INTRODUCTION TO ALGEBRAIC GEOMETRY

BY

J. G. SEMPLE

AND

L. ROTH

OXFORD
AT THE CLARENDON PRESS
1949

Oxford University Press, Amen House, London E.C.4
GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON
BOMBAY CALCUTTA MADRAS CAPE TOWN
Geoffrey Cumberlege, Publisher to the University

PRINTED IN GREAT BRITAIN

PREFACE

THE main object of this book is to provide a reasonably concise introduction to algebraic geometry, requiring no more background than the usual honours degree courses in projective geometry and algebra. We have tried especially to satisfy two needs: (i) to give the reader an adequate idea of the developments in the past hundred years, and (ii) to provide him with every opportunity, in the form of examples, for acquiring self-reliance and technical ability. We hope that in consequence it will be much easier for him afterwards to read more formal treatises and original memoirs on whatever particular branches he may elect to study.

Chapter I is introductory, but is also a convenient compendium of generalities and explanations to which we often refer later; the beginner is advised to read it only lightly at first, returning to it as occasion demands, and to pass quickly to Chapter II. Chapters II-V form the first main section of the book and constitute an elementary course on higher plane curves and rational correspondences. Chapters VI-VIII are a somewhat detailed exposition of the properties of linear systems and of the general technique of transformation and representation. Chapters IX and X are fairly self-contained accounts of the projective characters of surfaces and of line-geometry respectively; and Chapter XI exhibits the application of the condition-calculus to enumerative problems, while touching only briefly on the deeper theoretical questions which it involves. Finally, in Chapters XII and XIII we have given short introductory accounts of the invariantive geometry of curves and surfaces, with such applications as best serve to illustrate the general development of the subject.

It may be desirable to mention here one feature of our exposition which may trouble the minds of some readers: we refer to a certain degree of informality of language, sacrificing precision to brevity, which we have allowed ourselves, and which has long characterized most geometrical writing. Most particularly we refer to the recurrent use of such adjectives as 'general' or 'generic', or such phrases as 'in general', whose meaning, wherever they are used, depends always on the context and is invariably assumed to be capable of unambiguous interpretation by the reader. Our justification for such informality, in an introductory work, lies in the resulting economy of expression, which is fully worth while; the danger, at this stage, of serious ambiguity is slight. For example, in Theorem I of Chapter I, the phrase 'r general points' can only mean 'r points whose coordinate-vectors are linearly independent'. But in less simple cases, we should point out that every use of the word 'general' (or 'generic') implies, strictly speaking, the concept of a certain irreducible algebraic variety of objects, e.g. points; and that a 'general' point of this variety is one whose coordinates do *not* satisfy a certain set of relations (indicated by the context) which are not satisfied by all points of the variety.

In a work of this kind we cannot try to list all the sources from which we have drawn our material; but neither can we forebear to mention the long-standing debts of inspiration and instruction that both of us owe to Prof. H. F. Baker and to his great *Principles of Geometry*.

Our warm thanks are due also to Mr. D. B. Scott, of King's College, London, who read the page-proofs and made many useful last-minute suggestions; and to the officers of the Clarendon Press, who have been most helpful and efficient throughout.

LONDON,
May 1949

J. G. S.
L. R.

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CHAPTER I

INTRODUCTION

In this chapter our main purpose is to make a preliminary survey of the foundations and basic ideas of our subject before commencing the more systematic development. We begin, then, with a brief and rather informal introduction to projective space of any number of dimensions.

§ 1. The general projective space S_r

1. We shall suppose that the reader is already familiar with the algebraical basis of plane and three-dimensional projective geometry, so that the extension to the case of an r-dimensional space S_r is only formal. The basic algebraic entity of this geometry is the *homogeneous coordinate vector* $(x_0,..., x_r)$, this being defined as an ordered set of $r+1$ real or complex numbers, not all zero, whose ratios only are to be regarded as significant.

The points of S_r are to admit a definite class of coordinate representations, by any one of which they correspond, without exception, to the totality of effectively distinct homogeneous vectors $(x_0,..., x_r)$; and the different representations of the class are to be related to each other by the group of non-singular linear homogeneous transformations of the form

$$\rho y_i = a_{i0}x_0+a_{i1}x_1+...+a_{ir}x_r \quad (i = 0,..., r),$$

where ρ is a factor of proportionality and $|a_{ij}| \neq 0$.

It will then follow, exactly as in the three-dimensional case, for example, that any one coordinate representation is completely specified by (i) its *fundamental simplex*, and (ii) its *unit point*, the former being the set of $r+1$ points $A_0,..., A_r$ which are to have the basic coordinate vectors $(1, 0, 0,..., 0)$, $(0, 1, 0,..., 0)$,..., $(0, 0,..., 0, 1)$, and the latter being the point U whose coordinate vector is to be $(1, 1, 1,..., 1)$.

1.1. *Linear spaces.* The aggregate of ∞^{r-1} points whose coordinates satisfy a single linear equation

$$a_0x_0+a_1x_1+...+a_rx_r = 0 \tag{1}$$

is called a *prime*. We denote this by S_{r-1} or by $[r-1]$; and we call $a_0, a_1,..., a_r$ the *coordinates of the prime*. It appears, then, exactly as for S_3, that corresponding to any linear homogeneous transformation of point coordinates x_i there is an associated linear homogeneous transformation of prime coordinates a_i. This means that with the r-dimensional space of points there is associated an r-dimensional projective space of primes; and by interchange of point and prime coordinates we have a *principle of duality* for S_r.

THEOREM I. *There is a unique prime containing r general points.*

For if the r points have coordinate vectors $(x_0^{(1)}, x_1^{(1)},..., x_r^{(1)}),...,$ $(x_0^{(r)}, x_1^{(r)},..., x_r^{(r)})$, then there exists in general a unique set of ratios $a_0:a_1:...:a_r$ which satisfy the equations

$$\left.\begin{array}{c} a_0 x_0^{(1)}+a_1 x_1^{(1)}+...+a_r x_r^{(1)} = 0 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ a_0 x_0^{(r)}+a_1 x_1^{(r)}+...+a_r x_r^{(r)} = 0 \end{array}\right\}. \qquad (2)$$

Dually we have

THEOREM II. *In general r primes intersect in a unique point.*

For r simultaneous linearly independent equations of the form (1) will have a unique common solution $x_0:x_1:...:x_r$.

