equation. However, certain simple differential forms frequently occur, and it may help to be able to recognize integrating factors for these forms. The suggestions below are possible integrating factors of the differential equation when the differential equation contains the given differential form. That the suggested factors do render the differential form exact will be obvious after proper rearrangement or from the test for exactness.

If the differential equation contains the form	*try as an integrating factor*
$x\,dy + y\,dx$	xy or a function of xy
$x\,dx + y\,dy$	$x^2 + y^2$ or a function of $x^2 + y^2$
$x\,dy - y\,dx$	$\frac{1}{x^2}$ or $\frac{1}{y^2}$ or $\frac{1}{x^2 + y^2}$ or $\frac{1}{x^2}$ times a function of $\frac{y}{x}$

PROBLEMS

1. Solve the following equations by finding an integrating factor:

(a) $x\,dy + y\,dx + x^4y^2dx = 0.$
(b) $x\,dy - y\,dx = (x^2 + 1)\,dx.$
(c) $x\,dy - y\,dx = (x^2 + 4y^2)\,dx.$

(d) $y^2dx + x(x\,dy - y\,dx) = 0$.

(e) $dy + \frac{y}{x}\,dx = \sin x\,dx$.

(f) $x\,dy - y\,dx = x\,dx + y\,dy$.

(g) $x\,dy - y\,dx = xy^3(x\,dy + y\,dx)$.

(h) $(2x + y)(y^2 + 1)\,dx + (2y^2 + xy + 1)x\,dy = 0$

(i) $(\cos x - \sin x \tan y)(dx + dy) + dy = 0$.

(j) $dx = x^2 \sin xy\,dx + x^3 \cos xy(x\,dy + y\,dx)$.

(k) $dy = x^2y\,dx + x^3dy$.

(l) $x\,dy - y\,dx = x\sqrt{x^2 - y^2}\,dy$.

★**2.** Show that an integrating factor μ of $M\,dx + N\,dy = 0$ must satisfy the partial differential equation

$$M\frac{\partial\mu}{\partial y} - N\frac{\partial\mu}{\partial x} + \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 0.$$

★**3.** Using the result of the previous problem show that if

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

is independent of y, then $M\,dx + N\,dy = 0$ has an integrating factor μ which is a function of x alone. Show that μ then satisfies

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right).$$

Apply this result to find an integrating factor for $(y^3x + 1)\,dx + xy^2dy = 0$.

4. Use the result of Prob. 3 to obtain $e^{x^2/2}$ as an integrating factor of $(x^3 + xy^2)\,dx + 2y\,dy = 0$. Then solve the equation.

★**5.** Find a relation satisfied by an integrating factor which is a function of y alone.

5. Linear Equations. An extremely important type of first-order equation is the *linear equation*

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{17}$$

where $P(x)$ and $Q(x)$ are functions of x *alone*. If $Q(x) = 0$, (17) is said to be *linear homogeneous*, for then each non-zero term of the equation is of the first degree in y and dy/dx.

The simplest procedure for solving (17) perhaps is to observe that $e^{\int P dx}$ is an integrating factor, where $\int P\,dx$ is any indefinite integral of $P(x)$. Equation (17) after multiplying by $e^{\int P dx}$ becomes

$$e^{\int P dx}\frac{dy}{dx} + yPe^{\int P dx} = Qe^{\int P dx}.$$

Since the left member is the derivative of $ye^{\int P dx}$ and the right member is a function of x only, the general solution is

$$ye^{\int P dx} = \int Qe^{\int P dx} dx + C,$$

or

(18)
$$y = e^{-\int P dx}(\int Qe^{\int P dx} dx + C).$$

Equation (18) then furnishes a formula for solving the linear equation (17). However, (18) should *not* be used as a formula into which P and Q are substituted. Rather $e^{\int P dx}$ should be used as an integrating factor and the resulting equation integrated by inspection.

Example 7. Solve

$$\frac{dy}{dx} - \frac{1}{x} y = x.$$

The equation is linear and $P = -\frac{1}{x}$; so $\int P\, dx = -\log x$ and the integrating factor is

$$e^{\int P dx} = e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x}.$$

The differential equation now reads

$$\frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = 1$$

or

$$\frac{d}{dx}\left(\frac{y}{x}\right) = 1,$$

whence

$$\frac{y}{x} = x + c$$

and

$$y = x^2 + cx.$$

PROBLEMS

1. Solve

(a) $\frac{dy}{dx} + \frac{y}{x} = 1$.

(b) $y\, dx + x\, dy = \sin x\, dx$.

(c) $(y + y^3)\, dx + 4y^2 x\, dy = 2\, dy$ (Hint: Take x as the dependent variable).

(d) $x \frac{dy}{dx} - 4y = x^5$.

(e) $(1 + x^2)\,dy = dx + xy\,dx$.
(f) $d\rho + \rho\cos\theta\,d\theta = \sin 2\theta\,d\theta$.
(g) $dy = (2x + y)\,dx$.
(h) $(x^2 + x + 1)y' + 2(2x + 1)y = x$.
(i) $y' + 2y\sec 2x = 2\tan^2 x\cos 2x$.
(j) $\sin u\,dv + (v + u)\cos u\,du = 0$.
(k) $\sin\theta\,d\rho + 2\rho\cos\theta\,d\theta = -\sin 2\theta\,d\theta$.
(l) $y' + ay = e^{-ax}$.
(m) $x(x^2 - 1)\,dy = [5x^3 - (4x^2 - 2)y]\,dx$.

***2.** Show that the direction field of the linear equation $y' + Py = Q$ has the following property: The line elements on any vertical line $x = x_0$ all pass through a single point (ξ, η) with coordinates

$$\xi = x_0 + \frac{1}{P(x_0)}$$
$$\eta = \frac{Q(x_0)}{P(x_0)}.$$

3. Find integral curves of the following equations which pass through the point indicated:

(a) $x^2dy - y(x + 1)\,dx = -x\,dx$; $(1, 2)$.
(b) $2\,dy + 2y\cos x\,dx = \sin 2x\,dx$; $(\pi, 0)$.
(c) $y' + y = (x + 1)e^{-x} - x^3 - 3x^2$; $(0, -4)$.
(d) $(1 + x^2)(\arctan x)y' + y = x$; $(1, 1)$.
(e) $y' - y\tan x = \cos 2x$; $\left(\frac{\pi}{6}, \frac{1}{2\sqrt{3}}\right)$.
(f) $2(1 - x^2)y' + 3\alpha x - 3\alpha x^3 = 2xy$; $(0, \alpha)$.
(g) $(1 + x^2)\,dy + 2xy\,dx = \tan x\,dx$; $(0, 0)$.
(h) $L\frac{di}{dt} + Ri = E\sin\omega t$; $i = 0$ when $t = 0$.

***4.** Show that the equation

$$\frac{dy}{dx} + Py = Qy^n,$$

which is called "Bernoulli's equation," can be reduced to a linear equation by the change of variable $v = y^{1-n}$.

5. Use the result of Prob. 4 to find integral curves of the following equations. If a point is given find the curve passing through the point.

(a) $y' - xy = xy^{-1}$.
(b) $xy' + y = x^5y^4$.
(c) $3\frac{dx}{dy} + \frac{x}{y + 1} = \frac{3(y + 1)}{x^2}$.
(d) $y' - 2xy = 2xe^{x^2}\sqrt{y}$; $(0, 9)$.

(e) $xy' + y = y^2x^2 \sin x$; $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$.
(f) $y' + y = -6y^2e^{-2x}$; $(0, -1)$.

6. A suitable substitution will often render a differential equation linear. In the following equations make a suitable substitution and solve:

(a) $(\cos y)y' + x \sin y = Ax$.
(b) $y' + y \log y \operatorname{ctn} x = y$.
(c) $y' + 2 = 25e^{-y} \sin x$.
(d) $y' = 2y(1 - 2xy)$.
(e) $\sqrt{1 + x^2} (\sec^2 z)z' = 2x - \tan z$.

6. Homogeneous Equations.† A function of two variables‡ (x, y) is said to be *homogeneous of degree n* if

$$\varphi(tx, ty) = t^n\varphi(x, y) \tag{19}$$

identically in t, x, y. For example, $x^2 + xy + y^2$, $\sqrt{x^2 + y^2}$, and $\log 2y/x$ are homogeneous of degrees 2, 1, and 0, respectively, since

$$\begin{aligned} (tx)^2 + txty + (ty)^2 &\equiv t^2(x^2 + xy + y^2), \\ \sqrt{(tx)^2 + (ty)^2} &\equiv t\sqrt{x^2 + y^2} \qquad (t \geqq 0) \\ \log \frac{2ty}{tx} &\equiv t^0 \log \frac{2y}{x}. \end{aligned}$$

The differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{20}$$

is called *homogeneous* if M and N are both homogeneous functions of the same degree. Assuming then that (20) is homogeneous and observing that since (19) is an identity in x, y, t we may choose $t = x$. We have

$$M(x, y) = M\left(x \cdot 1, x\frac{y}{x}\right) = x^nM\left(1, \frac{y}{x}\right) \tag{21}$$

$$N(x, y) = N\left(x \cdot 1, x\frac{y}{x}\right) = x^nN\left(1, \frac{y}{x}\right) \tag{22}$$

and (20) reduces to

$$M\left(1, \frac{y}{x}\right)dx + N\left(1, \frac{y}{x}\right)dy = 0. \tag{23}$$

† The term *homogeneous equation* is used here in a sense different from that of Sec. 5 where *linear homogeneous equation* was defined. No confusion need arise however.

‡ The notion generalizes to n variables in an obvious way.

The form of (23) suggests making the change of variable $y/x = v$. Then $y = vx$ and $dy = v\,dx + x\,dv$. Substituting in (23) and rearranging we obtain

$$\frac{dx}{x} + \frac{N(1, v)\,dv}{M(1, v) + vN(1, v)} = 0. \tag{24}$$

Since in (24) the variables are separated, v is easily found and thus also $y = vx$. In solving homogeneous equations, (24) should *not* be used as a formula but rather the substitution $y = vx$ made.

Example 8. Solve the homogeneous equation

$$(x^2 + y^2)\,dx + xy\,dy = 0.$$

Letting $y = vx$ we have $(x^2 + v^2x^2)\,dx + xvx(v\,dx + x\,dv) = 0$ which reduces to

$$\frac{dx}{x} + \frac{v\,dv}{1 + 2v^2} = 0,$$

whence

$$\log x + \tfrac{1}{4}\log(1 + 2v^2) = \text{constant}$$

or

$$x^4(1 + 2v^2) = \text{constant} = k.$$

Therefore

$$x^4\left(1 + 2\frac{y^2}{x^2}\right) = k$$

and

$$x^4 + 2x^2y^2 = k.$$

PROBLEMS

1. Solve the following equations. When a point is specified, find the particular solution through that point.

(a) $2xy\,dx - (x^2 + y^2)\,dy = 0$.
(b) $(x - y)\,dx + (x + y)\,dy = 0$.
(c) $\left(x \tan\frac{y}{x} - y \sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\,dy = 0$.
(d) $xy^2dy = (x^3 + y^3)\,dx$; $(e, 2e)$.
(e) $\sqrt{x^2 + y^2}\,dx = x\,dy - y\,dx$; $(-1, 1)$.
(f) $x \sin\frac{y}{x}\,y' = y \sin\frac{y}{x} + x$.
(g) $x^4y' = 6x^2y(xy' - y)$
(h) $xyy' = x^2 + y^2 + 3xy - 2x^2y'$.
(i) $xy^2dx = y\,dx - x\,dy$.
(j) $x^2y' = y(x + \sqrt{x^2 + y^2})$

★2. Show that if $M\,dx + N\,dy = 0$ is homogeneous $1/(Mx + Ny)$ is an integrating factor.

3. Use the result of the preceding problem to solve Prob. 1.

7. Singular Points of the Direction Field. We return for a moment now to a consideration of the general first-order equation $p = f(x, y)$. In many cases $f(x, y)$ is the quotient of two functions $F(x, y)$ and $G(x, y)$ both of which can be expanded in Taylor series about some point (x_0, y_0), that is,

$$\frac{dy}{dx} = f(x, y) = \frac{F(x, y)}{G(x, y)} = \frac{a_0 + a_1(x - x_0) + a_2(y - y_0) + \cdots}{b_0 + b_1(x - x_0) + b_2(y - y_0) + \cdots} \tag{25}$$

where the dots (. . .) represent terms of at least the second degree in $(x - x_0)$ and $(y - y_0)$. Near (x_0, y_0), F and G are given to a first approximation by their linear terms so that we are led to study the differential equation

$$\frac{dy}{dx} = \frac{a_0 + a_1(x - x_0) + a_2(y - y_0)}{b_0 + b_1(x - x_0) + b_2(y - y_0)} \tag{26}$$

in the hope that we will get information as to the behavior of the general equation (25). Such actually is the case,† but we will not enter here into the relationship between solutions of (25) and (26), having introduced (25) solely to motivate the study of (26). The connections between (25) and (26) are of great importance in the study of nonlinear differential equations‡ which occur often in applications, so that the solution of (26) is of more than academic interest.

In our previous discussion of the direction field in Chap. II the function $f(x, y)$ was presumed to be continuous in x and y. Looking at (25) we observe that if $a_0 = 0 = b_0$ then both numerator and denominator vanish at (x_0, y_0) so that $f(x, y)$ is not even defined there, since 0/0 has no meaning. In this situation we say that (x_0, y_0) is a *singular point* of the direction field, that is, (x_0, y_0) is a singular point if $F(x_0, y_0) = G(x_0, y_0) = 0$, whereas not both $F(x, y) = 0$ and $G(x, y) = 0$ at every other point of some neighborhood of (x_0, y_0).

PROBLEM

1. Find the singular points of the direction fields defined by the following differential equations:

(a) $\dfrac{dy}{dx} = \dfrac{2x - 3y + 5}{x + 2y - 5}$.

† For a more complete discussion see, for example, L. Bieberbach, "Differentialgleichungen," pp. 68*ff*., Dover Publications, 1944.

‡ See N. Minorsky, "Non-linear Mechanics," Edwards Bros., 1947.

(b) $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 - 5}{2y^2 - x}$.

(c) $\dfrac{dy}{dx} = \dfrac{x^2 - y^2 - 9}{2y - x - 3}$.

8. The Linear Fractional Equation. Equation (26) is called the *linear fractional equation*. We will write it in the equivalent form

$$\frac{dy}{dx} = \frac{a + bx + cy}{d + ex + fy}. \tag{27}$$

There are two cases to consider:

Case 1. $\Delta = bf - ce = 0$.

Case 2. $\Delta = bf - ce \neq 0$.

In case 1 we may suppose that not both c and f are zero, for if they were the right member of (24) would be free from y and the solution could be obtained by a simple quadrature. Suppose first that $c \neq 0$. Since $\Delta = 0$, $ex + fy = k(bx + cy)$ where k is the constant f/c. Making the substitution $bx + cy = v$ we have

$$\frac{dy}{dx} = \frac{-b}{c} + \frac{1}{c}\frac{dv}{dx}$$

and (27) reduces to

$$-\frac{b}{c} + \frac{1}{c}\frac{dv}{dx} = \frac{a + v}{d + kv}$$

in which the variables are separable. If $c = 0$ then $f \neq 0$, and letting $v = ex + fy$ will produce the same simplification. This disposes of case 1.

Example 9. Solve

$$\frac{dy}{dx} = \frac{1 + x + y}{2 + x + y}.$$

Here $\Delta = 0$ and case 1 applies. Letting $v = x + y$ we have

$$-1 + \frac{dv}{dx} = \frac{1 + v}{2 + v}$$

and

$$\frac{dv}{dx} = \frac{3 + 2v}{2 + v},$$

whence we obtain easily

$$\frac{v}{2} + \frac{1}{4}\log(2v + 3) = x + \text{constant}$$

and

$$2(x + y) + \log(2x + 2y + 3) = k.$$

It is really case 2 in which we are interested and then only in the neighborhood of a singular point where the integral curves exhibit a variety of types.† We confine our attention to a technique for solving the equation. Since $\Delta \neq 0$ the two straight lines $0 = a + bx + cy$ and $0 = d + ex + fy$ are not parallel and intersect in a point (h, k). Translating the origin to (h, k) by means of the transformation $x' = x - h$, $y' = y - k$ we have $dy/dx = dy'/dx'$ and find that Equation (27) reduces to

$$\frac{dy'}{dx'} = \frac{bx' + cy'}{ex' + fy'}. \tag{28}$$

We will accordingly assume that this translation has been carried out and drop the primes. This is equivalent to assuming $a = d = 0$ in (27). In other words we can arrange it so that the origin is a singular point of the transformed differential equation which then is in the form

$$\frac{dy}{dx} = \frac{bx + cy}{ex + fy}. \tag{29}$$

Example 10. Translate the origin to the singular point of the equation

$$\frac{dy}{dx} = \frac{1 + x + y}{3 + x + 2y}.$$

Solving $1 + x + y = 0$ and $3 + x + 2y = 0$ simultaneously we find the point of intersection to be $(1, -2)$, whence $x = x' + 1$, $y = y' - 2$ and

$$\frac{dy'}{dx'} = \frac{x' + y'}{x' + 2y'}.$$

With our differential equation in the form (29) we observe that it is homogeneous so that its solution can be effected by the substitution $y = vx$. We make therefore this substitution and find that (29) becomes

$$v + x\frac{dv}{dx} = \frac{bx + cvx}{ex + fvx}$$

or

$$\frac{dv}{dx} = \frac{1}{x}\,\frac{b + (c - e)v - fv^2}{e + fv}.$$

In this last equation the variables are separable so that v and hence $y = vx$ can be found.

† See Appendix C.

Example 11. Solve

$$\frac{dy}{dx} = \frac{2x + y}{y}.$$

Since the equation is homogeneous we substitute $y = vx$ and have

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{2x + vx}{vx} = \frac{2 + v}{v}$$

and

$$x\frac{dv}{dx} = \frac{2 + v - v^2}{v},$$

or

$$\frac{v\,dv}{v^2 - v - 2} = -\frac{dx}{x}.$$

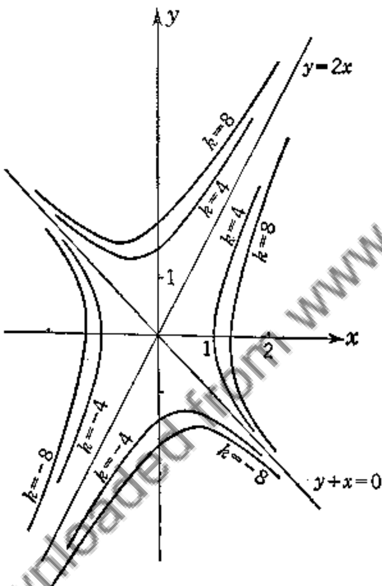


FIG. 14.

Splitting the left member by partial fractions gives

$$\frac{\frac{2}{3}}{v - 2}\,dv + \frac{\frac{1}{3}}{v + 1}\,dv = -\frac{dx}{x},$$

whence

$$\log (v - 2)^{\frac{2}{3}} + \log (v + 1)^{\frac{1}{3}} + \log x = \text{constant},$$

or

$$(v - 2)^2(v + 1)x^3 = \text{constant}$$

and

$$(y - 2x)^2(y + x) = \text{constant} = k.$$

A graph of the family of integral curves is shown in Fig. 14; observe that there are two curves through the origin, $y = 2x$ and $y = -x$.

It is not always an easy matter to sketch the family of integral curves. In some cases it helps to transform the equation to polar coordinates. In Appendix C the character of the family of integral curves of Equation (25) is discussed in detail. It is shown there that the character of the singular point may be predicted.

PROBLEMS

1. Solve $\frac{dy}{dx} = \frac{x + y}{x}$. Sketch the family of integral curves.

2. Solve $(5x - y + 9)\,dx + (x - 5y - 3)\,dy = 0$.

3. Solve $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}\cdot$ Sketch the family of integral curves. (Hint: Change to polar coordinates.)

4. Solve $(2x - y)\,dy = (2x - y + 2)\,dx$.

5. Solve $\dfrac{dy}{dx} = -\dfrac{x}{y}\cdot$ Sketch the family of integral curves.

6. Solve $(x + y + 2)\,dy + (4x - y - 2)\,dx = 0$.

7. Solve $(2x - y + 6)\,dy = (2y - x - 6)\,dx$.

8. Solve $(x - 3y + 3)\,dx = (2x - 6y + 1)\,dy$.

9. Solve $\dfrac{dy}{dx} = \dfrac{ay}{x}\cdot$ Sketch the family of integral curves for $a = 1, 2, -1$.

10. In which of the previous problems is there an integral curve passing through the singular point?

9. Miscellaneous Methods. The special types of first-order equations discussed so far by no means exhaust the possibilities, and in general we would be unable to get a solution of $y' = f(x, y)$ without using approximation methods. If the differential equation is very complicated, it might be necessary to resort to a graphical or numerical method such as sketched in Chap. II. In a large number of physical problems, however, one of the methods described above will apply; in fact the variables are likely to be separable. The following sections in this chapter are devoted to such applications. Some special types are treated in the problems at the end of the chapter.

When the equation is not one of the types so far discussed there are a number of tricks which *may* help. Among these are substitutions and power-series solution, this latter being discussed in Chap. VIII. For an extensive list of solvable types the reader is referred to the excellent compendium by E. Kamke (Ref. 7).

Example 12. Solve

$$\frac{dy}{dx} = \sin (x + y).$$

A little experimentation soon shows that our basic methods fail to work. However, the form of the equation suggests the substitution $u = x + y$, whence

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

and the differential equation becomes

$$\frac{du}{dx} = 1 + \sin u,$$

for which the variables are separable. Hence

$$\frac{du}{1 + \sin u} = dx,$$

$$\frac{du(1 - \sin u)}{(1 + \sin u)(1 - \sin u)} = \sec^2 u\, du - \sec u \tan u\, du = dx,$$

or

$$\tan u - \sec u = x + c$$

or

$$\frac{\sin u - 1}{\cos u} = x + c,$$

and

$$\frac{\sin (x + y) - 1}{\cos (x + y)} = x + c.$$

PROBLEMS

1. In the following, find a substitution which renders the equation solvable by our methods, and solve.

(a) $xy(x\, dy + y\, dx) = y^2 dy.$
(b) $(x + y)^2 dy = a^2 dx.$
(c) $(xy + 2)x\, dy + (2xy + 1)y\, dx = 0.$
(d) $(y - x + 1 + x\sqrt{y - x})\, dx = dy.$

2. In the following problems make the suggested substitutions and solve:

(a) $(x - y)^2 \frac{dy}{dx} = 1;\quad x - y = u.$

(b) $x \frac{dy}{dx} - 2y + ay^2 = bx^4;\quad y = vx^2.$

(c) $\left(1 - \frac{x}{y}\right) dy + (1 + e^{-\frac{x}{y}})\, dx = 0;\quad z = \frac{x}{y}.$

(d) $x\, dx + y\, dy + y\, dx - x\, dy = 0;\quad u = x^2 + y^2,\ v = \frac{y}{x}.$

APPLICATIONS

10. Geometrical Problems. Many geometrical problems involve tangents and normals to plane curves so that their analytical formulation naturally involves the derivative. A complete catalogue of such types of problems is clearly impossible even if desirable; we therefore content ourselves with some examples which will perhaps indicate the power inherent in the use of differential equations and demonstrate how some geometrical problems may be attacked analytically.

Example 13. Find the curve such that its normal at any point coincides in direction with the radius vector to the point from the origin.

Since the radius vector has slope y/x and the tangent the slope dy/dx, the problem states that

$$-\frac{dx}{dy} = \frac{y}{x}. \tag{30}$$

(We require here that $x \neq 0$ but y can be zero. Had we formulated the problem in the form $dy/dx = -x/y$, it would have been y which we would have had to assume different from zero. After the solution has been obtained, it will be clear geometrically what these restrictions mean. The point (0, 0) is excluded in either formulation. It is a singular point.)

Rewriting (30) in the (exact) form

$$x\,dx + y\,dy = 0,$$

we get a general solution

$$x^2 + y^2 = c = \text{constant}, \tag{31}$$

which for $c > 0$, is a family of circles with center at the origin.

It is now clear that when $x \to 0$ the slope of the normal at (x, y) to any member of the family (31) becomes infinite and Equation (30) fails to have meaning for $x = 0$. Nevertheless the geometric problem still makes sense because the radius vector is normal to the curve. By formulating our problem in the form

$$\frac{dy}{dx} = -\frac{x}{y} \tag{32}$$

we avoid the exceptional points $x = 0$ only to find that we have exchanged them for some others, namely, the points where $y = 0$. However the two formulations (30) and (32) taken together are valid (one or the other) wherever the geometric problem makes sense (that is, away from the origin), and if neither $x = 0$ nor $y = 0$ then (30) and (32) are equivalent. On the other hand, among the solutions of (31) there are some which are not solutions of the original geometric problem, namely, these for which $c \leqq 0$.

We will often encounter situations analogous to that treated above. That is, in giving an analytical form to a problem we may find that our formulation fails to be valid at certain exceptional points. It may then be possible to avoid some of these exceptional points by formulating the problem in a slightly different manner. In such cases one must always be sure that the two formulations are equivalent at points where both are valid. In the future we will not call attention to such difficulties and will allow the reader, if he chooses, to manage the exceptional points in any way that is convenient.

We have had occasion in Chap. II to discuss orthogonal trajectories in terms of rectangular coordinates. It is frequently necessary to discuss the orthogonal trajectories of a family of curves given in polar coordinates. If $\psi = \tau - \theta$, where (ρ, θ) are polar coordinates of a point on a curve and τ is the inclination of the tangent line, it is shown in calculus that

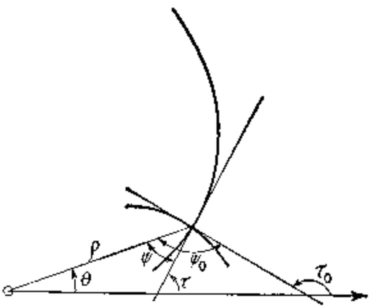


FIG. 15.

$$\rho\frac{d\theta}{d\rho} = \tan\psi. \tag{33}$$

Now consider the family of orthogonal trajectories to a given family of curves. If ψ_0 and τ_0 are corresponding angles on an orthogonal trajectory, see Fig. 15, we have

$$\tau_0 = \tau \pm \frac{\pi}{2}$$

$$\psi_0 = \tau_0 - \theta_0 = \tau \pm \frac{\pi}{2} - \theta = \psi \pm \frac{\pi}{2},$$

whence

$$\tan\psi_0 = \tan\left(\psi \pm \frac{\pi}{2}\right) = -\cot\psi = -\frac{1}{\tan\psi}. \tag{34}$$

In deriving Equation (34) it has been tacitly assumed that the point (ρ, θ) on the given curve was also given on the orthogonal trajectory by the same coordinates (ρ, θ). But one of the troublesome features of polar coordinates arises from the fact that a point has infinitely many sets of coordinates; for example, (ρ, θ) could also be given by $(\rho, \theta \pm 2k\pi)$ or $(-\rho, \theta \pm (2k + 1)\pi)$. We examine what, if any, change needs to be made in (34) when we take account of this multiplicity of coordinates. In any event we have

$$\theta_0 = \theta + k\pi \qquad (k = \text{integer})$$

then

$$\tau_0 - \theta_0 = \tau \pm \frac{\pi}{2} - \theta - k\pi$$

and

$$\tan\psi_0 = \tan\left[\left(\tau \pm \frac{\pi}{2} - \theta\right) - k\pi\right] = \tan\left(\tau \pm \frac{\pi}{2} - \theta\right)$$

since the tangent function has period π. Thus (34) is valid in all cases.

Example 14. Find the orthogonal trajectories of the family of cardioids $\rho = c(1 + \cos\theta)$.

Differentiating, we get

$$\frac{d\rho}{d\theta} = -c\sin\theta,$$

and eliminating the constant c gives

$$\rho\,\frac{d\theta}{d\rho} = -\frac{1+\cos\theta}{\sin\theta} = \tan\psi$$

for a differential equation satisfied by the given family. But from (34)

$$\rho_0\,\frac{d\theta_0}{d\rho_0} = \tan\psi_0 = -\frac{1}{\tan\psi}.$$

Hence

$$\rho_0\,\frac{d\theta_0}{d\rho_0} = \frac{\sin\theta_0}{1+\cos\theta_0}$$

is a differential equation for the orthogonal trajectories. Here we have taken $\theta_0 = \theta$. The variables are separable,

$$\frac{d\theta_0}{\sin\theta_0} + \operatorname{ctn}\theta_0\,d\theta_0 = \frac{d\rho_0}{\rho_0}$$

and

$$\log(\csc\theta_0 - \operatorname{ctn}\theta_0) + \log\sin\theta_0 = \log\rho_0 + \text{constant}$$

or

(35) $$\begin{aligned}\rho_0 &= c'\sin\theta_0\,(\csc\theta_0 - \operatorname{ctn}\theta_0),\\ \rho_0 &= c'(1 - \cos\theta_0).\end{aligned}$$

Since each member of the family (35) is also a member of the given family, the given family is self-orthogonal.

PROBLEMS

1. Find the orthogonal trajectories of the family $y^2 = cx^3$.

2. Find the orthogonal trajectory of the family $y^2 = kx$ which passes through the point $(-1, 2)$.

3. Find the family of curves with subnormal of constant length k.

4. Determine those curves for which the normal at (x, y) has an intercept on the x axis equal to $2x$.

5. Find the curve whose arc length from $x = 0$ to $x = x$ is $\sinh x$.

6. Find the orthogonal trajectories of the circles $(x - c)^2 + y^2 = c^2$.

7. The family of circles in Prob. 6 may be written in polar coordinates as $\rho = 2c \cos \theta$. Find the orthogonal trajectories in polar coordinates.

8. Find the orthogonal trajectories of the family $\rho^2(1 - \cos \theta) \sin \theta = k$.

9. Find all curves for which the radius vector from the origin makes a constant angle with the tangent line. If a particular curve passes through $\rho = 1, \theta = 0$ with inclination 45°, find its equation.

10. Find the curves (in polar coordinates) for which the polar angle θ is equal to the angle between the radius vector and the tangent line.

11. Find the curves (in polar coordinates) for which the angle between the radius vector and the tangent line is equal to one-half the inclination of the tangent line.

***12.** An airplane is searching for an enemy motorboat in a dense fog. For one instant the fog lifts and the boat is seen 5 miles away; then the fog descends again. If the speed of the plane is 240 miles per hour and the speed of the boat 60 miles per hour, what path could the plane follow to be certain to intercept the boat, assuming that, at the instant the boat is seen by the plane, the boat immediately starts out in a straight course in an arbitrary direction? (Use polar coordinates with pole at the point where the boat was when sighted.)

11. Rate Problems. The student is familiar with certain rate problems from calculus where the dependent variable x is given explicitly as a function of the time t, $x = f(t)$. In this case the rate of change is $dx/dt = f'(t)$ and finding x when the derivative is known reduces to a quadrature. In an important class of problems the rate of change of x is proportional to x, which yields a simple differential equation with the variables separable.

Interest on money, at the rate r per period, is *compounded* when the interest at the end of some fixed period is added to the principal at the end of that period. That is, if A is the amount of the money and t the period we have

$$\Delta A = rA \, \Delta t. \tag{36}$$

Interest is said to be *compounded continuously* if we have instead of (36) the following:

$$\frac{dA}{dt} = rA. \tag{37}$$

Example 15. Find the time required for one dollar to double when invested at the rate of 5 per cent per annum compounded continuously.

Let A denote the amount at the end of t years; then

$$\frac{dA}{dt} = 0.05A,$$

whence

$$\frac{dA}{A} = 0.05\,dt,$$

or

$$\log A = 0.05t + c,$$

and

$$A = Ce^{0.05t}.$$

When $t = 0$, $A = 1$, and therefore $C = 1$. If after time t (years) we have $A = 2$, then

$$2 = e^{0.05t},$$

or

$$t = \frac{\log 2}{0.05} = 20 \log 2 = 13.86.$$

In other words to double your money at 5 per cent compounded continuously would take 13.86 years.

Example 16. Radioactive elements decompose at a fixed rate.† That is, if A is the mass of a radioactive substance present at time t, then the rate of change per unit mass, $(1/A)\,dA/dt$, is constant $= r$. In this case r is of course negative. We are dealing therefore once more with Equation (37). The time required for A to decrease to one-half of its original value is called the *half-life* of the radioactive element. Since the general solution to (37) is

$$A = A_0e^{rt}, \tag{38}$$

the half-life is easily found by setting $A = A_0/2$ in (38) and solving for t_0. We obtain

$$\text{Half-life} = -\frac{\log 2}{r}.$$

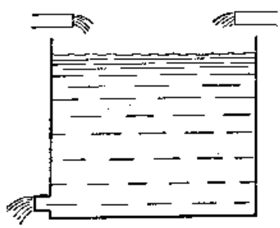


FIG. 16.

Example 17. A tank of 100-gallon capacity is initially full of water. Pure water is allowed to run into the tank at the rate of 1 gallon per minute, and at the same time brine containing $\frac{1}{4}$ pound of salt per gallon flows into the tank also at the rate of 1 gallon per minute. The mixture flows out at the rate of 2 gallons per minute. (It is assumed that there is perfect mixing.) Find the amount of salt in the tank after t minutes.

† This law has a statistical basis. It is true for large numbers of atoms but does not apply to individual atoms.

Let S be the amount of salt in pounds at time t. The concentration in pounds per gallon will be $S/100$. The rate of increase of S therefore is

$$\frac{dS}{dt} = \frac{1}{4} - 2\frac{S}{100}.$$

Whence

$$\frac{dS}{25 - 2S} = \frac{dt}{100},$$

and we find easily that

$$S = \tfrac{25}{2} - S_0 e^{-t/50},$$

where S_0 is the constant of integration. Since $S = 0$ when $t = 0$ we find that $S_0 = \frac{25}{2}$, and so

$$S = \tfrac{25}{2}(1 - e^{-t/50}).$$

PROBLEMS

1. If the half-life of radium is 1,600 years, how long will it take a mass of radium to disintegrate until but 10 per cent remains?

2. Find the value of \$100 compounded continuously for a period of 10 years at the rate of 4 per cent per year.

3. Newton's law of cooling asserts that the rate at which a body cools is proportional to the difference in temperature between the body and its surroundings. Write a differential equation which expresses this relation. If a body cools from 40°F to 30°F in 10 minutes when the ambient air is at 10°F, how long will it take to cool from 80°F to 50°F when the ambient air is at 30°F?

4. A man is fortunate enough to have a certain sum of money invested so as to compound continuously at 5 per cent per year. He withdraws money continuously at a constant rate (this could be approximately realized by a daily withdrawal) so as to draw \$1,000 per year. At the end of 10 years his fund is exhausted. How much did he start with?

5. A tank contains 100 gallons of brine containing 50 pounds of dissolved salt. Pure water runs in the tank at the rate of 1 gallon per minute, and brine flows out (assume continuous perfect mixing) at the rate of 1 gallon per minute. How long will it take to reduce the salt concentration to one-half of its initial value?

6. A large tank contains 100 gallons of pure water. A salt solution containing 1 pound of salt per gallon flows into the tank at the rate of 2 gallons per minute, while from another pipe a salt solution containing 2 pounds of salt per gallon flows in at the rate of 2 gallons per minute. The mixture flows out of the tank at the rate of 3 gallons per minute. Find the amount of salt and the concentration at the end of 100 minutes.

7. An insect colony has constant birth and death rates, that is, births (or deaths) per 1,000 of population per day. (In a human population such data would be per year.) Assuming that the population can vary continuously, describe the growth or decay of the population.

8. A material containing S pounds of salt is stirred with G gallons of pure water. The salt dissolves at a rate proportional to the product of the amount of undissolved salt and the difference between the concentration of the liquid and that of a saturated solution (3 pounds of salt per gallon). Find an expression giving the amount of undissolved salt as a function of the time.

12. Mechanical Problems. Newton's second law of motion asserts that force is proportional to mass (quantity of matter) times acceleration. Now force and acceleration are vector quantities so that it is necessary to consider both direction and magnitude. However in case the motion is in a straight line both force and acceleration are given by their components in the fixed direction. Newton's law then states that

$$\text{Force} = \text{constant} \times \text{mass} \times \text{acceleration}$$
$$F = kma$$

where the value of the constant of proportionality k depends on the units used to measure force, mass, distance, and time.

It is customary to choose a system of units so the constant k has the value 1. There are three such systems of units in common use. These

Distance	*Time*	*Mass*	*Force*
Foot	Second	Slug	Pound
Foot	Second	Pound	Poundal
Centimeter	Second	Gram	Dyne

systems are given in the accompanying table. In the examples and problems below the first of these systems of units (the foot-slug-second system) will be used exclusively. Note that at sea level a mass of one slug would weigh 32 pounds if we take the rough value of 32 feet per second per second for the acceleration of gravity.

In any of these systems of units, if a mass m moves in a straight line under the action of a force F, we have

$$\begin{aligned}\text{Force} = F = ma &= \text{mass} \times \text{acceleration}\\ &= m\frac{d^2s}{dt^2}\\ &= m\frac{dv}{dt}\end{aligned}$$

where s measures the distance of the particle from a fixed reference point and $v = ds/dt$ is the velocity. Observe that the signs of s, v, and a are vital.

Because of the second derivative d^2s/dt^2 occurring in Newton's law it is in general necessary to solve a second-order equation in order to determine the position of the particle. However if the force F is a function of the distance s, we may rewrite the acceleration

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}$$

and obtain a first-order equation in v (see Example 19 below).

In other cases the force is either constant or a known function of t and v. We then write the equation of motion in the form

$$F = m\frac{dv}{dt},$$

and we have a first-order equation. Frictional forces, for example, are often proportional to a power of the velocity.

Example 18. Consider a body of mass m falling with velocity v under the action of gravity and a frictional force proportional to v^α. Find v as a function of the time.

$$\begin{aligned}\text{Mass} \times \text{acceleration} &= \text{net force}\\ &= \text{gravitational force} - \text{frictional force},\end{aligned}$$

or

$$m\frac{dv}{dt} = mg - kv^\alpha$$

where k is a positive constant. Then

$$\frac{dv}{g - \frac{k}{m}v^\alpha} = dt$$

and the variables are separated. For "slow" motions α may often be taken equal to 1, rendering the integration easy. We get in this case

$$-\frac{m}{k}\log\left(g - \frac{k}{m}v\right) = t + \text{constant},$$

or

$$g - \frac{k}{m}v = Ae^{-kt/m}$$

and

$$v = \frac{mg}{k} - Be^{-kt/m}.$$

If $v = 0$ when $t = 0$ we find easily that $B = mg/k$, whence

$$v = \frac{mg}{k}(1 - e^{-kt/m}).$$

As $t \to \infty$, v approaches the *terminal velocity* mg/k. The analysis here in the case $\alpha = 1$ represents rather well the situation for small droplets of water or oil falling in air.

Example 19. Consider a horizontal weightless spring (Fig. 17) with one end fixed while at the other is attached a mass of m slugs. The spring force is proportional to the displacement (Hooke's law). If the mass is moving with velocity v_0 when the spring is unstretched, find v as a function of the stretch.

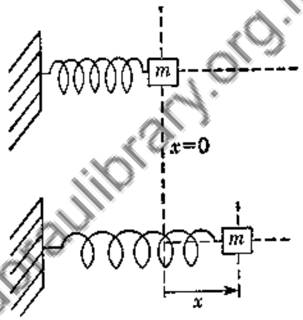


FIG. 17.

Choose coordinates so the mass is at $x = 0$ when the spring is unstretched. Then the spring force is $-kx$ where k is the "spring constant." The negative sign occurs because the force is always toward $x = 0$. Then

$$\begin{aligned} -kx &= m\frac{dv}{dt} \\ &= m\frac{dv}{dx}\frac{dx}{dt} \\ &= mv\frac{dv}{dx}. \end{aligned}$$

The variables are separable, and solving the differential equation gives

$$mv^2 = -kx^2 + c.$$

Since $v = v_0$ when $x = 0$ we find that $v_0^2 = c$, and hence

$$mv^2 + kx^2 = mv_0^2.$$

Observe that this equation asserts that the kinetic energy of the mass plus the potential energy stored in the spring is constant.

PROBLEMS

1. A boy and his sled weigh 64 pounds ($g = 32$ feet per second per second). They are being pulled on level ground by the boy's father at the constant speed

of 8 feet per second. At a command from his son the father exerts a constant force of 20 pounds on the towrope. If the resistance of the sled in pounds is equal to the speed in feet per second, find the speed after 5 seconds and also the terminal speed.

2. At what angle must a hill be sloped so the boy and his sled of Prob. 1 will slide downhill at the terminal speed of 8 feet per second? Determine the speed down such a hill if the sled starts from rest.

3. The spring of Example 19 now hangs vertically. Find the velocity of the mass if $v = v_0$ when $x = 0$.

4. A hole is drilled through the earth (assumed a sphere of radius 4,000 miles) from the north to the south pole. Using the fact that, inside the earth, the gravitational attraction is proportional to the distance from the center of the earth, with what velocity will a mass dropped from the north pole reach the center of the earth?

5. The earth exerts a gravitational pull on a mass m a distance r from its center which is proportional to m and inversely proportional to r^2, that is, force $= km/r^2$. Now this force is $mg(g = 32)$ when $r = 4{,}000$ miles. With what velocity must a projectile be fired vertically to escape the gravitational pull? (Centrifugal effects are neglected as well as friction.)

6. A mass of 2 slugs slides on a table and is subjected to the periodic force $10 \sin 2t$. The friction is equal to twice the velocity. If the motion starts from rest find v as a function of the time.

7. An airforce parachutist jumps from a plane. Before pulling the rip chord he has essentially reached a terminal velocity of 180 feet per second. Find the velocity as a function of the time he has fallen if his wind resistance is proportional to his velocity. What is his velocity 5 seconds after he jumps?

8. The drag of an airplane is equal to kv^2. Show that the terminal velocity in level flight, under a constant propeller thrust T, is $\sqrt{T/k}$.

9. A steamship weighs 64,000 tons and starts from rest under the impetus of a constant propeller thrust of 300,000 pounds. If the resistance in pounds is 10,000 v, where v is in feet per second, find its velocity as a function of the time and its terminal velocity in miles per hour.

10. A chain 64 feet long hangs over a frictionless pulley with negligible mass. Thirty-four feet of the chain hang on one side of the pulley, while the remaining 30 feet hang on the other. If initially chain and pulley are at rest, find the velocity as a function of the length of chain hanging on one side.

13. Simple Electric Circuits. Let q be the charge on the condenser in a circuit (see Fig. 18) containing a resistance R, inductance L, and capacitance C in series. A known e.m.f. (electromotive force) $E(t)$ is impressed across the circuit. The magnitudes q, E, L, R, C are in some physically consistent set of units which we will take as coulombs, volts, henrys, ohms, and farads, respectively. From elementary physics we have the following fundamental relation

$$E(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q, \tag{39}$$

which asserts that the e.m.f. is equal to the sum of the voltage drops across the components. Differentiating this equation yields, if we put dq/dt equal to the current i in amperes,

$$\frac{dE}{dt} = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i. \tag{40}$$

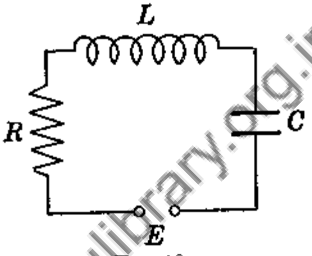


FIG. 18.

Now Equations (39) and (40) are both linear second-order equations. However in special cases either (39) or (40) will be a first-order equation in either q or i. For example, if $L = 0$, Equation (40) reduces to the equation

$$\frac{dE}{dt} = R\frac{di}{dt} + \frac{1}{C}i; \tag{41}$$

and if no condenser is in the circuit, Equation (39) reduces to

$$L\frac{di}{dt} + Ri = E. \tag{42}$$

Both (41) and (42) are first-order linear equations and therefore solvable by our methods.

Example 20. Find the current in the simple circuit of Fig. 18 with no capacitance C (condenser absent) and $E = E_0 \sin \omega t$. From (42) we have the linear equation

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{E_0}{L}\sin \omega t.$$

Using $e^{Rt/L}$ as an integrating factor we get

$$\begin{aligned} ie^{Rt/L} &= \int e^{Rt/L}\frac{E_0}{L}\sin \omega t\, dt + k \\ &= E_0\frac{R \sin \omega t - \omega L \cos \omega t}{R^2 + \omega^2L^2}e^{Rt/L} + k \end{aligned}$$

and

$$i = E_0 \frac{R \sin \omega t - \omega L \cos \omega t}{R^2 + \omega^2 L^2} + ke^{-Rt/L}.$$

If $i = i_0$ when $t = 0$ we have

$$i_0 = -\frac{E_0 \omega L}{R^2 + \omega^2 L^2} + k$$

and

$$(43) \qquad i = E_0 \frac{R \sin \omega t - \omega L \cos \omega t}{R^2 + \omega^2 L^2} + \left(i_0 + \frac{E_0 \omega L}{R^2 + \omega^2 L^2}\right) e^{-Rt/L}.$$

Equation (43) can be put in a more useful form as follows: Let φ be that acute angle for which

$$\tan \varphi = \frac{\omega L}{R}.$$

Then

$$\cos \varphi = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}, \qquad \sin \varphi = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$$

and (43) may be written in the simpler form

$$(44) \quad i = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin (\omega t - \varphi) + \left(i_0 + \frac{E_0 \omega L}{R^2 + \omega^2 L^2}\right) e^{-Rt/L}.$$

Several inferences may be made from (44). In the first place it is clear that after a long span of time the second term is very small and the current becomes a pure sine wave as it would have been from the beginning had there been no inductance present ($L = 0$). The amplitude of this wave however is $E_0/\sqrt{R^2 + \omega^2 L^2}$ instead of E_0/R as it would have been if no inductance were present.

In addition, the current and voltage are out of phase by the "phase angle" φ. The e.m.f. has its maximum amplitude at times $t = \pi/2\omega$, $3\pi/2\omega$, $5\pi/2\omega$, . . . , whereas the current will have maxima at $\omega t - \varphi = \pi/2$, $3\pi/2$, $5\pi/2$, . . . , that is, when $t = \left(\frac{\pi}{2} + \varphi\right)/\omega$, $\left(\frac{3\pi}{2} + \varphi\right)/\omega$, ... In other words the current *lags* behind the voltage by a time φ/ω.

The current in a simple circuit with $L = 0$, capacitance C, and e.m.f. $E = E_0 \sin \omega t$ may be found similarly from Equation (41). The results are much the same qualitatively except that the current is now found to *lead* the voltage. We omit the details.

While the simplest and most important e.m.f. is a simple sine wave, in many circuit problems it is necessary to consider more general e.m.f.'s and even discontinuous ones. One of the latter type, for example, could be generated by connecting a battery in the circuit and opening or closing the switch. If the battery has a constant e.m.f. $= E_0$ and the switch is closed at time $t = t_0$ after the initial time $t = 0$, then the graph of E appears as in Fig. 19. If now we have a simple circuit without a condenser, Equation (42) applies to the times where $E(t)$ is continuous, that is, away from $t = t_0$. Because $E(t)$ has a jump discontinuity at $t = t_0$, our earlier discussion does not explicitly cover this case. The function $E(t)$ considered here is a simple example of a type of function occurring often in electric circuits, namely, a "piecewise continuous" function.†

In Sec. 8, Chap. VII, it will be shown that, although $E(t)$ is but piecewise continuous, there is a *continuous* function $i(t)$ such that $i(t)$ has a continuous derivative wherever $E(t)$ is continuous and which satisfies (42) at those points. Observe that in this statement nothing is said about the derivative of $i(t)$ at points of discontinuity of $E(t)$, and in fact the derivative cannot exist at such points.

Our procedure in solving the battery problem is exactly the same as when E(t) is continuous. We have the differential equation

$$\frac{di}{dt} + \frac{Ri}{L} = \frac{E(t)}{L}$$

and the integrating factor $e^{Rt/L}$. Note that di/dt has discontinuities at the same points as $E(t)$. Then

$$e^{Rt/L}\left(\frac{di}{dt} + \frac{Ri}{L}\right) = \frac{d}{dt}\,(ie^{Rt/L}) = \frac{E}{L}\,e^{Rt/L}, \tag{45}$$

and integrating between the limits‡ 0 and t we have

$$\int_0^t \frac{d}{du}\,[i(u)e^{Ru/L}]\,du = i(t)e^{Rt/L} - i(0) = \frac{1}{L}\int_0^t E(u)e^{Ru/L}du.$$

Since $i(0) = 0$ we have

$$i(t) = \frac{e^{-Rt/L}}{L}\int_0^t E(u)e^{Ru/L}du. \tag{46}$$

† See Chap. I for the definition of piecewise continuity and Sec. 8, Chap. VII, for a more complete treatment of discontinuous e.m.f.'s.

‡ For the integral of a piecewise continuous function see Sec. 2, Chap. I.

In the integrand of (46), $E(u)$ is identically zero for $t < t_0$, and for $t > t_0$, $E(t) = E_0$; hence

$$(47) \quad i(t) = e^{-Rt/L}\frac{E_0}{L}\int_{t_0}^{t} e^{Ru/L}du = e^{-Rt/L}\frac{E_0}{R}(e^{Rt/L} - e^{Rt_0/L}) \qquad (t \geqq t_0)$$

or

$$(48) \qquad i(t) = \frac{E_0}{R}(1 - e^{-R(t-t_0)/L}) \qquad (\text{for } t \geqq t_0)$$

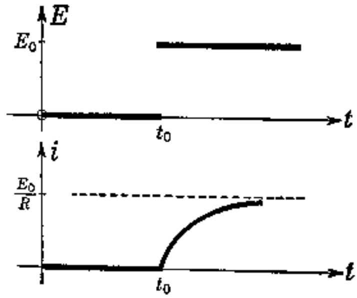


FIG. 19.

and

$$i(t) = 0 \qquad (\text{for } t < t_0)$$

The graph of (48) is drawn in Fig. 19. Observe that for large t, i is practically a constant $= E_0/R$. That is, E_0/R is the steady-state current and $(E_0/R)e^{-R(t-t_0)/L}$ is the transient. It should also be noticed that i is continuous at $t = t_0$, but di/dt does not exist at $t = t_0$.

If more complicated piecewise continuous e.m.f.'s are used, the procedure is exactly the same in principle. A more complete discussion will be found in Chap. VII.

PROBLEMS

1. Find the current in a simple series circuit with no inductance ($L = 0$), a resistance R, and a capacitance C if the e.m.f. is $E = E_0 \sin \omega t$, and $i = 0$ when $t = 0$. Show that the current leads the voltage.

2. In a simple series circuit without capacitance, $L = 0.2$ henry, $R = 10$ ohms, and E is the 60-cycle sine wave of amplitude 160 volts, that is $E = 160 \sin 120\,\pi t$. Find $i(t)$ if $i = 0$ when $t = 0$.

3. A simple series circuit contains no inductance, a resistance R, and a capacitance C. A constant e.m.f. E_0 is applied at time $t = t_0$ when $q = 0$. Find q as a function of t.

4. A series circuit contains a capacitance C and a resistance R, no inductance, and no e.m.f. The switch is closed at $t = 0$ when the charge on the condenser is q_0. Find $q(t)$.

5. Show, in a simple series circuit with inductance L, no resistance, no condenser, and e.m.f. $E = E_0 \sin \omega t$, that the current lags the voltage by a phase angle of 90°, disregarding the transient.

6. Show, in a simple series circuit with $L = 0$, $R = 0$, capacitance C, and e.m.f. $E = E_0 \sin \omega t$, that the current leads the voltage by a phase angle of 90°, disregarding the transient.

14. Flow from an Orifice. Consider the tank of Fig. 20 which is full of liquid whose depth is x. An orifice is located at the depth x. It is shown in physics that the velocity of efflux of the liquid from the orifice is $\sqrt{2gx}$ where g is the acceleration of gravity ($g = 32$ feet per second per second). If A is the cross-sectional area of the orifice, one would suppose that the volume rate of flow for depth x is $A\sqrt{2gx}$. However this is not the case as the stream contracts slightly on emerging from the orifice to form a somewhat narrower cross section called the *vena contracta.* The amount of contraction depends on the orifice shape. Thus the volume rate of flow is $kA\sqrt{2gx}$ where k is a constant depending on the orifice and usually is about 0.6, which is the figure we will use in our problems.

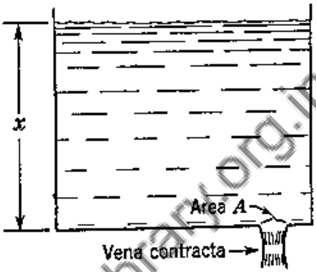


FIG. 20.

Example 21. A tank with a base 2 feet square has 4 feet of water in it. A 2-inch diameter circular orifice is located at the bottom. How long will it take for the tank to empty itself?

The volume at any instant is $4x = V$ and

$$\frac{dV}{dt} = -0.6A\sqrt{2gx}$$

where $A = \pi/144$ square feet. Then

$$4\frac{dx}{dt} = \frac{-0.6\pi}{144}\sqrt{64x} = \frac{-\pi\sqrt{x}}{30}$$

or

$$\frac{dx}{x^{\frac{1}{2}}} = \frac{-\pi}{120}\,dt,$$

and

$$2x^{\frac{1}{2}} = \frac{-\pi t}{120} + C.$$

Since $x = 4$ when $t = 0$, we have $C = 4$ and

$$x^{\frac{1}{2}} = \frac{-\pi t}{240} + 2$$

or

$$x = \left(2 - \frac{\pi t}{240}\right)^2.$$

Therefore $x = 0$ when $t = \frac{480}{\pi}$ seconds $= \frac{8}{\pi}$ minutes.

PROBLEMS

1. Find the time required to empty a cylindrical tank 4 feet in diameter, containing 6 feet of water, through a circular orifice in the bottom 3 inches in diameter.

2. A conical reservoir is 8 feet across the top and 10 feet deep, and contains 8 feet of water. There is a 2-inch diameter hole in the apex. How long will it take to empty the reservoir?

3. Find the final water level for the tank of Prob. 1 if in addition water is being piped into the tank at the rate of 0.1 cubic foot per second.

★4. A tank with a square base 5 feet on a side has 4 feet of water in it when a seam in its side fails forming a slit $\frac{1}{4}$ inch wide from top to bottom. Derive a formula giving the depth of water in terms of the time. (Assume that the coefficient of contraction is 0.8. The coefficient of contraction here is greater than in Example 21 because of the different character of the orifice.)

15. The Law of Mass Action. Under certain circumstances it is found that two substances X and Y react to form a third substance Z, and the rate at which Z is formed is proportional to the product of the masses of X and Y which are as yet untransformed. The conditions under which this situation prevails are discussed in chemistry textbooks, the law being referred to as *the law of mass action.* The law, formulated precisely, is as follows.

Suppose that initially we have x grams of the substance X and y grams of the substance Y and that α grams of X combine with β grams of Y to form $\alpha + \beta$ grams of Z. If z represents the number of grams of Z present at time t, then it contains $\alpha z/(\alpha + \beta)$ grams of X and $\beta z/(\alpha + \beta)$ grams of Y. Thus the amounts of X and Y remaining are $x - \alpha z/(\alpha + \beta)$ and $y - \beta z/(\alpha + \beta)$, respectively, so that

$$\frac{dz}{dt} = k\left(x - \frac{\alpha z}{\alpha + \beta}\right)\left(y - \frac{\beta z}{\alpha + \beta}\right), \tag{49}$$

where k is the constant of proportionality. Equation (49) may be written as

$$\frac{dz}{dt} = \frac{k\alpha\beta}{(\alpha + \beta)^2}\left(\frac{\alpha + \beta}{\alpha} x - z\right)\left(\frac{\alpha + \beta}{\beta} y - z\right),$$

or

$$\frac{dz}{dt} = K(A - z)(B - z), \tag{50}$$

where

$$K = \frac{k\alpha\beta}{(\alpha + \beta)^2}, \qquad A = \frac{\alpha + \beta}{\alpha} x, \qquad B = \frac{\alpha + \beta}{\beta} y.$$

In Equation (50) the variables are separable and (50) may be rewritten as

$$\frac{dz}{(A - z)(B - z)} = K\, dt.$$

There are two essentially different cases to be considered. If $A = B$, then the left member of the above equation integrates by the "power formula," whereas if $A \neq B$ exponential functions arise. Here we suppose $A \neq B$ and without loss of generality may suppose the substances labeled so that $A > B$. Then, expanding in partial fractions,

$$\frac{dz}{(A - z)(B - z)} = -\frac{1}{A - B}\frac{dz}{A - z} + \frac{1}{A - B}\frac{dz}{B - z},$$

whence

$$\frac{1}{A - B} \log \left(\frac{A - z}{B - z}\right) = Kt + C_1,$$

or

$$\frac{A - z}{B - z} = Ce^{K(A-B)t},$$

and

$$z = \frac{A - BCe^{K(A-B)t}}{1 - Ce^{K(A-B)t}}.$$

Since, further, $z = 0$ when $t = 0$ we find easily that $C = A/B$, so that

$$z = \frac{AB(1 - e^{-K(A-B)t})}{A - Be^{-K(A-B)t}}.$$

We see from this last equation that as $t \to \infty$, $z \to B$.

PROBLEMS

1. Work out the relation for z in terms of t if $A = B$ in Equation (50).

*2. Derive a formula for the amount u of a substance U formed from substances X, Y, and Z if α, β, γ grams of X, Y, Z, respectively, unite to form $\alpha + \beta + \gamma$ grams of U. Assume that at $t = 0$, we have x, y, z grams of X, Y, Z, respectively, and none of U is present.

16. Diffusion. If a block of salt is placed in water, the salt slowly dissolves and the salt solution gradually spreads, because of molecular action, through the water. In a like manner, if one or more hot bodies are placed in a colder medium, the heat from the bodies is disseminated by conduction† to the surrounding medium. In both cases we have examples of the process of *diffusion*. In this section we will confine our attention to the diffusion of heat.

If the bodies are kept at constant temperature, a steady state will be reached after a long period of time. In general the temperature at any point of the medium will be a function of the coordinates of the point and a description of the temperature variation will involve partial derivatives. In certain simple cases, however, the temperature will depend only on a single space coordinate x and only ordinary derivatives will arise. It is then found that

$$Q = -kA\,\frac{dT}{dx} \tag{51}$$

where Q is the quantity of heat flowing per unit time across a surface of area A perpendicular to the x direction, T is the temperature, and k is a constant of the medium, called the *thermal conductivity*. The negative sign occurs because the heat flows in the direction of decreasing temperature.

Since we are dealing with the steady-state condition, the heat flowing across one surface must be the same as that across any other surface so that the quantity Q in (51) is constant.

The constant k in (51) will of course depend on the units used. In the problems that follow we will suppose T to be in degrees Fahrenheit, Q in Btu (British thermal units), x in feet, and t in seconds.

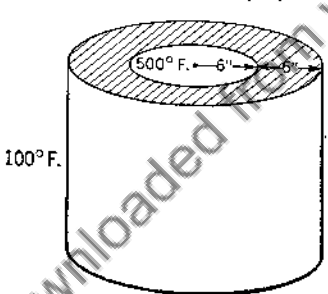


FIG. 21.

Example 22. A steam pipe (infinitely long) of diameter 1 foot has a cylindrical jacket 6 inches thick made of an insulating material ($k = 0.00033$). If the pipe is kept at 500°F and the outside of the jacket at 100°F, find the temperature distribution in the jacket. What is the heat loss per day per foot of pipe?

Here T is a function of the distance r from the center of the pipe alone and the area A per unit length is $2\pi r$.
Hence

$$Q = \text{constant} = -2\pi r k\,\frac{dT}{dr}$$

† Heat is of course also transferred by radiation, but this we neglect.

and

$$Q\frac{dr}{r} = -2\pi k\,dT. \tag{52}$$

Integrating (52) between $r = \frac{1}{2}$ to $r = 1$ and $T = 500$ to $T = 100$, we find

$$Q \log 2 = -2\pi k(100 - 500) = 800\pi k$$

and

$$Q = \frac{800\pi k}{\log 2} = 1.2(\text{Btu/sec})/\text{ft}.$$

The indefinite integral of (52) is

$$Q \log r = -2\pi kT + C.$$

Since at $r = 1$, $T = 100$ we find

$$0 = -2\pi k(100) + C,$$
$$C = 200\pi k.$$

Then solving for T and using our value of Q,

$$T = 100 - 400\frac{\log r}{\log 2}.$$

The heat loss per second per foot of pipe is $Q = 800\pi k/\log 2$. Hence the loss per day per foot of pipe is

$$Q(60)(60)(24) = 103{,}680 \text{ Btu per day}.$$

PROBLEMS

1. A hollow spherical shell of inner radius 1 foot and outer radius 2 feet has a source of heat inside it which keeps the interior at 400°F. If the conductivity k equals 0.0025, find the heat lost per hour when the exterior is kept at 100°F. Also find the temperature distribution as a function of the distance from the center of the shell.

2. A masonry wall is 2 feet thick ($k = 0.00075(2 + x)$, where x is the distance from the inner surface. If the inner surface is at 72°F and the outer surface at 32°F, find the heat loss per day per square foot of area.

***3.** A wall w feet thick ($k = 0.0005$) has its inner face at temperature T_1. The outer face is exposed to air at temperature T_2. The rate of heat loss to the air is proportional to the difference in the temperatures of the exposed face and the air. Find the heat lost per second per square foot in terms of T_1 and T_2. [Hint: Rate of heat loss to air $= c(T_3 - T_2)$ where T_3 is the temperature of the outer face and c is a constant of proportionality. Equate this loss to that through the wall.]

4. A pipe 4 inches in diameter containing a refrigerant at the constant temperature −30°F is covered by a 3-inch jacket of insulating material the thermal conductivity of which is 0.0002. If the temperature of the outside of the jacket is kept at 50°F, what is the temperature distribution in the insulating jacket and the heat gained by the refrigerant per day per foot of pipe?

MISCELLANEOUS PROBLEMS

Classify the following equations as to type and solve. (Some may possibly be done in several ways.) If an initial point is specified, find the integral curve through that point.

1. $x\dfrac{dy}{dx} + y = x.$

2. $(\sin x + y^2)\,dy = (x^2 - y\cos x)\,dx.$

3. $2x\dfrac{dy}{dx} = y + x^{\frac{3}{2}} - x^{\frac{1}{2}}.$

4. $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 + y^2}}{x};$ (2, 0).

5. $xy - x^2y' = -y.$

6. $y^2 + (x + y)^2 + xy = x^2\dfrac{dy}{dx}.$

7. $x^2\left(\dfrac{dy}{dx}\right)^2 - (x^2 + y^2)\dfrac{dy}{dx} + y^2 = 0.$

8. $(x + y)^2y' = 1.$

9. $(x + 2y + 3)\,dx + (x + 2y + 1)\,dy = 0.$

10. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx.$

11. $x\,dy + (y - \log x)\,dx = 0.$

12. $(a^2 + y^2)\,dx + (b^2 + x^2)\,dy = 0.$

13. $(x^2 + y^2)\,dx + 2xy\,dy = 0.$

14. $\left(y + \dfrac{y}{x^2 + xy}\right)dx = \left(\dfrac{1}{x + y} - x\right)dy.$

15. $\sin\left(\dfrac{dx}{dy}\right) = y;$ $(\frac{1}{2}\sqrt{3}, \frac{1}{2}).$

16. $(2x - y)\,dy + (x - 2y + 3)\,dx = 0;$ (−1, −1).

17. $y^2\,dx + x^2\,dy = xy\,dy;$ (1, 2).

18. $(x + 1)\dfrac{dy}{dx} - 2y = (x + 1)^4.$

19. $(xy^2 + y)\,dx + x\,dy = 0.$

20. $\dfrac{dy}{dx} = \dfrac{x - 2y - 5}{y - 2x + 4};$ (2, 1).

21. $y\,dy = x\,dy - y\,dx.$

22. $\cos x\,dy = y\sin x\,dx + \sin x\cos x\,dx.$

23. $\dfrac{dy}{dx} + 1 = e^{x-y}.$

24. $(x^2 + x)\dfrac{dy}{dx} = (x - y - 2xy)$.

25. $(x^2 - 1)^{\frac{3}{2}}\, dy + (x + 3xy\sqrt{x^2 - 1})\, dx = 0$.

26. $\dfrac{x^2y - 1}{y}\, dx + \dfrac{y^2 + x}{y^2}\, dy = 0$.

27. $x^2y' + 2xy = x^{-2}$.

28. $y'^2 - y' + 6 = 0$.

29. $(4x - y + 3)\, dx + (x + y + 2)\, dy = 0$.

30. $(x + y)^{-2}\dfrac{dy}{dx} = 1$.

31. Find the locus of the points of inflection of the integral curves of

$$y' = x^2 + y$$

without solving for y. Sketch the direction field and indicate the portions of the plane where the integral curves are concave up and concave down.

32. Find the orthogonal trajectory of the family $y = cx^3$ which passes through the point $(2, -1)$.

33. Show that if a mass particle slides down a curved frictionless wire through a vertical distance h the speed of the particle is $\sqrt{2gh}$.

34. How long will it take to empty a full spherical tank of radius 2 feet through a 2-inch diameter hole in the bottom? (Assume that the entire hole is at the bottom of the sphere.)

35. Determine the orthogonal trajectories of the family of hyperbolas $y = k + 1/x$.

36. Determine the orthogonal trajectories of the family $\rho = c \sin (\theta/2)$.

37. What curves have a subtangent of constant length?

38. Find the family of curves such that at every point (x, y) the tangent is perpendicular to the line joining the point (x, y) to the point $(1, -1)$.

39. Show that if the atmosphere is at rest

$$\frac{dp}{dh} = -\rho g$$

where p = pressure in pounds per square foot, h = altitude in feet, ρ = density in slugs per cubic foot at altitude h.

40. Use the result of Prob. 39 to determine the pressure as a function of altitude if (*a*) the atmosphere is isothermal: $p = k\rho$; (*b*) the atmosphere is adiabatic: $p = k\rho^{1.4}$.

★**41.** Show that if a mirror focuses a parallel beam on a point its shape is that of a paraboloid of revolution.

42. A simple circuit has inductance L, resistance R, and zero capacitance. It is connected to a battery of e.m.f. E and the current has attained its steady-state value E/R when the battery is disconnected. Determine the decay of the current.

43. Find the family of curves which cut the family of straight lines $y = cx$ at a constant angle α.

44. A boy, standing on the (straight) edge of a pond, has a boat on the end of a string of length l directly offshore from him. He walks along the bank pulling the boat. Find the path of the boat. (This is a plane problem, and the height of the boy above the water is taken as zero.)

45. A large tank contains 50 gallons of brine containing 20 pounds of dissolved salt. Pure water runs into the tank at the rate of 3 gallons per minute, and brine runs out at 2 gallons per minute. How much salt is in the tank at the end of 2 hours?

46. Bacteria when grown in a nutrient solution increase at a rate proportional to the number present. If in a particular culture the number doubles in 2 hours and there were 10^6 at the end of 10 hours, how many were there initially?

47. A room contains 5,000 cubic feet and a concentration of 0.5% CO_2 by volume. The CO_2 content is being increased at the constant rate of 0.1 cubic foot per minute by the occupants of the room. Blowers are turned on pumping air with 0.04% CO_2 by volume. How much air must be pumped in order to bring the CO_2 concentration eventually down to 0.05%? How long will it be before the CO_2 concentration is down to 0.1%?

48. A chain 50 feet long weighing 1 pound per foot hangs over a small frictionless pulley with 25 feet on either side. To each end is attached a 5-pound weight. One of the weights falls off. Find the velocity of the remaining weight as a function of its distance from the pulley.

49. A chain 50 feet long is on a smooth table with 5 feet hanging over the edge. The chain weighs 1 pound per foot and is initially at rest. Find the velocity of the end of the chain as a function of the length hanging over the edge.

CHAPTER IV

SECOND-ORDER DIFFERENTIAL EQUATIONS

1. Introduction. Second-order differential equations occur throughout the physical sciences. It is no overstatement to say that without an understanding of at least linear second-order equations with constant coefficients one cannot obtain a thorough grasp of the basic physical laws. One of the reasons for the frequency of second-order equations is the fact that Newton's law relates force to acceleration, and acceleration is the second derivative of distance with respect to time. Fortunately, the differential equations which arise in first approximations to physical problems are often linear, and it is just this class of equations which we mainly wish to discuss.

2. Numerical Solution. Existence Theorem. A second-order equation asserts that some functional relationship exists between d^2y/dx^2, dy/dx, y and x, that is,

$$F(x, y, y', y'') = 0, \tag{1}$$

where F is a function defined in some region of $xyy'y''$ space, for example, in the "four-dimensional box," $a < x < b$, $c < y < d$, $e < y' < f$, $g < y'' < h$. The function F is presumed to be continuous and to have as many derivatives as desired. Equation (1), however, is much too general to be easily discussed, and we will suppose that (1) has been solved for y'' in terms of x, y, y',

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \tag{2}$$

where f is a function defined over some region of xyy' space, is continuous (or piecewise continuous) and perhaps has continuous partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial y'$.

We saw in Chap. II that the solution of a first-order equation is determined if one specifies a point (x_0, y_0) through which the integral curve is to pass. For second-order equations one can require in addition that the integral curve have at (x_0, y_0) a given slope y_0'. (It is of course assumed that x_0, y_0, y_0' is a triple of numbers for which $f(x, y, y')$ is defined.)

Let us then try to piece together an approximate solution by starting out with our initial data x_0, y_0, y_0'. Choose arbitrarily an increment $\Delta x = h$. (Since this is an arbitrary choice it may happen that one will find it desirable later to rework the problem with a smaller value for h in order to achieve greater accuracy.)

From y_0' one can now find approximately the y coordinate of the point on the integral curve where $x = x_0 + h$, the change in y being approximately

$$\Delta y_0 = y_0'h.$$

The nearby point is P_1, namely, $(x_0 + h, y_0 + \Delta y_0)$. Now we cannot immediately repeat this process because we do not, as yet, know the value of the slope at P_1. However we can easily obtain this slope approximately from the differential equation. In going from P_0 to P_1 the slope will have changed by an amount given approximately by

$$\Delta y_0' = y_0''h,$$

as is easily seen from the differential relation

$$dy' = \frac{dy'}{dx}\,dx = \frac{d^2y}{dx^2}\,dx.$$

Thus the slope at P_1 is $y_1' = y_0' + \Delta y_0'$. We are now in possession of the same kind of data as we had initially and can obtain a second approximate point P_2, namely, $(x_1 + h, y_1 + \Delta y_1)$, where

$$\Delta y_1 = y_1'h.$$

The slope at P_2 is given by $y_1' + \Delta y_1'$ where

$$\Delta y_1' = y_1''h.$$

It is clear that the precision of this numerical technique depends greatly on the magnitude of h and the rate at which the first derivative changes, that is, the magnitude of the second derivative. Practice in actually working out problems soon will enable one to choose h small enough to yield reliable answers. There are, however, certain refinements† which materially lessen the labor and allow one to use larger values for h. We will not go into these techniques here.

Example 1. Find an approximate solution to

$$\frac{d^2y}{dx^2} = x + y^2 + \frac{dy}{dx}$$

which passes though (0, 0) with slope 0.

† See J. B. Scarborough, "Numerical Methods in Analysis," pp. 232–252, Johns Hopkins Press, 1930.

We choose $h = 0.5$ and find several approximate points on the integral curve. P_0, the initial point, is (0, 0).

$$\Delta y_0 = y_0'h = 0(0.5) = 0,$$

hence

$$y_1 = y_0 + \Delta y_0 = 0 + 0 = 0,$$

and P_1, the first approximate point, is (0.5, 0). Now

$$\Delta y_0' = y_0''h = (0 + 0^2 + 0)(0.5) = 0,$$

so

$$y_1' = y_0' + \Delta y_0' = 0 + 0 = 0.$$

Then

$$\Delta y_1 = y_1'h = 0(0.5) = 0,$$

so

$$y_2 = y_1 + \Delta y_1 = 0,$$

and P_2 is (1, 0). Similarly,

$$\begin{aligned} \Delta y_1' &= y_1''h = (0.5 + 0^2 + 0)(0.5) = 0.25, \\ y_2' &= y_1' + \Delta y_1' = 0 + 0.25 = 0.25, \\ \Delta y_2 &= y_2'h = (0.25)(0.5) = 0.13, \\ y_3 &= y_2 + \Delta y_2 = 0 + 0.13 = 0.13, \end{aligned}$$

and P_3 is (1.5, 0.13). The process may obviously be continued.

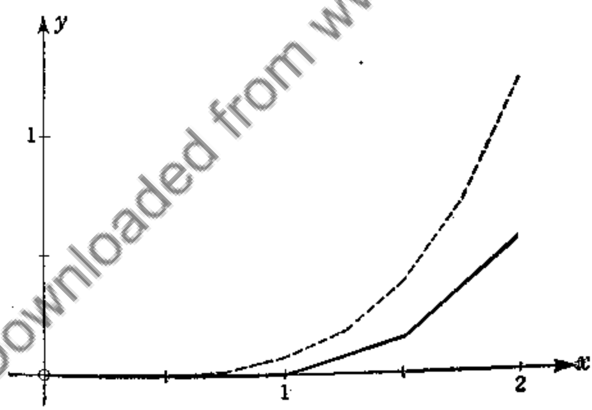


FIG. 22.

The graph of our approximate solution is drawn in Fig. 22. The approximate solution obtained using $h = 0.25$ is drawn as a dotted line.

Our approximate construction has now made plausible the following fundamental existence theorem for second-order equations. For a proof the reader is referred to more advanced texts, for example, Ref. 3, p. 71.

Theorem. Let $f(x, y, y')$ *be a continuous function for* $a_1 < x < a_2$, $b_1 < y < b_2$, $c_1 < y' < c_2$ *and have continuous partial derivatives* $\partial f/\partial y$ *and* $\partial f/\partial y'$ *in this same region. Then if* (x_0, y_0, y_0') *is a point of this region, the differential equation*

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

has a unique solution $y = y(x)$ *passing through* (x_0, y_0) *with slope* y_0' *and this solution* $y(x)$ *is defined over an interval* $\alpha_1 < x < \alpha_2$ *containing* x_0.

It should be observed that the interval (α_1, α_2) is not necessarily the same as (a_1, a_2). It may be much smaller.

The problem we have stated and whose solution is guaranteed by the existence theorem is known as an *initial-value problem.* This is a natural description since we are given data at an initial point. Most problems we will consider will be of this type. There is another type of problem, however, which sometimes occurs and is known as a *boundary-value problem.* For this problem we are given, instead of (x_0, y_0, y_0'), two points (x_0, y_0) and (x_1, y_1) for our integral curve to pass through. Here existence theorems become more difficult to state, and we will content ourselves with some intuitive remarks relating to Fig. 23.

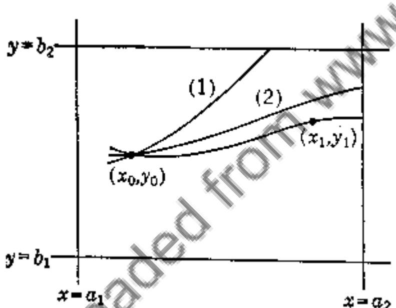


FIG. 23.

In the first place if we attempt to piece together a solution as before, starting from (x_0, y_0), we find that we do not know in what direction to move from (x_0, y_0). Suppose then we were to start out from (x_0, y_0) in the arbitrary direction y_0'. Our existence theorem assures us that a solution with initial values (x_0, y_0, y_0') exists. But it may happen that our solution would not extend as far as desired, as is exemplified by the curve (1) in the figure. Or even if the curve were to extend far enough, it might easily miss the point (x_1, y_1) as does curve (2). Clearly then we would have to be most lucky to find the curve passing through (x_1, y_1) by making an initial choice for y_0'.

PROBLEMS

1. If

$$\frac{d^2y}{dx^2} = \sqrt{x^2 + y}\,\sin\left(\frac{dy}{dx}\right)$$

and the integral curve passes through (1, 0) with slope equal to $\pi/6$, find roughly, the value of y and the slope at the point where $x = 1.5$.

2. If

$$y'' = \sin x \sin y(1 + y'^2)^{\frac{3}{2}}$$

find a point on the integral curve which passes through $(\pi/4, 0)$ with slope 1, on the line $x = \pi/2$.

3. Find, approximately, the value of y between $x = 1$ and $x = 2$ if $y = \frac{1}{2}$ and $y' = 0$ when $x = 1$, and if

$$y'' + y' - x = 0.$$

Use $\Delta x = 0.25$. Compare your answer with that obtained (between $x = 1$ and $x = 1.5$) when using $\Delta x = 0.1$. Then compare with the exact answer given by $y = 1 - x + x^2/2$.

4. Find approximately, between $x = 0$ and $x = 1$, the integral curve of

$$y'' - xy' + y = -1.5$$

if $y = 1$, $y' = 1$ when $x = 0$. Make a suitable choice for Δx.

5. If $y = 1$, $y' = 1$, $y'' = 0$ when $x = 1$, find approximately the point on the integral curve of

$$y''' + y'' + y' + y = x$$

with abscissa 1.2.

3. General Solutions and Families of Curves. It will be seen shortly that for certain types of second-order equations we can obtain solutions given simply in terms of known functions. Furthermore, these solutions will contain usually two arbitrary constants. We make the following definition: *A general solution of Equation* (2) *is a two-parameter family of solutions* $y = \varphi(x, c_1, c_2)$, or $g(x, y, c_1, c_2) = 0$.

It is possible to conclude from the existence theorem of the previous paragraph that there is such a two-parameter family. For, with the initial data x_0, y_0, y_0', there is a unique integral curve. We have only to hold x_0 fixed and allow y_0, y_0' to vary continuously in order to obtain such a two-parameter family.

If one can obtain explicitly a general solution $y = \varphi(x, c_1, c_2)$, it is possible to solve an initial-value problem or a boundary-value problem with relative ease. For an initial-value problem we have

$$(3)\qquad \begin{aligned} y_0 &= \varphi(x_0, c_1, c_2) \\ y_0' &= \frac{d\varphi(x_0, c_1, c_2)}{dx} \end{aligned}$$

which are two equations to be solved for the two arbitrary constants c_1 and c_2. For a boundary-value problem we have

$$(4)\qquad \begin{aligned} y_0 &= \varphi(x_0, c_1, c_2) \\ y_1 &= \varphi(x_1, c_1, c_2) \end{aligned}$$

which again are two equations in c_1 and c_2. We see then that if we have a general solution both initial-value and boundary-value problems reduce to the simultaneous solution of two equations in two unknowns.

PROBLEMS

1. A general solution of $y'' + 2y' + 5y = 0$ is $y = Ae^{-x} \sin 2(x + B)$. Find the particular solution for which $y = 0$, $y' = \frac{1}{2}$ when $x = 0$.

2. A general solution of $x^2y'' - 2xy' + 2y = 2$ is $y = 1 + kx + cx^2$.

(*a*) Find the integral curve passing through (1, 1) and (−1, 3).
(*b*) Find the integral curve passing through (−1, 0) with slope 0.

3. $y = c_1e^{-x} + c_2e^{x} + x$ is a general solution of $y'' - y = -x$. Show that there is always a unique integral curve of this family passing through the points (x_0, y_0) and (x_1, y_1) where $x_0 \neq x_1$.

***4.** The existence theorem of Sec. 2 asserts that an initial-value problem always has a solution. Boundary-value problems, on the other hand, need not always have a solution.

Show that the boundary-value problem

$$y'' + y = 0; \quad y = y_1 \text{ for } x = x_1,\ y = y_2 \text{ for } x = x_2; \quad x_1 \neq x_2$$

does not in general have a solution if $x_1 - x_2 = n\pi$, where n is an integer. Use the fact that the general solution of the differential equation is

$$y = A \sin x + B \cos x.$$

***5.** In Prob. 4 show that if $x_1 - x_2 = \pi$ there is a solution only if $y_1 = -y_2$, and in fact a one-parameter family of solutions.

4. Linear Equations with Constant Coefficients. By far the most important type of second-order equation is *linear*, that is, it has the form

$$(5)\qquad \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x),$$

where the coefficients a and b are functions of x alone. If, further, a and b are constant we have a *linear equation with constant coefficients,* which case will concern us for the major part of this chapter. If the right side of (5) vanishes, $f(x) \equiv 0$, the equation is said to be *linear homogeneous.*

The homogeneous equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{6}$$

has several very important properties:

(i) If $y_1(x)$ is a solution of (6) so is $c_1y_1(x)$, where c_1 is an arbitrary constant.

(ii) If $y_1(x)$ and $y_2(x)$ are solutions of (6) so is $y_1(x) + y_2(x)$.

(iii) If $y_1(x)$ and $y_2(x)$ are linearly independent† solutions of (6), then *every* solution of (6) is given by

$$c_1y_1(x) + c_2y_2(x) \tag{7}$$

where c_1 and c_2 are arbitrary constants. Hence in this case we are justified in speaking of (7) as *the* general solution.

The assertions (i) and (ii) can be verified by direct substitution, and this is left as an exercise for the reader; see also the discussion in Chap. V. Assertion (iii) is a special case of a general theorem given in Chap. V. It is also true that solutions of (6) are defined for all values of x for which the coefficients $a(x)$ and $b(x)$ are continuous, in particular then for all x if a and b are constants. It should be emphasized that (i), (ii), and (iii) apply only to the homogeneous equation.

With these results to go by it is clear that all we need to find is a pair of linearly independent solutions. We restrict ourselves now to the case of *constant coefficients* and observe that, because $(d/dx)e^{rx} = re^{rx}$, it is likely that (6) has a solution of the form e^{rx}. We therefore try $y = e^{rx}$ and obtain

$$r^2e^{rx} + are^{rx} + be^{rx} = 0.$$

Since $e^{rx} \neq 0$ we may divide it out and obtain a quadratic equation for r

$$r^2 + ar + b = 0, \tag{8}$$

which is called the *auxiliary equation.*

† Two functions $f(x)$ and $g(x)$ are said to be linearly independent if

$$\alpha f(x) + \beta g(x) \equiv 0$$

implies that $\alpha = \beta = 0$ (see Sec. 6, Chap. I).

Solving the quadratic equation (8) for r gives

(9) $$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

If we denote the two roots of (8) by r_1 and r_2, then e^{r_1x} and e^{r_2x} are linearly independent solutions if $r_1 \neq r_2$. (For a proof of their linear independence see Sec. 4, Chap. V.) The general solution in this case is then

(10) $$y = c_1e^{r_1x} + c_2e^{r_2x}.$$

In order that (10) be the general solution it is necessary that $r_1 \neq r_2$. However if the two roots of (8) are equal $r_1 = r_2 = r$, direct substitution shows that xe^{rx} is also a solution. For,

$$y = xe^{rx}$$
$$y' = xre^{rx} + e^{rx}$$
$$y'' = xr^2e^{rx} + 2re^{rx},$$

and substituting y, y', y'' in the differential equation gives

$$\begin{aligned} y'' + ay' + by &= xr^2e^{rx} + 2re^{rx} + axre^{rx} + ae^{rx} + bxe^{rx} \\ &= e^{rx}\,[x(r^2 + ar + b) + (2r + a)] = 0. \end{aligned}$$

The expression $r^2 + ar + b = 0$ because r is a root of the auxiliary equation (8), while $2r + a = 0$ because the roots are equal. Hence, if $r_1 = r_2 = r$, the general solution of (6) is

(11) $$y = c_1e^{rx} + c_2xe^{rx},$$

since e^{rx} and xe^{rx} are linearly independent. (For a proof see Sec. 4. Chap. V.)