1.11. For spaces of lower dimension we have first the

DEFINITION. The totality of points common to two primes is called a *secundum*, and is denoted by S_{r-2} or $[r-2]$.

If $A = 0$, $B = 0$ are the equations of the two primes in question, then the secundum evidently lies in every prime of the pencil whose equation is

$$A+\lambda B = 0; \qquad (3)$$

it is, in fact, the base of this *pencil of primes*, and from this remark there follows

THEOREM III. *There is a unique secundum through $r-1$ general points.*

For if the prime (1) is made to pass through $r-1$ general points, its coefficients are subjected to $r-1$ linearly independent conditions. We may therefore write (1) in the form $A+\lambda B = 0$, where A and B are fixed linear forms and λ is an arbitrary parameter, and the primes $A = 0$ and $B = 0$ themselves pass through the given points. The required secundum is the base of the pencil of primes thus determined.

1.12. *Linear sub-spaces in general*

DEFINITION. The totality of ∞^k points common to $r-k$ linearly independent primes $(0 \leqslant k < r)$ is a k-dimensional space S_k or $[k]$.

By immediate analogy with the result and method of proof of Theorem III, we have

THEOREM IV. *There is a unique $[k]$ through $k+1$ general points of S_r.*

The following particular cases are to be noted:

A space [1], called a *line*, is uniquely determined by two given points.

A space [2], called a *plane*, is uniquely determined by three non-collinear points.

THEOREM V. *If a prime is made to contain $k+1$ linearly independent points of a $[k]$, then it contains the $[k]$.*

For the $k+1$ given points impose $k+1$ linearly independent conditions on the general prime containing them, and the equation of such a prime must therefore be reducible to the form

$$\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_{r-k} A_{r-k} = 0,$$

where the equations $A_i = 0$ $(i = 1, 2, \ldots, r-k)$ represent $r-k$ fixed linearly independent primes through the given points. Since these intersect evidently in the unique $[k]$ defined by the given points, any prime through these points must contain the same $[k]$.

COROLLARY. *Any $[h]$ which contains $k+1$ points of general position in $[k]$ $(h > k)$ contains $[k]$ entirely.*

For the space $[h]$ is common to $r-h$ primes, each of which contains $k+1$ points of general position of $[k]$.

1.2. *Intersections and joins of spaces.* Given two or more spaces, we may require to know the space of greatest dimension common to them all, called the *intersection* of the spaces. Or we may wish to find the space of minimum dimension, termed the *join*, which contains all the given spaces. In the first place we have

THEOREM VI. *Two spaces $[h]$ and $[k]$ in general meet in a $[h+k-r]$ $(h+k \geqslant r)$.*

This follows from the definition of the spaces concerned. Through $[h]$ and $[k]$ respectively there pass sets of $r-h$ and $r-k$ linearly independent primes. The complete set of these $2r-h-k$ primes have in common a space $[h+k-r]$, provided that

$$h+k-r \geqslant 0.$$

COROLLARY 1. *A $[k]$ and a $[r-k]$ in general meet in a point.*

COROLLARY 2. *In general, n given spaces $[k_1], [k_2],..., [k_n]$ meet in a $[k_1+k_2+...+k_n-(n-1)r]$ $\left(\sum_1^n k_i \geqslant (n-1)r\right)$.*

This last is proved by applying the above theorem $n-1$ times.

As regards the join of two given spaces, we have

THEOREM VII. *Two spaces $[h]$ and $[k]$ $(h+k \leqslant r-2)$ in general lie in a $[h+k+1]$.*

If we take $h+1$ linearly independent points in $[h]$ and $k+1$ linearly independent points in $[k]$, we can draw through these a unique $[h+k+1]$, provided that $h+k+1 \leqslant r-1$; and, by Theorem V, this will contain both spaces entirely.

COROLLARY. *If the spaces have a space $[s]$ in common, they lie in a $[h+k-s]$.*

For we may take $s+1$ of the first set of $h+1$ points to be in $[s]$; and it is then sufficient to take $k-s$ further points of $[k]$ to define a space $[h+k-s]$ which contains both $[h]$ and $[k]$.

In the above formulae two spaces which do not intersect may be regarded, formally, as having an intersection space of dimension -1.

EXAMPLES

1. In space of four dimensions we have the following set of spaces: point, line, plane, prime.† For these we obtain the incidence relations shown in the table. Thus, in general, a line and a plane do not meet;

	prime	plane	line
prime	plane	line	point
plane	line	point	—
line	point	—	—

when they do meet, they lie in a prime. *A fortiori*, two lines do not in general intersect; when they do intersect they are coplanar.

2. Show that in [4] two general lines determine a unique prime containing them, and deduce that there is a unique line meeting three given lines of general position.

3. In [4], there is, in general, a unique plane meeting three given skew‡ planes in lines. To prove this we observe that such a plane must lie in a prime with each of the given planes, $\varpi_1, \varpi_2, \varpi_3$, say. Draw the unique

† A space [3] is frequently termed a *solid*.

‡ Two planes in [4] are *skew* if they meet only in a point.

prime through ϖ_1 which contains the point common to ϖ_2 and ϖ_3; this will meet ϖ_2 and ϖ_3 respectively in lines l_2 and l_3, whose join is the required plane.

4. In $[r]$, the spaces $[k]$ form a system of freedom $(r-k)(k+1)$. For a $[k]$ is determined by $k+1$ arbitrary points, and each of these, regarded as a point of $[r]$, depends on r distinct parameters; again each of them, regarded as a point of $[k]$, depends on k distinct parameters; and the first total, $(k+1)r$, must be diminished by the second, $(k+1)k$, since the $k+1$ points may be chosen anywhere in $[k]$.

In particular, the lines and planes of [4] form systems of freedom 6; and the lines of $[r]$ form a system of freedom $2(r-1)$.

5. Show that in [4] there is a duality

point,	line,	plane,	prime
prime,	plane,	line,	point.