Example 2. Find the general solution of

$$y'' - 2y' - 3y = 0.$$

The auxiliary equation is

$$r^2 - 2r - 3 = 0,$$

whence $r_1 = 3$ and $r_2 = -1$. The general solution is

$$y = c_1e^{3x} + c_2e^{-x}.$$

Example 3. Find the general solution of

$$y'' + 4y' + 4y = 0.$$

The auxiliary equation is

$$r^2 + 4r + 4 = 0,$$

and we have equal roots $-2, -2$. The general solution is

$$y = c_1e^{-2x} + c_2xe^{-2x}.$$

In Chap. V when operational methods for solving linear equations with constant coefficients are discussed, it will be convenient to use the symbolic operator "D" which denotes differentiation with respect to the independent variable; for example,

$$Dy = \frac{dy}{dx}.$$

We further define,

$$D^2y = D(Dy) = \frac{d^2y}{dx^2},$$

$$D^ny = D(D^{n-1}y) = \frac{d^ny}{dx^n},$$

and

$$(aD^n + bD^m)y = aD^ny + bD^my.$$

Our second-order equation (5) then may be written as

$$(D^2 + aD + b)y = f(x).$$

PROBLEMS

1. Find the general solutions of the following equations:

(a) $y'' - y' - 2y = 0.$
(b) $(D^2 + 3D + 2)y = 0.$
(c) $(D^2 + 4D + 2)y = 0.$
(d) $y'' + 2y' - 8y = 0.$
(e) $D^2y + 6Dy + 9y = 0.$
(f) $y'' = 9y.$
(g) $y'' - 2\sqrt{5}y' + 5y = 0.$
(h) $y'' - ky' - 3y = 0.$
(i) $2y'' - y' - 3y = 0.$
(j) $kly'' + (k - l)my' - m^2y = 0.$
(k) $y'' - 2my' + m^2y = 0.$
(l) $y'' = 15y - 2y'.$
(m) $4y = 12y' - 9y''.$
(n) $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0; \quad R^2 > 4\frac{L}{C}.$
(o) $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0; \quad R^2 = 4\frac{L}{C}.$

2. Find the particular integrals of the following equations which have the initial values or boundary values indicated.

(a) $y'' - y' - 2y = 0$; $y = 3, x = 0$; $y = 5, x = \log 2$.
(b) $(D^2 - 4)y = 0$; $y = 1, y' = \frac{1}{2}, x = 0$.
(c) $y'' - 4y' + 4y = 0$; $y = 3, y' = 4, x = 0$.
(d) $2y'' + y' - y = 0$; $y = 5, x = 0$; $y = 17, x = \log 4$.
(e) $y'' - 5y' + 6y = 0$; $y = e^2, y' = 3e^2, x = 1$.
(f) $y'' - 2y' - 3y = 0$; $y = 21.5, y' = 140, x = 1$.
(g) $y'' + 2y' + y = 0$; $y = 2.00, y' = -0.97, x = 1.2$.
(h) $y'' + 4y' + 2y = 0$; $y = -1, y' = 2 + 3\sqrt{2}, x = 0$.
(i) $y'' = y$; $y = 2.200, x = 0$; $y = 2.272, x = 0.5$.
(j) $y'' - 2y' - 15y = 0$; $y' = 10, y'' = 2, x = 0$.
(k) $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = 0$; $R = 3{,}000$, $L = 100 \times 10^{-6}$,

$C = 100 \times 10^{-12}$; $q = 1 \times 10^{-8}$, $\dfrac{dq}{dt} = 0$, $t = 0$.

★(l) $y'' - y' - 2y = 0$; $y = 5, x = 0$; minimum value of y is 4.09.

5. Complex Roots. In the previous discussion we tacitly assumed the roots r_1 and r_2 of Equation (8) were real numbers (the coefficients a and b are of course presumed real). We now inquire as to what meaning can be attached to our results if r_1 and r_2 are complex. Since a and b are real, complex roots of (8) are necessarily conjugate complex numbers which we denote by

(12)
$$r_1 = \alpha + \beta i$$
$$r_2 = \alpha - \beta i$$

with α and β real.

It will be necessary now to use the following relation, called "Euler's formula":

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

a derivation of which may be found in Sec. 9, Chap. I. The general solution of (6) then is

$$\begin{aligned} y &= c_1e^{r_1x} + c_2e^{r_2x} = c_1e^{(\alpha+\beta i)x} + c_2e^{(\alpha-\beta i)x} \\ &= e^{\alpha x}(c_1e^{\beta ix} + c_2e^{-\beta ix}) \\ &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]. \end{aligned}$$

If now we take c_1 and c_2 to be conjugate complex numbers,

$$c_1 = A + iB,$$
$$c_2 = A - iB,$$

we get

$$\begin{aligned} y &= e^{\alpha x}[(A + iB)(\cos \beta x + i \sin \beta x) + (A - iB)(\cos \beta x - i \sin \beta x)] \\ &= e^{\alpha x}[(2A \cos \beta x - 2B \sin \beta x) + i(0)]. \end{aligned}$$

For $A = \frac{1}{2}$, $B = 0$, we have

(13) $$y = e^{\alpha x} \cos \beta x,$$

and for $A = 0$, $B = -\frac{1}{2}$, we have

(14) $$y = e^{\alpha x} \sin \beta x.$$

Since the functions (13) and (14) are linearly independent solutions (for proof see Sec. 4, Chap. V), the general solution of (6) is

(15) $$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x),$$

because of property (iii), Sec. 4.

The reader unfamiliar with, or repelled by, complex exponents may want to verify independently that (13) and (14) are solutions. Direct substitution of these functions in (6) will verify that both functions are solutions of the differential equation. To carry this out it is necessary to recall the formulas for the sum and product of the roots of a quadratic equation:

$$r_1 + r_2 = 2\alpha = -a$$

and

$$r_1 r_2 = \alpha^2 + \beta^2 = b.$$

The details of the verification that (13) and (14) are solutions of (6) are left as an exercise for the reader.

Equation (15) may be given a somewhat different form as follows. Multiplying and dividing the right member by $\sqrt{c_1^2 + c_2^2}$ we have

$$y = \sqrt{c_1^2 + c_2^2}\, e^{\alpha x} \left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \cos \beta x + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \sin \beta x \right).$$

Let δ_1 be an angle such that

$$\sin \delta_1 = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}, \qquad \cos \delta_1 = \frac{c_2}{\sqrt{c_1^2 + c_2^2}},$$

and let $A = \sqrt{c_1^2 + c_2^2}$. Then the above solution becomes

$$\begin{aligned} y &= Ae^{\alpha x}(\sin \delta_1 \cos \beta x + \cos \delta_1 \sin \beta x) \\ &= Ae^{\alpha x} \sin (\beta x + \delta_1), \end{aligned}$$

or

(16) $$y = Ae^{\alpha x} \sin \beta(x + \delta),$$

where $\delta_1 = \beta\delta$. Equation (16) contains two arbitrary constants A and δ and hence is a general solution.

Example 4. Find the general solution of

$$(D^2 + 2D + 4)y = 0.$$

The auxiliary equation is

$$r^2 + 2r + 4 = 0,$$

which has the complex roots $r = -1 \pm i\sqrt{3}$, and therefore the general solution is

$$y = e^{-x}(c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x),$$

or

$$y = Ae^{-x} \sin \sqrt{3}\,(x + \delta).$$

Example 5. Find the motion of a particle of mass m which is attracted to a point 0 by a force proportional to the distance from 0 and directed toward 0.

Let x be the distance from 0, then the force on the particle is $-kx$ where k is a positive constant of proportionality. From Newton's law (force = mass × acceleration), we get

$$m\frac{d^2x}{dt^2} = -kx,$$

or†

$$\ddot{x} + \frac{k}{m}x = 0.$$

This linear equation is one of the most important differential equations of mechanics. The auxiliary equation is

$$r^2 + \frac{k}{m} = 0,$$

so that

$$r_1 = i\sqrt{\frac{k}{m}}, \qquad r_2 = -i\sqrt{\frac{k}{m}}.$$

The general solution then is

$$y = c_1 \cos \sqrt{\frac{k}{m}}\, t + c_2 \sin \sqrt{\frac{k}{m}}\, t = A \sin \sqrt{\frac{k}{m}}\,(t + \delta).$$

† We use the "dot" notation to denote differentiation with respect to time: $\frac{dx}{dt} = \dot{x}$, $\frac{d^2x}{dt^2} = \ddot{x}$, etc.

Inspection of this solution shows that the particle has a periodic motion with period $2\pi/\sqrt{k/m}$. We will return to this motion later.

To recapitulate, we now have three possible forms for the general solution of the homogeneous equation (6) with constant coefficients, depending on the character of the roots of the auxiliary equation.

$$
\begin{aligned}
&y = c_1e^{r_1x} + c_2e^{r_2x} && \text{(roots real and unequal)}\\
(17)\quad &y = c_1e^{rx} + c_2xe^{rx} && \text{(roots real and equal)}\\
&y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)\\
&\;\;= Ae^{\alpha x} \sin \beta(x + \delta) && \text{(conjugate complex roots, } \alpha \pm i\beta)
\end{aligned}
$$

If the equation does not have constant coefficients, certain special methods may be used. These methods are discussed in Chap. VII.

PROBLEMS

1. Find the general solutions of

(a) $(D^2 + 3D + 3)y = 0.$
(b) $(D^2 - 2D + 2)y = 0.$
(c) $y'' - 3y' + 2y = 0.$
(d) $y'' - y' + 3y = 0.$
(e) $2y'' - 5y' + 3y = 0.$
(f) $\dfrac{d^2s}{dt^2} - 3\dfrac{ds}{dt} - 4s = 0.$
(g) $y'' - 2ky' + (k^2 + 1)y = 0.$
(h) $(2D^2 - 3D + 2)y = 0.$
(i) $y'' - 4y' + (4 + k^2)y = 0.$
(j) $(4D^2 - 24D + 37)y = 0.$
(k) $(D^2 + 6D + 13)y = 0.$
(l) $(k^2D^2 - 4k^2D + 4k^2 + 1)y = 0.$
(m) $(k^2D^2 + 2kD + k^2 + 1)y = 0.$
(n) $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = 0;\quad R^2 < 4\dfrac{L}{C}.$
(o) $ay'' + 2y' + cy = 0;\quad ac > 1.$
(p) $ay'' + 2y' + cy = 0;\quad ac = 1.$
(q) $\dfrac{w}{g}\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0;\quad c^2 - 4\dfrac{kw}{g} < 0.$
(r) $\dfrac{w}{g}\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0;\quad c^2 - 4\dfrac{kw}{g} = 0.$
(s) $3\dfrac{d^2w}{dz^2} + k\dfrac{dw}{dz} + 5w = 0;\quad k^2 < 60.$

2. Find the integrals of the following equations satisfying the initial values or boundary values indicated:

(a) $y'' - y' + y = 0; \quad y = 1,\ y' = \frac{7}{2},\ x = 0.$

(b) $y'' + 4y' + 13y = 0; \quad y = 4,\ x = 0;\ y = 1.05,\ x = \frac{\pi}{6}.$

(c) $y'' + 2y' + 4y = 0; \quad y = 3,\ y' = 0,\ x = 0.$

(d) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0; \quad y = 3,\ t = 0; \quad y = 0.3,\ t = 1.5.$

(e) $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0; \quad C = 100 \times 10^{-12}, L = 100 \times 10^{-6}, R = 1{,}500;$
$q = 0, \frac{dq}{dt} = 0.01,\ t = 0.$

(f) $\frac{w}{g}\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0; \quad w = 20,\ g = 32,\ k = 10,\ c = 0.1;\ x = 2,$
$\frac{dx}{dt} = 0,\ t = 0.$

(g) $4y'' + 4y' + 5y = 0; \quad y = 0,\ x = \frac{\pi}{6}; \quad y = 1,\ x = 0.$

(h) $y'' + 2y' + 10y = 0; \quad y' = -0.59,\ x = \frac{\pi}{6}; \quad y = 0,\ x = 0.$

(i) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0; \quad x = 1.687,\ t = \frac{\pi}{6}; \quad x = 1.732,\ t = 0.$

(j) $y'' + 3y' + 2y = 0; \quad y = 2e^{-2},\ y' = -3e^{-2},\ x = 1.$

(k) $y'' + 2y' + 2y = 0; \quad y = 0,\ x = \frac{\pi}{2}; \quad y = e^{-2},\ x = 0.$

***3.** Find the general solution of $y''' - 2y'' + y' - 2y = 0.$

6. The Nonhomogeneous Equation. We have obtained the general solution to the homogeneous equation (6) in the case of constant coefficients. We turn now to the nonhomogeneous equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \tag{5}$$

and again assume in our first discussion that a and b are functions of x.

The corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{6}$$

will be referred to as the *reduced equation* in contrast to (5), which will be referred to as the *complete equation*.

Most methods of solution of the complete equation depend on knowing the general solution of the reduced equation. In addition, if we know a particular solution of the complete equation, then the general solution may be written down at once according to the following theorem.

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the reduced equation and $\varphi(x)$ is any solution of the complete equation, then

$$y = c_1y_1(x) + c_2y_2(x) + \varphi(x) \tag{18}$$

is a general solution of the complete equation.

Proof that (18) satisfies (5) may be obtained by direct substitution.

$$y'' + ay' + by$$

reduces to

$$c_1[y_1''(x) + ay_1'(x) + by_1(x)] + c_2[y_2''(x) + ay_2'(x) + by_2(x)] + [\varphi''(x) + a\varphi'(x) + b\varphi(x)].$$

Now the expressions in the first two brackets are zero since $y_1(x)$, $y_2(x)$ are solutions of the homogeneous equation. The last bracket is equal to $f(x)$ since $\varphi(x)$ is a solution of (5). Hence

$$y'' + ay' + by = f(x).$$

The expression (18) may be written in a somewhat different form as follows. Let $y_c = c_1y_1(x) + c_2y_2(x)$ denote the general solution of the reduced equation, called the *complementary function*, and let y_p denote the solution $\varphi(x)$, called a *particular integral* of the complete equation. Then

$$y = y_c + y_p. \tag{19}$$

Now expression (18) or (19) contains all solutions of the complete equation so that once again we are justified in using the term *the* general solution. In addition, the solutions exist for all values of the independent variable x for which $a(x)$, $b(x)$, and $f(x)$ are continuous.†

Since *any* solution of the complete equation suffices for writing down the general solution, provided that we can solve the reduced equation, we now bend our energies to the development of techniques for finding such a particular solution. Two elementary methods for finding particular integrals are given in the next sections.

PROBLEMS

1. Prove that if $y = \varphi(x)$ is a solution of $y'' + ay' + by = f(x)$, then $y = A\varphi(x)$ is a solution of $y'' + ay' + by = Af(x)$.

† These facts are consequences of the general existence theorems for linear equations given in Chap. V.

2. Prove that if $y = \varphi_1(x)$ is a solution of $y'' + ay' + by = f_1(x)$ and $y = \varphi_2(x)$ is a solution of $y'' + ay' + by = f_2(x)$, then $y = \varphi_1(x) + \varphi_2(x)$ is a solution of $y'' + ay' + by = f_1(x) + f_2(x)$.

3. Show that $y = x^2 + 2x$ and $y = \sin x$ are particular integrals for $y'' - y' = -2x$ and $y'' - y' = -\sin x - \cos x$, respectively. Hence find the general solution of $y'' - y' = x + \sin x + \cos x$.

4. What is the complementary function for the equation $(D^2 + D)y = \sin x$?

7. The Method of Undetermined Coefficients. This method is applicable only to equations with constant coefficients and when the right member $f(x)$ is such that the form of a particular integral may be guessed. We illustrate with an example.

Example 6. Find a particular integral of

$$(D^2 + 5D + 6)y = x^2 + 2x.$$

In this problem, since the right member is a polynomial, it is natural to seek a polynomial solution. Clearly the degree of the polynomial should be at least two. We try

$$y_p = Ax^2 + Bx + C.$$

Substituting in the differential equation yields

$$(2A) + 5(2Ax + B) + 6(Ax^2 + Bx + C) = x^2 + 2x.$$

Now this equation will be an identity in x if the coefficients of like powers are equal. We have

$$6Ax^2 + (10A + 6B)x + (2A + 5B + 6C) = x^2 + 2x$$

and

$$\begin{aligned} 6A &= 1 \\ 10A + 6B &= 2 \\ 2A + 5B + 6C &= 0. \end{aligned}$$

Solving these equations yields

$$A = \tfrac{1}{6}, \qquad B = \tfrac{1}{18}, \qquad C = -\tfrac{11}{108},$$

whence the desired particular integral is

$$y_p = \frac{18x^2 + 6x - 11}{108}.$$

Example 7. Find the general solution of

$$(D^2 - 3D + 2)y = 2 \sin x.$$

Here the auxiliary equation is $r^2 - 3r + 2 = 0$ and the complementary function is

$$y_c = c_1e^{2x} + c_2e^x.$$

To find a particular integral of the complete equation we set

$$y_p = A \sin x + B \cos x.$$

Then

$$\begin{aligned}(D^2 - 3D + 2)y_p &= -A \sin x - B \cos x - 3A \cos x + 3B \sin x \\ &\qquad + 2A \sin x + 2B \cos x \\ &= (A + 3B) \sin x + (B - 3A) \cos x \\ &= 2 \sin x.\end{aligned}$$

Hence, equating coefficients,

$$\begin{aligned}A + 3B &= 2 \\ B - 3A &= 0\end{aligned}$$

and solving we find $A = \frac{1}{5}$, $B = \frac{3}{5}$, whence

$$y_p = \tfrac{1}{5} \sin x + \tfrac{3}{5} \cos x.$$

The general solution is

$$y = c_1e^{2x} + c_2e^x + \tfrac{1}{5} \sin x + \tfrac{3}{5} \cos x.$$

Example 8. Find the general solution of

$$(D^2 + 1)y = \sin x.$$

The auxiliary equation is $r^2 + 1 = 0$ which has the roots $i, -i$. The complementary function therefore is

$$y_c = c_1 \sin x + c_2 \cos x.$$

Since the right member of the complete equation is $\sin x$, we are tempted to find a particular integral of the form $y_p = A \sin x + B \cos x$. But as we have seen this function solves the reduced equation. We must therefore look elsewhere for a particular integral.

We have seen that if the roots of the auxiliary equation are repeated the second linearly independent solution of the reduced equation is obtained by multiplying by the factor x. This suggests a possible solution to the problem. We try

$$y_p = Ax \sin x + Bx \cos x.$$

Then

$$\begin{aligned}y_p' &= A \sin x + B \cos x + Ax \cos x - Bx \sin x, \\ y_p'' &= 2A \cos x - 2B \sin x - Ax \sin x - Bx \cos x\end{aligned}$$

and substituting in the complete equation yields

$$2A \cos x - 2B \sin x = \sin x.$$

Equating coefficients of like functions we have

$$2A = 0$$
$$-2B = 1,$$

whence

$$A = 0, \qquad B = -\tfrac{1}{2}.$$

The particular integral therefore is

$$y_p = -\frac{x}{2}\cos x,$$

and the general solution of the complete equation is

$$y = c_1 \sin x + c_2 \cos x - \frac{x}{2}\cos x.$$

Example 9. Find the general solution of

$$(D^2 - 2D + 1)y = e^x.$$

The auxiliary equation has the equal roots 1, 1. The complementary function is

$$y_c = c_1 e^x + c_2 x e^x.$$

Since the right member of the complete equation is e^x, the first try for a particular integral would be $y_p = Ae^x$. But this function solves the reduced equation. In conformity with Example 8, therefore, we try $y_p = Axe^x$, but because of the equal roots of the auxiliary equation, this function, too, solves the reduced equation. We therefore multiply again by x and try

$$y_p = Ax^2e^x.$$

Then

$$y_p' = Ax^2e^x + 2Axe^x$$
$$y_p'' = Ax^2e^x + 4Axe^x + 2Ae^x,$$

and substituting in the complete equation yields

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = e^x,$$

or

$$2Ae^x = e^x$$

and

$$A = \tfrac{1}{2}.$$

The general solution of the complete equation is therefore

$$y = c_1e^x + c_2xe^x + \frac{x^2}{2}e^x.$$

From the above examples we see that the form of a particular integral may often be inferred from the form of the right member $f(x)$ of the complete equation. In general one may say that such is the case if by indefinitely repeated differentiation of $f(x)$ there are generated but a finite number of linearly independent functions. The accompanying table illustrates the procedure for some familiar functions.

$f(x)$	Trial y_p
$e^{\alpha x}$	$Ae^{\alpha x}$
$\cos \beta x$ or $\sin \beta x$	$A \sin \beta x + B \cos \beta x$
x^n	$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_n$
$x^ne^{\alpha x}$	$e^{\alpha x}(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
$e^{\alpha x} \sin \beta x$ or $e^{\alpha x} \cos \beta x$	$e^{\alpha x}(A \sin \beta x + B \cos \beta x)$

In case the trial y_p happens to have a component which solves the reduced equation, one should use the trial y_p multiplied by x. If this function also has a component which solves the reduced equation, multiply by x^2. This will be as far as one need go because we have a second-order equation.†

Of course if the right member $f(x)$ is a sum of several different functions, each function may be treated separately (see Prob. 2, Sec. 6).

PROBLEMS

1. Find general solutions to the following equations:

(a) $(D^2 - 4)y = \sin 2x$.
(b) $(D^2 - 4D + 3)y = 2e^{-x} - 2$.
(c) $(D^2 + 2D - 8)y = 16x$.
(d) $(D^2 - 2D + 2)y = e^x \sin x$.
(e) $(D^2 + 4)y = \sin 2x + x$.
(f) $(D^2 - 2D + 1)y = 3xe^x$.
(g) $(D^2 + 2D + 1)y = a + 25 \sin 2x$.
(h) $D^2y = a_0x^n + a_1x^{n-1} + \cdots + a_n$.

† Every solution of the reduced equation is an exponential function (with possible complex exponents) or x times such a function. No solution of the reduced equation therefore contains x^2 as a factor; hence it will never be necessary to multiply a trial y_p by more than x^2.

(*i*) $(D^2 + 16)y = A \cos^2 2x$.
(*j*) $(D^2 - 3D + 2)y = 2x^2 + 2x$.
(*k*) $y'' + 2y' = 4x$.
(*l*) $(D^2 + 4D + 5)y = A \sin 2x$.
(*m*) $(D^2 - 4D + 4)y = 2e^{2x} + xe^{2x}$.
(*n*) $(D^2 + D)y = \sin 2x$.
(*o*) $(D^2 - 2D)y = 4x^2e^{2x}$.

2. Find solutions to the following equations, satisfying the initial conditions indicated:

(*a*) $y'' - 3y' + 2y = -5 + 12e^{-x}$; $y = \frac{7}{2}$, $y' = 9$, $x = 0$.
(*b*) $y'' - 2y' = 2$; $y = 0$, $y' = 0$, $x = 0$.
(*c*) $(D^2 - D - 2)y = 5 \sin x$; $y = 1$, $y' = -1$, $x = 0$.
(*d*) $y'' + \lambda^2 y = A \sin \omega x$; $y = 0$, $y' = 0$, $x = 0$, $\omega \neq \lambda$.
(*e*) $y'' - 2y' - 3y = 2 \sin^2 x$; $y = -\frac{1}{3}$, $y' = 0$, $x = 0$.
(*f*) $y'' - 2y' - 3y = 3x$; $y = 0$, $y' = 1$, $x = 0$.

8. The Method of Variation of Parameters. This method of finding a particular integral is due to J. L. Lagrange (1736–1813). It consists in using the general solution of the reduced equation and replacing the arbitrary constants by functions so chosen that a particular integral is obtained. It is important to know that *this method is also applicable to linear equations with nonconstant coefficients.*

Consider the linear equation

$$y'' + a(x)y' + b(x)y = f(x), \tag{20}$$

and suppose that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the reduced equation

$$y'' + a(x)y' + b(x)y = 0. \tag{21}$$

The general solution of (21) then is

$$y = c_1y_1(x) + c_2y_2(x).$$

Now consider the function

$$y = c_1(x)y_1(x) + c_2(x)y_2(x) \tag{22}$$

where $c_1(x)$ and $c_2(x)$ are functions which are to be so chosen that (22) is a particular integral of (20). Differentiating (22) we obtain

$$\frac{dy}{dx} = c_1(x)y_1'(x) + c_2(x)y_2'(x) + c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{23}$$

Since we seek a solution of (20) and have two functions $c_1(x)$ and $c_2(x)$ at our disposal, we are free to impose one restriction on $c_1(x)$ and $c_2(x)$. We do this by setting the last two terms in (23) equal to zero

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0. \tag{24}$$

Note that these are just the terms that arise because c_1 and c_2 are not constant.

Differentiating (23), remembering that (24) holds, we obtain

$$\frac{d^2y}{dx^2} = c_1(x)y_1''(x) + c_2(x)y_2''(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x), \tag{25}$$

and substituting in the differential equation (20) we have

$$c_1y_1'' + c_2y_2'' + c_1'y_1' + c_2'y_2' + a(c_1y_1' + c_2y_2') + b(c_1y_1 + c_2y_2) = f,$$

or

$$c_1(y_1'' + ay_1' + by_1) + c_2(y_2'' + ay_2' + by_2) + c_1'y_1' + c_2'y_2' = f \tag{26}$$

Because y_1 and y_2 are solutions of the reduced equation the expressions in parentheses in (26) are zero; hence (26) reduces to

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = f(x). \tag{27}$$

Equations (24) and (27) constitute two linear equations in $c_1'(x)$ and $c_2'(x)$

$$\begin{aligned} c_1'y_1 + c_2'y_2 &= 0 \\ c_1'y_1' + c_2'y_2' &= f, \end{aligned}$$

and solving for c_1' and c_2' yields†

$$\begin{aligned} c_1' &= \frac{fy_2}{y_1y_2' - y_1'y_2} \\ c_2' &= \frac{-fy_1}{y_1y_2' - y_1'y_2}. \end{aligned} \tag{28}$$

From Equations (28) we obtain $c_1(x)$ and $c_2(x)$ by quadratures and our problem is solved.

We illustrate the method by solving an equation with constant

† Naturally $y_1'y_2 - y_1y_2'$ must not be identically zero. But this fact follows from the assumption that y_1 and y_2 are linearly independent. For suppose that $y_1 \neq 0$, then if $y_1y_2' - y_1'y_2 \equiv 0$ we have $(y_1y_2' - y_2y_1')/y_1^2 \equiv 0$ or $\frac{d}{dx}\left(\frac{y_2}{y_1}\right) \equiv 0$ or $y_2/y_1 = \text{constant} = c$, and $y_2 = cy_1$; so y_1 and y_2 are dependent on any interval where y_1 is never zero.

coefficients since only for this type of equation are we able at present to find the general solution of the reduced equation.

Example 10. Solve

$$(D^2 + 5D + 6)y = x^2 + 2x$$

by the method of variation of parameters.

The reduced equation is

$$(D^2 + 5D + 6)y = 0$$

and has the general solution

$$y_c = c_1e^{-2x} + c_2e^{-3x}.$$

Differentiating and regarding c_1 and c_2 as functions we get

$$y' = -2c_1e^{-2x} - 3c_2e^{-3x} + (c_1'e^{-2x} + c_2'e^{-3x}).$$

We set the term in parentheses equal to zero,

$$c_1'e^{-2x} + c_2'e^{-3x} = 0.$$

Thus

$$y' = -2c_1e^{-2x} - 3c_2e^{-3x}$$

and

$$y'' = 4c_1e^{-2x} + 9c_2e^{-3x} - 2c_1'e^{-2x} - 3c_2'e^{-3x}.$$

Substitution of y' and y'' in the differential equation yields

$$4c_1e^{-2x} + 9c_2e^{-3x} - 2c_1'e^{-2x} - 3c_2'e^{-3x} + 5(-2c_1e^{-2x} - 3c_2e^{-3x}) + 6(c_1e^{-2x} + c_2e^{-3x}) = x^2 + 2x$$

or

$$-2c_1'e^{-2x} - 3c_2'e^{-3x} = x^2 + 2x.$$

Solving for c_1' and c_2' gives us

$$c_1' = \frac{dc_1}{dx} = (x^2 + 2x)e^{2x}$$

$$c_2' = \frac{dc_2}{dx} = -(x^2 + 2x)e^{3x}.$$

These last equations are easily integrated to give†

$$c_1 = \frac{e^{2x}}{2}\left(x^2 + x - \frac{1}{2}\right)$$

$$c_2 = \frac{e^{3x}}{3}\left(-x^2 - \frac{4x}{3} + \frac{4}{9}\right).$$

† There is no need to add a constant of integration here since we seek a particular integral and any functions $c_1(x)$ and $c_2(x)$ satisfying the required conditions will do.

A particular integral of the complete equation then is

$$y_p = \frac{1}{2}\left(x^2 + x - \frac{1}{2}\right) + \frac{1}{3}\left(-x^2 - \frac{4}{3}x + \frac{4}{9}\right) = \frac{x^2}{6} + \frac{1}{18}x - \frac{11}{108}$$

and the general solution is

$$y = c_1e^{-2x} + c_2e^{-3x} + \frac{x^2}{6} + \frac{1}{18}x - \frac{11}{108},$$

which agrees with the solution obtained by the method of undetermined coefficients in Example 6.

Example 11. Find the general solution of

$$(D^2 + 1)y = \sec x.$$

The reduced equation has the general solution

$$y_c = c_1 \cos x + c_2 \sin x.$$

We try then for a particular integral

$$y_p = c_1(x) \cos x + c_2(x) \sin x.$$

Then

$$y_p' = -c_1 \sin x + c_2 \cos x + c_1' \cos x + c_2' \sin x$$

and we set

$$c_1' \cos x + c_2' \sin x = 0.$$

For the second derivative we get

$$y_p'' = -c_1 \cos x - c_2 \sin x - c_1' \sin x + c_2' \cos x.$$

On substitution of y_p'' and y_p in the differential equation and simplifying we obtain

$$-c_1' \sin x + c_2' \cos x = \sec x.$$

Solving simultaneously for c_1' and c_2' we obtain

$$c_1' = -\frac{\sin x}{\cos x}$$
$$c_2' = 1.$$

Integrating these equations gives

$$c_1 = \log \cos x$$
$$c_2 = x.$$

Then

$$y_p = x \sin x + (\log \cos x) \cos x.$$

A general solution of the complete equation is therefore

$$y = c_1 \cos x + c_2 \sin x + x \sin x + (\log \cos x) \cos x.$$

It should be observed that we could not have solved this example by the method of undetermined coefficients as we could not determine a proper form for a trial y_p.

PROBLEMS

1. Find a general solution of the following equations by the method of variation of parameters:

(a) $(D^2 - D - 2)y = e^{2x}$.
(b) $(D^2 + 2D + 1)y = e^{-x}/x$.
(c) $(D^2 + 1)y = \sin x$.
(d) $y'' + 4y = \tan 2x$.
(e) $y'' - y = \dfrac{1}{e^x - 1}$.
(f) $(D^2 - D - 2)y = e^{-2x} \cos e^{-x}$.
(g) $(D^2 + 4)y = \csc 2x$.
(h) $(D^2 + 4)y = \sin^2 x$.
(i) $y'' + 2y' = 5 + e^{-x}$.
(j) $y'' + 4y = \tan^2 2x$.

2. Find a particular integral of

(a) $(D^2 + 1)y = \sec x \csc x$.
(b) $(D^2 + 1)y = \text{ctn } 2x$.
(c) $(D^2 + 2D + 2)y = e^{-x} \sin x$.
(d) $(D^2 - 2D - 3)y = \frac{1}{2}x^2e^{-x}$.

3. If $y = c_1x^2 + c_2x^{-2}$ is the general solution of $x^2y'' + xy' - 4y = 0$, find the general solution of $x^2y'' + xy' - 4y = x^3$.

4. The general solution of $x^2y'' - 2y = 0$ is $y = c_1x^2 + c_2x^{-1}$. Find the general integral of $x^2y'' - 2y = x^2$.

5. The general solution of $x^2y'' + xy' - y = 0$ is $c_1x + c_2x^{-1}$. Find the general integral of $x^2y'' + xy' - y = x^2e^{-x}$.

6. If $y = c_1x + c_2(2x^2 - 1)$ is the general solution of

$$(2x^2 + 1)y'' - 4xy' + 4y = 0,$$

find a particular integral of $(2x^2 + 1)y'' - 4xy' + 4y = 2(1 + 6x^2)x^{-3}$.

9. Oscillatory Systems. In the remaining sections of this chapter we will be primarily concerned with vibration problems in which the independent variable is the time t. Accordingly, we will adopt the "dot" notation and write our second-order equation as

$$\ddot{x} + a\dot{x} + bx = f(t). \tag{29}$$

We consider only constant coefficients.

We discuss a mechanical system which gives rise to an equation of the form (29). Consider a particle of mass m attached to a weightless spring (Fig. 24) and subjected to a force $F(t)$ in the x direction. The neutral (unstretched spring) position of the mass is at $x = 0$, and the spring force is given by Hooke's law, that is, is proportional to the displacement x from the neutral position. If there is a frictional force retarding the motion proportional to the velocity $\dot{x}$, then Newton's law of motion asserts that

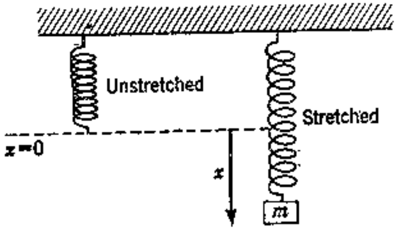


FIG. 24.

$$m\ddot{x} = -kx - r\dot{x} + F(t),$$

where k is the spring constant (positive) and r the coefficient of friction (positive). Then

$$m\ddot{x} + r\dot{x} + kx = F(t),$$

or

$$\ddot{x} + \frac{r}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m}. \tag{30}$$

Reference to (29) shows that in our problem $\ddot{x}$ arises from the inertia forces, $a\dot{x}$ from friction, and bx from the spring force. The function on the right $F(t)$ being in the nature of an impressed force is called the *forcing function*.

In addition to the simple spring problem above, second-order equations with constant coefficients are also encountered in simple electric-circuit problems. In Equation (39), Sec. 13, Chap. III, we encountered the following equation:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t), \tag{31}$$

where R, L, and C are the resistance, inductance, and capacitance in the circuit of Fig. 18, and E is the impressed e.m.f., which is supposed known. Thus comparing Equations (30) and (31) we see that, in an electrical circuit, inductance plays the role of mass in a mechanical system, resistance and coefficient of friction correspond, the spring constant corresponds to the reciprocal of the capacitance, and the charge on the condenser to the displacement.

The analogy between electrical and mechanical systems renders the mathematical treatments of the two systems identical and extends to much more complicated systems than those we have considered.

10. The Superposition Principle. We have found, so far, two methods for solving the complete equation (29), namely, "undetermined coefficients" and "variation of parameters." The theorem which follows enables us to write a particular integral whenever the right member of the complete equation is the sum of functions for which particular integrals are known. Equally important, the theorem often reflects a physical principle regarding the additivity of effects. For example, in a simple electrical circuit the effect of the sum of two impressed e.m.f.'s is equal to the sum of their individual effects.

Theorem (The Principle of Superposition). If $\varphi_i(t)$ are solutions of

$$\ddot{x} + a\dot{x} + bx = f_i(t)$$

for $i = 1, \ldots, n$, then

$$x = \varphi_1(t) + \cdots + \varphi_n(t),$$

is a solution of

$$\ddot{x} + a\dot{x} + bx = f_1(t) + \cdots + f_n(t).$$

The proof of this theorem is a simple exercise and is left to the reader. It should be emphasized that the theorem is valid for linear equations whether the coefficients a and b are constant or not.

11. Free Vibrations. If the forcing function is identically zero the differential equation (29) is homogeneous

$$\ddot{x} + a\dot{x} + bx = 0,$$

and we have the case of *free vibrations.* The character of the motion is determined by the nature of the roots of the auxiliary equation

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}$$

$$r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

There are three cases to consider.

Case 1. $a^2 - 4b > 0$. Then r_1 and r_2 are real and unequal, and if $b > 0$ (which we suppose), they have the same sign as $-a$. The general solution is

$$x = c_1e^{r_1t} + c_2e^{r_2t}.$$

If $a > 0$ (and we will assume this always), the motion is *aperiodic and overcritically damped.* The displacement can have at most one maximum, and then decays to zero asymptotically. This type of motion is called *subsidence.*

Case 2. $a^2 - 4b = 0$. Then r_1 and r_2 are equal to $-a/2$. The general solution is

$$x = c_1e^{-at/2} + c_2te^{-at/2}.$$

The motion is *aperiodic* and *critically damped.*

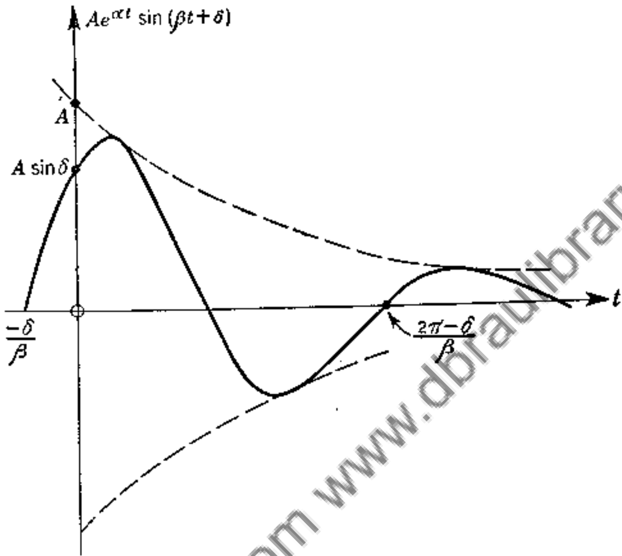


FIG. 25.

Case 3. $a^2 - 4b < 0$. Then r_1 and r_2 are conjugate complex

$$r_1 = \alpha + i\beta, \qquad r_2 = \alpha - i\beta,$$

where $\alpha = -a/2$ and $\beta = \sqrt{4b - a^2}/2$. The general solution is

$$x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) \tag{32}$$
$$= Ae^{\alpha t} \sin (\beta t + \delta).$$

The motion has an *oscillatory* character and is *undercritically damped* by the factor $e^{\alpha t} = e^{-at/2}$. The factor

$$c_1 \cos \beta t + c_2 \sin \beta t = A \sin (\beta t + \delta)$$

is a periodic function. Its period is $2\pi/\beta$, and its frequency $\beta/2\pi$ is called the *natural frequency* of the sytem, "natural" because the oscillations are not caused by any impressed force. A graph of the solution (32) appears in Fig. 25.

In any time interval equal to the period $2\pi/\beta$ the amplitude of the vibrations decreases in the constant ratio $e^{\alpha 2\pi/\beta}$. The logarithm of the amplitude thus decreases by the constant difference $\alpha 2\pi/\beta$. This difference is called the *logarithmic decrement*.

In the important case of *no damping* $a = 0 = \alpha$ one has $\beta = \sqrt{b}$, and (32) becomes

$$x = A \sin (\beta t + \delta).$$

We have then a simple sine wave. This motion is called *simple harmonic motion*. Observe that damping not only decreases the amplitude of the vibrations but also alters the natural frequency of the system.

PROBLEMS

1. A particle of mass 1 gram is attracted to a point 0 by a force proportional to the distance from 0. When the distance is 1 centimeter, the force is 100 dynes. What is the natural frequency of the system?

2. Suppose that the particle of Prob. 1 is also subject to a frictional force proportional to the speed of the particle and that this force is 5 dynes when the speed is 1 centimeter per second. What is the natural frequency of the system? The logarithmic decrement?

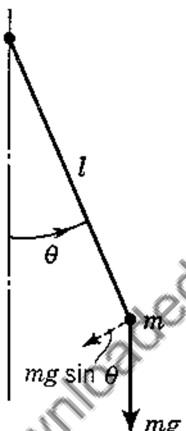


FIG. 26.

3. In a simple electric circuit there are in series a condenser with capacity of 200×10^{-6} farad, a resistance of 1 ohm, and an inductance of 0.1 henry. What is the natural frequency of the system? How large must the resistance be in order that the system be critically damped?

4. Show that the differential equation which governs the motion of the simple pendulum of Fig. 26, consisting of a mass m at the end of a weightless rod of length l, is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

Assuming θ is small, "linearize" this equation by replacing $\sin\theta$ by θ, and determine the period of the pendulum for small oscillations.

5. A cylindrical buoy 2 feet in diameter (Fig. 27) floats with it axis vertical. On depressing slightly and releasing it is found to bob with a period of 2 seconds. What is the weight of the buoy?

6. A 5-pound weight stretches a spring 6 inches. If the weight is pulled down 3 inches farther and released, find the equation of motion, its period and its frequency.

***7.** In Prob. 6 suppose that the top of the spring is given the periodic motion $x = \frac{1}{3}\sin \pi t$. Find the motion of the weight.

8. If the motion described by

$$\ddot{x} + a\dot{x} + bx = 0$$

has a natural frequency of 60 cycles per second and if the logarithmic decrement is $\frac{1}{10}$, find a and b.

12. Simple Forcing Functions. In most oscillatory systems (29) the forcing function $f(t)$ will be periodic. We therefore consider first the simplest periodic functions, namely the trigonometric functions sine and cosine. As will be seen later we will be able to resolve the general periodic forcing function in terms of these simple ones.

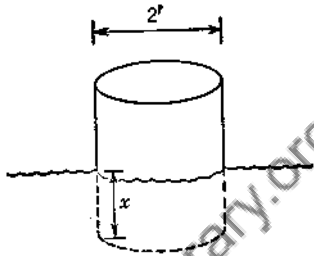


FIG. 27.

Consider the simple harmonic forcing function $f(t) = A \cos \omega t$. Differential equation (29) is then

$$\ddot{x} + a\dot{x} + bx = A \cos \omega t. \tag{33}$$

We seek a particular integral by the method of undetermined coefficients. Set

$$\begin{aligned} x_p &= A_1 \cos \omega t + B_1 \sin \omega t \\ \dot{x}_p &= -\omega A_1 \sin \omega t + \omega B_1 \cos \omega t \\ \ddot{x}_p &= -\omega^2 A_1 \cos \omega t - \omega^2 B_1 \sin \omega t. \end{aligned} \tag{34}$$

Substitution in differential equation (33) yields

$$-\omega^2 A_1 \cos \omega t - \omega^2 B_1 \sin \omega t + a(-\omega A_1 \sin \omega t + \omega B_1 \cos \omega t) + b(A_1 \cos \omega t + B_1 \sin \omega t) = A \cos \omega t.$$

Equating coefficients of $\sin \omega t$ and $\cos \omega t$ we get the equations

$$\begin{aligned} (b - \omega^2)A_1 + a\omega B_1 &= A \\ -a\omega A_1 + (b - \omega^2)B_1 &= 0, \end{aligned}$$

which when solved for A_1 and B_1 yield

$$\begin{aligned} A_1 &= \frac{(b - \omega^2)}{(b - \omega^2)^2 + a^2\omega^2} A, \\ B_1 &= \frac{a\omega}{(b - \omega^2)^2 + a^2\omega^2} A. \end{aligned} \tag{35}$$

It is assumed that the denominator in (35) does not vanish.†

† If the denominator vanishes, there is no friction ($a = 0$) and $b = \omega^2$ or $\omega = \sqrt{b}$. In other words the forcing function has the same frequency as the natural frequency of the system. In this case our particular integral would have the form $tA_1 \cos \omega t + tB_1 \sin \omega t$. We exclude this "resonance" case from consideration.

The particular integral then is

$$(36) \quad x_p = \frac{(b - \omega^2)}{(b - \omega^2)^2 + a^2\omega^2} A \cos \omega t + \frac{a\omega}{(b - \omega^2)^2 + a^2\omega^2} A \sin \omega t.$$

Now we may set

$$(37) \quad \cos \lambda = \frac{b - \omega^2}{\sqrt{(b - \omega^2)^2 + a^2\omega^2}}, \qquad \sin \lambda = \frac{a\omega}{\sqrt{(b - \omega^2)^2 + a^2\omega^2}},$$

whence (36) becomes

$$x_p = \frac{A}{\sqrt{(b - \omega^2)^2 + a^2\omega^2}} (\cos \lambda \cos \omega t + \sin \lambda \sin \omega t)$$

or

$$(38) \qquad x_p = \frac{A}{\sqrt{(b - \omega^2)^2 + a^2\omega^2}} \cos (\omega t - \lambda).$$

The general solution of (33) therefore is

$$(39) \quad x = Ce^{\alpha t} \sin (\beta t + \delta) + \frac{A}{\sqrt{(b - \omega^2)^2 + a^2\omega^2}} \cos (\omega t - \lambda)$$

if we have positive damping less than critical. Then $\alpha = -a/2 < 0$, and after a sufficiently long period of time the first term in (39) will be very small due to the damping factor $e^{\alpha t}$. Hence, after a long time the only motion will be that of the forced vibrations given by (38). These vibrations have period $2\pi/\omega$, frequency $\omega/2\pi$, amplitude $A/\sqrt{(b - \omega^2)^2 + a^2\omega^2}$, and lag the force by the phase angle λ.

The situation encountered here is described by saying that the first term in (39) represents the *transient phenomenon* while the last term in (39) is the *steady-state phenomenon*.

Now consider the simple harmonic forcing function $f(t) = B \sin \omega t$. Differential equation (29) is then

$$(40) \qquad \ddot{x} + a\dot{x} + bx = B \sin \omega t.$$

As before we seek a particular integral

$$(41) \qquad x_p = A_2 \cos \omega t + B_2 \sin \omega t.$$

An analogous procedure yields

$$(42) \qquad A_2 = \frac{-a\omega}{(b - \omega^2)^2 + a^2\omega^2} B$$

$$B_2 = \frac{(b - \omega^2)}{(b - \omega^2)^2 + a^2\omega^2} B,$$

and the particular integral

(43) $$x_p = \frac{-a\omega}{(b-\omega^2)^2 + a^2\omega^2} B \cos \omega t + \frac{(b-\omega^2)}{(b-\omega^2)^2 + a^2\omega^2} B \sin \omega t.$$

Using λ as defined by (37), Equation (43) becomes

$$x_p = \frac{B}{\sqrt{(b-\omega^2)^2 + a^2\omega^2}} [-\sin \lambda \cos \omega t + \cos \lambda \sin \omega t]$$

or

(44) $$x_p = \frac{B}{\sqrt{(b-\omega^2)^2 + a^2\omega^2}} \sin (\omega t - \lambda).$$

Just as before, (44) represents the forced vibrations or steady-state solution in the damped case.

PROBLEMS

1. Write down the steady-state solution for the current in the simple circuit of Fig. 18, using $E = E_0 \sin \omega t$. Show that the steady-state current may be written as

$$i = \frac{E_0}{Z}\left[\frac{-X}{Z} \cos \omega t + \frac{R}{Z} \sin \omega t\right]$$

where $X = L\omega - 1/C\omega$ is called the *reactance* and $Z = \sqrt{X^2 + R^2}$ is called the *impedance*. Thus show that the current leads the voltage by a time λ/ω where λ is given by

$$\sin \lambda = \frac{X}{Z}, \qquad \cos \lambda = \frac{R}{Z},$$

that is, the current maxima occur λ/ω seconds before the voltage maxima.

2. A simple circuit has an inductance of 0.1 henry, a resistance of 10 ohms, and a capacitance of 0.0002 farad. The impressed e.m.f. is the sum of two sinusoidal e.m.f.'s of equal phase, amplitudes 100 volts and 50 volts, and of frequencies $200/\pi$ and $100/\pi$, respectively. Find the current as a function of time if $i = 0$ and $q = 0$ when the e.m.f. $= 0$.

3. A weightless coil spring is stretched 2 inches by a force of 1 pound. It is hung by one end, and a mass weighing 8 pounds is attached to the other. A force equal to $4 \sin t + 2 \sin 2t$ is applied to the mass. If the system is at rest when $t = 0$, find the motion.

13. Resonance Phenomena. From (38), or (44), we see that a simple harmonic forcing function with amplitude A gives rise to forced vibrations with amplitude

(45) $$\frac{A}{\sqrt{(b-\omega^2)^2 + a^2\omega^2}}.$$

Thus the amplitude of the forced vibrations is proportional to the amplitude of the impressed force and depends also on a, b and the forcing frequency $\omega/2\pi$.

For $\omega = 0$ the forcing function is constant $= A$ and the steady-state solution is

$$x = \frac{A}{b}, \tag{46}$$

which might be called the "static amplitude." The ratio of the amplitudes (45) and (46) is given by

$$F = \frac{A}{\sqrt{(b-\omega^2)^2 + a^2\omega^2}} \Big/ \frac{A}{b} = \frac{b}{\sqrt{(b-\omega^2)^2 + a^2\omega^2}}. \tag{47}$$

F is called the *amplification factor*.

For $\omega = 0$, $F = 1$, and as $\omega \to \infty$, $F \to 0$. Hence, either F continually decreases between $\omega = 0$ and $\omega = \infty$, or F is not continually decreasing and attains a maximum somewhere between $\omega = 0$ and $\omega = \infty$. The forcing function $A \cos \omega t$ is said to be in *resonance* with the system at a value of ω for which the amplification factor F has a maximum. Observe that F has a maximum when the function

$$(b - \omega^2)^2 + a^2\omega^2$$

has a minimum. We differentiate this last function with respect to ω in order to find for what value of ω the factor F is a maximum. Setting the derivative equal to zero we get

$$2(b - \omega^2)(-2\omega) + 2a^2\omega = 0$$

or

$$-2\omega(2b - a^2 - \omega^2) = 0,$$

whence

$$\omega = 0,$$

or

$$\omega^2 = b - \frac{a^2}{2}, \tag{48}$$

$$\omega = \sqrt{b - \frac{a^2}{2}}.$$

It is readily verified that $\omega = 0$ makes F a minimum if $b - (a^2/2) > 0$ and a maximum if $b - (a^2/2) < 0$. It is also easily seen that ω as given by (48) gives a maximum. Note, however, that there will be *no resonance unless*

$$b - \frac{a^2}{2} > 0 \tag{49}$$

because otherwise ω would be a complex number. We observe also that condition (49) is more restrictive than our original assumption of less than critical damping, namely,

$$a^2 < 4b.$$

If we divide numerator and denominator in the expression (47) by b, we get

$$F = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{b}\right)^2 + \frac{a^2}{b}\frac{\omega^2}{b}}}. \tag{50}$$

From this equation we see that F is a function of the dimensionless quantities ω^2/b and a^2/b. Graphs of F versus $\omega/\sqrt{b}$ for various values

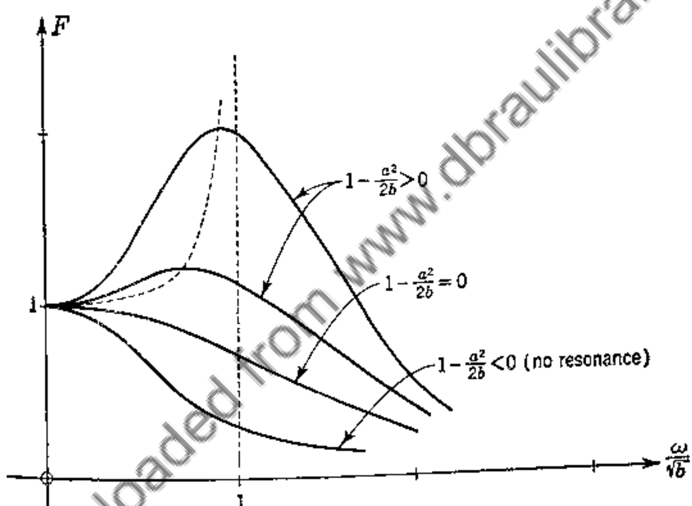


Fig. 28.

of a^2/b are shown in Fig. 28. The dotted curve in the figure passes through the maxima of F and, as $(a^2/b) \to 0$, approaches asymptotically the vertical line $\omega/\sqrt{b} = 1$, corresponding to the undamped natural frequency.

In the case of resonance the frequency of the forced vibrations is called the *resonance frequency*. It is

$$\text{Resonance frequency} = \frac{\omega}{2\pi} = \frac{\sqrt{b - \frac{a^2}{2}}}{2\pi}.$$

Since the natural frequency is equal to $\sqrt{b - \frac{a^2}{4}}/2\pi$ we see that the resonance frequency is always less than the natural frequency. When a^2/b is small, the resonance frequency differs little from the natural frequency. However when a^2/b is not small, the resonance frequency differs considerably from the natural frequency. This is, physically, a rather surprising result.

Example 12. A mass weighing 5 pounds is attached to one end of a damped spring, the other end being fixed. A 1-inch stretch of the spring gives a restoring force of 1 pound. The spring is subjected to varying simple harmonic forces, and it is found experimentally that the resonance frequency is 1 cycle per second. Assuming that the damping force is proportional to the velocity, so that our equations hold, find the damping coefficient.

Since a 1-pound force stretches the spring 1 inch, the spring constant is 12 pounds per foot. The mass is $5/g$ slugs. The differential equation then is

$$\frac{5}{g}\ddot{x} + r\dot{x} + 12x = f(t),$$

or

$$\ddot{x} + a\dot{x} + \frac{12g}{5}x = \frac{g}{5}f(t)$$

where

$$a = \frac{gr}{5}.$$

Since the resonance frequency is 1 cycle per second, we have

$$1 = \frac{1}{2\pi}\sqrt{b - \frac{a^2}{2}} = \frac{1}{2\pi}\sqrt{\frac{12g}{5} - \frac{a^2}{2}}$$

or

$$4\pi^2 = \frac{12g}{5} - \frac{a^2}{2}.$$

Thus

$$a^2 = 75.6,$$

and

$$a = 8.7.$$

Hence

$$r = \frac{5a}{g} = 1.4 \text{ slugs/sec.}$$

PROBLEMS

1. A mass of 1 slug is attached to a spring which has a restoring force of 1 pound for a stretch of 3 inches. If there is a damping force equal in magnitude to twice the velocity in feet per second, find the resonance frequency.

2. A simple series circuit containing R, L, C is said to be in resonance when the steady-state current has maximum amplitude. Show, using the result of Prob. 1, Sec. 12, that the resonance frequency is $\frac{1}{2\pi}\Big/\sqrt{LC}$.

3. In a simple circuit $L = 0.05$ henry, and R is 10 ohms. What should the capacitance be for the resonance frequency to be 200 cycles per second?

4. In the circuit of Prob. 2, what should the frequency of the impressed e.m.f. be in order that the amplitude of the oscillating charge on the condenser be a maximum?

5. Explain the discrepancy between the results of Probs. 2 and 4.

14. Superposition of Simple Solutions. Suppose now that the forcing function is periodic with period $2\pi/\omega$ and is the following sum of simple harmonic functions

$$f(t) = \sum_{k=1}^{n} (a_k \cos k\omega t + b_k \sin k\omega t). \tag{51}$$

In Sec. 12 we obtained particular integrals for simple harmonic forcing functions. Using (36) and (43) we see that particular integrals of

$$\ddot{x} + a\dot{x} + bx = a_k \cos k\omega t$$

and

$$\ddot{x} + a\dot{x} + bx = b_k \sin k\omega t$$

are, respectively,

$$x_p = \frac{(b - k^2\omega^2)}{(b - k^2\omega^2)^2 + a^2k^2\omega^2}\, a_k \cos k\omega t + \frac{ak\omega}{(b - k^2\omega^2)^2 + a^2k^2\omega^2}\, a_k \sin k\omega t$$

and

$$x_p = \frac{-ak\omega}{(b - k^2\omega^2)^2 + a^2k^2\omega^2}\, b_k \cos k\omega t + \frac{(b - k^2\omega^2)}{(b - k^2\omega^2)^2 + a^2k^2\omega^2}\, b_k \sin k\omega t.$$

Application now of the superposition principle yields as a particular integral of

$$\ddot{x} + a\dot{x} + bx = f(t)$$

where $f(t)$ is given by (51) the following function:

$$x_p = \sum_{k=1}^{n} \left[\frac{(b - k^2\omega^2)a_k - ak\omega b_k}{(b - k^2\omega^2)^2 + a^2k^2\omega^2} \cos k\omega t + \frac{ak\omega a_k + (b - k^2\omega^2)b_k}{(b - k^2\omega^2)^2 + a^2k^2\omega^2} \sin k\omega t \right]. \tag{52}$$

Example 13. Find, using (51) and (52), a particular integral of

$$\ddot{x} + \dot{x} + x = \sin 2\omega t + \cos 3\omega t.$$

Here $a = 1$, $b = 1$ and $b_2 = 1$, $a_3 = 1$, while all other a_i, b_j are zero. Hence,

$$\begin{aligned} x_p &= \frac{-2\omega}{(1 - 4\omega^2)^2 + 4\omega^2} \cos 2\omega t + \frac{0 + (1 - 4\omega^2)}{(1 - 4\omega^2)^2 + 4\omega^2} \sin 2\omega t \\ &\quad + \frac{(1 - 9\omega^2) - 0}{(1 - 9\omega^2)^2 + 9\omega^2} \cos 3\omega t + \frac{3\omega + 0}{(1 - 9\omega^2)^2 + 9\omega^2} \sin 3\omega t \\ &= \frac{-2\omega}{1 - 4\omega^2 + 16\omega^4} \cos 2\omega t + \frac{1 - 4\omega^2}{1 - 4\omega^2 + 16\omega^4} \sin 2\omega t \\ &\quad + \frac{1 - 9\omega^2}{1 - 9\omega^2 + 81\omega^4} \cos 3\omega t + \frac{3\omega}{1 - 9\omega^2 + 81\omega^4} \sin 3\omega t. \end{aligned}$$

Observe that the particular integrals we have obtained are (if there is damping) precisely the steady-state solutions. In addition they are the only periodic solutions. The periodic particular integrals will (in case there is damping) be referred to as *the* periodic solutions.

PROBLEMS

1. Find a particular integral of

$$\ddot{x} + \dot{x} + 2x = \sin t + \tfrac{1}{2} \sin 2t + \tfrac{1}{3} \sin 3t.$$

2. Find the periodic solution of

$$\ddot{x} + \frac{1}{2}\dot{x} - \frac{1}{2}x = \sin t + \cos \frac{t}{2}.$$

3. Find the periodic solution of

$$\ddot{x} + 2\dot{x} + 2x = 4 \sin t \cos t.$$

4. Find the general solution of

$$3\ddot{x} + 5x = 8 \cos^2 \frac{3t}{2} \sin 2t.$$

★5. If the differential equation of the forced vibrations of a mass is

$$\ddot{x} + c^2 x = A \cos \frac{t}{2} \cos \frac{3t}{2} \cos \frac{5t}{2} \cos \frac{7t}{2},$$

why is there resonance if c has one of the values 1, 2, 3, 4, 5, 7, 8, but no resonance if $c = 6$?

15. Approximation in the Mean. It is now possible for us to solve the differential equation (29) in case the forcing function is given by the sum (51) of simple harmonic functions. It is natural therefore to try to approximate the general periodic forcing function $f(t)$, of period $2\pi/\omega$, by the sum†

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos k\omega t + b_k \sin k\omega t).$$

Some measure therefore must be given which describes how well $S_n(t)$ approximates $f(t)$. One possibility, and a natural one, is to consider the maximum of the absolute value of the difference between $f(t)$ and $S_n(t)$,

(53) $$\text{Maximum } |f(t) - S_n(t)| \qquad \left(0 \leqq t \leqq \frac{2\pi}{\omega}\right)$$

As it happens though, (53) is not the best choice for a measure of the "difference between $f(t)$ and $S_n(t)$." A more convenient choice is

(54) $$\Delta_n = \int_0^{2\pi/\omega} [f(t) - S_n(t)]^2 dt.$$

The suitability of this choice will be borne out by the results that follow from it. It is possible however to give a physical argument for considering Δ_n. Think of $f(t)$ as a periodically alternating e.m.f. applied to a simple electrical circuit containing resistance only. The power consumed will be equal to the product of current and voltage. Since the current is proportional to the voltage, the power will be proportional to $f^2(t)$. Therefore, replacing the true e.m.f. by the approximation $S_n(t)$, an "error" e.m.f. $[f(t) - S_n(t)]$ is introduced whose power is proportional to $[f(t) - S_n(t)]^2$. The average power over one period is

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} [f(t) - S_n(t)]^2 dt.$$

Therefore if Δ_n is minimized, so is the average power contribution of the error e.m.f., and hence the physical effect of the approximation e.m.f. $S_n(t)$ should be close to that of the true e.m.f.

We thus choose Δ_n as a measure of the closeness of $S_n(t)$ to $f(t)$. If the a_k, b_k in $S_n(t)$ are so chosen that Δ_n is a minimum, we say that $S_n(t)$ is a *best approximation in the mean to* $f(t)$. The word "mean" is

† The reason for the factor $\frac{1}{2}$ in $\frac{1}{2}a_0$ will be apparent soon.

used because Δ_n is, except for a factor $\omega/2\pi$, equal to the average or mean of $[f(t) - S_n(t)]^2$ over one period.

We now pose the problem: How should the coefficients a_k and b_k in $S_n(t)$ be chosen in order that $S_n(t)$ be a best approximation in the mean to $f(t)$? To decide this question we actually compute Δ_n, then by rewriting it properly the answer to our problem will be evident.

$$(55) \quad \Delta_n = \int_0^{2\pi/\omega} \left[f(t) - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos k\omega t + b_k \sin k\omega t)\right]^2 dt$$

The expression in brackets in (55) is now squared and each term integrated separately. In evaluating the terms the following integrals are needed, whose evaluation is left to the reader.

$$(56) \quad \begin{aligned} &\int_0^{2\pi/\omega} \cos j\omega t \sin k\omega t \, dt = 0 \quad (\text{for } j, k = 0, 1, \ldots, n) \\ &\int_0^{2\pi/\omega} \cos j\omega t \cos k\omega t \, dt = \begin{cases} 0 & (\text{for } j \neq k) \\ \dfrac{\pi}{\omega} & (\text{for } j = k) \end{cases} \\ &\int_0^{2\pi/\omega} \sin j\omega t \sin k\omega t \, dt = \begin{cases} 0 & (\text{for } j \neq k) \\ \dfrac{\pi}{\omega} & (\text{for } j = k) \end{cases} \end{aligned}$$

With these integrals it is a straightforward matter to show that

$$(57) \quad \begin{aligned} \Delta_n = &\int_0^{2\pi/\omega} f^2(t)\, dt + \frac{\pi}{\omega}\left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right] \\ &- a_0 \int_0^{2\pi/\omega} f(t)\, dt - 2 \sum_{k=1}^{n} a_k \int_0^{2\pi/\omega} f(t) \cos k\omega t \, dt \\ &- 2 \sum_{k=1}^{n} b_k \int_0^{2\pi/\omega} f(t) \sin k\omega t \, dt. \end{aligned}$$

If we set

$$(58) \quad \begin{aligned} \alpha_k &= \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \cos k\omega t \, dt \quad (\text{for } k = 0, 1, \ldots, n) \\ \beta_k &= \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \sin k\omega t \, dt \quad (\text{for } k = 1, \ldots, n) \end{aligned}$$

then (57) becomes

$$\Delta_n = \int_0^{2\pi/\omega} f^2(t)\,dt + \frac{\pi}{\omega}\left[\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right] - \frac{\pi}{\omega}\left[a_0\alpha_0 + 2\sum_{k=1}^{n}(a_k\alpha_k + b_k\beta_k)\right],$$

which on completing the squares on the a_k, α_k and b_k, β_k, terms becomes

$$(59)\quad \Delta_n = \int_0^{2\pi/\omega} f^2(t)\,dt - \frac{\pi}{\omega}\left[\frac{\alpha_0^2}{2} + \sum_{k=1}^{n}(\alpha_k^2 + \beta_k^2)\right] + \frac{\pi}{\omega}\left\{\frac{(a_0 - \alpha_0)^2}{2} + \sum_{k=1}^{n}[(a_k - \alpha_k)^2 + (b_k - \beta_k)^2]\right\}.$$

Now the first two terms in (59) depend only on $f(t)$, while the last term is nonnegative and depends on the coefficients a_k and b_k. The minimum value of Δ_n will therefore be attained when each summand of the last term in (59) is zero, that is, when

$$a_k - \alpha_k = 0 \qquad (\text{for } k = 0, 1, \ldots, n)$$
$$b_k - \beta_k = 0 \qquad (\text{for } k = 1, \ldots, n)$$

The coefficients α_k and β_k as given by (58) are called the *Fourier*† *coefficients* of $f(t)$. The calculation of the Fourier coefficients belonging to a given periodic function thus requires the evaluation of the definite integrals in (58). If the function is simple, these definite integrals may be found from familiar integration formulas. If the given function is less simple, or if given graphically as a curve representing experimental data, it may be necessary to use numerical or mechanical methods for the evaluation of the definite integrals.

We have now shown that the periodic function $f(t)$, with period $2\pi/\omega$, is approximated best in the mean by the sum $S_n(t)$ if the coefficients in $S_n(t)$ are the Fourier coefficients of $f(t)$. It is important to observe that this property of the Fourier coefficients, of minimizing Δ_n, is the independent of the integer n. By this we mean that if we should obtain the best approximation in the mean to the function $f(t)$ by $S_n(t)$ with n, say equal to 5, and then should decide to increase the value of n, say to 10, the previously determined coefficients would

† After the French mathematician and physicist Jean B. J. Fourier, 1768–1830.

not change. It is this fact which is the main advantage of approximation in the mean.

So far we have assumed that the forcing function is continuous. In many applications, particularly in electricity, discontinuous forcing functions are required. The class of "piecewise continuous" or "sectionally continuous" functions includes a sufficiently wide variety of functions to satisfy most needs. These functions are continuous in any interval $\alpha \leqq t \leqq \beta$ except for a finite number of points where they have simple jump discontinuities (see Sec. 1, Chap. I). Since nowhere in our minimization of Δ_n have we assumed necessarily the continuity of $f(t)$, and since the operations involved in the proof are valid for piecewise continuous functions, we have the following stronger result.

Theorem. If $f(t)$ is periodic, with period $2\pi/\omega$, and piecewise continuous, then

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos k\omega t + b_k \sin k\omega t)$$

is the best approximation in the mean to $f(t)$ if

$$(60) \qquad \begin{aligned} a_k &= \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \cos k\omega t \, dt \quad (\text{for } k = 0, 1, \ldots, n) \\ b_k &= \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \sin k\omega t \, dt \quad (\text{for } k = 1, \ldots, n) \end{aligned}$$

It is also true that

$$(61) \qquad \lim_{n \to \infty} \int_0^{2\pi/\omega} [f(t) - S_n(t)]^2 dt = 0.$$

We will not prove the statement (61) as it is considerably deeper and more difficult than the statement of the theorem. It should be observed however that Equation (61) does not imply that when n is very large $S_n(t)$ is near $f(t)$. Rather, Equation (61) asserts that if $S_n(t)$ and $f(t)$ differ considerably they must so differ over a small enough range to contribute little to the integral.

16. Fourier Series. We know that we can approximate a periodic $f(t)$ in the mean by the trigonometric polynomial $S_n(t)$. It is natural to inquire whether the infinite series, called the *Fourier series,*

$$(62) \qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega t + b_k \sin k\omega t)$$

will actually converge to $f(t)$ if a_k and b_k are the Fourier coefficients of $f(t)$. An answer to this question for continuous functions is provided by the following theorem.

Theorem. If $f(t)$ is continuous, has period $2\pi/\omega$, and a piecewise-continuous derivative, then the Fourier series converges to $f(t)$, that is,

$$\lim_{n\to\infty} S_n(t) = f(t).$$

We will not give a proof of this theorem† since it would take us too far astray.

Some of the strangeness of the above theorem may be dispelled by a physical interpretation. Suppose $f(t)$ represents the periodic vibrations of a sound-producing medium. The Fourier series resolves these vibrations into their pure harmonic components which correspond to pure musical notes. For this

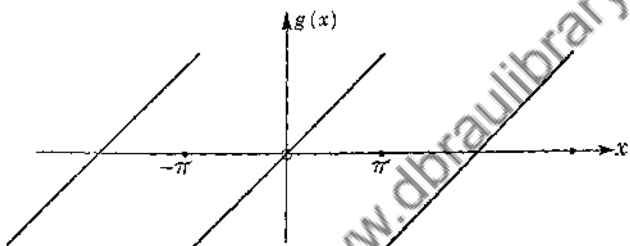


FIG. 29.

reason the terms of the $(k + 1)$th order in the Fourier series are called the kth harmonics. For $k = 1$ we have the ground tone, or fundamental. Observe that in this application the constant term $a_0/2$ is zero.

We observe finally that the Fourier series will converge to the function also for piecewise continuous functions with a piecewise continuous derivative except possibly at points of discontinuity of $f(t)$ where the series converges to the mean of the right- and left-hand limits.

In order to get nontrivial examples of periodic functions whose Fourier coefficients can be calculated, that is, for which the integrals (58) can be found, we define "artificial" functions from pieces of familiar ones. It happens that these artificial functions arise often in applications.

Example 14. Find the Fourier series of the function $g(x)$ defined as follows

$$\begin{aligned} g(x) &= x \qquad\qquad (\text{for } -\pi < x < \pi) \\ g(\pi) &= g(-\pi) = 0, \\ g(x + 2\pi) &= g(x). \end{aligned}$$

The graph of $g(x)$ is drawn in Fig. 29.

† For a proof see R. V. Churchill, "Fourier Series and Boundary Value Problems," pp. 67–72, McGraw-Hill, 1941. As a matter of fact the convergence is uniform with our hypotheses.

The Fourier coefficients are

$$a_k = \frac{1}{\pi}\int_0^{2\pi} g(x)\cos kx\,dx \qquad (\text{for } k = 0, 1, \ldots)$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} g(x)\sin kx\,dx \qquad (\text{for } k = 1, \ldots)$$

Because of the periodicity of $g(x)$, $\sin kx$, and $\cos kx$ we may integrate over any period interval, in particular over $(-\pi, \pi)$.

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx\,dx = 0 \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\,dx \\ &= \frac{1}{\pi}\left(x\,\frac{\cos kx}{-k}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos kx}{k}\,dx\right) \\ &= -\frac{2}{k}\cos k\pi. \\ &= (-1)^{k+1}\frac{2}{k}. \end{aligned}$$

The Fourier series for $g(x)$ therefore is

$$g(x) = 2\sum_1^\infty (-1)^{n+1}\frac{\sin nx}{n} = \begin{cases} x & (\text{for } -\pi < x < \pi) \\ 0 & (\text{for } x = \pm\pi) \end{cases}$$

PROBLEMS

1. Find the Fourier series of the following functions. In each case graph the function.

(a) $f(t) = 1; \quad 0 < t < \pi$
$\quad = -1; \quad -\pi < t < 0,$
$f(-\pi) = f(0) = f(\pi) = 0,$
$f(t + 2\pi) = f(t).$

This is called a "square wave," and has applications in electricity.

(b) $f(t) = |t|; \quad -1 \leqq t \leqq 1,$
$f(t + 2) = f(t).$

(c) $f(t) = \sin t; \quad 0 \leqq t \leqq \pi$
$f(t) = 0; \quad -\pi \leqq t \leqq 0$
$f(t + 2\pi) = f(t).$

(d) $f(t) = t^2; \quad -1 \leqq t \leqq 1$
$f(t + 2) = f(t).$

17. The General Forcing Function. In Chap. VII will be proved a theorem on linear equations with constant coefficients which in particular asserts that the differential equation

(29) $$\ddot{x} + a\dot{x} + bx = f(t)$$

has, for $a \neq 0$ and for continuous and periodic forcing functions $f(t)$, a unique particular solution $x_p(t)$ which has the same period as $f(t)$. If we accept this fact, we know, by the theorem of the preceding section, that the Fourier series of $x_p(t)$ will actually represent, that is, converge to, $x_p(t)$. That is,

$$x_p(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} (A_k \cos k\omega t + B_k \sin k\omega t). \tag{63}$$

By the superposition principle, remembering that $f(t)$ may be approximated (in the mean at least) by the sums (52), we would expect the particular integral to be given by Equation (52) with the sum running to infinity.

We will not however obtain the Fourier coefficients in the expansion (63) by the superposition principle. Rather we shall derive A_k and B_k in (63) directly from the differential equation (29) which $x_p(t)$ satisfies. This method has the advantage that no assumption has to be made as to whether the Fourier series of $x_p(t)$ and $f(t)$ actually converge to those functions or not. We assume only that $f(t)$ is piecewise continuous, $x_p(t)$ and $\dot{x}_p(t)$ are continuous, and $\ddot{x}_p(t)$ is piecewise continuous. All have period $2\pi/\omega$.

Multiply both sides of (29) by $\cos k\omega t$ and integrate from $t = 0$ to $t = 2\pi/\omega$. We get

$$\begin{aligned} \int_0^{2\pi/\omega} (\ddot{x}_p + a\dot{x}_p + bx_p) \cos k\omega t\, dt &= \int_0^{2\pi/\omega} f(t) \cos k\omega t\, dt \\ &= \frac{\pi}{\omega} a_k, \end{aligned} \tag{64}$$

where a_k is the kth Fourier cosine coefficient of $f(t)$. The left member of (64) may be transformed, using integration by parts.†

$$\begin{aligned} \int_0^{2\pi/\omega} \ddot{x}_p \cos k\omega t\, dt &= \dot{x}_p \cos k\omega t \Big|_0^{2\pi/\omega} + k\omega \int_0^{2\pi/\omega} \dot{x}_p \sin k\omega t\, dt \\ &= k\omega \left(x_p \sin k\omega t \Big|_0^{2\pi/\omega} - k\omega \int_0^{2\pi/\omega} x_p \cos k\omega t\, dt \right) \\ &= -k^2\omega^2 \int_0^{2\pi/\omega} x_p \cos k\omega t\, dt \\ &= -k^2\omega^2 \frac{\pi}{\omega} A_k, \end{aligned}$$

† Integration by parts is valid even if $\ddot{x}$ is but piecewise continuous (see Sec. 2, Chap. I).

where A_k is the kth Fourier cosine coefficient of $x_p(t)$. In a similar manner

$$\int_0^{2\pi/\omega} a\dot{x}_p \cos k\omega t\, dt = ak\omega \frac{\pi}{\omega} B_k.$$

Therefore (64) becomes

$$\frac{\pi}{\omega}(-k^2\omega^2 A_k + ak\omega B_k + bA_k) = \frac{\pi}{\omega} a_k$$

or

(65) $$(b - k^2\omega^2)A_k + ak\omega B_k = a_k.$$

In a precisely analogous fashion, multiplying (29) by sin $k\omega t$, we obtain

(66) $$-ak\omega A_k + (b - k^2\omega^2)B_k = b_k.$$

Solution of Equations (65) and (66) yields

(67) $$A_k = \frac{(b - k^2\omega^2)a_k - ak\omega b_k}{(b - k^2\omega^2)^2 + a^2k^2\omega^2}$$
$$B_k = \frac{ak\omega a_k + (b - k^2\omega^2)b_k}{(b - k^2\omega^2)^2 + a^2k^2\omega^2}.$$

As a check we note that Equations (67) agree with Equation (52).

Example 15. Find the first four terms of the Fourier series of the periodic solution of

$$\ddot{x} + 2\dot{x} + 2x = g(t),$$

where $g(t)$ is the function of Example 14.

From Example 14 we have, for $-\pi < t < \pi$,

$$g(t) = t = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n},$$

whence $\omega = 1$, and $a_k = 0$ for all k, while $b_k = \dfrac{2(-1)^{k+1}}{k}$. Since $b = 2$ and $a = 2$, we have from (67),

$$A_0 = 0$$
$$A_1 = \frac{-2(2)}{(2-1)^2 + 2^2} = -\frac{4}{5}.$$

$$B_1 = \frac{(2-1)(2)}{(2-1)^2 + 2^2} = \frac{2}{5}$$
$$A_2 = \frac{(-2)(2)(-\frac{2}{2})}{(2-4)^2 + (2^2)(2^2)} = \frac{1}{5}$$
$$B_2 = \frac{(2-4)(-\frac{2}{2})}{(2-4)^2 + (2^2)(2^2)} = \frac{1}{10}$$

Hence

$$x_p(t) = -\tfrac{1}{5}\cos t + \tfrac{2}{5}\sin t + \tfrac{1}{5}\cos 2t + \tfrac{1}{10}\sin 2t + \cdots .$$

PROBLEMS

1. Find several terms of the Fourier expansions of the periodic particular integrals of the following equations:

(i) $\ddot{x} + 2\dot{x} + 2x = f(t)$
(ii) $\ddot{x} + \dot{x} + 2x = f(t)$,

where $f(t)$ is in turn each of the functions of Prob. 1, Sec. 16.

2. Find the periodic solution of

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E(t)$$

where

$$E(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega t + b_k \sin k\omega t).$$

***3.** Use the result of Prob. 2 to calculate the integral

$$\int_0^{2\pi/\omega} i^2 dt$$

by integrating the series term by term.

18. Convergence of the Series Solution. We have obtained the Fourier series (63) of $x_p(t)$. That the series converges to $x_p(t)$ follows from the convergence theorem of Sec. 16 and the properties of $x_p(t)$. It is possible, however, to show without the use of the convergence theorem that the Fourier series (63) always converges to $x_p(t)$.

Suppose that first, $f(t)$ is piecewise continuous and periodic with period $2\pi/\omega$; second, the differential equation has a unique solution $x_p(t)$ with a continuous derivative in $0 \leqq t \leqq 2\pi/\omega$; third, $a \neq 0$.

We consider $x_p(t)$ in the interval $(0, 2\pi/\omega)$ and obtain as in Sec. 17 the Fourier expression (63) of $x_p(t)$.

Now

$$A_k \cos k\omega t + B_k \sin k\omega t = C_k \sin (k\omega t - \lambda_k)$$

where, by (67),

$$C_k = \sqrt{A_k^2 + B_k^2} = \frac{\sqrt{a_k^2 + b_k^2}}{\sqrt{(b - k^2\omega^2)^2 + a^2k^2\omega^2}}$$

and λ_k is defined by

$$\sin \lambda_k = \frac{-A_k}{\sqrt{A_k^2 + B_k^2}}, \qquad \cos \lambda_k = \frac{B_k}{\sqrt{A_k^2 + B_k^2}}.$$

Therefore,

$$|A_k \cos k\omega t + B_k \sin k\omega t| \leq C_k = \frac{1}{k^2} \cdot \frac{\sqrt{a_k^2 + b_k^2}}{\sqrt{\left(\frac{b}{k^2} - \omega^2\right)^2 + \frac{a^2\omega^2}{k^2}}}$$

The coefficient of $1/k^2$ is bounded, that is,

$$\frac{\sqrt{a_k^2 + b_k^2}}{\sqrt{\left(\frac{b}{k^2} - \omega^2\right)^2 + \frac{a^2\omega^2}{k^2}}} \leq M$$

where M is a constant independent of k.

Hence the Fourier series is termwise less in absolute value than M/k^2, which is a p series, $p = 2$, and hence convergent. Therefore the Fourier series is uniformly convergent to a function $S(t)$ which is of course periodic and continuous.

From our knowledge of approximation in the mean we have

$$\lim_{n \to \infty} \int_0^{2\pi/\omega} [S_n(t) - x_p(t)]^2 dt = 0.$$

Because the convergence of $S_n(t)$ to $S(t)$ is uniform, we may interchange the order of limit and integral and get

$$\int_0^{2\pi/\omega} [\lim_{n \to \infty} S_n(t) - x_p(t)]^2 dt = 0$$
$$\int_0^{2\pi/\omega} [S(t) - x_p(t)]^2 dt = 0$$

whence

$$S(t) \equiv x_p(t)$$

and the solution $x_p(t)$ is represented by its Fourier series.

19. A Boundary-value Problem. Boundary-value problems (see Secs. 2 and 3) are, in general, quite difficult to handle. We discuss in this section a particularly simple one which fortunately exhibits most of the typical features of such problems, except their difficulty. We seek the harmonic vibrations of a string under tension.

Consider a string situated with its ends on the x axis at $x = 0$ and $x = L$. The string is supposed to vibrate in the x, y plane. If $y(x, t)$ is the displacement in the y direction at x and the time t, we seek the differential equation for $y(x, t)$ which determines the motion. The small element of the string between x and $x + \Delta x$ (see Fig. 30) is, neglecting gravity, acted on only by the tensions on its ends. Since there is no motion in the x direction, the tension in the x direction is the same, T, at all points. Since $y_x(x, t)$, where the subscript denotes partial differentiation, is the slope of the deflection curve at any point, the tensions on the element are

$$-Ty_x(x, t) \qquad \text{(at the left end, } x)$$

and

$$+Ty_x(x + \Delta x, t) \qquad \text{(at the right end, } x + \Delta x)$$

The total force acting on the element therefore is

$$T[y_x(x + \Delta x, t) - y_x(x, t)] = Ty_{xx}(x', t)\,\Delta x, \tag{68}$$

where we have used the mean-value theorem, and $x < x' < x + \Delta x$.

The acceleration of the center of gravity of the element is $y_{tt}(x + \Delta x/2, t)$. The mass of the element is $\rho\,\Delta x$ where ρ is the linear density (mass per unit length), presumed constant. Hence, equating force to mass times acceleration we have

$$\rho y_{tt}\left(x + \frac{\Delta x}{2}, t\right)\Delta x = Ty_{xx}(x', t)\,\Delta x.$$

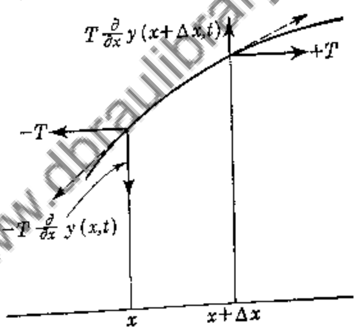


FIG. 30.

Dividing this equation by Δx and then letting $\Delta x \to 0$, there results

$$\frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}. \tag{69}$$

If we set

$$\frac{T}{\rho} = c^2,$$

Equation (69) becomes

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \tag{70}$$

which is a partial differential equation for the vibrating string. It has importance in other connections and is called the one-dimensional wave equation.

We are interested solely in harmonic vibrations of the string, that is, in vibrations of the form

$$y = u(x)\sin\omega t, \tag{71}$$

where $u(x)$ is the amplitude at the point x and $\omega/2\pi$ is the frequency of the vibrations. The problem is to determine both $u(x)$ and ω.

Substitution of (71) in (70) yields

$$-\frac{1}{c^2}\omega^2 u \sin \omega t = u'' \sin \omega t$$

or

$$u'' + \frac{\omega^2}{c^2} u = 0. \tag{72}$$

From this homogeneous linear equation we determine $u(x)$. Because the string is fixed at the ends, we have the boundary conditions

$$u(0) = 0 = u(L). \tag{73}$$

Equations (72) and (73) constitute a boundary-value problem whose solution is not difficult since we know the general solution of (72),

$$u(x) = A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}. \tag{74}$$

When (74) is substituted in (73), we get

$$A \cdot 0 + B \cdot 1 = 0$$
$$A \sin \frac{\omega L}{c} + B \cos \frac{\omega L}{c} = 0,$$

whence

$$B = 0$$
$$A \sin \frac{\omega L}{c} = 0. \tag{75}$$

One solution of (75) is $A = 0$, which corresponds to no motion of the string, $u \equiv 0$. This solution of the boundary-value problem is called the "trivial solution." However (75) will be satisfied for arbitrary A if

$$\sin \frac{\omega L}{c} = 0,$$

that is, if

$$\frac{\omega L}{c} = n\pi \qquad (n = 1, 2, 3, \ldots)$$

or

$$\omega = \frac{nc\pi}{L}.$$

Then

$$u(x) = A \sin \frac{n\pi x}{L} \tag{76}$$

and

$$y(x, t) = A \sin \frac{n\pi x}{L} \sin \frac{n\pi c}{L} t. \tag{77}$$

We have shown therefore that for the frequencies

$$\frac{\omega}{2\pi} = \frac{c}{2L}, \frac{2c}{2L}, \frac{3c}{2L}, \ldots, \tag{78}$$

and only for these frequencies do there exist nontrivial solutions of the boundary-value problems (72), (73). The frequencies (78) are the only ones for which harmonic vibrations of the string can exist. The lowest frequency is called the fundamental and the remaining ones, in order, the first, second, . . . , harmonics, or overtones. The corresponding amplitude functions (76) are shown in Fig. 31.

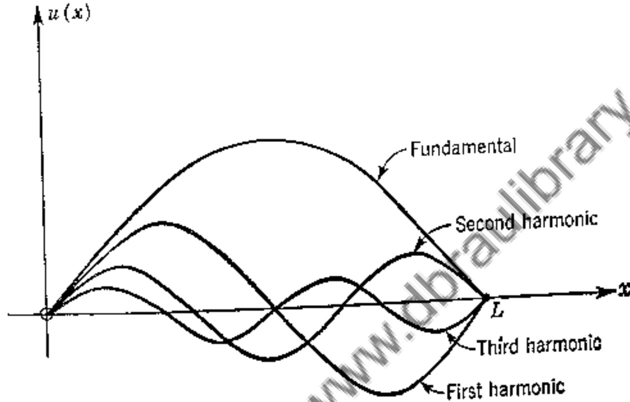


Fig. 31.

The boundary-value problem we have treated here is fairly typical of homogeneous boundary-value problems (that is, homogeneous linear equations with homogeneous boundary conditions) that contain an undetermined parameter, such as ω in (72). For such problems there is, usually, a sequence of values of the parameter called *characteristic values* (or *eigenvalues*) for which the boundary-value problem has a nontrivial solution. The solutions corresponding to characteristic values are called *characteristic functions* (or *eigenfunctions*). Characteristic functions corresponding to the same characteristic value are usually linearly dependent, while characteristic functions corresponding to different characteristic values are always linearly independent.

PROBLEMS

1. Determine the characteristic values and functions of the homogeneous boundary-value problem

$$u'' + \lambda^2 u = 0, \qquad 0 \leqq x \leqq L,$$
$$u(0) = 0 = u'(L).$$

Verify that the characteristic functions for different characteristic values are linearly independent.

2. Show that the only solution of the homogeneous boundary-value problem

$$u'' - \lambda^2 u = 0, \qquad 0 \leqq x \leqq L,$$
$$u(0) = 0, \qquad u(L) = 0,$$

is the trivial solution.

3. Find the characteristic values and characteristic functions for the homogeneous boundary-value problem

$$u'' + \lambda^2 u = 0, \qquad 0 \leqq x \leqq L,$$
$$u(0) = u(L), \qquad u'(0) = u'(L).$$

20. Special Nonlinear Equations. Except for our discussion of the general second-order equation (1) and the fundamental existence theorem we have confined our attention in this chapter to linear second-order equations, by far the most important type. Nevertheless, nonlinear equations do occur and some methods for attacking them are desirable. However, it should be understood that there is no technique that will enable one to find general solutions of arbitrary second-order equations. In this section we will give a few of the more frequently applicable devices. For a more comprehensive treatment the reader is referred to the excellent compendium of Kamke (Ref. 7). If the equation is intractable, it may be necessary to resort to a numerical solution.

If the *dependent variable is absent*, the general second-order equation (1) has the form

$$\frac{d^2y}{dx^2} = F\left(x, \frac{dy}{dx}\right). \tag{79}$$

The order of this equation may be reduced by the substitution

$$p = \frac{dy}{dx},$$
$$\frac{dp}{dx} = \frac{d^2y}{dx^2}.$$

Equation (79) then becomes

$$\frac{dp}{dx} = F(x, p),$$

which is of the first order and has a general solution

$$p = f(x, c_1). \tag{80}$$

A general solution of (79) may then be obtained from (80) by a quadrature

$$y = \int f(x, c_1)\, dx + c_2.$$

Example 16. Find a general solution of $(1 + x^2)y'' + y'^2 + 1 = 0$.

Setting $p = dy/dx$ gives

$$(1 + x^2)\frac{dp}{dx} + p^2 + 1 = 0.$$

The variables are separable in this equation and on integrating we obtain

$$\arctan p + \arctan x = c'.$$

Taking the tangent of both sides gives

$$\frac{p + x}{1 - px} = \tan c' = c'',$$

or

$$p = \frac{c'' - x}{c''x + 1} = \frac{1 - c_1x}{x + c_1} = -c_1 + \frac{c_1^2 + 1}{x + c_1},$$

whence

$$y = -c_1x + (c_1^2 + 1)\log(x + c_1) + c_2.$$

Example 17. Find y if $y' = xy'' + y''^2$ and $y = 0$, $y' = 2$ when $x = -1$.

Setting $p = dy/dx$ gives

$$p = x\frac{dp}{dx} + \left(\frac{dp}{dx}\right)^2$$

which is a Clairaut equation, with the general solution

$$p = c_1x + c_1^2.$$

Since $p = 2$ when $x = -1$ we readily find two values for c_1, namely, $c_1 = -1$, $c_1 = 2$. Hence,

$$\frac{dy}{dx} = -x + 1 \qquad \text{or} \qquad \frac{dy}{dx} = 2x + 4.$$

Then

$$y = -\frac{x^2}{2} + x + c_2 \qquad \text{or} \qquad y = x^2 + 4x + c_2'.$$

Since $y = 0$ when $x = -1$ we find $c_2 = \frac{3}{2}$, $c_2' = 3$ and the two possible solutions

$$y = -\frac{x^2}{2} + x + \frac{3}{2},$$
$$y = x^2 + 4x + 3.$$

The two solutions arise because the original differential equation is a quadratic in y''. Solving for y'' will yield two equations with the two solutions we have found.

If the *independent variable is absent*, the equation has the form

$$\frac{d^2y}{dx^2} = G\left(y, \frac{dy}{dx}\right). \tag{81}$$

In this case, too, the substitution $p = dy/dx$ will reduce the order of the equation. The second derivative d^2y/dx^2 is handled as follows. We may regard p as a function of either† x or y. Then

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\,\frac{dp}{dy},$$

whence (81) becomes

$$p\,\frac{dp}{dy} = G(y, p),$$

which is a first-order equation with a general solution

$$p = g(y, c_1).$$

A general solution of (81) is obtained by solving

$$\frac{dy}{dx} = g(y, c_1)$$

for which the variables are separable.

$$\int \frac{dy}{g(y, c_1)} = \int dx = x + c_2.$$

Example 18. Find y if $y'' = y'e^y$ and $y = -1$, $y' = 0$ when $x = 0$. Setting $dy/dx = p$ and $d^2y/dx^2 = p\,dp/dy$ reduces the equation to

$$p\,\frac{dp}{dy} = pe^y.$$

One solution of this equation is $p = 0$ or $y =$ constant. From the initial conditions it follows that this is not the required solution and we may divide by p.

† Consider the solution where $dy/dx \neq 0$, then it is possible to solve for x in terms of y and $p = dy/dx$ may be expressed in terms of y. The same transformation was used to write the acceleration as

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = \left(\frac{dv}{ds}\right)\left(\frac{ds}{dt}\right) = v\,\frac{dv}{ds}.$$

Then

$$\frac{dp}{dy} = e^y,$$

whence

$$p = e^y + c_1.$$

From the initial data c_1 is readily found to be $c_1 = -1$. So

$$p = \frac{dy}{dx} = e^y - 1$$

or

$$\frac{dy}{e^y - 1} = \frac{e^{-y}dy}{1 - e^{-y}} = dx.$$

Then

$$\log (1 - e^{-y}) = x + c_2$$

or

$$1 - e^{-y} = e^{x+c_2} = Ae^x.$$

From the initial data $A = 1 - e$ and the solution is

$$1 - e^{-y} = (1 - e)e^x.$$

PROBLEMS

1. Find a general solution of $y'' = xe^{2x}$.
2. If $yy'' + y'^2 - yy' = 0$, find the integral curve passing through the points $(0, 1)$ and $(\log \frac{1}{2}, 0)$.
3. If $y'' = 1 + y'^2$ and $y = y' = 0$ when $x = 0$, find y. Solve in two different ways.
4. Find a general solution of $y' - xy'' = 1 + y''^2$.
5. Find a general solution of $(x + y')y'' + y' = 0$.
6. If $y = y' = 0$ when $x = 0$ and $(1 + x^2)^2y'' + 2x(1 + x^2)y' = -4$, find y.
7. An integral curve of $y'^2 - 2y'' = -1$ passes through $(\pi, 0)$ with slope 0. Find its equation.
8. Find y if $y'' = e^{-2y}$ and $y = y' = 0$ when $x = -\pi/2$.

There are many tricks which will render tractable particular types of equations. No general rules may be given, but it often happens that a suitable change of variable will put the equation in a form where a familiar method is applicable. In some cases a simple substitution for the independent variable will prove adequate, while in others, a new dependent variable may be desirable. In the latter case if $y = y(z)$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dz}\frac{d^2z}{dx^2} + \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2.$$

Example 19. $y'' - \dfrac{2}{y}y'^2 - y = 0.$

Here the independent variable is absent, so we could reduce the order of the equation, but instead we choose to make the substitution $y = 1/z$. Then

$$y' = -\frac{1}{z^2}\frac{dz}{dx}$$
$$y'' = -\frac{1}{z^2}\frac{d^2z}{dx^2} + \frac{2}{z^3}\left(\frac{dz}{dx}\right)^2,$$

and the differential equation becomes

$$-\frac{1}{z^2}z'' + \frac{2}{z^3}z'^2 - 2z\frac{1}{z^4}z'^2 - \frac{1}{z} = 0$$

or

$$z'' + z = 0.$$

The general solution of this last equation is

$$z = A\sin x + B\cos x$$

so that

$$y = \frac{1}{A\sin x + B\cos x}.$$

PROBLEMS

9. Solve $yy'' - y'^2 = 2yy' - y^2\log y$ by putting $z = \log y$.

10. Solve $y'' - (\cos^2 x)y + (\tan x)y' + \cos^2 x = 0$, using the substitution $= \sin x$.

11. Solve $zz'' - 2z'^2 + 2\dfrac{zz'}{x} = 0$. Put $z = \dfrac{1}{y}$.

21. Application. Certain simple physical problems lead naturally to nonlinear equations of a type which we can handle. We illustrate with the problem of the loaded cable.

A cable hangs in a vertical plane (Fig. 32). Choose a rectangular coordinate system in the plane of the cable with the x axis horizontal. Let T be the tension in the cable, that is, the force exerted at P on the part of the cable to the left of P, and let X and Y be the horizontal and vertical components of T, respectively. Then $T = \sqrt{X^2 + Y^2}$ and

$$X = T\cos\theta$$
$$Y = T\sin\theta,$$

whence

(82) $$\frac{dy}{dx} = \frac{Y}{X} = \tan\theta.$$

Now the only reason for the tension to vary from point to point is a variation in the loading, that is, in Y. Hence X is constant. It is an arbitrary constant, or parameter, of the problem and determines the tautness of the cable.

Differentiating (82) we get

(83) $$\frac{d^2y}{dx^2} = \frac{1}{X}\frac{dY}{dx}.$$

Equation (83) is our basic equation. As we will see in the example below, it will be possible to express dY/dx in terms of x and y.

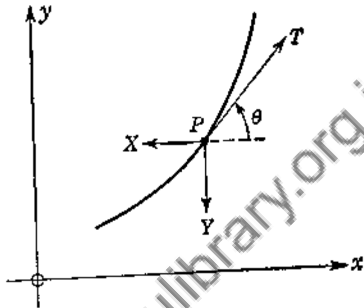


FIG 32

Example 20. Determine the shape of a uniform cable that hangs under its own weight.

If w is the weight of the cable per unit length, then $Y = ws$, where s is the arc length measured from the lowest part of the cable, and

$$\frac{dY}{dx} = w\frac{ds}{dx} = w\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Equation (83) now becomes

$$\frac{d^2y}{dx^2} = \frac{w}{X}\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

which is a nonlinear second-order equation with dependent variable absent. Setting $p = dy/dx$ reduces the order giving

$$\frac{dp}{dx} = \frac{w}{X}\sqrt{1 + p^2},$$

which on solving gives

$$\log\left(p + \sqrt{1 + p^2}\right) = \frac{wx}{X} + C$$

or

$$p + \sqrt{1 + p^2} = Ae^{wx/X} \qquad (A > 0)$$

and

$$p = \frac{A}{2}e^{wx/X} - \frac{1}{2A}e^{-wx/X}.$$

Now if coordinates are so chosen that for $x = 0$ we have $dy/dx = 0$ (the lowest point on the cable), we find from the last equation

$$0 = \frac{A}{2} - \frac{1}{2A},$$

whence $A = \pm 1$. Thus, choosing $A = 1$,

$$p = \frac{e^{wx/X} - e^{-wx/X}}{2} = \sinh \frac{wx}{X}$$

and integrating,

$$y = \frac{X}{w} \cosh \frac{wx}{X} + B.$$

This curve is called the *catenary.*

PROBLEMS

1. Find the suspension curve of a weightless cable which is uniformly loaded, that is, $Y = kx$.

***2.** Find the suspension curve of a weightless cable if $dY/dx = ky$. What physical loading would approximately realize this problem?

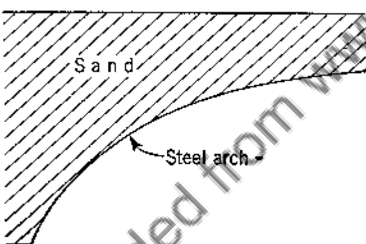


FIG. 33.

3. The differential equation of the simple pendulum is

$$\ddot{\theta} = -\frac{g}{l} \sin \theta.$$

Determine t as a function of θ if $\theta = \theta_0$, $\dot{\theta} = 0$ when $t = 0$. Note: Your answer will be expressed as a definite integral. This definite integral is called an "elliptic integral" and is not expressible in terms of elementary functions.

4. An arch made of thin sheet steel supports a level pile of sand (see Fig. 33). The stress at any point is due entirely to the weight of the sand. What must be the shape of the arch if there are no bending stresses, that is, if the resultant stress at any point is a pure compression.

MISCELLANEOUS PROBLEMS

1. Find the general solution of the following equations:

(*a*) $(D^2 + D - 2)y = x^2 + 3e^x$.
(*b*) $\ddot{x} + 4x = A \sin^2 t$.
(*c*) $(D^2 + 4)y = \cos x + \sin 2x$.
(*d*) $(D^2 + 2D + 1)y = A + B \sin 2x$.
(*e*) $y'' + 4y' + 5y = A \sin x$.

(*f*) $y'' - y = 4x^{-3} - 2x^{-1}$.
(*g*) $y'' - 2y' + 2y = (x^2 - x + 1)e^{x^2}$.
(*h*) $y'' + \sqrt{2}\, y' + 2y = 2\sqrt{2} \log x - 2x^{-3}$.
(*i*) $y''' - y' = 2x$.
(*j*) $(D^3 - 2D^2 - D + 2)y = \sin x$.

2. Find solutions of the following equations which satisfy the conditions indicated:

(*a*) $(D^2 - 4D + 13)y = 0$; $y = a,\ y' = -a,\ x = 0$.
(*b*) $(D^2 - 3D + 2)y = 0$; $y = e^{-1},\ y' = 2e^{-1},\ x = -\frac{1}{2}$.
(*c*) $(D^2 - 2D + 4)y = 0$; $y = 0,\ x = 0$; $y = 1.353,\ x = \dfrac{\pi}{6\sqrt{3}}$.
(*d*) $(D^2 - 2D + 3)y = 3x^2 - 4x - 4$; $y = 0,\ y' = 2 - \sqrt{2},\ x = 0$.
(*e*) $y'' + y' + y = (2 + x)\cos x + \sin x$; $y = 2,\ x = 0$; $y = 0.846$, $x = \dfrac{\pi}{\sqrt{3}}$.
(*f*) $y'' - 2y' - y = 0$; $y = -1,\ y'' = -3 + 4\sqrt{2},\ x = 0$.
(*g*) $y'' + 2y' + 2y = 5\cos x$; $y = -1,\ x = 0$; $y = 1.792,\ x = \dfrac{\pi}{2}$.

3. A cylindrical buoy 2 feet in diameter and weighing 100 pounds floats vertically in water. Find the period of oscillation when it is depressed slightly and released.

4. A cubical block of wood 28 inches on an edge floats in water with a face down. It is depressed slightly and released, whereupon it bobs with a period of 1.35 seconds. Find the specific gravity of the wood.

5. A weightless spring supports a weight of 20 pounds. The natural frequency of the system is found to be 1.27 cycles per second. Find the spring constant k.

6. A 5-pound weight stretches a weightless spring 2 inches. It is found experimentally that the spring system has a natural frequency of $5/\pi$ cycles per second. If there is damping proportional to the velocity, find the time required for the amplitude of the oscillations to decrease to one-half their original value.

7. Find numerically, between $x = 0$ and $x = 1$, using $\Delta x = 0.1$, the solution of $y'' = \sin y - y'^2 + x$ if $y = 0$ and $y' = 0$ when $x = 0$.

8. Find numerically, between $x = 1$ and $x = 2$, the solution of

$$y'' = 1 - y'^2$$

for which $y = 1,\ y' = 1$ when $x = 1$. Why is your solution exact?

9. A simple series circuit has an inductance of 0.2 henry, a resistance of 10 ohms, and a capacitance of 10^{-4} farad. The impressed e.m.f. is

$$E = 100 \sin 200t.$$

Find q, if $q = 0.005$ and $i = 0$ when $t = 0$.

10. Consider the function $f(x) = 1 - |x|$ in the interval $-1 \leqq x \leqq 1$. What should a_0, a_1, b_1, a_2, b_2 be in order that

$$a_0/2 + a_1 \cos \pi x + b_1 \sin \pi x + a_2 \cos 2\pi x + b_2 \sin 2\pi x$$

be the best approximation in the mean to $f(x)$ in the interval $-1 \leqq x \leqq 1$?

*__11.__ A dog is running due west at a speed of 30 feet per second when he sees, 200 feet straight ahead of him, a rabbit running due north at a speed of 20 feet per second. If the rabbit continues in the same direction at the same speed, and if the dog runs so as to point always at the rabbit, find the path of the dog.

CHAPTER V

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER. CONSTANT COEFFICIENTS

1. Introduction. The general linear differential equation of order n (n may be any of the integers 1, 2, 3, . . .) is of the form

$$(1) \qquad \frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x).$$

The coefficients $a_1(x)$, $a_2(x)$, . . . , $a_n(x)$ and the right member $f(x)$ may be functions of the independent variable x or constants. Not all of them are necessarily present in every linear differential equation of nth order; any number of them may be identically zero. To start with, the coefficient of the derivative of highest order in a linear differential equation is not necessarily unity as in Equation (1), but it is not identically zero and, therefore, the equation may be divided by this coefficient (isolated points at which the coefficient of d^ny/dx^n vanishes are called *singular points* of the differential equation and are not dealt with in this chapter). Thus, it is no restriction of generality to assume the equation to be cast in form (1).

The term $f(x)$ in Equation (1) is isolated from the remaining terms of the equation and is written as the right member because it is the only term of the equation that is free from the unknown function $y(x)$. If $f(x)$ is identically zero, the equation is said to be *homogeneous*, otherwise *nonhomogeneous*. In connection with the nonhomogeneous equation (1) one often studies the homogeneous equation that has the same left-hand member as (1). The latter is then referred to as the *reduced equation* to distinguish it from the former, the *complete equation*.

For the left-hand member of Equation (1) we will sometimes use the symbol $L[y(x)]$ or $L[y]$, which may be read as "a linear differential expression L formed with the function $y(x)$." Then Equation (1) takes the shorthand form

$$(2) \qquad L[y(x)] = f(x).$$

From the results in earlier chapters on differential equations of the first and second order it is to be expected that a general solution of

Equation (1) contains n essential constants. To single out a particular solution, additional conditions must be imposed. In this chapter these conditions will usually be *initial conditions* of the form

$$(3) \qquad \begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ y''(x_0) &= y_2 \\ &\cdots\cdots\cdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}$$

where $y_0, y_1, y_2, \ldots, y_{n-1}$ are any given numbers. These conditions specify the value of the solution $y(x)$ and its first $n-1$ derivatives at an initial point $x = x_0$. If $y_0 = y_1 = y_2 = \cdots = y_{n-1} = 0$, then the initial conditions are said to be *homogeneous*.

PROBLEMS

1. Let $L_1[y]$, $L_2[y]$ stand for the following differential expressions:

$$L_1[y] = \sin^2 x \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \frac{1}{1+x}y,$$

$$L_2[y] = \cos^2 x \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + \frac{1}{1-x}y.$$

Write in expanded form and simplify:

(*a*) $L_1[ay]$, where a is a constant.
(*b*) $L_1[ay]$, where a is a function of x.
(*c*) $L_2[a_1y_1 + a_2y_2]$, where a_1, a_2 are constants.
(*d*) $L_1[y] + L_2[y]$.
(*e*) $L_1[\log \sin x]$.
(*f*) $L_1[y \log \sin x]$.

2. If L is a differential expression like the left-hand term of Equation (1) and

$$L[y_1(x)] = f_1(x),\ L[y_2(x)] = f_2(x),\ \ldots,\ L[(y_k(x)] = f_k(x),$$

show that

$$L[y_1(x) + y_2(x) + \cdots + y_k(x)] = f_1(x) + f_2(x) + \cdots + f_k(x).$$

(This is the "principle of superposition," see Sec. 7.)

3. If L is a differential expression like the left-hand term of Equation (1), and $c_1, c_2, \ldots, c_k$ are constants show that

$$L[c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)] \\ = c_1L[y_1(x)] + c_2L[y_2(x)] + \cdots + c_kL[y_k(x)].$$

2. Existence and Uniqueness of Solutions. The theory of linear differential equations of order n parallels the theory of algebraic equations of degree n in many ways. Just as the latter is based on the "fundamental theorem of algebra," which asserts the existence of at least one real or complex root for any algebraic equation of degree n, so the theory of linear differential equations of order n is based on a fundamental theorem which deals with the existence of solutions under certain conditions. Since the existence of a solution of Equation (1) requires the existence of all the derivatives that occur in this equation, it is clear that some restrictions must be imposed on the functions $a_1(x), a_2(x), \ldots, a_n(x), f(x)$. The following theorem states that it is sufficient to assume these functions as continuous.

Fundamental Theorem. If $a_1(x), a_2(x), \ldots, a_n(x), f(x)$ are continuous in the interval $a \leqq x \leqq b$ that includes the point $x = x_0$, then Equation (1) *has one and only one solution in $a \leqq x \leqq b$ that satisfies conditions* (3).

The proof for this theorem is not simple and is omitted at this point. (It is given in Appendix B.) If the solution in question could be "exhibited," that is, constructed from the data of the equation and the initial conditions by known operations, the proof would simply consist in a check. This will actually be carried out below for the special case where the coefficients $a_1(x), a_2(x), \ldots, a_n(x)$ are constants. But, in general, the solution cannot be exhibited, and our fundamental theorem is an example of a pure "existence theorem." It can be shown† that the solution of Equation (1) can, in general, not be obtained by a finite sequence of algebraic operations plus differentiations and quadratures. To solve Equation (1) infinite sequences of successive approximations, or infinite series, definite integrals, and other infinite processes are resorted to, of which but little can be taken up in this chapter.

A general method of proof will suggest itself from the following treatment of a special case. This discussion will also serve to clarify the purpose of the initial conditions. Let us assume that the functions $a_1(x), a_2(x), \ldots, a_n(x), f(x)$ and every solution $y(x)$ can be expanded in series of positive integral powers of $(x - x_0)$, all of which are convergent for $a \leqq x \leqq b$. Then it follows that all these functions have derivatives of every order in $a \leqq x \leqq b$, and the power series are Taylor series. In particular,

$$(4) \quad y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots \quad (a \leqq x \leqq b)$$

† See J. F. Ritt, "Integration in Finite Terms," Columbia University Press, 1948.

Therefore, $y(x)$ will be determined in the interval $a \leqq x \leqq b$ if the values of $y(x)$ and of all its derivatives at the one point $x = x_0$ are known. The derivatives of order 0, 1, 2, . . . , $n - 1$ are given by the initial conditions (3). Substituting these in differential equation (1), written for $x = x_0$, the nth derivative is obtained:

$$y^{(n)}(x_0) = -a_1(x_0)y_{n-1} - a_2(x_0)y_{n-2} - \cdot \cdot \cdot - a_n(x_0)y_0 + f(x_0).$$

If Equation (1) is differentiated and $x = x_0$ is substituted, the $(n + 1)$st derivative is obtained in terms of the already known derivatives of order 0, 1, 2, . . . , n. By our hypothesis the equation can be differentiated any number of times, and in this way the derivatives of $y(x)$ of any order at $x = x_0$ are obtained in terms of previously determined derivatives. When the values of these derivatives are substituted in expansion (4), a power series is obtained which, according to our assumption, converges for $a \leqq x \leqq b$ to a function that is, by construction, an integral of Equation (1) satisfying the initial conditions. If any other solution of the same problem existed, it could also be expanded in a series of the form (4), and the coefficients in this expansion would necessarily be determined by the same relations as those of the former solution. Therefore, two such solutions could not differ in the interval $a \leqq x \leqq b$, or in other words, the solution is unique.

The presented argument does not constitute a proof of the above theorem, because use was made of the unproven hypothesis that the functions $a_1(x)$, $a_2(x)$, . . . , $a_n(x)$, $f(x)$ and every integral can be expanded in series of powers of $(x - x_0)$ convergent for $a \leqq x \leqq b$. Now it can be proved that every integral can be expanded in such a power series if the functions $a_1(x)$, $a_2(x)$, . . . , $a_n(x)$, $f(x)$ can (see also Sec. 1, Chap. VIII). But since the only restrictions of our theorem on these functions are that they be continuous, it is in general not true that they can be expanded in power series convergent in $a \leqq x \leqq b$. However, continuous functions can be uniformly approximated, to any desired degree of accuracy, by functions which are so expandable.† The integrals of those equations obtained from Equation (1) by replacing the functions $a_1(x)$, $a_2(x)$, . . . , $a_n(x)$, $f(x)$ by the approximating expandable functions can be proved to converge to the desired integral of Equation (1). In the manner thus sketched a complete proof of the fundamental theorem can be carried out.

† For every function $F(x)$ that is continuous in the interval $a \leqq x \leqq b$ there exists a sequence of *polynomials* that converges uniformly, for $a \leqq x \leqq b$, to $F(x)$. This is "Weierstrass' approximation theorem."

3. Remarks on the Fundamental Theorem. If a function $y(x)$ is to be a solution of Equation (1), it obviously must have derivatives of order 1, 2, . . . , n. Since differentiability of a function implies its continuity, it follows that the derivatives of order 0, 1, 2, . . . , $n - 1$ are continuous. Moreover, by (1)

$$y^{(n)}(x) = f(x) - a_1(x)y^{(n-1)}(x) - a_2(x)y^{(n-2)}(x) - \cdots - a_n(x)y(x),$$

and since the right-hand member of this equation is continuous, this is also the case for $y^{(n)}(x)$.

The fundamental theorem does not only assert that there exists an integral of Equation (1) satisfying initial conditions (3), but also that there is but one such solution. For this reason it is said to be an existence and uniqueness theorem. There are many important applications of the uniqueness part of the fundamental theorem. Suppose two different methods are used to solve Equation (1) with the given initial conditions. Then the two obtained expressions, although they may widely differ in form, must represent identical functions, for $a \leqq x \leqq b$. In this way power series, definite integrals, and other infinite expansions can often be "evaluated," that is, identified as expansions of known functions. The following two examples may serve to illustrate this point.

Example 1. Consider the homogeneous differential equation

$$L[y(x)] = 0$$

with the homogeneous initial conditions

$$y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0.$$

One integral satisfying these conditions is readily guessed, namely, $y(x) \equiv 0$. By the uniqueness theorem there is no other solution, and it is useless to try another method to find one.

Example 2. By direct substitution it can be verified that each of the three functions

$$y_1(x) = 2(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots)$$

$$y_2(x) = \int_0^1 \frac{t^x - t^{-x}}{\log t}\, dt$$

$$y_3(x) = \log \frac{1 + x}{1 - x}$$

is, for $|x| < 1$, a solution of the equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = 0$$

and satisfies the initial conditions

$$y(0) = 0, \qquad y'(0) = 2.$$

Hence, by the uniqueness part of the fundamental theorem, these three functions are identical. The series $y_1(x)$ is a power-series expansion, and the definite integral $y_2(x)$ is an integral representation of the function $y_3(x)$.

PROBLEMS

1. Show that both

$$y_1(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots$$

and

$$y_2(x) = \arcsin x$$

are, for $|x| < 1$, solutions of the equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$. How does it follow that $y_1(x) = y_2(x)$?

2. Show that no integral curve of the equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad (a_1(x), a_2(x) \text{ continuous})$$

can be tangent to the x axis.

3. Show that if $y = y_1(x)$, $y = y_2(x)$ are two integral curves of the equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x) = 0 \quad (a_1(x), a_2(x) \text{ continuous})$$

that intersect the x axis in the same point, then $y_1(x)$, $y_2(x)$ differ only by a constant factor. Hint: Suppose $y_1(x_0) = y_2(x_0) = 0$ and $y_1'(x_0) = m_1$, $y_2'(x_0) = m_2$. Then apply the result of Prob. 2 to the integral curve $y = m_2y_1(x) - m_1y_2(x)$.

★**4.** Show that both

$$y_1(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$$

and

$$y_2(x) = \frac{1}{\pi}\int_0^\pi \cos(x \cos\theta)\, d\theta$$

are solutions of the equation ("Bessel's")

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$. These two functions are identical (equal to the Bessel function $J_0(x)$, see Sec. 5, Chap. VIII), but this fact cannot be deduced from the fundamental theorem. Why not?

★5. Verify that $y = x \sin x$ is a solution of the equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + 2)y = 0$$

and of the initial conditions $y(0) = 0$, $y'(0) = 0$. Another solution is $y \equiv 0$. Is this a contradiction to the theorem of Sec. 2?

6. Show that if $y = y_1(x)$ is a solution of the equation

$$\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = f(x)$$

and the functions $a_1(x)$, $a_2(x)$, . . . , $a_n(x)$, $f(x)$ possess derivatives of every order in the interval $a \leqq x \leqq b$, then $y_1(x)$ possesses derivatives of every order. Hint: Since $y = y_1(x)$ is a solution of the equation, we have

$$y_1^{(n)} = -a_1(x)y_1^{(n-1)} - \cdots - a_n(x)y_1 + f(x).$$

Likewise, obtain $y_1^{(n+1)}$, $y_1^{(n+2)}$, ...

4. Linear Combinations, Linear Independence, Linear Systems. Before we turn to the discussion of the general solution of a linear differential equation, we restate some of the definitions concerning linear dependence and independence of functions given in Chap. I and explore their general consequences.

A sum of the form $c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$, formed from given functions $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$ and from arbitrary constants c_1, c_2, . . . , c_k, is called a *linear combination* of the functions $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$. It is understood that the constants c_1, c_2, . . . , c_k may be complex numbers. The function $y(x)$ is said to be *linearly independent* of the functions $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$ if $y(x)$ cannot be represented as a linear combination of $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$.

A finite number of functions $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$ are said to be linearly independent (of one another) if none of them can be represented as a linear combination of the remaining ones.

It is readily seen that linear independence of the functions $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$ is equivalent to the statement: there is no linear combination of $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$ that is identically zero except the trivial one, namely, $0 \cdot y_1(x) + 0 \cdot y_2(x) + \cdots + 0 \cdot y_k(x)$.†

† A necessary and sufficient condition for the linear independence of k functions that are integrals of a linear differential equation of order k is the nonvanishing of the so-called Wronskian determinant (see, for example, Ref. 3, p. 116).

In this book special methods are used whenever the linear independence of functions is to be examined.

The multitude of functions that are linear combinations of k fixed linearly independent functions $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$ are said to form a *linear system of functions of rank* k, and the functions $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$ are said to be a *basis* of this system. We shall use the symbol $S[y_1(x), y_2(x), \ldots, y_k(x)]$ to denote the linear system formed from the basis $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$.

The following examples will help to clarify these definitions. At the same time several important sets of functions will be shown to be linearly independent.

Example 3. $3x^2 - 2x + 1$ is a linear combination of $(x^2 + x)$ and $(x^2 + 6x - 1)$; for $3x^2 - 2x + 1 = 4(x^2 + x) - 1(x^2 + 6x - 1)$.

Example 4. $\log(1 + \sqrt{2x - 1})$ is a linear combination of 1 and $\log(x + \sqrt{2x - 1})$; for

$$\log(1 + \sqrt{2x - 1}) = (\tfrac{1}{2}\log 2)\cdot 1 + \tfrac{1}{2}\log(x + \sqrt{2x - 1}).$$

Example 5. The functions $1, x, x^2, \ldots, x^k$ are linearly independent. For, assume that there is a linear combination of them that is identically zero, namely,

$$(5) \qquad c_0 1 + c_1 x + c_2 x^2 + \cdots + c_k x^k = 0.$$

If not all the coefficients in Equation (5) are zero, then it is an algebraic equation of degree no higher than k which can be satisfied by at most k values of x, but not identically.

Example 6. $S[1, x, x^2, \ldots, x^n]$ is the system of all polynomials of degrees 0, 1, 2, . . . , n.

Example 7. All the harmonic vibrations $A \sin(\nu t + \delta)$ of fixed frequency ν, but arbitrary amplitude A and arbitrary phase δ, form a linear system of rank 2. As the basis of this system we may choose the two linearly independent functions $\sin \nu t$, $\cos \nu t$; for,

$$A \sin(\nu t + \delta) = (A \cos \delta) \sin \nu t + (A \sin \delta) \cos \nu t.$$

Conversely,

$$c_1 \sin \nu t + c_2 \cos \nu t = A \sin(\nu t + \delta),$$

where $A = \sqrt{c_1^2 + c_2^2}$, $\delta = \arctan(c_2/c_1)$ (see also Sec. 5, Chap. IV).

Theorem. The functions

$$e^{r_1x}, xe^{r_1x}, \ldots, x^{n_1}e^{r_1x}; e^{r_2x}, xe^{r_2x}, \ldots, x^{n_2}e^{r_2x}; \ldots; e^{r_kx}, xe^{r_kx}, \ldots, x^{n_k}e^{r_kx},$$

where $n_1, n_2, \ldots, n_k$ are any nonnegative integers and $r_1, r_2, \ldots, r_k$ are any distinct real or complex numbers, are linearly independent.

To prove this important theorem assume that there is an identically vanishing linear relation between those functions. Then it would have the form

(6) $$P_1(x)e^{r_1x} + P_2(x)e^{r_2x} + \cdots + P_k(x)e^{r_kx} = 0,$$

where $P_1(x), \ldots, P_k(x)$ are polynomials or constants of which at least one, say $P_k(x)$, is not identically zero. Dividing (6) by e^{r_1x} we obtain

(7) $$P_1(x) + P_2(x)e^{(r_2-r_1)x} + \cdots + P_k(x)e^{(r_k-r_1)x} = 0.$$

If this equation is differentiated a sufficient number of times, the first summand $P_1(x)$ is reduced to zero since this is a polynomial. But none of the other terms is reduced to zero since none of the differences $r_2 - r_1, r_3 - r_1, \ldots, r_k - r_1$ is zero.†

Hence, a relation of the form

(8) $$Q_2(x)e^{(r_2-r_1)x} + \cdots + Q_k(x)e^{(r_k-r_1)x} = 0$$

results, where $Q_2(x), \ldots, Q_k(x)$ are certain polynomials or constants and $Q_k(x)$ is not identically zero. Dividing (8) by $e^{(r_2-r_1)x}$ we obtain

$$Q_2(x) + Q_3(x)e^{(r_3-r_2)x} + \cdots + Q_k(x)e^{(r_k-r_2)x} = 0.$$

This equation is now differentiated a sufficient number of times so as to reduce $Q_2(x)$ to zero. This procedure is continued until finally a relation of the form

$$Z_k(x)e^{(r_k-r_{k-1})x} = 0$$

is derived, where $Z_k(x)$ is a polynomial or a nonvanishing constant. But this last relation cannot be an identity, and therefore the assumption that a relation of form (6) exists is proved to be wrong.

It is not difficult to see that among the functions of a linear system of rank k there are no more than k linearly independent ones, which means that by forming all linear combinations of certain given functions no more linearly independent functions are obtained than there had been originally. It follows from this remark that, if $Y_1(x)$, $Y_2(x), \ldots, Y_k(x)$ are any k linearly independent functions from the system $S[y_1(x), y_2(x), \ldots, y_k(x)]$, every other function of S may be obtained as a linear combination of $Y_1(x), Y_2(x), \ldots, Y_k(x)$.

† It is left to the student to verify that the n-th derivative of a function

$$y = (ax^n + bx^{n-1} + \cdots)e^{rx},$$

where $a \neq 0$ and $r \neq 0$ is of the form $y^{(n)} = (ar^nx^n + b_1x^{n-1} + \cdots)e^{rx}$ (this holds also when $n = 0$, that is, $y = ae^{rx}$). Hence, none of the derivatives $y', y'', y''', \cdots$ can reduce to zero.

Therefore, these latter functions are also a basis of system S, and we have the equation

$$S[Y_1(x), Y_2(x), \ldots, Y_k(x)] = S[y_1(x), y_2(x), \ldots, y_k(x)],$$

expressing the fact that any k linearly independent functions taken from a linear system of rank k may serve as a basis of this system.

Example 8. As an example consider the system of all polynomials of degrees 0, 1, 2, . . . , k, for which the functions $1, x, x^2, \ldots, x^k$ form a basis (see Example 5 above). Among all these polynomials are also the $(k + 1)$ linearly independent ones, namely, $1, (x - x_0), (x - x_0)^2, \ldots, (x - x_0)^k$. Therefore,

$$S[1, x, x^2, \ldots, x^k] = S[1, (x - x_0), (x - x_0)^2, \ldots, (x - x_0)^k]. \tag{9}$$

Example 9. Another system of great importance in the theory of linear differential equations is

$$S[e^{(\alpha+\beta i)x}, e^{(\alpha-\beta i)x}] \qquad (i = \sqrt{-1}) \tag{10}$$

As is well known (see Sec. 9, Chap. I),

$$e^{(\alpha\pm\beta i)x} = e^{\alpha x}e^{\pm i\beta x} = e^{\alpha x}\cos\beta x \pm ie^{\alpha x}\sin\beta x.$$

Therefore, both $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ are linear combinations of $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$. On the other hand we also have the identities

$$e^{\alpha x}\cos\beta x = e^{\alpha x}\frac{e^{i\beta x} + e^{-i\beta x}}{2} = \frac{1}{2}e^{(\alpha+\beta i)x} + \frac{1}{2}e^{(\alpha-\beta i)x}$$

$$e^{\alpha x}\sin\beta x = e^{\alpha x}\frac{e^{i\beta x} - e^{i\beta x}}{2i} = -\frac{i}{2}e^{(\alpha+\beta i)x} + \frac{i}{2}e^{(\alpha-\beta i)x}.$$

Therefore, both $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are linear combinations of $e^{(\alpha+\beta i)x}$, $e^{(\alpha-\beta i)x}$. The two results may be stated in one equation:

$$S[e^{(\alpha+\beta i)x}, e^{(\alpha-\beta i)x}] = S[e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x]. \tag{11}$$

PROBLEMS

1. Represent $\cos 2x$ as a linear combination of

(a) $\sin^2 x$, $\cos^2 x$.
(b) 1, $\cos^2 x$.
(c) -3, $\sin^2 x$.

2. Represent $2x^2 + 8x + 3$ as a linear combination of

(a) $1, x, x^2$.
(b) $1, (x - 1), (x - 1)^2$.
(c) $5, (x + 2)^2$.
(d) $1, x + 1$.

3. Show that if $y_1(x)$, $y_2(x)$ are linear combinations of $v_1(x)$, $v_2(x)$ and $v_1(x)$, $v_2(x)$ are linear combinations of $u_1(x)$, $u_2(x)$, then $y_1(x)$, $y_2(x)$ are linear combinations of $u_1(x)$, $u_2(x)$.

4. Represent $A \sinh (ax + b)$ and $A \cosh (ax + b)$ as linear combinations of e^{ax}, e^{-ax}.

5. Represent e^{ax} and e^{-ax} as linear combinations of $\sinh (ax)$, $\cosh (ax)$.

6. Show that

$$S[e^{ax}, e^{-ax}] = S[\sinh (ax), \cosh (ax)].$$

7. Show that

$$S[e^{(\alpha+\beta i)x}, e^{(\alpha-\beta i)x}, e^{(-\alpha+\beta i)x}, e^{(-\alpha-\beta i)x}] \\ = S[\sinh \alpha x \sin \beta x, \sinh \alpha x \cos \beta x, \cosh \alpha x \sin \beta x, \cosh \alpha x \cos \beta x].$$

***8.** Represent $\cos^n x$ $(n = 1, 2, \ldots)$ as a linear combination of $1, \cos x, \cos 2x, \ldots, \cos nx$. Hint: Use

$$\cos^n x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^n.$$

***9.** Can $\sin^n x$ be represented as a linear combination of $\sin x, \sin 2x, \ldots, \sin nx$ for every positive integer n?

10. Show that the functions $1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots$ are linearly independent. Hint: Use the theorem of this section.

***11.** Show that if $y_1(x)$, $y_2(x)$, $y_3(x)$ are linearly dependent then the determinant

$$\begin{vmatrix} y_1(x), & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix}$$

vanishes for all x.

12. Show that any set of functions is linearly dependent if (*a*) $y = 0$ is one of the functions; (*b*) two of the functions are identical; (*c*) one of the functions is a constant multiple of another; (*d*) a subset of those functions is linearly dependent.

5. General Solution of the Homogeneous Equation.

By the use of the terminology introduced in the preceding section it is possible to characterize the multitude of solutions of a linear homogeneous differential equation in a very concise manner. This is done in the following theorem:

Theorem. The integrals of the linear homogeneous differential equation of order n, $L[y] = 0$, form a linear system of rank n.

The proof of this theorem is carried out in three steps. At first, it is shown that any linear combination of particular integrals is itself an integral. Then n particular integrals $Y_0(x), Y_1(x), \ldots, Y_{n-1}(x)$ are singled out, which are shown to be linearly independent. Finally, it is proved that every integral is a linear combination of $Y_0(x)$, $Y_1(x)$, $\ldots$, $Y_{n-1}(x)$.

The first step is the easiest. Assume that $y(x)$ is a linear combination of the integrals $y_1(x)$, $y_2(x)$, $\ldots$, $y_k(x)$, namely,

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x). \tag{12}$$

To verify that $y(x)$ itself is an integral consider a typical term of $L[y]$, say $a_m(x)y^{(m)}(x)$. By (12) we have

$$a_m(x)y^{(m)}(x) = c_1a_m(x)y_1^{(m)}(x) + c_2a_m(x)y_2^{(m)}(x) + \cdots + c_ka_m(x)y_k^{(m)}(x).$$

The sum of all these terms ($m = 0, 1, 2, \ldots, n$) is $L[y]$. Hence,†

$$\begin{aligned} L[y] &= c_1L[y_1] + c_2L[y_2] + \cdots + c_kL[y_k] \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_k \cdot 0 \\ &= 0. \end{aligned} \tag{13}$$

Hence, every linear combination of integrals of the equation $L[y] = 0$ is itself an integral of this equation.

Next, let us single out n particular integrals $Y_0(x)$, $Y_1(x)$, $\ldots$, $Y_{n-1}(x)$, distinguished by the different initial conditions that they satisfy. At the initial point $x = x_0$, $Y_k(x)$ and its first $(n - 1)$ derivatives are to vanish, except the kth derivative which is to be unity, or

$$\begin{aligned} Y_k^{(i)}(x_0) &= 0 \qquad (\text{for } i \neq k) \\ &= 1 \qquad (\text{for } i = k) \end{aligned} \tag{14}$$

By the fundamental theorem of Sec. 2, there are such solutions and they are uniquely determined. Of these solutions it is readily seen that they are linearly independent. For, assume one of them, say $Y_k(x)$, is a linear combination of the remaining ones:

$$Y_k(x) = c_0Y_0(x) + c_1Y_1(x) + \cdots + c_{k-1}Y_{k-1}(x) + c_{k+1}Y_{k+1}(x) + \cdots + c_{n-1}Y_{n-1}(x). \tag{15}$$

This equation may be differentiated k times (since $k \leqq n - 1$), and then the initial value x_0 may be substituted for x, whereby the impossible equation

$$1 = c_0 \cdot 0 + c_1 \cdot 0 + \cdots + c_{n-1} \cdot 0$$

results. Hence, the assumption that $Y_0(x)$, $Y_1(x)$, $\ldots$, $Y_{n-1}(x)$ are not linearly independent must be rejected. Among the integrals of

† On account of the "distributive" property, $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$, $L[y]$ is said to be a *linear* differential expression.

$L[y] = 0$ there are, at least, n linearly independent ones, in particular the integrals $Y_0(x), Y_1(x), \ldots, Y_{n-1}(x)$.

To complete the proof let us consider a linear combination of these particular integrals:

$$Y(x) = c_0Y_0(x) + c_1Y_1(x) + \cdots + c_{n-1}Y_{n-1}(x), \tag{16}$$

where $c_0, c_1, \ldots, c_{n-1}$ are arbitrary constants. When Equation (16) is differentiated i times ($i = 0, 1, 2, \ldots, n-1$), and x_0 is then substituted for x, one obtains, because of (14),

$$\begin{aligned} Y^{(i)}(x_0) &= c_0 \cdot 0 + c_1 \cdot 0 + \cdots + c_i \cdot 1 + \cdots + c_{n-1} \cdot 0 \\ &= c_i. \end{aligned}$$

Hence, $Y(x)$ is a solution of the equation $L[y] = 0$ satisfying the initial conditions

$$Y(x_0) = c_0,\ Y'(x_0) = c_1,\ \ldots,\ Y^{(n-1)}(x_0) = c_{n-1},$$

and because of the fundamental theorem this is the only such solution. Since the numbers $c_0, c_1, \ldots, c_{n-1}$ are arbitrary, any solution whatsoever of $L[y] = 0$ can be obtained in the form (16). Hence the n particular integrals $Y_0(x), Y_1(x), \ldots, Y_{n-1}(x)$ form a linear basis of all the solutions, and the theorem is proved.

It should be observed that the particular integrals $Y_0(x), Y_1(x), \ldots, Y_{n-1}(x)$ considered in the above proof do not constitute the only linear basis of the system of integrals of $L[y] = 0$. Any other set of n linearly independent integrals, say $y_1(x), y_2(x), \ldots, y_n(x)$, form such a basis, and every solution of the equation can be obtained as a linear combination of the basis integrals. Hence, the content of the theorem may also be stated as follows: *The general solution of the homogeneous linear differential equation of order n $L[y] = 0$ is*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

where $y_1(x), y_2(x), \ldots, y_n(x)$ are any n linearly independent particular solutions and $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example 10. The linear homogeneous differential equation of order n

$$\frac{d^ny}{dx^n} = 0 \tag{17}$$

obviously has the n particular integrals $1, x, x^2, \ldots, x^{n-1}$, which are linearly independent (see Example 3, Sec. 4). Hence, these functions form a basis of the linear system of all integrals of Equation (17), and the class of all solutions is the class of polynomials of degrees 0, 1, 2,

$\ldots, n-1$. The particular integrals satisfying conditions (14) are in this case

$$Y_0(x) = 1,\ Y_1(x) = \frac{x - x_0}{1!},\ Y_2(x) = \frac{(x-x_0)^2}{2!},\ \ldots,\ Y_{n-1}(x) = \frac{(x-x_0)^{n-1}}{(n-1)!}.$$

Thus, we have two expressions for the general integral of Equation (17):

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1}$$

and

$$C_0 + C_1\frac{x-x_0}{1!} + C_2\frac{(x-x_0)^2}{2!} + \cdots + C_{n-1}\frac{(x-x_0)^{n-1}}{(n-1)!}.$$

PROBLEMS

1. The equation

$$x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

has three linearly independent integrals of the form x^r. Determine these and then establish the general solution.

2. The equation

$$\frac{d^4x}{dt^4} + 4a\frac{d^3x}{dt^3} + 6a^2\frac{d^2x}{dt^2} + 4a^3\frac{dx}{dt} + a^4x = 0$$

has four linearly independent integrals of the form t^re^{-at}. Determine these and then establish the general solution.

3. Show that both the functions

$$1,\ \sin x,\ \cos x,\ \sin 2x,\ \cos 2x$$

and the functions

$$\sin x,\ \cos x,\ \sin^2 x,\ \sin x\cos x,\ \cos^2 x$$

form a linear basis of the solutions of the equation

$$\frac{d^5y}{dx^5} + 5\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = 0.$$

6. General Solution of the Nonhomogeneous Equation. For a nonhomogeneous linear differential equation it is no longer true that a linear combination of particular integrals is itself an integral. Another property takes its place. Suppose that $y_f(x)$ is a particular integral of the nonhomogeneous equation

$$L[y] = f(x) \tag{18}$$

and $y_0(x)$ is an integral of the reduced equation

(19) $$L[y] = 0,$$

that is,

(20) $$L[y_f] = f(x), \qquad L[y_0] = 0.$$

Then, by adding the last two equations, we obtain

(21) $$L[y_f] + L[y_0] = L[y_f + y_0] = f(x).$$

Equation (21) obviously means that $y_f(x) + y_0(x)$ is also a solution of Equation (18). Hence, by adding integrals of the reduced equation to a particular integral of the complete equation new integrals of the complete equation are obtained. The question arises whether, starting with one particular integral of the complete equation, one can obtain all of them by just adding integrals of the reduced equation.

This is indeed the case. For, let $\tilde{y}_f(x)$ be any other integral of the complete equation. That is

(22) $$L[\tilde{y}_f] = f(x).$$

Then one obtains by subtracting (22) from (20):

$$L[y_f - \tilde{y}_f] = 0.$$

Hence, the difference of the two particular integrals of the complete equation is a solution of the reduced equation. In other words, the arbitrary integral $\tilde{y}_f(x)$ of the complete equation can differ from the particular integral $y_f(x)$ of the same equation only by some integral of the reduced equation. We state this result as a theorem.

Theorem. The general solution of the nonhomogeneous linear differential equation $L[y] = f(x)$ *is obtained from any particular integral of this equation by adding the general solution of the reduced equation* $L[y] = 0$.

The general solution of the reduced equation is called the *complementary function.* Hence, this theorem may be expressed by the symbolic equation

General integral = particular integral + complementary function.

Example 11. Consider the nonhomogeneous equation

(23) $$\frac{d^n y}{dx^n} = Ae^x.$$

It is easy to guess a particular integral, namely, $y = Ae^x$. The reduced equation is $y^{(n)} = 0$, whose general solution was found in Example 10, Sec. 5. Therefore, the general solution of Equation (23) is

$$y(x) = Ae^x + c_0 + c_1x + c_2x^2 + \cdot \cdot \cdot + c_{n-1}x^{n-1}.$$

Example 12. An example of general importance is supplied by the equation

$$\frac{d^ny}{dx^n} = f(x), \tag{24}$$

where $f(x)$ is an unspecified continuous function. The problem to find the general solution of this equation may be considered as an extension of the problem to find the general solution of the equation

$$\frac{dy}{dx} = f(x),$$

which is solved by the indefinite integral of $f(x)$.

To solve Equation (24) we may interpret it as a differential equation of first order in the unknown function $y^{(n-1)}(x)$:

$$\frac{d}{dx} y^{(n-1)} = f(x).$$

Hence

$$y^{(n-1)}(x) = \int^x f(x_1)\, dx_1 + C_1 = F_1(x) + C_1, \tag{25}$$

where $F_1(x)$ denotes an indefinite integral of $f(x)$. Proceeding with equation (25) as we proceeded with Equation (24), we find

$$y^{(n-2)}(x) = \int^x [F_1(x_1) + C_1]\, dx_1 = F_2(x) + C_1x + C_2,$$

where $F_2(x)$ is an indefinite integral of $F_1(x)$, and therefore, $F_2(x)$ is an iterated indefinite integral of $f(x)$. When this process is repeated n times, one obtains

$$y(x) = F_n(x) + c_0 + c_1x + c_2x^2 + \cdot \cdot \cdot + c_{n-1}x^{n-1}, \tag{26}$$

where $F_n(x)$ is an n times repeated indefinite integral of $f(x)$, and c_0, c_1, . . . , c_{n-1} are arbitrary constants. In expression (26), $F_n(x)$ is a particular integral and $(c_0 + c_1x + \cdot \cdot \cdot + c_{n-1}x^{n-1})$ is the complementary function.

Any n times repeated indefinite integral of $f(x)$ may be used as $F_n(x)$ in Equation (26). An important special case is obtained if all the lower limits of the n integrals are chosen to be equal, say x_0. Then we have the following particular solution of Equation (24):

$$F_n(x) = \int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_3} dx_2 \int_{x_0}^{x_2} f(x_1)\,dx_1. \tag{27}$$

This particular integral is characterized by the fact that it satisfies the initial conditions

$$F_n(x_0) = F_n'(x_0) = F_n''(x_0) = \cdots = F_n^{(n-1)}(x_0) = 0. \tag{28}$$

In the following we shall make repeated use of a formula that gives the n times iterated integral (27) as a simple integral

$$\int_{x_0}^{x} dx_n \int_{x_0}^{x_n} dx_{n-1} \cdots \int_{x_0}^{x_2} f(x_1)\,dx_1 = \int_{x_0}^{x} \frac{(x-x_1)^{n-1}}{(n-1)!} f(x_1)\,dx_1. \tag{29}$$

To prove this identity we might use repeated integration by parts to reduce the iterated integral to a simple integral. We choose another method which serves us also as an application of the fundamental theorem. Let us differentiate the right-hand member of Equation (29) 1, 2, . . . , n times, keeping in mind that the variable x occurs both in the integrand and as upper limit of the integral:

$$\begin{aligned}\frac{d}{dx}\int_{x_0}^{x} \frac{(x-x_1)^{n-1}}{(n-1)!} f(x_1)\,dx_1 &= (n-1)\int_{x_0}^{x} \frac{(x-x_1)^{n-2}}{(n-1)!} f(x_1)\,dx_1 \\ &\qquad + \left[\frac{(x-x_1)^{n-1}}{(n-1)!} f(x_1)\right]_{x_1=x} \\ &= \int_{x_0}^{x} \frac{(x-x_1)^{n-2}}{(n-2)!} f(x_1)\,dx_1.\end{aligned}$$

Likewise,

$$\frac{d^2}{dx^2}\int_{x_0}^{x} \frac{(x-x_1)^{n-1}}{(n-1)!} f(x_1)\,dx_1 = \int_{x_0}^{x} \frac{(x-x_1)^{n-3}}{(n-3)!} f(x_1)\,dx_1,$$

. . . .

$$\frac{d^{n-1}}{dx^{n-1}}\int_{x_0}^{x} \frac{(x-x_1)^{n-1}}{(n-1)!} f(x_1)\,dx_1 = \int_{x_0}^{x} f(x_1)\,dx_1,$$

and

$$\frac{d^n}{dx^n}\int_{x_0}^{x} \frac{(x-x_1)^{n-1}}{(n-1)!} f(x_1)\,dx_1 = f(x).$$

The last of these equations shows that the right-hand member of Equation (29) is a solution of Equation (24). The preceding equations with x_0 substituted for x show that this solution vanishes together with its first $(n - 1)$ derivatives at x_0. Since, by the fundamental theorem, there can be but one such solution, the two members of Equation (29) are proved to be identical.

PROBLEMS

Find the general solutions of the following equations. Use both repeated integrations and formula (29).

1. $\dfrac{d^3y}{dx^3} = Ae^{ax}$.

2. $\dfrac{d^2x}{dt^2} = At \cos \omega t$.

3. $\dfrac{d^4y}{dx^4} = A \log x$.

4. $\dfrac{d^2y}{dx^2} = (a^2 - x^2)^{-\frac{3}{2}}$.

5. $\dfrac{d^4x}{dt^4} = Ae^{-a^2t^2}$.

Find the general solutions of the following differential equations. Make use of the suggested functions, which by proper choice of the constants are either integrals of the complete or of the reduced equation.

6. $\dfrac{d^2y}{dx^2} - 13\dfrac{dy}{dx} + 36y = 3 \sin 6x$; $y_1 = Ae^{4x}$, $y_2 = Be^{9x}$, $y_3 = C \cos 6x$.

7. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = \dfrac{5}{x^2}$; $y_1 = Ax^{-2}$, $y_2 = Bx^{-3}$, $y_3 = Cx^3$.

8. $(x^2 + 1)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x(x^2 + 3)$; $y_1 = Ax$, $y_2 = Bx^3$, $y_3 = C + Dx^2$.

9. $x^4\dfrac{d^3y}{dx^3} - x^3\dfrac{d^2y}{dx^2} + x^2\dfrac{dy}{dx} = k$; $y_1 = A + Bx + Cx^2$; $y_2 = Dx^2 \log x$, $y_3 = Ex^{-1}$.

10. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = ke^{ax}$; $y_1 = (A + Bx) \sin 2x$, $y_2 = (C + Dx) \cos 2x$, $y_3 = Ee^{ax}$.

7. Principle of Superposition. This principle is often helpful in the case of nonhomogeneous linear differential equations whose nonhomogeneous term is the sum of several distinct functions.

Assume that $y_a(x)$, $y_b(x)$, $y_c(x)$, . . . are any integrals of the equations $L[y(x)] = f_a(x)$, $L[y(x)] = f_b(x)$, $L[y(x)] = f_c(x)$, . . . , respectively, that is, of nonhomogeneous equations with the same homogeneous part but different nonhomogeneous terms. Then

$$\begin{aligned} L[y_a(x)] &= f_a(x) \\ L[y_b(x)] &= f_b(x) \\ L[y_c(x)] &= f_c(x) \\ &\cdots\cdots \end{aligned} \tag{30}$$

and summation of these equations gives

$$L[y_a(x) + y_b(x) + y_c(x) + \cdots] = f_a(x) + f_b(x) + f_c(x) + \cdots,$$

or in other words, the function $y(x) = y_a(x) + y_b(x) + y_c(x) + \cdots$ is an integral of the equation

$$L[y(x)] = f_a(x) + f_b(x) + f_c(x) + \cdots \tag{31}$$

whose homogeneous part is the same as that of Equations (30) and whose nonhomogeneous term is the sum of those of Equations (30). Conversely, in order to find a solution of Equation (31) we may first find integrals of the various Equations (30) and then sum these integrals. This method goes by the name *principle of superposition*.

It should be remarked that this method is also made use of in the case of nonhomogeneous terms that are sums of *infinitely* many terms (power series, Fourier series, etc.), but in such cases additional conditions as to the convergence of these sums are necessary for the method to be valid. For a discussion of such conditions see Sec. 18, Chap. IV.

Example 13. In a linear automatic control mechanism the value of the output (or response) variable $x_o(t)$ at the time t is related to the value of the input (or signal) variable $x_i(t)$ at the time t by an equation of the form

$$L[x_o(t)] = x_i(t),$$

where L is some linear differential expression. Hence, by the principle of superposition, the response to a composite signal is the sum of responses to the components of the signal.

PROBLEMS

1. Find a particular solution of the form $x = A \cos \beta t + B \sin \beta t$ for the equation $(d^4x/dt^4) + cx = a \cos \beta t + b \sin \beta t$, and use the result to find a particular solution for each of the following equations.

(a) $\dfrac{d^4x}{dt^4} + x = \cos^2 \beta t.$

(b) $\dfrac{d^4x}{dt^4} + x = \sin^2 \beta t \cos^2 \beta t.$

(c) $\dfrac{d^4x}{dt^4} + cx = \dfrac{a_0}{2} + \displaystyle\sum_{k=1}^{n} (a_k \cos k\beta t + b_k \sin k\beta t); \quad c > 0.$

(d) $\dfrac{d^4x}{dt^4} + 8x = 4 \sin t \sin 2t \sin 3t.$

2. The equation $4x^2(d^2y/dx^2) + y = Ax^k$ has an integral of the form ax^k. Determine this and use the result to find a particular integral of the equation

$$4x^2 \frac{d^2y}{dx^2} + y = \frac{x^n - 1}{x - 1}.$$

Hint: Recall that $\dfrac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}$.

8. Algebra of Differential Operators. In the remainder of this chapter only linear differential equations with *constant* coefficients will be considered. These are equations of the form

$$(32) \qquad a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x),$$

where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are independent of x. As we shall see the solution of such equations can be constructed by algebraic operations and a finite number of quadratures (involving the function $f(x)$) alone.

The treatment of these equations is facilitated by an *operator algebra* which is the subject of this section. It concerns the differential operators $d/dx, d^2/dx^2, \ldots$, which will symbolically be denoted by $D, D^2, \ldots$. Their use is governed by a number of rules.

Definition 1. If the function $f(x)$ has a derivative of order k it may be "multiplied" from the left by the operator D^k, and the result $D^kf(x)$ represents d^kf/dx^k. D^0 is to represent the "identity operator," that is, $D^0f(x) = f(x)$.

This symbolic multiplication must be carefully distinguished from the actual multiplication of numbers by numbers or functions by functions, but its usefulness lies in the fact that it shares many properties with actual multiplication.

Rule 1. If c is a constant

$$D^k(cf) = cD^kf.$$

Rule 2. $$D^k(f_1 + f_2) = D^kf_1 + D^kf_2.$$

Rules 1 and 2 follow immediately from the most elementary properties of derivatives. Generally, operators for which rules 1 and 2 are valid are said to be *linear operators*. The operators D^k are special linear operators.

At this point a decisive difference between actual multiplication and the defined symbolic multiplication may be observed. Whereas it is true that $D(cf) = c\,Df$ if c is a constant, it is not true that $D(gf) = g\,Df$ if g is a function of x. For example,

$$D(e^x f) = \frac{d}{dx}(e^x f) = e^x\,Df + e^x f,$$

whereas

$$e^x\,Df = e^x\,Df.$$

Definition 2. *Polynomials* in the operator D may be formed:

$$P(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n D^0$$

whose meaning is defined by the equation

$$\begin{aligned} P(D)f(x) &= (a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n D^0)f(x) \\ &= a_0 \frac{d^n f}{dx^n} + a_1 \frac{d^{n-1}f}{dx^{n-1}} + \cdots + a_{n-1}\frac{df}{dx} + a_n f. \end{aligned}$$

Example 14

$$\begin{aligned} (D-1)^3 y &= (D - D^0)^3 y \\ &= (D^3 - 3D^2 + 3D - D^0)y \\ &= \frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y. \end{aligned}$$

Example 15

$$\begin{aligned} P(D)e^{qx} &= (a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n D^0)e^{qx} \\ &= a_0 \frac{d^n}{dx^n} e^{qx} + a_1 \frac{d^{n-1}}{dx^{n-1}} e^{qx} + \cdots + a_{n-1}\frac{d}{dx}e^{qx} + a_n e^{qx} \\ &= (a_0 q^n + a_1 q^{n-1} + \cdots + a_{n-1}q + a_n)e^{qx} \\ &= P(q)e^{qx}. \end{aligned}$$

Rule 3. Suppose $P_1(q)$, $P_2(q)$ are two polynomials, and

$$P_1(q)P_2(q) = Q(q).$$

Then

$$P_1(D)P_2(D)f = Q(D)f.$$

We shall prove this rule by an example which makes clear the procedure in the general case. Suppose

$$P_1(q) = a_1 q + b_1, \qquad P_2(q) = a_2 q + b_2.$$

Then

$$Q(q) = a_1 a_2 q^2 + (a_1 b_2 + a_2 b_1)q + b_1 b_2.$$

On the other hand,

$$\begin{aligned} P_1(D)P_2(D)f &= (a_1D + b_1)(a_2Df + b_2f) \\ &= (a_1D + b_1)\left(a_2\frac{df}{dx} + b_2f\right) \\ &= a_1D\left(a_2\frac{df}{dx} + b_2f\right) + b_1\left(a_2\frac{df}{dx} + b_2f\right) \\ &= a_1a_2\frac{d^2f}{dx^2} + a_1b_2\frac{df}{dx} + b_1a_2\frac{df}{dx} + b_1b_2f \\ &= (a_1a_2D^2 + (a_1b_2 + b_1a_2)D + b_1b_2)f \\ &= Q(D)f. \end{aligned}$$

Since every polynomial can be written as a product of linear factors the presented argument readily becomes a proof of rule 3 in general.

If an operator polynomial $P(D)$ is to be applied to a product of two functions $f(x)g(x)$, the result is, in general, quite complicated since every term a_kD^k in the polynomial requires k differentiations of the product $f(x)g(x)$. But for one kind of product, which will frequently occur in our applications, namely, $e^{rx}f(x)$, the result is surprisingly simple.

Rule 4

$$P(D)(e^{rx}f) = e^{rx}P(D + r)f.$$

To prove this rule we start again with a linear polynomial, namely, $P(D) = aD + b$. Then

$$\begin{aligned} P(D)(e^{rx}f) &= (aD + b)(e^{rx}f) \\ &= aD(e^{rx}f) + be^{rx}f \\ &= ae^{rx}Df + are^{rx}f + be^{rx}f \\ &= e^{rx}[a(D + r) + b]f \\ &= e^{rx}P(D + r)f. \end{aligned}$$

Now assume that $P(D)$ is a product of two linear operators, namely, $P(D) = P_1(D)P_2(D)$. Then, by rule 3 and by what we have proved so far,

$$\begin{aligned} P(D)(e^{rx}f) &= P_1(D)P_2(D)[e^{rx}f] = P_1(D)[e^{rx}P_2(D + r)f] \\ &= e^{rx}P_1(D + r)[P_2(D + r)f] \\ &= e^{rx}P_1(D + r)P_2(D + r)f \\ &= e^{rx}P(D + r)f. \end{aligned}$$

This can obviously be done for any number of linear factors; hence rule 4 is proved.

The formula in rule 4 can also be written as

$$e^{-rx}P(D)e^{rx}f = P(D + r)f.$$

The rule is often referred to as the *shifting rule.*

Example 16. Find $\dfrac{d^3}{dx^3}(e^{\alpha x}\cos\beta x)$.

By the shifting rule

$$\begin{aligned} D^3(e^{\alpha x}\cos\beta x) &= e^{\alpha x}(D + \alpha)^3\cos\beta x \\ &= e^{\alpha x}(D^3 + 3\alpha D^2 + 3\alpha^2 D + \alpha^3 D^0)\cos\beta x \\ &= e^{\alpha x}(\beta^3 - 3\alpha^2\beta)\sin\beta x + (\alpha^3 - 3\alpha\beta^2)\cos\beta x. \end{aligned}$$

Example 17. Find $(D + r)^k e^{qx}$, and $P(D + r)e^{qx}$ for any polynomial P. By the shifting rule

$$\begin{aligned} (D + r)^k e^{qx} &= e^{-rx}D^k(e^{rx}e^{qx}) \\ &= e^{-rx}\frac{d^k}{dx^k}e^{(r+q)x} \\ &= (r + q)^k e^{qx}. \end{aligned}$$

Since any polynomial $P(D + r)$ is a sum of powers $(D + r)^k$ multiplied by constants, it follows from the last result that

$$P(D + r)e^{qx} = P(q + r)e^{qx}.$$

Example 18. Find $e^{-sx}(D + r)^k(e^{sx}f(x))$ and $e^{-sx}P(D + r)(e^{sx}f(x))$ for any polynomial P.

By the shifting rule

$$\begin{aligned} e^{-sx}(D + r)^k(e^{sx}f) &= e^{-sx}e^{-rx}D^k(e^{rx}e^{sx}f) \\ &= e^{-(r+s)x}D^k(e^{(r+s)x}f) \\ &= (D + r + s)^k f. \end{aligned}$$

As in Example 17 we may conclude

$$e^{-sx}P(D + r)e^{sx}f = P(D + r + s)f.$$

9. Application to the Solution of Homogeneous Equations. We now return to the problem of solving the homogeneous differential equation with constant coefficients

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0 \qquad (a_0 \neq 0) \tag{33}$$

This equation may be written in the form

$$P(D)y = 0, \tag{34}$$

where $P(D)$ is the operator polynomial

$$P(D) = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_nD^0.$$

If we replace the operator D by an algebraic variable q, we obtain an ordinary polynomial $P(q)$ of degree n. Let us assume this polynomial is factored into two factors, say

$$P(q) = P_1(q)P_2(q).$$

Then, by rule 3 of the preceding section, Equation (34) may be replaced by either one of the equations

$$P_1(D)P_2(D)y = 0,$$
$$P_2(D)P_1(D)y = 0.$$

Now if $P_2(D)y = 0$, then $P_1(D)P_2(D)y = P_1(D)0 = 0$. Likewise, if $P_1(D)y = 0$, then $P_2(D)P_1(D)y = 0$. Hence, if $y(x)$ is a solution of either $P_1(D)y = 0$ or of $P_2(D)y = 0$, then $y(x)$ is a solution of the original equation $P(D)y = 0$. Thus, we have the important result.

If the polynomial $P_1(D)$ is a factor of $P(D)$, and if $P_1(D)y = 0$, then $P(D)y = 0$.

The practical use of this result lies, of course, in the fact that the equation $P_1(D)y = 0$ is of lower order than $P(D)y = 0$. Thus, integrals of the original equation $P(D)y = 0$ are obtained by solving differential equations of lower order, whose operators are factors of the operator polynomial $P(D)$. The method is illustrated by the following example.

Example 19. Find the general solution of the equation

(35) $$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$$

The polynomial $P(q) = q^3 - 2q^2 + q$ can be factored as

$$P(q) = q(q^2 - 2q + 1) = q(q - 1)^2.$$

Hence solutions of Equation (35) can be found by solving the equations

(36) $$Dy = 0,$$
(37) $$(D - 1)^2y = 0.$$

The equation $Dy = 0$ has the solution $y(x) = A$, where A is an arbitrary constant. Equation (37) can, by the shifting rule, be written as

$$e^xD^2(e^{-x}y) = 0$$

or

$$\frac{d^2}{dx^2}(e^{-x}y) = 0$$

whose general solution is

$$e^{-x}y(x) = Bx + C$$

or

$$y(x) = (Bx + C)e^x,$$

where B, C are arbitrary constants. Now, since the sum of any two solutions of a linear homogeneous equation is again a solution, we have the solution

$$y(x) = A + (Bx + C)e^x. \tag{38}$$

Since (38) contains the three linearly independent solutions $e^{0x} = 1$, e^x, xe^x (see the theorem of Sec. 4), it is the general solution of the third-order differential equation (35).

To make the procedure applied in the preceding example a general method for solving linear homogeneous differential equations with constant coefficients we first take up the question of how to factor the polynomial $P(q)$. By the *fundamental theorem of algebra*, a polynomial $P(q)$ of degree n whose highest coefficient is a_0 can be factored as follows:

$$P(q) = a_0(q - r_1)(q - r_2) \cdot \cdot \cdot (q - r_n), \tag{39}$$

where $r_1, r_2, \ldots, r_n$ are the roots of the equation $P(q) = 0$. Not all the roots are necessarily distinct, nor are all or any of them necessarily real numbers. If the same root c occurs k times in product (39), then the corresponding factors can be combined into the one factor

$$(q - c)^k. \tag{40}$$

If the polynomial $P(q)$ has real coefficients (as we shall always assume in the following), then if $(a + bi)$ is a complex root of $P(q) = 0$, so is $(a - bi)$, and the two factors $(q - a - bi)$ and $(q - a + bi)$ can be combined into the one quadratic factor $(q - a)^2 + b^2$. If the same pair of conjugate complex roots $(a + bi)$, $(a - bi)$ occurs l times in product (39), the corresponding quadratic factors can be combined into the one factor

$$[(q - a)^2 + b^2]^l. \tag{41}$$

Therefore, the *polynomial* $P(q)$ *can be written as a product of factors of the form* (40) *and* (41), *where these factors correspond to real roots and pairs of conjugate complex roots, respectively.*

After having solved the algebraic problem of factoring $P(q)$ it remains to solve the differential equations corresponding to each of the factors of form (40) or (41), that is, differential equations of the form

(42) $$(D - c)^k y = 0 \qquad (k = 1, 2, \ldots)$$

and

(43) $$[(D - a)^2 + b^2]^l y = 0 \qquad (l = 1, 2, \ldots)$$

By applying the shifting rule to Equation (42), this equation is changed into

$$e^{cx} D^k (e^{-cx} y) = 0,$$

or

$$\frac{d^k}{dx^k} (e^{-cx} y) = 0,$$

whose general solution is obtained by k-fold integration:

$$e^{-cx} y(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_{k-1} x^{k-1},$$

or

(44) $$y(x) = (C_0 + C_1 x + \cdots + C_{k-1} x^{k-1}) e^{cx},$$

where $C_0, C_1, \ldots, C_{k-1}$ are k arbitrary constants.

Equation (43) can be written as

$$[D - (a + bi)]^l [D - (a - bi)]^l y = 0,$$

and, by the preceding paragraph, we have the following solutions of this equation

$$(\alpha_0 + \alpha_1 x + \cdots + \alpha_{l-1} x^{l-1}) e^{(a+bi)x}$$

and

$$(\beta_0 + \beta_1 x + \cdots + \beta_{l-1} x^{l-1}) e^{(a-bi)x}.$$

Hence, the following linear combinations of these solutions are also solutions:

(45) $$e^{ax}[(A_0 + A_1 x + \cdots + A_{l-1} x^{l-1}) \cos bx + (B_0 + B_1 x + \cdots + B_{l-1} x^{l-1}) \sin bx],$$

where $A_0, A_1, \ldots, A_{l-1}, B_0, B_1, \ldots, B_{l-1}$ are arbitrary constants.

In summary, *for each factor of* $P(q)$ *of the form* (40) *we have the solution* (44), *and for each factor of* $P(q)$ *of the form* (41) *we have the solution* (45). It is seen that in each case the number of linearly independent integrals contained in a solution is equal to the degree of the corresponding factor. Hence, if for each factor of $P(q)$ the corresponding solution (44) or (45) is formed, and if these solutions are added, a solution of $P(D)y = 0$ is obtained which is a linear combination of n linearly independent solutions (see the theorem of Sec. 4) and, therefore, constitutes the general solution of this equation.

The problem of finding the general solution of a linear homogeneous equation with constant coefficients is so common and important in the theory of differential equations that the following theorem should be considered as basic for the whole theory.

Theorem. Let the polynomial $P(q)$ be the product of factors of the form

$$(q - c_j)^{k_j}$$

and

$$[(q - a_j)^2 + b_j^2]^{l_j},$$

where the c_j are the distinct real roots and the $a_j \pm ib_j$ are the distinct complex roots of $P(q) = 0$. For each factor of the first kind form the solution

$$(C_0 + C_1x + \cdot \cdot \cdot + C_{k_j-1}x^{k_j-1})e^{c_jx},$$

and for each factor of the second kind form the solution

$$e^{a_jx}[(A_0 + A_1x + \cdot \cdot \cdot + A_{l_j-1}x^{l_j-1}) \cos b_jx + (B_0 + B_1x + \cdot \cdot \cdot + B_{l_j-1}x^{l_j-1}) \sin b_jx],$$

where the A_i, B_i, C_i are arbitrary constants. Then the sum of all these particular solutions is the general solution of the linear homogeneous equation $P(D)y = 0$.

The algebraic equation $P(q) = 0$ is called the *auxiliary equation* belonging to the differential equation $P(D)y = 0$. By the foregoing theorem the problem of finding the general solution of a linear homogeneous differential equation with constant coefficients is reduced to the problem of determining the roots of the auxiliary equation. This is an algebraic problem and can be solved by any of the various algebraic, numerical, or graphical methods studied in the "Theory of Equations."†

Example 20. Find the general solution of the equation

$$\frac{d^6y}{dx^6} + 8\frac{d^3y}{dx^3} = 0. \tag{46}$$

† See, for example, Ref. 8, Chap. VIII.

In this case $P(q) = q^6 + 8q^3$, and this polynomial is easily factored:

$$P(q) = q^3(q^3 + 8) = q^3(q + 2)(q^2 - 2q + 4)$$
$$= q^3(q + 2)[(q - 1)^2 + (\sqrt{3})^2].$$

To the factor $(q - 0)^3$ belongs the solution $(C_0 + C_1x + C_2x^2)e^{0x}$, to $(q + 2)$ belongs C_3e^{-2x}, to $[(q - 1)^2 + (\sqrt{3})^2]$ belongs $[C_4 \cos (\sqrt{3}\,x) + C_5 \sin (\sqrt{3}\,x)]e^x$. Hence, the general solution of Equation (46) is

$$(47)\quad y(x) = C_0 + C_1x + C_2x^2 + C_3e^{-2x} + [C_4 \cos (\sqrt{3}\,x) + C_5 \sin (\sqrt{3}\,x)]e^x.$$

PROBLEMS

Find the general solutions of the following equations:

1. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} = 0.$

2. $\dfrac{d^3y}{dx^3} + 5\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = 0.$

3. $D^4y + 2D^3y + D^2y = 0.$

4. $(D^4 - 1)y = 0.$

5. $(D^4 + 1)y = 0.$

6. $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0.$

7. $(D^4 - D^3 - 3D^2 + 5D - 2)y = 0.$

8. $(D^4 + 2D^2 + 1)y = 0.$

9. $(D^5 + D^3 + D^2 + 1)\,y = 0.$

10. $(D^6 + 3D^4 + 3D^2 + 1)y = 0.$

11. $(D^n + aD^{n-1})y = 0.$

12. $(D^2 + 2aD + 2a^2)^n y = 0.$

13. Show that the solution of the equation $[(D - a)^2 + b^2]^k y = 0$ can be written in the form

$$y = e^{ax}[A_0 \sin (bx + \alpha_0) + A_1x \sin (bx + \alpha_1) + \cdots + A_{k-1}x^{k-1} \sin (bx + \alpha_{k-1})],$$

where $A_0, \ldots, A_{k-1}; \alpha_0, \ldots, \alpha_{k-1}$ are arbitrary constants.

14. Show that the solution of the equation $[(D - a)^2 - b^2]^k y = 0$ can be written in the form

$$y = e^{ax}[A_0 \sinh (bx + a_0) + A_1x \sinh (bx + a_1) + \cdots + A_{k-1}x^{k-1} \sinh (bx + a_{k-1})],$$

where $A_0, \ldots, A_{k-1}; a_0, \ldots, a_{k-1}$ are arbitrary constants.

10. Application to the Solution of Nonhomogeneous Equations. We next consider the solution of the nonhomogeneous equation

$$(48)\quad P(D)y = \frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = f(x),$$

where $f(x)$ is a given continuous function. In this case, too, the rules of operational calculus derived in Sec. 8 are of great help.

We first recall from Sec. 6 that the general solution of a nonhomogeneous linear equation can be obtained as the sum of any one particular solution and of the general solution of the reduced equation. Since we have dealt with the latter in the preceding section it remains to study methods for finding particular integrals of nonhomogeneous equations.

One device that is found convenient at times is an application of the principle of superposition (see Sec. 6). This is best illustrated by an example.

Example 21. Find a particular solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = x^2 - 17\cos 3x. \tag{49}$$

We split the right member into the parts x^2 and $-17\cos 3x$ and, correspondingly, put $y(x) = y_1(x) + y_2(x)$, where

$$\frac{d^2y_1}{dx^2} + 2\frac{dy_1}{dx} - y_1 = x^2$$
$$\frac{d^2y_2}{dx^2} + 2\frac{dy_2}{dx} - y_2 = -17\cos 3x.$$

To solve these equations one may use one of the methods to be developed below or, less formally, one may set up the trial solutions with undetermined coefficients

$$y_1(x) = ax^2 + bx + c$$
$$y_2(x) = A\cos 3x + B\sin 3x,$$

which suggest themselves on a little thought. Substitution in the respective equations and comparison of coefficients of like terms yield

$$y_1(x) = -x^2 - 4x - 10$$
$$y_2(x) = \tfrac{5}{4}\cos 3x - \tfrac{3}{4}\sin 3x.$$

Hence,

$$y(x) = -(x^2 + 4x + 10) + \tfrac{1}{4}(5\cos 3x - 3\sin 3x)$$

is a particular solution of Equation (49).

The method applied in the foregoing example can be used only when the right-hand term $f(x)$ of the nonhomogeneous equation is of a special type, that is, consists of sums and products of $x^n (n = 0, 1, 2, \ldots)$, e^{ax}, $\sin bx$, $\cos bx$ (see Sec. 7, Chap. IV).

A *general* method for determining a particular solution starts with factoring the operator polynomial $P(D)$, as it was done in the preceding section. Then Equation (48) takes the form

$$(50)\quad (D - c_1)^{k_1}(D - c_2)^{k_2} \cdots [(D - a_1)^2 + b_1^2]^{l_1}[(D - a_2)^2 + b_2^2]^{l_2} \cdots y = f(x),$$

where $c_1, c_2, \ldots$ are the distinct real roots; $a_1 \pm b_1 i, a_2 \pm b_2 i, \ldots$ are the distinct complex roots of the auxiliary equation $P(q) = 0$; and $k_1, k_2, \ldots, l_1, l_2, \ldots$ are their respective multiplicities. Let us write Equation (50) as

$$(51)\qquad (D - c_1)^{k_1} z_1 = f(x),$$

where $z_1(x)$ is the factor remaining of the left member of Equation (50) after $(D - c_1)^{k_1}$ is split off, that is,

$$(52)\quad (D - c_2)^{k_2} \cdots [(D - a_1)^2 + b_1^2]^{l_1}[(D - a_2)^2 + b_2^2]^{l_2} \cdots y = z_1(x)$$

We first determine $z_1(x)$ as a solution of Equation (51) and then substitute it in Equation (52). This equation is then similar to the original Equation (51), but it is of lower order (its order is $n - k_1$). After $z_1(x)$ has been determined the above process may be applied to Equation (52). We may, for example, rewrite (52) as

$$(53)\qquad [(D - a_1)^2 + b_1^2]^{l_1} z_2 = z_1(x),$$

where $z_2(x)$ is the factor remaining of the left member of Equation (52) after $[(D - a_1)^2 + b_1^2]^{l_1}$ is split off, that is,

$$(54)\quad (D - c_2)^{k_2} \cdots [(D - a_2)^2 + b_2^2]^{l_2} \cdots y = z_2(x).$$

After $z_2(x)$ is determined as a solution of Equation (53) and substituted in Equation (54), another equation is obtained of the general form of the original Equation (50), but its order is still further decreased (in our case it is $n - k_1 - 2l_1$). Continuing in this way, we eventually obtain a particular integral $y(x)$.

It is seen that each step in this reductive process requires a particular integral of an equation like (51) or of an equation like (53). We now consider these special equations.

The simplest equation of type (51) is obtained for $k_1 = 1$:

$$(55)\qquad (D - c)z = f(x),$$

Using the shifting rule, this equation may be written as

$$e^{cx}\,De^{-cx}z = f(x)$$

or

$$\frac{d}{dx}\,(e^{-cx}z) = e^{-cx}f(x).$$

Hence,

$$e^{-cx}z(x) = \int^x e^{-cx_1}f(x_1)\,dx_1$$

and

(56)
$$z(x) = e^{cx}\int^x e^{-cx_1}f(x_1)\,dx_1 = \int^x e^{c(x-x_1)}f(x_1)\,dx_1$$

is a solution. Since we are looking for a particular solution only, any lower limit may be chosen in the integral of (56).

Turning to the more general equation

(57)
$$(D - c)^k z = f(x)$$

we obtain, again by the shifting theorem,

$$e^{cx}D^k e^{-cx}z = f(x)$$

or

$$\frac{d^k}{dx^k}\,(e^{-cx}z) = e^{-cx}f(x).$$

Hence,

$$e^{-cx}z(x) = \int^x dx_k \int^{x_k} dx_{k-1} \int^{x_{k-1}} \cdots dx_2 \int^{x_2} e^{-cx_1}f(x_1)\,dx_1$$

and

(58)
$$z(x) = \int^x dx_k \int^{x_k} dx_{k-1} \int^{x_{k-1}} \cdots dx_2 \int^{x_2} e^{c(x-x_1)}f(x_1)\,dx_1$$

is a solution. If we choose, in particular, the same lower limits in all the integrals of (58), then the k-fold integral can be converted into a simple integral [see formula (29), Sec. 6]:

(59)
$$z(x) = \int^x \frac{(x - x_1)^{k-1}}{(k-1)!}\,e^{c(x-x_1)}f(x_1)\,dx_1.$$

Next, we consider equations of the type (53), starting with the case $l_1 = 1$,

(60)
$$[(D - a)^2 + b^2]z = f(x).$$

This equation may be rewritten as

$$(D - a - bi)(D - a + bi)z = f(x)$$

and becomes on application of the shifting rule

$$e^{(a+bi)x}D[e^{-(a+bi)x}e^{(a-bi)x}D(e^{-(a-bi)x}z)] = f(x)$$

or

$$D[e^{-2bix}D(e^{-(a-bi)x}z)] = e^{-(a+bi)x}f(x).$$

Hence,

$$e^{-2bix}D(e^{-(a-bi)x}z) = \int^x e^{-(a+bi)x_1}f(x_1)\,dx_1$$

or

$$D(e^{-(a-bi)x}z) = e^{2bix}\int^x e^{-(a+bi)x_1}f(x_1)\,dx_1.$$

Hence

$$e^{-(a-bi)x}z(x) = \int^x e^{2bix_2}dx_2\int^{x_2} e^{-(a+bi)x_1}f(x_1)\,dx_1$$

or

$$z(x) = e^{(a-bi)x}\int^x e^{2bix_2}dx_2\int^{x_2} e^{-(a+bi)x_1}f(x_1)\,dx_1.$$

In this iterated integral the first integration is done over the range $x_1 \leqq x_2$, the second integration over $x_2 \leqq x$. If we invert the order of integration, then the first integral extends over $x_1 \leqq x_2 \leqq x$, and the second integral over $x_1 \leqq x$. Hence, we obtain

$$z(x) = e^{(a-bi)x}\int^x e^{-(a+bi)x_1}f(x_1)\,dx_1\int_{x_1}^x e^{2bix_2}dx_2$$

and, since

$$\int_{x_1}^x e^{2bix_2}dx_2 = \frac{1}{2bi}\,(e^{2bix} - e^{2bix_1})$$

we find

(61) $$z(x) = \frac{1}{2bi}\int^x (e^{(a+bi)(x-x_1)} - e^{(a-bi)(x-x_1)})f(x_1)\,dx_1$$
$$= \frac{1}{b}\int^x e^{a(x-x_1)}\sin b(x - x_1)f(x_1)\,dx_1.$$

The more general equation

(62) $$[(D - a)^2 + b^2]^l z = f(x)$$

can be solved by applying the method outlined in the preceding paragraph l times in succession. To illustrate the procedure we work out the solution for $l = 2$.