Establish a similar result for a space of any number of dimensions.

6. In general, a $[k]$ and a $[r-k-1]$ do not meet; if they do meet, a single condition is implied, namely, the concurrence of the complete set of $r+1$ primes which serve to define them. Similarly, if a $[k]$ and a $[r-k-2]$ intersect, two conditions are implied, and so on. Thus in particular if a $[k]$ is to pass through a given point, then $r-k$ distinct conditions are imposed on it.

§ 2. Collineations and Correlations of S_r

2. We next introduce the reader to linear transformations of S_r, this being a subject which must plainly be of fundamental importance throughout the whole of projective geometry. The algebra of such transformations is most conveniently expressed in the notation of matrices; and we shall therefore employ this notation, without assuming, however, any more than the elementary facts about matrix products and inverses. We shall also use a simple operational symbolism for transformations.

We use Clarendon type for matrices. Also, if $\mathbf{A}$ is a matrix, we shall denote its transposed matrix by $\tilde{\mathbf{A}}$, and its determinant (if it is a square matrix) by $|\mathbf{A}|$. Thus we shall have, for example,

$$\widetilde{\mathbf{AB}} = \tilde{\mathbf{B}}\tilde{\mathbf{A}}, \qquad |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|,$$

$$(\tilde{\mathbf{A}})^{-1} = (\widetilde{\mathbf{A}^{-1}}), \qquad |\mathbf{A}^{-1}| = |\mathbf{A}|^{-1}.$$

The point $(x_0, x_1,..., x_r)$ of S_r will always be represented by the column matrix

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_r \end{pmatrix},$$

and the prime whose equation is

$$l_0x_0+l_1x_1+...+l_rx_r = 0 \tag{1}$$

will be represented by the row matrix

$$\mathbf{l} = (l_0, l_1,..., l_r).$$

The equation of the prime may then be written in the form

$$\mathbf{lx} = 0. \tag{2}$$

If $\mathbf{A} = (a_{rs})$ is a square matrix of $r+1$ rows and columns, then $\mathbf{Ax}$ is the column matrix $\mathbf{y}$ whose elements are the $r+1$ linear functions

$$y_i = \sum_j a_{ij}x_j \quad (i = 0,..., r)$$

of those of $\mathbf{x}$; a (scalar) product such as $\tilde{\mathbf{x}}\mathbf{Ax}'$ is a bilinear form

$$\sum_{r,s} a_{rs}x_r x'_s,$$

in the elements of $\mathbf{x}$ and $\mathbf{x}'$; and $\tilde{\mathbf{x}}\mathbf{Ax}$ represents the quadratic form

$$\sum_{r,s} a_{rs}x_r x_s.$$

We begin then with the following

DEFINITION. If $\mathbf{x}$ and $\mathbf{y}$ represent general points of two r-dimensional spaces Σ and Σ', and if $\mathbf{A}$ is any non-singular square matrix of order $r+1$, then the linear transformation defined by the equation

$$\mathbf{y} = \mathbf{Ax} \tag{3}$$

is called a *collineation* of Σ on Σ'.

To any point $\mathbf{x}$ of Σ there corresponds plainly a unique point $\mathbf{y}$ of Σ'; and conversely, since $\mathbf{A}$ is non-singular, we may write

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}, \tag{4}$$

which shows that the reverse transformation is also a collineation carrying any point $\mathbf{y}$ of Σ' into a unique point $\mathbf{x}$ of Σ.

Equations (3) and (4) are respectively the equations of the forward correspondence (from Σ to Σ') and of the backward correspondence (from Σ' to Σ).

If T denotes the forward transformation, we may write, symbolically,

$$\mathbf{y} = T(\mathbf{x}), \qquad \mathbf{x} = T^{-1}(\mathbf{y}). \tag{5}$$

If the point $\mathbf{x}$ of Σ describes the prime $\mathbf{l}$ whose equation is (2), then, by (4), the corresponding point $\mathbf{y}$ of Σ' describes the prime

$\mathbf{m}$ whose equation is $\mathbf{l}\mathbf{A}^{-1}\mathbf{y} = 0$: so that the correspondence between primes $\mathbf{l}$ and $\mathbf{m}$ of Σ and Σ' is given by the equation

$$\mathbf{m} = \mathbf{l}\mathbf{A}^{-1} \quad \text{or} \quad \tilde{\mathbf{m}} = \tilde{\mathbf{A}}^{-1}\tilde{\mathbf{l}}; \tag{6}$$

and, conversely,

$$\mathbf{l} = \mathbf{m}\mathbf{A} \quad \text{or} \quad \tilde{\mathbf{l}} = \tilde{\mathbf{A}}\tilde{\mathbf{m}}. \tag{7}$$

Hence: *A collineation between two spaces can equally well be regarded as a linear transformation of primes into primes.*

In fact, if $\mathbf{l}$ passes through the fixed point $\mathbf{x}$ of Σ, then

$$0 = \mathbf{l}\mathbf{x} = \mathbf{m}\mathbf{A}\mathbf{x} = \mathbf{m}\mathbf{y},$$

so that $\mathbf{m}$ passes through the corresponding point $\mathbf{y}$ of Σ'; and vice versa. Thus the duality is complete.

2.1. *Products of collineations.* If we now assume that the space Σ' is superposed on Σ, we obtain a collineation of Σ into itself. The transformation T will then in general carry a point $\mathbf{x}$ into a different point $\mathbf{y}$, and the repeated transformation, which is represented by

$$\mathbf{z} = T(\mathbf{y}) = T^2(\mathbf{x}), \tag{8}$$

will in general give rise to a point $\mathbf{z}$ different from both $\mathbf{y}$ and $\mathbf{x}$. However, it may happen that, for a given transformation T, $\mathbf{z}$ always coincides with $\mathbf{x}$; such a transformation is called *involutory*.

THEOREM VIII. *The product of any number of collineations is a collineation.*

This follows at once from the fact that if $\mathbf{y} = \mathbf{A}\mathbf{x}$ and $\mathbf{z} = \mathbf{B}\mathbf{y}$, then $\mathbf{z} = \mathbf{B}\mathbf{A}\mathbf{x}$, the square matrix $\mathbf{B}\mathbf{A}$ having determinant equal to the product of those of $\mathbf{A}$ and $\mathbf{B}$.

2.2. *Correlations.* With the previous notation we have the following

DEFINITION. If $\mathbf{x}$ is a generic point of Σ, and $\mathbf{m}$ is a prime of Σ', then the transformation

$$\tilde{\mathbf{m}} = \mathbf{A}\mathbf{x} \quad \text{or} \quad \mathbf{m} = \tilde{\mathbf{x}}\tilde{\mathbf{A}}, \tag{9}$$

defines a *correlation* between Σ and Σ'; in other words, a linear transformation of the points of the first space into the primes of the second.

From (9) we derive the reverse transformation

$$\mathbf{x} = \mathbf{A}^{-1}\tilde{\mathbf{m}}. \tag{10}$$

Suppose now that $\mathbf{x}$ describes a given prime whose equation is $\mathbf{lx} = 0$. Then $\tilde{\mathbf{x}}\tilde{\mathbf{l}} = 0$; whence, by (9), and the rule for transposing a reciprocal, we obtain $\mathbf{m}\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{l}} = 0$. This relation shows that all primes of Σ' corresponding to points of $\mathbf{l}$ pass through the point $\mathbf{y}$ defined by

$$\mathbf{y} = \tilde{\mathbf{A}}^{-1}\tilde{\mathbf{l}}. \tag{11}$$

This point is said to correspond to the prime $\mathbf{l}$. We may write (11) in the form $\tilde{\mathbf{y}} = \mathbf{lA}^{-1}$; so that

$$\mathbf{l} = \tilde{\mathbf{y}}\mathbf{A}. \tag{12}$$

Equations (9) and (11) are those of the forward correspondence T from Σ to Σ'; and equations (10) and (12) are those of the backward correspondence. They may be written as

$$T(\mathbf{x}) = \tilde{\mathbf{x}}\tilde{\mathbf{A}}, \qquad T(\mathbf{l}) = \tilde{\mathbf{A}}^{-1}\tilde{\mathbf{l}}, \tag{13}$$

$$T^{-1}(\mathbf{y}) = \tilde{\mathbf{y}}\mathbf{A}, \qquad T^{-1}(\mathbf{m}) = \mathbf{A}^{-1}\tilde{\mathbf{m}}. \tag{14}$$

2.3. *The coincidence quadrics.* We now assume that the spaces Σ and Σ' coincide. If $\mathbf{x}^*$ is any point of the prime corresponding, by the forward transformation T, to $\mathbf{x}$, we have

$$\tilde{\mathbf{x}}\tilde{\mathbf{A}}\mathbf{x}^* = 0, \tag{15}$$

which is a general bilinear relation between the coordinates of $\mathbf{x}$ and $\mathbf{x}^*$. By transposing the product on the left we obtain

$$\tilde{\mathbf{x}}^*\mathbf{A}\mathbf{x} = 0, \tag{16}$$

which means, by (12), that $\mathbf{x}$ lies in the prime corresponding by T^{-1} to $\mathbf{x}^*$.

If the point $\mathbf{x}$ lies in its own corresponding prime $T(\mathbf{x})$, then

$$\tilde{\mathbf{x}}\mathbf{A}\mathbf{x} = \sum a_{rs} x_r x_s = 0. \tag{17}$$

Thus the locus of points having this property is a quadric (§ 3), termed the *first coincidence quadric* Q_1 of the correlation. Points of this quadric lie on the primes which correspond to them either by T or by T^{-1}.

In the same way, the envelope of primes $\mathbf{l}$ which pass through their corresponding points $T(\mathbf{l})$ or $T^{-1}(\mathbf{l})$ has the equation

$$\mathbf{l}\mathbf{A}^{-1}\tilde{\mathbf{l}} = \sum a^{rs} l_r l_s = 0. \tag{18}$$

This again defines a quadric (envelope) which we call the *second coincidence quadric* Q_2 of the correlation.

2.4. *Polarities.* It is clear that the effect of operating twice with T is to give a collineation; for, by § 2.2,

$$\mathbf{x}' = T^2(\mathbf{x}) = T(\tilde{\mathbf{x}}\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}^{-1}\mathbf{A}\mathbf{x}. \tag{19}$$

Thus the collineation T^2 has matrix $\tilde{\mathbf{A}}^{-1}\mathbf{A}$. The reverse transformation $(T^2)^{-1}$ has matrix $\mathbf{A}^{-1}\tilde{\mathbf{A}}$, which is easily seen to be identical with that of $(T^{-1})^2$.

The correlation T may be said to be *involutory* if the collineation T^2 is simply the identical transformation $\rho\mathbf{x}' = \mathbf{x}$, where ρ is a scalar. This implies that if a point $\mathbf{x}^*$ lies in the prime $T(\mathbf{x})$, then $\mathbf{x}$ lies in the prime $T(\mathbf{x}^*)$.

If T is involutory, we must have

$$\tilde{\mathbf{A}}^{-1}\mathbf{A} = \rho\mathbf{I},$$

where $\mathbf{I}$ is a unit matrix. Thus

$$\mathbf{A} = \rho\tilde{\mathbf{A}}, \tag{20}$$

so that $$a_{rs} = \rho a_{sr} \quad \text{(for all suffixes } r \text{ and } s).$$

We assume now that a_{rs} is not zero for some particular pair of the suffixes, say, the pair r_0, s_0.

If $r_0 = s_0$, then it follows at once that $\rho = 1$.

If $r_0 \neq s_0$, then it follows first that $a_{s_0 r_0}$ is also not zero; so that, besides $a_{r_0 s_0} = \rho a_{s_0 r_0}$, we have also the relation $a_{s_0 r_0} = \rho a_{r_0 s_0}$. Combining these we obtain $\rho^2 = 1$, which therefore holds in either case.

If $\rho = 1$, we have $a_{rs} = a_{sr}$: the matrix $\mathbf{A}$ is symmetrical, and the coincidence quadrics Q_1 and Q_2 are one and the same quadric Q. The correlation T is then a *polarity*, namely, the relation between a point and its polar prime with respect to the quadric Q.