Writing the equation as

(63) $$[(D - a)^2 + b^2]\{[(D - a)^2 + b^2]z\} = f(x)$$

it takes the form of Equation (60) with the expression in braces as unknown. Hence, by (61),

$$[(D - a)^2 + b^2]z = \frac{1}{b}\int^x e^{a(x-x_1)} \sin b(x - x_1) f(x_1)\, dx_1. \tag{64}$$

This is another equation of type (60), with $f(x)$ replaced by the integral on the right side. Therefore, by (61),

$$z(x) = \frac{1}{b^2}\int^x e^{a(x-x_2)} \sin b(x - x_2)\, dx_2 \int^{x_2} e^{a(x_2-x_1)} \sin b(x_2 - x_1) f(x_1)\, dx_1. \tag{65}$$

By inverting the order of integration as we did above, we obtain

$$z(x) = \frac{1}{b^2}\int^x e^{a(x-x_1)} f(x_1)\, dx_1 \int_{x_1}^x \sin b(x - x_2) \sin b(x_2 - x_1)\, dx_2,$$

and since

$$\int_{x_1}^x \sin b(x - x_2) \sin b(x_2 - x_1)\, dx_2 = \frac{1}{2b}[\sin b(x - x_1) - b(x - x_1) \cos b(x - x_1)],$$

we have

$$z(x) = \frac{1}{2b^2}\int^x e^{a(x-x_1)} [\sin b(x - x_1) - b(x - x_1) \cos b(x - x_1)] f(x_1)\, dx_1. \tag{66}$$

An examination of the procedure shows that a particular integral of the general Equation (62) is

$$z(x) = \frac{1}{b^l}\int^x e^{a(x-x_1)} S(x - x_1) f(x_1)\, dx_1, \tag{67}$$

where $S(x - x_1)$ is an abbreviation for the $(l - 1)$ times repeated integral

$$S(x - x_1) = \int_{x_1}^x dx_2 \int_{x_2}^x dx_3 \cdots \int_{x_{l-1}}^x dx_l \sin b(x - x_l) \sin b(x_l - x_{l-1}) \cdots \sin b(x_2 - x_1).$$

Example 22. Find the general solution of the equation

$$\frac{d^6y}{dx^6} + 8\frac{d^3y}{dx^3} = Ae^x. \tag{68}$$

The complementary function for this equation was found in Example 20, Sec. 9:

$$y_0(x) = C_0 + C_1x + C_2x^2 + C_3e^{-2x} + (C_4 \cos \sqrt{3}\, x + C_5 \sin \sqrt{3}\, x)e^x.$$

To find a particular solution of (68) we write the left member in factored form:

$$(D + 2)D^3[(D - 1)^2 + 3]y = Ae^x. \tag{69}$$

Applying (59), we have

$$D^3[(D - 1)^2 + 3]y = A \int^x e^{-2(x-x_1)} e^{x_1} dx_1$$
$$= \frac{A}{3} e^x.$$

Applying (59) once more, we find

$$[(D - 1)^2 + 3]y = \frac{A}{3} \int^x \frac{(x - x_1)^2}{2} e^{x_1} dx_1$$
$$= \frac{A}{3} e^x.$$

Now we apply (61) and obtain

$$y(x) = \frac{A}{3\sqrt{3}} \int^x \sin \sqrt{3}(x - x_1) e^{x-x_1} e^{x_1} dx_1$$
$$= \frac{A}{9} e^x.$$

Hence, the general solution of Equation (68) is

$$y(x) = C_0 + C_1 x + C_2 x^2 + C_3 e^{-2x} + \left(\frac{A}{9} + C_4 \cos \sqrt{3}\, x + C_5 \sin \sqrt{3}\, x\right) e^x.$$

Example 23. Find the general solution of the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = \frac{3x^2 - 8x + 3}{(1 - x)^3}. \tag{70}$$

The factored form of this equation is

$$(D - 3)(D + 1)y = \frac{3x^2 - 8x + 3}{(1 - x)^3}.$$

Applying (59), we have

$$(D + 1)y = \int^x e^{3(x-x_1)} \frac{3x_1^2 - 8x_1 + 3}{(1 - x_1)^3} dx_1. \tag{71}$$

It would be futile to try to carry out this quadrature. In spite of this there exists a simple solution of Equation (70) as will be seen in a moment. Applying (59) again to Equation (71), we find

$$y(x) = \int^x e^{-(x-x_2)} dx_2 \int^{x_2} e^{3(x_2-x_1)} \frac{3x_1^2 - 8x_1 + 3}{(1 - x_1)^3} dx_1.$$

If we now invert the order of integration, we have

$$y(x) = \int^x e^{-3x_1} \frac{3x_1^2 - 8x_1 + 3}{(1 - x_1)^3} \, dx_1 \int_{x_1}^x e^{4x_2 - x} dx_2$$
$$= \frac{1}{4} \int^x \frac{3x_1^2 - 8x_1 + 3}{(1 - x_1)^3} \left(e^{3(x-x_1)} - e^{-(x-x_1)}\right) dx_1.$$

This integral can be evaluated. To do this, the fraction is written as a sum of partial fractions

$$\frac{3x_1^2 - 8x_1 + 3}{(1 - x_1)^3} = -\frac{2}{(1 - x_1)^3} + \frac{2}{(1 - x_1)^2} + \frac{3}{1 - x_1},$$

and integration by parts gives then

$$y(x) = \frac{1}{4}\left[-\frac{1}{(1 - x)^2} + \frac{2}{1 - x}\right](e^0 - e^0)$$
$$- \frac{1}{4}\int^x \left[\frac{1}{(1 - x_1)^2} - \frac{2}{1 - x_1} + \frac{3}{1 - x_1}\right][3e^{3(x-x_1)} + e^{-(x-x_1)}] \, dx_1.$$

Another integration by parts finishes the job

$$y(x) = -\frac{1}{4}\frac{1}{1 - x}(3e^0 + e^0)$$
$$+ \frac{1}{4}\int^x \left(-\frac{1}{1 - x_1} + \frac{1}{1 - x_1}\right)(9e^{3(x-x_1)} - e^{-(x-x_1)}) \, dx_1$$
$$= \frac{1}{x - 1}.$$

Adding the complementary function to this particular integral, we have as the general solution of Equation (70)

$$y(x) = \frac{1}{x - 1} + C_1 e^{3x} + C_2 e^{-x}.$$

It is apparent from the general discussion as well as from the examples that each linear factor in the operator polynomial $P(D)$ gives rise to one quadrature. The repeated integrals that occur in the above method can always be reduced to a simple integral in the same way as it was done for the repeated integrals (58) and (65). This procedure is important in cases where the integrals cannot easily be evaluated in closed form (for example, when $f(x)$ is given in tabulated or graphical form only).

We carry out the process for the case of three distinct real roots. Then the equation is of the form

$$(D - r_1)(D - r_2)(D - r_3)y = f(x) \qquad (r_1 \neq r_2 \neq r_3) \tag{72}$$

Applying formula (57) three times in succession, we find

$$(D - r_2)(D - r_3)y = \int^x e^{r_1(x-x_1)}f(x_1)\,dx_1$$

$$(D - r_3)y = \int^x e^{r_2(x-x_2)}dx_2 \int^{x_2} e^{r_1(x_2-x_1)}f(x_1)\,dx_1$$

$$y(x) = \int^x e^{r_3(x-x_3)}dx_3 \int^{x_3} e^{r_2(x_3-x_2)}dx_2 \int^{x_2} e^{r_1(x_2-x_1)}f(x_1)\,dx_1.$$

By inverting the order of integration we find

$$y(x) = e^{r_3x}\int^x e^{-r_1x_1}f(x_1)\,dx_1 \int_{x_1}^x e^{(r_1-r_2)x_2}dx_2 \int_{x_2}^x e^{(r_2-r_3)x_3}dx_3.$$

But

$$e^{r_3x}\int_{x_1}^x e^{(r_1-r_2)x_2}dx_2 \int_{x_2}^x e^{(r_2-r_3)x_3}dx_3 = e^{r_1x}\left[\frac{e^{r_1(x-x_1)}}{(r_1 - r_2)(r_1 - r_3)} + \frac{e^{r_2(x-x_1)}}{(r_2 - r_1)(r_2 - r_3)} + \frac{e^{r_3(x-x_1)}}{(r_3 - r_1)(r_3 - r_2)}\right].$$

Hence, we have the solution

$$y(x) = \int^x \left[\frac{e^{r_1(x-x_1)}}{(r_1 - r_2)(r_1 - r_3)} + \frac{e^{r_2(x-x_1)}}{(r_2 - r_1)(r_2 - r_3)} + \frac{e^{r_3(x-x_1)}}{(r_3 - r_2)(r_3 - r_1)}\right] f(x_1)\,dx_1. \tag{73}$$

It is easily seen how this result is generalized to the case of the equation

$$(D - r_1)(D - r_2) \cdots (D - r_n)y = f(x) \qquad (r_1 \neq r_2 \neq r_3 \neq \cdots \neq r_n) \tag{74}$$

The corresponding solution is

$$y(x) = \int^x \left[\frac{e^{r_1(x-x_1)}}{(r_1 - r_2)(r_1 - r_3) \cdots (r_1 - r_n)} + \frac{e^{r_2(x-x_1)}}{(r_2 - r_1)(r_2 - r_3) \cdots (r_2 - r_n)} + \cdots + \frac{e^{r_n(x-x_1)}}{(r_n - r_1)(r_n - r_2) \cdots (r_n - r_{n-1})}\right] f(x_1)\,dx_1. \tag{75}$$

PROBLEMS

By the method of this section find a particular or general solution to each of the following equations:

1. $(D + 1)^3y = x^3e^{-x}$.
2. $(D - 1)^3y = x^{-3}e^x$.
3. $(D^3 + 3D^2 - 4D - 12)y = x^2e^{2x}$.
4. $(D^3 + 1)y = e^{-x}\sin x$.

5. $(D^4 - 1)y = e^x \cos x$.
6. $(D^4 + 2D^2 + 1)y = x \sin x$.
7. $(D - a)(D - b)(D - c)y = Ae^{ax}$; $a \neq b \neq c$.
8. $(D - a)^2(D - b)y = Ae^{ax}$; $a \neq b$.
9. $(D - a)^2(D - b)^2y = Ae^{ax}$; $a \neq b$.
10. $(D - r_1)(D - r_2) \cdots (D - r_n)y = Ae^{sx}$; $r_1 \neq r_2 \neq r_3 \neq \cdots \neq r_n \neq s$.
11. $(D^2 + a^2)(D^2 + b^2)y = A \sin ax$; $|a| \neq |b|$ and $a \neq 0$, $b \neq 0$.
12. $(D^2 + a^2)^2y = A \sin ax$; $a \neq 0$.

Find a particular solution to each of the following equations in the form of a simple integral:

13. $(D^3 - D)y = f(x)$.
14. $(D^3 + D)y = f(x)$.
*15. $(D - a)^2(D - b)^2y = f(x)$; $a \neq b$.
*16. $(D^2 + a^2)(D^2 + b^2)y = f(x)$; $|a| \neq |b|$ and $a \neq 0$, $b \neq 0$.
*17. $(D^n - aD^{n-1})y = f(x)$; $a \neq 0$.

11. Partial Fraction Decomposition of $1/P(D)$. In the following an alternative method for determining a particular integral of a nonhomogeneous equation is described, which has the advantage that the successive integrations involved in the first method are replaced by a sum of simple integrals. We consider again Equation (48) in its factored form (50), but for simplicity we consider a more specific example:

$$(76) \qquad (D - a)^\alpha (D - b)^\beta [(D - c)^2 + d^2]^\gamma y = f(x).$$

We apply the methods of *partial fraction decomposition* studied in algebra and calculus to the fraction

$$\frac{1}{(q - a)^\alpha (q - b)^\beta [(q - c)^2 + d^2]^\gamma}.$$

Those methods enable us to find constants $A_0, A_1, \ldots, A_{\alpha-1}, B_0, B_1, \ldots, B_{\beta-1}, C_0, C_1, \ldots, C_{2\gamma-1}$ such that

$$\frac{1}{(q - a)^\alpha (q - b)^\beta [(q - c)^2 + d^2]^\gamma} = \frac{A_0 + A_1 q + \cdots + A_{\alpha-1} q^{\alpha-1}}{(q - a)^\alpha} + \frac{B_0 + B_1 q + \cdots + B_{\beta-1} q^{\beta-1}}{(q - b)^\beta} + \frac{C_0 + C_1 q + \cdots + C_{2\gamma-1} q^{2\gamma-1}}{[(q - c)^2 + d^2]^\gamma}.$$

This last equation when cleared of fractions becomes

$$(77) \quad \begin{aligned} 1 = {} & (A_0 + A_1 q + \cdots + A_{\alpha-1} q^{\alpha-1})(q - b)^\beta [(q - c)^2 + d^2]^\gamma \\ & + (B_0 + B_1 q + \cdots + B_{\beta-1} q^{\beta-1})(q - a)^\alpha [(q - c)^2 + d^2]^\gamma \\ & + (C_0 + C_1 q + \cdots + C_{2\gamma-1} q^{2\gamma-1})(q - a)^\alpha (q - b)^\beta. \end{aligned}$$

This is an identity in polynomials of q and, therefore, also holds for the corresponding operator polynomials (see rule 3, Sec. 8). Therefore,

$$(78) \qquad 1 \cdot f(x) = f_a(x) + f_b(x) + f_c(x),$$

where

$$
\begin{aligned}
&f_a(x) = (A_0 + A_1D + \cdots + A_{\alpha-1}D^{\alpha-1})(D-b)^\beta[(D-c)^2 + d^2]^\gamma f(x),\\
(79)\quad &f_b(x) = (B_0 + B_1D + \cdots + B_{\beta-1}D^{\beta-1})(D-a)^\alpha[(D-c)^2 + d^2]^\gamma f(x),\\
&f_c(x) = (C_0 + C_1D + \cdots + C_{2\gamma-1}D^{2\gamma-1})(D-a)^\alpha(D-b)^\beta f(x).
\end{aligned}
$$

Thus, the right member of Equation (76) is now decomposed into three parts. By the principle of superposition, if we put

$$(80)\qquad y(x) = y_a(x) + y_b(x) + y_c(x),$$

Equation (76) breaks up into the three equations

$$
\begin{aligned}
(D-a)^\alpha(D-b)^\beta[(D-c)^2 + d^2]^\gamma y_a &= f_a(x)\\
&= (A_0 + A_1D + \cdots + A_{\alpha-1}D^{\alpha-1})(D-b)^\beta[(D-c)^2 + d^2]^\gamma f(x)\\
(D-a)^\alpha(D-b)^\beta[(D-c)^2 + d^2]^\gamma y_b &= f_b(x)\\
&= (B_0 + B_1D + \cdots + B_{\beta-1}D^{\beta-1})(D-a)^\alpha[(D-c)^2 + d^2]^\gamma f(x)\\
(D-a)^\alpha(D-b)^\beta[(D-c)^2 + d^2]^\gamma y_c &= f_c(x)\\
&= (C_0 + C_1D + \cdots + C_{2\gamma-1}D^{2\gamma-1})(D-a)^\alpha(D-b)^\beta f(x).
\end{aligned}
$$

The first of these equations becomes, after dropping the factors

$$(D-b)^\beta[(D-c)^2 + d^2]^\gamma$$

on both sides,

$$(D-a)^\alpha y_a(x) = (A_0 + A_1D + \cdots + A_{\alpha-1}D^{\alpha-1})f(x)$$

and a solution of it is, by (59),

$$y_a(x) = \int^x \frac{(x-x_1)^{\alpha-1}e^{a(x-x_1)}}{(\alpha-1)!}(A_0 + A_1D + \cdots + A_{\alpha-1}D^{\alpha-1})f(x_1)\,dx_1.$$

The remaining two equations are solved in the same manner.

Example 24. Solve Equation (69) of Example 22 by the partial fractions method. We first find the coefficients in the expansion

$$\frac{1}{q^6 + 8q^3} = \frac{A_0 + A_1q + A_2q^2}{q^3} + \frac{B_0}{q+2} + \frac{C_0 + C_1q}{(q-1)^2 + 3}.$$

Multiplication of this equation by $q^6 + 8q^3$ and equating coefficients of like powers of q yield

$$A_0 = \tfrac{1}{8},\ A_1 = A_2 = 0,\ B_0 = -\tfrac{1}{96},\ C_0 = -\tfrac{1}{24},\ C_1 = \tfrac{1}{96}.$$

Therefore,

$$1 = \frac{1}{8}(q+2)[(q-1)^2 + 3] - \frac{1}{96}q^3[(q-1)^2 + 3] + q^3(q+2)\left(\frac{q}{96} - \frac{1}{24}\right).$$

This identity also holds when q is replaced by the differential operator D. When applied to the right member of Equation (69), this becomes

$$
\begin{aligned}
Ae^x = \tfrac{1}{8}(D+2)[(D-1)^2 + 3]Ae^x - \tfrac{1}{96}D^3[(D-1)^2 + 3]Ae^x&\\
+ D^3(D+2)(\tfrac{1}{96}D - \tfrac{1}{24})Ae^x&.
\end{aligned}
$$

Putting $y(x) = y_a(x) + y_b(x) + y_c(x)$, we then have to solve

$$D^3(D + 2)[(D - 1)^2 + 3]y_a = \tfrac{1}{8}(D + 2)[(D - 1)^2 + 3]Ae^x,$$

that is,

$$D^3y_a = \tfrac{1}{8}Ae^x,$$

one of whose solutions is $y_a(x) = \frac{1}{8}Ae^x$.

Likewise,

$$D^3(D + 2)[(D - 1)^2 + 3]y_b = -\tfrac{1}{96}D^3[(D - 1)^2 + 3]Ae^x,$$

that is,

$$(D + 2)y_b = -\tfrac{1}{96}Ae^x,$$

one of whose solutions is

$$y_b(x) = -\frac{A}{288}e^x.$$

Finally,

$$D^3(D + 2)[(D - 1)^2 + 3]y_c = D^3(D + 2)(\tfrac{1}{96}D - \tfrac{1}{24})Ae^x,$$

that is,

$$[(D - 1)^2 + 3]y_c = (\tfrac{1}{96}D - \tfrac{1}{24})Ae^x = -\tfrac{1}{32}Ae^x,$$

one of whose solutions is

$$y_c(x) = -\frac{A}{96}e^x.$$

Thus, we find the particular integral of Equation (69)

$$y(x) = \left(\frac{1}{8} - \frac{1}{288} - \frac{1}{96}\right)Ae^x = \frac{A}{9}e^x,$$

the same particular integral that we found in Example 22.

PROBLEMS

By the method of this section solve Probs. 3–11 and 13–16 of Sec. 10.

12. Particular Integrals in Special Cases. In some special cases that are not uncommon in applications, particular solutions of nonhomogeneous equations with constant coefficients can be found without any integrations. First, this can always be done when the right member of the equation is a polynomial function. We then deal with a differential equation of the form

$$(81)\quad P(D)y \equiv (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = A_0x^k + A_1x^{k-1} + \cdots + A_k.$$

A little reflection leads one to expect that this equation has a polynomial among its solutions and that the degree of this polynomial is

k provided that $a_n \neq 0$. Let us make, for the moment, the assumption $a_n \neq 0$ and let us put

$$y(x) = C_0x^k + C_1x^{k-1} + \cdots + C_{k-1}x + C_k. \tag{82}$$

We then have to determine the coefficients $C_0, C_1, \ldots, C_k$ such that (82) becomes a solution of Equation (81). This is done by carrying out all the differentiations required in Equation (81), substituting in (81), and equating the coefficients of like powers of x. For illustration, we carry out a few steps:

$$\begin{aligned} a_n y &= a_nC_0x^k + a_nC_1x^{k-1} + \cdots + a_nC_k \\ a_{n-1}Dy &= a_{n-1}kC_0x^{k-1} + \cdots + a_{n-1}C_{k-1} \\ a_{n-2}D^2y &= a_{n-2}k(k-1)C_0x^{k-2} + \cdots + 2a_{n-2}C_{k-2}, \end{aligned}$$

etc. Substitution in Equation (81) and equating the coefficients of $x^k, x^{k-1}, x^{k-2}, \ldots$ yield

$$\begin{aligned} a_nC_0 &= A_0 \\ a_nC_1 + a_{n-1}kC_0 &= A_1 \\ a_nC_2 + a_{n-1}(k-1)C_1 + a_{n-2}k(k-1)C_0 &= A_2, \end{aligned}$$

etc. Since, by assumption, $a_n \neq 0$ it is evident from these equations that $C_0, C_1, C_2, \ldots$ can be successively determined. This means that a solution of form (82) can actually be found.

Example 25. Find a particular integral of the equation

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + 16y = 16x^2 + 258. \tag{83}$$

Here $a_n = 16 \neq 0$. Hence, there must exist a solution of the form

$$y(x) = C_0x^2 + C_1x + C_2.$$

Then

$$\begin{aligned} Dy &= 2C_0x + C_1 \\ D^2y &= 2C_2 \\ D^3y &= D^4y = 0. \end{aligned}$$

Hence, substitution in (83) results in

$$16C_0x^2 + 16C_1x + (16C_2 + 2C_0) = 16x^2 + 258.$$

Therefore,

$$\begin{aligned} 16C_0 &= 16, & C_0 &= 1 \\ 16C_1 &= 0, & C_1 &= 0 \\ 16C_2 + 2C_0 &= 258, & C_2 &= 16. \end{aligned}$$

Hence, $y(x) = x^2 + 16$ is the desired integral.

If it happens that $a_n = 0$, the procedure must be slightly modified. Then Equation (81) is written in the form

$$(84)\quad (a_0D^{n-1} + a_1D^{n-2} + \cdots + a_{n-1})Dy = A_0x^k + A_1x^{k-1} + \cdots + A_k.$$

This is an equation just like (81), but now for the function Dy. Hence, if $a_{n-1} \neq 0$, then a solution

$$Dy = C_0x^k + C_1x^{k-1} + \cdots + C_k$$

is found as before, and $y(x)$ itself is obtained by a simple quadrature. If a_{n-1} is also zero, then Equation (81) can be considered as an equation in D^2y, and so forth.

Example 26. Find a particular integral of the equation

$$(85)\quad \frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = x^6 - 360x^2.$$

This equation may be written as

$$(86)\quad (D^2 - 1)D^2y = x^6 - 360x^2.$$

Hence, there must be a solution in which

$$D^2y = A_0x^6 + A_1x^5 + A_2x^4 + A_3x^3 + A_4x^2 + A_5x + A_6.$$

Then

$$(87)\quad (D^2 - 1)(D^2y) = -A_0x^6 - A_1x^5 + (30A_0 - A_2)x^4 + (20A_1 - A_3)x^3 + (12A_2 - A_4)x^2 + (6A_3 - A_5)x + (2A_4 - A_6).$$

Equating the coefficients of the right members of (86) and (87) yields

$$\begin{aligned} -A_0 &= 1, & A_0 &= -1 \\ -A_1 &= 0, & A_1 &= 0 \\ 30A_0 - A_2 &= 0, & A_2 &= -30 \\ 20A_1 - A_3 &= 0, & A_3 &= 0 \\ 12A_2 - A_4 &= -360, & A_4 &= 0 \\ 6A_3 - A_5 &= 0, & A_5 &= 0 \\ 2A_4 - A_6 &= 0, & A_6 &= 0. \end{aligned}$$

Therefore,

$$D^2y = -x^6 - 30x^4,$$

and two integrations give the particular solution of Equation (85)

$$y(x) = -\tfrac{1}{56}x^8 - x^6.$$

Another class of nonhomogeneous equations for which a particular integral can be easily determined are those of the form

$$(88)\qquad P(D)y = (A_0x^k + A_1x^{k-1} + \cdots + A_{k-1}x + A_k)e^{cx}$$

where c is a real or complex constant. Multiplying Equation (88) by e^{-cx} and applying the shifting rule, we get

$$(89)\qquad P(D + c)(e^{-cx}y) = A_0x^k + A_1x^{k-1} + \cdots + A_k.$$

This is an equation of the form (81) for the unknown function $e^{-cx}y(x)$. Therefore, it can be solved in the same way.

If the right-hand member of the nonhomogeneous equation consists of several summands of the form $Q_i(x)e^{c_ix}$, where $Q_i(x)$ is a polynomial, then a solution is found for each summand and the sum of the solutions thus obtained is, by the principle of superposition, a solution of the given equation.

Next, we turn to equations of the types

$$(90)\qquad P(D)y = (A_0x^k + A_1x^{k-1} + \cdots + A_k)e^{ax}\cos bx,$$
$$(91)\qquad P(D)y = (A_0x^k + A_1x^{k-1} + \cdots + A_k)e^{ax}\sin bx.$$

These equations can be written as†

$$(90)\quad P(D)y = \operatorname{Re}\,(A_0x^k + A_1x^{k-1} + \cdots + A_k)e^{(a+bi)x},$$
$$(91)\quad P(D)y = \operatorname{Im}\,(A_0x^k + A_1x^{k-1} + \cdots + A_k)e^{(a+bi)x},$$

respectively. Therefore, if the equation

$$P(D)y = (A_0x^k + A_1x^{k-1} + \cdots + A_k)e^{cx}\quad (c = a + bi)$$

is solved by the above method, the real part of this solution is an integral of Equation (90), and the imaginary part is an integral of Equation (91).

Example 27. Find a particular integral of Equation (69) of Example 22, Sec. 10, by the method of this section.

Dividing the equation

$$(D^6 + 8D^3)y = Ae^x$$

by e^x, we have

$$e^{-x}(D^6 + 8D^3)y = A$$

or, by the shifting rule,

$$[(D + 1)^6 + 8(D + 1)^3](e^{-x}y) = A.$$

† Re (read: real part), Im (read: imaginary part) of a complex number $u = a + bi$ are defined as

$$\operatorname{Re} u = \operatorname{Re}\,(a + bi) = a,\ \operatorname{Im} u = \operatorname{Im}\,(a + bi) = b.$$

Since the right-hand member is a polynomial of degree zero, there must be a solution of the form

$$e^{-x}y = C.$$

Substitution gives

$$9C = A, \qquad C = \frac{A}{9}.$$

Hence, $y = (A/9)e^x$ is a particular solution of Equation (69), the same as found in Example 22, Sec. 10, and Example 24, Sec. 11. It is apparent that the method of this section is preferable to the other methods when it can be applied.

Example 28. Find a particular integral of the equation

(92) $$(D^3 - D)y = A \sin bx \qquad (b \neq 0)$$

Since $\sin bx = \text{Im } e^{ibx}$, we consider the equation

$$(D^3 - D)z = Ae^{ibx}.$$

Multiplication by e^{-ibx} leads to

$$[(D + ib)^3 - (D + ib)](e^{-ibx}z) = A.$$

There must be a solution of the form

$$e^{-ibx}z(x) = C.$$

Substitution gives

$$C = \frac{A}{b(b^2 + 1)}\, i.$$

Hence,

$$z(x) = \frac{A}{b(b^2 + 1)}\, ie^{ibx} = \frac{A}{b(b^2 + 1)}\,(i \cos bx - \sin bx)$$

and

$$y(x) = \text{Im } z(x) = \frac{A}{b(b^2 + 1)} \cos bx$$

is a particular integral of Equation (92).

PROBLEMS

By the method of this section find particular integrals to the following equations:

1. $(D^3 + D^2 - D - 1)y = x^4 + 4x^3.$
2. $(a_0D^n + a_1D^{n-1} + \cdots + a_{n-3}D^3 + D^2 - 3D + 1)y = x^2 - 2.$
3. $(D^6 + 8D^3)y = 960x^3.$
4. $(D^6 + 8D^3)y = Ae^x.$

5. $(D + r)^n y = Ae^{sx}$.
6. $(D + 1)^3 y = x^3 e^{-x}$.
7. $(D^3 + 1)y = x^3 e^{-x}$.
8. $(D^3 + D^2 + 9D + 1)y = 6 \sin 3x$.
9. $(D^4 - 1)y = e^x \cos x$.
10. $(D^4 + 2D^2 + 1)y = x \sin x$.

Solve Probs. 7–12 of Sec. 10 by the method of this section.

13. Integrals Satisfying Given Initial Conditions. In many differential equation problems, especially those arising from applications, it is not the general solution that is required, but a particular solution that satisfies given initial conditions. In such problems the general solution may be used as an auxiliary tool for obtaining the desired particular solution. By making use of the initial conditions, linear algebraic equations are derived for the constants of integration that occur in the general solution. From these equations the constants are determined and the desired solution is then found. This procedure is illustrated by the following example.

Example 29. Find the solution of

$$\frac{d^6y}{dx^6} + 8\,\frac{d^3y}{dx^3} = Ae^x \tag{93}$$

that satisfies the initial conditions

$$y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = 0, \qquad y^{(5)}(0) = -A. \tag{94}$$

The general solution of Equation (93) was found in Example 22, Sec. 10. The necessary differentiations become easier if this solution is written in exponential form

$$y(x) = C_0 + C_1x + C_2x^2 + C_3e^{-2x} + C_4e^{(1+i\sqrt{3})x} + C_5e^{(1-i\sqrt{3})x} + \frac{A}{9}\,e^x.$$

Then on differentiating five times in succession and on substituting $x = 0$ in each of the equations obtained, conditions (94) become

$$\begin{aligned}
C_0 + C_3 + C_4 + C_5 &= -\frac{A}{9}\\
C_1 + 2C_3 + (1 + i\sqrt{3})C_4 + (1 - i\sqrt{3})C_5 &= -\frac{A}{9}\\
2C_2 + 4C_3 - 2(1 - i\sqrt{3})C_4 - 2(1 + i\sqrt{3})C_5 &= -\frac{A}{9}\\
-8C_3 - 8C_4 - 8C_5 &= -\frac{A}{9}\\
16C_3 - 8(1 + i\sqrt{3})C_4 - 8(1 - i\sqrt{3})C_5 &= -\frac{A}{9}\\
-32C_3 + 16(1 - i\sqrt{3})C_4 + 16(1 + i\sqrt{3})C_5 &= -\frac{10A}{9}.
\end{aligned}$$

Elimination of C_3 from the last two equations gives

$$C_4 - C_5 = -\frac{i\sqrt{3}}{72} A.$$

When C_3 is eliminated from the fourth and fifth equations, and use is made of the preceding equation, one obtains

$$C_4 + C_5 = 0.$$

Hence,

$$-C_4 = C_5 = \frac{i\sqrt{3}}{144} A.$$

Successive substitutions in the fourth, third, second, and first equations then yield

$$C_3 = \frac{A}{72}, \qquad C_2 = -\frac{A}{24}, \qquad C_1 = -\frac{A}{8}, \qquad C_0 = -\frac{A}{8}.$$

Therefore, the desired solution is

$$\begin{aligned} y(x) &= -\frac{A}{8} - \frac{A}{8}x - \frac{A}{24}x^2 + \frac{A}{72}e^{-2x} + \frac{A}{9}e^x \\ &\qquad - \frac{i\sqrt{3}}{144}Ae^{(1+i\sqrt{3})x} + \frac{i\sqrt{3}}{144}Ae^{(1-i\sqrt{3})x} \\ &= -\frac{A}{24}(x^2 + 3x + 3) + \frac{A}{72}e^{-2x} + \frac{A}{9}e^x + \frac{A\sqrt{3}}{72}e^x \sin\sqrt{3}\,x. \end{aligned}$$

The reader should answer to himself the question why the system of equations for the constants of integration is always a determinate and consistent system. The key to the answer is found in the theorems of Sec. 2 and 5.

The determination of the constants of integration so as to fit the initial conditions is a laborious task. The labor involved can be shortened somewhat by the following observations. In Sec. 10 particular solutions of the special equations

$$(95) \qquad (D - c)^k u(x) = f(x)$$

and

$$(96) \qquad [(D - a)^2 + b^2]^l v(x) = f(x)$$

were derived. It is of interest to know what the initial conditions are that are satisfied by those particular solutions. Writing x_0 as lower limit in (59), we have the following particular integral of Equation (95):

$$(97) \qquad u(x) = \int_{x_0}^{x} \frac{(x - x_1)^{k-1}}{(k-1)!} e^{c(x-x_1)} f(x_1)\, dx_1.$$

By successive differentiation and substitution of x_0 for x it is readily checked that

(98) $$u(x_0) = u'(x_0) = \cdots = u^{(k-1)}(x_0) = 0.$$

Likewise, writing x_0 as lower limit in (67), the following particular integral of Equation (96) is obtained:

$$v(x) = \int_{x_0}^{x} S_l(x - x_1)e^{a(x-x_1)}f(x_1)\,dx_1.$$

As before, it can be readily checked that

(99) $$v(x_0) = v'(x_0) = \cdots + v^{(2l-1)}(x_0) = 0.$$

For the more general equation

(100) $$P(D)y \equiv (D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = f(x)$$

a particular solution was found in Sec. 11, which may be written symbolically as

(101) $$y(x) = \frac{1}{P(D)} f(x),$$

where the evaluation of the right-hand term involves the partial fractions expansion of the fraction $1/P(q)$. If in all the integrals arising from the partial fractions the same lower limit x_0 is used, then the obtained solution satisfies the conditions.

(102) $$y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0.$$

This cannot be checked so easily as the above equations (98), (99). The proof of this fact is postponed (see Sec. 1, rule 2, Chap. VI).

How use can be made of the aforegoing remarks is illustrated in the following examples:

Example 30. Find the solution of

(103) $$\frac{d^ny}{dx^n} = f(x)$$

that satisfies the initial conditions

(104) $$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}.$$

By (97), the integral

(105) $$\int_{x_0}^{x} \frac{(x - x_1)^{n-1}}{(n-1)!} f(x_1)\,dx_1$$

is the solution of Equation (103) that vanishes together with its first $(n-1)$ derivatives at $x = x_0$. Hence, in order to obtain the desired solution we add to (105) the solution of the reduced equation $y^{(n)}(x) = 0$ that satisfies conditions (104). This solution is immediately seen to be

$$(106)\quad y_0 + y_1 \frac{(x-x_0)}{1!} + y_2 \frac{(x-x_0)^2}{2!} + \cdots + y_{n-1} \frac{(x-x_0)^{n-1}}{(n-1)!}.$$

Hence, the desired solution is the sum of (105) and (106). If we substitute $y^{(n)}(x)$ for $f(x)$, we obtain

$$(107)\quad y(x) = y(x_0) + y'(x_0)\frac{(x-x_0)}{1!} + y''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + y^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + \int_{x_0}^{x} y^{(n)}(x_1)\frac{(x_1-x_0)^n}{n!}\,dx_1$$

and this formula holds, as can be seen from the derivation, for any function $y(x)$ that has a (piecewise) continuous derivative of nth order in the interval between x_0 and x. This is the well-known *Taylor's formula* with the integral form of the remainder term.

Example 31. Find the solution of

$$(108)\qquad [(D-a)^2 + b^2]y = f(x) \qquad (b \neq 0)$$

that satisfies the initial conditions

$$(109)\qquad y(x_0) = y_0, \qquad y'(x_0) = y_1.$$

By (98), the integral

$$(110)\qquad \frac{1}{b}\int_{x_0}^{x} e^{a(x-x_1)} \sin b(x-x_1) f(x_1)\,dx_1$$

is the solution of Equation (108) that vanishes together with its first derivative at $x = x_0$. Hence, in order to obtain the desired solution we add to (110) the solution of the reduced equation

$$[(D-a)^2 + b^2]y(x) = 0$$

that satisfies conditions (109). The general solution of this equation can be written as

$$e^{a(x-x_0)}[A \cos b(x-x_0) + B \sin b(x-x_0)]$$

and we obtain for A and B the equations

$$A = y_0, \qquad aA + bB = y_1$$

or

$$A = y_0, \qquad B = \frac{y_1 - ay_0}{b}.$$

Therefore

$$y(x) = e^{a(x-x_0)}\left[y_0 \cos b(x - x_0) + \frac{y_1 - ay_0}{b} \sin b(x - x_0)\right] + \frac{1}{b}\int_{x_0}^{x} e^{a(x-x_1)} \sin b(x - x_1) f(x_1)\, dx_1$$

is the desired solution.

PROBLEMS

To each of the following equations find the particular solution satisfying the given initial conditions:

1. $(D + 1)^3y = x^3e^{-x}$; $y = Dy = D^2y = 0$ when $x = 0$.
2. $(D - 1)^3y = x^{-3}e^x$; $y = Dy = D^2y = 0$ when $x = 1$.
3. $(D^3 + 3D^2 - 4D - 12)y = xe^{2x}$; $y = Dy = D^2y = 0$ when $x = 0$.
***4.** $(D + r)^ny = x^ke^{-rx}$; $y = Dy = D^2y = \cdots = D^{n-1}y = 0$ when $x = 0$.
5. $(D + r)^ny = f(x)$; $y = Dy = D^2y = \cdots = D^{n-1}y = 0$ when $x = 0$.
6. $(D + r)^2(D + s)^2y = f(x)$, $r \neq s$; $y = Dy = D^2y = D^3y = 0$ when $x = 0$.
7. $(D^2 + a^2)(D^2 + b^2)y = f(x)$, $|a| \neq |b|$ and $a \neq 0$, $b \neq 0$; $y = Dy = D^2y = D^3y = 0$ when $x = 0$.
8. $(D^3 - D)y = f(x)$; $y = Dy = D^2y = 1$ when $x = 0$.
9. $(D^3 + D)y = f(x)$; $y = D^2y = 0$, $Dy = 1$ when $x = 0$.
***10.** $(D^n - aD^{n-1})y = f(x)$, $a \neq 0$; $y = 1$, $Dy = D^2y = \cdots = D^{n-1}y = 0$ when $x = 0$.

CHAPTER VI

ALGEBRA OF INVERSE OPERATORS. SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

In this chapter the algebra of differential operators introduced in the preceding chapter is further developed. Whereas in the preceding chapter only positive integral powers and polynomials of the differential operator were considered, we now take up negative powers and rational functions of operators. These are studied mainly for their application to the solution of systems of simultaneous linear differential equations with constant coefficients. Such systems are frequently used in many fields of applied science, especially in the analysis of vibrating mechanical systems with several degrees of freedom and of multiloop electric networks.

1. Inverse Operators. In Sec. 10, Chap. V, we found that the differential equation

$$(D - c)^k y(x) = f(x) \qquad \left(D = \frac{d}{dx}\right) \tag{1}$$

has the particular solution

$$y(x) = \int_{x_0}^{x} \frac{(x - x_1)^{k-1}}{(k-1)!} e^{c(x-x_1)} f(x_1)\, dx_1, \tag{2}$$

where x_0 can be arbitrarily chosen. The symbolic form of Equation (1) suggests a symbolic solution:

$$y(x) = \frac{1}{(D - c)^k} f(x) = (D - c)^{-k} f(x). \tag{3}$$

We are free to identify the thus far undefined symbol $(D - c)^{-k} f(x)$ with the expression given by the right-hand term of (2) and will then be able to say that differential Equation (1) is solved like an algebraic equation.

Definition 1. The *inverse operator*

$$(D - c)^{-k} \text{ or } \frac{1}{(D - c)^k} \qquad (k = 1, 2, \ldots)$$

is defined as the integral

$$(4) \qquad (D - c)^{-k}f(x) = \int_{x_0}^{x} \frac{(x - x_1)^{k-1}}{(k-1)!} e^{c(x-x_1)}f(x_1)\,dx_1,$$

where x_0 is an arbitrary but fixed number.

In particular, if $c = 0$, then (4) becomes

$$(5) \quad D^{-k}f(x) = \int_{x_0}^{x} \frac{(x - x_1)^{k-1}}{(k-1)!} f(x_1)\,dx_1$$
$$= \int_{x_0}^{x} dx_k \int_{x_0}^{x_k} dx_{k-1} \cdots \int_{x_0}^{x_3} dx_2 \int_{x_0}^{x_2} f(x_1)\,dx_1$$

(see Example 12, Sec. 6, Chap. V)

that is, $D^{-k}f(x)$ is a k times repeated integral of $f(x)$, which is as it should be since $D^k f(x)$ is the kth derivative of $f(x)$.

The fact that (2) is a solution of Equation (1) can be expressed by the equation

$$(6) \qquad (D - c)^k (D - c)^{-k} f(x) = f(x),$$

that is, the original function $f(x)$ is restored if operated upon in succession by the operators $(D - c)^{-k}$ and $(D - c)^k$. In other words, the operator consisting of the integrations indicated by $(D - c)^{-k}$ followed by the differentiations indicated by $(D - c)^k$ is the "identity operator" which leaves every function unchanged.

$$(7) \qquad (D - c)^k (D - c)^{-k} = (D - c)^0 = 1$$

Equation (7) expresses the fundamental property of the operator $(D - c)^{-k}$, whose definition was designed so as to make this equation valid.

From Equation (7) one might expect the equation

$$(8) \qquad (D - c)^{-k}(D - c)^k = 1$$

to hold, that is,

$$(9) \qquad (D - c)^{-k}(D - c)^k f(x) = f(x).$$

That this is, in general, not true can be seen from an example. Let $c = 0$ and $k = 1$, then Equation (9) states

$$\int_{x_0}^{x} \frac{d}{dx_1} f(x_1)\,dx_1 = f(x),$$

which is true if and only if $f(x_0) = 0$. In general, it can be easily seen that Equation (9) holds [for functions $f(x)$ with a piecewise continuous derivative of kth order] if and only if

$$f(x_0) = f'(x_0) = \cdots = f^{(k-1)}(x_0) = 0.$$

Hence, Equation (8) cannot be accepted as universally valid. When this result is compared with the result of the preceding paragraph, one arrives at the conclusion that

$$(D - c)^k(D - c)^{-k} \neq (D - c)^{-k}(D - c)^k.$$

Hence, the combination of a differential operator with an integral operator, although it has the appearance of an ordinary product, is not commutative as the ordinary product of numbers is. Clearly, conclusions by analogy are not reliable and a careful examination of all operations is necessary.

Equation (4) may also be written in the form

$$(10) \qquad (D - c)^{-k}f(x) = e^{cx}\int_{x_0}^{x} \frac{(x - x_1)^{k-1}}{(k-1)!}(e^{-cx_1}f(x_1))\,dx_1 = e^{cx}D^{-k}(e^{-cx}f(x))$$

or, replacing c by $-r$,

$$(11) \qquad D^{-k}(e^{rx}f(x)) = e^{rx}(D + r)^{-k}f(x).$$

Hence, the *shifting rule* (rule 4, Sec. 8, Chap. V) *applies to integral operators in the same way as to differential operators.*

Example 1. Show that

$$(D + r + s)^{-k}f(x) = e^{-sx}(D + r)^{-k}(e^{sx}f(x)).$$

By the shifting rule $(D + r)^{-k} = e^{-rx}D^{-k}e^{rx}$; hence

$$e^{-sx}(D + r)^{-k}(e^{sx}f(x)) = e^{-sx}(e^{-rx}D^{-k}e^{rx})(e^{sx}f(x)) = e^{-(r+s)x}D^{-k}(e^{(r+s)x}f(x)).$$

We now discuss the result of applying two operators $(D + r)^{\pm k}$, $(D + r)^{\pm l}$ $(k, l = 1, 2, \ldots)$ in succession. It suffices to do this for the operators $D^{\pm k}$, $D^{\pm l}$. By use of the shifting rule the results can then easily be extended to the more general operators.

From definition it is clear that $D^kD^l = D^{k+l}$. It is also true that $D^kD^{-l} = D^{k-l}$. For,

$$D^kD^{-l}f(x) = \frac{d^k}{dx^k}\int_{x_0}^{x} dx_l \int_{x_0}^{x_l} dx_{l-1} \cdot \cdot \cdot \int_{x_0}^{x_2} f(x_1)\, dx_1.$$

The k times repeated differentiation reduces the l integrals to $l - k$ integrals if $l > k$, to $f(x)$ if $l = k$, and to $f^{(k-l)}(x)$ if $k > l$. Hence, in all cases, $D^kD^{-l}f(x) = D^{k-l}f(x)$. The equation $D^{-k}D^{-l} = D^{-k-l}$ is also true; for the $(k + l)$ times repeated integral of a function $f(x)$ can be obtained by integrating the l times repeated integral k times. The remaining equation $D^{-k}D^l = D^{-k+l}$ is, in general, not true as was shown above for the special case $k = l$. But if

$$f(x_0) = f'(x_0) = \cdot \cdot \cdot = f^{(l-1)}(x_0) = 0,$$

then $D^{-l}D^lf(x) = f(x)$, as was observed above. Hence, under these conditions

$$D^{-k+l}f(x) = D^{-k+l}(D^{-l}D^lf(x)) = D^{-k+l-l}(D^lf(x)) = D^{-k}(D^lf(x)).$$

By applying the shifting rule twice in succession we have

$$\begin{aligned}(D + r)^{\pm k}(D + r)^{\pm l}f(x) &= (D + r)^{\pm k}[e^{-rx}D^{\pm l}(e^{rx}f(x))] \\ &= e^{-rx}D^{\pm k}\{e^{+rx}[e^{-rx}D^{\pm l}(e^{rx}f(x))]\} \\ &= e^{-rx}D^{\pm k}D^{\pm l}(e^{rx}f(x)).\end{aligned}$$

Hence, the case of the operators $(D + r)^{\pm k}$, $(D + r)^{\pm l}$ is reduced to the case of the operators $D^{\pm k}$, $D^{\pm l}$ treated above.

The results may be summarized in the following rule:

Rule 1. *If k, l are positive integers, then*

$$\begin{aligned}(D + r)^k(D + r)^l &= (D + r)^{k+l} \\ (D + r)^k(D + r)^{-l} &= (D + r)^{k-l} \\ (D + r)^{-k}(D + r)^{-l} &= (D + r)^{-k-l}\end{aligned}$$

The equation $(D + r)^{-k+l}f(x) = (D + r)^{-k}(D + r)^lf(x)$ also holds true if $f(x_0) = f'(x_0) = \cdot \cdot \cdot = f^{(l-1)}(x_0) = 0$.

Even more general operators than the differential and integral operators considered so far may now be defined. Let $R(q)$ be a *rational function* of the variable q, that is, $R(q)$ is the quotient of two polynomials in q:

$$R(q) = \frac{B(q)}{A(q)} = \frac{b_0q^m + b_1q^{m-1} + \cdot \cdot \cdot + b_{m-1}q + b_m}{a_0q^n + a_1q^{n-1} + \cdot \cdot \cdot + a_{n-1}q + a_n}.$$

The difference $(m - n)$ of the degrees of the numerator and denominator is said to be the *degree of the rational function* $R(q)$. The degree may be positive, zero, or negative. If the degree is positive or zero, then the numerator $B(q)$ may be divided by the denominator $A(q)$, the quotient being a polynomial of degree $(m - n)$ (a constant if $m = n$), and the remainder being a rational function of negative degree. From algebra and calculus it is known that a rational function of negative degree can be decomposed into a sum of *partial fractions* (see also Sec. 11, Chap. V). The partial fractions are all of the form $C/(q - \alpha)$ if α is a simple (that is, nonrepeated) zero of the denominator polynomial $A(q)$. If α is a k times repeated zero, then the corresponding sum of partial fractions is of the form

$$\frac{C_k}{(q - \alpha)^k} + \frac{C_{k-1}}{(q - \alpha)^{k-1}} + \cdots + \frac{C_1}{q - \alpha}.$$

Summarizing we may say that every rational function $R(q)$ may be cast in the form

$$R(q) = P(q) + \sum \frac{C}{(q - \alpha)^k}, \tag{12}$$

where $P(q)$ is a polynomial whose degree is equal to the degree of $R(q)$ if this degree is nonnegative [if $R(q)$ is of negative degree then $P(q)$ is absent in (12)] and the other term is a sum of partial fractions. This decomposition can be made in one way only.

Equation (12) together with the earlier definitions of differential and integral operators suggest the following:

Definition 2. If $R(q)$ is a rational function whose decomposition is

$$R(q) = P(q) + \sum \frac{C}{(q - \alpha)^k},$$

then the operator $R(D)$, where $D = \frac{d}{dx}$, is defined by the equation

$$R(D)f(x) = P(D)f(x) + \Sigma C(D - \alpha)^{-k}f(x).$$

It should be observed that this definition is in agreement with the definition of differential operators (Definition 2, Sec. 8, Chap. V) and that of integral operators (Definition 1 of this section); as a matter of fact, it includes these definitions as special cases.

The following example illustrates the use of Definition 2.

Example 2. Evaluate

$$\frac{D^3 + 2D^2 + 3}{D^2 + D - 2}\, e^{3x} \qquad \text{(for } x_0 = 0\text{)}$$

The decomposition of $\dfrac{q^3 + 2q^2 + 3}{q^2 + q - 2}$ is

$$\frac{q^3 + 2q^2 + 3}{q^2 + q - 2} = q + 1 + \frac{2}{q - 1} - \frac{1}{q + 2}.$$

Hence,

$$\begin{aligned}
\frac{D^3 + 2D^2 + 3}{D^2 + D - 2} e^{3x} &= [D + 1 + 2(D - 1)^{-1} - (D + 2)^{-1}]e^{3x} \\
&= \frac{d}{dx} e^{3x} + e^{3x} + 2 \int_0^x e^{(x-x_1)} e^{3x_1} dx_1 \\
&\qquad - \int_0^x e^{-2(x-x_1)} e^{3x_1} dx_1 \\
&= 3e^{3x} + e^{3x} + 2e^x \left(\frac{e^{2x}}{2} - \frac{1}{2}\right) - e^{-2x} \left(\frac{e^{5x}}{5} - \frac{1}{5}\right) \\
&= \tfrac{24}{5} e^{3x} - e^x + \tfrac{1}{5} e^{-2x}.
\end{aligned}$$

To complete our operator algebra it now remains to study the result of the successive application of two operators $R_1(D)$, $R_2(D)$, where $R_1(q)$, $R_2(q)$ are rational functions. An example will illuminate the problems involved.

Example 3. Evaluate $\dfrac{D}{D + 1}\left(\dfrac{D}{D - 1} f(x)\right)$ and $\dfrac{D^2}{D^2 - 1} f(x)$, and compare the results.

The decompositions of the occurring rational functions are

$$\frac{q}{q - 1} = 1 + \frac{1}{q - 1}, \qquad \frac{q}{q + 1} = 1 - \frac{1}{q + 1},$$
$$\frac{q^2}{q^2 - 1} = 1 + \frac{\frac{1}{2}}{q - 1} - \frac{\frac{1}{2}}{q + 1}.$$

Therefore,

$$\begin{aligned}
\frac{D}{D + 1}\left(\frac{D}{D - 1} f(x)\right) &= \frac{D}{D + 1}\left[\left(1 + \frac{1}{D - 1}\right) f(x)\right] \\
&= \frac{D}{D + 1}\left(f(x) + \int_{x_0}^x e^{(x-x_1)} f(x_1)\, dx_1\right) \\
&= \left(1 - \frac{1}{D + 1}\right)\left(f(x) + \int_{x_0}^x e^{(x-x_1)} f(x_1)\, dx_1\right) \\
&= f(x) + \int_{x_0}^x e^{(x-x_1)} f(x_1)\, dx_1 \\
&\quad - \int_{x_0}^x e^{-(x-x_1)} f(x_1)\, dx_1 - \int_{x_0}^x e^{-(x-x_2)} dx_2 \int_{x_0}^{x_2} e^{(x_2-x_1)} f(x_1)\, dx_1.
\end{aligned}$$

Through integration by parts,

$$\int_{x_0}^{x} e^{2x_2}dx_2 \int_{x_0}^{x_2} e^{-x_1}f(x_1)\,dx_1 = \left[\frac{e^{2x_2}}{2}\int_{x_0}^{x_2} e^{-x_1}f(x_1)\,dx_1\right]_{x_2=x_0}^{x_2=x} - \int_{x_0}^{x}\frac{e^{2x_1}}{2}e^{-x_1}f(x_1)\,dx_1$$

$$= \frac{e^{2x}}{2}\int_{x_0}^{x} e^{-x_1}f(x_1)\,dx_1 - \frac{1}{2}\int_{x_0}^{x} e^{x_1}f(x_1)\,dx_1.$$

Hence,

$$\frac{D}{D+1}\left(\frac{D}{D-1}f(x)\right) = f(x) + \frac{1}{2}\int_{x_0}^{x} e^{(x-x_1)}f(x_1)\,dx_1 - \frac{1}{2}\int_{x_0}^{x} e^{-(x-x_1)}f(x_1)\,dx_1.$$

On the other hand,

$$\frac{D^2}{D^2-1}f(x) = \left(1 + \frac{\frac{1}{2}}{D-1} - \frac{\frac{1}{2}}{D+1}\right)f(x)$$

$$= f(x) + \frac{1}{2}\int_{x_0}^{x} e^{(x-x_1)}f(x_1)\,dx_1 - \frac{1}{2}\int_{x_0}^{x} e^{-(x-x_1)}f(x_1)\,dx_1.$$

It is seen that the results are identical although they were arrived at in different ways.

In general, we want to compare $R_1(D)[R_2(D)f(x)]$ with $[R_1(D)R_2(D)]f(x)$. In the first expression $R_1(q)$, $R_2(q)$ are decomposed, and after the integrations corresponding to the fractions of $R_2(q)$ are done those corresponding to the fractions of $R_1(q)$ are performed. In the second expression the rational function $R_1(q) \cdot R_2(q)$ is decomposed, and then the integrations corresponding to the fractions of this function are performed.

Since $R_1(q)$ is a sum of terms like $(q-\alpha)^k$ and $(q-\alpha)^{-k}$ and $R_2(q)$ is a sum of terms like $(q-\beta)^l$, $(q-\beta)^{-l}$, $(k, l = 0, 1, 2, \ldots)$, it suffices to consider the combinations $(D-\alpha)^{\pm k}(D-\beta)^{\pm l}$. This is done in the subsequent cases, 1–4.

Case 1. $(D-\alpha)^k[(D-\beta)^l f(x)]$. By rule 3, Sec. 8, Chap. V, this is equal to $[(D-\alpha)^k(D-\beta)^l]f(x)$.

Case 2. $(D-\alpha)^k[(D-\beta)^{-l}f(x)]$. $(D-\alpha)^k$ may be written as $(D-\beta+\beta-\alpha)^k$. Hence, $(D-\alpha)^k$ may be expanded as a sum of terms like $(D-\beta)^m$ $(m = 0, 1, \ldots, k)$. But

$$(D-\beta)^m[(D-\beta)^{-l}f(x)] = (D-\beta)^{m-l}f(x),$$

by rule 1 of this section. Hence,

$$(D - \alpha)^k[(D - \beta)^{-l}f(x)] = [(D - \alpha)^k(D - \beta)^{-l}]f(x).$$

From cases 1 and 2 it follows that

$$R_1(D)[R_2(D)f(x)] = [R_1(D)R_2(D)]f(x)$$

if $R_1(q)$ is a *polynomial* and $R_2(q)$ is any rational function. From this result we obtain two interesting conclusions.

First, let $R_1(q)$ be the polynomial $P(q) = a_0q^n + a_1q^{n-1} + \cdots + a_n$ and let $R_2(q)$ be the rational function $1/P(q)$. Then, we have

$$P(D)\left[\frac{1}{P(D)}f(x)\right] = \left[\frac{P(D)}{P(D)}\right]f(x) = f(x).$$

In other words,

$$y(x) = \frac{1}{P(D)}f(x)$$

is a solution of the differential equation

$$P(D)y = f(x).$$

This is the same solution that was found in Sec. 11, Chap. V.

Second, let $R_1(q)$ be q^k $(k = 0, 1, 2, \ldots)$ and let $R_2(q)$ be a rational function $R(q)$ of *negative* degree $-n$. Then

$$D^k[R(D)f(x)] = [D^kR(D)]f(x).$$

Now if k is one of the numbers $0, 1, 2, \ldots, n - 1$, then the rational function $q^kR(q)$ is still of negative degree, hence in its decomposition there are only fractions. Therefore, $[D^kR(D)]f(x)$ consists of a sum of integrals all having the same lower limit x_0. If x_0 is substituted for x in this sum it vanishes. Hence, we have proved the following rule:

Rule 2. *If $R(q)$ is a rational function of negative degree $-n$, then $R(D)f(x)$ is a function whose derivatives of order $0, 1, 2, \ldots, n - 1$ vanish for $x = x_0$.*

In particular, it follows from rule 2 that $y(x) = \frac{1}{P(D)}f(x)$ is that solution of the equation $P(D)y = f(x)$ which satisfies the initial conditions

$$y(x_0) = y'(x_0) = y''(x_0) = \cdots = y^{(n-1)}(x_0) = 0.$$

This result was stated without proof in Sec. 13, Chap. V.

We now take up the remaining combinations of $(D-\alpha)^{\pm k}(D-\beta)^{\pm l}f(x)$.

Case 3. $[(D-\alpha)^{-k}(D-\beta)^{-l}]f(x)$. By rule 2 this is a function whose derivatives of order $0, 1, 2, \ldots, k+l-1$ vanish at $x = x_0$. Therefore, [see the discussion following Equation (9)]

$$(D-\alpha)^{-k}(D-\alpha)^{k}\{[(D-\alpha)^{-k}(D-\beta)^{-l}]f(x)\} = [(D-\alpha)^{-k}(D-\beta)^{-l}]f(x).$$

By case 2, the factor $(D-\alpha)^k$ may be combined with the operator in the bracket, giving $(D-\beta)^{-l}$. Hence,

$$(D-\alpha)^{-k}[(D-\beta)^{-l}f(x)] = [(D-\alpha)^{-k}(D-\beta)^{-l}]f(x).$$

Case 4. $(D-\alpha)^{-k}[(D-\beta)^{l}f(x)]$. In this case we expand $(D-\beta)^l$ in powers of $(D-\alpha)$; the highest power occurring is $(D-\alpha)^l$. Hence, we need consider only

$$(D-\alpha)^{-k}(D-\alpha)^{m}f(x) \qquad \text{for} \qquad m = 0, 1, 2, \ldots, l.$$

By rule 1 this is $(D-\alpha)^{-k+m}f(x)$ provided that

$$f(x_0) = f'(x_0) = \cdots = f^{(m-1)}(x_0) = 0.$$

Hence,

$$(D-\alpha)^{-k}[(D-\beta)^{l}f(x)] = [(D-\alpha)^{-k}(D-\beta)^{l}]f(x)$$

if $f(x_0) = f'(x_0) = \cdots = f^{(l-1)}(x_0) = 0$.

Summarizing the results of cases 1–4, we have the following rule:

Rule 3.

$$R_1(D)[R_2(D)f(x)] = [R_1(D)R_2(D)]f(x)$$

if the degree of the rational function $R_2(q)$ is negative or zero. The equation is also true if the degree l of $R_2(D)$ is positive provided that $f(x_0) = f'(x_0) = \cdots = f^{(l-1)}(x_0) = 0$.

Example 4. Write the iterated integral

$$y(x) = \int_{x_0}^{x} e^{r_3(x-x_3)}dx_3 \int_{x_0}^{x_3} e^{r_2(x_3-x_2)}dx_2 \int_{x_0}^{x_2} e^{r_1(x_2-x_1)}f(x_1)\,dx_1 \qquad (r_1 \neq r_2 \neq r_3)$$

as a simple integral.

This problem was solved in Sec. 10, Chap. V, by the method of integration by parts. We now solve it by the use of rule 3. Using the definition of integral operators, $y(x)$ can be written as

$$y(x) = (D-r_3)^{-1}\{(D-r_2)^{-1}[(D-r_1)^{-1}f(x)]\}$$

and, by rule 3 this is

$$y(x) = [(D - r_3)^{-1}(D - r_2)^{-1}(D - r_1)^{-1}]f(x)$$
$$= \frac{1}{(D - r_1)(D - r_2)(D - r_3)} f(x).$$

The partial fractions expansion of the occurring rational function is

$$\frac{1}{(q - r_1)(q - r_2)(q - r_3)} = \frac{1/(r_1 - r_2)(r_1 - r_3)}{q - r_1} + \frac{1/(r_2 - r_1)(r_2 - r_3)}{q - r_2} + \frac{1/(r_3 - r_1)(r_3 - r_2)}{q - r_3}.$$

Hence,

$$y(x) = \left[\frac{1}{(r_1 - r_2)(r_1 - r_3)}(D - r_1)^{-1} + \frac{1}{(r_2 - r_1)(r_2 - r_3)}(D - r_2)^{-1} + \frac{1}{(r_3 - r_1)(r_3 - r_2)}(D - r_3)^{-1}\right] f(x)$$
$$= \int_{x_0}^{x} \left[\frac{e^{r_1(x - x_1)}}{(r_1 - r_2)(r_1 - r_3)} + \frac{e^{r_2(x - x_1)}}{(r_2 - r_1)(r_2 - r_3)} + \frac{e^{r_3(x - x_1)}}{(r_3 - r_1)(r_3 - r_2)}\right] f(x_1)\, dx_1.$$

PROBLEMS

★**1.** Show that

$$(D + r)^{-k}(D + r)^k f(x) = f(x)$$

if and only if $f(x_0) = f'(x_0) = \cdots = f^{(k-1)}(x_0) = 0$.

Evaluate, using $x_0 = 0$, the following expressions:

2. $\dfrac{2D^2 - 5D + 1}{D^3 - D}(1 + xe^x)$.

3. $\dfrac{D^3 - D^2 - 5D - 2}{D^4 + 3D^3 + 2D^2}\sin x$.

★**4.** $\dfrac{D^4}{D^4 - 1}\cosh x$.

★**5.** $\dfrac{D^3}{D^2 + 1}\cos 2x$.

★**6.** Suppose that $P_1(q)$, $P_2(q)$ are polynomials of degree n_1, n_2, respectively. Show that if $y_1(x)$ is a particular solution of the equation $P_1(D)y = f(x)$ satisfying the initial conditions $y = Dy = \cdots = D^{n_1-1}y = 0$ when $x = x_0$ then $y_2(x) = [1/P_2(D)]y_1(x)$ solves the equation $P_1(D)P_2(D)y = f(x)$ and satisfies the conditions $y = Dy = \cdots = D^{(n_1+n_2-1)}y = 0$.

2. Systems of Simultaneous Linear Differential Equations. As stated at the beginning of this chapter a great many problems in science and engineering lead to systems of simultaneous differential equations with several unknown functions. By this we understand a set of equations involving some unknown functions $x(t)$, $y(t)$, $z(t)$, . . . and some of their derivatives.† The system is said to be ***linear*** if all the equations of the set are of the form

$$(13)\quad (a_0D^k + a_1D^{k-1} + \cdots + a_k)x + (b_0D^l + b_1D^{l-1} + \cdots + b_l)y + \cdots = f(t).$$

The coefficients $a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_l, \ldots$ may be functions of t. But in the following we shall only consider *linear systems with constant coefficients*, where all the mentioned coefficients are constants. The right-member terms $f(t), \ldots$, representing the terms free of the unknowns, may depend on t. If all the right-member terms are zero, the system is said to be *linear homogeneous*.

Using the operator notation the coefficient of x in Equation (13) may be written as an operator polynomial $P(D)$, the coefficient of y as some other operator polynomial $Q(D)$, and so forth. To distinguish the several equations of the system subscripts 1, 2, . . . may be employed. The system then takes the form

$$(14)\quad \begin{array}{l} P_1(D)x + Q_1(D)y + \cdots = f_1(t) \\ P_2(D)x + Q_2(D)y + \cdots = f_2(t). \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

By an *integral* (or *solution*) of the system we mean a set of functions $x(t), y(t), z(t), \ldots$ which when substituted in the equations of the system turn these equations into identities in t. As in the case of a single differential equation there exist, in general, many integrals of the same system, and initial (or other) conditions are used to single out a particular one. But as to the number of required conditions no such simple rules as for single differential equations can be given. This will become apparent from an examination of a few examples.

Example 5.

$$\begin{array}{l} (D^2 - 1)x - D^2y = -2 \sin t \\ (D^2 + 1)x + D^2y = 0. \end{array}$$

Adding these equations we eliminate y and find, for x, the equation

$$2D^2x = -2 \sin t,$$

whose general solution is

$$(15)\quad x(t) = \sin t + A_1t + A_2,$$

† In the following sections t denotes the independent variable, while $x, y, z, \ldots$ denote the dependent variables. The symbol D denotes the differential operator d/dt.

where A_1, A_2 are arbitrary constants. When this is substituted in the first equation we obtain

$$D^2y = -A_1t - A_2,$$

whose general solution is

$$y(t) = \frac{-A_1}{6}t^3 - \frac{A_2}{2}t^2 + B_1t + B_2, \tag{16}$$

where B_1, B_2 are additional arbitrary constants. It is immediately checked that the second equation of the system is also satisfied by these functions. Hence, we have found a solution with four arbitrary constants. From the form of this solution it is apparent that a particular integral can be singled out by specifying

$$x = x_0,\ Dx = x_1,\ y = y_0,\ Dy = y_1 \qquad (\text{when } t = t_0)$$

where x_0, x_1, y_0, y_1 are arbitrary numbers. Another possible set of initial conditions is

$$y = y_0,\ Dy = y_1,\ D^2y = y_2,\ D^3y = y_3 \qquad (\text{when } t = t_0)$$

and no condition for x, Dx, ... For, then A_1, A_2, B_1, B_2 can be determined from Equation (16) and its first three derivatives. On the other hand, it would not do to specify initial values only for the function $x(t)$ and its derivatives, since by (15) the expression for $x(t)$ involves only the constants A_1, A_2.

Example 6.

$$(D^2 - 1)x + D^2y = -2 \sin t$$
$$(D^2 + 1)x + D^2y = 0.$$

To eliminate y we subtract the two equations, obtaining

$$2x = 2 \sin t$$

or

$$x(t) = \sin t.$$

On substituting this in the first equation there results

$$D^2y = 0.$$

Hence,

$$y(t) = B_1t + B_2,$$

where B_1, B_2 are arbitrary constants.

Although the system is quite similar to the one of the preceding example (only a sign is changed) we see that in this case the general solution contains but two arbitrary constants. A particular solution is singled out by the initial conditions

$$y = y_0, \qquad Dy = y_1 \qquad\qquad (\text{when } t = t_0)$$

No initial conditions can be imposed on $x(t)$.

Example 7.

$$(D^2 - 1)x + (D^2 - 2)y = -2 \sin t$$
$$(D^2 + 1)x + D^2y = 0.$$

By subtracting the two equations we find

$$x + y = \sin t. \tag{17}$$

Substitution of this result in the second equation yields

$$x(t) = \sin t,$$

and because of (17),

$$y(t) = 0.$$

There are no arbitrary constants in this solution. Therefore, no initial conditions can be imposed on either $x(t)$ or $y(t)$.

Example 8.

$$(D^2 - 1)x + (D^2 - D)y = -2 \sin t$$
$$(D^2 + D)x + D^2y = 0.$$

By subtracting the two equations one finds

$$(D + 1)x + Dy = 2 \sin t,$$

and this equation, when differentiated, yields

$$(D^2 + D)x + D^2y = 2 \cos t.$$

This last equation is inconsistent with the second of the equations of the system. Hence, this system has no solution whatsoever.

The four discussed examples show that, as far as the number of possible solutions are concerned, systems of differential equations may vary widely although in their form there appears to be little difference. Because of this, the formal appearance of the equations in a system is not much of a clue as to how many constants of integration there are in the general solution and by what sort of initial conditions they can be determined.

There exist general criteria as to the number of arbitrary constants in the general solution. These criteria involve the determinant formed by the coefficients of the system (see the footnote on page 207). Th method of solution to be taken up in Sec. 4 supplies an answer to the above questions in each individual case.

When one tries to solve a system of equations the first step that suggests itself is to derive from the given equations a new equation from which all but one unknown are eliminated. In the case of algebraic linear systems the elimination of unknowns is done by forming linear combinations of the given equations with suitably chosen numerical factors. In dealing with differential systems like (14), where the coefficients are operator polynomials, elimination of unknowns is done by forming linear combinations of the equations of the system using operator polynomials as factors. For example, if the unknown $x(t)$ is to be eliminated from the first two equations of system (14), one may "multiply" the first equation by $P_2(D)$, the second equation by $P_1(D)$, and then subtract the two equations. It should be kept in mind that "multiplication" by operator polynomials actually involves differentiations of the equations. For this reason the resulting equations, although derived from the given equations and, therefore, necessarily satisfied by any solution of the given system, may have more solutions with more constants of integration than the original equations. To eliminate the extraneous solutions that result from this procedure *it is necessary to substitute the general solution of the derived equations in all the equations of the original system.* This usually leads to a reduction of the arbitrary constants in the solution. It is hardly necessary to mention that if an equation of the system contains an unknown function but none of its derivatives then this equation should be solved for this unknown like an algebraic equation and the result should be substituted in the remaining equations.

Methods that avoid extraneous solutions are discussed in subsequent sections.

Example 9. Find the general solution of the system

$$\begin{aligned} (D^2 - 2D + 3)x + (D - 1)y + Dz &= 0 \\ (3D + 1)x - 3Dy - Dz &= 1 \\ 2x - 2y - z &= -4. \end{aligned}$$

We may first solve the last equation for z, obtaining

$$z = 2x - 2y + 4.$$

When this is substituted in the first two equations, there results the system

$$\begin{aligned} (D^2 + 3)x - (D + 1)y &= 0 \\ (D + 1)x - Dy &= 1. \end{aligned}$$

To eliminate y from these equations we operate with D on the first and with $-(D+1)$ on the second equation and then add, obtaining

$$(D^3 - D^2 + D - 1)x + 0y = D0 - (D+1)1$$

or

$$(D-1)(D^2+1)x = -1.$$

The general solution of this equation is

$$x = 1 + c_1 \sin t + c_2 \cos t + c_3 e^t.$$

To determine y one may similarly eliminate x from the above two equations, or one may substitute the established solution for x in one of the equations, thus obtaining an equation for y alone. The latter method is preferable since the same substitution must also be done for the purpose of eliminating extraneous solutions. On substitution for x the second of the above equations becomes

$$\begin{aligned} Dy &= (D+1)x - 1 \\ &= (c_1 + c_2)\cos t + (c_1 - c_2)\sin t + 2c_3 e^t, \end{aligned}$$

whose general solution is

$$y = (c_1 + c_2)\sin t - (c_1 - c_2)\cos t + 2c_3 e^t + c_4.$$

If these tentative solutions for x and y are substituted in the first of the above equations one finds

$$\begin{aligned} (-c_1 + 3c_1 - c_1 + c_2 - c_1 - c_2)\sin t \\ + (-c_2 + 3c_2 - c_1 - c_2 + c_1 - c_2)\cos t \\ + (c_3 + 3c_3 - 2c_3 - 2c_3)e^t + 3 - c_4 = 0. \end{aligned}$$

Hence c_4 is not an arbitrary constant, but $c_4 = 3$. Since

$$z = 2x - 2y + 4$$

the general solution of the system is

$$\begin{aligned} x &= 1 + c_1 \sin t + c_2 \cos t + c_3 e^t \\ y &= 3 + (c_1 + c_2)\sin t - (c_1 - c_2)\cos t + 2c_3 e^t \\ z &= -2c_2 \sin t + 2c_1 \cos t - 2c_3 e^t, \end{aligned}$$

containing the three arbitrary constants c_1, c_2, c_3.

PROBLEMS

To each of the following systems the general solution can be easily determined by trial or by elimination of unknowns. Discuss the number of arbitrary constants of integration and possible initial conditions.

1. $(D+1)x = 2e^t$
$Dx - y = e^t.$

2. $(D-2)x + (D-2)y = 2t^2 - 2t$
$(D+1)x - (D+1)y = t^2 + 2t.$

3. $(D+1)x + y = \dfrac{1}{t}$
$(D-1)x + y = -\dfrac{1}{t} + 2e^{-t}.$

4. $(D+1)x + y = \dfrac{1}{t}$
$(D+1)x - y = -\dfrac{1}{t} + e^{-t}.$

5. $(D^2+4)x = 3\sin t$
$Dx - (D^2-1)y = 2\cos t.$

6. $(D^2-1)x + (D^2+1)y = f(t)$
$(D-1)x + (D-1)y = 0.$

7. $(D-1)x = 0$
$3x + 2(D+1)y = 0$
$2y + (2D-3)z = 0.$

8. $(D^2+D+1)x + (D^2+1)y = e^{2t}$
$(D+1)x + Dy = 1.$

9. $(D^2+D+1)x + (D^2-D+1)y + Dz = t^2$
$Dx + (D+1)y = t$
$x - 2y + z = 0.$

***10.** Show that the system

$$x + Q_1(D)y = f_1(t)$$
$$P_2(D)x + Q_2(D)y = f_2(t)$$

has one and only one solution satisfying the initial conditions

$$y = Dy = D^2y = \cdots = D^ky = 0 \text{ when } t = t_0,$$

where k is the degree of the polynomial $Q_2(q) - Q_1(q)P_2(q)$ (which we assume to be not identically vanishing). What is this solution if $f_1(t) = f_2(t) \equiv 0$?

11. Show that the (diagonal) system

$$P_1(D)x = f_1(t)$$
$$Q_2(D)y + P_2(D)x = f_2(t)$$
$$R_3(D)z + Q_3(D)y + P_3(D)x = f_3(t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

has one and only one solution satisfying the initial conditions

$$\begin{aligned} x = Dx = \cdots = D^{k_1-1}x = 0 \\ y = Dy = \cdots = D^{k_2-1}y = 0 \\ z = Dz = \cdots = D^{k_3-1}z = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad (\text{when } t = t_0)$$

where $k_1, k_2, k_3, \ldots$ are the degrees of the polynomials $P_1(q)$, $Q_2(q)$, $R_3(q)$, $\ldots$, respectively. What is this solution if $f_1(t) = f_2(t) = \cdots \equiv 0$?

3. Solution by Operational Methods. In order to avoid cumbersome notation we shall discuss only systems of three equations with three unknowns, the generalization to the case of n equations with n unknowns is trivial and requires no further elaboration. The system to be considered is

$$\begin{aligned} P_1(D)x + Q_1(D)y + R_1(D)z &= f_1(t) \\ P_2(D)x + Q_2(D)y + R_2(D)z &= f_2(t) \\ P_3(D)x + Q_3(D)y + R_3(D)z &= f_3(t). \end{aligned} \tag{18}$$

First we assume the equations to be of first order. Then the occurring operator polynomials are of degree 1, that is, of the form $aD + b$, where a and/or b may possibly be zero. If all the operator polynomials are of degree 0, then system (18) is an ordinary linear system as studied in algebra. For such systems two cases must be distinguished. The system is either *independent* or *dependent*, according to whether the determinant of the system is different from or equal to zero (see Sec. 12, Chap. I). If all or some of the operator polynomials in Equation (18) are of degree 1, then there is a greater variety of possibilities. According to the discussion of the preceding section one may expect that there are general solutions with three, two, one, or no arbitrary constants, or the system may be dependent. In this case, too, the "characteristic determinant" formed from the coefficients of the system, that is, the operator polynomial

$$\Delta(D) = \begin{vmatrix} P_1(D) & Q_1(D) & R_1(D) \\ P_2(D) & Q_2(D) & R_2(D) \\ P_3(D) & Q_3(D) & R_3(D) \end{vmatrix} \tag{19}$$

tells the whole story. This polynomial may be of degree 3, 2, 1, it may be a nonvanishing constant, or it may be identically zero. It can be shown that these alternatives correspond to the above-mentioned cases of three, two, one, no arbitrary constants in the general solution, or the system being dependent, respectively.† We shall consider here only the most common case, where the degree of $\Delta(D)$ is 3.

If we disregard, for the moment, that the coefficients of system (18) are operators and consider them as ordinary numbers, then (18) appears like an algebraic system of three linear equations in the three unknowns x, y, z. Its solution is, by Cramer's rule (see Sec. 12, Chap. I),

$$x = \frac{\begin{vmatrix} f_1 & Q_1 & R_1 \\ f_2 & Q_2 & R_2 \\ f_3 & Q_3 & R_3 \end{vmatrix}}{\begin{vmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} P_1 & f_1 & R_1 \\ P_2 & f_2 & R_2 \\ P_3 & f_3 & R_3 \end{vmatrix}}{\begin{vmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} P_1 & Q_1 & f_1 \\ P_2 & Q_2 & f_2 \\ P_3 & Q_3 & f_3 \end{vmatrix}}{\begin{vmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{vmatrix}},$$

† For any linear system of differential equations with constant coefficients it is true that the number of arbitrary constants in the general solution is equal to the degree in D of the characteristic determinant $\Delta(D)$ if the latter does not vanish identically. If $\Delta \equiv 0$ the system is dependent (see, for example, Ref. 3, p. 144).

which can also be written in the form

$$(20)\qquad \begin{aligned} x(t) &= \frac{\Delta_{P_1}(D)}{\Delta(D)} f_1(t) + \frac{\Delta_{P_2}(D)}{\Delta(D)} f_2(t) + \frac{\Delta_{P_3}(D)}{\Delta(D)} f_3(t) \\ y(t) &= \frac{\Delta_{Q_1}(D)}{\Delta(D)} f_1(t) + \frac{\Delta_{Q_2}(D)}{\Delta(D)} f_2(t) + \frac{\Delta_{Q_3}(D)}{\Delta(D)} f_3(t) \\ z(t) &= \frac{\Delta_{R_1}(D)}{\Delta(D)} f_1(t) + \frac{\Delta_{R_2}(D)}{\Delta(D)} f_2(t) + \frac{\Delta_{R_3}(D)}{\Delta(D)} f_3(t), \end{aligned}$$

where $\Delta(D)$ is the characteristic determinant (19) and $\Delta_{P_i}(D)$, $\Delta_{Q_i}(D)$, $\Delta_{R_i}(D)$ are cofactors of the elements $P_i(D)$, $Q_i(D)$, $R_i(D)$ in the determinant $\Delta(D)$. Equations (20) were tentatively derived by handling the symbol D like a number, but they were put in a form that makes a correct interpretation possible. By Definition 2, Sec. 1, each of the fractions in (20) is a sum of integrals extended over the interval (t_0, t), where t_0 is an arbitrary chosen number. We are going to show that expressions (20), thus interpreted, actually represent a solution of system (18). This fortuitous result is a convincing demonstration of the power of operational calculus.

Theorem 1. *If the operator polynomials in system* (18) *are all of degree* $\leqq 1$ *and if the determinant formed from them is of degree* 3, *then system* (18) *has one and only one solution satisfying the initial conditions*

$$(21)\qquad x(t) = y(t) = z(t) = 0 \qquad (when\ t = t_0)$$

and it is given by expressions (20) *obtained by formally applying Cramer's rule to system* (18).

Proof: Since $\Delta(D)$ is of degree 3 and each of the cofactors $\Delta_{P_i}(D)$, $\Delta_{Q_i}(D)$, $\Delta_{R_i}(D)$ is the degree $\leqq 2$, the rational functions $\Delta_{P_i}(D)/\Delta(D)$, $\Delta_{Q_i}(D)/\Delta(D)$, $\Delta_{R_i}(D)/\Delta(D)$ are of degree $\leqq -1$. Hence, by rule 2, Sec. 1, all the terms of the right members of Equations (20) vanish for $t = t_0$. Therefore, initial conditions (21) are satisfied.

Now substitute expressions (20) in the first of Equations (18) and obtain

$$(22)\quad P_1(D)x + Q_1(D)y + R_1(D)z = \sum_{i=1}^{3} \left[P_1(D) \frac{\Delta_{P_i}(D)}{\Delta(D)} f_i(t) + Q_1(D) \frac{\Delta_{Q_i}(D)}{\Delta(D)} f_i(t) + R_1(D) \frac{\Delta_{R_i}(D)}{\Delta(D)} f_i(t) \right].$$

By rule 3, Sec. 1, we have

$$P_1(D) \frac{\Delta_{P_i}(D)}{\Delta(D)} = \frac{P_1(D)\Delta_{P_i}(D)}{\Delta(D)}, \text{ etc.}$$

Furthermore, we have the relations

$$P_1(D)\Delta_{P_1}(D) + Q_1(D)\Delta_{Q_1}(D) + R_1(D)\Delta_{R_1}(D) = 1$$
$$P_1(D)\Delta_{P_2}(D) + Q_1(D)\Delta_{Q_2}(D) + R_1(D)\Delta_{R_2}(D) = 0$$
$$P_1(D)\Delta_{P_3}(D) + Q_1(D)\Delta_{Q_3}(D) + R_1(D)\Delta_{R_3}(D) = 0,$$

which are well-known relations for determinants whose elements are ordinary numbers (see Sec. 12, Chap. I,) and therefore hold also true for operator polynomials. Hence, Equation (22) becomes

$$P_1(D)x + Q_1(D)y + R_1(D)z = 1 \cdot f_1(t) + 0 \cdot f_2(t) + 0 \cdot f_3(t),$$

that is, expressions (20) satisfy the first of Equations (18). Likewise the other two equations of system (18) are seen to be satisfied.

It remains to prove the uniqueness of the solution. Assume that $x^*(t)$, $y^*(t)$, $z^*(t)$ is any solution of system (18) satisfying initial conditions (21) [possibly differing from solution (20)]. Operating on the three equations of (18) by Δ_{P_1}/Δ, Δ_{P_2}/Δ, Δ_{P_3}/Δ (the argument D is omitted for brevity) and summing we obtain

$$\sum_{i=1}^{3} \frac{\Delta_{Pi}}{\Delta} (P_i x^*) + \sum_{i=1}^{3} \frac{\Delta_{Pi}}{\Delta} (Q_i y^*) + \sum_{i=1}^{3} \frac{\Delta_{Pi}}{\Delta} (R_i z^*) = \sum_{i=1}^{3} \frac{\Delta_{Pi}}{\Delta} f_i. \tag{23}$$

By rule 3, Sec. 1, since each of the P_i, Q_i, R_i is an operator polynomial of degree ≤ 1 and since $x^*(t_0) = y^*(t_0) = z^*(t_0) = 0$,

$$\sum_{i=1}^{3} \frac{\Delta_{Pi}}{\Delta} (P_i x^*) = \left(\sum_{i=1}^{3} \frac{P_i \Delta_{Pi}}{\Delta} \right) x^* = 1 \cdot x^*$$
$$\sum_{i=1}^{3} \frac{\Delta_{Pi}}{\Delta} (Q_i y^*) = \left(\sum_{i=1}^{3} \frac{Q_i \Delta_{Pi}}{\Delta} \right) y^* = 0 \cdot y^*$$
$$\sum_{i=1}^{3} \frac{\Delta_{Pi}}{\Delta} (R_i z^*) = \left(\sum_{i=1}^{3} \frac{R_i \Delta_{Pi}}{\Delta} \right) z^* = 0 \cdot z^*.$$

Hence, Equation (23) becomes

$$x^* = \sum_{i=1}^{3} \frac{\Delta_{Pi}}{\Delta} f_i,$$

and, therefore, $x^*(t)$, $y^*(t)$, $z^*(t)$ is no different from solution (20).

In order to evaluate solution (20) it is necessary to factor the characteristic determinant, that is, to find the roots of the equation

$$\Delta(q) = 0.$$

This equation is known as the *auxiliary* (or *characteristic*) *equation* of the system. After having factored $\Delta(q)$, the partial fraction decompositions of the rational functions $\Delta_{P_i}(q)/\Delta(q)$, $\Delta_{Q_i}(q)/\Delta(q)$, $\Delta_{R_i}(q)/\Delta(q)$ must be found, and the indicated quadratures involving $f_1(t)$, $f_2(t)$, $f_3(t)$ must be performed.

If a solution is to be found satisfying the more general initial conditions

$$x(t) = x_0,\ y(t) = y_0,\ z(t) = z_0 \qquad \text{(when } t = t_0) \tag{24}$$

then the system of equations is first transformed by the substitution

$$\begin{aligned} x(t) &= x_0 + X(t) \\ y(t) &= y_0 + Y(t) \\ z(t) &= z_0 + Z(t). \end{aligned} \tag{25}$$

The new unknowns $X(t)$, $Y(t)$, $Z(t)$ vanish, for $t = t_0$, hence, can be found by the above method.

Example 10. Find the solution of the system

$$\begin{aligned} (D + 2)x + y &= 1 \\ (D - 1)y + (D + 2)z &= 0 \\ (D - 1)x + Dz &= -1 \end{aligned}$$

for which

$$x = 1, \qquad y = z = 0 \qquad \text{(when } t = 0)$$

With the transformation $x(t) = 1 + X(t)$ the system becomes

$$\begin{aligned} (D + 2)X + y &= -1 \\ (D - 1)y + (D + 2)z &= 0 \\ (D - 1)X + Dz &= 0. \end{aligned}$$

The characteristic determinant is

$$\Delta = \begin{vmatrix} D + 2 & 1 & 0 \\ 0 & D - 1 & D + 2 \\ D - 1 & 0 & D \end{vmatrix} = (D + 1)(D - 1)(D + 2).$$

Since it is of degree 3, Theorem 1 applies. By (20),

$$X = \frac{\Delta_{P_1}}{\Delta}(-1), \qquad y = \frac{\Delta_{Q_1}}{\Delta}(-1), \qquad z = \frac{\Delta_{R_1}}{\Delta}(-1).$$

But

$$\begin{aligned} \frac{\Delta_{P_1}}{\Delta} &= \frac{D}{(D + 1)(D + 2)} = -\frac{1}{D + 1} + \frac{2}{D + 2} \\ \frac{\Delta_{Q_1}}{\Delta} &= \frac{1}{D + 1} \\ \frac{\Delta_{R_1}}{\Delta} &= -\frac{D - 1}{(D + 1)(D + 2)} = \frac{2}{D + 1} - \frac{3}{D + 2}. \end{aligned}$$

Hence,

$$X(t) = \int_0^t (-e^{t_1-t} + 2e^{2(t_1-t)})(-1)\,dt_1 = -e^{-t} + e^{-2t}$$
$$y(t) = \int_0^t e^{t_1-t}(-1)\,dt_1 = -1 + e^{-t}$$
$$z(t) = \int_0^t (2e^{t_1-t} - 3e^{2(t_1-t)})(-1)\,dt_1 = -\tfrac{1}{2} + 2e^{-t} - \tfrac{3}{2}e^{-2t}.$$

Therefore,

$$x(t) = 1 - e^{-t} + e^{-2t},\ y(t) = -1 + e^{-t},\ z(t) = -\tfrac{1}{2} + 2e^{-t} - \tfrac{3}{2}e^{-2t}$$

is the desired solution.