If $\rho = -1$, we have $a_{rs} = -a_{sr}$, and $a_{rr} = 0$: the matrix $\mathbf{A}$ is skew-symmetric; and the quadrics Q_1 and Q_2 are non-existent, for the condition $\sum a_{rs} x_r x_s = 0$ is satisfied identically. This type of correlation, for which every point lies in its corresponding prime, is called a *null-polarity*. Since any skew-symmetric determinant of odd order vanishes, a null-polarity in space of any even dimension is necessarily singular (i.e. degenerate).

§ 3. Algebraic manifolds. Primals

3. So far we have considered only those manifolds which are represented by one or more linear equations; we turn now to the discussion of non-linear manifolds.

DEFINITION. A *primal* in $[r]$ is the locus of points whose coordinates satisfy an equation of the form

$$F(x_0, x_1, ..., x_r) = 0, \tag{1}$$

where F is a polynomial.

If F is of degree n, the primal is said to be of *order* n and is denoted by a symbol of the form V^n_{r-1}. If F is of degree 2, the primal is called a *quadric*.

The number of disposable constants on which a V^n_{r-1} depends is $\binom{n+r}{r}-1$, this being one less than the number of coefficients in (1). Hence, since the condition on V^n_{r-1} to pass through a given point is linear, we deduce

THEOREM IX. *A unique primal V^n_{r-1} can be made to pass through a generic† set of $\binom{n+r}{r}-1$ points.*

Thus, for example, a quadric can be made to pass through $\frac{1}{2}r(r+3)$ arbitrary points.

3.1. *Polar primals.* The fundamental property of the primal is expressed as follows:

THEOREM X. *An arbitrary line meets a V^n_{r-1} in n points.*

Let P and Q be points whose coordinates are $(x_0, x_1, ..., x_r)$ and $(y_0, y_1, ..., y_r)$ respectively. A general point of the line PQ has coordinates $(x_0+\lambda y_0, ..., x_r+\lambda y_r)$ for a suitable value of λ; and this point lies on V^n_{r-1} if

$$F(x_0+\lambda y_0, ..., x_r+\lambda y_r) = 0. \tag{2}$$

This is an equation of degree n in λ, so that PQ meets V^n_{r-1} in n points, as stated.

We now expand (2) explicitly, as an equation in λ, in the form

$$F(x)+\lambda\Delta_x F(x)+\tfrac{1}{2}\lambda^2\Delta_x^2 F(x)+...+\lambda^n F(y) = 0, \tag{3}$$

where $F(x) \equiv F(x_0, x_1, ..., x_r)$, and the symbol Δ_x is the differential operator

$$y_0\frac{\partial}{\partial x_0}+y_1\frac{\partial}{\partial x_1}+...+y_r\frac{\partial}{\partial x_r}.$$

If P lies on V^n_{r-1}, then $F(x) = 0$, and one root of (3) is zero. If

† Except where the contrary is expressly stated, the word *generic* will be used throughout this book in a sense equivalent to that of the phrase *of general position*. See preface.

PQ is a tangent to V_{r-1}^n at P, then a second root of (3) must be zero, the condition for this being

$$\Delta_x F(x) \equiv y_0 \frac{\partial F}{\partial x_0} + y_1 \frac{\partial F}{\partial x_1} + \dots + y_r \frac{\partial F}{\partial x_r} = 0. \tag{4}$$

By regarding the y_i in this equation as current coordinates, we obtain

THEOREM XI. *If P is a generic point of a primal V_{r-1}^n, then the locus of lines tangent to V_{r-1}^n at P is a prime.*

This is called the *tangent prime* to V_{r-1}^n at P. An exceptional case occurs when $\partial F/\partial x_0, \dots, \partial F/\partial x_r$ all vanish at P: the tangent prime is then illusory, and P is called a *singular point* of V_{r-1}^n.

We may now interpret (4) quite differently by keeping Q fixed and regarding the x_i as current coordinates. The equation then represents a primal of order $n-1$, which we call the *first polar* of Q with respect to V_{r-1}^n; and any point P of V_{r-1}^n which lies on this polar primal is such that PQ has two of its intersections with V_{r-1}^n coincident at P. From this we deduce

THEOREM XII. *The first polar of an arbitrary point Q with respect to a primal V_{r-1}^n is a primal of order $n-1$, which meets V_{r-1}^n in the locus of points of contact of tangent lines from Q. Tangent primes from Q have the same contact locus.*

Returning now to (3) which we suppose already to have two zero roots, we observe that this equation will have a third zero root if

$$\Delta_x^2 F(x) \equiv \left(y_0 \frac{\partial}{\partial x_0} + \dots + y_r \frac{\partial}{\partial x_r}\right)^2 F(x) = 0. \tag{5}$$

In this case PQ will have three-point contact with V_{r-1}^n at P. Still regarding the x_i as current coordinates, equation (5) defines a primal of order $n-2$, which we call the *second polar* of Q with respect to V; and we can assert

THEOREM XIII. *The second polar of an arbitrary point Q with respect to V_{r-1}^n meets the intersection of this primal with the first polar of Q in the locus of points of contact of inflexional tangents drawn from Q to V_{r-1}^n.*

3.11. *Multiple points.* A point of multiplicity k of V_{r-1}^n is a point P such that a generic line through it meets the primal in only $n-k$ points elsewhere.

If (x_i) is such a point P, then it appears by (3) that all th expressions

$$F(x), \quad \Delta_x F(x), \quad \Delta_x^2 F(x), \quad \ldots, \quad \Delta_x^{k-1} F(x)$$

must vanish for arbitrary $(y_i) \equiv Q$; and for this it is both necessary and sufficient, in virtue of the homogeneity of the functions involved, that all the partial derivatives of order $k-1$ of F should vanish at P.

Furthermore, the first polar of any point Q has then a $(k-1)$-fold point at P; for every $(k-2)$th x-derivative of $\Delta_x F(x)$ is a linear combination of $(k-1)$th x-derivatives of F; and by exactly similar reasoning we derive the general result:

Every k-fold point of V_{r-1}^n is $(k-s)$-fold at least on the s-th polar of every point of S_r.

EXAMPLES

1. Show that an arbitrary plane meets V_{r-1}^n in a curve of order n, and that an arbitrary S_3 meets it in a surface of order n.

2. The *class* of V_{r-1}^n, defined as the number of its tangent primes which belong to an arbitrary pencil, is in general $n(n-1)^{r-1}$.

3. Prove that, in general, V_{r-1}^n contains $\infty^{(2r-n-3)}$ lines, provided $2r \geqslant n+3$.

3.2. *The intersection of primals. Bézout's Theorem.*

In S_r, r given primals have in general a finite number of common points whose coordinates are found by solving their equations simultaneously for $x_0 : x_1 : \ldots : x_r$. When the primals are generally situated with respect to each other, we obtain the following result:

THEOREM XIV. *In S_r, r generic irreducible primals $V_{r-1}^{n_i}$ $(i = 1, 2, \ldots, r)$, of orders $n_1, n_2, \ldots, n_r$, have $n_1 n_2 \ldots n_r$ common points.*

This is the simplest case of Bézout's Theorem. We give two proofs, one based on the theory of elimination, the other on a specialization principle.

FIRST PROOF: Consider two curves V_1^m and V_1^n which lie in the same plane S_2; let their equations, when arranged in powers of x_2, be

$$f(x_2) \equiv a_0 x_2^m + a_1 x_2^{m-1} + \ldots + a_m = 0, \tag{6}$$

$$\phi(x_2) \equiv b_0 x_2^n + b_1 x_2^{n-1} + \ldots + b_n = 0, \tag{7}$$

where a_r, b_r are homogeneous of degree r in x_0 and x_1.

If $\alpha_2, \beta_2, \gamma_2, \ldots$ are the roots of (6), the condition that (6) and (7) should have a common root is

$$R \equiv \phi(\alpha_2)\phi(\beta_2)\phi(\gamma_2)\ldots = 0.$$

Now this resultant R is of weight mn in a_i and b_i; that is, R is a sum of terms such as
$$a_0^{\lambda_0} a_1^{\lambda_1} \ldots b_0^{\mu_0} b_1^{\mu_1} \ldots,$$
where $\lambda_1 + 2\lambda_2 + \ldots + \mu_1 + 2\mu_2 + \ldots = mn$.

Hence R is homogeneous and of degree nm in x_0 and x_1, so that the condition $R = 0$ will in general determine mn values of the ratio $x_1 : x_0$. The curves V_1^m and V_1^n therefore meet in mn points.

Consider next the case of three surfaces in S_3, V_2^m, V_2^n, and V_2^p whose equations, arranged in powers of x_2 and x_3, are
$$f(x_2, x_3) = 0, \quad \phi(x_2, x_3) = 0, \quad \psi(x_2, x_3) = 0.$$

We have seen that the first two equations have in general mn solutions, say (α_2, α_3), (β_2, β_3), (γ_2, γ_3),.... Then the condition that the three equations should have a common solution is
$$R \equiv \psi(\alpha_2, \alpha_3)\psi(\beta_2, \beta_3)\psi(\gamma_2, \gamma_3)\ldots = 0.$$

This condition is expressible as a homogeneous equation of degree mnp in x_0 and x_1, so that the surfaces therefore meet in mnp points.

Proceeding thus by successive stages, we may establish Bézout's Theorem, in the simplest case, for primals of any dimension $r-1$.

SECOND PROOF: Let us assume that the r equations defining the primals $V_{r-1}^{n_1}$, $V_{r-1}^{n_2}$,... have a finite number of solutions and, further, that, if we vary continuously the coefficients in those equations, this number will remain unaltered, provided always that it does not become infinite. We then choose the coefficients so as to make each of the equations resoluble into a set of distinct linear equations such that the linear equations of one set are all different from those of any other set. Thus we take each primal $V_{r-1}^{n_i}$ to be a set of n_i primes, generically situated with respect to each other and with respect to the remaining sets. The points common to the r primals are now the intersections of r primes, one from each set, taken in all possible ways; so that the total number is $n_1 n_2 n_3 \ldots n_r$.

§ 4. THE GENERAL IRREDUCIBLE ALGEBRAIC MANIFOLD

4. From a consideration of primals, we pass on now to algebraic manifolds in general; and more particularly, since the concept is fundamental, to a consideration of those which are *irreducible* in a sense about to be explained.

If (x_i) is such a point P, then it appears by (3) that all the expressions

$$F(x),\quad \Delta_x F(x),\quad \Delta_x^2 F(x),\quad \ldots,\quad \Delta_x^{k-1} F(x)$$

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3. Prove that, in general, V_{r-1}^n contains $\infty^{(2r-n-3)}$ lines, provided $2r \geqslant n+3$.

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We have seen that the first two equations have in general mn solutions, say (α_2, α_3), (β_2, β_3), (γ_2, γ_3),.... Then the condition that the three equations should have a common solution is
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Proceeding thus by successive stages, we may establish Bézout's Theorem, in the simplest case, for primals of any dimension $r-1$.

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§ 4. THE GENERAL IRREDUCIBLE ALGEBRAIC MANIFOLD

4. From a consideration of primals, we pass on now to algebraic manifolds in general; and more particularly, since the concept is fundamental, to a consideration of those which are *irreducible* in a sense about to be explained.

A primal, being given by the vanishing of a polynomial, is irreducible if the polynomial in question is irreducible.

On this then we base the following definition:

DEFINITION: A point-locus in S_r is said to be an irreducible algebraic manifold V_k of dimension k if its points can be shown to be in algebraic $(1,1)$ correspondence with the points of an irreducible primal M_k of a space S_{k+1}.

Algebraically this means that if $(x) = (x_0,..., x_r)$ is a general point of S_r and if $(y) = (y_0,..., y_{k+1})$ is a general point of S_{k+1}, then there exists a set of $r+1$ polynomials $F_0, F_1,..., F_r$, homogeneous and of the same degree in $y_0, y_1,..., y_{k+1}$, and a further irreducible homogeneous polynomial $M(y_0,..., y_{k+1})$, such that as (y) describes the primal $M = 0$ the point (x) given by $\rho x_i = F_i$ describes V_k; and the correspondence is such that the generic point of V_k arises from only one point of M_k.