In applications to mechanical and electrical systems the equations are mostly of second order. The above method applies to such systems, too.

Theorem 2. *If the operator polynomials in system* (18) *are of degree* $\leqq 2$ *and if the determinant formed from them is of degree* 6, *then system* (18) *has one and only one solution satisfying the initial conditions*

(26) $x(t) = Dx(t) = y(t) = Dy(t) = z(t) = Dz(t) = 0$ (*when* $t = t_0$)

and it is given by expressions (20) *obtained by formally applying Cramer's rule to system* (18).

Proof: Since $\Delta(D)$ is of degree 6 and each of the cofactors $\Delta_{P_i}(D)$, $\Delta_{Q_i}(D)$, $\Delta_{R_i}(D)$ is of degree $\leqq 4$, the rational functions $\Delta_{P_i}(D)/\Delta(D)$, $\Delta_{Q_i}(D)/\Delta(D)$, $\Delta_{R_i}(D)/\Delta(D)$ are of degree $\leqq -2$. Hence, by rule 2, Sec. 1, all the terms of the right members of Equations (20) vanish, together with their first derivatives, for $t = t_0$. Therefore, initial conditions (21) are satisfied.

The remainder of the proof is virtually the same as that of Theorem 1.

If a solution is to be found satisfying the more general initial conditions

(27) $$\begin{aligned} x(t) &= x_0, \quad y(t) = y_0, \quad z(t) = z_0 \\ Dx(t) &= x_1,\ Dy(t) = y_1,\ Dz(t) = z_1 \end{aligned} \qquad (\text{when } t = t_0)$$

then the system of equations is first transformed by the substitution

(28) $$\begin{aligned} x(t) &= x_0 + x_1 \cdot (t - t_0) + X(t) \\ y(t) &= y_0 + y_1 \cdot (t - t_0) + Y(t) \\ z(t) &= z_0 + z_1 \cdot (t - t_0) + Z(t). \end{aligned}$$

The new unknowns $X(t)$, $Y(t)$, $Z(t)$ vanish, together with their first derivatives, for $t = t_0$, hence, can be found by the above method.

Example 11. Find the solution of the system

$$\begin{aligned} x + D^2y + z &= 1 \\ (D^2 - 2)x + 6y &= 0 \\ 6y + (D^2 + 2)z &= 0 \end{aligned}$$

for which

$$x = 1, \qquad Dx = y = Dy = z = Dz = 0 \quad (\text{when } t = t_0)$$

With the transformation $x(t) = 1 + X(t)$ the system becomes

$$\begin{aligned} X + D^2y + z &= 0 \\ (D^2 - 2)X + 6y &= 2 \\ 6y + (D^2 + 2)z &= 0. \end{aligned}$$

The characteristic determinant is

$$\Delta = \begin{vmatrix} 1 & D^2 & 1 \\ D^2 - 2 & 6 & 0 \\ 0 & 6 & D^2 + 2 \end{vmatrix} = -D^2(D^4 - 16).$$

Since it is of degree 6, Theorem 2 applies. By (20),

$$X = \frac{\Delta_{P_2}}{\Delta}2, \qquad y = \frac{\Delta_{Q_2}}{\Delta}2, \qquad z = \frac{\Delta_{R_2}}{\Delta}2.$$

But

$$\begin{aligned} \frac{\Delta_{P_2}}{\Delta} &= \frac{D^4 + 2D^2 - 6}{D^2(D^4 - 16)} = \frac{\frac{3}{8}}{D^2} + \frac{\frac{9}{16}}{D^2 - 4} + \frac{\frac{1}{16}}{D^2 + 4} \\ \frac{\Delta_{Q_2}}{\Delta} &= -\frac{D^2 + 2}{D^2(D^4 - 16)} = \frac{\frac{1}{8}}{D^2} - \frac{\frac{3}{16}}{D^2 - 4} + \frac{\frac{1}{16}}{D^2 + 4} \\ \frac{\Delta_{R_2}}{\Delta} &= \frac{6}{D^2(D^4 - 16)} = \frac{-\frac{3}{8}}{D^2} + \frac{\frac{3}{16}}{D^2 - 4} + \frac{\frac{3}{16}}{D^2 + 4}. \end{aligned}$$

Hence,

$$X(t) = \tfrac{3}{8}t^2 + \tfrac{9}{32}(\cosh 2t - 1) - \tfrac{1}{32}(\cos 2t - 1)$$

and

$$\begin{aligned} x(t) &= \tfrac{3}{4} + \tfrac{3}{8}t^2 + \tfrac{9}{32}\cosh 2t - \tfrac{1}{32}\cos 2t \\ y(t) &= \tfrac{1}{8}t^2 - \tfrac{3}{32}(\cosh 2t - 1) - \tfrac{1}{32}(\cos 2t - 1) \\ &= \tfrac{1}{8} + \tfrac{1}{8}t^2 - \tfrac{3}{32}\cosh 2t - \tfrac{1}{32}\cos 2t \\ z(t) &= -\tfrac{3}{8}t^2 + \tfrac{3}{32}(\cosh 2t - 1) - \tfrac{3}{32}(\cos 2t - 1) \\ &= -\tfrac{3}{8}t^2 + \tfrac{3}{32}\cosh 2t - \tfrac{3}{32}\cos 2t. \end{aligned}$$

When dealing with more general systems of form (18) to which neither Theorem 1 nor Theorem 2 applies, the following procedure is often used. Let $\Delta_{P_1}(D)$, $\Delta_{Q_1}(D)$, . . . be, as before, the differential operators corresponding to the cofactors of the elements $P_1(D)$, $Q_1(D)$, . . . , respectively, in the characteristic determinant $\Delta(D)$. Then applying $\Delta_{P_1}(D)$, $\Delta_{P_2}(D)$, $\Delta_{P_3}(D)$ to the three equations of system (18) we obtain

(29) $$\Delta(D)x = \Delta_{P_1}(D)f_1(t) + \Delta_{P_2}(D)f_2(t) + \Delta_{P_3}(D)f_3(t).$$

In a similar fashion we can obtain

(30) $$\Delta(D)y = \Delta_{Q_1}(D)f_1(t) + \Delta_{Q_2}(D)f_2(t) + \Delta_{Q_3}(D)f_3(t)$$
(31) $$\Delta(D)z = \Delta_{R_1}(D)f_1(t) + \Delta_{R_2}(D)f_2(t) + \Delta_{R_3}(D)f_3(t).$$

These are three linear differential equations† with constant coefficients. Each of them contains but one unknown. The auxiliary equation is the same for each of the three equations, namely, $\Delta(q) = 0$. The three equations differ only in their right-hand members, which are easily calculated from the given functions $f_1(t)$, $f_2(t)$, $f_3(t)$. From our derivation it follows that every solution $x(t)$, $y(t)$, $z(t)$ of system (18) [more precisely, every solution that has as many derivatives as occur in $\Delta(D)$] must satisfy Equations (29), (30), (31). But it does not follow that every solution of the latter three equations is a solution to system (18). In order to solve system (18) one determines the general solutions of Equations (29), (30), (31) and substitutes the found solutions in the equations of system (18). The result of the substitution is a number of relations between the constants of integrations occurring in the general integral of Equations (29), (30), (31).

Example 12. Solve

$$\begin{aligned}(D+1)x + (D+1)y + (D^2-1)z &= 0\\ (D+1)x - (D+1)y + (D+1)^2z &= \sinh t\\ x + y - (D-1)z &= 0.\end{aligned}$$

The characteristic determinant is

$$\Delta(D) = \begin{vmatrix} D+1 & D+1 & D^2-1 \\ D+1 & -(D+1) & (D+1)^2 \\ 1 & 1 & -(D-1) \end{vmatrix} = 4(D+1)^2(D-1).$$

† In the special case when $\Delta(D)$ is a constant different from 0, Equations (29), (30), (31) are algebraic linear equations and are solved simply by dividing through by the constant Δ. If $\Delta(D)$ vanishes identically, then system (18) is dependent, and Equations (29), (30), (31) constitute conditions on $f_1(t)$, $f_2(t)$, $f_3(t)$ for the system to be consistent.

Since $f_1(t)$ and $f_3(t)$ are identically zero we need only the following cofactors:

$$\Delta_{P_2}(D) = 2(D^2 - 1), \qquad \Delta_{Q_2}(D) = -2(D^2 - 1), \qquad \Delta_{R_2}(D) = 0.$$

Hence, Equations (29), (30), (31) become

$$4(D + 1)^2(D - 1)x = 2(D^2 - 1) \sinh t = 0$$
$$4(D + 1)^2(D - 1)y = -2(D^2 - 1) \sinh t = 0$$
$$4(D + 1)^2(D - 1)z = 0 \sinh t = 0.$$

The general solutions of these equations are

$$x = A_1e^t + (A_2 + A_3t)e^{-t}$$
$$y = B_1e^t + (B_2 + B_3t)e^{-t}$$
$$z = C_1e^t + (C_2 + C_3t)e^{-t}.$$

When these expressions are substituted in the three equations of the system, the following equations result:

$$2(A_1 + B_1)e^t + (A_3 + B_3 - 2C_3)e^{-t} = 0$$
$$2(A_1 - B_1 + 2C_1)e^t + (A_3 - B_3)e^{-t} = \sinh t = \tfrac{1}{2}e^t - \tfrac{1}{2}e^{-t}$$
$$(A_1 + B_1)e^t + [A_2 + B_2 + 2C_2 - C_3 + (A_3 + B_3 + 2C_3)t]e^{-t} = 0.$$

Since these equations must be identities in t they imply

$$A_1 + B_1 = 0, \qquad A_1 - B_1 + 2C_1 = \tfrac{1}{2}$$
$$A_3 + B_3 - 2C_3 = 0, \qquad A_3 - B_3 = -\tfrac{1}{2}, \qquad A_3 + B_3 + 2C_3 = 0$$
$$A_2 + B_2 + 2C_2 - C_3 = 0$$

or

$$A_1 \text{ arbitrary}, \qquad B_1 = -A_1, \qquad C_1 = \tfrac{1}{8} - A_1$$
$$A_3 = -\tfrac{1}{4}, \qquad B_3 = \tfrac{1}{4}, \qquad C_3 = 0$$
$$A_2 \text{ arbitrary}, \qquad B_2 \text{ arbitrary}, \qquad C_2 = -\frac{A_2 + B_2}{2}.$$

Hence, the general solution of the given system is

$$x = A_1e^t + (A_2 - \tfrac{1}{4}t)e^{-t}$$
$$y = -A_1e^t + (B_2 + \tfrac{1}{4}t)e^{-t}$$
$$z = (\tfrac{1}{8} - A_1)e^t - \frac{A_2 + B_2}{2}e^{-t}.$$

One notices that the number of arbitrary constants in the general solution is 3, which is also the degree of the characteristic determinant. This is in agreement with the general result mentioned in the footnote on page 207.

PROBLEMS

Solve the following systems with the given initial conditions:

1. $(D-1)x + 2y = 0$
$3x + (D-2)y = 0.$

(a) $x = 1,\ y = 0$ when $t = 0$
(b) $x = 0,\ y = 1$ when $t = 0.$

2. $Dx + (D-1)y = 3e^t$
$2Dx + (D+2)y = 0$
$x = 1,\ y = 0$ when $t = 0.$

3. $(D+a)x - by = A$
$bx + (D-a)y = B;\quad a^2 - b^2 = 1$
$x = y = 0$ when $t = 0.$

4. $(D+3)x + 16y + 14z = 0$
$12x - (D-40)y + 33z = 0$
$12x + 38y + (D+31)z = 0$
$x = 1,\ y = z = 0$ when $t = 0.$

5. $(D^2 - D)x + Dy = e^t$
$(D^2 - 1)x + D^2y = e^t$
$x = Dx = y = Dy = 1$ when $t = 0.$

6. $(D^2+2)x + 3y = \cos 3t$
$7x + (D^2+6)y = -\cos 3t$
$x = Dx = y = Dy = 0$ when $t = 0.$

7. $D^2x + a\,Dy = A\cos at$
$-a\,Dx + D^2y = 0;\quad a \neq 0$
$x = Dx = y = Dy = 0$ when $t = 0.$

8. $D^2x + y = 2\cos t$
$(D^2-1)y + D^2z = 0$
$(D^2-1)x + D^2z = 0$
$x = Dx = y = Dy = 0,\ z = z_0,\ Dz = z_1$ when $t = 0.$

Find the general solutions of the following systems:

9. $Dx + 4y = 2e^t$
$(2D-3)x + (D^2 - D)y = 0.$

10. $(D^2+3)x + 2Dy = 2e^{-t}$
$(D-1)x + y = -e^{-t}.$

11. $(D^3+1)x + Dy = \cos 2t$
$(D^2-1)x + y = -\cos 2t.$

12. $(D^2+2)x + Dy + Dz = 0$
$Dx + y = 0$
$2x - Dy - Dz = 0.$

***13.** $Dx + ay - bz = 0$
$-ax + Dy + cz = 0$
$bx - cy + Dz = 0;\quad a^2 + b^2 + c^2 = 1.$

4. Reduction to Diagonal Form. The method of solving a system of differential equations by reducing it to a diagonal form is quite general and applies to any system of simultaneous linear differential equations with constant coefficients. There is no restriction as to the number of unknowns, number of equations, orders of equations, nor is there any need for the hypothesis that the number of unknowns be equal to the number of equations.

The method consists essentially in this: the given system is transformed into an *equivalent* one with the property that each of its equations contains at least one unknown that does not occur in the preceding equations. Once this is accomplished the solution is at hand. The equations of the transformed system are solved one by one for the unknowns that do not occur in the preceding equations. An example where the described transformation can be achieved simply by rearrangement of the equations will illustrate the method of solution.

Example 13. Find the general solution of the system

$$\begin{aligned} x + (D - 1)y + (D + 2)z &= 0 \\ Dy + Dz - u &= 0 \\ (D + 1)x - y &= 0. \end{aligned}$$

We have three equations with the four unknowns $x(t)$, $y(t)$, $z(t)$, $u(t)$. If we write them in the order

$$\begin{aligned} (D + 1)x - y &= 0 \\ x + (D - 1)y + (D + 2)z &= 0 \\ Dy + Dz - u &= 0 \end{aligned}$$

then the system is in diagonal form, that is, has the property that each equation contains an unknown that does not occur in the preceding equations. The first equation can now be solved without regard to the remaining equations. Since it contains two unknowns, one of them, say $x(t)$, may be chosen arbitrarily. Hence, we put

$$x = f(t) \qquad \text{(arbitrary)}$$

then

$$y = (D + 1)f(t).$$

Substitution of these expressions in the second of the above equations yields for $z(t)$ the equation

$$\begin{aligned} (D + 2)z &= -f(t) - (D - 1)(D + 1)f(t) \\ &= -D^2f(t). \end{aligned}$$

Hence,

$$z = -\int^t e^{2(t_1-t)}D^2f(t_1)\,dt_1 + Ae^{-2t},$$

where A is an arbitrary constant. Finally, u is obtained by substitution in the last of the above equations:

$$\begin{aligned} u &= D(D + 1)f(t) + D\left(-\int^t e^{2(t_1-t)}D^2f(t_1)\,dt_1 + Ae^{-2t}\right) \\ &= (D^2 + D)f(t) - D^2f(t) + 2\int^t e^{2(t_1-t)}D^2f(t_1)\,dt_1 - 2Ae^{-2t} \\ &= Df(t) + 2\int^t e^{2(t_1-t)}D^2f(t_1)\,dt_1 - 2Ae^{-2t}. \end{aligned}$$

Two systems of simultaneous differential equations are said to be *equivalent* if each solution of one system is also a solution of the other system. We now show, first, how a pair of equations containing the same unknown can be transformed into an equivalent pair one of whose equations is free of this unknown. Let the two equations be written symbolically as

$$L_1 = 0, \qquad L_2 = 0. \tag{32}$$

(This notation does not imply that the equations are assumed to be homogeneous.) Assume the common unknown is $x(t)$ and $P_1(D)$, $P_2(D)$ are the coefficients of $x(t)$ in L_1, L_2, respectively. Then the equation

$$P_2(D)L_1 - P_1(D)L_2 = 0 \tag{33}$$

which is derived from Equations (32) is obviously free from $x(t)$, but Equation (33) together with one of Equations (32) does not necessarily constitute a pair of equations that is equivalent to the original pair (32). For, since Equation (33) is, in general, of higher order than either of Equations (32), the new pair of equations has solutions that are not solutions of system (32). The proper way of procedure is first to determine the *least common multiple* $P(q)$ of the two polynomials $P_1(q)$, $P_2(q)$ (that is, the polynomial of lowest degree that contains $P_1(q)$, $P_2(q)$ as factors). Then there must be factors $Q_1(q)$, $Q_2(q)$ such that

$$Q_1(q)P_1(q) = Q_2(q)P_2(q) = P(q).$$

Therefore, the equation

$$Q_1(D)L_1 - Q_2(D)L_2 = 0, \tag{34}$$

which is derived from Equations (32), is free from $x(t)$.

By their choice, the polynomials $Q_1(q)$, $Q_2(q)$ have no common factor. For a pair of such polynomials it is possible† to find two other polynomials $Q_1^*(q)$, $Q_2^*(q)$ such that

$$\begin{vmatrix} Q_1(q) & Q_2(q) \\ Q_1^*(q) & Q_2^*(q) \end{vmatrix} = Q_1(q)Q_2^*(q) - Q_2(q)Q_1^*(q) \equiv 1. \tag{35}$$

† For a proof see A. A. Albert, "College Algebra," p. 102, McGraw-Hill, 1946. In most cases the polynomials $Q_1(q)$, $Q_2(q)$ are so simple that the polynomials $Q_1^*(q)$, $Q_2^*(q)$ satisfying identity (35) can be guessed. In other cases the method of undetermined coefficients can be used to find $Q_1^*(q)$, $Q_2^*(q)$.

With the operator polynomials $Q_1^*(D)$, $Q_2^*(D)$ formed from the polynomials $Q_1^*(q)$, $Q_2^*(q)$, we can derive another equation from (32),

$$(36) \qquad Q_1^*(D)L_1 - Q_2^*(D)L_2 = 0.$$

We can now show that the two equations (34), (36) are equivalent to Equations (32). Since they were derived from (32) every solution of the latter must be necessarily a solution of (34), (36). On the other hand, operating with $Q_2^*(D)$ on (34) and with $-Q_2(D)$ on (36), we obtain, because of (35), $L_1 = 0$. Likewise, by operating with $Q_1^*(D)$ on (34) and with $-Q_1(D)$ on (36) we obtain $L_2 = 0$. Hence, Equations (32) are derivable from (34), (36), and consequently, every solution of the latter must be a solution of (32).

Thus we have a method of transforming a pair of equations containing the same unknown, say $x(t)$, into an equivalent pair of equations one of which is free from $x(t)$. We use operators $Q_1(D)$, $Q_2(D)$, which have no common factor, to derive the equation free from $x(t)$, and another pair of operators $Q_1^*(D)$, $Q_2^*(D)$, which we may call *adjoint operators* and which are related to $Q_1(D)$, $Q_2(D)$ by identity (35),† to derive the other equation. By this method any system can be reduced to an equivalent *diagonal system*, which has the property that each of its equations contains at least one unknown that does not occur in the preceding equations.‡ To do this, pairs of adjoint operators are applied to all pairs of equations containing the same unknown, say $x(t)$, until this unknown is successively eliminated from all but (at most) one equation, say equation (a). Leaving this equation aside and applying adjoint operators to all pairs of the remaining equations containing the same unknown, say $y(t)$, this unknown is successively eliminated from all but (at most) one equation, say equation (b). When this procedure is carried on, an equivalent system of equations (a), (b), (c), . . . is obtained where each equation has at least one unknown that does not occur in the following ones. Hence, the inverted system . . . (c), (b), (a) is in diagonal form.

If the system in its diagonal form is such that the first equation contains one unknown and each succeeding equation contains exactly one more unknown, then the unknowns can be determined one by one from these equations. The system is then *determinate* since the unknowns are, except for arbitrary constants of integration, uniquely determined.

† It is clear that in place of unity we may choose any constant different from zero as the right member of identity (35).

‡ It may happen that one or more of the first equations contain no unknowns at all.

If, however, the first equation of the diagonalized system contains more than one unknown, or if any of the other equations contain more than one unknown above those occurring in the preceding equations, then some of the unknowns remain arbitrary and the remaining ones are expressed in terms of these arbitrary ones. The system is *indeterminate*.

Finally, it may happen that the first or some of the first equations of the diagonalized system contain no unknown at all. Then these equations are either identities and can be disregarded. In this case the system is *dependent* and *consistent*. Or the equations without unknowns are not identities, then they cannot be solved. In this case the system is *dependent* and *inconsistent*.

Example 14. Find the general solution of the system

$$(a)\qquad (D^2 - 1)x + 2(D + 1)y + (D + 1)z = 0$$
$$(b)\qquad (D - 1)^2x + 4Dy + (2D - 1)z = 0$$
$$(c)\qquad (D - 1)x + 2y - (D - 1)z = 0.$$

The unknown y is easily eliminated from Equations (b), (c) by applying the operators 1, $2D$. The adjoint operators are, obviously, 0, 1. Leaving (a) unchanged we then have the equivalent system

$$(a)\qquad (D^2 - 1)x + 2(D + 1)y + (D + 1)z = 0$$
$$(b')\qquad -(D^2 - 1)x + (2D^2 - 1)z = 0$$
$$(c)\qquad (D - 1)x + 2y - (D - 1)z = 0.$$

Now y is eliminated from Equations (a), (c) by applying the operators 1, $(D + 1)$. The adjoint operators are 0, 1, and we obtain

$$(a')\qquad (D^2 + D)z = 0$$
$$(b')\qquad -(D^2 - 1)x + (2D^2 - 1)z = 0$$
$$(c)\qquad 2y + (D - 1)x - (D - 1)z = 0.$$

The system is now in diagonal form, and it is seen to be a determinate system. The general solution of (a') is

$$z = C_1 + C_2e^{-t}.$$

Substitution in (b') leads to the following equation for x:

$$(D^2 - 1)x = -C_1 + C_2e^{-t},$$

whose general solution is

$$x = C_1 - \frac{C_2}{2}te^{-t} + A_1e^t + A_2e^{-t}.$$

Finally, from equation (c),

$$y = \left[A_2 - \frac{C_2}{2}\left(t + \frac{3}{2}\right)\right] e^{-t}.$$

Example 15. How must the function $h(t)$ be chosen to make the following system consistent, and what is then its general solution?

$$(a) \qquad (D^2 - 1)x - (D^2 - D)y = h(t)$$
$$(b) \qquad (D^2 + D)x - D^2y = 0.$$

The coefficients of y have the common factor D. Hence, the operators used to eliminate y from equations (a), (b) are D, $(D - 1)$, and the adjoint operators are 1, 1. The transformed equations are

$$(a') \qquad 0x + 0y = Dh(t)$$
$$(b') \qquad -(D + 1)x + Dy = h(t).$$

Since Equation (a') contains neither x nor y, system (a), (b) is dependent. Equation (a') is an identity only if $Dh(t) \equiv 0$; hence system (a), (b) is consistent if and only if $h(t)$ is constant, say $h(t) = k$. Then only Equation (b') remains to be satisfied. Since this equation contains two unknowns one of them can be chosen arbitrarily, say

$$x = f(t) \qquad \text{(arbitrary)}.$$

Then we have for y the equation

$$Dy = k + (D + 1)f(t),$$

whose general solution is

$$y = kt + f(t) + \int^t f(t_1)\, dt_1 + A,$$

where A is an arbitrary constant.

As was stated before and as can be seen from the above examples, in many cases it is quite easy to guess adjoint operators. Moreover, for the very common case where the two equations from which the unknown is to be eliminated are of first order in this unknown, adjoint operators are found by a simple general rule. Let the two equations be

$$(37) \qquad \begin{aligned} (a_1D + a_2)x + A &= 0 \\ (b_1D + b_2)x + B &= 0, \end{aligned}$$

where x is the unknown to be eliminated, and A, B denote the terms free from x. If the polynomials $(a_1q + a_2)$, $(b_1q + b_2)$ have no factor in common, that is, if $a_1b_2 - a_2b_1 \neq 0$, then the operators

$$Q_1(D) = b_1D + b_2, \qquad Q_2(D) = a_1D + a_2$$

are used to eliminate x. Adjoint operators are

$$Q_1{}^*(D) = b_1, \qquad Q_2{}^*(D) = a_1,$$

because

$$Q_1(q)Q_2{}^*(q) - Q_2(q)Q_1{}^*(q) = a_1b_2 - a_2b_1 \neq 0.$$

The result can be stated as follows: If $a_1b_2 - a_2b_1 \neq 0$, then the two equations equivalent to Equations (37) are obtained by first eliminating both x and Dx, and second eliminating Dx, but not x.

If $a_1b_2 - a_2b_1 = 0$, then the two polynomials $a_1q + a_2$, $b_1q + b_2$ differ only by a numerical factor, say, $b_1q + b_2 = k(a_1q + a_2)$. Then we may choose the operators k, 1 and the adjoint operators 0, 1 or 1, 0. Hence, if

$$a_1b_2 - a_2b_1 = 0,$$

then one equation is left unchanged and the other equation is obtained by eliminating x and Dx using constants as operators.

PROBLEMS

Reduce to diagonal systems and solve:

1. $(D^2 + 1)x + Dy = h(t)$
$D^3x + (D^2 - 1)y = 0.$

2. $(D^2 - 3D)x + (2D - 1)y = 0$
$(D + 1)x - Dy = e^{3t}.$

3. $Dx = y - z$
$Dy = z - x$
$Dz = x - y.$

4. $(D^2 - 1)x + (D + 1)y + (D + 1)z = 0$
$(D - 1)^2x + 2Dy + (2D - 1)z = 0$
$(D - 1)x + y - (D - 1)z = 0$
$D(D + 1)z = 0.$

5. $D^2x + y = 0$
$(D^2 - 1)y + D^2z = 0$
$x + y + z = h(t).$

6. $(D^2 + D)x + Dz - u = 0$
$Dx - Dy - (D + 2)z = 0$
$x - y + 2z + u = 0.$

7. How must the constant a be chosen so as to make the following system consistent, and what is then its general solution?

$$(D^4 - 1)x + (D^2 + 1)y = 2e^t$$
$$(D^2 - 1)^2x + (D^2 - 1)y = ae^t.$$

8. Reduce the systems of Probs. 9–13, Sec. 3, to diagonal systems and solve them.

APPLICATIONS

5. Applications to Mechanical Systems. We consider mechanical systems in which the motion is restricted either to translations along fixed straight lines (*translational systems*) or to rotations about fixed axes (*rotational systems*). We assume that the moving bodies are rigid and that the guides constraining the motion are of negligible mass.

The elementary components of the *translational* systems to be considered are masses constrained to move along fixed lines, springs, dampers, and sources of force (acceleration) or of velocity. Although one and the same physical component of the system may act as a moving mass, as a spring, and as a damper, we shall assume that masses, springs, and dampers are separate elements of the system. Each of these elements results in a force acting along the line of motion.

Masses provide *reaction* or *inertia* forces. If x is the displacement of mass m with respect to a reference frame that is at rest or in uniform motion, and $v = Dx$ is the velocity, then the force necessary to impart the acceleration D^2x to mass m is $mDv = mD^2x$. The negative of this force, that is, $-mD^2x$, is the force resulting from the motion of the inert mass. It is called *reaction force due to inertia.*

Springs provide *restoring forces.* If stretched they tend to contract, thus exerting a pull; if compressed they tend to expand, thus exerting a push. If x is the displacement difference of the ends of the spring, then the restoring force is assumed to be $-kx$, where k is independent of x (that is, the spring is "linear") and is called the stiffness constant of the spring.

Dampers provide *resistance forces.* They tend to slow down the motion. If $v = Dx$ is the velocity difference of the parts of the damper that are in frictional contact, then the resistance force is assumed to be $-bv$, where b is independent of v (that is, the damper is "linear") and is called the resistance constant of the damper. Viscous friction as provided in a dashpot between piston and cylinder can be considered, within wide limits, as a linear resistance force.

The masses m, stiffness constants k, and resistance constants b are assumed to be invariant in time. With these assumptions the equations of motion of the mechanical systems to be considered are linear differential equations with constant coefficients. Their nonhomogeneous parts represent the sources of force or of velocity, which may be arbitrary functions of time. The equations of motion may be established by the use of *D'Alembert's principle*,† which may be expressed as follows:

The sum of all the instantaneous forces acting on a body including the reaction force due to inertia is zero.

This is but another version of Newton's second law of motion. By this principle the equations of dynamics are formally the same as those of statics, the difference consisting only in the inclusion of reactive forces.

An example will illustrate the proper use of the introduced concepts.

Example 16. A mechanical system (see Fig. 34) consists of two masses m_1, m_2, connected by a spring of stiffness k. The masses are

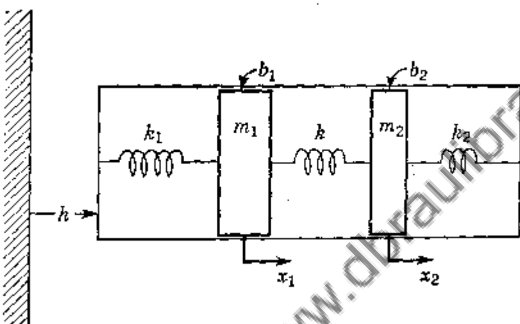


FIG. 34.

constrained by a frame to move only in a horizontal direction. There is viscous friction between the masses and the frame, the respective resistance constants being b_1, b_2. Mass m_1 is restrained to one end of the frame by a spring of stiffness k_1; mass m_2 is restrained to the other end by a spring of stiffness k_2. Determine the motion of the masses resulting from a given horizontal motion of the frame, assuming that at time $t = 0$ the springs are not stressed and the masses are at rest.

Let $h(t)$, $x_1(t)$, $x_2(t)$ be the displacements of the frame and of the masses, at time t, from fixed reference lines which are chosen such that when $h = x_1 = x_2 = 0$ there is no stress in the springs. Then the elongations of the springs k_1, k, k_2 at time t are, respectively, $x_1(t) - h(t)$, $x_2(t) - x_1(t)$, $h(t) - x_2(t)$. Therefore, the restoring forces acting on m_1, m_2 are, respectively,

$$k(x_2 - x_1) - k_1(x_1 - h), \qquad k_2(h - x_2) - k(x_2 - x_1).$$

† Named after the French mathematician and encyclopedist Jean le Rond d'Alembert, 1717–1783.

Since the velocities of the masses with respect to the frame at time t are $D(x_1(t) - h(t))$, $D(x_2(t) - h(t))$, the resistance forces acting on m_1, m_2 are, respectively,

$$-b_1D(x_1 - h), \qquad -b_2D(x_2 - h).$$

Hence, by D'Alembert's principle, the equations of motion are

$$\begin{aligned}-m_1D^2x_1 - b_1D(x_1 - h) + k(x_2 - x_1) - k_1(x_1 - h) &= 0\\ -m_2D^2x_2 - b_2D(x_2 - h) + k_2(h - x_2) - k(x_2 - x_1) &= 0\end{aligned}$$

or

$$\begin{aligned}(m_1D^2 + b_1D + k + k_1)x_1 - kx_2 &= (b_1D + k_1)h\\ -kx_1 + (m_2D^2 + b_2D + k + k_2)x_2 &= (b_2D + k_2)h.\end{aligned}$$

If we denote the known right members of these equations by $f_1(t), f_2(t)$, then the solution of this system for which $x_1 = Dx_1 = x_2 = Dx_2 = 0$ when $t = 0$ can be written symbolically as (see Theorem 2, Sec. 3)

$$\begin{aligned}x_1(t) &= \frac{m_2D^2 + b_2D + k + k_2}{\Delta(D)} f_1(t) + \frac{k}{\Delta(D)} f_2(t)\\ x_2(t) &= \frac{m_1D^2 + b_1D + k + k_1}{\Delta(D)} f_2(t) + \frac{k}{\Delta(D)} f_1(t),\end{aligned}$$

where

$$\Delta(D) = (m_1D^2 + b_1D + k + k_1)(m_2D^2 + b_2D + k + k_2) - k^2.$$

We next consider *rotational* systems. Their elementary components are masses constrained to rotate about fixed axes, torsional springs, dampers, and sources of torque (angular acceleration) and of angular velocity. Each of these elements results in a torque acting about the axis of rotation.

Masses provide *reaction* or *inertia* torques. If θ is the angular displacement of mass m about the axis of rotation from a reference line that is fixed or in uniform rotation, and if $\omega = D\theta$ is the angular velocity, then the torque necessary to impart the angular acceleration $D^2\theta = D\omega$ to mass m is $ID\omega = ID^2\theta$, where I is the polar moment of inertia of mass m with respect to the axis of rotation. The negative of this torque, that is $-ID^2\theta$, is the torque resulting from the rotation of the inert mass. It is called *reaction torque due to inertia*.

Torsional springs provide *restoring torques* as a result of angular twist. If θ is the angular displacement difference of the ends of the spring, then the restoring torque is assumed to be $-k\theta$, where k is independent of θ and is called the torsional stiffness constant of the spring.

Dampers provide *resistance torques.* If $\omega = D\theta$ is the angular velocity difference of the surfaces of the damper that are in frictional contact, then the resistance torque is assumed to be $-b\omega$, where b is independent of ω and is called the torsional resistance constant of the damper.

As for translational systems, the equations of motion for rotational systems are linear differential equations with constant coefficients if we assume the inertia moments I, the stiffness constants k, and the resistance constants b to be invariant in time. D'Alembert's principle, stated above for translational systems, applies to rotational systems, too, provided that in its statement forces are replaced by torques.

Example 17. A shaft (see Fig. 35) carries three flywheels of the same polar moment of inertia I. One of its ends is free, the other

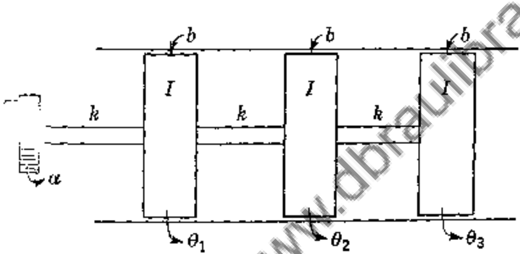


FIG. 35.

end carries a driving wheel. The three pieces of shaft between the driving wheel and flywheels are of the same torsional stiffness k. The flywheels are in frictional contact with a stationary surface, the torsional resistance constant for each being b. Determine the angular motion of the flywheels resulting from a given angular motion of the driving wheel, assuming that at time $t = 0$ the shaft is not twisted and the flywheels are at rest.

Let $\alpha(t)$, $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$ be the angular displacements, at time t, of the driving wheel and flywheels from fixed reference lines which are such that when $\alpha = \theta_1 = \theta_2 = \theta_3 = 0$ there is no torsion in the shaft. Then the twists of the three pieces of shaft at time t are $\theta_1(t) - \alpha(t)$, $\theta_2(t) - \theta_1(t)$, $\theta_3(t) - \theta_2(t)$. Therefore, the restoring torques acting on the three rotors are

$$k[(\theta_2 - \theta_1) - (\theta_1 - \alpha)],\ k[(\theta_3 - \theta_2) - (\theta_2 - \theta_1)],\ -k(\theta_3 - \theta_2).$$

The resistance torques are

$$-bD\theta_1, \qquad -bD\theta_2, \qquad -bD\theta_3.$$

Therefore, the equations of motion are

$$\begin{aligned} -ID^2\theta_1 - bD\theta_1 + k(\theta_2 - 2\theta_1 + \alpha) &= 0 \\ -ID^2\theta_2 - bD\theta_2 + k(\theta_3 - 2\theta_2 + \theta_1) &= 0 \\ -ID^2\theta_3 - bD\theta_3 - k(\theta_3 - \theta_2) &= 0, \end{aligned}$$

or

$$\begin{aligned} (ID^2 + bD + 2k)\theta_1 - k\theta_2 &= k\alpha \\ -k\theta_1 + (ID^2 + bD + 2k)\theta_2 - k\theta_3 &= 0 \\ -k\theta_2 + (ID^2 + bD + k)\theta_3 &= 0. \end{aligned}$$

By Theorem 2, Sec. 3, the solution of this system for which $\theta_1 = D\theta_1 = \theta_2 = D\theta_2 = \theta_3 = D\theta_3 = 0$ when $t = 0$ is, symbolically,

$$\begin{aligned} \theta_1(t) &= k\,\frac{(ID^2 + bD + 2k)(ID^2 + bD + k) - k^2}{\Delta(D)}\,\alpha(t) \\ \theta_2(t) &= k^2\,\frac{ID^2 + bD + k}{\Delta(D)}\,\alpha(t) \\ \theta_3(t) &= k^3\,\frac{1}{\Delta(D)}\,\alpha(t), \end{aligned}$$

where

$$\Delta(D) = \begin{vmatrix} ID^2 + bD + 2k & -k & 0 \\ -k & ID^2 + bD + 2k & -k \\ 0 & -k & ID^2 + bD + k \end{vmatrix}.$$

PROBLEMS

1. A train is made up of a locomotive of mass m_1 and one car of mass m_2. The connecting coupling is a spring of stiffness constant k together with a shock absorber that is a viscous friction damper of resistance constant b. Determine the motions of locomotive and car as functions of time if the train starts from rest at time $t = 0$ and the driving force of the locomotive is $f(t)$.

2. Determine the motion of the train of Prob. 1 if the velocity of the locomotive is v_0 at time $t = 0$ and there is no driving force. What is the smallest value of the resistance constant b so that the relative motion of locomotive and car is not oscillatory?

3. A train is made up of a locomotive and two cars, each having a mass m. The connecting couplings are springs of stiffness constant k together with shock absorbers of resistance constant b. Determine the motions of the three units if the locomotive has velocity v_0 at time $t = 0$ and there is no driving force. What is the smallest value of the resistance constant b so that the relative motion of the units is not oscillatory?

4. A uniform shaft free to rotate in bearings carries three disks. The polar moment of inertia of each of the two end disks is I, that of the center disk is $2I$. The torsional stiffness constant of the shaft between each two disks is k.

Determine the motion of one of the end disks if the shaft starts from rest and an alternating torque $T_0 \sin \omega t$ is applied to the center disk. For what values of ω is there resonance?

5. A shaft is fixed at the point O and carries disks of the same polar moment of inertia at the points A, B, C. If the torsional stiffness constants of the portions OA, AB, BC of the shaft are $k_1 = 11k$, $k_2 = 8k$, $k_3 = 19k$, respectively, determine the natural frequencies of the vibrating shaft.

6. A shaft carries at one end a disk, whose polar moment of inertia is I_1. At the other end it carries a vibration damper consisting of a drum that is rigidly attached to the shaft and an inner flywheel, whose rotation relative to the drum produces damping represented by the torsional resistance constant b. The polar moments of inertia of drum and flywheel are I_2 and I_3, respectively, and the torsional stiffness constant of the shaft is k. Determine the motion of the disk if a torque $T(t)$ is applied to it, assuming that the shaft is at rest when $t = 0$. What is the frequency of the free damped oscillations (that is, with no torque applied) if in a system of consistent units $I_1 = 111$, $I_2 = 1$, $I_3 = 8$; $k = 111$, $b = 16$.

7. A mechanical system consists of two pendulums of equal length l carrying equal masses m suspended from points A, B lying on a horizontal. The pendulums are connected by a spring $A'B'$ of stiffness constant k, where $AA' = BB' = h$. Determine the natural frequencies of small vibrations of the system. *Hint:* For small angles the sine can be replaced by the angle itself and the cosine by unity. In this way the problem is "linearized."

6. Application to Electric Systems. The electric systems that we are going to consider are one-dimensional networks whose physical dimensions are small as compared with the main wave lengths of the currents under consideration. With this restriction we may assume that all current and voltage changes take place practically instantaneously throughout the system, and hence we may disregard their dependence on space variables. The elements of the network appear then as "lumped" rather than as "distributed."

The elementary components of the networks to be considered are resistors, capacitors, inductors, and sources of voltage (electromotive force) or of current. Physically, these are commonly represented by resistance coils, condensers, inductance coils, and batteries or generators. Although one and the same physical component may combine the properties of a resistor, capacitor, and inductor, we shall assume that these are separate elements of the system.

Resistors cause voltage drops. If i is the current flowing through the resistor, then the voltage drop across the resistor is Ri, where R is independent of i (that is, the resistor is "linear") and is called the resistance. Since i is the rate of change of the charge q flowing through any cross section of the resistor, $i = Dq$, the voltage drop across the resistor may also be written as RDq. Capacitors when charged act like voltage sources. If q is the charge on the capacitor, then the voltage drop across the capacitor is assumed to be $(1/C)q$, where C is independent of q and is called the capacitance. Inductors tend to resist current changes. The voltage necessary to induce the rate of change Di in the inductor is assumed to be LDi, where L is independent of Di and is called (self-) inductance.

The resistances R, capacitances C, inductances L are assumed to be invariant in time. With these assumptions the network equations of the electric systems to be considered are linear differential equations with constant coefficients. Their nonhomogeneous parts represent the sources of voltage or current, which may be arbitrary functions of time.

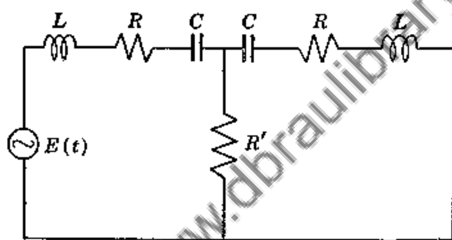


FIG. 36.

The network equations are readily established by the use of Kirchhoff's laws:

1. The sum of the instantaneous branch currents flowing to or from a junction point (node) in the network is zero.

2. Around any closed circuit in the network the sum of the instantaneous voltage drops in a specific direction is zero.

For branches that contain capacitors one uses the charge flowing through any cross section as the dependent variable; for branches without capacitors one preferably uses the current as the dependent variable since the equation in the current is of lower order than in the charge.

An example will illustrate the concepts introduced here.

Example 18. Determine the currents through the inductance coils of the network of Fig. 36 if these currents and the charges on the condensers are zero at time $t = 0$.

As the dependent variables we choose the charges $q_1(t)$, $q_2(t)$ on the two condensers. Then the currents through the inductance coils are $Dq_1(t)$, $Dq_2(t)$ and the current through the resistor R' is, by the first of the Kirchhoff laws, $D[q_1(t) - q_2(t)]$. Hence, using the second Kirchhoff law, we obtain the following network equations:

$$\left(LD^2 + RD + \frac{1}{C}\right) q_1 + R'D(q_1 - q_2) = E(t)$$
$$\left(LD^2 + RD + \frac{1}{C}\right) q_2 - R'D(q_1 - q_2) = 0,$$

or

$$\left[LD^2 + (R + R')D + \frac{1}{C}\right] q_1 - R'Dq_2 = E(t)$$
$$-R'Dq_1 + \left[LD^2 + (R + R')D + \frac{1}{C}\right] q_2 = 0.$$

By Theorem 2, Sec. 3, the solution of the system for which

$$q_1 = Dq_1 = q_2 = Dq_2 = 0$$

when $t = 0$ is, symbolically,

$$q_1(t) = \frac{LD^2 + (R + R')D + C^{-1}}{\Delta(D)} E(t)$$
$$q_2(t) = \frac{R'D}{\Delta(D)} E(t),$$

where

$$\Delta(D) = \left[LD^2 + (R + R')D + \frac{1}{C}\right]^2 - R'^2D^2$$
$$= \left(LD^2 + RD + \frac{1}{C}\right)\left[LD^2 + (R + 2R')D + \frac{1}{C}\right].$$

Finally, we have

$$i_1(t) = Dq_1(t) = \frac{LD^3 + (R + R')D^2 + C^{-1}D}{\Delta(D)} E(t)$$
$$i_2(t) = Dq_2(t) = \frac{R'D^2}{\Delta(D)} E(t).$$

PROBLEMS

1. If in the network represented by Fig. 37 there is, at time $t = 0$, no charge on the condenser and no current flowing, determine the charge as a function of time, using $C = 10^{-5}$ farad, $L_1 = 0.1$ henry, $R_1 = 50$ ohms, $R_2 = 200$ ohms, $E = 100$ volts.

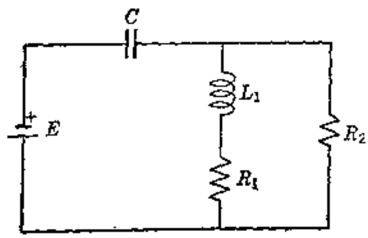


FIG. 37.

2. If in the network represented by Fig. 38 there are, at time $t = 0$, no charges on the condensers and no currents flowing, determine the currents through the coils L_1, L_2 as functions of time.

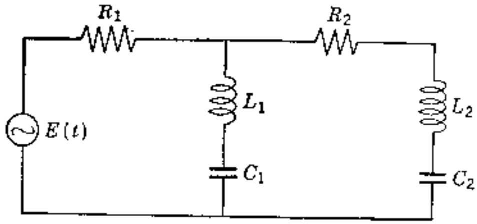


FIG. 38.

3. Each side of a triangular circuit contains a capacitance C, and each vertex is connected to a common central point by an inductance L. Show that the natural frequency of the oscillations of this network is $1/2\pi\sqrt{3LC}$. Also consider the network with capacitances and inductances interchanged.

4. Assuming there is initially no charge on the condenser of the symmetric T network of Fig. 39*a* and no currents flowing, determine the voltage $v(t)$ across the terminal resistance R if $L = 0.01$ henry, $C = 2 \times 10^{-6}$ farad, $R = 100$ ohms.

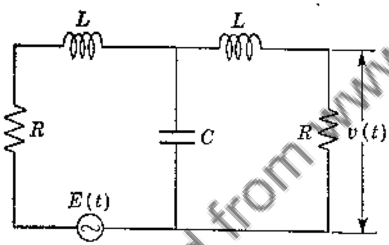


FIG. 39*a*.

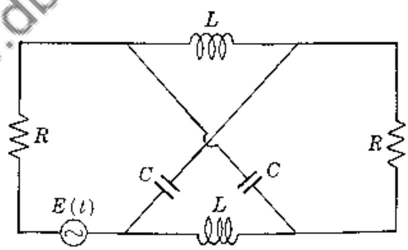


FIG. 39*b*.

5. Assuming there are initially no charges on the condensers of the symmetric lattice network of Fig. 39*b* and no currents flowing, show that the sum of the charges at time t is

$$\frac{1}{R}\int_0^t e^{(t_1-t)/RC}E(t_1)\,dt_1.$$

CHAPTER VII

LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

In this chapter a few methods are taken up that apply to the solution of linear differential equations with variable coefficients. The methods are, of necessity, restricted both as to their scope and their practicality. A more general treatment of this subject would require the use of advanced parts of the theory of functions, whose knowledge is not assumed in this elementary exposition.

It should be recalled at this point that the results established in Secs. 1–7 of Chap. V are quite general and apply to linear equations with variable coefficients as well as to those with constant coefficients. A brief summary of the most important of these results follows:

In the neighborhood of an "ordinary" point there exists one and only one solution of the equation satisfying given initial conditions.

There exists a linear basis for the general solution of the reduced equation.

The general solution of the complete equation can be obtained as the sum of any particular solution and of the general solution of the reduced equation.

Solutions of various equations belonging to the same reduced equation add up to a solution of a similar equation with a nonhomogeneous term that is the sum of the various nonhomogeneous terms (principle of superposition).

These results will be used extensively in the following sections.

1. Equations of Euler-Cauchy.† These equations are closely related to those with constant coefficients. They are of the form

$$(1)\quad x^n \frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_ny = f(x),$$

where $a_1, a_2, \ldots, a_n$ are constants. Equations of this type may also be written in the form

$$(2)\quad \frac{d^ny}{dx^n} + \frac{a_1}{x}\frac{d^{n-1}y}{dx^{n-1}} + \frac{a_2}{x^2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + \frac{a_{n-1}}{x^{n-1}}\frac{dy}{dx} + \frac{a_n}{x^n}y = g(x),$$

† Named varyingly, after the Swiss mathematician Leonhard Euler, 1707–1783, or the French mathematician Augustin Cauchy, 1789–1857.

and since the "degrees" of x and of dx add up to the same number $-n$ in each of the terms of the left-hand side of Equation (2), such equations may be said to be of *homogeneous dimension.*

In order to solve Equation (1) we introduce a new independent variable t defined by the equation

$$(3) \qquad x = e^t, \qquad t = \log x$$

Since $\log x$ is real only when $x > 0$, this substitution is, strictly speaking, valid only for $x > 0$. The point $x = 0$ is not an ordinary point of the equation, as can be seen from (2) (see also Sec. 3, Chap. VIII), and is, therefore, excluded from our considerations. For $x < 0$ one may employ the substitution

$$(4) \qquad -x = e^t, \qquad t = \log(-x)$$

and then proceed as with substitution (3).

To examine the effect of substituting $x = e^t$ in an equation of the above type consider the equation of second order

$$(5) \qquad x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = f(x).$$

By the chain rule of differentiation,

$$(6) \qquad \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt}$$

or

$$(7) \qquad \frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}.$$

Hence, Equation (5) becomes

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + a_1\frac{dy}{dt} + a_2 y = f(e^t)$$

or

$$(8) \qquad \frac{d^2y}{dt^2} + (a_1 - 1)\frac{dy}{dt} + a_2 y = f(e^t).$$

Equation (8) is a linear equation with constant coefficients and can be dealt with by the methods developed in Chaps. IV and V. If in the resulting solution $t = \log x$ is substituted, the solution of Equation (5) is obtained.

Thus it is seen that the substitution $x = e^t$ has the effect of transforming Equation (5) into one with constant coefficients. To see that this is also true for the general Equation (1), we first prove the differentiation formula

$$(9) \qquad x^r \frac{d^r}{dx^r} = \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right) \cdots \left(\frac{d}{dt} - r + 1\right)$$

for $r = 1, 2, \ldots$.

For $r = 1$ and $r = 2$ this has been proved by Equations (6) and (7). To prove it generally to be true, assume formula (9) has been proved for $r = 1, 2, \ldots, n - 1$. Then

$$\frac{d^{n-1}}{dx^{n-1}} = \frac{1}{x^{n-1}} \frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - n + 2\right).$$

Hence, by the product rule for differentiation,

$$\frac{d^n}{dx^n} = -\frac{n-1}{x^n} \frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - n + 2\right) + \frac{1}{x^{n-1}} \frac{d}{dx}\left[\frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - n + 2\right)\right].$$

If in the last expression $\frac{d}{dx}$ is replaced by $\frac{dt}{dx}\frac{d}{dt} = \frac{1}{x}\frac{d}{dt}$, one obtains

$$\frac{d^n}{dx^n} = -\frac{n-1}{x^n} \frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - n + 2\right) + \frac{1}{x^n} \frac{d}{dt}\left[\frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - n + 2\right)\right]$$

$$= \frac{1}{x^n}\left(-n + 1 + \frac{d}{dt}\right) \frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - n + 2\right)$$

or

$$x^n \frac{d^n}{dx^n} = \frac{d}{dt}\left(\frac{d}{dt} - 1\right) \cdots \left(\frac{d}{dt} - n + 2\right)\left(\frac{d}{dt} - n + 1\right),$$

which is formula (9) for $r = n$. Hence, this formula is proved by mathematical induction.

If in Equation (1) substitutions are made for each term

$$x^r \frac{d^r y}{dx^r} \; (r = 1, 2, \ldots, n)$$

according to formula (9), a linear equation of the same order with constant coefficients is obtained. Writing D for the operator d/dt the transformed equation reads

$$(10) \quad [D(D-1)\cdots(D-n+1) + a_1D(D-1)\cdots(D-n+2) + \cdots + a_{n-1}D + a_n]y = f(e^t).$$

Example 1. Find the general solution of the equation

$$(11) \qquad x^4\frac{d^4y}{dx^4} + 4x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = -\log x \quad (x > 0).$$

Using formula (9) or (10),† this equation transforms into

$$[D(D-1)(D-2)(D-3) + 4D(D-1)(D-2) + D(D-1) + D - 1]y = -t$$

or

$$(D^4 - 2D^3 + 2D - 1)y = -t,$$

where $x = e^t$ and $D = d/dt$. The last equation can be factored as follows:

$$(D+1)(D-1)^3y = -t$$

and is easily seen to have the particular solution $y = t + 2$. Hence, its general solution is

$$y = Ae^{-t} + (B + Ct + Dt^2)e^t + t + 2,$$

and the general solution of Equation (11) is

$$y(x) = \frac{A}{x} + (B + C\log x + D\log^2 x)x + e^x + 2.$$

In this example it is almost more laborious to check the solution than to derive it.

The auxiliary equation belonging to the transformed Equation (10) is (see Sec. 9, Chap. V)

$$r(r-1)\cdots(r-n+1) + a_1r(r-1)\cdots(r-n+2) + \cdots + a_{n-1}r + a_n = 0.$$

This equation is also said to be the auxiliary equation of the untransformed equation (1). If r_1 is a root of the auxiliary equation, then e^{r_1t} is

† It is not necessary to memorize formula (9), since the substitution $x = e^t$ can be carried out in each individual case. But as the example shows, the use of formula (9) saves considerable labor.

a solution of the reduced equation (10), and therefore, x^r is a solution of the reduced equation (1). If r_1 is a repeated root of multiplicity m_1, then $(A_0 + A_1 t + \cdots + A_{m-1}t^{m-1})e^{r_1 t}$ is a solution of the reduced equation (10), and

$$(12) \qquad (A_0 + A_1 \log x + \cdots + A_{m-1} \log^{m-1} x)x^{r_1}$$

is a solution of the reduced equation (1). If $r_1 = \alpha + \beta i$, $r_2 = \alpha - \beta i$ are conjugate complex roots of the auxiliary equation, then $e^{\alpha t}(A \cos \beta t + B \sin \beta t)$ is a solution of the reduced equation (10), and therefore,

$$(13) \qquad [A \cos (\beta \log x) + B \sin (\beta \log x)]\, x^\alpha$$

is a solution of the reduced equation (1).

Equations of the type

$$(14) \quad (Ax + B)^n \frac{d^n y}{dx^n} + a_1(Ax + B)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(Ax + B) \frac{dy}{dx} + a_n y = f(x),$$

where A, B, a_1, a_2, . . . , a_n are constants, are not essentially different from equations of type (1). The transformation

$$(15) \qquad Ax + B = e^t$$

reduces Equation (14) to one with constant coefficients.

PROBLEMS

Find the general solutions of the following equations:

1. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 0.$
2. $2x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + y = Ax + B.$
3. $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = Ax^2 + Bx + C.$
4. $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = x \log x.$
5. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = (1 - \log x)^2.$
6. $x^4 \frac{d^4y}{dx^4} + 5x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = 2 \sin (2 \log x) + 11 \cos (2 \log x).$
7. $x^3 \frac{d^3y}{dx^3} + (3 - 3m)x^2 \frac{d^2y}{dx^2} + (3m^2 - 3m + 1)x \frac{dy}{dx} - m^3 y = 0.$

8. $(2x-1)^2\dfrac{d^2y}{dx^2}+(2-4x)\dfrac{dy}{dx}-12y=6(x^2-x+1).$

9. Establish the Euler-Cauchy equation of third order whose general solution is $y = Ax + Bx^2 + Cx^3$.

10. Establish the Euler-Cauchy equation of third order whose general solution is $y = Ax + Bx \log x + Cx (\log x)^2$.

2. Reduction by Known Integrals. The order of a linear differential equation can be reduced if any nonidentically vanishing solution of the reduced equation is known. This corresponds to the well-known device in algebra where the degree of an equation can be reduced if one of its roots is known.

Consider first the linear equation of second order

$$\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=f(x) \tag{16}$$

and assume $y_0(x)$ is a known integral of the reduced equation, that is,

$$\frac{d^2y_0}{dx^2}+a_1(x)\frac{dy_0}{dx}+a_2(x)y_0=0. \tag{17}$$

Suppose the substitution

$$y(x)=y_0(x)u(x),\qquad u(x)=\frac{y(x)}{y_0(x)}\quad(\text{wherever } y_0(x)\neq 0) \tag{18}$$

is made, where $u(x)$ is the new unknown function that is to replace $y(x)$ in Equation (16). Differentiating (18), one has

$$\begin{aligned} y' &= y_0'u + y_0u' \\ y'' &= y_0''u + 2y_0'u' + y_0u''. \end{aligned} \tag{19}$$

On substituting (18) and (19) in Equation (16), one obtains

$$y_0u''+(2y_0'+a_1y_0)u'+(y_0''+a_1y_0'+a_2y_0)u=f(x). \tag{20}$$

But, because of (17), the coefficient of u in this equation vanishes, and this equation becomes

$$y_0u''+(2y_0'+a_1y_0)u'=f(x). \tag{21}$$

This can be considered as a *first-order equation* in the unknown $u'(x)$. When this equation is solved, $u(x)$ is found by a quadrature and then $y(x)$ is found by (18).

The procedure for the general linear equation of order n

$$\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=f(x) \tag{22}$$

is the same. Suppose that $y_0(x)$ is a known solution of the reduced equation, that is,

$$(23) \quad \frac{d^n y_0}{dx^n} + a_1(x) \frac{d^{n-1} y_0}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{dy_0}{dx} + a_n(x) y_0 = 0.$$

As in (18) one puts $y = y_0 u$ and substitutes in Equation (22). This requires the knowledge of the first n derivatives of the product $y_0 u$, the first two of which are worked out in (19).† However, to see what the final result will be it is sufficient to write down but the first terms in the expansions of those derivatives, namely,

$$\begin{aligned} y' &= y_0' u + \cdots \\ y'' &= y_0'' u + \cdots \\ y''' &= y_0''' u + \cdots \\ &\cdots\cdots\cdots\cdots \\ y^{(n)} &= y_0^{(n)} u + \cdots . \end{aligned}$$

Because of these terms the coefficient of u in the transformed equation is

$$y_0^{(n)} + a_1 y_0^{(n-1)} + \cdots + a_{n-1} y_0' + a_n y_0,$$

which vanishes, by (23). Consequently, the transformed equation contains $u^{(n)}, u^{(n-1)}, \ldots, u'$, but not u itself; hence it is a differential equation of order $(n - 1)$ in the unknown $u'(x)$.

If besides $y_0(x)$ another solution $y_1(x)$ of the reduced equation is known and $y_1(x)$ is linearly independent of $y_0(x)$, then

$$u_1'(x) = \frac{d}{dx}\left(\frac{y_1(x)}{y_0(x)}\right)$$

is a known solution of the transformed equation, whose order is by one lower than the original equation. Repeating then the above procedure, the order of the latter equation can again be reduced by one. Continuing in this way it is seen that if k linearly independent solutions of Equation (23) are known its order may be reduced by k.

The practical value of this rule lies in the fact that sometimes one or more particular solutions of the reduced equation are known for some reason, or can be easily found. When this happens the task of finding the general solution is greatly simplified.

† By repeated use of the product rule for differentiation it is easily seen that

$$\frac{d^r}{dx^r}(y_0 u) = y_0^{(r)} u + \frac{r}{1} y_0^{(r-1)} u' + \frac{r(r-1)}{1 \cdot 2} y_0^{(r-2)} u'' + \cdots + y_0 u^{(r)}.$$

Example 2. Suppose it is required to find the general solution of the equation

$$\sin x \frac{d^3y}{dx^3} + \cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} + (\cos x)y = 0. \tag{24}$$

In order to exclude points at which the coefficient of the highest derivative vanishes, let x be limited to the interval $0 < x < \pi$.

The equation may be written as

$$\sin x(y''' + y') + \cos x(y'' + y) = 0$$

or

$$\sin x \frac{d}{dx}(y'' + y) + \cos x(y'' + y) = 0$$

or

$$\left(\sin x \frac{d}{dx} + \cos x\right)(y'' + y) = 0.$$

Hence, it is seen that $y_0(x) = \sin x$ and $y_1(x) = \cos x$ are linearly independent solutions of this equation. Putting

$$y(x) = y_0(x)u(x) = \sin x\, u(x)$$

one has

$$\begin{aligned} y' &= (\cos x)u + (\sin x)u' \\ y'' &= -(\sin x)u + 2(\cos x)u' + (\sin x)u'' \\ y''' &= -(\cos x)u - 3(\sin x)u' + 3(\cos x)u'' + (\sin x)u''', \end{aligned}$$

and Equation (24) becomes

$$(\sin^2 x)u''' + 4(\sin x \cos x)u'' + 2(\cos^2 x - \sin^2 x)u' + 0u = 0$$

or

$$u''' + 4(\text{ctn } x)u'' + 2(\text{ctn}^2 x - 1)u' = 0. \tag{25}$$

Since $y_1(x) = \cos x$ is a solution of Equation (24),

$$u_1(x) = \frac{y_1(x)}{\sin x} = \text{ctn } x$$

must be a solution of Equation (25). Then $u_1' = -\csc^2 x$, and to repeat the above procedure one puts

$$u'(x) = u_1'(x)v(x) = -(\csc^2 x)v(x), \qquad v(x) = -(\sin^2 x)u'(x). \tag{26}$$

Then

$$\begin{aligned} u'' &= 2(\csc^2 x \text{ ctn } x)v - (\csc^2 x)v' \\ u''' &= -(4 \csc^2 x \text{ ctn}^2 x + 2 \csc^4 x)v + 4(\csc^2 x \text{ ctn } x)v' - (\csc^2 x)v'' \end{aligned}$$

and Equation (25) becomes

$$-(\csc^2 x)v'' + 0v' + 0v = 0$$

(that the coefficient of v is zero is a necessary result of the method; not so for the coefficient of v'; its vanishing is fortuitous). Hence,

$$v'' = 0,$$
$$v(x) = A + Bx.$$

Then, by (26),

$$u'(x) = -(A + Bx)\csc^2 x$$

and integration gives

$$u(x) = (A + Bx) \operatorname{ctn} x - B \log \sin x + C.$$

Therefore,

$$y(x) = A \cos x + C \sin x + B(x \cos x - \sin x \log \sin x),$$

and this is the general solution of Equation (24).

As a special result of the method of reduction by known integrals we have the important conclusion that if n linearly independent integrals of the reduced linear equation of order n are known then the general solution of the complete equation can be obtained by quadratures alone. If only $(n - 1)$ linearly independent solutions of the reduced equation are known, the complete equation can be reduced to a linear equation of order $n - (n - 1) = 1$. From Chap. III it is known that a linear equation of first order can always be solved by quadratures alone. Hence, we have the following theorem:

Theorem. If $(n - 1)$ linearly independent integrals of the reduced linear differential equation of order n are known, then the general solution of the complete equation can be found by quadratures alone.

PROBLEMS

For the following equations one or more particular integrals of the corresponding reduced equation are given. Find the general solution of the complete equations.

1. $x \dfrac{d^2y}{dx^2} - (1 + x) \dfrac{dy}{dx} + y = 0; \quad y_1 = 1 + x.$

2. $4x(1 - x) \dfrac{d^2y}{dx^2} + (6 - 8x) \dfrac{dy}{dx} - y = 0; \quad y_1 = x^{-\frac{1}{2}}.$

3. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = 2x^2; \quad y_1 = x^{-1}.$

4. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right) y = 0; \quad y_1 = x^{-\frac{1}{2}} \sin x.$

5. $(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 6y = 0; \quad y_1 =$ polynomial of degree 2.

6. $x^3 \frac{d^3y}{dx^3} + x \frac{dy}{dx} - y = \frac{1}{x}; \quad y_1 = x, y_2 = x \log x.$

3. Method of Variation of Parameters. As shown in the preceding section the order of a linear equation can be reduced if some integrals of the reduced equation are known. The procedure described there consisted in reducing the order step by step through successive substitutions, each of which employs another of the known integrals. An alternative method will be described now in which only one substitution is made that uses all the known k integrals and reduces the order of the equation by the number k at once. The method is essentially the same as that of Sec. 9, Chap. IV.

Not to complicate matters by cumbersome notation, assume that three linearly independent integrals $y_1(x)$, $y_2(x)$, $y_3(x)$ of the reduced equation (22) are known; hence

$$\frac{d^ny_i}{dx^n} + a_1(x) \frac{d^{n-1}y_i}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{dy_i}{dx} + a_n(x)y_i = 0 \qquad (i = 1, 2, 3) \tag{27}$$

We introduce three unknown functions $u_1(x)$, $u_2(x)$, $u_3(x)$ and make the substitution

$$y(x) = y_1(x)u_1(x) + y_2(x)u_2(x) + y_3(x)u_3(x). \tag{28}$$

Then

$$y' = y_1'u_1 + y_2'u_2 + y_3'u_3 + \underline{y_1u_1' + y_2u_2' + y_3u_3'}. \tag{29}$$

This derivative will be simplified if we put the underlined part equal to zero, that is,

$$y_1u_1' + y_2u_2' + y_3u_3' = 0. \tag{30}$$

Since there are three functions $u_1(x)$, $u_2(x)$, $u_3(x)$ at our disposal to replace the one function $y(x)$, it is possible to impose two conditions on these functions, and (30) is one of them.

Now (29) is replaced by

$$y' = y_1'u_1 + y_2'u_2 + y_3'u_3 \tag{31}$$

and another differentiation gives

$$y'' = y_1''u_1 + y_2''u_2 + y_3''u_3 + \underline{y_1'u_1' + y_2'u_2' + y_3'u_3'} \tag{32}$$

where the underlined terms can be omitted if we impose on u_1, u_2, u_3 the further condition

$$y_1'u_1' + y_2'u_2' + y_3'u_3' = 0. \tag{33}$$

In the succeeding derivatives of $y(x)$ no more simplifications like those above are possible; otherwise too many equations would be obtained for the three unknowns u_1, u_2, u_3. Hence,

$$\begin{aligned} y''' &= y_1'''u_1 + y_2'''u_2 + y_3'''u_3 + y_1''u_1' + y_2''u_2' + y_3''u_3' \\ y^{(4)} &= y_1^{(4)}u_1 + y_2^{(4)}u_2 + y_3^{(4)}u_3 + \cdots + y_1''u_1'' + y_2''u_2'' + y_3''u_3'' \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y^{(n)} &= y_1^{(n)}u_1 + y_2^{(n)}u_2 + y_3^{(n)}u_3 + \cdots + y_1''u_1^{(n-2)} + y_2''u_2^{(n-2)} \\ &\qquad + y_3''u_3^{(n-2)}. \end{aligned}$$

When all these derivatives are substituted in Equation (23), and use is made of (27), then a transformed equation is obtained that contains u_1', u_2', u_3', u_1'', u_2'', . . . , $u_1^{(n-2)}$, $u_2^{(n-2)}$, $u_3^{(n-2)}$, but not the functions u_1, u_2, u_3 themselves. From the previously established two equations (30), (33), two of the functions u_1', u_2', u_3' can be eliminated, that is, two of them, for example, u_2', u_3', can be expressed in terms of the third one, in this case u_1'. When these expressions are substituted in the transformed equation, a linear equation results that contains only u_1', u_1'', . . . , $u_1^{(n-2)}$. Hence it is a linear differential equation of order $(n - 3)$ in the unknown u_1', and thus the order of the original equation is reduced by 3. When the transformed equation is solved, then $u_1(x)$ is obtained by a quadrature and $u_2(x)$, $u_3(x)$ by two more quadratures. The solution $y(x)$ itself is then given by (28).

The description of the method is somewhat involved, but the process itself is quite simple. Summarized for the case of k known integrals it is as follows. If k linearly independent integrals $y_1(x)$, $y_2(x)$, . . . , $y_k(x)$ of the reduced equation are known, then any linear combination

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x) \tag{34}$$

with constant coefficients is also an integral of the reduced equation. The substitution to be made is obtained by replacing the constants c_1, c_2, . . . , c_k by variables $u_1(x)$, $u_2(x)$, . . . , $u_k(x)$:

$$y(x) = y_1(x)u_1(x) + y_2(x)u_2(x) + \cdots + y_k(x)u_k(x) \tag{35}$$

[compare (28)]. But the variables $u_1, u_2, \ldots, u_k$ are treated like constants in the first $(k-1)$ derivatives of (35), which amounts to putting

$$
\begin{aligned}
y_1u_1' + y_2u_2' + \cdots + y_ku_k' &= 0 \\
y_1'u_1' + y_2'u_2' + \cdots + y_k'u_k' &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_1^{(k-2)}u_1' + y_2^{(k-2)}u_2' + \cdots + y_k^{(k-2)}u_k' &= 0
\end{aligned} \tag{36}
$$

[compare (30) and (33)]. The first $(k-1)$ derivatives of (35) are then simplified as follows [compare (31), (32)]:

$$
\begin{aligned}
y' &= y_1'u_1 + y_2'u_2 + \cdots + y_k'u_k \\
y'' &= y_1''u_1 + y_2''u_2 + \cdots + y_k''u_k \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y^{(k-1)} &= y_1^{(k-1)}u_1 + y_2^{(k-1)}u_2 + \cdots + y_k^{(k-1)}u_k.
\end{aligned} \tag{37}
$$

No further simplifications are made in the remaining derivatives $y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}$.

On substituting the expressions for y and its derivatives in the original equation a transformed equation is obtained that contains $u_1', u_2', \ldots, u_k', u_1'', u_2'', \ldots, u_k'', \ldots, u_1^{(n-k-1)}, u_2^{(n-k-1)}, \ldots, u_k^{(n-k-1)}$. If from this transformed equation together with equations (36) all but one of the unknowns $u_1, u_2, \ldots, u_k$ are eliminated, a linear equation is obtained that is of order $(n-k)$ in the derivative of the one unknown that is not eliminated.

Because of the formal replacement of the constants $c_1, c_2, \ldots, c_k$ in (34) by the variables $u_1(x), u_2(x), \ldots, u_k(x)$, this method is known as *variation of parameters*. It is widely used to find the general solution of a nonhomogeneous equation of order n after n linearly independent solutions of the reduced equation (that is, the complementary function) have been determined. By the remarks at the end of the preceding section this can always be done by quadratures alone (see also Sec. 9, Chap. IV, where this method was used for the solution of nonhomogeneous linear equations of order 2 with constant coefficients).

Example 3. To illustrate this method we apply it to the same equation

$$(\sin x)y''' + (\cos x)y'' + (\sin x)y' + (\cos x)y = 0 \tag{38}$$

that was solved by a different method in the example of the preceding section. The equation has the integrals $y_1 = \cos x$, $y_2 = \sin x$. Hence we put

$$y(x) = (\cos x)u_1 + (\sin x)u_2. \tag{39}$$

Then

$$
\begin{aligned}
y' &= -(\sin x)u_1 + (\cos x)u_2 \\
y'' &= -(\cos x)u_1 - (\sin x)u_2 - (\sin x)u_1' + (\cos x)u_2' \qquad (40)\\
y''' &= (\sin x)u_1 - (\cos x)u_2 - 2(\cos x)u_1' - 2(\sin x)u_2' \\
&\qquad - (\sin x)u_1'' + (\cos x)u_2''
\end{aligned}
$$

and

(41) $$(\cos x)u_1' + (\sin x)u_2' = 0.$$

Substitution of (39) and (40) in (38) gives

(42) $$-(\sin^2 x)u_1'' + (\sin x \cos x)u_2'' - 3(\sin x \cos x)u_1' + (\cos^2 x - 2\sin^2 x)u_2' = 0.$$

From (41) we have

(43) $$u_2' = -(\operatorname{ctn} x)u_1'$$

and, therefore

(44) $$u_2'' = (\csc^2 x)u_1' - (\operatorname{ctn} x)u_1''.$$

Substitution of (43) and (44) in (42) gives

$$-(\sin^2 x + \cos^2 x)u_1'' + (\operatorname{ctn} x - 3\sin x \cos x - \cos^2 x \operatorname{ctn} x + 2\sin x \cos x)u_1' = 0$$

or

$$u_1'' = 0.$$

Hence,

(45) $$u_1(x) = A + Bx$$

and, by (43),

$$u_2' = -B \operatorname{ctn} x$$

(46) $$u_2(x) = -B \log \sin x + C.$$

Substitution of (45) and (46) in (39) leads to the general solution

$$y(x) = A\cos x + C\sin x + B(x\cos x - \sin x \log \sin x).$$

PROBLEMS

For the following equation some particular integrals of the corresponding reduced equations are given. Find, by the method of this section, the general solution of the complete equation.

1. $(1 + x^2)^2 \dfrac{d^2y}{dx^2} + 2x(1 + x^2)\dfrac{dy}{dx} + y = A \arctan x;\quad y_1 = 1/\sqrt{1 + x^2},$
$y_2 = x/\sqrt{1 + x^2}.$

2. $x^3 \dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} - y = \dfrac{1}{x};\quad y_1 = x,\ y_2 = x \log x.$

3. $\frac{d^3y}{dx^3} - \left(\frac{3}{x} + 2x\right)\frac{d^2y}{dx^2} + \left(\frac{3}{x^2} + 2 - 4x^2\right)\frac{dy}{dx} + 8x^3y = x^3(x^2 - 1);$
$y_1 = e^{-x^2}, \quad y_2 = e^{x^2}, \quad y_3 = x^2e^{x^2}.$

4. Removal of the Second Highest Derivative. The transformation

$$y(x) = y_0(x)u(x), \qquad u(x) = \frac{y(x)}{y_0(x)} \tag{47}$$
(for all x for which $y_0(x) \neq 0$)

used when $y_0(x)$ is a known integral of the reduced equation, is often useful even if $y_0(x)$ is not an integral. In particular, by such a transformation it is always possible to remove the second highest derivative term from a linear differential equation. This corresponds to the common device in algebra of removing the second highest power term from an algebraic equation by a substitution of the form $x = u + h$.

Let the given differential equation be

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x). \tag{48}$$

When the substitution (47) is made, a new equation in $u(x)$ of the same order is obtained. Its $u^{(n-1)}$ term will originate from $y^{(n)}$ and $y^{(n-1)}$, since none of the lower derivatives can result in a $u^{(n-1)}$ term. But

$$y^{(n)} = y_0u^{(n)} + ny_0'u^{(n-1)} + \cdots .$$
$$y^{(n-1)} = y_0u^{(n-1)} + \cdots .$$

Hence, the coefficient of $u^{(n-1)}$ in the transformed equation is

$$ny_0' + a_1(x)y_0.$$

If this is to vanish, then we must have

$$\frac{y_0'}{y_0} = -\frac{1}{n}a_1(x)$$

or integrated,

$$y_0(x) = \exp\left(-\frac{1}{n}\int^x a_1(x')\,dx'\right).$$

Hence, by the substitution

$$y(x) = u(x)\exp\left(-\frac{1}{n}\int^x a_1(x')\,dx'\right), \tag{49}$$

Equation (48) is transformed into a linear equation of the same order with the derivative of order $(n - 1)$ absent. This, by itself, need not make the solution any easier, but sometimes the transformed equation is of a "solvable" type.

Example 4. To find the general solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \left[1 + \frac{2}{(1+3x)^2}\right]y = 0 \tag{50}$$

one may try the substitution

$$\begin{aligned} y(x) &= u(x) \exp\left(-\tfrac{1}{2}\int^x a_1(x')\,dx'\right) \\ &= u(x)e^{-x}. \end{aligned}$$

Then

$$\begin{aligned} y' &= (-u + u')e^{-x} \\ y'' &= (u - 2u' + u'')e^{-x} \end{aligned}$$

and Equation (50) becomes

$$u'' + \frac{2}{(1+3x)^2}u = 0. \tag{51}$$

This is an Euler-Cauchy equation. Hence, putting

$$1 + 3x = e^t, \qquad t = \log(1 + 3x) \tag{52}$$

we have

$$\begin{aligned} \frac{du}{dx} &= \frac{3}{1+3x}\frac{du}{dt} \\ \frac{d^2u}{dx^2} &= -\left(\frac{3}{1+3x}\right)^2\frac{du}{dt} + \left(\frac{3}{1+3x}\right)^2\frac{d^2u}{dt^2}, \end{aligned}$$

and Equation (51) becomes

$$9\frac{d^2u}{dt^2} - 9\frac{du}{dt} + 2u = 0.$$

The general solution of this equation with constant coefficients is

$$u = Ae^{\frac{1}{3}t} + Be^{\frac{2}{3}t}.$$

Hence, by (52),

$$u(x) = A(1+3x)^{\frac{1}{3}} + B(1+3x)^{\frac{2}{3}},$$

and

$$y(x) = [A(1+3x)^{\frac{1}{3}} + B(1+3x)^{\frac{2}{3}}]e^{-x}.$$

PROBLEMS

The following equations can be simplified by removing the second highest derivative. Obtain the general solution.

1. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + (x^2 + 2)y = 0.$
2. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} - \dfrac{2}{(x+1)^2}y = 0.$

3. $x^2 \frac{d^4y}{dx^4} + 8x \frac{d^3y}{dx^3} + 12 \frac{d^2y}{dx^2} + x^2 y = 2e^x.$

4. $(\sin x) \frac{d^3y}{dx^3} + 3(\cos x) \frac{d^2y}{dx^2} - 6(\sin x) \frac{dy}{dx} + (2 \sin x - 4 \cos x)y = \sin x - 3 \cos x.$

5. Show that in the case of linear differential equations of *first* order the method of this section is identical with the method of the integrating factor (see Sec. 5, Chap. III).

6. The equation

$$\frac{d^2x}{dt^2} + b(t) \frac{dx}{dt} + k(t)x = f(t)$$

represents the motion of a vibrating system with variable damping coefficient $b(t)$, variable spring coefficient $k(t)$, and forcing function $f(t)$. Show that $x(t) = e^{-B(t)}X(t)$, where $B(t) = \frac{1}{2} \int^t b(t')\, dt'$ and where $X(t)$ is the motion of an undamped vibrating system with spring coefficient

$$K = k - \frac{1}{4} b^2 - \frac{1}{2} \frac{db}{dt}$$

and forcing function $F = e^B f$.

5. Equations of Riccati.† Every homogeneous linear differential equation of second order can be reduced to a differential equation of first order of a special type. This is achieved by the substitution

(53) $$u(x) = \frac{y'(x)}{y(x)}, \qquad y(x) = \exp\left(\int^x u(x')\, dx'\right),$$

where $u(x)$ is the new dependent variable replacing y. Then

(54) $$\begin{aligned} y' &= yu \\ y'' &= yu' + y'u \\ &= y(u' + u^2). \end{aligned}$$

Substituting (53), (54) in the homogeneous linear equation of second order

(55) $$\frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = 0,$$

we obtain

$$y(u' + u^2) + a_1 yu + a_2 y = 0,$$

or after dropping the factor y,

(56) $$\frac{du}{dx} + u^2 + a_1(x)u + a_2(x) = 0.$$

† Named after the Italian mathematician Count Riccati, 1676–1754.

This nonlinear equation of first order is said to be of Riccati's type, the most general *Riccati equation* being defined as

$$\text{(57)} \qquad \frac{du}{dx} + b_0(x)u^2 + b_1(x)u + b_2(x) = 0,$$

where $b_0(x)$ is not identically zero (otherwise the equation would be linear).

That, conversely, every Riccati equation can be converted into a homogeneous linear equation of second order can be seen by reversing the above procedure. Put

$$\text{(58)} \qquad u(x) = \frac{1}{b_0(x)}\frac{y'(x)}{y(x)}, \qquad y(x) = \exp\left(\int^x b_0(x')u(x')\,dx'\right)$$

[this is (53) when $b_0(x) \equiv 1$]. Then

$$u' = \frac{b_0yy'' - b_0'yy' - b_0y'^2}{b_0^2y^2}$$

and substitution in (57) yields

$$\frac{b_0yy'' - b_0'yy' - b_0y'^2}{b_0^2y^2} + b_0\frac{y'^2}{b_0^2y^2} + b_1\frac{y'}{b_0y} + b_2 = 0,$$

or, after multiplication by b_0y,

$$\text{(59)} \qquad y'' + \left(b_1 - \frac{b_0'}{b_0}\right)y' + b_2b_0y = 0.$$

This is a homogeneous linear equation of second order.

We have seen that every integral $y(x)$ of Equation (55) leads to an integral $u(x) = y'(x)/y(x)$ of Equation (56), and every integral $u(x)$ of (56) leads to an integral $y(x) = \exp\left(\int^x u(x')\,dx'\right)$ of (55). Hence, the two equations (55), (56) are completely equivalent, and their general solutions are related to each other by the equation $u = y'/y$.

Since the general solution of the second-order equation (55) contains two arbitrary constants, it appears that the relation $u = y'/y$ would result in a solution of the first-order equation (56) also containing two arbitrary constants. This, of course, cannot be true. The apparent paradox is easily dispelled. Since (55) is a homogeneous linear equation its general solution is of the form

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where $y_1(x)$, $y_2(x)$ are two linearly independent integrals. Therefore

$$u = \frac{y'}{y} = \frac{c_1y_1' + c_2y_2'}{c_1y_1 + c_2y_2} = \frac{y_1' + (c_2/c_1)y_2'}{y_1 + (c_2/c_1)y_2}.$$

and this expression contains essentially but one constant c_2/c_1.

It should not be assumed that by converting a homogeneous linear equation of second order into a first-order equation of Riccati's type the former is brought closer to its solution. There exists no general method for the solution of a Riccati equation. Even of the so-called *special Riccati equation*

(60) $$\frac{du}{dx} + u^2 + ax^m = 0, \qquad (a = \text{constant})$$

it is known that it cannot be solved by quadratures alone except if m is a number that can be written in the form

$$-\frac{4k}{2k \pm 1}$$

where k is a positive integer.† The importance of the relationship between linear equations of second order and Riccati equations lies in the fact that it makes all the methods developed for one type of equation available to the other type. Thus, the aforementioned result on the integrability by quadratures of the special Riccati equation (60) applies equally well to the corresponding linear equation of second order

(61) $$\frac{d^2y}{dx^2} + ax^my = 0.$$

On the other hand, we derive an important result for Riccati equations from a result derived earlier for linear equations. By the Theorem of Sec. 3, a homogeneous linear differential equation of second order can be solved by quadratures if any particular integral of the equation is known. Because of the relationship established above the same must hold for Riccati equations. Thus, assume that $u_0(x)$ is any particular integral of Equation (57), that is,

(62) $$\frac{du_0}{dx} + b_0u_0^2 + b_1u_0 + b_2 = 0.$$

Then introduce a new dependent variable v by the relation

(63) $$u(x) = u_0(x) + \frac{1}{v(x)}.$$

† For details see Ref. 4, p. 143.

Hence

$$u' = u_0' - \frac{v'}{v^2}$$

and substitution in (62) gives

$$u'_0 - \frac{v'}{v^2} + b_0\left(u_0^2 + 2\frac{u_0}{v} + \frac{1}{v^2}\right) + b_1u_0 + \frac{b_1}{v} + b_2 = 0.$$

Multiplication by $-v^2$ and cancellation of those terms that add up to zero, by (62), result in

(64) $$v' - (2b_0u_0 + b_1)v - b_0 = 0.$$

This is a *linear* equation of first order and, therefore, can be solved by quadratures.

Example 5. For illustration, let it be required to find the general solution of the Riccati equation

(65) $$\frac{dy}{dx} + y^2 + (2x + 1)y + (1 + x + x^2) = 0.$$

By trial one finds the particular integral $y_0(x) = -x$. Hence, in order to find the general solution one puts

$$y = -x + \frac{1}{v}$$

Then

$$y' = -1 - \frac{v'}{v^2}$$

and Equation (65) becomes

$$-1 - \frac{v'}{v^2} + x^2 - \frac{2x}{v} + \frac{1}{v^2} - 2x^2 - x + \frac{2x + 1}{v} + 1 + x + x^2 = 0,$$

or

$$v' - v - 1 = 0.$$

The general solution of this simple equation is $v = -1 + Ce^x$. Hence, the general solution of (65) is

$$y = -x + \frac{1}{Ce^x - 1}.$$

PROBLEMS

1. Check that $u = \operatorname{ctn} x$ is a solution of the Riccati equation

$$\frac{du}{dx} + u^2 + u \sin 2x = \cos 2x$$

and find its general solution.

2. Find the general solution of the Riccati equation

$$\frac{du}{dx} + u^2 = A^2,$$

where A is a constant. What is the corresponding linear equation of second order?