The equations

$$\rho x_i = F_i(y_0,..., y_{k+1}) \quad (i = 0,..., r), \tag{1}$$

$$M(y_0, y_1,..., y_{k+1}) = 0, \tag{2}$$

are called the *parametric equations* of V_k. The parameters are the $k+1$ ratios $y_i : y_0$ $(i = 1, 2,..., k+1)$, of which, however, only k are independent in virtue of the relation (2). It is in general impossible to dispense with the redundant parameter without introducing functions other than polynomials into the representation.

As in S_3, a V_1 is called a curve and a V_2 is called a surface.

4.1. ***General composite algebraic manifold.*** By eliminating all the parameters y_i/y_0 from the equations (1) and (2) of an irreducible V_k, we should obtain various homogeneous relations between the x_i representing primals of S_r passing through V_k; and the final aim of this elimination is the discovery of a *fundamental set* of primals through V_k, such, namely, as have no common points other than those of V_k. Such a fundamental set of primals, since it does not involve an auxiliary primal M_k, is often more convenient as a description of V_k than the parametric representation.

On the other hand, the totality of points common to a finite set of primals of S_r, while it must be regarded as an algebraic manifold, may break up into many irreducible components which may even be of different dimensions.

We may agree then to interpret the term *algebraic manifold*, in the broad sense, as referring to any combination of irreducible algebraic manifolds in the same space, such manifolds being allowed to count multiply in the combination and being not necessarily all of the same dimension.

The *dimension* of an algebraic manifold is the greatest dimension of any one of its irreducible components.

An algebraic manifold is said to be *pure* if all its components are of the same dimension.

EXAMPLES

1. In S_3, the points common to two quadrics may form (i) an irreducible quartic curve, or (ii) a line and a twisted cubic curve, or (iii) a single conic (of contact); and so on.

2. In S_3, the points common to three quadrics may form (i) a set of eight isolated points, or (ii) a line and four isolated points, or (iii) a twisted cubic curve; and so on. Case (ii) is an example of the occurrence of a mixed or impure algebraic manifold of dimension 1. Case (iii) shows that a twisted cubic can be specified by a fundamental set of three quadrics passing through it. It will appear later, however, that, in general, a curve in S_3 requires a fundamental set of four surfaces to specify it completely.

4.2. *Order of a manifold.* In what follows the manifolds considered are supposed to be irreducible; but the extension to algebraic manifolds in general is immediate and can be left to the reader.

The system of equations formed by combining (1) and (2) with an arbitrary system of k linear equations of the form

$$\sum_{i=0}^{r} a_{ij} x_i = 0 \quad (j = 1, 2, \ldots, k) \tag{3}$$

will in general have a finite number n of solutions; and this number n cannot vary with the coefficients a_{ij} except by becoming infinite,† which would mean that the S_{r-k} defined by (3) had some special relation with V_k. Hence:

THEOREM XV. *A generic S_{r-k} meets V_k in a finite number n of points.*

This number n is called the *order* of the manifold. We shall write V_k^n to denote a manifold of dimension k and order n.

COROLLARY 1. *A generic S_{r-k+1} meets V_k^n in a curve of order n.*

For any S_{r-k} lying in S_{r-k+1} is a prime in that space and meets V_k^n in n points; hence the locus of these points is a curve of order n.

† See Ch. XI, § 6.

By analogous reasoning, we deduce

COROLLARY 2. *If $k > h$, a generic S_{r-k+h} meets V_k^n in a V_h^n.*

In particular we have

COROLLARY 3. *The section of a primal V_{r-1}^n by a space S_h is a primal of order n in S_h.*

Note. The above results refer of course only to generic sections of V_k^n. Any particular space may happen to meet V_k^n in a manifold of greater dimension than we should normally expect, even to the extent, in special cases, of lying entirely on V_k^n.

4.21. *Multiple points*

DEFINITION. A point P of V_k^n such that a generic S_{r-k} passing through it meets V_k^n in $n-i$ points not at P is a *simple point* if $i = 1$, and a *multiple point of order i* (or i-fold point) if $i > 1$.

A locus of ∞^t such points on V_k^n constitutes an i-fold manifold V_t of V_k^n.

4.3. *Intersections of manifolds.* In S_r, let a point P be a function of k independent parameters, so that it describes a k-dimensional manifold V; and in the same way let Q be a function of h parameters, so that it describes an h-dimensional manifold W. If we require P and Q to coincide, we impose r conditions on the $k+h$ parameters, thereby reducing the number of free parameters to $k+h-r$; so that, in general, the totality of points common to V and W is a manifold of dimension $k+h-r$. This proves

THEOREM XVI. *In S_r, the points common to two manifolds, of dimensions k and h respectively, form a manifold which is in general of dimension $k+h-r$ ($h+k \geqslant r$).*

At first sight it would seem natural to define the *intersection* of two manifolds merely as the totality of points common to them both; but in algebraic geometry, where manifolds are considered less as individuals than as members of families, the concept of intersection multiplicity enters, and the above definition is sufficient only in straightforward instances. In the first place, if V_k and V_l are two algebraic manifolds in S_r, they are regarded as having a *normal intersection* only if the totality of their common points has the normal (minimum) dimension $k+l-r$; a dimension higher than this makes the intersection *abnormal* or *improper*. And in the second place, even when the manifold common to V_k

and V_l is a V_{k+l-r}, certain of its components may have to count as *multiple components* of the intersection, in order to secure, for example, that when V_k and V_l are made to vary continuously, at least the *order* of their intersection manifold shall not alter. We cannot at this stage give a rule for assigning multiplicities—that ranks as one of the major problems of the subject; all we can do here is to state, without attempting to prove it, the theorem fixing the order which a normal intersection of two algebraic manifolds must possess.