3. Show that the Riccati equation (57) can be solved by elementary methods if

$$b_1 - \frac{b_0'}{b_0} = Ax^{-1}, \qquad b_0b_2 = Bx^{-2},$$

where A, B are constants. Hint: The corresponding linear differential equation of second order is an Euler-Cauchy equation.

★4. Let $u_1(x)$, $u_2(x)$, $u_3(x)$, $u_4(x)$ be any four different integrals of a Riccati equation. Show that their "cross ratio" $\dfrac{u_3 - u_1}{u_4 - u_1} \div \dfrac{u_3 - u_2}{u_4 - u_2}$ is constant, independent of x. Hint: Let $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ be the corresponding integrals of the corresponding linear differential equation of second order. Then $y_3(x)$ and $y_4(x)$ must be linear combinations of $y_1(x)$, $y_2(x)$. From this fact the statement follows by straightforward calculation.

5. The space factor of the quantum mechanical wave equation for a particle of mass m vibrating along the x axis under the influence of forces derivable from a potential energy function $V(x)$ is

$$\frac{h^2}{8\pi^2 m}\frac{d^2y}{dx^2} + (E - V(x))y = 0,$$

where the constant E is the total energy of the particle and h is Planck's constant.† Show that by putting

$$y(x) = A \exp\left(\frac{2\pi i}{h}\int^x u(x')\,dx'\right)$$

the above equation is transformed into a Riccati equation for the function $u(x)$. This leads to an important approximation method for the solution of the above equation.

6. Transformation of Variables. In all the classes of differential equations treated in this chapter the solution is based on some transformation of the variables involved. Thus, in the case of Euler-Cauchy equations the transformation used is $x = e^t$, which is a transformation of the independent variable. To eliminate the second highest derivative from a linear equation of order n the substitution $y = u \exp\left(-\dfrac{1}{n}\displaystyle\int^x a_1(x')\,dx'\right)$ is made, which is a transformation of

† See, for example, E. C. Kemble, "The Fundamental Principles of Quantum Mechanics," p. 82, McGraw-Hill, 1937.

the dependent variable. Again, to transform a homogeneous linear equation of second order into a Riccati equation one makes the substitution $y'(x)/y(x) = u(x)$, which involves both the dependent variable and its derivative.

Many other transformations are used to convert differential equations to a form in which their solution becomes apparent. General rules as to which transformation is to be used in any given case cannot be given here.† In this section a few rules concerning the proper use of substitutions are presented. They are based on general rules for differentiation of functions of one or several variables.

If a new independent variable t is introduced to replace x, and the equations of transformation are

$$x = \varphi(t), \qquad t = \psi(x), \tag{66}$$

then all derivatives are changed as follows ($'$, $''$, . . . stand for d/dx, d^2/dx^2, . . .),

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} = \psi'\frac{dy}{dt} \\ \frac{d^2y}{dx^2} &= \psi'^2\frac{d^2y}{dt^2} + \psi''\frac{dy}{dt} \\ \frac{d^3y}{dx^3} &= \psi'^3\frac{d^3y}{dt^3} + 3\psi'\psi''\frac{d^2y}{dt^2} + \psi'''\frac{dy}{dt}, \end{aligned} \tag{67}$$

etc.

The corresponding transformations for a change from the dependent variable y to the new variable u, given by the equation

$$y = \psi(x, u), \tag{68}$$

are (ψ_u, ψ_x, ψ_{uu}, . . . stand for the partial derivatives $\partial\psi/\partial u$, $\partial\psi/\partial x$, $\partial^2\psi/\partial u^2$, . . .):

$$\begin{aligned} \frac{dy}{dx} &= \psi_u\frac{du}{dx} + \psi_x \\ \frac{d^2y}{dx^2} &= \psi_u\frac{d^2u}{dx^2} + \psi_{uu}\left(\frac{du}{dx}\right)^2 + 2\psi_{ux}\frac{du}{dx} + \psi_{xx} \\ \frac{d^3y}{dx^3} &= \psi_u\frac{d^3u}{dx^3} + 3\psi_{uu}\frac{d^2u}{dx^2}\frac{du}{dx} + 3\psi_{ux}\frac{d^2u}{dx^2} + \psi_{uuu}\left(\frac{du}{dx}\right)^3 \\ &\qquad + 3\psi_{uux}\left(\frac{du}{dx}\right)^2 + 3\psi_{uxx}\frac{du}{dx} + \psi_{xxx}, \end{aligned} \tag{69}$$

etc.

† A systematic treatment of substitutions and their effect on differential equations is part of the so-called "theory of Lie groups," initiated by the Norwegian mathematician Sophus Lie, 1842–1899.

From these formulas it is seen that a change of the independent variable transforms a linear equation into another linear equation. A change of the dependent variable, however, transforms a linear equation into a nonlinear equation unless the substitution itself is linear in the new variable (that is, $\psi_{uu} \equiv 0$). In either case the order of the differential equation remains unchanged.

However if the substitution involves the derivative of the old dependent variable or the derivative of the new dependent variable, the order of the differential equation may be decreased or increased (compare the transformation of a linear equation of order two into a Riccati equation). For example, if the substitution is

$$y(x) = \chi\left(\frac{du}{dx}, u, x\right), \tag{70}$$

then

$$\frac{dy}{dx} = \chi_{u'}u'' + \chi_u u' + \chi_x$$

$$\frac{d^2y}{dx^2} = \chi_{u'}u''' + \chi_{u'u'}u''^2 + 2\chi_{u'u}u''u' + 2\chi_{u'x}u'' + \chi_u u'' + \chi_{uu}u'^2 + 2\chi_{ux}u' + \chi_{xx}$$

etc. In this case the order of the transformed equation will be one higher than that of the original equation.

In the absence of general rules, skill, experience, and thorough acquaintance with the standard types of "solvable" equations are the only guides for the choice of a suitable transformation. In some cases the form of the equation itself suggests the substitution.

The following examples illustrate the use of various substitutions.

Example 6

$$(\sin^2 x)\frac{d^2y}{dx^2} + (\tan x)\frac{dy}{dx} - k^2(\cos^2 x)y = 0. \tag{71}$$

Here the substitution

$$\sin x = t$$

suggests itself. Then

$$\frac{dy}{dx} = (\cos x)\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = (\cos^2 x)\frac{d^2y}{dt^2} - (\sin x)\frac{dy}{dt}$$

and the differential equation becomes

$$(\sin^2 x \cos^2 x) \frac{d^2y}{dt^2} + (\sin x - \sin^3 x) \frac{dy}{dt} - k^2(\cos^2 x)y = 0,$$

or, after dropping the factor $\cos^2 x$,

$$t^2 \frac{d^2y}{dt^2} + t \frac{dy}{dt} - k^2y = 0.$$

This is an Euler-Cauchy equation. Its solution is easily found to be

$$y = At^k + Bt^{-k}.$$

Hence, the general solution of Equation (71) is

$$y(x) = A \sin^k x + B \sin^{-k} x.$$

Example 7

(72)
$$x \frac{d^2y}{dx^2} + (x + 3) \frac{dy}{dx} + 2y = 0.$$

If the two summands $xy'' + 3y'$ were $xy''' + 3y''$ instead, then this sum would be the third derivative of xy. Hence, the substitution

$$y = \frac{du}{dx}$$

is suggested. Then Equation (72) becomes

$$x \frac{d^3u}{dx^3} + (x + 3) \frac{d^2u}{dx^2} + 2 \frac{du}{dx} = 0,$$

which can be written as

$$\frac{d^3(xu)}{dx^3} + \frac{d^2}{dx^2}(xu) = 0,$$

or, putting $xu = v$,

(73)
$$\frac{d^3v}{dx^3} + \frac{d^2v}{dx^2} = 0.$$

The general solution of this equation with constant coefficients is

$$v = A + Bx + Ce^{-x}.$$

Hence,

$$u = \frac{A}{x} + B + \frac{C}{x} e^{-x}$$

and

$$y = \frac{A_1}{x^2} + C_1 \left(\frac{1}{x} + \frac{1}{x^2}\right) e^{-x}.$$

By checking this solution we verify that the substitution $y = u'$ has not introduced any extraneous solutions.

PROBLEMS

In the following problems transformations are suggested that will simplify the equations. Find the general solution.

1. $x \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + x^3y = 0; \quad t = x^2.$

2. $x(1 + x^2)^2 \dfrac{d^2y}{dx^2} - (1 - 3x^2)(1 + x^2) \dfrac{dy}{dx} - 8x^3y = 4x^3(1 + x^2);$
$t = 1 + x^2.$

3. $x^2 \dfrac{d^2y}{dx^2} + 2x^2 \tan y \left(\dfrac{dy}{dx}\right)^2 + x \dfrac{dy}{dx} - \sin y \cos y = 0; \quad u = \tan y.$

4. $x(x + 1)^2 \dfrac{d^2y}{dx^2} + (3x + 2)(x + 1) \dfrac{dy}{dx} + y = \log (x + 1); \quad u = xy.$

5. $x \dfrac{d^2y}{dx^2} + x \left(\dfrac{dy}{dx}\right)^2 + \dfrac{1}{2} \dfrac{dy}{dx} = \dfrac{1}{4}; \quad t = \sqrt{x}, u = e^y.$

6. $(1 + x^2)x \dfrac{d^2y}{dx^2} + 2(1 + x)^2 \dfrac{dy}{dx} + 4y = 0; \quad u = y + x \dfrac{dy}{dx}.$

7. $x^3 \dfrac{d^3y}{dx^3} - 36x \dfrac{dy}{dx} - 48y = 0; \quad y = \dfrac{du}{dx}; x^2u = v.$

8. Show that a transformation of the form $t = ax + b (a \neq 0)$ transforms a linear equation with constant coefficients into another such equation.

9. Show that a transformation of the form $t = x^r (r \neq 0)$ transforms an Euler-Cauchy equation into another such equation.

10. The special Riccati equation

$$\frac{dy}{dx} + y^2 = A^2x^{-4}$$

can be transformed by the substitutions

$$x = t^{-1}, \qquad y = x^{-1} - ux^{-2}$$

into a special Riccati equation with constant right-hand term. Hence find its general solution.

11. What transformation of y transforms the equation

$$\frac{d^2y}{dx^2} - x \frac{dy}{dx} + ay = 0$$

into the equation

$$\frac{d^2u}{dx^2} + \left(a + \frac{1}{2} - \frac{x^2}{4}\right) u = 0.$$

Hint: Recall method of Sec. 4.

7. Exact Differential Equations and Integrating Factors. A differential equation is said to be *exact* if it is the derivative of another equation. For example, the equation

$$\frac{d^3y}{dx^3} + \sin x \frac{d^2y}{dx^2} + \cos x \frac{dy}{dx} + \frac{dy}{dx} = f(x) \tag{74}$$

is exact since it is the derivative of the equation

$$\frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} + y = \int^x f(x')\, dx' + C. \tag{75}$$

When a differential equation is recognized as an exact equation, one integration is readily done and the obtained equation, which contains one arbitrary constant of integration, is said to be a *first integral* of the differential equation. Thus, Equation (75) is a first integral of Equation (74). A first integral is *not* itself a solution of the differential equation, but it is an important step toward the solution since it is a differential equation of order one lower than the original equation. In many applications, first integrals represent by themselves significant end results. For example, the equations in mechanics expressing conservation of energy and of momentum are first integrals of the corresponding equations of motion.

For linear differential equations a criterion for exactness is readily obtained. Let the differential equation be

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x). \tag{76}$$

If this equation is to be exact, then the term $a_0y^{(n)}$ must arise from differentiation of the product $a_0y^{(n-1)}$. But the differentiation of this product gives $a_0y^{(n)} + a_0'y^{(n-1)}$. The remainder obtained by subtracting this from the left side of Equation (76), that is,

$$(-a_0' + a_1)y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_ny \tag{77}$$

must also be exact. Therefore, the term $(-a_0' + a_1)y^{(n-1)}$ must arise from differentiation of $(-a_0' + a_1)y^{(n-2)}$. But the derivative is $(-a_0' + a_1)y^{(n-1)} + (-a_0'' + a_1')y^{(n-2)}$. Subtracting this from Expression (77), the remainder

$$(a_0'' - a_1' + a_2)y^{(n-2)} + \cdots + a_ny \tag{78}$$

must again be exact. Continuing in this way we come to the conclusion that

$$[a_0^{(n)} - a_1^{(n-1)} + a_2^{(n-2)} - \cdots + (-1)^n a_n]y \tag{79}$$

must be exact. But Expression (79) can be the derivative of an expression only if the coefficient of y is zero, that is,

$$a_0^{(n)} - a_1^{(n-1)} + a_2^{(n-2)} - \cdots + (-1)^n a_n = 0. \tag{80}$$

This equation was derived as a necessary condition for the exactness of Equation (76). But, by retracing the steps of the above reasoning, it is seen that this condition is also sufficient. Hence, Equation (80) is the desired criterion for exactness of Equation (76).

If a linear equation of second order is exact, it is readily solved. For, then, the first integral is a linear equation of first order, whose general solution can always be found by quadratures.

Example 8. To illustrate the use of first integrals the equation

$$(\sin x)\frac{d^3y}{dx^3} + (\cos x)\frac{d^2y}{dx^2} + (\sin x)\frac{dy}{dx} + (\cos x)y = 0, \tag{81}$$

which was treated in Secs. 2 and 3, is taken up again. Criterion (80) is seen to be satisfied; hence this equation is exact. The sum $(\sin x)y''' + (\cos x)y''$ is the derivative of $(\sin x)y''$, and the sum $(\sin x)y' + (\cos x)y$ is the derivative of $(\sin x)y$; hence a first integral of Equation (81) is

$$(\sin x)(y'' + y) = B$$

or

$$y'' + y = B \csc x.$$

The general solution of this equation with constant coefficients is readily found to be

$$y(x) = A \cos x + C \sin x + B(\cos x - \sin x \log \sin x).$$

If a differential equation is not exact it is possible at times to find an *integrating factor*, that is, a factor that renders the equation exact. Suppose that $\mu(x)$ is such a factor for Equation (76). Then the multiplied equation

$$a_0(x)\mu(x)y^{(n)} + a_1(x)\mu(x)y^{(n-1)} + \cdots + a_{n-1}(x)\mu(x)y' + a_n(x)\mu(x)y = \mu(x)f(x) \tag{82}$$

must be exact. By criterion (80) this is true if and only if

$$\frac{d^n}{dx^n}(a_0\mu) - \frac{d^{n-1}}{dx^{n-1}}(a_1\mu) + \cdots + (-1)^n a_n\mu = 0. \tag{83}$$

This equation may be used to check whether a suggested function $\mu(x)$ is an integrating factor or not. However, in most cases it would be impractical to try to solve Equation (83) so as to find an integrating factor, since (83) is itself a differential equation of order n and, in general, is no less difficult to solve than the original equation. Equation (83) is said to be the *adjoint equation* of the reduced equation (76). It has many other applications besides being a criterion for integrating factors.

No general treatment of the subject of integrating factors can be given here, and this method is suggested here only for those cases where an integrating factor can be found by inspection or by a few systematic trials.

Example 9. In the case of the equation

$$x^3 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = 0 \tag{84}$$

one may conjecture a power of x to be an integrating factor. Trying $\mu(x) = x^k$, we must have, by (83),

$$\frac{d^2}{dx^2}(x^{3+k}) - \frac{d}{dx}(2x^{1+k}) - 2x^k = 0$$

or

$$(3 + k)(2 + k)x^{1+k} - 2(2 + k)x^k = 0.$$

This equation is satisfied if $k = -2$. Hence, x^{-2} is an integrating factor for Equation (84), and the multiplied equation

$$x \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - \frac{2}{x^2} y = 0$$

is exact. This last equation is now written as

$$\frac{d}{dx}(xy') + \frac{d}{dx}\left[\left(\frac{2}{x} - 1\right)y\right] + \left(\frac{2}{x^2} - \frac{2}{x^2}\right)y = 0$$

from which it follows that

$$xy' + \left(\frac{2}{x} - 1\right)y = A$$

is a first integral. This is a linear equation of first order, and its general solution is readily found:

$$y(x) = A_1x + Bxe^{2/x}.$$

PROBLEMS

Show that the following equations are exact and find their general solutions. (In several cases the first integrals are also exact equations.)

1. $(1 + x^2) \dfrac{d^2y}{dx^2} - 2y = 2x.$

2. $(x + x^3) \dfrac{d^2y}{dx^2} + (1 + 7x^2) \dfrac{dy}{dx} + 8xy = 0.$

3. $(x - 2y) \dfrac{d^2y}{dx^2} + 2 \dfrac{dy}{dx}\left(1 - \dfrac{dy}{dx}\right) = 12x^2.$

4. $y \dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 1 = 0.$

5. $\dfrac{1}{x} \dfrac{d^3y}{dx^3} + \left(1 - \dfrac{3}{x^2}\right) \dfrac{d^2y}{dx^2} + \dfrac{6}{x^3} \dfrac{dy}{dx} - \dfrac{6}{x^4} y = 0.$

6. Show that the linear equation with constant coefficients

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)y = f(x); \quad D = \frac{d}{dx}$$

is exact if and only if $a_n = 0$.

7. Show that the Euler-Cauchy equation

$$(a_0x^nD^n + a_1x^{n-1}D^{n-1} + \cdots + a_{n-1}xD + a_n)y = f(x); \quad D = \frac{d}{dx}$$

is exact if and only if

$$a_0 - \frac{a_1}{n} + \frac{a_2}{n(n-1)} - \frac{a_3}{n(n-1)(n-2)} + \cdots + (-1)^n \frac{a_n}{n!} = 0.$$

8. Show that the equation

$$(a_0x^{m_0}D^n + a_1x^{m_1}D^{n-1} + \cdots + a_{n-1}D)y = f(x); \quad D = \frac{d}{dx}$$

where $m_k < n - k$ $(k = 0, 1, \ldots)$ is always exact.

***9.** The equation

$$x^6 \frac{d^2y}{dx^2} + (3x^5 - 16x) \frac{dy}{dx} - (3x^4 - 16)y = 0$$

has an integrating factor of the form x^m. Determine m and then find the general solution of the equation.

10. The equation of motion of a particle of mass m moving along the x axis under the influence of a force $f(x)$ that depends only on the distance of the particle from the fixed point 0 ("central force") is

$$m \frac{d^2x}{dt^2} = f(x).$$

Show that dx/dt is an integrating factor of this equation and that the corresponding first integral is the equation for conservation of energy.

*11. Show that the adjoint of the adjoint of a homogeneous linear equation is the latter equation itself. How does it follow from this that every integrating factor of the adjoint equation is an integral of the original equation, and vice versa?

8. Step Functions as Forcing Functions. Indicial and Weighting Functions. The fundamental existence and uniqueness theorem of Sec. 2, Chap. V, states that the equation

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = f(t) \tag{85}$$

has one and only one solution $x(t)$ that is continuous, has continuous derivatives of order 1, 2, . . . , n, and satisfies given initial conditions, provided that the given functions $a_1(t)$, . . . $a_n(t)$, $f(t)$ are continuous.

The condition that the right-hand term $f(t)$ of Equation (85) be continuous can readily be relaxed. This is of importance for theoretical reasons and also because in many applications to problems in mechanical or electrical vibrations, where the right-hand term $f(t)$ represents an impressed force or voltage (see Sec. 9, Chap. IV), this function is discontinuous. This is, in particular, the case when the force or voltage are of the switch-on or intermittent type. The discontinuities here involved are of the simplest kind, so-called *jump discontinuities*.† The most elementary function possessing a jump discontinuity is the *unit step function* defined by

$$\mathbf{1}(t) = \begin{cases} 0 & (\text{for } t < 0) \\ 1 & (\text{for } t \geqq 0) \end{cases} \tag{86}$$

From this definition it follows that

$$\lim_{t\to 0-} \mathbf{1}(t) = 0, \qquad \lim_{t\to 0+} \mathbf{1}(t) = 1.$$

The difference between these two one-sided limits is 1, which is the magnitude of the jump of the function at the point (or time) $t = 0$. At all other points the function $\mathbf{1}(t)$ is continuous.

Obviously, $h\mathbf{1}(t)$ is a step function like $\mathbf{1}(t)$ except that the magnitude of the jump is h. It is also clear that $\mathbf{1}(t - t_0)$ is a unit step function like $\mathbf{1}(t)$ except that the jump occurs at the point $t = t_0$:

$$\mathbf{1}(t - t_0) = \begin{cases} 0 & (\text{for } t < t_0) \\ 1 & (\text{for } t \geqq t_0) \end{cases} \tag{87}$$

† A piecewise continuous function $f(t)$ is said to have a jump discontinuity at $t = t_0$ of magnitude h if $h = f(t_0+) - f(t_0-) \neq 0$.

More general functions can easily be formed by combinations of step functions. For example, the *escalator function* of Fig. 40 can be represented by the equation

$$y = h[\mathbf{1}(t) + \mathbf{1}(t - \tau) + \mathbf{1}(t - 2\tau) + \cdots]. \tag{88}$$

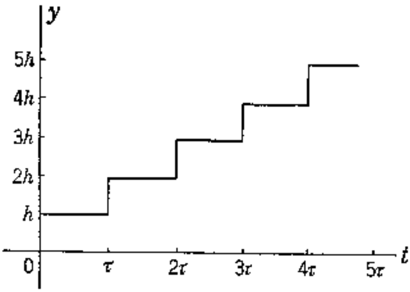


FIG. 40.

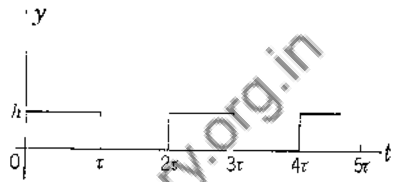


FIG. 41.

The *meander function* of Fig. 41 has the equation

$$y = h[\mathbf{1}(t) - \mathbf{1}(t - \tau) + \mathbf{1}(t - 2\tau) - \cdots]. \tag{89}$$

The *saw-tooth function* of Fig. 42 has the equation

$$y = h\left[\frac{t}{\tau}\,\mathbf{1}(t) - \mathbf{1}(t - \tau) - \mathbf{1}(t - 2\tau) - \cdots\right]. \tag{90}$$

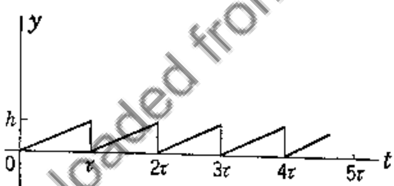


FIG. 42.

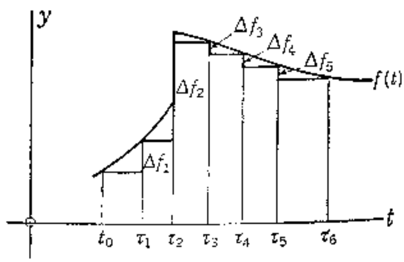


FIG. 43.

For the following it is useful to note that every continuous or piecewise-continuous function (that is, a function that is continuous except for a finite number of jump discontinuities) can be approximated by step functions. The method is apparent from the example of Fig. 43. The equation for the approximating step function is

$$y = f(t_0)\mathbf{1}(t - t_0) + \Delta f_1 \cdot \mathbf{1}(t - \tau_1) + \Delta f_2 \cdot \mathbf{1}(t - \tau_2) + \cdots, \tag{91}$$

where

$$\Delta f_1 = f(\tau_1) - f(t_0), \qquad \Delta f_2 = f(\tau_2) - f(\tau_1), \quad \ldots$$

If the function $f(t)$ is to be approximated in the finite interval $t_0 \leqq t \leqq t_1$ then for any given positive number $\epsilon > 0$ a step function can be found which differs from the given function $f(t)$ by no more than ϵ in the whole interval $t_0 \leqq t \leqq t_1$.

Now assume that we deal with a mechanical or electrical system whose vibrations are described by Equation (85). Assume that the system is at rest up to the time $t = t_0$ and that at this time a force (or voltage) of constant magnitude 1 is impressed. The problem then is to find the solution of the differential equation

$$(92) \qquad L[x] \equiv \frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n(t)x = \mathbf{1}(t - t_0)$$

with the initial conditions

$$(93) \qquad x(t_0) = x'(t_0) = \cdots = x^{(n-1)}(t_0) = 0.$$

The meaning of the initial conditions is that the values of the function $x(t)$ and of its first $(n - 1)$ derivatives approach 0 both as t approaches t_0 from the left and as t approaches t_0 from the right.

Although the Theorem of Sec. 2, Chap. V, does not apply to this problem without changes because of the discontinuity of the forcing function, the basic results as to existence and uniqueness of the solution can be easily derived from that theorem. Since the right-hand term of Equation (92) vanishes for $t < t_0$ and since conditions (93) hold, it follows from that theorem that

$$(94) \qquad x(t) = 0 \qquad (\text{for } t \leqq t_0)$$

For $t \geqq t_0$, the right-hand term of Equation (92) is equal to unity. By the same theorem of Chap. V there exists one and only one solution which satisfies the equation

$$(95) \qquad L[x] = 1$$

and the initial conditions (93). Let us assume this solution is found and let it be denoted by

$$(96) \qquad x = x_1(t, t_0) \qquad (\text{for } t \geqq t_0)$$

where t_0 is written as a second variable to indicate the dependence of the solution on the choice of the initial point t_0.

In summary, we have found there is for all values of t a unique solution of Equation (92) satisfying initial conditions (93), and it is

$$(97) \qquad x = K(t, t_0) = \begin{cases} 0 & (\text{for } t \leqq t_0) \\ x_1(t, t_0) & (\text{for } t \geqq t_0) \end{cases}$$

This composite function $K(t, t_0)$ is called the *indicial function* belonging to the differential expression $L[x]$. This function and its first $(n - 1)$ derivatives are continuous for all values of t. This is obvious for $t \neq t_0$. It is also true for $t = t_0$ since, by construction, the solution $K(t, t_0)$ and its first $(n - 1)$ derivatives approach the limit 0 both as t approaches t_0 from the left and as t approaches t_0 from the right. However, the nth order derivative of $K(t, t_0)$ is no longer continuous. From Equation (92) we have, since $K(t, t_0)$ satisfies this equation for $t \neq t_0$,

$$(98) \quad \frac{\partial^n K}{\partial t^n} = -a_1(t) \frac{\partial^{n-1} K}{\partial t^{n-1}} - a_2(t) \frac{\partial^{n-2} K}{\partial t^{n-2}} - \cdots - a_n(t)K + \mathbf{1}(t - t_0).$$

Since the functions on the right-hand side of this equation are continuous for all values of t except that $\mathbf{1}(t - t_0)$ has a jump of magnitude 1 at $t = t_0$, it follows that the *nth derivative of the indicial function* $K(t, t_0)$ *is continuous for all values of* t *except for* $t = t_0$, *where it has a jump of magnitude* 1.

It is clear that, if the forcing function in Equation (92) were

$$h\mathbf{1}(t - t_0) \qquad (h = \text{constant})$$

and nothing else were changed, the solution of the initial value problem would be $x = hK(t, t_0)$.

Now consider the case of the equation

$$(99) \quad L[x] = \frac{d^n x}{dt^n} + a_1(t) \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n(t)x = \begin{cases} 0 & (\text{for } t < t_0) \\ f(t) & (\text{for } t \geqq t_0) \end{cases}$$

where $f(t)$ is an arbitrary function. Let us try to find a solution of this equation that satisfies the same initial conditions (93) that we had before. If we replace the right-hand term in Equation (99) by the approximating step function (91), then the right-hand term becomes a sum of terms of the general form

$$\Delta f_k \cdot \mathbf{1}(t - \tau_k).$$

The solution for this forcing function satisfying initial conditions (93) is

$$\Delta f_k \cdot K(t, \tau_k).$$

Hence, by the principle of superposition (see Sec. 7, Chap. V) the solution of the equation whose right member is the step function (91) is

$$(100) \qquad f(t_0)K(t, t_0) + \Delta f_1 K(t, \tau_1) + \Delta f_2 K(t, \tau_2) + \cdots$$

As the points $\tau_1, \tau_2, \ldots$ at which the jumps of the approximating step functions occur move closer together the sum in (100) tends to the integral

$$x(t) = K(t, t_0)f(t_0) + \int_{t_0}^{t} K(t, \tau)f'(\tau)\,d\tau \tag{101}$$

provided that the function $f(t)$ has a (piecewise) continuous derivative $f'(t)$. Therefore, we expect the function (101) to be the solution of differential equation (99) which satisfies initial conditions (93). This can be checked independently of the above derivation of formula (101). The verification is left as an exercise to the reader. The expression on the right-hand side of formula (101) is called *Duhamel's integral*. The preceding results are summarized in the following theorem:

Theorem. If $f(t)$ has a (piecewise) continuous derivative, then the solution of the differential equation

$$L[x] = \frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n(t)x = \begin{cases} 0 & (\text{for } t < t_0) \\ f(t) & (\text{for } t \geqq t_0) \end{cases}$$

which satisfies the initial conditions

$$x(t_0) = x'(t_0) = \cdots = x^{(n-1)}(t_0) = 0$$

is

$$x(t) = K(t, t_0)f(t_0) + \int_{t_0}^{t} K(t, \tau)f'(\tau)\,d\tau,$$

where $K(t, \tau)$ is the indicial function belonging to $L[x]$.

Formula (101) can be transformed through integration by parts:

$$\begin{aligned}\int_{t_0}^{t} K(t, \tau)f'(\tau)\,d\tau &= -\int_{t_0}^{t} \frac{\partial}{\partial\tau} K(t, \tau)f(\tau)\,d\tau + K(t, t)f(t) \\ &\qquad - K(t, t_0)f(t_0) \\ &= -\int_{t_0}^{t} \frac{\partial}{\partial\tau} K(t, \tau)f(\tau)\,d\tau - K(t, t_0)f(t_0),\end{aligned} \tag{102}$$

since $K(t, t) = 0$ by definition of the indicial function. Substitution of (102) in (101) yields

$$x(t) = -\int_{t_0}^{t} \frac{\partial}{\partial\tau} K(t, \tau)f(\tau)\,d\tau. \tag{103}$$

If we put

$$G(t, \tau) = -\frac{\partial}{\partial\tau} K(t, \tau), \tag{104}$$

then formula (103) becomes

$$x(t) = \int_{t_0}^{t} G(t, \tau) f(\tau)\, d\tau. \tag{105}$$

Formula (105) does not involve the derivative of the function $f(t)$ and holds true for any continuous function $f(t)$. For this weakened condition the result is more difficult to prove than the above theorem.†

The function $G(t, \tau) = -\frac{\partial}{\partial \tau} K(t, \tau)$ which occurs in the integral of formula (105) is called the *weighting function*, or *Green's function*, belonging to the differential expression $L[x]$. It is readily seen that $G(t, t_0)$ satisfies the same initial conditions (93) as the indicial function $K(t, t_0)$. Moreover, by the definition of a derivative

$$\begin{aligned} G(t, t_0) &= -\frac{\partial K}{\partial \tau}(t, t_0) \\ &= -\lim_{h \to 0} \frac{K(t, t_0 + h) - K(t, t_0)}{h} \\ &= \lim_{h \to 0} \frac{K(t, t_0) - K(t, t_0 + h)}{h}. \end{aligned}$$

Hence, for sufficiently small values of h, the function

$$\frac{K(t, t_0) - K(t, t_0 + h)}{h} \tag{106}$$

may be considered as a good approximation to $G(t, t_0)$. If it is remembered that $K(t, t_0)$ is a solution of differential equation $L[x] = y$ with $y = \mathbf{1}(t - t_0)$ as forcing function, then it is apparent that (106) is a solution of the same equation with

$$y = \frac{\mathbf{1}(t - t_0) - \mathbf{1}(t - t_0 - h)}{h} \tag{107}$$

as forcing function. The graph of this function is shown in Fig. 44. Considered as a mechanical or electromotive force it is an *impulse* of duration h and of intensity $1/h$, hence of "moment" $h \times 1/h = 1$. From this consideration results the following interpretation of Green's function:

If Equation (99) represents the excitation of a mechanical or electrical system, then the Green's function $G(t, t_0)$ belonging to it repre-

† A proof can be found in Ref. 3, Sec. 11.1.

sents the excitation at time t due to an impulse at time t_0 of "infinitesimal" duration, of "infinite" intensity, and of unit moment.

Indicial and weighting functions are particularly useful in cases where it is required to find the responses of an oscillating system to a variety of impressed forces. Once the indicial or weighting function for the system is known, the solution for any impressed force is found by evaluating the integral of formula (101) or (105), which may often most conveniently be done by numerical, graphical, or mechanical methods. In practical work, especially in problems that lead to difficult differential equations with variable coefficients, the indicial or weighting function is sometimes determined by experiment, namely, as the response of the system to a unit-step or unit-impulse force, respectively.

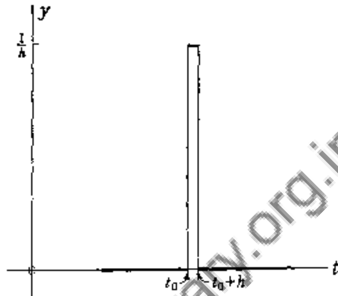


FIG. 14.

Example 10. Find the indicial and the weighting functions belonging to the differential expression

$$L[x] = \frac{d^4x}{dt^4} + k^4x \tag{108}$$

and then find the solution of the equation

$$L[x] = \begin{cases} f(t) & (\text{for } t \geqq t_0) \\ 0 & (\text{for } t < t_0) \end{cases} \tag{109}$$

that satisfies the initial conditions

$$x(t_0) = x'(t_0) = x''(t_0) = x'''(t_0) = 0. \tag{110}$$

We must first find the solution of the problem

$$\frac{d^4x}{dt^4} + k^4x = 1 \tag{111}$$

$$x(\tau) = x'(\tau) = x''(\tau) = x'''(\tau) = 0. \tag{112}$$

Obviously, $x = 1/k^4$ is a particular solution of Equation (111). The complementary function of this equation can be written as

$$x = A \cos k(t - \alpha) + B \cosh k(t - \beta)$$

where A, α, B, β are the constants of integration. Therefore, the general solution of Equation (111) is

$$x = \frac{1}{k^4} + A \cos k(t - \alpha) + B \cosh k(t - \beta).$$

Initial conditions (112) lead to the following equation for the constants A, α, B, β:

$$\begin{aligned} \frac{1}{k^4} + A \cos k(\tau - \alpha) + B \cosh k(\tau - \beta) &= 0 \\ -Ak \sin k(\tau - \alpha) + Bk \sinh k(\tau - \beta) &= 0 \\ -Ak^2 \cos k(\tau - \alpha) + Bk^2 \cosh k(\tau - \beta) &= 0 \\ Ak^3 \sin k(\tau - \alpha) + Bk^3 \sinh k(\tau - \beta) &= 0, \end{aligned}$$

which are solved by

$$\alpha = \beta = \tau, \qquad A = B = -\frac{1}{2k^4}.$$

Therefore, the indicial function belonging to $L[x]$ is

$$K(t, \tau) = \begin{cases} 0 & (\text{for } t \leqq \tau) \\ \dfrac{1}{k^4}\left[1 - \dfrac{1}{2}\cos k(t - \tau) - \dfrac{1}{2}\cosh k(t - \tau)\right] & (\text{for } t \geqq \tau) \end{cases}$$

and the weighting function is

$$G(t, \tau) = \begin{cases} 0 & (\text{for } t \leqq \tau) \\ \dfrac{1}{2k^3}[\sin k(t - \tau) - \sinh k(t - \tau)] & (\text{for } t \geqq \tau) \end{cases}$$

Using the weighting function the solution of Equation (109) satisfying conditions (110) may be written as

$$x(t) = \frac{1}{2k^3}\int_{t_0}^{t} [\sin k(t - \tau) - \sinh k(t - \tau)]f(\tau)\, d\tau \quad (\text{for } t \geqq t_0)$$

Indicial and weighting functions are also used for initial-value problems of *systems* of linear differential equations. The following special case will suffice to indicate the general procedure. Let us consider the system

$$\text{(113)} \qquad \begin{aligned} L_1[x_1, x_2] &= P_1(D)x_1 + P_2(D)x_2 = y_1(t) \\ L_2[x_1, x_2] &= Q_1(D)x_1 + Q_2(D)x_2 = y_2(t), \end{aligned}$$

where $P_1(D)$, $P_2(D)$, $Q_1(D)$, $Q_2(D)$ are linear differential expressions with constant or variable coefficients and $y_1(t)$, $y_2(t)$ are given continuous functions. Assume that the initial conditions

$$\text{(114)} \qquad \begin{aligned} x_1(t_0) &= x_1'(t_0) = \cdots = x_1^{(m)}(t_0) = 0 \\ x_2(t_0) &= x_2'(t_0) = \cdots = x_2^{(n)}(t_0) = 0 \end{aligned}$$

are such that there exists one and only one solution $x_1(t)$, $x_2(t)$ of system (113) satisfying initial conditions (114). Then in order to solve the system

$$L_1[x_1, x_2] = \begin{cases} 0 & (t < t_0) \\ f_1(t) & (t \geqq t_0) \end{cases} \tag{115}$$
$$L_2[x_1, x_2] = \begin{cases} 0 & (t < t_0) \\ f_2(t) & (t \geqq t_0) \end{cases}$$

with initial conditions (114), let us write $x_1 = K_{11}(t, t_0)$, $x_2 = K_{12}(t, t_0)$ for the solution of the system

$$L_1[x_1, x_2] = \mathbf{1}(t - t_0), \qquad L_2[x_1, x_2] = 0, \tag{116}$$

and $x_1 = K_{21}(t, t_0)$, $x_2 = K_{22}(t, t_0)$ for the solution of the system

$$L_1[x_1, x_2] = 0, \qquad L_2[x_1, x_2] = \mathbf{1}(t - t_0), \tag{117}$$

always subject to initial conditions (114). Then the solution of system (115) satisfying initial conditions (114) is

$$\begin{aligned} x_1(t) &= K_{11}(t, t_0)f_1(t_0) + \int_{t_0}^t K_{11}(t, \tau)f_1'(\tau)\, d\tau \\ &\qquad + K_{12}(t, t_0)f_2(t_0) + \int_{t_0}^t K_{12}(t, \tau)f_2'(\tau)\, d\tau \\ x_2(t) &= K_{21}(t, t_0)f_1(t_0) + \int_{t_0}^t K_{21}(t, \tau)f_1'(\tau)\, d\tau \\ &\qquad + K_{22}(t, t_0)f_2(t_0) + \int_{t_0}^t K_{22}(t, \tau)f_2'(\tau)\, d\tau. \end{aligned} \tag{118}$$

The same solution can be written by the use of the weighting functions

$$\begin{aligned} G_{11}(t, \tau) &= -\frac{\partial}{\partial \tau} K_{11}(t, \tau), \qquad & G_{12}(t, \tau) &= -\frac{\partial}{\partial \tau} K_{12}(t, \tau) \\ G_{21}(t, \tau) &= -\frac{\partial}{\partial \tau} K_{21}(t, \tau), \qquad & G_{22}(t, \tau) &= -\frac{\partial}{\partial \tau} K_{22}(t, \tau) \end{aligned} \tag{119}$$

as

$$\begin{aligned} x_1(t) &= \int_{t_0}^t [G_{11}(t, \tau)f_1(\tau) + G_{12}(t, \tau)f_2(\tau)]\, d\tau \\ x_2(t) &= \int_{t_0}^t [G_{21}(t, \tau)f_1(\tau) + G_{22}(t, \tau)f_2(\tau)]\, d\tau. \end{aligned} \tag{120}$$

The four functions $K_{11}(t)$, $K_{12}(t)$, $K_{21}(t)$, $K_{22}(t)$ and $G_{11}(t)$, $G_{12}(t)$, $G_{21}(t)$, $G_{22}(t)$ form the *indicial matrix* and the *weighting* (or *Green's*) *matrix* belonging to the differential system L_1, L_2.

Example 11. Find the indicial matrix and the weighting matrix belonging to the differential system

(121) $$L_1[x_1, x_2] = \frac{d^2x_1}{dt^2} + k_1^2x_2, \qquad L_2[x_1, x_2] = \frac{d^2x_2}{dt^2} + k_2^2x_1.$$

We must first find the solution of the problem

$$\frac{d^2x_1}{dt^2} + k_1^2x_2 = 1, \qquad \frac{d^2x_2}{dt^2} + k_2^2x_1 = 0$$
$$x_1(\tau) = x_1'(\tau) = x_2(\tau) = x_2'(\tau) = 0.$$

One finds after some calculation

(122) $$\begin{aligned} x_1 &= -\frac{1}{2k_1k_2}\cos\sqrt{k_1k_2}\,(t-\tau) + \frac{1}{2k_1k_2}\cosh\sqrt{k_1k_2}\,(t-\tau) = K_{11}(t,\tau) \\ x_2 &= -\frac{1}{2k_1^2}\cos\sqrt{k_1k_2}\,(t-\tau) - \frac{1}{2k_1^2}\cosh\sqrt{k_1k_2}\,(t-\tau) + \frac{1}{k_1^2} \\ &= K_{12}(t,\tau). \end{aligned}$$

Next we must find the solution of the problem

$$\frac{d^2x_1}{dt^2} + k_1^2x_2 = 0, \qquad \frac{d^2x_2}{dt^2} + k_2^2x_1 = 1$$
$$x_1(\tau) = x_1'(\tau) = x_2(\tau) = x_2'(\tau) = 0.$$

The result is similar to the above:

(123) $$\begin{aligned} x_1 &= -\frac{1}{2k_2^2}\cos\sqrt{k_1k_2}\,(t-\tau) - \frac{1}{2k_2^2}\cosh\sqrt{k_1k_2}\,(t-\tau) + \frac{1}{k_2^2} \\ &= K_{21}(t,\tau) \\ x_2 &= -\frac{1}{2k_1k_2}\cos\sqrt{k_1k_2}\,(t-\tau) + \frac{1}{2k_1k_2}\cosh\sqrt{k_1k_2}\,(t-\tau) = K_{22}(t,\tau). \end{aligned}$$

From (122), (123), one finds immediately the four functions of the weighting matrix:

$$\begin{aligned} G_{11}(t,\tau) &= \frac{1}{2\sqrt{k_1k_2}}\sin\sqrt{k_1k_2}\,(t-\tau) + \frac{1}{2\sqrt{k_1k_2}}\sinh\sqrt{k_1k_2}\,(t-\tau) \\ G_{12}(t,\tau) &= \frac{1}{2k_1}\sqrt{\frac{k_2}{k_1}}\sin\sqrt{k_1k_2}\,(t-\tau) - \frac{1}{2k_1}\sqrt{\frac{k_2}{k_1}}\sinh\sqrt{k_1k_2}\,(t-\tau) \\ G_{21}(t,\tau) &= \frac{1}{2k_2}\sqrt{\frac{k_1}{k_2}}\sin\sqrt{k_1k_2}\,(t-\tau) - \frac{1}{2k_2}\sqrt{\frac{k_1}{k_2}}\sinh\sqrt{k_1k_2}\,(t-\tau) \\ G_{22}(t,\tau) &= \frac{1}{2\sqrt{k_1k_2}}\sin\sqrt{k_1k_2}\,(t-\tau) + \frac{1}{2\sqrt{k_1k_2}}\sinh\sqrt{k_1k_2}\,(t-\tau). \end{aligned}$$

PROBLEMS

1. Represent the functions graphed in Fig. 45 by the use of the unit step function.

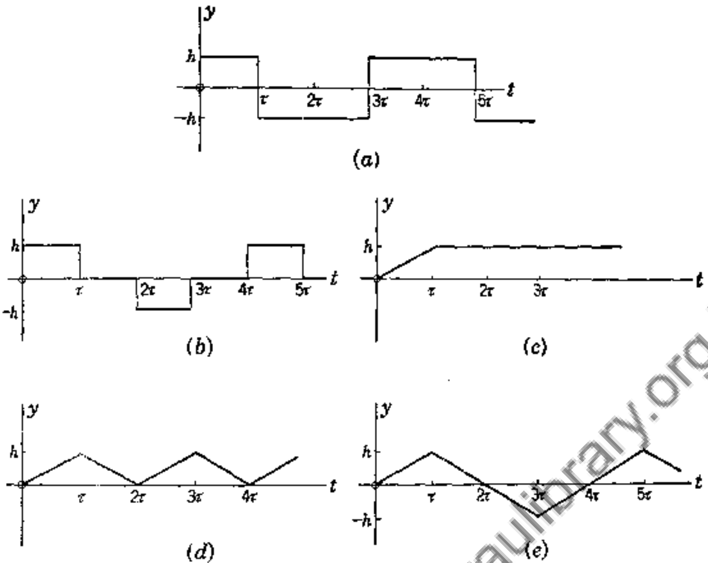


FIG. 45.

2. Find the indicial and weighting functions belonging to the differential expressions

(a) $\dfrac{d^2x}{dt^2} + k^2x; \quad k \neq 0.$

(b) $\dfrac{d^2x}{dt^2} - k^2x; \quad k \neq 0.$

(c) $\dfrac{d^2x}{dt^2} + a\,\dfrac{dx}{dt}; \quad a \neq 0.$

(d) $\dfrac{d^2x}{dt^2} + a\,\dfrac{dx}{dt} + bx; \quad b \neq 0,\ a^2 - 4b \neq 0.$

(e) $t^2\,\dfrac{d^2x}{dt^2} + at\,\dfrac{dx}{dt}; \quad a \neq 1.$

(f) $t^2\,\dfrac{d^2x}{dt^2} + at\,\dfrac{dx}{dt} + bx; \quad b \neq 0,\ (a-1)^2 - 4b \neq 0.$

★(g) $\dfrac{d^2x}{dt^2} + at\,\dfrac{dx}{dt} + ax; \quad a \neq 0.$

3. By the use of the results of Prob. 2 find the solution of the equation

$$L[x] = \begin{cases} f(t); & \text{for } t \geqq 0 \\ 0; & \text{for } t < 0 \end{cases}$$

subject to the initial conditions

$$x(0) = x'(0) = 0,$$

where $L[x]$ is any of the differential expressions $2a \ldots g$.

4. A sinusoidal voltage $V_0 \sin \omega t$ is switched on at time $t = 0$ in a simple circuit of inductance L and resistance R. By the use of the results of Prob. 2, find the current at any time $t > 0$.

5. A square-wave voltage of amplitude V_0 and period 2τ, as graphed in Fig. 45a, is switched on in a simple circuit with an initially uncharged capacitance C, resistance R, and negligible inductance. Find the current at any time $t > 0$ by the method of this section.

★**6.** A trailer of mass M is hitched to an automobile of mass m by a spring of negligible mass, whose spring constant is k. Automobile and trailer being initially at rest, the engine starts the car by exerting on it a force that builds up linearly from the value 0 at time $t = 0$ to the maximal value F_0 at time $t = \tau$, as in graph c, Fig. 45. What is the maximum force in the spring? What would it be if the driving force was applied suddenly at time $t = 0$? Hint: The equation for the extension x of the spring is

$$m \frac{d^2x}{dt^2} + k \frac{M + m}{M} x = \frac{t}{\tau} F_0[\mathbf{1}(t) - \mathbf{1}(t - \tau)] + F_0\mathbf{1}(t - \tau).$$

★**7.** Two resistanceless circuits L_1, C_1 and L_2, C_2 are coupled by the mutual inductance M. If at time $t = 0$, when the currents and charges are zero, a battery of e.m.f. E_0 is applied in the primary, find the current in the secondary at any time $t > 0$.

★**8.** Show that

$$K(t, \tau) = K(t - \tau, 0)$$
$$G(t, \tau) = G(t - \tau, 0)$$

if these indicial and weighting functions belong to a differential expression that has constant coefficients. Hint: If $x(t)$ is a solution of a linear differential equation with constant coefficients satisfying certain initial conditions for $t = 0$, then $x(t - \tau)$ is a solution of the same differential equation and satisfies the same initial conditions for $t = \tau$.

9. Solve Probs. 2, 4, and 5, Sec. 6, Chap. VI, with the external e.m.f. $E(t)$ replaced by $\mathbf{1}(t)$.

9. Periodic Coefficients. Periodic Solutions. Linear differential equations with coefficients that are functions with the same period are, next to equations with constant coefficients, probably the most important linear equations as far as applications are concerned. Special cases of such equations were considered in Sec. 17, Chap. IV, and use was there made of an important theorem whose proof will be given in this section.

The equations to be considered are of the form

$$(124) \quad \frac{d^nx}{dt^n} + a_1(t) \frac{d^{n-1}x}{dt^{n-1}} + a_2(t) \frac{d^{n-2}x}{dt^{n-2}} + \cdots + a_n(t)x = f(t),$$

where the coefficients are continuous, the forcing functions are piecewise continuous, and all are periodic with period τ. Coefficients that are constant also may be considered to be periodic, with arbitrary period τ, since if $a(t)$ is constant it satisfies the identity $a(t + \tau) = a(t)$ for arbitrary τ.

First, we remark that if $x = x(t)$ is a solution of Equation (124) whose coefficients have period τ then $x = x(t + \tau)$ is also a solution. For let us rewrite Equation (124) as

$$D^n x(t) + a_1(t)D^{n-1}x(t) + \cdots + a_n(t)x(t) = f(t), \tag{125}$$

where the differential operator D is used for the derivative with respect to t. Since Equation (125) must hold for all values of t, we may replace t by $t + \tau$. Then, making use of periodicity of $a_1(t)$, . . . , $a_n(t)$ and $f(t)$, we obtain

$$D^n x(t + \tau) + a_1(t)D^{n-1}x(t + \tau) + \cdots + a_n(t)x(t + \tau) = f(t),$$

and this equation expresses that $x = x(t + \tau)$ satisfies the original differential equation. In the same way we could show that if $x = x(t)$ is a solution of the equation with periodic coefficients, then all the functions $x = x(t + k\tau)$, $k = \pm 1, \pm 2, \pm 3, \ldots$ are solutions. If the solution $x = x(t + \tau)$ happens to be identically the same function as the solution $x = x(t)$, then this solution is itself periodic with the same period as the coefficients. This, however, cannot be expected, in general. For example, the coefficients of the equation

$$D^2 x + 2x = \sin t \tag{126}$$

have the period $\tau = 2\pi$. It is easily checked that $x = \sin \sqrt{2}\, t + \sin t$ is a solution of this equation. But this solution is not periodic at all. By the above result all the functions

$$\begin{aligned} x &= \sin \sqrt{2}\,(t + 2\pi k) + \sin (t + 2\pi k) \\ &= \sin \sqrt{2}\,(t + 2\pi k) + \sin t \qquad (k = \pm 1, \pm 2, \ldots) \end{aligned}$$

must also be solutions of Equation (126).

However it is true that, with mild restrictions, every linear differential equation whose coefficients have a period τ has exactly one solution possessing the same period τ. The restriction is that the reduced equation (that is, the equation in which the forcing function is replaced by 0) should have no solution of period τ, except the identically vanishing solution. This important result will now be stated as a theorem, and its proof will indicate how the one solution of period τ can be determined.

Theorem. If all the coefficients and the forcing function of a nonhomogeneous linear differential equation are periodic with period τ, then it has exactly one solution that is periodic with period τ, provided the reduced equation has no solution of period τ other than the identically vanishing solution.

Proof: That there can be no more than one solution of period τ is obvious. For, if there were two different solutions of period τ, their difference would be a nonidentically vanishing function of period τ that satisfies the reduced equation. This is impossible by hypothesis.

Now assume, for the moment, that there exists a solution of Equation (124), say $x = x^*(t)$, which satisfies the boundary conditions

$$(127)\quad x^*(0) = x^*(\tau),\ Dx^*(0) = Dx^*(\tau),\ \dots,\ D^{n-1}x^*(0) = D^{n-1}x^*(\tau).$$

By a previous remark $x = x^*(t + \tau)$ is also a solution, for which we write $x^{**}(t)$. Then because of (127),

$$\begin{aligned} x^{**}(0) &= x^*(0 + \tau) = x^*(0), \\ Dx^{**}(0) &= Dx^*(0),\ \dots,\ D^{n-1}x^{**}(0) = D^{n-1}x^*(0). \end{aligned}$$

Hence, the values of the two solutions $x^*(t)$, $x^{**}(t) = x^*(t + \tau)$ and their first $(n - 1)$ derivatives are equal for $t = 0$. By the fundamental theorem of Sec. 2, Chap. V, the two solutions $x^*(t)$, $x^{**}(t)$ must, therefore, be identical, that is, $x^*(t) = x^*(t + \tau)$ for all values of t. Hence, in order to find a periodic solution of Equation (124) we need only to find a solution that satisfies the special boundary conditions (127).

From Sec. 6, Chap. V, we know that the general solution of Equation (124) is of the form

$$(128)\qquad x = x_f(t) + c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t),$$

where $x_f(t)$ is a particular solution of this equation and $x_1(t)$, $x_2(t)$, . . . , $x_n(t)$ are n linearly independent solutions of the reduced equation. The constants $c_1, c_2, \dots, c_n$ are arbitrary numbers. We try to determine them so that boundary conditions (127) are satisfied. This leads to the equations

$$(129)\quad \begin{aligned} &c_1[x_1(\tau) - x_1(0)] + c_2[x_2(\tau) - x_2(0)] \\ &\qquad + \cdots + c_n[x_n(\tau) - x_n(0)] = -[x_f(\tau) - x_f(0)] \\ &c_1D[x_1(\tau) - x_1(0)] + c_2D[x_2(\tau) - x_2(0)] \\ &\qquad + \cdots + c_nD[x_n(\tau) - x_n(0)] = -D[x_f(\tau) - x_f(0)] \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &c_1D^{n-1}[x_1(\tau) - x_1(0)] + c_2D^{n-1}[x_2(\tau) - x_2(0)] \\ &\qquad + \cdots + c_nD^{n-1}[x_n(\tau) - x_n(0)] = D^{n-1}[x_f(\tau) - x_f(0)]. \end{aligned}$$

These are n homogeneous linear equations for the n unknowns $c_1, c_2, \ldots, c_n$. It is a well-known result of algebra (see Sec. 12, Chap. I) that such a system has exactly one solution provided that the reduced system (that is, where the right-hand terms are zero) has no solution different from $c_1 = c_2 = \cdots = c_n = 0$. In our case this last condition is satisfied; otherwise the reduced equation belonging to (124) would have the nonidentically vanishing solution

$$x = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t)$$

of period τ, against our hypothesis. Therefore, a solution of Equation (124) satisfying boundary conditions (127) can be found, and this is a solution of period τ.

In the special case where the coefficients of the nonhomogeneous equation are constant, the hypotheses of the above theorem are easily checked. All the solutions of the reduced equation are then of the form

$$x_0 = c_1e^{r_1t} + c_2e^{r_2t} + \cdots + c_ne^{r_nt},$$

where $r_1, r_2, \ldots, r_n$ are the roots of the auxiliary equation. There are nontrivial solutions of period τ among these if and only if at least one of the roots is equal to $2k\pi\sqrt{-1}/\tau$, $(k = 0, \pm 1, \pm 2, \ldots)$. Thus we derive from the above theorem the following special result.

Corollary. If the forcing function of a nonhomogeneous linear differential equation with constant coefficients has period τ, then it has exactly one solution of period τ provided that none of the roots of its auxiliary equation is equal to $2k\pi\sqrt{-1}/\tau$ $(k = 0, \pm 1, \pm 2, \ldots)$.

In the case where the forcing function is either $A \cos pt$ or $A \sin pt$ the periodic solution (if it exists) can be found with a minimum of effort. By considering $A \cos pt$ and $A \sin pt$ as the real and imaginary parts of Ae^{ipt}, respectively, the given equation is the real or imaginary part of

(130) $$P(D)x = Ae^{ipt}.$$

The periodic solution must be of the form $x = Ce^{ipt}$. To determine C we substitute in (130) and obtain

$$P(D)Ce^{ipt} = Ae^{ipt}$$

or

$$CP(D)e^{ipt} = Ae^{ipt}.$$

But $P(D)e^{ipt} = P(ip)e^{ipt}$; hence

(131) $$C = \frac{A}{P(ip)}.$$

Example 12. Find the periodic solution of the equation

$$(D^3 + 1)x = \sin^3 pt. \tag{132}$$

The function $\sin^3 pt$ may be written as

$$\begin{aligned} \sin^3 pt &= \tfrac{3}{4} \sin pt - \tfrac{1}{4} \sin 3pt \\ &= \operatorname{Im}\left(\tfrac{3}{4}e^{ipt} - \tfrac{1}{4}e^{3ipt}\right). \end{aligned}$$

Therefore, we consider the equations

$$\begin{aligned} (D^3 + 1)x_1 &= \tfrac{3}{4}e^{ipt} \\ (D^3 + 1)x_2 &= -\tfrac{1}{4}e^{3ipt}. \end{aligned}$$

Putting

$$x_1 = C_1 e^{ipt}, \qquad x_2 = C_2 e^{3ipt}$$

we have immediately

$$\begin{aligned} C_1 &= \frac{\frac{3}{4}}{(ip)^3 + 1} = \frac{\frac{3}{4}}{1 - ip^3} = \frac{3}{4}\,\frac{1 + ip^3}{1 + p^6} \\ C_2 &= \frac{-\frac{1}{4}}{(3ip)^3 + 1} = -\frac{\frac{1}{4}}{1 - 27ip^3} = -\frac{1}{4}\,\frac{1 + 27ip^3}{1 + 729p^6}. \end{aligned}$$

Hence, the periodic solution of Equation (132) is

$$\begin{aligned} x &= \operatorname{Im}\left(\frac{3}{4}\,\frac{1 + ip^3}{1 + p^6}e^{ipt} - \frac{1}{4}\,\frac{1 + 27ip^3}{1 + 729p^6}e^{3ipt}\right) \\ &= \frac{3}{4}\,\frac{p^3 \cos pt + \sin pt}{1 + p^6} - \frac{1}{4}\,\frac{27p^3 \cos 3pt + \sin 3pt}{1 + 729p^6}. \end{aligned}$$

PROBLEMS

1. Assuming that the two functions $f(t)$ and $g(t)$ have the same period τ, investigate the periodicity of the following functions:

(*a*) $af(t) + bg(t)$; (*b*) $f(t)g(t)$; (*c*) $f(t)/g(t)$; (*d*) $F(f(t), g(t))$ where $F(x, y)$ is an arbitrary function; (*e*) $f(t + a)$; (*f*) $f(at)$; (*g*) $f(t^2)$; (*h*) $\dfrac{df}{dt}$.

2. Establish the condition under which $F(t) = \int_a^t f(t')\,dt'$ has the same period as $f(t)$.

The following differential equations have periodic coefficients. Find their general integrals and look for periodic solutions among them. Explain the results by the use of the theorem of this section.

3. $(2 + \sin t)\dfrac{d^2x}{dt^2} + 2\cos t\,\dfrac{dx}{dt} - (\sin t)x = A; \quad A \neq 0.$

4. $(2 + \sin t)\dfrac{d^2x}{dt^2} + 2\cos t\,\dfrac{dx}{dt} - (\sin t)x = A\cos t.$

5. $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 10x = 37 \sin 3t.$

6. $\frac{d^4x}{dt^4} + 6\frac{d^2x}{dt^2} + 8x = 12 \sin 4t.$

7. $\frac{d^4x}{dt^4} + 6\frac{d^2x}{dt^2} + 8x = A \sin 2t; \quad A \neq 0.$

★8. $\frac{d^2x}{dt^2} + 2(1 + \sin t)\frac{dx}{dt} + (1 + \cos t + 2 \sin t + \sin^2 t)x = 1 + \sin t.$

Hint: $e^{t - \cos t}$ is an integrating factor.

Determine the periodic solutions to the following equations:

9. $\frac{d^2x}{dt^2} + 8x = \sin^2 t \cos^2 t.$

10. $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + 3x = \cos^4 \frac{t}{2}$

11. $\frac{d^4x}{dt^4} + 16\frac{d^2x}{dt^2} + x = \sin \frac{3}{8} t \cos \frac{5}{8} t.$

12. Show that the equation $(d^2x/dt^2) + a^2x = A \cos \frac{1}{2}t \cos \frac{3}{2}t \cos \frac{5}{2}t \cos \frac{7}{2}t$ has a unique periodic solution for all positive values of a that are irrational, and determine this solution.

10. Steady State. Stability. Linear physical systems, mechanical, electrical, and others, like those discussed in Secs. 5 and 6 of Chap. VI, are usually made up of elements that are invariant in time or change in a periodic rhythm. These systems are described by differential equations of the kind treated in the preceding section. The coefficients of these equations are constants or functions of the time that have a common period τ, and their nonhomogeneous parts correspond to the external sources of mechanical or electric energy. In general, such systems dissipate energy and, therefore, cannot sustain excitations indefinitely without a continually functioning external source of energy. We shall refer to this common type as *dissipative systems.*

Mathematically speaking, a system is dissipative if every solution of the reduced equation describing it tends to zero as time increases indefinitely. In particular, the reduced equation can have no nontrivial periodic solution, and therefore, the equations for such systems satisfy the hypothesis of the theorem of Sec. 9. Consequently a dissipative system can sustain, for a given impressed energy source of period τ, one and only one excitation of period τ, called the *steady state* of the system. If the initial conditions are not those of this one periodic solution, then the steady state will, at least theoretically, never be attained. Actual experiments teach us that every dissipative system, no matter what its initial state happens to be, will, when driven by a periodic force, tend to the steady state asymptotically. For this reason, any other state can be considered as temporary and is called a *transient state*. The mathematical explanation for this is not difficult. Since every solution to a nonhomogeneous linear equation can be obtained by adding to any particular solution a suitably chosen integral of the reduced equation, every state of a forced system can be considered as a superposition of the steady state and of some state of the system with no external energy source in it. For dissipative systems the latter kind of states tend to zero as time increases indefinitely, hence are transitory.

Because of the property that they return to the steady state from any initial condition, dissipative systems are also said to be *stable*. For systems that are described by equations with constant coefficients one can easily determine whether they are stable or not. The general solution of a reduced equation with constant coefficients is of the form

$$C_1e^{r_1t} + C_2e^{r_2t} + \cdots + C_ne^{r_nt}, \tag{133}$$

where the numbers $r_1, r_2, \ldots, r_n$ are the roots of the corresponding auxiliary equation. If the complex number $r = a + bi$ is one of these roots, then

$$e^{rt} = e^{at}e^{bit} = e^{at}(\cos bt + i \sin bt).$$

Hence, it is seen that solution (133) converges to 0 as $t \to \infty$ for every choice of the constants $C_1, C_2, \ldots, C_n$ if and only if the real parts of all the roots $r_1, r_2, \ldots, r_n$ are negative.

The general solution of a homogeneous linear differential equation with constant coefficients has the above form (133) only if the corresponding auxiliary equation has no repeated roots. If $r = a + bi$ is a k-fold repeated root, then the corresponding term in the general solution is

$$e^{rt}(c_0 + c_1t + c_2t^2 + \cdots + c_{k-1}t^{k-1})$$
$$= e^{at}(c_0 + c_1t + c_2t^2 + \cdots + c_{k-1}t^{k-1})(\cos bt + i \sin bt).$$

It is seen that in this case, too, the solution converges to 0 as $t \to \infty$ if and only if $a < 0$. The results may be stated in the following theorem:

Theorem. A physical system described by a linear differential equation with constant coefficients is stable if and only if the real parts of all the roots of the corresponding auxiliary equation are negative.

Usefulness is added to this theorem by the fact that it is possible to decide whether the roots of an algebraic equation all have negative real parts or not without actually solving for the roots. This is accomplished by the following criterion for whose proof the reader is referred to Ref. 8, p. 304.

Hurwitz's Criterion. The algebraic equation with real coefficients

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where it is assumed that $a_0 > 0$, has all roots with negative real part if and only if the n determinants

$$a_1, \quad \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \quad \ldots, \quad \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-1} & a_{2n-2} & & \cdots & a_n \end{vmatrix}$$

are positive.

It is understood that coefficients with subscripts $> n$ in these determinants are to be replaced by zero.

For example, the general cubic equation with real coefficients

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0,$$

which is first multiplied by ± 1 so as to make $a_0 > 0$, has all roots with negative real part if and only if

$$a_1 > 0, \qquad a_1a_2 - a_0a_3 > 0, \qquad a_3(a_1a_2 - a_0a_3) > 0.$$

In this case, the last condition may be replaced by $a_3 > 0$.

PROBLEMS

The following equations describe linear physical systems. Verify that they are stable and find their steady states.

1. $a\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = A \sin \omega t;\quad a > 0, b > 0, c > 0.$

2. $\dfrac{d^3x}{dt^3} + a\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + \dfrac{ab}{2}x = A \cos \sqrt{b}\, t;\quad a > 0, b > 0.$

3. $\dfrac{d^4x}{dt^4} + a\dfrac{d^3x}{dt^3} + (b + 2)\dfrac{d^2x}{dt^2} + a\dfrac{dx}{dt} + bx = A \sin t + B \cos t;\quad a > 0, b > 0.$

***4.** Show that in Hurwitz' criterion the condition

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-1} & a_{2n-2} & & \cdots & a_n \end{vmatrix} > 0$$

can be replaced by $a_n > 0$.

5. Prove without the use of Hurwitz' criterion that the two roots of

$$a_2x^2 + a_1x + a_0 = 0; \quad a_0 > 0$$

have negative real parts if and only if $a_1 > 0$, $a_2 > 0$.

6. Prove that if the equation

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0; \quad a_0 > 0$$

has only roots with negative real parts, then all the coefficients are positive. Hint: Factor the equation into real factors, that is, use linear factors for real roots and quadratic factors for pairs of conjugate complex roots. Then use the result of Prob. 5.

CHAPTER VIII

SOLUTION IN POWER SERIES. SOME CLASSICAL EQUATIONS

All the methods described in the preceding chapter are designed to reduce a given differential equation to a form where some integrals can be found by quadratures. The scope of such methods is necessarily narrow since even among linear differential equations of second order those which can be solved by quadratures form but a very special type.

The method of solution in series to be described in this chapter applies to a wide class of linear differential equations that cannot be solved by quadratures. This method is of great practical and theoretical value and has, in the case of linear equations with variable coefficients, wider use than any other method.

A complete treatment of solution in series must make use of the fundamentals of the theory of functions of a complex variable. Since knowledge of that theory is not assumed in this book, a few theorems will have to be presented without proof. However, all definitions and explanations necessary for an understanding of the methods will be given.

1. Method of Successive Differentiations. At the beginning of Chap. V a method was outlined by which the solution of a linear differential equation of order n can be obtained in the form of a Taylor series. If the solution $y(x)$ can be expanded in a Taylor series which is valid in a neighborhood† of the initial point x_0, this series must be of the form

$$(1)\quad y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots .$$

The first n coefficients in this series, $y(x_0), \frac{y'(x_0)}{1!}, \ldots, \frac{y^{(n-1)}(x_0)}{(n-1)!}$, are given by the initial conditions. The remaining coefficients are found by differentiating the given equation once, twice, three times, etc., and substituting x_0 for x.

† By *neighborhood* of a point we shall always understand an interval centered about the point.

To illustrate this method let it be required to find the first five terms in the expansion in powers of $(x - 1)$ of the integral of the equation

$$(2) \qquad x\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - 2y = 0$$

that satisfies the conditions

$$(3) \qquad y(1) = 0, \qquad y'(1) = 1.$$

The desired expansion is of the form

$$(4) \quad y(x) = y(1) + \frac{y'(1)}{1!}(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3 + \cdots .$$

The first two coefficients in this series are given by conditions (3). To find the remaining coefficients Equation (2) and its first two derivatives are used

$$\begin{aligned} xy'' + x^2y' - 2y &= 0 \\ xy''' + (1 + x^2)y'' + (2x - 2)y' &= 0 \\ xy^{(4)} + (2 + x^2)y''' + (4x - 2)y'' + 2y' &= 0. \end{aligned}$$

Substitution of $x = 1$, $y = 0$, $y' = 1$ gives

$$\begin{aligned} y''(1) &= -1 \\ y'''(1) &= 2 \\ y^{(4)}(1) &= -6. \end{aligned}$$

Therefore, expansion (4) becomes

$$(5) \quad y(x) = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots .$$

The result has been derived under the assumption that the solution can be expanded as a Taylor series which is valid in some neighborhood of the initial point $x = 1$. Only if this assumption is correct can one be certain that an expansion carried sufficiently far will approximate the solution to any desired degree of accuracy. Therefore, the following theorem, by which the validity of the expansion can immediately be ascertained, is of greatest importance. For convenient formulation of the theorem two definitions are first introduced.

Definition 1. A function $f(x)$ is *analytic* at $x = x_0$ if $f(x)$ can be expanded in a Taylor series valid† in some neighborhood of this point.

† "Valid" means that the series converges and its limit value is $f(x)$.

Definition 2. The point $x = x_0$ is an *ordinary* point of the linear differential equation

$$(6) \quad \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + a_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n(x) y = f(x)$$

if the coefficients $a_1(x), a_2(x), \ldots, a_n(x)$ and the right-hand member $f(x)$ are analytic at $x = x_0$.

It should be noticed that the coefficient of the highest derivative in Equation (6) is unity. If there is a general coefficient $a_0(x)$, then the definition should be applied only to the equation that is obtained after one has divided through by $a_0(x)$. In this connection it should be recalled that, if $a_0(x)$ is analytic and does not vanish at x_0, $1/a_0(x)$ can be expanded in a power series that is valid in some neighborhood of x_0 and, therefore, is analytic at x_0.

Theorem 1. *At an ordinary point every solution of the equation is analytic.*

For a proof of this theorem the reader is referred to more advanced texts (see, for example, Ref. 4, p. 100).

For illustration, let the theorem be applied to Equation (2). Since the coefficients x, x^2, -2 are analytic everywhere, and since the coefficient of the highest derivative vanishes only at $x = 0$, it follows from the theorem that every solution of Equation (2) can be expanded in a Taylor series about any point $x_0 \neq 0$.

The theorem asserts that the resulting Taylor series converges in some interval (neighborhood) about the initial point x_0, but it does not say how large this interval of convergence is. Actually, the size of the interval of convergence can be determined by inspection of the coefficients of the equation, but this requires knowledge of the singularities of the coefficients considered as functions of a complex variable. If the resulting power series is simple enough, the convergence can be tested by one of the familiar convergence tests. However, the following general result, whose proof is closely related to the proof of Theorem 1, is found useful in the determination of the interval of convergence.

Theorem 2. *If the expansions of all the coefficient functions* $a_1(x)$, $a_2(x), \ldots, a_n(x)$, $f(x)$ *are valid for* $|x - x_0| < R$, *then the expansion of every solution of Equation* (6) *is valid for* $|x - x_0| < R$.

In particular, it follows from Theorem 2 that if the functions $a_1(x), \ldots, a_n(x)$, $f(x)$ are polynomials, then every power-series expansion of every solution of Equation (6) is valid for all values of x.

PROBLEMS

Expand the solutions of the following initial-value problems in power series. Obtain at least three nonvanishing coefficients beyond those given by the initial conditions.

1. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2y = 0; \quad y = 1, \frac{dy}{dx} = 0 \text{ for } x = 0.$

2. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2y = 0; \quad y = 0, \frac{dy}{dx} = 1 \text{ for } x = 0.$

3. $\frac{d^2y}{dx^2} + e^x\frac{dy}{dx} - xy = 0; \quad y = 1, \frac{dy}{dx} = 0 \text{ for } x = 0.$

4. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\log x)y = 0; \quad y = 0, \frac{dy}{dx} = \frac{1}{2} \text{ for } x = 1.$

5. $\frac{d^3x}{dt^3} - 4t^2\frac{dx}{dt} + 12tx = 0; \quad x = 0, \frac{dx}{dt} = 0, \frac{d^2x}{dt^2} = 2 \text{ for } t = 0.$

6. $\frac{d^3x}{dt^3} - 4t^2\frac{dx}{dt} + 12tx = 0; \quad x = 1, \frac{dx}{dt} = 0, \frac{d^2x}{dt^2} = 0 \text{ for } t = 0.$

Determine the points on the x axis that are not ordinary points of the following differential equations:

7. $\frac{d^3y}{dx^3} + \frac{x-1}{x+2}\frac{d^2y}{dx^2} + \frac{x}{x^2-1}\frac{dy}{dx} + \frac{x}{x^2+1}y = 0.$

8. $(x^2 + x)\frac{d^2y}{dx^2} + (2x+1)\frac{dy}{dx} + \frac{x}{x-2}y = 0.$

9. $\sin x\frac{d^2y}{dx^2} - e^x\frac{dy}{dx} + (\sec x)y = 0.$

10. $x\frac{d^2y}{dx^2} - \sin x\frac{dy}{dx} + (e^x - 1)y = 0.$

11. $\frac{d^2y}{dx^2} + x^{\frac{1}{3}}\frac{dy}{dx} + (\arctan x)y = 0.$

12. If the solutions of Equations 7–11 were to be expanded in powers of $(x - \frac{1}{2})$, for what values of x could validity of the expansions be predicted?

***13.** Establish the differential equation whose general integral is

$$y = Ay_1(x) + By_2(x),$$

where $y_1(x)$, $y_2(x)$ have the expansions

$$y_1(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad y_2(x) = \sum_{k=0}^{\infty} b_k x^k$$

valid in some neighborhood of $x = 0$. Show that $x = 0$ is an ordinary point of the obtained equation if and only if $a_0b_1 - a_1b_0 \neq 0$.

*14. A linear homogeneous differential equation of third order has among its solutions the function $y = x(1 - \cos x)$. Show, by the use of general principles, that $x = 0$ cannot be an ordinary point of the equation. Hint: Notice that $y = 0, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 0$ for $x = 0$.

2. Method of Undetermined Coefficients. Ordinary Points. A more systematic way of finding the Taylor series expansion of a solution than that outlined in the preceding section is the so-called *method of undetermined coefficients*. This method enables one not only to find a few terms in the expansion but often also to find the general term.

If x_0 is an ordinary point of the differential equation, the solution can be expanded in a series of the form

$$(7)\quad y(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \cdots = \sum_{k=0}^{\infty} c_k(x - x_0)^k,$$

where $c_0, c_1, c_2, \ldots$ are coefficients yet to be determined. Then the derivatives of $y(x)$ can also be expanded:

$$y'(x) = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + 4c_4(x - x_0)^3 + \cdots = \sum_{k=0}^{\infty} kc_k(x - x_0)^{k-1}$$

$$y''(x) = 2c_2 + 3 \cdot 2c_3(x - x_0) + 4 \cdot 3c_4(x - x_0)^2 + \cdots = \sum_{k=0}^{\infty} k(k-1)c_k(x - x_0)^{k-2},$$

etc.

Now all the coefficient functions in the equation are expanded, and then the products of these coefficients by the corresponding derivatives of $y(x)$ are expanded. Finally, all the terms containing the same power of $(x - x_0)$ are combined, and thus an equation of the form

$$(8)\quad \sum_{k=0}^{\infty} d_k(x - x_0)^k = 0$$

is obtained, where d_0 is a linear function of c_0; d_1 is a linear function of c_0, c_1; d_2 is a linear function of c_0, c_1, c_2; etc. Equation (8) can hold for all values x in some neighborhood of x_0 only if

$$d_0 = d_1 = d_2 = \cdots = 0.$$

These are linear equations in $c_0, c_1, c_2, \ldots,$ from which these coefficients can successively be determined.

Example 1. Let this method be applied to find the two solutions $y_1(x)$, $y_2(x)$ of the equation

$$(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0 \tag{9}$$

for which

$$y_1(0) = 0, \qquad y_1'(0) = 1 \tag{10}$$
$$y_2(0) = 1, \qquad y_2'(0) = 0. \tag{11}$$

At the initial point $x_0 = 0$ the coefficients of Equation (9) are analytic, and the coefficient of y'' is not zero there. Hence, $x_0 = 0$ is an ordinary point of the differential equation and each solution can be expanded in a series

$$y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots \tag{12}$$

valid in some neighborhood of $x_0 = 0$. Then

$$y'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \cdots$$
$$y''(x) = 2x_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \cdots$$

and substitution in Equation (9) gives

$$\begin{aligned}(1 + x^2)(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \cdots)\\ + 2x(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \cdots)\\ - 2(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots) = 0,\end{aligned}$$

or, after collection of like terms,

$$\begin{aligned}(2c_2 - 2c_0) + 6c_3x + (12c_4 + 4c_2)x^2 + (20c_5 + 10c_3)x^3\\ + (30c_6 + 18c_4)x^4 + \cdots = 0.\end{aligned}$$

Hence,

$$\begin{aligned}c_2 &= c_0\\ c_3 &= 0\\ c_4 &= -\tfrac{1}{3}c_2 = -\tfrac{1}{3}c_0\\ c_5 &= -\tfrac{1}{2}c_3 = 0\\ c_6 &= -\tfrac{3}{5}c_4 = \tfrac{1}{5}c_0.\end{aligned}$$

For the solution $y_1(x)$, by (10), $c_0 = 0$, $c_1 = 1$. Hence

$$c_2 = c_4 = c_6 = \cdots = 0;\ c_3 = c_5 = c_7 = \cdots = 0,$$

and

$$y_1(x) = x.$$

For the solution $y_2(x)$, by (11), $c_0 = 1$, $c_1 = 0$. Hence

$$c_2 = 1,\ c_3 = 0,\ c_4 = -\tfrac{1}{2},\ c_5 = 0,\ c_6 = \tfrac{1}{5},\ \ldots$$

and

$$y_2(x) = 1 + x^2 - \tfrac{1}{3}x^4 + \tfrac{1}{5}x^6 + \cdots .$$

A more efficient procedure, which in many cases yields the general term in the expansion, is as follows:
Put

$$y(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Then

$$y'(x) = \sum_{k=0}^{\infty} k c_k x^{k-1}$$

$$y''(x) = \sum_{k=0}^{\infty} k(k-1) c_k x^{k-2},$$

and substitution in Equation (9) results in

$$\sum_{k=0}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=0}^{\infty} k(k-1)c_k x^k + 2\sum_{k=0}^{\infty} k c_k x^k - 2\sum_{k=0}^{\infty} c_k x^k = 0.$$

The coefficient of $x^{k-2} (k \geqq 2)$ is

$k(k-1)c_k,$	in the first sum
$(k-2)(k-3)c_{k-2},$	in the second sum
$2(k-2)c_{k-2},$	in the third sum
$-2c_{k-2},$	in the fourth sum.

Since the total coefficient of x^{k-2} must be zero, one obtains

$$k(k-1)c_k + [(k-2)(k-3) + 2(k-2) - 2]c_{k-2} = 0,$$

or

$$k(k-1)c_k + k(k-3)c_{k-2} = 0,$$

or

(13) $$c_k = -\frac{k-3}{k-1}\, c_{k-2}.$$

By this formula any coefficient c_k can be computed if the coefficient c_{k-2} is already known. Putting consecutively $k-2$, $k-4$, . . . for k in formula (13) one obtains

$$c_{k-2} = -\frac{k-5}{k-3}c_{k-4}$$

$$c_{k-4} = -\frac{k-7}{k-5}c_{k-6},$$

etc., until one reaches

$$c_3 = -\frac{0}{2}c_1 \tag{14a}$$

or

$$c_2 = -\frac{-1}{1}c_0 \tag{14b}$$

depending on whether one starts with an odd number k or an even number k. Then c_k is expressed in terms of c_1 (if k is an odd number) or in terms of c_0 (if k is an even number). A formula like (13) that permits one to calculate the numbers of a sequence step-by-step is called *recursion formula.*

Since, by (14a), $c_3 = 0$ it follows from the recursion formula (13) that

$$c_3 = c_5 = c_7 = \cdots = 0. \tag{15}$$

Hence, it remains to determine c_1 and the coefficients with even subscripts. For $y_1(x)$, by (10), $c_1 = 1$, $c_0 = 0$. Then, by formula (13),

$$c_2 = c_4 = c_6 = \cdots = 0.$$

Therefore, $y_1(x) = x$.

For $y_2(x)$, by (11), $c_1 = 0$, $c_0 = 1$. Then, by formula (13)

$$\begin{aligned} c_{2k} &= -\frac{2k-3}{2k-1}c_{2k-2} \\ &= +\frac{2k-3}{2k-1}\frac{2k-5}{2k-3}c_{2k-4} \\ &= -\frac{2k-3}{2k-1}\frac{2k-5}{2k-3}\frac{2k-7}{2k-5}c_{2k-6} \\ &= \cdots\cdots\cdots\cdots\cdots \\ &= (-1)^k\frac{2k-3}{2k-1}\frac{2k-5}{2k-3}\frac{2k-7}{2k-5}\cdots\frac{-1}{1}1 \\ &= (-1)^{k+1}\frac{1}{2k-1}. \end{aligned}$$

Therefore,

$$y_2(x) = 1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \cdots + (-1)^{k+1}\frac{x^{2k}}{2k-1} + \cdots.$$

This power series can easily be related to the well-known expansion of an elementary function. For

$$y_2(x) = 1 + x\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{k+1}\frac{x^{2k-1}}{2k-1} + \cdots\right)$$
$$= 1 + x \arctan x.$$

Since $y_1(x)$, $y_2(x)$ are two linearly independent solutions of equation (9), its general solution is

$$y(x) = Ax + B(1 + x \arctan x).$$

Only in exceptional cases will it be possible to identify an obtained series as the expansion of a known function. There are many instances in the history of mathematics where the solution of a differential equation led to a series of simple construction that could not be identified as the expansion of any of the then known functions and was then accepted as the expression of a new function. Most of the "higher" functions found their way into mathematics and science by their series expansions derived from differential equations. A few examples of such functions will be discussed in succeeding sections.

PROBLEMS

Expand in powers of x the general solution of the following equations. In each case try to identify the obtained series as an expansion of a known function.

1. $(1 + x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0.$

2. $(x^2 - 1)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0.$

3. $\dfrac{d^3y}{dx^3} - x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} - 2y = 0.$

4. Obtain the general solution of the equation

$$\frac{d^2y}{dx^2} - (x - 1)y = 0$$

in powers of $(x - 1)$.

Find the series solutions of the following initial-value problems:

5. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 2y = 0; \quad y = 1, \dfrac{dy}{dx} = 0$ for $x = 0.$

(Compare with Prob. 1, Sec. 1.)

6. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 2y = 0; \quad y = 0, \dfrac{dy}{dx} = 1$ for $x = 0.$

(Compare with Prob. 2, Sec. 1.)

7. $\frac{d^3x}{dt^3} - 4t^2\frac{dx}{dt} + 12tx = 0; \quad x = 0, \frac{dx}{dt} = 0, \frac{d^2x}{dt^2} = 2$ for $t = 0$.

(Compare with Prob. 5, Sec. 1.)

8. $\frac{d^3x}{dt^3} - 4t^2\frac{dx}{dt} + 12tx = 0; \quad x = 1, \frac{dx}{dt} = 0, \frac{d^2x}{dt^2} = 0$ for $t = 0$.

(Compare with Prob. 6, Sec. 1.)

3. Regular Singular Points. Ordinary points are not the only ones about which solutions can be expanded in series. This can also be done for various singular points, that is, points about which not all the coefficient functions of the differential equation can be expanded in Taylor series or at which the coefficient of the highest derivative vanishes. A specially important subclass of these points is the so-called *regular-singular points.*

Definition. The point $x = x_0$ is a *regular-singular* point if the differential equation can be written in the form

$$(16) \quad (x - x_0)^n\frac{d^ny}{dx^n} + (x - x_0)^{n-1}b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + (x - x_0)^{n-2}b_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + (x - x_0)b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = 0,$$

where† $b_1(x), b_2(x), \ldots, b_n(x)$ are analytic at $x = x_0$.

In the neighborhood of regular-singular points there are series solutions that are not necessarily Taylor series, but simple modifications of such series. To derive the actual form of these solutions it would be necessary to use arguments from the theory of functions of a complex variable. To avoid this the following theorem is offered without proof:

Theorem. If x_0 is a regular-singular point of a linear differential equation, then there exists at least one solution of the form

$$(17) \qquad y(x) = (x - x_0)^r \sum_{k=0}^{\infty} c_k(x - x_0)^k,$$

† If all the functions $b_1(x), b_2(x), \ldots, b_n(x)$ are constants, then this is an Euler-Cauchy equation (see Sec. 1, Chap. VII). Hence, the point $x = x_0$ in Euler-Cauchy equations is a regular singular point.

where the expansion is valid in some neighborhood of x_0. More specifically, the series expansion in formula (17) *is valid for $|x - x_0| < R$ if the series expansions for the coefficient functions $b_1(x), \ldots, b_n(x)$ in Equation* (16) *are valid for $|x - x_0| < R$.*

For a proof of this theorem the reader is referred to more advanced texts (see, for example, Ref. 4, Sec. 50).

If in the above sum the coefficient c_0 vanishes, then some power of $(x - x_0)$ [at least $(x - x_0)^1$] can be factored out and combined with the factor $(x - x_0)^r$. In the following it will always be assumed that the highest possible power of $(x - x_0)$ is factored out from the sum and is combined with the factor $(x - x_0)^r$. With this understanding it will always be true that $c_0 \neq 0$ in expression (17), and the exponent r is then uniquely determined. It is called the *exponent of the solution* $y(x)$ at the point x_0. (At an ordinary point the exponent of every not identically vanishing solution is one of the numbers $0, 1, 2, \ldots, n - 1$; see Problem 6 below.)

To determine the coefficients in expansion (17) one proceeds very much as in the case of ordinary points, the one important difference being that now the exponent r has to be determined, too. First, one expands the coefficient functions $b_1(x), b_2(x), \ldots, b_n(x)$ of Equation (16) in powers of $(x - x_0)$. Then this equation takes the form

$$(18) \quad (x - x_0)^n \frac{d^n y}{dx^n} + [b_{10}(x - x_0)^{n-1} + \cdots] \frac{d^{n-1}y}{dx^{n-1}} + [b_{20}(x - x_0)^{n-2} + \cdots] \frac{d^{n-2}y}{dx^{n-2}} + \cdots + [b_{n0} + \cdots]y = 0,$$

where $b_{10}, b_{20}, \ldots, b_{n0}$ are the values of $b_1(x), b_2(x), \ldots, b_n(x)$ at $x = x_0$.

Then $y(x)$ and its derivatives are expanded. By (17),

$$(19) \quad \begin{aligned} y(x) &= \sum_{k=0}^{\infty} c_k(x - x_0)^{k+r} \\ y'(x) &= \sum_{k=0}^{\infty} (k + r)c_k(x - x_0)^{k+r-1} \\ y''(x) &= \sum_{k=0}^{\infty} (k + r)(k + r - 1)c_k(x - x_0)^{k+r-2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y^{(n)}(x) &= \sum_{k=0}^{\infty} (k+r)(k+r-1) \cdots (k+r-n+1)c_k(x-x_0)^{k+r-n}, \end{aligned}$$

Now expressions (19) are substituted in Equation (18), like terms are collected, and the coefficients of $(x - x_0)^r$, $(x - x_0)^{r+1}$, . . . are equated to zero. The equation resulting from equating the coefficient of $(x - x_0)^r$ to zero is

$$(20)\quad [r(r-1)\cdots(r-n+1) + b_{10}r(r-1)\cdots(r-n+2) + b_{20}r(r-1)\cdots(r-n+3) + \cdots + rb_{n-1,0} + b_{n,0}]c_0 = 0.$$

Since, according to the chosen procedure, c_0 cannot be zero, the expression in the bracket must vanish. This expression is a polynomial of degree n in the unknown r, which shall be designated by $g(r)$. Hence

$$(21)\quad g(r) = r(r-1)\cdots(r-n+1) + b_{10}r(r-1)\cdots(r-n+2) + b_{20}r(r-1)\cdots(r-n+3) + \cdots + rb_{n-1,0} + b_{n,0} = 0.$$

This equation is called the *indicial equation*. Each of its n roots can be used as the exponent in expression (17) (for exceptions see below). Once an exponent has been decided upon, one proceeds to determine the coefficients c_k. The coefficient c_0 remains undetermined; it will appear as a factor of the solution $y(x)$. Of course, its value is entirely arbitrary since any constant multiple of a solution is itself a solution of Equation (16).

The coefficients of $(x - x_0)^{r+1}$, $(x - x_0)^{r+2}$, . . . are now equated to zero, and thus equations are obtained from which the coefficients c_1, c_2, . . . can be obtained successively. It is easily checked that the equation resulting from equating the coefficient of $(x - x_0)^{r+k}$ to zero starts as follows:

$$(22)\quad [(k+r)(k+r-1)\cdots(k+r-n+1) + b_{10}(k+r)(k+r-1)\cdots(k+r-n+2) + \cdots + (k+r)b_{n-1,0} + b_{n,0}]c_k + \cdots = 0,$$

where the terms that are not written out contain c_{k-1}, c_{k-2}, . . . , c_0. Comparing Equation (22) with Equation (21) it is seen that (22) may also be written as

$$(23)\qquad g(r+k)c_k + \cdots = 0.$$

Having determined c_1, c_2, . . . , c_{k-1} from the previous equations, c_k can be determined from Equation (23) unless $g(r + k) = 0$. This exceptional case will be discussed below (see Sec. 6).

The outlined program can, in general, be carried out for each of the n roots of the indicial equation, and thus n linearly independent solutions expanded about the regular-singular point can be obtained.

Example 2. To illustrate the method let it be required to find the general solution of the equation

$$x^2 \frac{d^2y}{dx^2} + \left(x^2 + \frac{x}{2}\right)\frac{dy}{dx} + xy = 0 \tag{24}$$

expanded about the point $x = 0$.

It is immediately checked that the point $x = 0$ is a regular-singular point of this equation, the functions $b_1(x)$, $b_2(x)$ of the general equation (16) being $(x + \frac{1}{2})$ and x, respectively, in this case. Hence, putting

$$y(x) = \sum_{k=0}^{\infty} c_k x^{k+r},$$

then

$$y'(x) = \sum_{k=0}^{\infty} (k + r)c_k x^{k+r-1},$$

$$y''(x) = \sum_{k=0}^{\infty} (k + r)(k + r - 1)c_k x^{k+r-2},$$

and Equation (24) becomes

$$\sum_{k=0}^{\infty} (r + k)(r + k - 1)c_k x^{r+k} + \sum_{k=0}^{\infty} (r + k)c_k x^{r+k+1} + \frac{1}{2}\sum_{k=0}^{\infty} (r + k)c_k x^{r+k} + \sum_{k=0}^{\infty} c_k x^{r+k+1} = 0.$$

Equating the coefficient of x^{r+k} to zero,

$$(r + k)(r + k - 1)c_k + (r + k - 1)c_{k-1} + \tfrac{1}{2}(r + k)c_k + c_{k-1} = 0$$

or

$$(r + k)(r + k - \tfrac{1}{2})c_k + (r + k)c_{k-1} = 0. \tag{25}$$

For $k = 0$, there results the indicial equation

$$r(r - \tfrac{1}{2}) = 0.$$

Hence,

$$r_1 = \tfrac{1}{2}, \qquad r_2 = 0$$

are the exponents of the solutions.

For the root $r_1 = \frac{1}{2}$, Equation (25) becomes, after dropping the factor $(\frac{1}{2} + k)$,

$$kc_k + c_{k-1} = 0$$

or

$$c_k = -\frac{1}{k} c_{k-1}.$$

Using this recursion formula for $k - 1, k - 2, \ldots, 1$,

$$c_k = \frac{-1}{k} \frac{-1}{k-1} \cdots \frac{-1}{1} c_0$$
$$= \frac{(-1)^k}{k!} c_0,$$

and if c_0 is put equal to 1,

$$c_k = \frac{(-1)^k}{k!}.$$

Hence, the solution belonging to the exponent $r_1 = \frac{1}{2}$ is

(26)
$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{k+\frac{1}{2}}$$
$$= x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$$
$$= \sqrt{x}\, e^{-x}.$$

Turning, next, to the exponent $r_2 = 0$, Equation (25) becomes, after dropping the factor k,

$$(k - \tfrac{1}{2})c_k + c_{k-1} = 0,$$

or

$$c_k = -\frac{1}{k - \frac{1}{2}} c_{k-1}.$$

Using this recursion formula for $k - 1, k - 2, \ldots, 1$, and putting $c_0 = 1$,

$$c_k = \frac{(-1)^k}{(k - \frac{1}{2})(k - \frac{3}{2}) \cdots (\frac{3}{2})(\frac{1}{2})}.$$

Hence, the solution belonging to the exponent $r_2 = 0$ is

(27)
$$y_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k - \frac{1}{2})(k - \frac{3}{2}) \cdots (\frac{3}{2})(\frac{1}{2})} x^k.$$

This series is not the expansion of an elementary function.

The general solution of differential equation (24), expanded about the point $x_0 = 0$, is

$$y(x) = Ax^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k + B \sum_{k=0}^{\infty} \frac{(-1)^k}{(k - \frac{1}{2})(k - \frac{3}{2}) \cdots (\frac{1}{2})} x^k.$$

Since the coefficient functions in Equation (24) are polynomials, the expansions in the above expression must be valid for all values of x.

PROBLEMS

Expand in powers of x the general solution of the following equations. In each case try to identify the obtained series as expansions of known functions.

1. $4x \dfrac{d^2y}{dx^2} + 2 \dfrac{dy}{dx} + y = 0.$

2. $(2x^2 + x) \dfrac{d^2y}{dx^2} + 3 \dfrac{dy}{dx} - 4y = 0.$

3. $\left(x^2 + \dfrac{x}{2}\right) \dfrac{d^2y}{dx^2} + 2 \dfrac{dy}{dx} - 2y = 0.$

4. $4x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + (x^2 - 1)y = 0.$

5. Find the solution of the equation

$$(x^2 - 1) \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 8y = 0$$

subject to the condition $y = 3$ for $x = 1$. Why is this one condition sufficient to specify a unique solution?

***6.** If the solution $y(x)$ of Equation (16) has the form

$$y(x) = (x - x_0)^r \sum_{k=0}^{\infty} c_k (x - x_0)^k, \quad \text{where} \quad c_0 \neq 0,$$

then r is said to be the exponent of $y(x)$ at $x = x_0$. Show that r must be one of the numbers $0, 1, 2, \ldots, n - 1$ if x_0 is an ordinary point of the equation. Hint: Use the uniqueness theorem of Sec. 2, Chap. V.

4. Gauss's† Hypergeometric Equation. This equation has the form

$$x(1 - x) \frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x] \frac{dy}{dx} - \alpha\beta y = 0, \tag{28}$$

where α, β, γ are given constants. Its characteristic feature is that the coefficients of y'', y', y are polynomials of degree 2, 1, 0, respectively. The seemingly more general equation

$$(x^2 + ax + b) \frac{d^2y}{dx^2} + (cx + d) \frac{dy}{dx} + ey = 0 \tag{29}$$

can be reduced to the form (28) by a linear transformation of the independent variable (see Prob. 4 below). Many equations in applied

† Named after the German mathematician Karl Friedrich Gauss, 1777–1855.

science are special cases of the hypergeometric equation or can be transformed into it by suitable substitutions.

It is immediately recognized that $x = 0$ and $x = 1$ are regular-singular points, whereas all other values of x are ordinary points of the hypergeometric equation. In the following the solutions will be expanded about the point $x = 0$.

To conform with the general procedure outlined in the preceding section, Equation (28) is first multiplied by x:

$$x^2(1 - x)y'' + [\gamma x - (\alpha + \beta + 1)x^2]y' - \alpha\beta xy = 0. \tag{30}$$

Then (19) is substituted, yielding

$$\sum_{k=0}^{\infty} (r + k)(r + k - 1)c_k x^{r+k} - \sum_{k=0}^{\infty} (r + k)(r + k - 1)c_k x^{r+k+1} + \gamma \sum_{k=0}^{\infty} (r + k)c_k x^{r+k} - (\alpha + \beta + 1) \sum_{k=0}^{\infty} (r + k)c_k x^{r+k+1} - \alpha\beta \sum_{k=0}^{\infty} c_k x^{r+k+1} = 0.$$

Equating the coefficient of x^{r+k} to zero,

$$(r + k)(r + k - 1)c_k - (r + k - 1)(r + k - 2)c_{k-1} + \gamma(r + k)c_k - (\alpha + \beta + 1)(r + k - 1)c_{k-1} - \alpha\beta c_{k-1} = 0,$$

or

$$(r + k)(r + k + \gamma - 1)c_k - [(r + k - 1)(r + k + \alpha + \beta - 1) + \alpha\beta]c_{k-1} = 0. \tag{31}$$

For $k = 0$ one obtains the indicial equation

$$r(r + \gamma - 1) = 0,$$

whose roots are

$$r_1 = 0, \qquad r_2 = 1 - \gamma. \tag{32}$$

Using at first the root $r_1 = 0$, then (31) becomes

$$k(\gamma + k - 1)c_k = (\alpha + k - 1)(\beta + k - 1)c_{k-1}.$$

Applying this recursion formula for $k - 1, k - 2, \ldots, 1$, and putting $c_0 = 1$, one obtains the general coefficient

$$c_k = \frac{(\alpha + k - 1)(\alpha + k - 2) \cdots \alpha \cdot (\beta + k - 1)(\beta + k - 2) \cdots \beta}{k(k - 1) \cdots 1 \cdot (\gamma + k - 1)(\gamma + k - 2) \cdots \gamma}. \tag{33}$$

This result is valid except when γ is one of the numbers $0, -1, -2, \ldots$. Hence, except for one of these cases, the solution belonging to the exponent $r_1 = 0$ is

$$(34)\quad y_1(x) = 1 + \frac{\alpha\beta}{1 \cdot \gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)}x^3 + \cdots .$$

This series is known as the *hypergeometric series*, and the function defined by it is the *hypergeometric function*, commonly designated as $F(\alpha, \beta, \gamma; x)$. Only for special values of the parameters α, β, γ, is this an elementary function (see Prob. 6 below). Series (34) converges for $|x| < 1$. This follows from the remark after the theorem of Sec. 3 and can also be readily verified by the ratio test.

To find the solution belonging to the other exponent let $r_2 = 1 - \gamma$ be substituted in (31):

$$(35)\quad k(1 - \gamma + k)c_k = (\alpha - \gamma + k)(\beta - \gamma + k)c_{k-1}.$$

Applying this recursion formula for $k - 1, k - 2, \ldots, 1$, and putting $c_0 = 1$, one obtains the general coefficient

$$(36)\quad c_k = \frac{(\alpha - \gamma + k)\cdots(\alpha - \gamma + 1)(\beta - \gamma + k)\cdots(\beta - \gamma + 1)}{k(k-1)\cdots 1 \cdot (1 - \gamma + k)\cdots(1 - \gamma + 1)}$$

This result is valid except when γ is one of the numbers $2, 3, 4, \ldots$. Hence, except for one of these cases, the solution belonging to the exponent $r_2 = 1 - \gamma$ is

$$(37)\quad y_2(x) = x^{1-\gamma}\left[1 + \frac{(\alpha - \gamma + 1)(\beta - \gamma + 1)}{1 \cdot (-\gamma + 2)}x + \frac{(\alpha - \gamma + 1)(\alpha - \gamma + 2)(\beta - \gamma + 1)(\beta - \gamma + 2)}{1 \cdot 2 \cdot (-\gamma + 2)(-\gamma + 3)}x^2 + \cdots\right].$$

Comparing series (37) with series (34) it is seen that the series in (37) can be obtained from the hypergeometric series by replacing α by $\alpha - \gamma + 1$, β by $\beta - \gamma + 1$, and γ by $-\gamma + 2$. Hence, (37) may also be written as

$$(38)\quad y_2(x) = x^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x).$$

If $\gamma = 1$ then $y_2(x)$ is identical with $y_1(x)$. Otherwise, $y_2(x)$ is linearly independent of $y_1(x)$. (To check this, simply substitute $x = 0$ in an assumed relation of the form $c_1y_1(x) + c_2y_2(x) = 0$.)

Therefore,

$$(39)\quad y(x) = AF(\alpha, \beta, \gamma; x) + Bx^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x)$$

is the general solution of the hypergeometric equation except when γ is an integer.

If γ is one of the integers $0, -1, -2, \ldots$, then $y_1(x)$ does not exist, but $y_2(x)$ does. If γ is one of the integers $2, 3, \ldots$, then $y_2(x)$ does not exist, but $y_1(x)$ does. If, finally, $\gamma = 1$ then $y_1(x)$ and $y_2(x)$ exist, but are not linearly independent. Hence, in all cases, at least one integral is obtained expanded in a series of the form (16) about the regular-singular point $x_0 = 0$, in accordance with the theorem of the preceding section.

If γ is an integer, then the present procedure does not lead to the general solution. This case is discussed in Sec. 6.

PROBLEMS

Solve, in terms of the hypergeometric function, the following equations:

1. $4x(1 - x)\dfrac{d^2y}{dx^2} + 2(1 - 4x)\dfrac{dy}{dx} - y = 0.$

2. $x(1 - x)\dfrac{d^2y}{dx^2} + (cx + d)\dfrac{dy}{dx} + ey = 0.$

3. $x(a + x)\dfrac{d^2y}{dx^2} + (cx + d)\dfrac{dy}{dx} + ey = 0; \quad a \neq 0.$

4. $(x^2 + ax + b)\dfrac{d^2y}{dx^2} + (cx + d)\dfrac{dy}{dx} + ey = 0; \quad a^2 > 4b.$

Hint: let $x^2 + ax + b = (x - s_1)(x - s_2)$. Make the substitution

$$X = \frac{(x - s_1)}{(s_2 - s_1)}$$

and the above equation takes the form of the hypergeometric equation.

5. $(x^2 - 5x + 4)\dfrac{d^2y}{dx^2} - \dfrac{3}{5}x\dfrac{dy}{dx} + \dfrac{16}{25}y = 0.$ Hint: Proceed as in Prob. 4.

6. Name the functions to which the hypergeometric function reduces in the following special cases:

(*a*) $\alpha = 1, \beta = \gamma.$
(*b*) $\alpha = -n, \beta = \gamma.$
(*c*) $\alpha = \beta = 1, \gamma = 2.$
(*d*) α or β a negative integer.
★(*e*) $\alpha = \beta = \frac{1}{2}, \gamma = \frac{3}{2}.$
★(*f*) $\alpha = \frac{1}{2}, \beta = 1, \gamma = \frac{3}{2}.$

★7. When $a^2 = 4b$ in the equation of Prob. 4 show that $x = -a/2$ is a regular-singular point if and only if $d/c = a/2$.

8. Find the solution of the equation

$$2x(1-x)\frac{d^2y}{dx^2} + (3-5x)\frac{dy}{dx} - y = 0$$

for which $(dy/dx) = 1$ when $x = 0$. Why is this one condition sufficient for the determination of a unique solution?

5. Bessel's† Differential Equation. Bessel's equation arises in innumerable problems of applied science, particularly in boundary-value problems for right circular cylinders. It has the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0, \tag{40}$$

where n^2 is a given nonnegative constant.

It is immediately recognized that $x = 0$ is a regular-singular point of Equation (40), whereas all other values of x are ordinary points. In the following, the solution will be expanded about $x = 0$.

If formulas (19) are substituted in Equation (40), there results

$$\sum_{k=0}^{\infty}(r+k)(r+k-1)c_k x^{r+k} + \sum_{k=0}^{\infty}(r+k)c_k x^{r+k} + \sum_{k=0}^{\infty}c_k x^{r+k+2} - n^2\sum_{k=0}^{\infty}c_k x^{r+k} = 0.$$

Equating the coefficient of x^{r+k} to zero yields

$$[(r+k)(r+k-1) + (r+k) - n^2]c_k + c_{k-2} = 0,$$

or

$$[(r+k)^2 - n^2]c_k + c_{k-2} = 0. \tag{41}$$

For $k = 0$, one obtains the indicial equation

$$r^2 - n^2 = 0 \tag{42}$$

whose roots are

$$r_1 = n \geqq 0, \qquad r_2 = -n.$$

For $k = 1$, Equation (41) becomes

$$[(r+1)^2 - n^2]c_1 = 0, \tag{43}$$

† Named after the German astronomer and mathematician Friedrich Wilhelm Bessel, 1784–1846.

and since for neither of the roots (42) the bracket expression in (43) vanishes, we must have

$$c_1 = 0 \tag{44}$$

for both $r_1 = n$ and $r_2 = -n$.

Let us first find the solution belonging to the exponent $r_1 = n$. Then (41) becomes

$$c_k = -\frac{c_{k-2}}{(n+k)^2 - n^2} = -\frac{c_{k-2}}{k(2n+k)}. \tag{45}$$

Since, by (44), $c_1 = 0$, it follows from recursion formula (45) that all coefficients whose subscripts are odd numbers vanish. It remains to calculate the coefficients with even subscripts. Writing $2k$ for k in formula (45), we have

$$c_{2k} = -\frac{c_{2k-2}}{2^2k(n+k)}. \tag{46}$$

Applying this recursion formula for $2k-2$, $2k-4$, . . . , 2, and putting $c_0 = 1$, one obtains the general coefficient with even subscript:

$$c_{2k} = \frac{(-1)^k}{2^{2k}k!(n+1)(n+2)\cdots(n+k)}. \tag{47}$$

Therefore, the solution belonging to the exponent n ($\geqq 0$) is

$$y_1(x) = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}k!(n+1)(n+2)\cdots(n+k)} x^{2k}. \tag{48}$$

Except for special values of n (see Prob. 9 below), this is not the expansion of an elementary function. Multiplied by the constant factor $1/2^n n!$, this† is the expansion of what is known as *Bessel's function‡ of order n*, and is commonly designated as $J_n(x)$:

$$J_n(x) = \frac{x^n}{2^n n!} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}k!(n+1)(n+2)\cdots(n+k)} x^{2k} \tag{49}$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{n!} - \frac{(x/2)^2}{1!(n+1)!} + \frac{(x/2)^4}{2!(n+2)!} - \frac{(x/2)^6}{3!(n+3)!} + \cdots\right].$$

† For definition and tables of the factorial function $n!$ for values other than 0, 1, 2, . . . see Ref. 6, Sec. II. In this case $n!$ is often denoted as $\Gamma(n+1)$.

‡ More precisely, $J_n(x)$ is a Bessel function of the "first kind." For other Bessel functions see Sec. 6, in particular the footnote on p. 305.

This is one solution of Bessel's equation. Most frequently used are the Bessel functions of order 0 and 1. They are

$$(50)\qquad \begin{aligned} J_0(x) &= 1 - \frac{(x/2)^2}{1!^2} + \frac{(x/2)^4}{2!^2} - \frac{(x/2)^6}{3!^2} + \cdots \\ J_1(x) &= x/2 - \frac{(x/2)^3}{1!2!} + \frac{(x/2)^5}{2!3!} - \frac{(x/2)^7}{3!4!} + \cdots. \end{aligned}$$

Graphs of these functions are shown in Fig. 46. For other graphs and tables see Ref. 6, Sec. VIII.

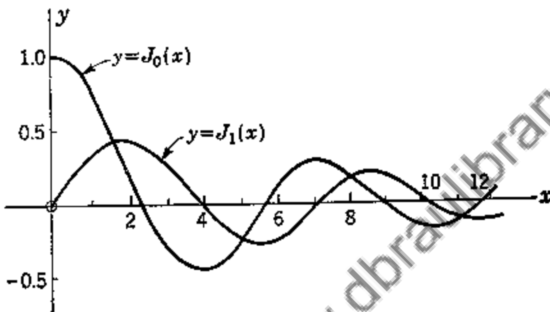


FIG. 46.

Turning to the exponent $r_2 = -n$, (41) becomes

$$(51)\qquad \begin{aligned} c_k &= -\frac{c_{k-2}}{(-n+k)^2 - n^2} \\ &= -\frac{c_{k-2}}{k(-2n+k)}. \end{aligned}$$

This is the same recursion formula as (45) except that n is replaced by $-n$ and that formula (51) becomes invalid if n is a positive integer. Hence, except for this latter case, a solution belonging to the exponent $-n$ is obtained by replacing n by $-n$ in (49):

$$(52)\quad J_{-n}(x) = (x/2)^{-n}\left[\frac{1}{(-n)!} - \frac{(x/2)^2}{1!(-n+1)!} + \frac{(x/2)^4}{2!(-n+2)!} - \frac{(x/2)^6}{3!(-n+3)!} + \cdots\right].$$

If $n = 0$, then this solution is identical with the previously found solution (50). In all other cases $J_{-n}(x)$ is linearly independent of $J_n(x)$. (To check this simply substitute $x = 0$ in an assumed relation of the form $c_1J_n(x) + c_2J_{-n}(x) = 0$.) Therefore,

$$y(x) = AJ_n(x) + BJ_{-n}(x)$$

is the general solution of Bessel's equation except when n is an integer.

If n is one of the integers 1, 2, . . . , then $J_n(x)$ exists, but $J_{-n}(x)$ as defined in (52) cannot be formed.† If $n = 0$, then $J_{-n}(x)$ and $J_n(x)$ exist, but are linearly dependent. Hence, in all cases, at least one integral is obtained expanded in a series of the form (16) about the regular-singular point $x_0 = 0$, in accordance with the theorem of Sec. 3.

If n is an integer, then the present procedure does not lead to the general solution. This case is discussed in the following section.

PROBLEMS

Solve, in terms of Bessel functions (of the first kind) the following equations. In cases where the Bessel functions of the first kind do not supply the general solution state the fact and the reasons.

1. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0.$

2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (a^2x^2 - n^2)y = 0; \quad a \neq 0.$

3. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x-s}\dfrac{dy}{dx} + \left(a^2 - \dfrac{n^2}{(x-s)^2}\right)y = 0; \quad a \neq 0.$

4. $x\dfrac{d^2y}{dx^2} + (1 + 2n)\dfrac{dy}{dx} + xy = 0.$

Hint: Make substitution $y = x^{-n}Y$.

5. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + xy = 0.$

Hint: Make substitution $y = x^{-\frac{1}{2}}Y$ as suggested by Prob. 4; solve also by the use of substitution $y = x^{-1}Y$.

6. $\dfrac{d^2y}{dx^2} + a^2xy = 0; \quad a \neq 0.$

Hint: Make substitutions $x = \left(\dfrac{3}{2a}X\right)^{\frac{2}{3}}$, $y = x^{\frac{1}{2}}Y$.

† If n is one of the numbers 1, 2, . . . , then $J_{-n}(x)$ still can be defined formally by (52), if one agrees to put

$$\frac{1}{(-n)!} = \frac{1}{(-n+1)!} = \cdots = \frac{1}{(-1)!} = 0.$$

The thus defined $J_{-n}(x)$, for $n = 1, 2, \ldots$, is not linearly independent of $J_n(x)$, but it is easily seen that

$$J_{-n}(x) = (-1)^nJ_n(x).$$

7. $\dfrac{d^2y}{dx^2} + a^2x^2y = 0; \quad a \neq 0.$

Hint: Make substitutions $x = \left(\dfrac{2}{a} X\right)^{\frac{1}{2}}$, $y = x^{\frac{1}{2}}Y$.

8. $\dfrac{d^2y}{dx^2} + (e^{2x} - n^2)y = 0.$

Hint: Make substitution $x = \log X$.

***9.** Comparing the results obtained by the two different substitutions suggested in Prob. 5 show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Hint: Compare first terms in power-series expansions. Use the values

$$(\tfrac{1}{2})! = (\tfrac{1}{2}) \sqrt{\pi}, \qquad (-\tfrac{1}{2})! = \sqrt{\pi}.$$

6. Roots of Indicial Equation Differing by Integer. As was pointed out in Sec. 3, the coefficients of expansion (16) can be successively determined from recursion formulas of the form (23) except if $g(r + k) = 0$ for some positive integer k. This cannot happen if r is the algebraically largest root of the indicial equation (in the case of complex roots, if r is the root with the algebraically largest real part). Therefore, the procedure for finding the coefficients, described in Sec. 3, can always be carried out for, at least, one root of the indicial equation, and thus, at least, one solution can be obtained of form (16), as asserted in the theorem of that section. But if there are roots of the indicial equation which differ by an integer, then for the smaller root it will happen that $g(r + k) = 0$ for some positive integer k. For such a root the general procedure cannot be carried out, and there will be fewer than n solutions found in the form (16). This, obviously, will also happen if the indicial equation has repeated roots (which can be considered as roots differing by the integer 0).

Thus, for the hypergeometric equation of Sec. 4 the indicial equation was

$$r(r + \gamma - 1) = 0,$$

and only one solution of form (16), the one belonging to the algebraically larger of the two roots 0, $1 - \gamma$, is obtained if γ is an integer. Again, for Bessel's equation of Sec. 5 the indicial equation was

$$r^2 - n^2 = 0,$$

and only one solution of form (16), the one belonging to the nonnegative exponent n, is obtained if n is an integer.

The general solution in these exceptional cases is somewhat more complicated. It will be discussed here only for differential equations of second order. If $x = 0$ is a regular-singular point of such an equation, then it has the form

(53) $$x^2 \frac{d^2y}{dx^2} + x(b_{10} + \cdots)\frac{dy}{dx} + (b_{20} + \cdots)y = 0,$$

where the dots indicate terms of at least first degree in x. The indicial equation is, by (20) and (21),

$$g(r) = r(r-1) + b_{10}r + b_{20} = 0,$$

or

(54) $$r^2 + (b_{10} - 1)r + b_{20} = 0.$$

In the exceptional case the two roots of the indicial equation are r and $r - m$, where m is one of the numbers 0, 1, 2, ... Since by (54) the sum of the two roots must be equal to $-(b_{10} - 1)$, we have

(55) $$2r - m = 1 - b_{10}.$$

The solution $y_1(x)$ belonging to the exponent r, which is the algebraically larger of the two roots r, $r - m$ is of the form

(56) $$y_1(x) = x^r(1 + c_1x + c_2x^2 + \cdots),$$

where the coefficients can be determined by the method described in Sec. 3 (for convenience, the value 1 is chosen for c_0). In the preceding chapter (see Sec. 2) it is shown that if one solution of a homogeneous linear equation is known, then the order of the equation can be reduced by one. The appropriate substitution is

$$y(x) = y_1(x)u(x).$$

Then

$$y' = y_1u' + y_1'u$$
$$y'' = y_1u'' + 2y_1'u' + y_1''u,$$

and Equation (53) becomes, if account is taken of y_1 being a solution,

$$x^2y_1u'' + 2x^2y_1'u' + x(b_{10} + \cdots)y_1u' = 0$$

or

(57) $$u'' + \left(2\frac{y_1'}{y_1} + \frac{b_{10}}{x} + \cdots\right)u' = 0.$$

Now, by (56)

$$\frac{y_1'}{y_1} = \frac{r}{x} + \cdots$$

where the dots stand for terms that do not contain negative powers of x. Hence, (57) can be written as

$$u'' + \left(\frac{2r + b_{10}}{x} + \cdots\right)u' = 0.$$

When use is made of (55) this equation may also be written as

$$\frac{u''}{u'} = -\left(\frac{m+1}{x} + \cdots\right)$$

and integration gives

$$\log u' = -(m + 1) \log x + \cdots,$$

hence

$$u' = x^{-(m+1)} \exp (\cdots) \tag{58}$$

where the dots stand for some series of nonnegative powers of x. When $\exp (\cdots)$ is itself expanded as a power series of the form

$$1 + a_1x + a_2x^2 + \cdots,$$

then (58) becomes

$$u' = x^{-(m+1)} + a_1x^{-m} + \cdots + a_mx^{-1} + a_{m+1} + a_{m+2}x + \cdots. \tag{59}$$

If $m = 0$ (case of *repeated* roots of indicial equation), then (59) reads

$$u' = x^{-1} + a_1 + a_2x + \cdots. \tag{60}$$

Integration of (59) and (60) gives

$$u(x) = \begin{cases} \log x + a_1x + \dfrac{a_2}{2}x^2 + \cdots & (\text{if } m = 0) \\ \dfrac{x^{-m}}{-m} + \dfrac{a_1x^{-m+1}}{-m+1} + \cdots + a_m \log x + a_{m-1}x + \cdots & (\text{if } m = 1, 2, \cdots) \end{cases} \tag{61}$$

Therefore, the desired solution $y(x) = y_1(x) \cdot u(x)$ is of the form

$$y_2(x) = \begin{cases} y_1(x) \log x + x^r(C_1x + C_2x^2 + \cdots) & (\text{if } m = 0) \\ y_1(x) \log x + x^{r-m}\left(-\dfrac{1}{m} + C_1x + C_2x^2 + \cdots\right) & (\text{if } m = 1, 2, \cdots) \end{cases} \tag{62}$$

where $C_1, C_2, \ldots$ are coefficients to be determined in each individual case.

The result, thus derived, is that if the indicial equation has repeated roots or roots that differ by an integer, then one integral (belonging to the algebraically larger exponent) has form (56) and a second integral has form (62). The coefficients $c_1, c_2, \ldots$ and $C_1, C_2, \ldots$, respectively, are found by the method of undetermined coefficients. That the resulting series are valid expansions in the neighborhood of the regular-singular point is proved in more advanced texts (see, for example, Ref. 4, Sec. 50).

To illustrate the use of formula (62) let it be required to find the general solution of Bessel's equation of order zero, expanded about $x = 0$. The equation is

$$x^2y'' + xy' + x^2y = 0. \tag{63}$$

In Sec. 5 one integral was found:

$$y_1(x) = J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}. \tag{64}$$

The indicial equation is [see (42)] $r^2 = 0$. Hence, $r = 0$ is a repeated root, and to find another integral one uses (62):

$$y(x) = y_1(x) \log x + C_1 x + C_2 x^2 + \cdots$$
$$= J_0(x) \log x + \sum_{m=1}^{\infty} C_m x^m.$$

Then

$$y'(x) = J_0'(x) \log x + \frac{J_0(x)}{x} + \sum_{m=1}^{\infty} m C_m x^{m-1}$$
$$y''(x) = J_0''(x) \log x + \frac{2J_0'(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{m=1}^{\infty} m(m-1) C_m x^{m-2}.$$

When we substitute these expansions in (63) and remember that $J_0(x)$ satisfies this equation we find

$$2xJ_0'(x) - J_0(x) + \sum_{m=1}^{\infty} m(m-1)C_m x^m + J_0(x) + \sum_{m=1}^{\infty} mC_m x^m + \sum_{m=1}^{\infty} C_m x^{m+2} = 0,$$

or, since

$$J_0'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!k!} \left(\frac{x}{2}\right)^{2k-1},$$

(65) $$\sum_{m=1}^{\infty} m^2 C_m x^m + \sum_{m=1}^{\infty} C_m x^{m+2} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!k!2^{2k-2}} x^{2k}.$$

Since on the right-hand side of this equation there are only even powers of x, it is clear that it can be satisfied only if the coefficients C with odd subscripts are zero. Hence, it remains to find the coefficients C with even subscripts. Putting $m = 2k$ and equating the coefficients of x^{2k} in Equation (65) we find

(66) $$4k^2 C_{2k} + C_{2k-2} = \frac{(-1)^{k-1}}{(k-1)!k!2^{2k-2}}.$$

From this recursion formula the general coefficient C_{2k} can be determined.† For the first three coefficients one obtains

† The result is

$$C_{2k} = \frac{(-1)^{k-1}}{(k!)^2 2^{2k}} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right).$$

$$\begin{array}{lll} k = 1: & 4C_2 = 1, & \text{or } C_2 = \frac{1}{4} \\ k = 2: & 16C_4 + C_2 = -\frac{1}{8}, & \text{or } C_4 = -\frac{3}{128} \\ k = 3: & 36C_6 + C_4 = \dfrac{1}{192}, & \text{or } C_6 = \dfrac{11}{13{,}824}. \end{array}$$

Therefore,

$$y_2(x) = J_0(x) \log x + \left(\frac{x^2}{4} - \frac{3}{128}x^4 + \frac{11}{13{,}824}x^6 - \cdots\right) \tag{67}$$

is the desired solution of Equation (63). This function is called (*Neumann's*†) *Bessel function of the second kind* and is commonly designated by $K_0(x)$. Hence, the general solution of equation (63) may be written as

$$y(x) = AJ_0(x) + BK_0(x).$$

7. Point at Infinity. For many purposes both of theoretical and applied science the solution of a differential equation is required for large values of the independent variable. In particular, if the independent variable is time, one may want to know the solution at a distant time when the disturbances due to temporary causes are sufficiently weakened. For such applications the series expansions of the preceding sections would be unsuited since, even if those series converge for all values of the independent variable, they become impractical for numerical calculation when the values of the variable involved are large.

In this section expansions "about the point at infinity" are discussed. These are expansions that are valid for all sufficiently large values of the variable. No new theory is necessary for such expansions. For, the substitution

$$x = \frac{1}{z}, \qquad z = \frac{1}{x} \tag{68}$$

transforms every neighborhood of the point $x = \infty$ into a neighborhood of the point $z = 0$. Hence, all that is necessary is to transform the differential equation by this substitution and then to expand the solution in the neighborhood of $z = 0$ by the methods described in the preceding sections. On replacing z by $1/x$ in the expansions thus found, the desired expansions about the point at infinity are then obtained.

In accordance with this explanation the point at infinity will be said to be an ordinary point or a regular-singular point with the

† Named after the German mathematician Karl Neumann, 1832–1925. Various other "Bessel functions of the second kind" are in use. They are all solutions of Bessel's differential equation and form together with the Bessel functions of the first kind a fundamental system of linearly independent solutions.

exponents r_1, r_2, if, for the transformed equation, the point $z = 0$ is an ordinary point or a regular-singular point with the exponents r_1, r_2, respectively.

Example 3. Let it be required to expand the general solution of the equation

$$4x^3 \frac{d^2y}{dx^2} + 6x^2 \frac{dy}{dx} + y = 0$$

in the neighborhood of $x = \infty$ (that is, for large values of x).

We make substitution (68) and obtain

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} = -z^2\frac{dy}{dz}$$
$$\frac{d^2y}{dx^2} = \frac{1}{x^4}\frac{d^2y}{dz^2} + \frac{2}{x^3}\frac{dy}{dz} = z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz}.$$

Then the equation becomes

$$4z\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + y = 0.$$

It is seen that $z = 0$ is a regular-singular point. Hence, we put

$$y = \sum_{k=0}^{\infty} c_k z^{k+r}.$$

Then

$$\frac{dy}{dz} = \sum_{k=0}^{\infty} (k + r)c_k z^{k+r-1}$$
$$\frac{d^2y}{dz^2} = \sum_{k=0}^{\infty} (k + r)(k + r - 1)c_k z^{k+r-2}.$$

With these expansions the above equation becomes

$$4\sum_{k=0}^{\infty} (k + r)(k + r - 1)c_k z^{k+r-1} + 2\sum_{k=0}^{\infty} (k + r)c_k z^{k+r-1} + \sum_{k=0}^{\infty} c_k z^{k+r} = 0.$$

Equating the coefficient of z^{k+r-1} to zero,

$$2(k + r)(2k + 2r - 1)c_k + c_{k-1} = 0.$$

For $k = 0$ the indicial equation is obtained,

$$r(2r - 1) = 0,$$

whose roots are $r_1 = 0$, $r_2 = \frac{1}{2}$.

For $r_1 = 0$ we have the recursion formula

$$c_k = -\frac{c_{k-1}}{2k(2k - 1)}.$$

Applying this formula for $k - 1$, $k - 2$, . . . , 1, and setting $c_0 = 1$, we find

$$c_k = \frac{(-1)^k}{(2k)!}$$

and the solution is

$$y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^k.$$

Replacing z by $1/x$, this becomes

$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{-k},$$

which is readily identified as the expansion of

$$y_1(x) = \cos (x^{-\frac{1}{2}}).$$

For $r_2 = \frac{1}{2}$ we obtain the recursion formula

$$c_k = -\frac{1}{2k(2k + 1)} c_{k-1}.$$

If we proceed as above we find

$$c_k = \frac{(-1)^k}{(2k + 1)!}$$

and

$$y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} z^k$$

or

$$y_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{-k}$$
$$= x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{-\frac{2k+1}{2}},$$

which is readily identified as the expansion of

$$y_2(x) = x^{\frac{1}{2}} \sin (x^{-\frac{1}{2}}).$$

PROBLEMS

Expand in series that are valid in the neighborhood of $x = \infty$ the solutions of the following equations. Try to identify the obtained series as expansions of known functions.

1. $x^2(x^2 - 1) \frac{d^2y}{dx^2} + x(2x^2 - 3) \frac{dy}{dx} - y = 0.$

2. $x^6 \frac{d^3y}{dx^3} + x^2(6x^3 + 1) \frac{d^2y}{dx^2} + 2x(3x^3 - 1) \frac{dy}{dx} + 2y = 0.$

3. $x^2(x + 2) \frac{d^2y}{dx^2} - x(x - 4) \frac{dy}{dx} - 4y = 0.$

4. $4x^4 \frac{d^2y}{dx^2} + 4x^3 \frac{dy}{dx} - (x^2 + 1)y = 0.$

5. $x^4 \frac{d^2y}{dx^2} + x^3 \frac{dy}{dx} + (1 - n^2x^2)y = 0.$

6. $x^2(x - 1) \frac{d^2y}{dx^2} + x[\alpha + \beta - 1 + (1 - \delta)x] \frac{dy}{dx} - \alpha\beta y = 0.$

Find the solutions of the following "terminal-value" problems. Name the interval of convergence, and try to identify the solutions.

7. $t^4 \frac{d^2x}{dt^2} + t(2t^2 - 1) \frac{dx}{dt} - 2x = 0;\quad x \to 1,\ \frac{dx}{dt} \to 0$ as $t \to \infty$.

8. $t^4 \frac{d^2x}{dt^2} + t(2t^2 - 1) \frac{dx}{dt} - 2x = 0;\quad x \to 0,\ t^2 \frac{dx}{dt} \to 1$ as $t \to \infty$.

9. $t^7 \frac{d^3x}{dt^3} + 6t^6 \frac{d^2x}{dt^2} + 2t(3t^4 - 2) \frac{dx}{dt} - 12x = 0;\quad x \to 1,\ \frac{dx}{dt} \to 0,\ \frac{d^2x}{dt^2} \to 0$ as $t \to \infty$.

★**10.** Show that $x = \infty$ is an ordinary point of the equation

$$\frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = f(x)$$

if the functions $x^2a_1(x) - 2x$, $x^4a_2(x)$, $x^4f(x)$ are analytic in the neighborhood of $x = \infty$.

*11. Show that $x = \infty$ is a regular-singular point of the differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

if $xa_1(x)$ and $x^2a_2(x)$ are analytic in the neighborhood of $x = \infty$.

12. What kind of point (ordinary, regular-singular, nonregular-singular) is $x = \infty$ (*a*) in Gauss's equation? (*b*) in Bessel's equation?

8. Legendre's† Differential Equation. Legendre's equation arises in numerous problems of applied science, particularly in boundary-value problems for spheres. It has the form

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \tag{69}$$

where n is a given constant, mostly a positive integer, which is the only case that will be considered here. We propose to expand its solution in the neighborhood of the point at infinity. Making the transformation $x = 1/z$, we have as in the preceding section

$$\frac{dy}{dx} = -\frac{1}{x^2}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz}.$$

With these substitutions Equation (69) becomes

$$(z^4 - z^2)\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz} + n(n+1)y = 0. \tag{70}$$

It is immediately recognized that $z = 0$ is a regular-singular point of this equation. Hence, the point at infinity is a regular-singular point of Legendre's equation. Putting, as in Sec. 3,

$$y = \sum_{k=0}^{\infty} c_k z^{k+r},$$

we have

$$\frac{dy}{dz} = \sum_{k=0}^{\infty}(k+r)c_k z^{k+r-1}$$

$$\frac{d^2y}{dz^2} = \sum_{k=0}^{\infty}(k+r)(k+r-1)c_k z^{k+r-2}.$$

† Named after the French mathematician Adrien-Marie Legendre, 1752–1833.

With these expansions Equation (70) becomes

$$\sum_{k=0}^{\infty} (k + r)(k + r - 1)c_k z^{k+r+2} - \sum_{k=0}^{\infty} (k + r)(k + r - 1)c_k z^{k+r} + 2\sum_{k=0}^{\infty} (k + r)c_k z^{k+r+2} + n(n + 1) \sum_{k=0}^{\infty} c_k z^{k+r} = 0.$$

Equating the coefficient of z^{k+r} to zero,

$$[(k + r - 2)(k + r - 3) + 2(k + r - 2)]c_{k-2} - [(k + r)(k + r - 1) - n(n + 1)]c_k = 0$$

or

$$(71) \quad [(k + r)(k + r - 1) - n(n + 1)]c_k - (k + r - 2)(k + r - 1)c_{k-2} = 0.$$

For $k = 0$ the indicial equation is obtained,

$$(72) \qquad r(r - 1) - n(n + 1) = 0,$$

whose roots are

$$r_1 = -n, \qquad r_2 = n + 1.$$

For $k = 1$, formula (71) becomes

$$[r(r + 1) - n(n + 1)]c_1 = 0,$$

and since for neither of the roots r_1, r_2 the bracket factor vanishes, c_1 must be 0 in both cases. Then, because of recursion formula (71), also $c_3 = 0 = c_5 = c_7 = \cdots$. It remains to calculate the coefficients c with even subscripts. Replacing k by $2k$ in (71) we have

$$(73) \qquad c_{2k} = \frac{(2k + r - 2)(2k + r - 1)}{(2k + r)(2k + r - 1) - n(n + 1)} c_{2k-2}.$$

Let us consider, at first, the solution belonging to the exponent $r_1 = -n$. Then (73) becomes

$$(74) \qquad c_{2k} = \frac{-(n + 2 - 2k)(n + 1 - 2k)}{2k(2n + 1 - 2k)} c_{2k-2}.$$

Applying this formula for $2k - 2$, $2k - 4$, . . . , 2, and setting $c_0 = 1$, we find

(75)

$$c_{2k} = (-1)^k \frac{n(n - 1)(n - 2) \cdots (n - 2k + 1)}{2 \cdot 4 \cdot 6 \cdots 2k(2n - 1)(2n - 3) \cdots (2n - 2k + 1)}.$$

It follows from this formula that $c_{n+1} = c_{n+2} = c_{n+3} = \cdots = 0$. Hence, the only coefficients different from zero in this solution are $c_0, c_2, c_4, \ldots c_n$ (or c_{n-1} if n is odd), and the solution is

$$y_1 = \sum_{k=0}^{\frac{n}{2} \text{ or } \frac{(n-1)}{2}} (-1)^k \frac{n(n-1)\cdots(n-2k+1)}{2\cdot 4 \cdots 2k(2n-1)(2n-3)\cdots(2n-2k+1)} z^{2k-n}.$$

Replacing z by $1/x$, this becomes

$$(76)\quad y_1(x) = \sum_{k=0}^{\frac{n}{2} \text{ or } \frac{(n-1)}{2}} (-1)^k \frac{n(n-1)\cdots(n-2k+1)}{2\cdot 4 \cdots 2k(2n-1)(2n-3)\cdots(2n-2k+1)} x^{n-2k}$$
$$= x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)} x^{n-4} - \cdots .$$

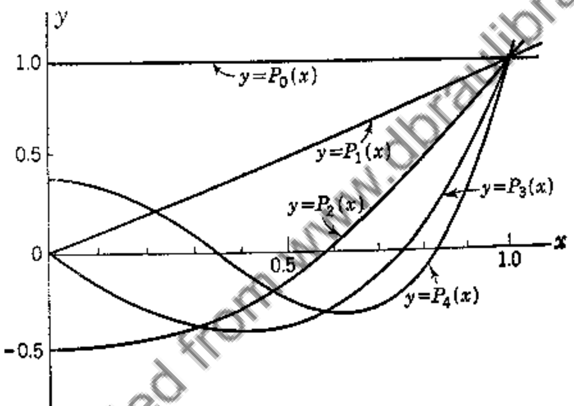


FIG. 47.

This is a polynomial of degree n, and when multiplied† by the constant factor $(2n)!/2^n(n!)^2$ it is known as a *Legendre polynomial*, of degree n, commonly designated as $P_n(x)$:

$$(77)\qquad P_n(x) = \frac{(2n)!}{2^n(n!)^2} y_1(x).$$

The first five Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x), \qquad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3).$$

Graphs of these functions are shown in Fig. 47. For further graphs and tables see, for example, Ref. 6, Sec. VII.

† This factor is chosen so as to make $P_n(1) = 1$ for all n.

For the exponent $r_2 = n + 1$, formula (73) becomes

$$c_{2k} = \frac{(n + 2k)(n + 2k - 1)}{2k(2n + 2k + 1)} c_{2k-2}. \tag{78}$$

Applying this recursion formula for $2k - 2, 2k - 4, \ldots, 2$, and setting $c_0 = 1$, we find

$$c_{2k} = \frac{(n + 1)(n + 2) \cdots (n + 2k)}{2 \cdot 4 \cdot 6 \cdots 2k(2n + 3)(2n + 5) \cdots (2n + 2k + 1)}. \tag{79}$$

Hence, the integral belonging to the exponent r_2 is

$$y_2 = \sum_{k=0}^{\infty} \frac{(n + 1)(n + 2) \cdots (n + 2k)}{2 \cdot 4 \cdots 2k \cdot (2n + 3)(2n + 5) \cdots (2n + 2k + 1)} z^{2k+n+1}.$$

Replacing z by $1/x$ this becomes

(80) $y_2(x) =$

$$\sum_{k=0}^{\infty} \frac{(n + 1)(n + 2) \cdots (n + 2k)}{2 \cdot 4 \cdot 6 \cdots 2k(2n + 3)(2n + 5) \cdots (2n + 2k + 1)} x^{-2k-n-1}$$

$$= x^{-(n+1)}\left[1 + \frac{(n + 1)(n + 2)}{2 \cdot (2n + 3)} x^{-2} + \frac{(n + 1)(n + 2)(n + 3)(n + 4)}{2 \cdot 4(2n + 3)(2n + 5)} x^{-4} + \cdots\right].$$

From the remark after the theorem in Sec. 3 it follows that this expansion is valid for $|x| > 1$. This can also be readily verified by the ratio test. The function thus obtained, when multiplied by the constant factor $\frac{2^n(n!)^2}{(2n + 1)!}$ is known as a *Legendre function* of the *second kind* and is commonly denoted as $Q_n(x)$,

$$Q_n(x) = \frac{2^n(n!)^2}{(2n + 1)!} y_2(x). \tag{81}$$

The general solution of Legendre's equation can now be written as

$$y(x) = AP_n(x) + BQ_n(x).$$

PROBLEMS

Solve, in terms of Legendre polynomials and functions the following equations:

1. $(1 - x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 12y = 0.$

2. $(a^2 - x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0; \quad a \neq 0.$

3. $\dfrac{d}{dx}\left[(x^2 + ax + b)\dfrac{dy}{dx}\right] - n(n+1)y = 0; \quad a^2 - 4b > 0.$

Hint: Make substitution $x = \sqrt{\dfrac{a^2}{4} - b}\, X - \dfrac{a}{2}.$

4. Prove that

$$P_{2m+1}(0) = 0$$
$$P_{2m}(0) = (-1)^m \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m m!}.$$

5. Show that $x = 1$ and $x = -1$ are regular-singular points of the Legendre equation and that the indicial equation for either point is $r^2 = 0$.

6. Prove that the only solution of Legendre's equation that remains finite at $x = 1$ (or $x = -1$) is $cP_n(x)$, where c is an arbitrary constant.

7. Expand in the neighborhood of $x = 1$ the solution of Legendre's equation for which $y = 1$ when $x = 1$ [this solution is, by the result of Prob. 6 and by the footnote, page 311, $P_n(x)$ itself].

8. Show that

$$P_n(x) = x^n F\left(\frac{-n}{2}, \frac{1-n}{2}, \frac{1-2n}{2}; x^{-2}\right)$$
$$Q_n(x) = x^{-n-1} F\left(\frac{n+1}{2}, \frac{n+2}{2}, \frac{2n+3}{2}; x^{-2}\right),$$

where $F(\alpha, \beta, \gamma; x)$ is the hypergeometric function.

***9.** Prove that

$$y = \frac{d^n}{dx^n}(x^2 - 1)^n$$

is a solution of Legendre's equation.

Then from the result of Prob. 6 and from the footnote, page 311, it follows that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n \qquad (\textit{Rodrigues' formula})$$

10. Show that the equation

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{dY}{d\theta}\right) + n(n+1)\sin\theta\, Y = 0$$

is satisfied by $Y = P_n(\cos\theta)$ and by $Y = Q_n(\cos\theta)$.

APPENDIX A

THE EXISTENCE THEOREM FOR FIRST-ORDER EQUATIONS

The proof presented here of the theorem stated below is in essence that given by the French mathematician E. Picard in 1890. Various refinements in the proof have since been made, and the hypotheses may be considerably weakened. For example, it is possible to prove the *existence* of a solution assuming only continuity of $f(x, y)$ even though the proof of Picard uses also the existence and continuity of $\partial f/\partial y$. However, in order to prove that the solution is unique, more than the continuity of $f(x, y)$ is required.

Theorem. If $f(x, y)$ and $\partial f/\partial y$ are continuous in the rectangle R: $a_1 \leqq x \leqq a_2$, $b_1 \leqq y \leqq b_2$, then through each point (x_0, y_0) interior to R there passes a unique integral curve of the equation

(1) $$\frac{dy}{dx} = f(x, y).$$

Before proceeding with the proof we establish some facts which will be useful later. We first note that since $f(x, y)$ is continuous in R it is bounded, that is, there is a constant A such that, for all points (x, y) in R,

(2) $$|f(x, y)| < A.$$

We next observe that, by the mean-value theorem,

(3) $$|f(x, y_1) - f(x, y_2)| = |y_2 - y_1| \left|\frac{\partial}{\partial y} f(x, y_1 + \theta(y_2 - y_1)\right|,$$

where $0 < \theta < 1$ and (x, y_1) and (x, y_2) are in R. Now the function $\partial f/\partial y$ is continuous and is therefore bounded; hence there is a constant B such that for all (x, y) in R

(4) $$\left|\frac{\partial}{\partial y} f(x, y)\right| < B.$$

From (3) and (4) we have,† for all (x, y_1) and (x, y_2) in R,

(5) $$|f(x, y_1) - f(x, y_2)| \leqq B|y_2 - y_1|.$$

† The existence and continuity of $\partial f/\partial y$ is used only to derive condition (5). Thus the existence and continuity of $\partial f/\partial y$ may be replaced by (5), the latter being known as a "Lipschitz condition" after the German mathematician R. Lipschitz.

Finally, for convenience we restrict our attention to the subrectangle R_0 of R with center at (x_0, y_0) (see Fig. 48)

$$R_0: |x - x_0| < a, \qquad |y - y_0| < b. \tag{6}$$

It will be necessary later to choose a suitable value for a.

To establish the *existence* of an integral curve of (1) passing through (x_0, y_0) we transform our problem into an equivalent problem involv-

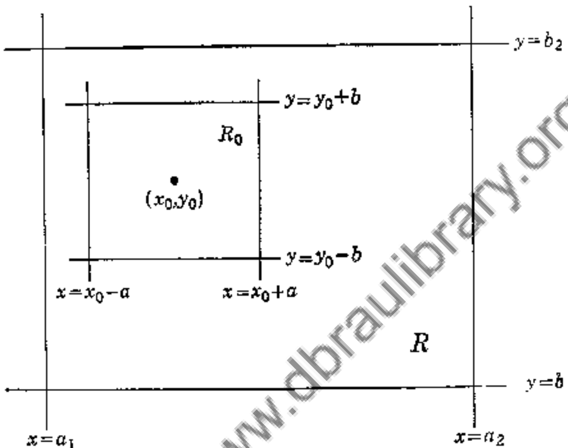


Fig. 48.

ing an *integral equation*. Suppose that $y = y(x)$ satisfies (1) and $y(x_0) = y_0$. Integrating (1) between x_0 and x we obtain

$$y(x) - y(x_0) = \int_{x_0}^{x} f(s, y(s))\, ds$$

or

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds. \tag{7}$$

Hence if a solution exists it satisfies (7). Conversely if $y(x)$ satisfies the integral equation (7) it also satisfies the differential equation (1), as may be seen by differentiating (7). Clearly $y(x_0) = y_0$. The solution of the given initial-value problem is thus reduced to the solution of the integral equation (7).

The solution of the integral equation is effected by a simple iteration process which yields a sequence $y_n(x)$ of approximating functions. This sequence of functions $y_n(x)$ possesses a limit function $y(x)$ which satisfies (7). For the initial approximation we choose

$$y_0(x) \equiv y_0$$

and define $y_1(x)$ as follows:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0)\, ds.$$

The second approximation is defined by

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))\, ds.$$

The process may be continued, the nth iteration giving

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds \quad (n = 1, 2, 3, \cdots) \tag{8}$$

In order to be sure that (8) is a valid definition it is necessary to be sure that the point $(x, y_{n-1}(x))$ remains in the rectangle R where $f(x, y)$ is defined. Now if $(x, y_{n-1}(x))$ lies in R_0, then

$$|y_n(x) - y_0| = \left|\int_{x_0}^{x} f(s, y_{n-1}(s))\, ds\right| < A\,|x - x_0|.$$

Hence $|y_n(x) - y_0|$ will be less than b if $A|x - x_0| < b$. This can be assured if we choose $a = b/A$. It follows therefore that for every n Equation (8) yields a continuous function in the interval $x_0 - a < x < x_0 + a$.

We now show that the sequence of functions $y_n(x)$ approaches a limit function $y(x)$. Observe that

$$y_n(x) = y_0 + [y_1(x) - y_0] + [y_2(x) - y_1(x)] + \cdots + [y_n(x) - y_{n-1}(x)].$$

Therefore, to show that $y_n(x)$ approaches a limit, it suffices to show that the infinite series

$$y_0 + [y_1(x) - y_0] + \cdots + [y_n(x) - y_{n-1}(x)] + \cdots \tag{9}$$

converges. To show that (9) converges we examine the magnitude of the nth term

$$|y_n(x) - y_{n-1}(x)| = \left|\int_{x_0}^{x} [f(s, y_{n-1}(s)) - f(s, y_{n-2}(s))]\, ds\right|.$$

From Equation (5) it follows that

$$|y_n(x) - y_{n-1}(x)| \leqq B \int_{x_0}^{x} |y_{n-1}(s) - y_{n-2}(s)|\, ds. \tag{10}$$

Now

$$|y_1(x) - y_0| = \left|\int_{x_0}^{x} f(s, y_0)\, ds\right| < A|x - x_0|$$

and we get from (10) in the case $n = 2$

$$|y_2(x) - y_1(x)| \leqq B \left| \int_{x_0}^{x} A|s - x_0|\, ds \right| = AB \frac{|x - x_0|^2}{2}.$$

By mathematical induction we obtain for the general case

$$|y_n(x) - y_{n-1}(x)| \leqq AB^{n-1} \frac{|x - x_0|^n}{n!}. \tag{11}$$

From (11) we see that the terms in (9) are in absolute value less than the terms of the series

$$|y_0| + A|x - x_0| + AB \frac{|x - x_0|^2}{2} + \cdots + AB^{n-1} \frac{|x - x_0|^n}{n!} + \cdots$$

which is convergent. Hence the series (9) converges for $|x - x_0| < a$, and in fact absolutely and uniformly (see Ref. 1, pp. 386–392). Therefore, the function

$$y(x) = \lim_{n \to \infty} y_n(x)$$

exists and is continuous (Ref. 1, p. 393).

To verify that $y(x)$ satisfies the integral equation (7) we take the limit of both sides of (8):

$$\begin{aligned} \lim y_n(x) &= y_0 + \lim_{n \to \infty} \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds \\ &= y_0 + \int_{x_0}^{x} \lim_{n \to \infty} f(s, y_{n-1}(s))\, ds \\ &= y_0 + \int_{x_0}^{x} f(s, \lim_{n \to \infty} y_{n-1}(s))\, ds. \end{aligned}$$

Hence,

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds.$$

The interchange of limit and integral is permissible here because of the uniform convergence (Ref. 1, p. 395).

The existence of a solution having been established we turn now to the uniqueness proof. We observe first that it suffices to show uniqueness in any interval containing x_0 in its interior since (x_0, y_0) is an arbitrary point of R. We restrict our attention therefore to a closed interval I for which

$$B|x - x_0| < 1, \tag{12}$$

where B is the constant in (5).

Now suppose that $y(x)$ and $y^*(x)$ are two solutions of the integral equation (7). If $y(x) \not\equiv y^*(x)$ in I, then $|y(x) - y^*(x)|$ has a positive maximum M attained at some point $x = x_1$ of the interval I. Then

$$\begin{aligned} M = |y(x_1) - y^*(x_1)| &= \left| \int_{x_0}^{x_1} [f(s, y(s)) - f(s, y^*(s))] \, ds \right| \\ &\leqq \left| \int_{x_0}^{x_1} B|y(s) - y^*(s)| \, ds \right| \\ &\leqq B|x_1 - x_0|M \\ &< M, \end{aligned}$$

the last inequality being valid because of the restriction (12) for the interval I. This contradiction completes the proof.

APPENDIX B

EXISTENCE THEOREMS FOR LINEAR EQUATIONS

The existence theorem of Appendix A applies to the linear equation

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where $P(x)$ and $Q(x)$ are continuous for $a \leqq x \leqq b$. Writing (1) in the form

$$\frac{dy}{dx} = Q - Py$$

we see that the function $f(x, y)$ of Appendix A is $Q - Py$ and $\partial f/\partial y = -P$. These functions are continuous for $a \leqq x \leqq b$ and arbitrary y. The approximating functions $y_n(x)$ are defined and continuous for $a \leqq x \leqq b$, and hence $y(x)$ is also defined for $a \leqq x \leqq b$.

For systems of equations a similar theorem may be proved. For linear systems the proof is considerably simplified. In order not to complicate matters we consider here a system of two linear equations.

Theorem 1. *If $A(x)$, $B(x)$, $C(x)$, $D(x)$, $E(x)$, $F(x)$ are continuous in the interval $a \leqq x \leqq b$ and if x_0, y_0, z_0 are given, where $a \leqq x_0 \leqq b$, then there is a unique pair of functions $y(x)$ and $z(x)$ defined for $a \leqq x \leqq b$ which satisfy the system of equations*

$$\begin{aligned} \frac{dy}{dx} &= Ay + Bz + C \\ \frac{dz}{dx} &= Dy + Ez + F \end{aligned} \tag{2}$$

and for which $y(x_0) = y_0$, $z(x_0) = z_0$.

The proof follows that given in Appendix A, and hence only a sketch of the proof will be given. The solution of the initial-value problem is equivalent to the solution of the following system of integral equations.

$$\begin{aligned} y(x) &= y_0 + \int_{x_0}^{x} A(s)y(s)\,ds + \int_{x_0}^{x} B(s)z(s)\,ds + \int_{x_0}^{x} C(s)\,ds \\ z(x) &= z_0 + \int_{x_0}^{x} D(s)y(s)\,ds + \int_{x_0}^{x} E(s)z(s)\,ds + \int_{x_0}^{x} F(s)\,ds. \end{aligned} \tag{3}$$

Sequences of functions $y_n(x)$ and $z_n(x)$ are defined as follows:

$$(4)\quad \begin{aligned} y_n(x) &= y_0 + \int_{x_0}^{x} A(s)y_{n-1}(s)\,ds + \int_{x_0}^{x} B(s)z_{n-1}(s)\,ds + \int_{x_0}^{x} C(s)\,ds \\ z_n(x) &= z_0 + \int_{x_0}^{x} D(s)y_{n-1}(s)\,ds + \int_{x_0}^{x} E(s)z_{n-1}(s)\,ds + \int_{x_0}^{x} F(s)\,ds \end{aligned}$$

for $n = 1, 2, \ldots$, and $y_0(x) \equiv y_0$, $z_0(x) \equiv z_0$.

The convergence of the sequences of functions $y_n(x)$ and $z_n(x)$ follows from the convergence of the infinite series

$$(5)\quad \begin{aligned} &y_0 + [y_1(x) - y_0] + \cdots + [y_n(x) - y_{n-1}(x)] + \cdots \\ &z_0 + [z_1(x) - z_0] + \cdots + [z_n(x) - z_{n-1}(x)] + \cdots. \end{aligned}$$

Now

$$(6)\quad \begin{aligned} |y_1(x) - y_0| &= \left|\int_{x_0}^{x} A(s)y_0\,ds + \int_{x_0}^{x} B(s)y_0\,ds + \int_{x_0}^{x} C(s)\,ds\right| \\ &\leqq \alpha|y_0||x - x_0| + \beta|y_0||x - x_0| + \gamma|x - x_0| \\ &\leqq M|x - x_0|, \end{aligned}$$

where α, β, γ are upper bounds for $|A(x)|$, $|B(x)|$, and $|C(x)|$, respectively, in $a \leqq x \leqq b$, and $M = \alpha|y_0| + \beta|y_0| + \gamma$. For the general term we have

$$(7)\quad |y_n(x) - y_{n-1}(x)| = \left|\int_{x_0}^{x} A(s)[y_{n-1}(s) - y_{n-2}(s)]\,ds + \int_{x_0}^{x} B(s)[z_{n-1}(s) - z_{n-2}(s)]\,ds\right|.$$

From (6) and (7) it follows after some calculation that

$$(8)\quad |y_n(x) - y_{n-1}(x)| \leqq MN^{n-1}\frac{|x - x_0|^n}{n!},$$

where $N = \alpha + \beta$. Similarly

$$|z_n(x) - z_{n-1}(x)| = RS^{n-1}\frac{|x - x_0|^n}{n!}$$

where R and S are suitably chosen constants. Therefore the series in (5) both converge absolutely and uniformly to the continuous functions $y(x)$ and $z(x)$. It is an easy matter to prove that the integral equations (3) are satisfied.

The existence proof is thus completed. We omit the proof of uniqueness, noting only that it may be handled along the same lines as that given in Appendix A.

Remark 1. The technique of the proof obviously permits extension to the case of n equations in n unknown functions.

Remark 2. If the coefficients $A, \ldots, F$ are analytic the approximating functions also are analytic and $y(x)$ and $z(x)$ therefore are analytic because they are limits of uniformly convergent series of analytic functions.

Remark 3. The existence theorem for systems of equations will give an existence theorem for higher order equations. This we will illustrate for the case of the second-order linear equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = c(x). \tag{9}$$

Setting $dy/dx = z$ we see that Equation (9) is equivalent to the system

$$\frac{dy}{dx} = z$$
$$\frac{dz}{dx} = -a(x)z - b(x)y + c(x).$$

The following theorem then is a corollary of the existence theorem for systems of two equations. Its generalization to nth-order equations is obvious.

Theorem 2. If $a(x)$, $b(x)$, $c(x)$ are continuous in the interval $\alpha \leqq x \leqq \beta$ and if x_0, y_0, y_0' are given, where $\alpha \leqq x_0 \leqq \beta$, then the equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = c(x)$$

has a unique solution $y(x)$ in the interval $\alpha \leqq x \leqq \beta$ such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$.

APPENDIX C

THE FAMILY OF INTEGRAL CURVES OF THE LINEAR FRACTIONAL EQUATION

The linear fractional equation is

$$\text{(1)} \qquad \frac{dy}{dx} = \frac{ax + by}{cx + dy}$$

where a, b, c, and d are constants. We confine our attention to the case where $ad - bc \neq 0$ since otherwise the right member of (1) reduces to a constant. In Sec. 8, Chap. III, it was seen that Equation (1) could always be solved as a homogeneous equation. We are less interested here in obtaining a general solution than in discovering the character of the family of integral curves in the neighborhood of the origin which is a singular point. The results of our investigation are shown in Fig. 49, where the various possible types are drawn and their relations to the values of a, b, c, d are indicated.

Instead of solving (1) directly we change the problem into one involving a system of equations. This change is made because the new problem permits easier classification of the various possible types.

Consider a parametric representation of a curve

$$\text{(2)} \qquad x = x(t), \qquad y = y(t),$$

where the parameter t may be thought of as the time. Equations (2) describe the plane motion of a point whose x and y components of velocity are dx/dt and dy/dt. If these components of velocity depend linearly on x and y according to the equations

$$\text{(3)} \qquad \begin{aligned} \frac{dx}{dt} &= cx + dy \\ \frac{dy}{dt} &= ax + by \end{aligned}$$

then the point moves in a path whose slope at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{ax + by}{cx + dy},$$

that is, the path is an integral curve of (1). The solutions of Equation

(1) may therefore be obtained by eliminating t between the solution $x = x(t)$, $y = y(t)$ of the system (3).

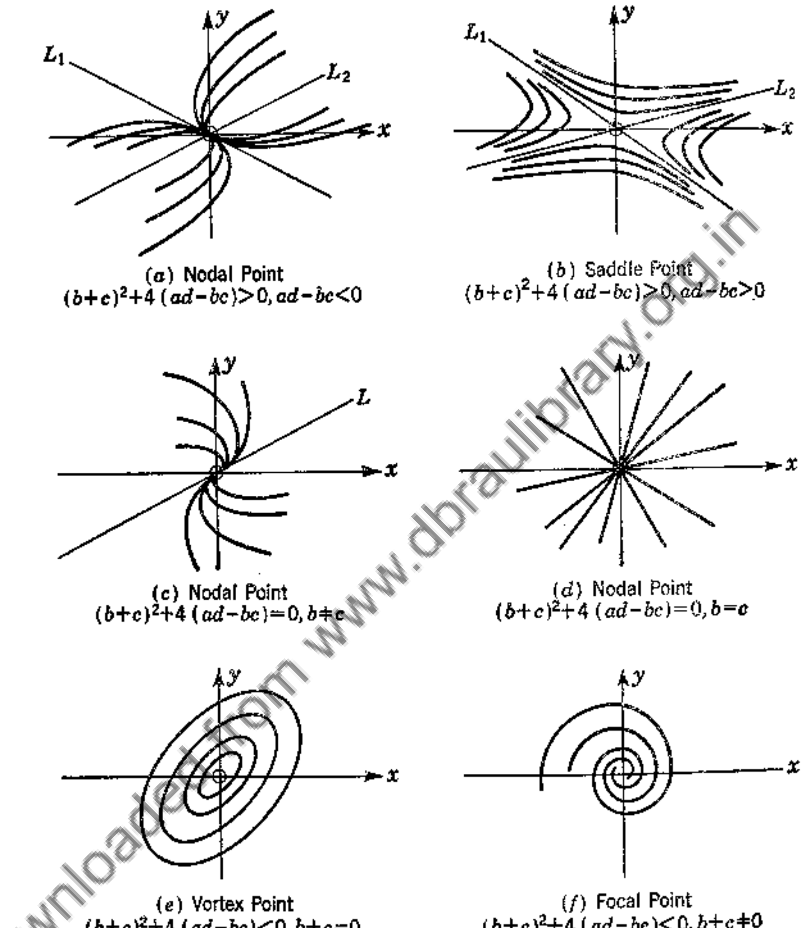


Fig. 49.

Because the coefficients in (3) are constant it is natural to attempt the exponential solution

$$x = Ae^{\lambda t}, \qquad y = Be^{\lambda t}. \tag{4}$$

Substitution of (4) in (3) yields

$$\begin{aligned} A\lambda e^{\lambda t} &= (cA + dB)e^{\lambda t} \\ B\lambda e^{\lambda t} &= (aA + bB)e^{\lambda t}, \end{aligned}$$

or

$$(5)\qquad \begin{aligned}(c - \lambda)A + dB &= 0\\ aA + (b - \lambda)B &= 0.\end{aligned}$$

Equations (5) are linear and homogeneous in A and B. In order that a nontrivial solution exist it is necessary and sufficient that the determinant of the coefficients vanish.

$$(6)\qquad \begin{vmatrix} c - \lambda & d \\ a & b - \lambda \end{vmatrix} = 0 = \lambda^2 - (b + c)\lambda - (ad - bc).$$

The quadratic equation (6) is called the *characteristic equation* of the system (3). The roots of (6) are denoted by λ_1 and λ_2,

$$\lambda_1 = \frac{(b + c) + \sqrt{(b + c)^2 + 4(ad - bc)}}{2}$$

$$\lambda_2 = \frac{(b + c) - \sqrt{(b + c)^2 + 4(ad - bc)}}{2}$$

Because of the assumption $ad - bc \neq 0$ neither root can be zero. There are three cases to be considered depending on the nature of the roots λ_1 and λ_2.

Case 1. $(b + c)^2 + 4(ad - bc) > 0$; λ_1, λ_2 real, $\lambda_1 \neq \lambda_2$.
Case 2. $(b + c)^2 + 4(ad - bc) = 0$; λ_1, λ_2 real, $\lambda_1 = \lambda_2$.
Case 3. $(b + c)^2 + 4(ad - bc) < 0$; λ_1, λ_2 conjugate complex.

Case 1. Let $A = A_1$, $B = B_1$ be a nontrivial solution of (5) for $\lambda = \lambda_1$, and let $A = A_2$, $B = B_2$ be a nontrivial solution for $\lambda = \lambda_2$. Then $A_1B_2 - A_2B_1 \neq 0$ and

$$(7)\qquad x = A_1e^{\lambda_1 t}, \qquad y = B_1e^{\lambda_1 t}$$

and

$$(8)\qquad x = A_2e^{\lambda_2 t}, \qquad y = B_2e^{\lambda_2 t}$$

are solutions of the system (3). The general solution, which by the results of Chap. VI can contain but two arbitrary constants, is

$$(9)\qquad \begin{aligned} x &= c_1A_1e^{\lambda_1 t} + c_2A_2e^{\lambda_2 t}\\ y &= c_1B_1e^{\lambda_1 t} + c_2B_2e^{\lambda_2 t}, \end{aligned}$$

where c_1 and c_2 are arbitrary.

To describe the family of curves (9) it is necessary to consider the signs of λ_1 and λ_2. However, regardless of the signs of λ_1 and λ_2 there are two rectilinear solutions given by (7) and (8). Denote these straight lines by L_1 and L_2, respectively. The equations of L_1 and L_2 are

$$L_1: \quad A_1y = B_1x$$
$$L_2: \quad A_2y = B_2x.$$

To discuss the family of curves (9) [apart from the rectilinear solutions (7) and (8) which are obtained for $c_2 = 0$ and $c_1 = 0$, respectively], it is convenient to regard (9) as a transformation.

Consider the transformation

$$\text{(10)} \qquad \begin{aligned} x &= c_1A_1\xi + c_2A_2\eta \\ y &= c_1B_1\xi + c_2B_2\eta \end{aligned}$$

from the $\xi\eta$ plane to the xy plane. The determinant of the transformation is $c_1c_2(A_1B_2 - A_2B_1)$ and is different from zero if $c_1c_2 \neq 0$. Such a transformation (called an *affine transformation*) maps the $\xi\eta$ plane on the xy plane in a one-to-one manner and sends straight lines into straight lines. It is shown in works on geometry that (10) is the result of a rotation of the $\xi\eta$ plane followed by a projection of the $\xi\eta$ plane on the xy plane.

Now consider the curve in the $\xi\eta$ plane whose equation is

$$\text{(11)} \qquad \xi = e^{\lambda_1 t}, \qquad \eta = e^{\lambda_2 t}.$$

The curve (9) is the image in the xy plane of the curve (11) under the transformation (10). In order to discuss this curve it is necessary to consider the signs of λ_1 and λ_2. There are two cases to be considered.

Case 1*a*. λ_1 and λ_2 have the same sign ($ad - bc < 0$). The curve (11) then is the "parabolalike" curve

$$\eta = \xi^{\lambda_2/\lambda_1}.$$

If $\lambda_2 < \lambda_1 < 0$ this curve is tangent to the ξ axis and its image in the xy plane is a parabolalike curve tangent to the line L_1 at the origin. This family of curves is drawn in Fig. 49*a*. If $0 < \lambda_2 < \lambda_1$ the family of curves in the xy plane is tangent to L_2 at the origin. In each of these cases the origin is called a *nodal point*.

Case 1*b*. λ_1 and λ_2 have opposite signs ($ad - bc > 0$). The curve (11) is then the "hyperbolalike" curve

$$\xi\eta^{-\lambda_1/\lambda_2} = 1.$$

Its image in the xy plane under the transformation (10) is also like a hyperbola and has L_1 and L_2 as asymptotes. The family of solutions (9) for this case is drawn in Fig. 49*b*. The origin is called a *saddle point*.

Case 2. $\lambda_1 = \lambda_2 = \lambda$. The procedure of case 1 fails to produce a solution with two arbitrary constants, but as with second-order equations with constant coefficients another solution with t as a multiplier may be found. Omitting the details, the general solution of (3) is found to be

$$x = [c_1A_1 + c_2(A_2 + A_3t)]e^{\lambda t}$$
$$y = [c_1B_1 + c_2(B_2 + B_3t)]e^{\lambda t}, \tag{12}$$

where $A_1, \ldots, B_3$ are definite constants and c_1 and c_2 are arbitrary. Now if A_3 and B_3 are not both zero there is but one rectilinear solution L (obtained for $c_2 = 0$), whose equation is

$$L: \qquad B_1x = A_1y.$$

As t becomes infinite the point (x, y) either approaches the origin tangent to L or recedes to infinity, depending on the sign of λ. The family of curves (12) for this case is shown in Fig. 49*c*.

On the other hand if A_3 and B_3 are both zero, the family (12) has a particularly simple form. Examination of the derivation of Equations (12) shows that A_3 and B_3 can both be zero only in case $b = c$. For this case the general solution of (3) turns out to be

$$x = c_1e^{\lambda t}$$
$$y = c_2e^{\lambda t} \tag{13}$$

where c_1 and c_2 are arbitrary. Equations (13) include all lines through the origin and are drawn in Fig. 49*d*.

In both cases the origin is called a *nodal point* (degenerate).

Case 3. λ_1 and λ_2 are conjugate complex. Set $\lambda_1 = \mu + i\nu$, $\lambda_2 = \mu - i\nu$. Then the solutions of (5), $A = A_1$, $B = B_1$ for $\lambda = \lambda_1$ are the complex conjugates of the solution $A = A_2$, $B = B_2$ for $\lambda = \lambda_2$. If $A_1 = \alpha_1 + i\alpha_2$, $B_1 = \beta_1 + i\beta_2$ then $A_2 = \alpha_1 - i\alpha_2$, $B_2 = \beta_1 - i\beta_2$, and $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. As in (7) one solution of the system (3) is

$$\begin{aligned} x &= A_1e^{\lambda_1 t} = (\alpha_1 + i\alpha_2)e^{(\mu+i\nu)t} \\ &= e^{\mu t}[\alpha_1 \cos \nu t - \alpha_2 \sin \nu t + i(\alpha_2 \cos \nu t + \alpha_1 \sin \nu t)] \\ y &= B_1e^{\lambda_1 t} = (\beta_1 + i\beta_2)e^{(\mu+i\nu)t} \\ &= e^{\mu t}[\beta_1 \cos \nu t - \beta_2 \sin \nu t + i(\beta_2 \cos \nu t + \beta_1 \sin \nu t)]. \end{aligned} \tag{14}$$

Another solution is obtained, for A_2, B_2, by replacing i by $-i$ in (14). By taking one-half the sum of these two solutions we obtain a *real* solution of the system (3)

$$x = e^{\mu t}(\alpha_1 \cos \nu t - \alpha_2 \sin \nu t)$$
$$y = e^{\mu t}(\beta_1 \cos \nu t - \beta_2 \sin \nu t). \tag{15}$$

Similarly, by taking $-i$ times one-half the difference of the above solutions, a second real solution, similar to (15), can be obtained.

With the two real solutions obtained above it can be seen that all real solutions are of the form

$$\begin{aligned} x &= e^{\mu t}(c_1 \cos \nu t + c_2 \sin \nu t) \\ y &= e^{\mu t}(k_1 \cos \nu t + k_2 \sin \nu t), \end{aligned} \tag{16}$$

where c_1, c_2, k_1, k_2 are real constants for which $c_1k_2 - c_2k_1 \neq 0$.

As in case 1 the curve (16) is most easily described as the image in the xy plane of the curve

$$\xi = e^{\mu t} \cos \nu t, \qquad \eta = e^{\mu t} \sin \nu t \tag{17}$$

in the $\xi\eta$ plane under the transformation

$$\begin{aligned} x &= c_1\xi + c_2\eta \\ y &= k_1\xi + k_2\eta. \end{aligned} \tag{18}$$

To determine the nature of the curve (16) it is necessary to consider two cases.

Case 3*a*. $\mu = 0$, $(b + c = 0)$. In this case the curve (17) is the circle $\xi^2 + \eta^2 = 1$. Since the affine transformation (18) is a rotation followed by a projection, the image of the circle is an ellipse in the xy plane. The family (16) for this case is drawn in Fig. 49*e*. The origin is called a *vortex point*.

Case 3*b*. $\mu \neq 0$, $(b + c \neq 0)$. The curve (17) is a logarithmic spiral, and its image under the affine transformation (18) is also a spiral. The family of curves (16) for this case is drawn in Fig. 49*f*. The origin is called a *focal* (or *spiral*) *point*.

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ANSWERS

Chap. I, Sec. 1

2. (*a*) Defined and continuous for all x. (*b*) Defined and continuous for $-1 < x < 1$. (*c*) Defined and continuous for $(n - \frac{1}{4})\pi \leqq \theta \leqq (n + \frac{1}{4})\pi$.

Chap. I, Sec. 3

3. Yes. $\int \sin \sqrt{x}\, dx$ is not elementary but this is not easy to prove. **4.** The number x (real or complex) is an "algebraic number" if x satisfies an equation $a_0x^n + \cdots + a_n = 0$ where the coefficients $a_0, \ldots, a_n$ are *integers*. **5.** $4 \int_0^{\sqrt{2}} \sqrt{(4 - x^2)/(4 - 2x^2)}\; dx$. **6.** $\arcsin x = \int_0^x dt/\sqrt{1 - t^2}$; and $y = 1/\sqrt{1 - x^2}$ satisfies the equation $(1 - x^2)y^2 - 1 = 0$.

Chap. I, Sec. 4

1. $y = cx + k$.

Chap. I, Sec. 5

1. (*a*) $y = \pm a$. (*b*) $x^2 + y^2 = a^2$. (*c*) $y = \pm x$. (*d*) $y^2 - x^2 = p^2$.

Chap. I, Sec. 6

3. Hint: Excluding the trivial case $y_1 \equiv y_2 \equiv 0$, suppose the notation chosen so that $y_1(x) \neq 0$. Then $(y_1y_2' - y_2y_1')/y_1^2 = d(y_2/y_1)/dx$.

Chap. I, Sec. 12

1. The cofactors of the second row are 8, 8, -8. Theorem 1 gives, as the expansion by cofactors of the second row, $(2)(8) + (1)(8) + (2)(-8) = 8$. Theorem 2 applied to the elements of the first row and cofactors of the second row, $(-1)(8) + (-2)(8) + (-3)(-8) = 0$. **2.** $(1)(1) + (2)(-2) + (3)(1) = 0$. **3.** The determinant A is zero, but not all the determinants B_i are zero. **4.** Yes, because $A = 0$.

Chap. II, Sec. 2

1. (*a*) 3. (*b*) 4. (*c*) 1. (*d*) 2. (*e*) 1. (*f*) 1.

Chap. II, Sec. 4

1. (*a*), (*b*), (*d*), (*f*), (*g*), (*h*), (*i*). **2.** (*a*) $x^2y'^2 + x^2 = r^2y'^2$.
(*b*) $x^2y'' - 2xy' + 2y = 0$. (*c*) $y^2 = 2xyy' - x^2 + 1$. (*d*) $y'' - y = 2 - x^2$.
(*e*) $u'' + 9u = 0$. (*f*) $v'' + 9v = 0$.
(*g*) $xyy'' - xy'^2 + 2xy^3y'' + 2xy^2y'^2 - yy' - 2y^3y' = 0$.
3. $(x - y)y'' = 1 + y' + y'^2 + y'^3$. **4.** $xyy'' + xy'^2 - yy' = 0$.

Chap. II, Sec. 5

1. (*e*) Hint: $p = \pm\sqrt{x}$ and there are two integral curves through each point. **2.** $\alpha^2 - 4\beta \geqq 0$. **3.** Two. Yes, points for which $x^2 + 4y < 0$.

Chap. II, Sec. 6

1. $x^2 + y$ is continuous for all x, y and $\partial(x^2 + y)/\partial y = 1$ is continuous. **2.** The region of existence is the whole plane. The region of uniqueness excludes the x axis. **3.** $x \geqq 0$. Yes.

Chap. II, Sec. 7

1. $y = 2xy'$ and $yy' + 2x = 0$. **3.** Hints: One method is to find the points of intersection of two curves corresponding to different values of λ, say λ_1 and λ_2, and show that at such a point the slopes are negative reciprocals. Another, and perhaps easier, method is to show that the given family is a general solution of $yy'^2 + 2xy' - y = 0$ and that the two values of y' given by the differential equation are negative reciprocals. **4.** $3yy' + 2x = 0$.

Chap. II, Sec. 8

1. $2y = x + 2$. Because the slope is constant $dy = \Delta y$. **2.** Since the slope will change appreciably between $x = 0$ and $x = 1$, a small value of h is necessary for accuracy. However, using $h = 0.25$ gives the approximate points (0.25, 0.56), (0.5, 0.64), (0.75, 0.74), (1, 0.88). The correct values are (to the nearest hundredth) (0.25, 0.57), (0.5, 0.67), (0.75, 0.8), (1, 1). The refined process of the text gives the correct values also. **3.** Approximate points using $h = 0.1$ are (1.1, 0.1), (1.2, 0.21), (1.3, 0.33), (1.4, 0.46), (1.5, 0.60), (1.6, 0.75), (1.7, 0.91), (1.8, 1.08), (1.9, 1.26), (2.0, 1.45). **4.** Approximate points using $h = 0.25$ are (0.25, 1.82), (0.5, 2.06), (0.75, 2.28), (1, 2.47).

Chap. II, Sec. 9

1. $y = cx - c^3$. **2.** $y = px + \dfrac{1}{2p}$. The parabola is also an integral curve of the equation. **3.** $y = px \pm r\sqrt{1 + p^2}$. Two lines through each point.

Chap. II, Sec. 10

1. Singular solutions $y = \pm 1$. **2.** $x(x - 1)^2 = 0$. The envelope is $x = 0$. $x = 1$ is an extraneous locus. **3.** $xy'^2 - 2yy' + 4x = 0$. Singular solutions $y = \pm 2x$. **4.** Singular solutions $y = \pm 1$.

Chap. II, Miscellaneous Problems

4. $2x^2 - xy = y$. **5.** The locus is the second-degree curve $(x - x_0)(ax + by + c) = y - y_0$. **6.** $y = 2xy' - yy'^2$. Singular solutions $y = \pm x$. **7.** Since, between $x = 1$ and $x = 1.5$, y' increases from 1 to more than 2.25 it is desirable to use a small value for h. Using $h = 0.1$ gives $f(1.5) = 0.67$. **8.** $yy' = xy'^2 + 1$. The singular solution is $y^2 = 4x$. **9.** $x^2 + y^2 = 1$. **10.** The region of existence is $2n\pi \leqq y \leqq (2n + 1)\pi$. The region of uniqueness is $2n\pi < y < (2n + 1)\pi$. **11.** $y'' = 0$. **12.** $(1 + y'^2)y''' - 3y'y''^2 = 0$. **13.** $(xy' - y)(yy' + x) = 0$. **14.** $y = \frac{1}{2}x^2$.

Chap. III, Sec. 2

1. Parts (*b*) and (*d*) are exact. **2.** (*a*) $xy + x^2 = c$. (*b*) $\frac{x^2}{2} + \frac{x}{y} + 2 \log y = c$. (*c*) $x^3 - 3xy^2 = c$. (*d*) $xy = c(y+1)(x-1)$. (*e*) $ax^2 + 2bxy + cy^2 = k$. (*f*) Not exact. (*g*) $(x+1)^2 = (c-2x)(y-1)$. (*h*) $y \sin x - x \cos y + \cos y = c$. (*i*) $\sin (x+y) - \cos x \tan y - x = c$. (*j*) $e^y \sin x + y^2 = c$. (*k*) $x \log (x^2 + y^2) + xy = c$. (*l*) $2y + \log \frac{1+xy}{1-xy} = c$. (*m*) $(\sin 2x + 2) \tan y = c$. (*n*) $\sin^2 x \cos y = c$. (*o*) $xy = c$. (*p*) $(x+y-1)/(x-y+1) = c$. (*q*) $\arctan (x+y) + y - x = c$.

Chap. III, Sec. 3

1. (*a*) $x = A(y-1)e^y$. (*b*) $\frac{1}{x^2} + \frac{1}{y^2} = c$. (*c*) $y - 1 = Axe^{x-y}$.
(*d*) $y = x \cos c + \sqrt{1-x^2} \sin c$. (*e*) $xy = c(x+1)(y-1)$.
(*f*) $\sin y = Axe^{-x^2}$. (*g*) $x^2y^2(x-1) = c(x+1)$. (*h*) $i = i_0e^{-Rt/L}$.
(*i*) $e^{2x} + 2 \arctan \sqrt{y} = c$. (*j*) $(1+x) \sin^2 y = c(1-x)$.
(*k*) $(x+y) = c(a^2 - xy)$. (*l*) $e^{-x} + e^{-y} = c$. (*m*) $\cos y = c(x^2+1)$.
(*n*) $1 + y = c(1-y)e^{2x}$.
(*o*) $y[x^2 + 2x - c(x+1)] = x^2 + 6x + 4 + c(x^2 + x - 1)$.
2. (*a*) $y = 2x^2/(1 + 2x - 2x^2)$. (*b*) $\sec x + \tan y = -1$. (*c*) $\rho = e^{\theta/k}$.
(*d*) $(1 + e^x) \sin y = \sqrt{2}$. (*e*) $x^2 - y^2 = 3$. (*f*) $xy^2 = 2x^2 + x - 2$.
(*g*) $e^{2x} - 2e^x + \log (1+e^x)^2 = \log y^2$. (*h*) $x = 2ye^{(-1+1/y)}$.
(*i*) $x + y + \log (x-1)^2(y-1)^2 = 5 + e$. (*j*) $\tan y = 2 \sin x$.
(*k*) $e^{x^2} + e^{-y^2} = 3$. (*l*) $2xe^{-2x} + e^{-2x} - 2 \log (y^2+1) = 1$. **3.** $y = ce^x$.
4. Either $f(x) \equiv 0$ or $f(x) = e^{kx}$.

Chap. III, Sec. 4

1. (*a*) $x^4y = cxy + 3$. (*b*) $y = x^2 + cx - 1$. (*c*) $2y = x \tan (2x + c)$.
(*d*) $y = x/(\log x - c)$. (*e*) $xy = \sin x - x \cos x + c$.
(*f*) $2 \arctan (y/x) = \log (x^2 + y^2) + c$. (*g*) $2x + x^2y^3 + cy = 0$.
(*h*) I.F. $= \frac{1}{x(x+y)(y^2+1)}$; $(y^2+1)x^2(x+y)^2 = c$.
(*i*) I.F. $= \cos y$; $\sin (x+y) + \sin y = c$. (*j*) I.F. $= 1/x^2$; $x^2 \sin xy + 1 = cx$.
(*k*) I.F. $= y^2$; $y^3 = \frac{c}{(1-x^3)}$. (*l*) $\arcsin \frac{y}{x} = y + c$.

3. If we set $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \varphi(x)$, then $e^{\int \varphi(x)dx}$ is an integrating factor. In the differential equation this integrating factor is e^{3x}/x. **4.** $(y^2 + x^2 - 2)e^{x^2/2} = c$.

5. $\frac{1}{\mu}\frac{d\mu}{dy} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$.

Chap. III, Sec. 5

1. (*a*) $2y = x + cx^{-1}$. (*b*) $xy = -\cos x + c$. (*c*) $(1+y^2)^2x = 2 \log y + y^2 + c$.
(*d*) $y = x^5 + cx^4$. (*e*) $y = x + c\sqrt{1+x^2}$. (*f*) $\rho = 2 \sin \theta - 2 + ce^{-\sin\theta}$.

(g) $y = -2x - 2 + ce^x$. (h) $(x^2 + x + 1)^2 y = x^4/4 + x^3/3 + x^2/2 + c$.
(i) $(\sec 2x + \tan 2x)y = 2 \tan x - 2x - 4 \log \cos x - 2 \sin^2 x + c$.
(j) $v \sin u + u \sin u + \cos u = c$. (k) $3\rho \sin^2 \theta = -2 \sin^3 \theta + c$.
(l) $y = (x + c)e^{-ax}$. (m) $x^2(x^2 - 1)y = x^5 + c$. **3.** (a) $y = x(1 + e^{(1-1/x)})$.
(b) $y = \sin x - 1 + e^{-\sin x}$. (c) $2y = e^{-x}(x^2 + 2x - 8) - 2x^3$.
(d) $2y \arctan x = \pi/2 + \log [(1 + x^2)/2]$.
(e) $6y \cos x = 6 \sin x - 4 \sin^3 x - 1$. (f) $(2y + \alpha x^2 - \alpha) \sqrt{1 - x^2} = \alpha$.
(g) $(1 + x^2)y = \log \sec x$. (h) $i = E \dfrac{R \sin \omega t - \omega L \cos \omega t}{R^2 + \omega^2 L^2} + \dfrac{E\omega L}{R^2 + \omega^2 L^2} e^{-Rt/L}$.

5. (a) $y^2 = ce^{-x^2} - 1$. (b) $y^3 = 2/(cx^3 - 3x^5)$. (c) $x^3 = (y + 1)^2 + c(y + 1)^{-1}$.
(d) $y = e^{x^2}(e^{x^2/2} + 2)^3$. (e) $y^{-1} = x \cos x + \dfrac{9 - \pi^2}{2\pi^2} x$. (f) $y = \dfrac{1}{(e^x - 2e^{-2x})}$.
6. (a) $\sin y = A + ce^{-x^2/2}$. (b) $\log y = -\operatorname{ctn} x + c \csc x$.
(c) $e^y = 5(2 \sin x - \cos x) + ce^{-2x}$. (d) $y^{-1} = 2x - 1 + ce^{-2x}$.
(e) $(x + \sqrt{1 + x^2}) \tan z = x^2 + x\sqrt{1 + x^2} - \log (x + \sqrt{1 + x^2}) + c$.

Chap. III, Sec. 6

1. (a) $(y - x)(y + x) = cy$. (b) $\log (x^2 + y^2) + 2 \arctan (y/x) = c$.
(c) $y = x \arctan (c/x)$. (d) $y^3 = x^3 \log e^5 x^3$.
(e) $y = \frac{1}{2}[(1 + \sqrt{2})x^2 + (1 - \sqrt{2})]$. (f) $y = x \arccos \log (c/x)$.
(g) $x^2 \log y = cx^2 + 6xy$. (h) $x^2 = c(y + x)e^{y/x}$. (i) $y = 2x/(x^2 + c)$.
(j) $y = 2cx^2/(c^2 - x^2)$. **2.** Hint: Use Euler's theorem for homogeneous functions which is as follows: If $f(x, y)$ is homogeneous of degree n, then $xf_x + yf_y = nf$.

Chap. III, Sec. 7

1. (a) $(\frac{5}{7}, \frac{15}{7})$. (b) $(2, \pm 1)$. (c) $(-3, 0)$, $(5, 4)$.

Chap. III, Sec. 8

1. $y = cx + x \log x$. **2.** $(x - y + 1)^2(x + y + 3)^3 = c$.
3. $y = x \tan (\log \sqrt{x^2 + y^2} + c)$. **4.** $\log (2x - y - 2)^2 = y - x + c$.
5. $x^2 + y^2 = c$. **6.** $\log [4x^2 + (y + 2)^2] + \arctan (y + 2)/2x = c$.
7. $(x + y)^3 = c(x - y + 4)$. **8.** $x - 2y + c = 5 \log (x - 3y + 8)$. **9.** $y = cx^n$.
10. In numbers 1, 2, 7, 9 an integral curve passes through the singular point. In numbers 4, 8 there is no singular point.

Chap. III, Sec. 9

1. (a) $3x^2y^2 = 2y^3 + c$. (b) $y - a \arctan (x + y)/a = c$.
(c) $(xy - 1)^3 = cxy^2$. (d) $y = x + (ce^{x/2} - x - 2)^2$. **2.** (a) $y = x + \dfrac{ce^{2y} + 1}{ce^{2y} - 1}$.
(b) $\sqrt{b}\, x^2 + \sqrt{a}\, y = (\sqrt{b}\, x^2 - \sqrt{a}\, y)ce^{\sqrt{ab}x^2}$. (c) $x + ye^{x/y} = c$.
(d) $x^2 + y^2 = ce^{2 \arctan y/x}$.

Chap. III, Sec. 10

1. $3y^2 + 2x^2 = c$. **2.** $y^2 + 2x^2 = 6$. **3.** $y^2 = 2kx + c$. **4.** $y^2 - x^2 = c$.
5. $y = \pm \cosh x + c$. **6.** $x^2 + y^2 = cy$. **7.** $\rho = c \sin \theta$. **8.** $\rho(1 + 2 \cos \theta) = c$.

9. $\rho = ce^{k\theta}$, $\rho = e^{\theta}$. **10.** $\rho = c \sin \theta$ or $\rho \sin \theta = c$. **11.** $\rho = c \sin \theta$.
12. $\theta = 0 (5 \geqq \rho \geqq 1)$ and $\rho = e^{\theta/\sqrt{15}}$, also other answers.

Chap. III, Sec. 11

1. 5,315 years. **2.** \$149.18. **3.** (a) $dT/dt = k(T - T_0)$. (b) 22.6 min.
4. \$7,869.40. **5.** 1 hr 9 min. **6.** 281.25 lb; 1.406 lb/gal.
7. $P = P_0 e^{(b-d)t/1,000}$, where b and d are births and deaths per 1,000 per day and t is in days. **8.** $x = \dfrac{S(3G - S)e^{k(3G-S)t/G}}{3G - Se^{k(3G-S)t/G}}$.

Chap. III, Sec. 12

1. 19 ft/sec; 20 ft/sec. **2.** 7.2° from the horizontal; $v = 8(1 - e^{-t/2})$.
3. $mv^2 = -kx^3 + 2mgx + mv_0^2$. **4.** 26,000 ft/sec. **5.** 36,750 ft/sec.
6. $v = \sin 2t - 2 \cos 2t + 2e^{-t}$. **7.** $v = 180(1 - e^{-8t/45})$; 106 ft/sec.
9. $v = 30(1 - e^{-0.0025t})$; 20.5 mph. **10.** $v^2 = (x - 30)(x - 34)$, where x is the length of the longest side.

Chap. III, Sec. 13

1. $$i = \frac{\omega E_0}{R} \frac{1}{1/R^2C^2 + \omega^2} \left(\frac{1}{RC} \cos \omega t + \omega \sin \omega t \right) - \frac{\omega E_0}{R^2C(1/R^2C^2 + \omega^2)} e^{-t/RC};$$

steady-state current $= \dfrac{\omega E_0}{R \sqrt{1/R^2C^2 + \omega^2}} \sin (\omega t + \delta)$ where $\tan \delta = 1/RC\omega$.
2. $i = 2.10 \sin (120\pi t - 1.44) + 2.08e^{-50t}$. **3.** $q = E_0C(1 - e^{-(t-t_0)/RC})$.
4. $q = q_0e^{-t/RC}$.

Chap. III, Sec. 14

1. 4 min 21 sec. **2.** 5 min 48 sec. **3.** 21.3 in. **4.** $x = 4/(1 + 0.00356t)^2$.

Chap. III, Sec. 15

1. $z = A^2Kt/(1 + AKt)$. **2.** $(A - u)^a(B - u)^b(C - u)^c = A^aB^bC^ce^{-Kt}$, where $K = \alpha\beta\gamma k/(\alpha + \beta + \gamma)^3$, $A = (\alpha + \beta + \gamma)/\alpha$, $B = (\alpha + \beta + \gamma)/\beta$, $C = (\alpha + \beta + \gamma)/\gamma$, A, B, C distinct, and $a = 1/(B - A)(C - A)$, $b = 1/(C - B)(A - B)$, $c = 1/(A - C)(B - C)$.

Chap. III, Sec. 16

1. Heat loss $21{,}600\pi$ Btu/hr; $T = 600/r - 200$. **2.** 3,740 Btu; $T = 112 - 57.7 \log (2 + x)$. **3.** $Q = ck(T_1 - T_2)/(cw + k)$. **4.** $T = -30 + 87.3 \log 6r$; 9,870 Btu.

Chap. III, Miscellaneous Problems

1. $y = x/2 + c/x$. **2.** $3y \sin x - x^3 + y^3 = c$. **3.** $2y = x^{\frac{1}{2}} - x^{\frac{1}{2}} \log x + cx^{\frac{1}{2}}$.
4. $4y = x^2 - 4$. **5.** $y = cxe^{-1/x}$. **6.** $2y = -x + x \tan (\log x + c)$.
7. $y = x + c$ and $y = x/(1 + cx)$. **8.** $y = \arctan (x + y) + c$.
9. $x + y + c = 2 \log (x + 2y + 5)$. **10.** $y = (x^2 - c^2)/2c$.
11. $y = \log x - 1 + c/x$. **12.** $y = a \dfrac{c - \tan (a/b \arctan x/b)}{1 + c \tan (a/b \arctan x/b)}$.

13. $x(x^2 + 3y^2) = c$. **14.** $xy + \log \dfrac{x}{x+y} = c$.

15. $y \arcsin y + \sqrt{1 - y^2} = x + \pi/12$. **16.** $(x + y - 3)^3 = 125(y - x - 1)$.
17. $y = 2e^{(y-2x)/x}$. **18.** $3y = (x + 1)^5 + c(x + 1)^2$. **19.** $y = 1/(x \log cx)$.
20. $(y + x)^3 = 27(y - x)$. **21.** $x = cy - y \log y$. **22.** $2y = -\cos x + c \sec x$.

23. $y = \log \dfrac{ce^{-x} + e^x}{2}$. **24.** $2y(x^2 + x) = x^2 + c$. **25.** $2y(x^2 - 1)^{\frac{3}{2}} = c - x^2$.

26. $x^3y - 3x + 3y^2 = cy$. **27.** $y = cx^{-2} - x^{-3}$. **28.** No real solution.

29. $\log [4(x + 1)^2 + (y + 1)^2] + \arctan \dfrac{y+1}{2(x+1)} = c$. **30.** $x = \tan (x + c) - y$.

31. Flex point locus $x^2 + 2x + y = 0$. **32.** $3y^2 + x^2 = 7$.
33. Hint: The tangential acceleration is d^2s/dt^2 where s is the arc length. Then $md^2s/dt^2 = -mg \sin \theta = -mg\, dy/ds$ and multiplication by ds/dt renders the equation integrable. **34.** 256 sec. **35.** $3y = x^3 + c$. **36.** $\rho = k \cos^4 (\theta/2)$.
37. $y = ce^{x/k}$, where k is the length of the subtangent.
38. $x^2 + y^2 - 2x + 2y = c$. **40.** (a) $p = p_0e^{-gh/k}$; (b) $p^{\frac{2}{7}} = p_0^{\frac{2}{7}} - \frac{2}{7}ghk^{-\frac{5}{7}}$, where p_0 is the pressure at sea level. **41.** Hint: By symmetry we may confine ourselves to two dimensions. Use polar coordinates with the beam in the direction of the polar axis. Then $\rho = c/(1 + \cos \theta)$. **42.** $i = (E/R)e^{-Rt/L}$. **43.** $\rho = ce^{a\theta}$, where $a = \operatorname{ctn} \alpha$. **44.** $d = l \log \dfrac{l + \sqrt{l^2 - y^2}}{y}$ where d is the distance walked along the shore and y is the distance of the boat from shore. **45.** 1.73 lb. **46.** 31,300.
47. 1,000 cu ft/min; 11 min.
48. $v^2 = 2g(x - 25)(x - 20)/55$ where x is the distance of the weight below the pulley. **49.** $v^2 = 16(x^2 - 25)/25$.

Chap. IV, Sec. 2

1. $y = 0.324$, $y' = 0.904$ using $\Delta x = 0.1$. **2.** $y = 0.850$ using $\Delta x = \pi/16$.
3. (1.25, 0.5), (1.5, 0.562), (1.75, 0.687), (2.0, 0.875), using $\Delta x = 0.25$; (1.1, 0.5), (1.2, 0.51), (1.3, 0.53), (1.4, 0.56), (1.5, 0.60) using $\Delta x = 0.1$. **4.** (0.1, 1.1), (0.2, 1.175), (0.3, 1.225), (0.4, 1.249), (0.5, 1.246), (0.6, 1.216), (0.7, 1.157), (0.8, 1.067), (0.9, 0.945), (1.0, 0.787). **5.** (1.2, 1.2).

Chap. IV, Sec. 3

1. $y = \frac{1}{4}e^{-x} \sin 2x$. **2.** (a) $y = 1 - x + x^2$. (b) $y = 1 + 2x + x^2$.

Chap. IV, Sec. 4

1. (a) $y = c_1e^{2x} + c_2e^{-x}$. (b) $y = c_1e^{-2x} + c_2e^{-x}$.
(c) $y = c_1e^{(-2+\sqrt{2})x} + c_2e^{(-2-\sqrt{2})x}$. (d) $y = c_1e^{-4x} + c_2e^{2x}$.
(e) $y = c_1e^{-3x} + c_2xe^{-3x}$. (f) $y = c_1e^{3x} + c_2e^{-3x}$. (g) $y = c_1e^{\sqrt{5}x} + c_2xe^{\sqrt{5}x}$.
(h) $y = c_1e^{(k+\sqrt{k^2+12})x/2} + c_2e^{(k-\sqrt{k^2+12})x/2}$. (i) $y = c_1e^{3x/2} + c_2e^{-x}$.
(j) $y = c_1e^{mx/k} + c_2e^{-mx/l}$. (k) $y = c_1e^{mx} + c_2xe^{mx}$. (l) $y = c_1e^{-5x} + c_2e^{3x}$.
(m) $y = c_1e^{2x/3} + c_2xe^{2x/3}$.

(n) $$q = c_1 \exp \left(\frac{-R + \sqrt{R^2 - 4L/C}}{2L}\, t \right) + c_2 \exp \left(\frac{-R - \sqrt{R^2 - 4L/C}}{2L}\, t \right)$$

(o) $q = c_1e^{-Rt/2L} + c_2te^{-Rt/2L}$. **2.** (a) $y = 2e^{-x} + e^{2x}$. (b) $y = \frac{5}{8}e^{2x} + \frac{3}{8}e^{-2x}$.
(c) $y = 3e^{2x} - 2xe^{2x}$. (d) $y = 9e^{x/2} - 4e^{-x}$. (e) $y = e^{3x-1}$.
(f) $y \doteq 2.02e^{3x} - 51e^{-x}$. (g) $y \doteq 2.95e^{-x} + 3.1xe^{-x}$.
(h) $y = e^{(-2+\sqrt{2})x} - 2e^{(-2-\sqrt{2})x}$. (i) $y \doteq 0.9e^{x} + 1.3e^{-x}$.
(j) $y = \frac{4}{5}e^{3x} - 2e^{-3x}$. (k) $q = 10^{-9}(11.70e^{(-0.383\times 10^7)t} - 1.70e^{(-2.618\times 10^7)t})$.
(l) $y = 0.5e^{2x} + 4.5e^{-x}$.

Chap. IV, Sec. 5

1. (a) $y = e^{-3x/2}[c_1 \sin(\sqrt{3}\,x/2) + c_2 \cos(\sqrt{3}\,x/2)]$.
(b) $y = e^{x}(c_1 \sin x + c_2 \cos x)$. (c) $y = c_1e^{2x} + c_2e^{x}$.
(d) $y = e^{x/2}(c_1 \sin \sqrt{11}\,x/2 + c_2 \cos \sqrt{11}\,x/2)$. (e) $y = c_1e^{x} + c_2e^{3x/2}$.
(f) $s = c_1e^{4t} + c_2e^{-t}$. (g) $y = e^{kx}(c_1 \sin x + c_2 \cos x)$.
(h) $y = e^{3x/4}(c_1 \sin(\sqrt{7}\,x/4) + c_2 \cos(\sqrt{7}\,x/4))$.
(i) $y = e^{2x}(c_1 \sin kx + c_2 \cos kx)$. (j) $y = e^{3x}(c_1 \sin x/2 + c_2 \cos x/2)$.
(k) $y = e^{-3x}(c_1 \sin 2x + c_2 \cos 2x)$. (l) $y = e^{2x}(c_1 \sin x/k + c_2 \cos x/k)$.
(m) $y = e^{-x/k}(c_1 \sin x + c_2 \cos x)$.

(n) $q = e^{-Rt/2L}\left(c_1 \sin \frac{\sqrt{4L/C - R^2}}{2L}t + c_2 \cos \frac{\sqrt{4L/C - R^2}}{2L}t\right)$.

(o) $y = e^{-x/a}\left(c_1 \sin \frac{\sqrt{ac - 1}}{a}x + c_2 \cos \frac{\sqrt{ac - 1}}{a}x\right)$.

(p) $y = c_1e^{-x/a} + c_2xe^{-x/a}$.

(q) $x = e^{-\frac{cgt}{2w}}\left(c_1 \sin \frac{\sqrt{4kwg - g^2c^2}}{2w}t + c_2 \cos \frac{\sqrt{4kwg - g^2c^2}}{2w}t\right)$.

(r) $x = e^{-\frac{cgt}{2w}}(c_1 + c_2t)$. (s) $w = e^{-kz/6}\left(c_1 \sin \frac{\sqrt{60 - k^2}}{6}z + c_2 \cos \frac{\sqrt{60 - k^2}}{6}z\right)$.

2. (a) $y = e^{x/2}[\cos(\sqrt{3}\,x/2) + 2\sqrt{3} \sin(\sqrt{3}\,x/2)]$.
(b) $y = e^{-2x}(4 \cos 3x + 3 \sin 3x)$. (c) $y = e^{-x}(\sqrt{3} \sin \sqrt{3}\,x + 3 \cos \sqrt{3}\,x)$.
(d) $y = e^{-2t}(3 + 2.017t)$. (e) $q = 1.51 \times 10^{-9}e^{(-0.75\times 10^7)t} \sin (0.661 \times 10^7)t$.
(f) $x = e^{-0.08t}(0.04 \sin 4t + 2 \cos 4t)$. (g) $y = 2e^{-x/2} \cos(x + \pi/3)$.
(h) $y = e^{-x} \cos(3x - \pi/2)$. (i) $x = 2e^{t} \cos(t + \pi/6)$. (j) $y = e^{-2x} + e^{-x-1}$.
(k) $y = e^{-x-2} \cos x$. **3.** $y = c_1e^{2x} + c_2 \sin x + c_3 \cos x$.

Chap. IV, Sec. 6

3. $y = c_1e^{x} + c_2 - \frac{x^2}{2} - x - \sin x$. **4.** $y = c_1e^{-x} + c_2$.

Chap. IV, Sec. 7

1. (a) $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{8} \sin 2x$. (b) $y = c_1e^{3x} + c_2e^{x} + \frac{1}{4}e^{-x} - \frac{2}{3}$.

(c) $y = c_1e^{-4x} + c_2e^{2x} - 2x - \frac{1}{2}$. (d) $y = e^{x}\left(c_1 \sin x + c_2 \cos x - \frac{x}{2} \cos x\right)$.

(e) $y = c_1 \sin 2x + c_2 \cos 2x + \frac{x}{4} - \frac{x}{4} \cos 2x$. (f) $y = e^{x}\left(c_1 + c_2x + \frac{x^3}{2}\right)$.

(g) $y = e^{-x}(c_1 + c_2 x) + a - 3 \sin 2x - 4 \cos 2x$.

(h) $y = \dfrac{a_0 x^{n+2}}{(n+2)(n+1)} + \dfrac{a_1 x^{n+1}}{(n+1)n} + \cdots + \dfrac{a_n x^2}{2.1} + c_1 x + c_2$.

(i) $y = c_1 \sin 4x + c_2 \cos 4x + \dfrac{A}{32} + \dfrac{Ax}{16} \sin 4x$.

(j) $y = c_1 e^x + c_2 e^{2x} + x^2 + 4x + 5$. (k) $y = c_1 + c_2 e^{-2x} + x^2 - x$.

(l) $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + \dfrac{A}{65} \sin 2x - \dfrac{8A}{65} \cos 2x$.

(m) $y = e^{2x}(c_1 + c_2 x + x^2 + x^3/6)$. (n) $y = c_1 + c_2 e^{-x} - \frac{1}{5} \sin 2x - \frac{1}{10} \cos 2x$.
(o) $y = c_1 + c_2 e^{2x} + e^{2x}(x - x^2 + 2x^3/3)$. **2.** (a) $y = -\frac{5}{2} + 7e^{2x} - 3e^x + 2e^{-x}$.
(b) $y = \frac{1}{2}e^{2x} - \frac{1}{2} - x$. (c) $y = \frac{1}{3}e^{2x} + \frac{1}{6}e^{-x} - \frac{3}{2} \sin x + \frac{1}{2} \cos x$.

(d) $y = \dfrac{A}{\lambda(\lambda^2 - \omega^2)} (\lambda \sin \omega x - \omega \sin \lambda x)$.

(e) $y = -\frac{3}{52}e^{3x} - \frac{1}{20}e^{-x} + \frac{4}{65} \sin 2x + \frac{7}{65} \cos 2x - \frac{1}{3}$.
(f) $y = \frac{1}{3}e^{3x} - e^{-x} - x + \frac{2}{3}$.

Chap. IV, Sec. 8

1. (a) $y = c_1 e^{2x} + c_2 e^{-x} + xe^{2x}/3$. (b) $y = e^{-x}(c_1 + c_2 x + x \log x)$.

(c) $y = c_1 \sin x + c_2 \cos x - \dfrac{x}{2} \cos x$.

(d) $y = c_1 \sin 2x + c_2 \cos 2x - \frac{1}{4} \cos 2x \log (\sec 2x + \tan 2x)$.

(e) $y = c_1 e^x + c_2 e^{-x} + \dfrac{1}{2} + \dfrac{e^x}{2} \log (1 - e^{-x}) + \dfrac{e^{-x}}{2} \log (e^x - 1)$.

(f) $y = c_1 e^{2x} + c_2 e^{-x} - \cos e^{-x} + 2e^x \sin e^{-x} + 2e^{2x} \cos e^{-x}$.

(g) $y = c_1 \sin 2x + c_2 \cos 2x + \dfrac{1}{4} \sin 2x \log \sin 2x - \dfrac{x}{2} \cos 2x$.

(h) $y = c_1 \sin 2x + c_2 \cos 2x + \dfrac{1}{8} - \dfrac{x}{8} \sin 2x$.

(i) $y = c_1 + c_2 e^{-2x} - e^{-x} + 5x/2$.
(j) $y = c_1 \sin 2x + c_2 \cos 2x + \frac{1}{4} \sin 2x \log (\sec 2x + \tan 2x) - \frac{1}{2}$.
2. (a) $y_p = \sin x \log (\csc x - \operatorname{ctn} x) - \cos x \log (\sec x + \tan x)$.
(b) $y_p = \frac{1}{2} \sin x \log (\csc x - \operatorname{ctn} x) + \frac{1}{2} \cos x \log (\sec x + \tan x)$.
(c) $y_p = -\frac{1}{2}xe^{-x} \cos x$.

(d) $y_p = -e^{-x}\left(\dfrac{1}{128} + \dfrac{x}{32} + \dfrac{x^2}{16} + \dfrac{x^3}{12}\right)$. **3.** $y = \dfrac{x^3}{5} + c_1 x^2 + c_2 x^{-2}$.

4. $y = c_1 x^2 + c_2 x^{-1} + \dfrac{x^2}{3} \log x$. **5.** $y = c_1 x + c_2 x^{-1} + e^{-x}(1 + x^{-1})$.

6. $y_p = \dfrac{1}{x}$.

Chap. IV, Sec. 11

1. 1.59 cycles/sec. **2.** 1.54 cycles/sec; log decrement is 1.62.
3. 35.5 cycles/sec; 44.7 ohms. **4.** $2\pi \sqrt{l/g}$. **5.** 637 lb.

6. $x = \frac{1}{4}\cos 8t + \frac{1}{2}$, (feet). 7. $x = 0.5 - 0.155\sin 8t + 0.394\sin \pi t$.
8. $a = 12$, $b = 142{,}000$.

Chap. IV, Sec. 12

2. $i = e^{-50t}(-5.53\sin 218t + 1.212\cos 218t)$
$+ 1.168\sin 400t - 3.212\cos 400t + 4\sin 200t + 2\cos 200t$.
3. $x = -0.305\sin 4.90t + 1.333\cos 4.90t + 0.696\sin t + 0.4\sin 2t$.

Chap. IV, Sec. 13

1. 0.225 cycle/sec. 3. 12.5×10^{-6} farad. 4. $\dfrac{1}{2\pi}\sqrt{1/LC - R^2/2L^2}$.

Chap. IV, Sec. 14

1. $x_p = -\frac{1}{2}\cos t + \frac{1}{2}\sin t - \frac{1}{8}\cos 2t - \frac{1}{8}\sin 2t - \frac{1}{58}\cos 3t - \frac{7}{174}\sin 3t$.
2. $x_p = 2\cos\dfrac{t}{2} + 2\sin\dfrac{t}{2} - \cos t - \sin t$. 3. $x_p = -\frac{2}{5}\cos 2t - \frac{1}{5}\sin 2t$.
4. $x = c_1\sin\sqrt{\frac{5}{3}}\,t + c_2\cos\sqrt{\frac{5}{3}}\,t - \sin t - \frac{4}{7}\sin 2t - \frac{1}{35}\sin 5t$.
5. Hint: Show that the forcing function may be written as
$\dfrac{A}{8}(1 + \cos t + \cos 2t + \cos 3t + \cos 4t + \cos 5t + \cos 7t + \cos 8t)$.

Chap. IV, Sec. 16

1. (a) $\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{1-(-1)^n}{n}\sin nt$. (b) $\dfrac{1}{2} - \dfrac{4}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos(2n-1)\pi t$.

(c) $\dfrac{1}{\pi} + \dfrac{1}{2}\sin t - \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{4n^2-1}\cos 2nt$. (d) $\dfrac{1}{3} + \dfrac{4}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos n\pi t$.

Chap. IV, Sec. 17

1. (ia) $x_p = -\dfrac{8}{5\pi}\cos t + \dfrac{4}{5\pi}\sin t - \dfrac{8}{85}\cos 3t - \dfrac{28}{255\pi}\sin 3t + \cdots$.

(ib) $x_p = \dfrac{1}{4} - \dfrac{4(2-\pi^2)}{\pi^2(4+\pi^4)}\cos\pi t - \dfrac{8}{\pi(4+\pi^4)}\sin\pi t - \dfrac{4(2-9\pi^2)}{9\pi^2(4+81\pi^4)}\cos 3\pi t$
$- \dfrac{8}{3\pi(4+81\pi^4)}\sin 3\pi t + \cdots$.

(ic) $x_p = \dfrac{1}{2\pi} - \dfrac{1}{5}\cos t + \dfrac{1}{5}\sin t + \dfrac{1}{15\pi}\cos 2t - \dfrac{2}{15\pi}\sin 2t + \cdots$.

(id) $x_p = \dfrac{1}{6} - \dfrac{4(2-\pi^2)}{\pi^2(4+\pi^4)}\cos\pi t - \dfrac{8}{\pi(4+\pi^4)}\sin\pi t + \dfrac{1-2\pi^2}{2\pi^2(1+4\pi^4)}\cos 2\pi t$
$+ \dfrac{1}{\pi(1+4\pi^4)}\sin 2\pi t + \cdots$.

(iia) $x_p = -\dfrac{2}{\pi}\cos t + \dfrac{2}{\pi}\sin t - \dfrac{2}{29\pi}\cos 3t - \dfrac{14}{87\pi}\sin 3t + \cdots$.

(ii*b*) $x_p = \frac{1}{4} - \frac{4(2-\pi^2)}{\pi^2(4-3\pi^2+\pi^4)}\cos \pi t - \frac{4}{\pi(4-3\pi^2+\pi^4)}\sin \pi t$

$- \frac{4(2-9\pi^2)}{9\pi^2(4-27\pi^2+81\pi^4)}\cos 3\pi t - \frac{4}{3\pi(4-27\pi^2+81\pi^4)}\sin 3\pi t + \cdots.$

(ii*c*) $x_p = \frac{1}{2\pi} - \frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{6\pi}\cos 2t - \frac{1}{6\pi}\sin 2t + \cdots.$

(ii*d*) $x_p = \frac{1}{6} - \frac{4(2-\pi^2)}{\pi^2(4-3\pi^2+\pi^4)}\cos \pi t - \frac{4}{\pi(4-3\pi^2+\pi^4)}\sin \pi t$

$+ \frac{(1-2\pi^2)}{2\pi^2(1-3\pi^2+\pi^4)}\cos 2\pi t + \frac{1}{2\pi(1-3\pi^2+\pi^4)}\sin 2\pi t + \cdots.$

2. $q = \frac{a_0 C}{2} + \sum_{k=1}^{\infty} \frac{1}{k\omega}\left[\frac{-a_k(Lk\omega - 1/Ck\omega) - b_k R}{(Lk\omega - 1/Ck\omega)^2 + R^2}\cos k\omega t\right.$

$\left. + \frac{a_k R - b_k(Lk\omega - 1/Ck\omega)}{(Lk\omega - 1/Ck\omega)^2 + R^2}\sin k\omega t\right].$

3. $\frac{\pi}{\omega}\sum_{k=1}^{\infty}\frac{a_k^2 + b_k^2}{(Lk\omega - 1/Ck\omega)^2 + R^2}.$

Chap. IV, Sec. 19

1. $\lambda_n = n\pi/2L$, $n = 1, 3, \ldots$; $u_n = \sin(n\pi x/2L)$.
3. $\lambda_n = 2n\pi/L$, $n = 0, 1, 2, \ldots$; $u_n = A_n \cos(2n\pi x/L) + B_n \sin(2n\pi x/L)$.

Chap. IV, Sec. 20

1. $y = \frac{x}{4}e^{2x} - \frac{1}{4}e^{2x} + c_1 x + c_2$. **2.** $y^2 + 1 = 2e^x$. **3.** $y = \log \sec x$.

4. $y = \frac{1}{2}c_1 x^2 + (c_1^2 + 1)x + c_2$. **5.** $y = -\frac{x^2}{2} \pm \log(x + \sqrt{x^2 + c_1}) + c_2$.

6. $y = -2(\arctan x)^2$. **7.** $y = -2 \log \sin(x/2)$. **8.** $y = \log \cosh\left(x + \frac{\pi}{2}\right)$.

9. $\log y = Ae^x + Bxe^x$. **10.** $y = Ae^{\sin x} + Be^{-\sin x} + 1$. **11.** $z = Ax/(1 + Bx)$.

Chap. IV, Sec. 21

1. $y = \frac{k}{2X}x^2 + c_1 x + c_2$. **2.** $y = c_1 e^{x\sqrt{k/X}} + c_2 e^{-x\sqrt{k/X}}$. Load the cable with a rope fringe such that the lower ends of the ropes lie in a horizontal line.

3. $\sqrt{\frac{l}{2g}}\int_{\theta_0}^{\theta}\frac{dx}{\sqrt{\cos x - \cos\theta_0}} = t$. **4.** Choose the x axis in the top of the sand and the positive y axis downward. If the top of the arch is at $x = 0$, $y = A \cosh \sqrt{w}\, x$ where w is the weight of the sand per unit volume.

Chap. IV, Miscellaneous Problems

1. (*a*) $y = c_1 e^x + c_2 e^{-x} - \frac{x^2}{2} - \frac{x}{2} - \frac{3}{4} + xe^x$.

(*b*) $x = c_1 \cos 2t + c_2 \sin 2t + \dfrac{A}{8}(1 - t \sin 2t)$.

(*c*) $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{3}\cos x - \dfrac{x}{4}\cos 2x$.

(*d*) $y = c_1 e^{-x} + c_2 x e^{-x} + A - \dfrac{4B}{25}\cos 2x - \dfrac{3B}{25}\sin 2x$.

(*e*) $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + \dfrac{A}{8}(\sin x - \cos x)$.

(*f*) $y = c_1 e^{x} + c_2 e^{-x} + 2x^{-1}$. (*g*) $y = e^{x}(c_1 \cos x + c_2 \sin x) + \frac{1}{4}e^{x^2}$.
(*h*) $y = e^{-t/\sqrt{2}}(c_1 \cos \sqrt{\frac{3}{2}}\, t + c_2 \sin \sqrt{\frac{3}{2}}\, t) + \sqrt{2}\log x - x^{-1}$.
(*i*) $y = c_1 + c_2 e^{x} + c_3 e^{-x} - x^2$. (*j*) $y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + \frac{1}{5}\sin x + \frac{1}{10}\cos x$.
2. (*a*) $y = a e^{2x}(\cos 3x - \sin 3x)$. (*b*) $y = e^{2x}$. (*c*) $y = e^{x}\sin \sqrt{3}\, x$.
(*d*) $y = e^{x}(2 \cos \sqrt{2}\, x - \sin \sqrt{2}\, x) + x^2 - 2$.
(*e*) $y = e^{-x/2}(2 \cos \sqrt{\frac{3}{2}}\, x - \sin \sqrt{\frac{3}{2}}\, x) + x \sin x$. (*f*) $y = e^{(1+\sqrt{2})x} - 2e^{(1-\sqrt{2})x}$.
(*g*) $y = e^{-x}(\sin x - 2 \cos x) + 2 \sin x + \cos x$. **3.** 0.77 sec. **4.** 0.686.
5. $k = 40$. **6.** 0.0722 sec. **7.** $x = 0.5$, $y = 0.0100$; $x = 1.0$, $y = 0.1142$.
8. Exact solution is $y = x$.
9. $q = e^{-25t}(0.03 \cos 222t - 0.0191 \sin 222t) - 0.025 \cos 200t + 0.025 \sin 200t$.
10. $a_0 = 1$, $a_1 = 4/\pi^2$, $b_1 = a_2 = b_2 = 0$. **11.** Choose the origin at the point where the rabbit was when sighted with the x axis east and the y axis north.

Then $y = 240 + 60\left[\left(\dfrac{x}{200}\right)^{\frac{3}{2}} - 5\left(\dfrac{x}{200}\right)^{\frac{1}{2}}\right]$.

Chap. V, Sec. 1

1. (*d*) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + \dfrac{2}{1 - x^2}\, y$. (*e*) $2 \operatorname{ctn} x + \dfrac{\log \sin x}{1 + x} - 1$.

(*f*) $\log \sin x L_1[y] + 2 \cos x \sin x \dfrac{dy}{dx} + (2 \operatorname{ctn} x - 1)y$.

Chap. V, Sec. 4

8. $$\cos^n x = 2^{1-n}\left[\frac{1}{2}\,\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} + \frac{n!}{\left(\frac{n-2}{2}\right)!\left(\frac{n+2}{2}\right)!}\cos 2x + \frac{n!}{\left(\frac{n-4}{2}\right)!\left(\frac{n+4}{2}\right)!}\cos 4x + \cdots + \frac{n!}{0!n!}\cos nx\right] \quad \text{(if } n \text{ even)}$$

$$= 2^{1-n}\left[\frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}\cos x + \frac{n!}{\left(\frac{n-3}{2}\right)!\left(\frac{n+3}{2}\right)!}\cos 3x + \frac{n!}{\left(\frac{n-5}{2}\right)!\left(\frac{n+5}{2}\right)!}\cos 5x + \cdots + \frac{n!}{0!n!}\cos nx\right] \quad \text{(if } n \text{ odd)}$$

Chap. V, Sec. 5

1. $y = c_1x + c_2x^2 + c_3x^{-1}$. **2.** $y = (c_0 + c_1t + c_2t^2 + c_3t^3)e^{-\omega t}$.

Chap. V, Sec. 6

1. $y = \frac{A}{a^3} e^{ax} + c_0 + c_1x + c_2x^2$; if $a \neq 0$.

$= \frac{A}{6} x^3 + c_0 + c_1x + c_2x^2$; if $a = 0$.

2. $x = \frac{A}{\omega^3} (2 \sin \omega t - \omega t \cos \omega t) + c_0 + c_1t$; if $\omega \neq 0$.

$= \frac{A}{6} t^3 + c_0 + c_1t$; if $\omega = 0$.

3. $y = \frac{Ax^4}{288} (-25 + 12 \log x) + c_0 + c_1x + c_2x^2 + c_3x^3$.

4. $y = -\frac{\sqrt{a^2 - x^2}}{a^2} + c_0 + c_1x$.

5. $y = \frac{A}{12a^4} (1 + a^2t^2)e^{-a^2t^2} + \frac{A}{12a^2} t(3 + 2a^2t^2) \int^t e^{-a^2t_1^2}dt_1 + c_0 + c_1t + c_2t^2 + c_3t^3$; if $a \neq 0$.

$= \frac{A}{24} t^4 + c_0 + c_1t + c_2t^2 + c_3t^3$; if $a = 0$.

6. $y = \frac{1}{26} \cos 6x + c_1e^{4x} + c_2e^{9x}$. **7.** $y = -x^{-2} + c_1x^3 + c_2x^{-3}$.

8. $y = \frac{1}{2}x^3 + c_1x + c_2(1 - x^2)$. **9.** $y = -\frac{k}{9x} + c_0 + (c_1 + c_2 \log x)x^2$.

10. $y = (c_0 + c_1x) \sin 2x + (c_2 + c_3x) \cos 2x + \frac{k}{(a^2 + 4)^2} e^{ax}$.

Chap. V, Sec. 7

1. (*a*) $x = \frac{1}{2} + \frac{1}{2(1 + 16\beta^4)} \cos 2\beta t$. (*b*) $x = \frac{1}{8} - \frac{1}{8(1 + 256\beta^4)} \cos 4\beta t$

(*c*) $x = \frac{a_0}{2c} + \sum_{k=1}^{n} \left(\frac{a_k}{c + k^4\beta^4} \cos k\beta t + \frac{b_k}{c + k^4\beta^4} \sin k\beta t \right)$.

(*d*) $x = \frac{1}{24} \sin 2t + \frac{1}{264} \sin 4t - \frac{1}{1{,}304} \sin 6t$.

2. $y = A \left[1 + \frac{x}{1^2} + \frac{x^2}{3^2} + \frac{x^3}{5^2} + \cdots + \frac{x^{n-1}}{(2n - 3)^2} \right]$.

Chap. V, Sec. 9

1. $y = c_0 + c_1x + c_2e^{3x}$. **2.** $y = c_0 + c_1e^{-x} + c_2e^{-4x}$.

3. $y = c_0 + c_1x + (c_2 + c_3x)e^{-x}$. **4.** $y = c_1e^x + c_2e^{-x} + c_3 \sin x + c_4 \cos x$.

5. $y = [c_1 \sin(\frac{1}{2}\sqrt{2}\,x) + c_2 \cos(\frac{1}{2}\sqrt{2}\,x)]e^{\frac{1}{2}\sqrt{2}x} + [c_3 \sin(\frac{1}{2}\sqrt{2}\,x) + c_4 \cos(\frac{1}{2}\sqrt{2}\,x)]e^{-\frac{1}{2}\sqrt{2}x}$. 6. $y = (c_0 + c_1x)e^{-x} + (c_3 + c_4x)e^{2x}$.
7. $y = (c_0 + c_1x + c_2x^2)e^x + c_3e^{-2x}$. 8. $y = (c_0 + c_1x)\sin x + (c_2 + c_3x)\cos x$.
9. $y = c_1e^{-x} + c_2 \sin x + c_3 \cos x + [c_4 \sin(\frac{1}{2}\sqrt{3}\,x) + c_5 \cos(\frac{1}{2}\sqrt{3}\,x)]e^{\frac{x}{2}}$.
10. $y = (c_0 + c_1x + c_2x^2)\sin x + (c_3 + c_4x + c_5x^2)\cos x$.
11. $y = c_0 + c_1x + c_2x^2 + \cdots + c_{n-2}x^{n-2} + c_{n-1}e^{-ax}$.
12. $y = (c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1})\sin ax + (c_n + c_{n+1}x + \cdots + c_{2n-1}x^{n-1})\cos ax$.

Chap. V, Sec. 10

1. $y = \left(\frac{x^6}{120} + c_0 + c_1x + c_2x^2\right)e^{-x}$. 2. $y = \left(\frac{1}{2}\log x + c_0 + c_1x + c_2x^2\right)e^x$.

3. $y = \left(\frac{1}{60}x^3 - \frac{9}{400}x^2 + \frac{61}{4{,}000}x + c_1\right)e^{2x} + c_2e^{-2x} + c_3e^{-3x}$.

4. $y = \left(\frac{3\sin x - 2\cos x}{13} + c_1\right)e^{-x} + \left[c_2 \sin\left(\frac{1}{2}\sqrt{3}\,x\right) + c_3 \cos\left(\frac{1}{2}\sqrt{3}\,x\right)\right]e^{-\frac{1}{2}x}$.

5. $y = (-\frac{1}{5}\cos x + c_1)e^x + c_2e^{-x} + c_3 \sin x + c_4 \cos x$.

6. $y = \left(\frac{-1}{24}x^3 + c_0 + c_1x\right)\sin x + \left(-\frac{1}{8}x^2 + c_2 + c_3x\right)\cos x$.

7. $y = \left[\frac{A}{(a-b)(a-c)}x + c_1\right]e^{ax} + c_2e^{bx} + c_3e^{cx}$.

8. $y = \left[\frac{A}{2(a-b)}x^2 + c_0 + c_1x\right]e^{ax} + c_2e^{bx}$.

9. $y = \left[\frac{A}{2(a-b)^2}x^2 + c_0 + c_1x\right]e^{ax} + (c_2 + c_3x)e^{bx}$.

10. $$y = -A\left[\frac{1}{(r_1 - r_2)(r_1 - r_3)\cdots(r_1 - r_n)(r_1 - s)} + \frac{1}{(r_2 - r_1)\cdots(r_2 - r_n)(r_2 - s)} + \cdots + \frac{1}{(r_n - r_1)\cdots(r_n - r_{n-1})(r_n - s)}\right]e^{sx} + c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx}.$$

11. $y = c_1 \sin ax + \left[\frac{A}{2a(a^2 - b^2)}x + c_2\right]\cos ax + c_3 \sin bx + c_4 \cos bx$.

12. $y = \left(-\frac{A}{8a^2}x^2 + c_0 + c_1x\right)\sin ax + (c_2 + c_3x)\cos ax$.

13. $y = \int^x [-1 + \cosh(x - x_1)]f(x_1)\,dx_1$.

14. $y = \int^x [1 - \cos(x - x_1)]f(x_1)\,dx_1$.

15. $$y = \frac{1}{(a-b)^2}\int^x \left[\left(x - x_1 - \frac{2}{a-b}\right)e^{a(x-x_1)} + \left(x - x_1 - \frac{2}{b-a}\right)e^{b(x-x_1)}\right] f(x_1)\,dx_1.$$

16. $$y = \frac{1}{b^2-a^2}\int^x \left[\frac{1}{a}\sin a(x-x_1) - \frac{1}{b}\sin b(x-x_1)\right] f(x_1)\,dx_1.$$

17. $$y = -\int^x \left[\frac{(x-x_1)^{n-2}}{a(n-2)!} + \frac{(x-x_1)^{n-3}}{a^2(n-3)!} + \cdots + \frac{1}{a^{n-1}} - e^{a(x-x_1)}\right] f(x_1)\,dx_1.$$

Chap. V, Sec. 12

1. $y = -x^4 - 12x^2 - 24.$ **2.** $y = x^2 + 6x + 14.$ **3.** $y = x^6 - 15x^3.$

4. $y = \frac{A}{9}e^x.$

5. $y = \frac{A}{(r+s)^n}e^{sx};$ if $r + s \neq 0.$

$= \frac{A}{n!}x^n e^{sx};$ if $r + s = 0.$

6. $y = \frac{x^6}{120}e^{-x}.$ **7.** $y = \frac{1}{12}(x^4 + 4x^3 + 8x^2 + 8x)e^{-x}.$ **8.** $y = -\frac{3}{4}\sin 3x.$

9. $y = -\frac{1}{5}e^x \cos x.$ **10.** $y = -\frac{1}{24}(x^3 \sin x + 3x^2 \cos x).$

Chap. V, Sec. 13

1. $y = \frac{x^6}{120}e^{-x}.$ **2.** $y = \left(\frac{1}{2}\log x + \frac{3}{4} - x + \frac{1}{4}x^2\right)e^x.$

3. $y = \left(\frac{1}{40}x^2 - \frac{9}{400}x + \frac{61}{8{,}000}\right)e^{2x} - \frac{1}{64}e^{-2x} + \frac{1}{125}e^{-3x}.$

4. $y = \frac{k!}{(n+k)!}x^{n+k}e^{-rx}.$

5. $$y = \frac{1}{(n-1)!}\int_0^x (x-x_1)^{n-1}e^{-r(x-x_1)}f(x_1)\,dx_1.$$

6. $$y = \frac{1}{(r-s)^2}\int_0^x \left[\left(x - x_1 + \frac{2}{r-s}\right)e^{-r(x-x_1)} + \left(x - x_1 + \frac{2}{s-r}\right)e^{-s(x-x_1)}\right] f(x_1)\,dx_1.$$

7. $$y = \frac{-1}{a^2-b^2}\int_0^x \left[\frac{1}{a}\sin a(x-x_1) - \frac{1}{b}\sin b(x-x_1)\right] f(x_1)\,dx_1.$$

8. $$y = e^x + \int_0^x [-1 + \cosh(x-x_1)]f(x_1)\,dx_1.$$

9. $$y = \sin x + \int_0^x [1 - \cos(x-x_1)]f(x_1)\,dx_1.$$

10. $$y = 1 - \int_0^x \left[\frac{(x-x_1)^{n-2}}{a(n-2)!} + \frac{(x-x_1)^{n-3}}{a^2(n-3)!} + \cdots + \frac{1}{a^{n-1}} - e^{a(x-x_1)}\right] f(x_1)\,dx_1.$$

Chap. VI, Sec. 1

2. $6 + x - \frac{9}{4}e^{-x} + (-\frac{15}{4} + \frac{5}{2}x - x^2)e^x$.
3. $-1 - x + \frac{1}{2}e^{-x} + \frac{1}{5}e^{-2x} + \frac{19}{10}\sin x + \frac{3}{10}\cos x$.
4. $-\frac{1}{4}\cos x + \frac{1}{4}x\sinh x + \frac{3}{4}\cosh x$. **5.** $-\frac{1}{3}\sin x - \frac{4}{3}\sin 2x$.

Chap. VI, Sec. 2

1. $x = e^t + ce^{-t}$, $y = -ce^{-t}$. **2.** $x = c_1e^{2t} + c_2e^{-t}$, $y = -t^2 + c_1e^{2t} - c_2e^{-t}$.

3. $x = t^{-1} - e^{-t}$, $y = t^{-2}$. **4.** $x = \left(c + \frac{t}{2}\right)e^{-t}$, $y = t^{-1} - \frac{1}{2}e^{-t}$.

5. $x = \sin t - c_1 \sin 2t + c_2 \cos 2t$,

$y = \frac{1}{2}\cos t - \frac{2c_1}{5}\cos 2t + \frac{2c_2}{5}\sin 2t + c_3e^t + c_4e^{-t}$.

6. $x = -\frac{1}{2}f(t) + c_1e^t$, $y = \frac{1}{2}f(t)$. **7.** $x = c_1e^t$, $y = -\frac{3c_1}{4}e^t + c_2e^{-t}$,

$z = -\frac{3c_1}{2}e^t + \frac{2c_2}{5}e^{-t} + c_3e^{\frac{3}{2}t}$. **8.** $x = 1 - 2e^{2t}$, $y = -1 + 3e^{2t}$.

9. $x = t^2 + t - 1$, $y = -t$, $z = -t^2 - 3t + 1$.

Chap. VI, Sec. 3

1. (a) $x = \frac{2}{5}e^{4t} + \frac{3}{5}e^{-t}$, $y = -\frac{3}{5}e^{4t} + \frac{3}{5}e^{-t}$. (b) $x = -\frac{2}{5}e^{4t} + \frac{2}{5}e^{-t}$,
$y = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}$. **2.** $x = -\frac{1}{2} + 3e^t - \frac{3}{2}e^{4t}$, $y = -2e^t + 2e^{4t}$.
3. $x = A\sinh t + (aA - bB)(1 - \cosh t)$, $y = B\sinh t + (bA - aB)(1 - \cosh t)$.
4. $x = 3e^t - 2e^{3t}$, $y = -6e^t + 6e^{3t}$, $z = 6e^t - 6e^{3t}$. **5.** $x = e^t$, $y = e^t$.

6. $x = -\frac{1}{10}\cos 3t + \frac{1}{10}\cosh t$, $y = \frac{1}{10}\cos 3t - \frac{1}{10}\cosh t$. **7.** $x = \frac{A}{2a}t\sin at$,

$y = \frac{A}{2a^2}(\sin at - at\cos at)$. **8.** $x = y = t\sin t$, $z = z_1t - 2t\sin t + 2(1 - \cos t) + z_0$.

9. $x = (c_0 + c_1t)e^{2t} + c_2e^{-3t}$, $y = \frac{1}{2}e^t - \frac{1}{4}(2c_0 + c_1 + 2c_1t)e^{2t} + \frac{3}{4}c_2e^{-3t}$.
10. $x = c_1e^{-t} + c_2e^{3t}$, $y = (2c_1 - 1)e^{-t} - 2c_2e^{3t}$. **11.** $x = \cos 2t + c_1e^{-t}$,
$y = 4\cos 2t$. **12.** $x = c_1 \sin 2t + c_2 \cos 2t$, $y = 2c_2 \sin 2t - 2c_1 \cos 2t$,
$z = c_0 - c_2 \sin 2t + c_1 \cos 2t$. **13.** $x = A + B\sin(t + \delta)$,

$y = \frac{b}{c}A - \frac{B}{1 - c^2}[bc\sin(t + \delta) + a\cos(t + \delta)]$,

$z = \frac{a}{c}A - \frac{B}{1 - c^2}[ac\sin(t + \delta) - b\cos(t + \delta)]$.

Chap. VI, Sec. 4

1. $x = (1 - D^2)h(t)$, $y = D^3h(t)$. **2.** $x = c_1 \sin t + c_2 \cos t + c_3e^t$,
$y = (c_1 + c_2)\sin t - (c_1 - c_2)\cos t + 2c_3e^t - 2e^{\frac{1}{2}t}$.

3. $x = A - \frac{B}{2}\sin\sqrt{3}(t + \delta) - \frac{\sqrt{3}}{2}B\cos\sqrt{3}(t + \delta)$,

$y = A + B\sin\sqrt{3}(t + \delta)$,

$z = A - \frac{B}{2}\sin\sqrt{3}(t + \delta) + \frac{\sqrt{3}}{2}B\cos\sqrt{3}(t + \delta)$.

4. $x = A_1 + A_2te^{-t} + A_3e^t + A_4e^{-t}$, $y = (2A_2t + 3A_2 + 2A_4)e^{-t}$. $z = A_1 - 2A_2e^{-t}$. **5.** $x = f(t)$ arbitrary, $y = -D^2f(t)$, $z = h(t) + (D^2 - 1)f(t)$; where the system is consistent if and only if $D^2h(t) = 0$.
6. $x = f(t)$ arbitrary, $y = c_1e^{-t} + (D + 1)f(t)$,

$$z = c_1e^{-t} + c_2e^{-2t} - \int^t e^{2(t_1-t)}D^2f(t_1)\,dt_1,$$

$$u = -c_1e^{-t} - 2c_2e^{-2t} + Df(t) + 2\int^t e^{2(t_1-t)}D^2f(t_1)\,dt_1.$$

7. $a = 0$; $x = f(t)$ arbitrary, $y = e^t + (1 - D^2)f(t)$.

Chap. VI, Sec. 5

1. $x_1 = \dfrac{m_2D^2 + bD + k}{\Delta(D)}f(t)$, $x_2 = \dfrac{bD + k}{\Delta(D)}f(t)$,

where $\Delta(D) = D^2[m_1m_2D^2 + (m_1 + m_2)(bD + k)]$.

2. $x_1 = V_0\left[t - \dfrac{m_2}{m_1m_2D^2 + (m_1 + m_2)(bD + k)}(b + kt)\right]$,

$x_2 = V_0\dfrac{m_1}{m_1m_2D^2 + (m_1 + m_2)(bD + k)}(b + kt)$; $b_{\min} = 2\left(\dfrac{m_1m_2}{m_1 + m_2}k\right)^{\frac{1}{2}}$.

3. $x_1 = V_0\left[t - \dfrac{mD^2 + 2D'}{(mD^2 + D')(mD^2 + 3D')}(b + kt)\right]$,

$x_2 = V_0\dfrac{1}{mD^2 + 3D'}(b + kt)$, $x_3 = V_0\dfrac{D'}{(mD^2 + D')(mD^2 + 3D')}(b + kt)$,

where $D' = bD + k$; $b_{\min} = 2\sqrt{mk}$.

4. $\theta = \dfrac{T_0}{4I}\left[\dfrac{t}{\omega} + \dfrac{\omega_0^3 \sin \omega t - \omega^3 \sin \omega_0 t}{\omega_0\omega^2(\omega^2 - \omega_0^2)}\right]$, where $\omega_0 = \sqrt{\dfrac{2k}{I}}$.

Resonance for $\omega^2 = \omega_0^2$ and $\omega^2 = \frac{1}{2}\omega_0^2$. **5.** $\sqrt{2k/I}$, $\sqrt{19k/I}$, $\sqrt{44k/I}$.

6. $$\frac{1}{I_1}\frac{D^3 + b(1/I_2 + 1/I_3)D^2 + (k/I_2)D + bk/I_2I_3}{D^2[D^3 + b(1/I_2 + 1/I_3)D^2 + k(1/I_1 + 1/I_2)D + bk(1/I_1I_2 + 1/I_2I_3 + 1/I_3I_1)]}T(t);$$

frequency $= \dfrac{1}{\pi}$. **7.** $\dfrac{1}{2\pi}\sqrt{\dfrac{g}{l}}$, $\dfrac{1}{2\pi}\sqrt{\dfrac{g}{l} + 2\dfrac{kh^2}{ml^2}}$.

Chap. VI, Sec. 6

1. $q = 10^{-3}[1 - e^{-500t}(\cos 1{,}000t + \frac{1}{2}\sin 1{,}000t)]$ coulomb.

2. $i_1 = \dfrac{L_2D^3 + R_2D^2 + C_2^{-1}D}{\Delta(D)}E(t)$, $i_2 = \dfrac{L_1D^2 + C_1^{-1}D}{\Delta(D)}E(t)$, where

$\Delta(D) = (L_1D^2 + R_1D + C_1^{-1})(L_2D^2 + R_2D + C_2^{-1}) + (L_1D^2 + C_1^{-1})R_1D$.

4. $$V = 10^4\int_0^t\left[\frac{1}{2}e^{10^4(t_1-t)} + \frac{1}{\sqrt{3}}e^{\frac{1}{2}10^4(t_1-t)}\sin\left(\frac{\sqrt{3}}{2}10^4(t - t_1) - \frac{\pi}{3}\right)\right]E(t_1)\,dt_1.$$

Chap. VII, Sec. 1

1. $y = c_1x^{-3} + c_2x^2$. **2.** $y = \dfrac{A}{6}x + B + c_1x^{-\frac{1}{2}} + c_2x^{-1}$.

3. $y = \frac{A}{9}x^2 + \frac{B}{4}x + C + (c_0 + c_1 \log x)x^{-1}$.

4. $y = \frac{x}{4}(\log x - 1) + (c_0 + c_1 \log x)x^{-1}$.

5. $y = 1 + (\log x)^2 + (c_0 + c_1 \log x)x + c_2x^{-1}$.
6. $y = \frac{1}{4}\sin(2\log x) + \frac{1}{4}\cos(2\log x) + (c_0 + c_1 \log x + c_2 \log^2 x)x + c_3x^{-2}$.
7. $y = (c_0 + c_1 \log x + c_2 \log^2 x)x^m$.
8. $y = -\frac{1}{2}(\frac{3}{2} - x + x^2) + c_1(2x - 1)^{-1} + c_2(2x - 1)^3$.
9. $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$. 10. $x^3y''' + xy' - y = 0$.

Chap. VII, Sec. 2

1. $y = c_0(1 + x) + c_1e^x$.
2. $y = x^{-\frac{1}{2}}[c_0 + c_1 \log(x - \frac{1}{2} + \sqrt{x^2 - x})]$. 3. $y = \frac{2}{3}x^2 + c_1x + c_2x^{-1}$.
4. $y = x^{-\frac{1}{2}}(c_1 \sin x + c_2 \cos x)$.
5. $y = c_1(3x^2 - 1) + c_2\left(\frac{3x^2 - 1}{2}\log\frac{1 + x}{1 - x} - 3x\right)$.
6. $y = -\frac{1}{8x} + x(c_0 + c_1 \log x + c_2 \log^2 x)$.

Chap. VII, Sec. 3

1. $y = A \arctan x + (c_0 + c_1x)(1 + x^2)^{-\frac{1}{2}}$. 2. $y = -\frac{1}{8x} + x(c_0 + c_1 \log x + c_2 \log^2 x)$.

3. $y = \frac{x^2}{8} + (c_0 + c_1x^2)e^{x^2} + c_2e^{-x^2}$.

Chap. VII, Sec. 4

1. $y = x(c_1 \sin x + c_2 \cos x)$. 2. $y = c_1x^{-1}(x + 1)^2 + c_2x^{-1}(x + 1)^{-1}$.
3. $y = x^{-2}\left[e^x + e^{-\frac{x}{\sqrt{2}}}\left(c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right) + e^{\frac{x}{\sqrt{2}}}\left(c_3 \cos\frac{x}{\sqrt{2}}\right.\right.$
$\left.\left. + c_4 \sin\frac{x}{\sqrt{2}}\right)\right]$.
4. $y = \frac{7}{10} - \frac{1}{10}\operatorname{ctn} x + [(c_0 + c_1x)e^x + c_2e^{-x}]\csc x$.

Chap. VII, Sec. 5

1. $y = \operatorname{ctn} x + 1 \Big/ \int^x \frac{\sin^2 x}{\sin^2 x'} e^{\frac{1}{2}(\cos 2x' - \cos 2x)}\,dx'$.
2. $u = A \tanh(Ax + c)$; $y'' - A^2y = 0$. 3. $u = \frac{1}{b_0x}\frac{r_1 + r_2cx^{r_2 - r_1}}{1 + cx^{r_2 - r_1}}$ and $u = \frac{r_2}{b_0x}$, where $r_{1,2} = \frac{1}{2}[1 - A \pm \sqrt{(1 - A)^2 - 4B}]$. 5. $u' + \frac{2\pi i}{h}u^2 - \frac{4\pi mi}{h}(E - V) = 0$.

Chap. VII, Sec. 6

1. $y = c_1 \sin\frac{x^2}{2} + c_2 \cos\frac{x^2}{2}$.

2. $y = \frac{1+x^2}{3}\log(1+x^2) + c_1(1+x^2) + c_2(1+x^2)^{-2}$. **3.** $y = \arctan\left(c_1x + \frac{c_2}{x}\right)$.
4. $y = -x^{-1}[\log(x+1) + c_1(x+1) + c_2(x+1)^{-1}]$.
5. $y = \log(c_1e^{\sqrt{x}} + c_2e^{-\sqrt{x}})$. **6.** $y = c_1x^{-1} + c_2x^{-1}\int^x e^{-4\arctan x'}dx'$.
7. $y = c_1x^{-2} + c_2x^{-3} + c_3x^8$. **10.** $y = x^{-1} - Ax^{-2}\tanh(Ax^{-1}+c)$.
11. $y = ue^{x^2/4}$.

Chap. VII, Sec. 7

1. $y = (c_1 - 1)x + c_1(1+x^2)\arctan x + c_2(1+x^2)$.
2. $y = (1+x^2)^{-2}\left[c_0 + c_1\left(\frac{x^2}{2} + \log x\right)\right]$. **3.** $y^2 - xy + x^4 + c_1x + c_2 = 0$.
4. $(x-c_1)^2 + y^2 = c_2$. **5.** $y = c_1x + c_2xe^{-x^2/2} + c_3x\int^x e^{(x'^2-x^2)/2}dx'$.
9. $y = c_1x + c_2xe^{-4/x^4}$.

Chap. VII, Sec. 8

1. (a) $y = 2h[\frac{1}{2}\mathbf{1}(t) - \mathbf{1}(t-\tau) + \mathbf{1}(t-3\tau) - \mathbf{1}(t-5\tau) + \cdots]$.
(b) $y = h[\mathbf{1}(t) - \mathbf{1}(t-\tau) - \mathbf{1}(t-2\tau) + \mathbf{1}(t-3\tau) + \mathbf{1}(t-4\tau) - \cdots]$.
(c) $y = \frac{h}{\tau}[t\mathbf{1}(t) - (t-\tau)\mathbf{1}(t-\tau)]$.
(d) $y = \frac{2h}{\tau}\left[\frac{t}{2}\mathbf{1}(t) - (t-\tau)\mathbf{1}(t-\tau) + (t-2\tau)\mathbf{1}(t-2\tau) - (t-3\tau)\mathbf{1}(t-3\tau) + \cdots\right]$.
(e) $y = \frac{2h}{\tau}\left[\frac{t}{2}\mathbf{1}(t) - (t-\tau)\mathbf{1}(t-\tau) + (t-3\tau)\mathbf{1}(t-3\tau) - (t-5\tau)\mathbf{1}(t-5\tau) + \cdots\right]$.

2. (a) $K(t,\tau) = \frac{1}{k^2}[1 - \cos k(t-\tau)]$, $G(t,\tau) = \frac{1}{k}\sin k(t-\tau)$; for $t \geqq \tau$.
(b) $K(t,\tau) = \frac{1}{k^2}[\cosh k(t-\tau) - 1]$, $G(t,\tau) = \frac{1}{k}\sinh k(t-\tau)$; for $t \geqq \tau$.
(c) $K(t,\tau) = \frac{t-\tau}{a} + \frac{1}{a^2}(e^{-a(t-\tau)} - 1)$, $G(t,\tau) = \frac{1}{a}(1 - e^{-a(t-\tau)})$; for $t \geqq \tau$.
(d) $K(t,\tau) = \frac{1}{b}\left(1 - \frac{r_2e^{r_1(t-\tau)} - r_1e^{r_2(t-\tau)}}{r_2 - r_1}\right)$, $G(t,\tau) = \frac{e^{r_1(t-\tau)} - e^{r_2(t-\tau)}}{r_1 - r_2}$; for $t \geqq \tau$,
where $r_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$.
(e) $K(t,\tau) = \frac{1}{a-1}\log\frac{t}{\tau} + \frac{1}{(a-1)^2}\left[\left(\frac{t}{\tau}\right)^{1-a} - 1\right]$,
$G(t,\tau) = \frac{1}{(a-1)\tau}\left[1 - \left(\frac{t}{\tau}\right)^{1-a}\right]$; for $t \geqq \tau$.

(f) $K(t,\tau) = \frac{1}{b}\left[1 - \frac{r_2(t/\tau)^{r_1} - r_1(t/\tau)^{r_2}}{r_2 - r_1}\right]$, $G(t,\tau) = \frac{\left(\frac{t}{\tau}\right)^{r_1} - \left(\frac{t}{\tau}\right)^{r_2}}{(r_1 - r_2)\tau}$; for $t \geqq \tau$, where $r_{1,2} = \frac{1}{2}(1 - a \pm \sqrt{(1-a)^2 - 4b})$.

(g) $K(t, \tau) = \frac{1}{a}\left(1 - e^{-\frac{1}{2}a(t^2-\tau^2)}\right) - \tau \int_\tau^t e^{\frac{1}{2}a(t'^2-t^2)}dt'$,

$G(t, \tau) = \int_\tau^t e^{\frac{1}{2}a(t'^2-t^2)}dt'$; for $t \geqq \tau$.

4. $i = \frac{V}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - \phi) + e^{-(R/L)t} \sin \phi$, where $\tan \phi = L\omega/R$.

5. $i = \frac{V_0}{R}[e^{-\nu t}\mathbf{1}(t) - 2e^{-\nu(t-\tau)}\mathbf{1}(t-\tau) + 2e^{-\nu(t-3\tau)}\mathbf{1}(t-3\tau) - \cdots]$, where $\nu = 1/RC$.

6. $F_{\max} = \frac{F_0}{1 + m/M}\left(1 + \frac{|\sin \nu\tau/2|}{\nu\tau/2}\right)$; $F_{\max} = \frac{2F_0}{1 + m/M}$, where $\nu^2 = k(1/M + 1/m)$. **7.** $i_2 = -\frac{M}{L_1L_2 - M^2} E_0 \left(\frac{\sqrt{r_1} \sin \sqrt{r_1}\, t - \sqrt{r_2} \sin \sqrt{r_2}\, t}{r_1 - r_2}\right)$,

where $r_{1,2} = \left[\left(\frac{L_1}{C_1} + \frac{L_2}{C_2}\right) \pm \sqrt{\left(\frac{L_1}{C_1} - \frac{L_2}{C_2}\right)^2 + \frac{4M^2}{C_1C_2}}\right]/2(L_1L_2 - M^2)$.

Chap. VII, Sec. 9

2. $\int_a^{a+\tau} f(t')\, dt' = 0$. **3.** $x = \frac{\frac{1}{2}At^2 + c_1t + c_0}{2 + \sin t}$; no periodic solution.

4. $x = \frac{-A \cos t + c_1t + c_0}{2 + \sin t}$; all solutions for which $c_1 = 0$ are periodic.

5. $x = 6 \cos 3t + \sin 3t + e^t(c_1 \sin 3t + c_2 \cos 3t)$; unique periodic solution is $y = 6 \cos 3t + \sin 3t$.

6. $x = \frac{1}{14} \sin 4t + c_1 \sin \sqrt{2}\, t + c_2 \cos \sqrt{2}\, t + c_3 \sin 2t + c_4 \cos 2t$; unique solution of period $\frac{\pi}{2}$ is $x = \frac{1}{14} \sin 4t$.

7. $x = \frac{A}{8} t \cos 2t + c_1 \sin \sqrt{2}\, t + c_2 \cos \sqrt{2}\, t + c_3 \sin 2t + c_4 \cos 2t$; no periodic solution. **8.** $x = e^{-(t - \cos t)} \int^t e^{t_1 - \cos t_1}dt_1 + (c_0 + c_1t)e^{-(t - \cos t)}$; unique periodic solution is $x = e^{-(t-\cos t)}\left(\int_0^t e^{t_1 - \cos t_1}dt_1 + c_0\right)$, where $c_0 = \frac{e^{2\pi}}{e^{2\pi} - 1}\int_{-2\pi}^0 e^{t_1 - \cos t_1}dt_1$.

9. $x = \frac{1}{64} + \frac{1}{64} \cos 4t$. **10.** $x = \frac{1}{8} + \frac{1}{6} \cos t + \frac{1}{120} \cos 2t$.

11. $x = -128 \sin \frac{1}{4}t - \frac{1}{28} \sin t$.

12. $x = \left(\frac{1}{a^2} + \frac{1}{a^2 - 1} \cos t + \frac{1}{a^2 - 4} \cos 2t + \frac{1}{a^2 - 9} \cos 3t + \frac{1}{a^2 - 16} \cos 4t + \frac{1}{a^2 - 25} \cos 5t + \frac{1}{a^2 - 49} \cos 7t + \frac{1}{a^2 - 64} \cos 8t\right) \frac{A}{8}$.

Chap. VII, Sec. 10

1. $x = \frac{A}{(c - a\omega^2)^2 + b^2\omega^2} [(c - a\omega^2) \sin \omega t - b\omega \cos \omega t]$. **2.** $x = -\frac{2A}{ab} \cos \sqrt{b}\, t$.

3. $x = -A \sin t - B \cos t$.

Chap. VIII, Sec. 1

1. $y = 1 + x^2$. **2.** $y = x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{3x^7}{7!} \cdots$.

3. $y = 1 + \frac{x^3}{3!} - \frac{x^4}{4!} - \frac{2x^5}{5!} \cdots$.

4. $y = \frac{1}{2}(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{24}(x-1)^4 + \frac{1}{48}(x-1)^5 \cdots$

5. $x = t^2 - \frac{t^6}{30} - \frac{t^{10}}{1{,}800} \cdots$. **6.** $x = 1 - \frac{t^4}{2} - \frac{t^8}{168} - \frac{t^{12}}{11{,}088} \cdots$.

7. $x = -2, -1, 1$. **8.** $x = -1, 0, 2$. **9.** $x = \frac{n\pi}{2}$ $(n = 0, \pm 1, \pm 2, \cdots)$.

10. None. **11.** $x = 0$.

Chap. VIII, Sec. 2

1. $y = A + B \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = A + B \arctan x$.

2. $y = Ax + B \sum_{n=0}^{\infty} \frac{(2n-3)^2(2n-5)^2 \cdots (-1)^2}{(2n)!} x^{2n}$.

3. $y = Ae^{x^3/3} + Bx\left(1 + \frac{x^3}{4} + \frac{x^6}{4 \cdot 7} + \frac{x^9}{4 \cdot 7 \cdot 10} + \cdots\right)$

$+ Cx^2\left(1 + \frac{x^3}{5} + \frac{x^6}{5 \cdot 8} + \frac{x^9}{5 \cdot 8 \cdot 11} + \cdots\right)$.

4. $y = A\left[1 + \frac{(x-1)^3}{2 \cdot 3} + \frac{(x-1)^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{(x-1)^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \cdots\right]$

$+ B(x-1)\left[1 + \frac{(x-1)^3}{3 \cdot 4} + \frac{(x-1)^6}{3 \cdot 4 \cdot 6 \cdot 7} + \frac{(x-1)^9}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \cdots\right]$.

5. $y = 1 + x^2$. **6.** $y = x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{3}{7!}x^7 - \frac{3 \cdot 5}{9!}x^9 + \frac{3 \cdot 5 \cdot 7}{11!}x^{11} \cdots$.

7. $x = t^2 + 2^2 \frac{-1}{4 \cdot 5 \cdot 6} t^6 + 2^4 \frac{-1 \cdot 3}{4 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 10} t^{10}$

$+ 2^6 \frac{-1 \cdot 3 \cdot 7}{4 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 10 \cdot 12 \cdot 13 \cdot 14} t^{14} \cdots$.

8. $x = 1 + 2^2 \frac{-3}{2 \cdot 3 \cdot 4} t^4 + 2^4 \frac{-3 \cdot 1}{2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8} t^8$

$+ 2^6 \frac{-3 \cdot 1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 10 \cdot 11 \cdot 12} t^{12} \cdots$.

Chap. VIII, Sec. 3

1. $y = A \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^n + B \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+\frac{1}{2}} = A \cos \sqrt{x} + B \sin \sqrt{x}$.

2. $y = A(1 + \frac{4}{3}x + \frac{2}{3}x^2) + Bx^{-2}(1 + 8x + 24x^2 + 32x^3 + 16x^4)$.
3. $y = A(1 + x + \frac{2}{5}x^2) + Bx^{-3}(1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5)$.

4. $$y = Ax^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} x^{2n} + Bx^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!} x^{2n}$$

$$= Ax^{-\frac{1}{2}} \sin \frac{x}{2} + Bx^{-\frac{1}{2}} \cos \frac{x}{2}.$$

5. $y = 4x^2 - 1$.

Chap. VIII, Sec. 4

1. $y = AF(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; x) + Bx^{\frac{1}{2}}F(1, 1, \frac{3}{2}; x)$.
2. $y = AF(\alpha, \beta, d; x) + Bx^{1-d}(\alpha - d + 1, \beta - d + 1, 2 - d; x)$, where $\alpha = -\frac{1}{2}(c + 1) + \frac{1}{2}\sqrt{(c + 1)^2 + 4e}$, $\beta = -\frac{1}{2}(c + 1) - \frac{1}{2}\sqrt{(c + 1)^2 + 4e}$.

3. $$y = AF\left(\alpha, \beta, \frac{d}{a}; -\frac{x}{a}\right) + Bx^{1-d/a}F\left(\alpha - \frac{d}{a} + 1, \beta - \frac{d}{a} + 1, 2 - \frac{d}{a}; -\frac{x}{a}\right),$$

$\alpha = -\frac{1}{2}(1 - c) + \frac{1}{2}\sqrt{(1 - c)^2 - 4e}$, $\beta = -\frac{1}{2}(1 - c) - \frac{1}{2}\sqrt{(1 - c)^2 - 4e}$.

4. $$y = AF\left(\alpha, \beta, \gamma; \frac{x - s_1}{s_2 - s_1}\right) + B\left(\frac{x - s_1}{s_2 - s_1}\right)^{1-\gamma} F\left(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; \frac{x - s_1}{s_2 - s_1}\right)$$

where $\alpha = -\frac{1}{2}(1 - c) + \frac{1}{2}\sqrt{(1 - c)^2 - 4e}$, $\beta = -\frac{1}{2}(1 - c) - \frac{1}{2}\sqrt{(1 - c)^2 - 4e}$, $\gamma = (cs_1 + d)/(s_1 - s_2)$.

5. $$y = AF\left(-\frac{4}{5}, -\frac{4}{5}, -\frac{4}{5}; \frac{4 - x}{3}\right) + B\left(\frac{4 - x}{3}\right)^{\frac{9}{5}} F\left(1, 1, \frac{14}{5}; \frac{4 - x}{3}\right)$$

$$= A_1(x - 1)^{\frac{4}{5}} + B\left(\frac{4 - x}{3}\right)^{\frac{9}{5}} F\left(1, 1, \frac{14}{5}; \frac{4 - x}{3}\right).$$

6. (*a*) $(1 - x)^{-1}$; (*b*) $(1 - x)^n$; (*c*) $-x^{-1} \log (1 - x)$; (*d*) a polynomial; (*e*) $\frac{1}{2} x^{-\frac{1}{2}} \arccos (1 - 2x)$; (*f*) $\frac{1}{2} x^{-\frac{1}{2}} \log \frac{1 + x^{\frac{1}{2}}}{1 - x^{\frac{1}{2}}}$.

8. $$y = 3F\left(\frac{1}{2}, 1, \frac{3}{2}; x\right) = \frac{3}{2} x^{-\frac{1}{2}} \log \frac{1 + x^{\frac{1}{2}}}{1 - x^{\frac{1}{2}}}.$$

Chap. VIII, Sec. 5

1. $y = AJ_0(x)$. **2.** $y = AJ_n(ax) + BJ_{-n}(ax)$.
3. $y = AJ_n[a(x - s)] + BJ_{-n}[a(x - s)]$. **4.** $y = Ax^{-n}J_n(x) + Bx^{-n}J_{-n}(x)$.
5. $y = Ax^{-\frac{1}{2}}J_{\frac{1}{2}}(x) + Bx^{-\frac{1}{2}}J_{-\frac{1}{2}}(x) = Cx^{-1}\sin x + Dx^{-1}\cos x$.
6. $y = Ax^{\frac{1}{2}}J_{\frac{1}{3}}\left(\frac{2a}{3} x^{\frac{3}{2}}\right) + Bx^{\frac{1}{2}}J_{-\frac{1}{3}}\left(\frac{2ax^{\frac{3}{2}}}{3}\right)$. **7.** $y = Ax^{\frac{1}{2}}J_{\frac{1}{4}}\left(\frac{a}{2} x^2\right) + Bx^{\frac{1}{2}}J_{-\frac{1}{4}}\left(\frac{a}{2} x^2\right)$.
8. $y = AJ_n(e^x) + BJ_{-n}(e^x)$.

Chap. VIII, Sec. 7

1. $$y = Ax^{-1} + B \sum_{n=0}^{\infty} \frac{(2n - 3)^2(2n - 5)^2 \cdots (-1)^2}{(2n)!} x^{-2n}.$$

2. $y = Ae^{\frac{1}{3}x^{-3}} + Bx^{-1}\left(1 + \frac{x^{-3}}{4} + \frac{x^{-6}}{4 \cdot 7} + \frac{x^{-9}}{4 \cdot 7 \cdot 10} + \cdots\right)$

$$+ Cx^{-2}\left(1 + \frac{x^{-3}}{5} + \frac{x^{-6}}{5 \cdot 8} + \frac{x^{-9}}{5 \cdot 8 \cdot 11} + \cdots\right).$$

3. $y = A(1 + \frac{4}{3}x^{-1} + \frac{2}{3}x^{-2}) + Bx^2(1 + 8x^{-1} + 24x^{-2} + 32x^{-3} + 16x^{-4})$.

4. $y = Ax^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} x^{-2n} + Bx^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!} x^{-2n}$

$$= Ax^{\frac{1}{2}} \sin\left(\frac{1}{2x}\right) + Bx^{\frac{1}{2}} \cos\left(\frac{1}{2x}\right).$$

5. $y = AJ_n(1/x) + BJ_{-n}(1/x)$.

6. $y = AF(\alpha, \beta, 1 + \delta; 1/x) + Bx^{\delta}F(\alpha - \delta, \beta - \delta, 1 - \delta; 1/x)$.

7. $x = 1 + t^{-2}$.

8. $x = -t^{-1} - \frac{1}{3!}t^{-3} + \frac{1}{5!}t^{-5} - \frac{3}{7!}t^{-7} + \frac{3 \cdot 5}{9!}t^{-9} - \frac{3 \cdot 5 \cdot 7}{11!}t^{-11} \cdots$.

9. $x = 1 + 2^2 \frac{-3}{2 \cdot 3 \cdot 4} t^{-4} + 2^4 \frac{-3 \cdot 1}{2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8} t^{-8}$

$$+ 2^6 \frac{-3 \cdot 1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 10 \cdot 11 \cdot 12} t^{-12} \cdots.$$

12. (*a*) Regular singular. (*b*) Nonregular singular.

Chap. VIII, Sec. 8

1. $y = AP_3(x) + BQ_3(x)$. **2.** $y = AP_n(x/a) + BQ_n(x/a)$.

3. $y = AP_n\left(\frac{x + a/2}{\sqrt{a^2/4 - b}}\right) + BQ_n\left(\frac{x + a/2}{\sqrt{a^2/4 - b}}\right)$.

7. $y = P_n(x) = 1 + \frac{n(n+1)}{2 \cdot 1^2}(x - 1) + \frac{(n-1)n(n+1)(n+2)}{2^2 \cdot 1^2 \cdot 2^2}(x - 1)^2$

$$+ \frac{(n-2)n(n+1)(n+2)(n+3)}{2^3 \cdot 1^2 \cdot 2^2 \cdot 3^2}(x - 1)^3 + \cdots + \frac{(2n)!}{2^n \cdot 1^2 \cdot 2^2 \cdots n^2}(x - 1)^n.$$

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