THEOREM XVII (*Generalized Theorem of Bézout*). *If two algebraic manifolds intersect normally and have orders m and n respectively, then their intersection is an algebraic manifold of order mn.*

Broadly we may regard this theorem as asserting (i) that when V_k^m and V_l^n are *generically situated* (a term requiring special explanation here), their intersection, taken as their totality of common points, is of order mn; while (ii), when V_k^m and V_l^n are not restricted as above, their intersection, duly interpreted according to some universal fixed rule, is still always of order mn. As particular cases of the theorem, which will be required later, we note the following:

(1) In S_3, a generically situated curve V_1^m and surface V_2^n meet in mn points.

If we suppose, namely, that the curve, by varying continuously, can break up into m distinct lines, then each of these would meet the surface in n points; and the result would follow at once.

(2) In S_4, two surfaces V_2^m and V_2^n, generically situated, meet in mn points.

Here again the result would follow at once if, by continuous variation, the surface V_2^m could be made to break up into m distinct planes, each of which would meet V_2^n in n points.

In each case, therefore, the theorem follows if we assume that such variations are possible. But the proof of these assumptions is exceedingly difficult and at present incomplete.

4.4. *Cones in S_r*. Corresponding to the type of surface in ordinary space which we term a cone, there exist in S_r various types of conical manifolds, namely, point-cones, line-cones, plane-cones, and generally S_h-cones ($h \leqslant r-2$), whose definitions and modes of generation we now proceed to lay down.

DEFINITION. Any manifold in S_r which is generated by a variable S_{h+1} through a fixed S_h is called an S_h-cone of S_r.

To generate an S_h-cone, the general procedure is to join a fixed space S_h to every point of a fixed k-dimensional manifold V_k ($k \leqslant r-h-2$): the joining $[h+1]$'s generate an S_h-cone which we denote by $S_h(V_k)$, and we call S_h the *vertex* of this cone, V_k its *directrix*, and the $[h+1]$'s its *generators*.

In general, the dimension of $S_h(V_k)$ is $h+k+1$; so that, if $k = r-h-2$, $S_h(V_k)$ is a *conical primal*.

As to the order of $S_h(V_k)$, we now prove

THEOREM XVIII. *If S_h does not meet V_k, then the order of $S_h(V_k)$ is equal to the order of V_k.*

Let V_k be of order n, so that an arbitrary S_{r-k} meets it in n points. Since $S_h(V_k)$ has dimension $h+k+1$, its order is the number of points in which it meets an arbitrary space Π, of dimension $r-h-k-1$; in other words, its order is the number of its generators which meet Π. A generator of $S_h(V_k)$ meets Π, however, if and only if it lies in the space Π', of dimension $r-k$, joining Π to S_h; which means that it must be one of the n generators joining S_h to the n intersections of Π' with V_k, none of these intersections being situated in S_h since, by hypothesis, S_h does not meet V_k. Thus $S_h(V_k)$ is of order n, as was to be proved.

In the above, we note that if Π is made to meet S_h in a point, then it will not, in general, meet $S_h(V_k)$ in any further points at all; so that every point of S_h is an n-fold point of $S_h(V_k)$. Thus the cone has its vertex S_h as n-fold space.

EXAMPLES

1. In S_4 there exist surfaces which are point-cones, primals which are point-cones, and primals which are line-cones.

2. Show that in S_r the equation $f(x_0, x_1,..., x_{r-h-1}) = 0$ is that of an S_h-cone whose vertex has equations $x_0 = 0$, $x_1 = 0$,..., $x_{r-h-1} = 0$, and whose directrix has equations $x_{r-h} = 0$,..., $x_r = 0$, $f = 0$.

3. Prove that $\frac{1}{2}(h+1)(h+2)$ linear conditions must be imposed on a quadric of S_r in order that it should be an S_h-cone.

4.5. *Projection.*

DEFINITION. The projection of a point P from a vertex S_h ($0 \leqslant h \leqslant r-2$) on to a space S_{r-h-1} is the intersection of that space with the S_{h+1} joining P to the vertex.

Assuming that S_h and S_{r-h-1} are generically situated with respect to each other, the projection of P is a point P'. This leads to

THEOREM XIX. *Projection from a vertex S_h is equivalent to $h+1$ successive projections from point vertices.*

Let X_0, X_1,..., X_h be $h+1$ general points of S_h, and let S_{r-h-1} be the space on which the projection is to be made. Then there is a unique prime S_{r-1} containing S_{r-h-1} and $X_1, X_2,..., X_h$: let P_0 be the projection of P from X_0 on S_{r-1}. Similarly, in S_{r-1} there is a unique S_{r-2} containing S_{r-h-1} and X_2, X_3,..., X_h: let P_1 be the projection of P_0 from X_1 on S_{r-2}. Continuing thus, we obtain finally a point P_h which is the projection of P_{h-1} from X_h on S_{r-h-1}.

Now the space S_{h+1} containing X_0, X_1,..., X_h and P contains also P_0, which is collinear with $X_0 P$, and hence also P_1, which is collinear with $X_1 P_0$,..., and finally P_h, which is collinear with $X_h P_{h-1}$. Thus P_h is the projection of P from S_h on S_{r-h-1}.

DEFINITION. The projection of a manifold V_k^n from S_h ($h \leqslant r-k-2$) on S_{r-h-1} is the manifold U_k described by the projection P' of a variable point P of V_k^n. The projection is *proper* if the generic point of U_k is the projection of a unique point of V_k^n.

THEOREM XX. *Any proper projection U_k of V_k^n, from a vertex S_h which does not meet V_k^n, has order n. The sections of U_k by primes of its own space are projections of prime sections of V_k^n which pass through S_h.*

Let $S_{r-h-1-k}$ be a generic space, of the dimension indicated, in S_{r-h-1}; the points P' where it meets U_k are projections of the points P in which V_k^n is met by the S_{r-k} passing through S_h and $S_{r-h-1-k}$. If the projection is proper, to a point P' there corresponds a single point P; and if S_h does not meet V_k^n, there are in general n points P'.

Again, an arbitrary prime of S_{r-h-1} lies in a unique S_{r-1} passing through S_h, so that the manifold in which it meets U_k is the projection of the manifold in which S_{r-1} meets V_k^n.

4.51. *Note.* The reason for the restriction $h \leqslant r-k-2$ now appears; for if $h = r-k-1$, every S_{h+1} through the vertex S_h meets V_k^n in n points, so that each point P' is the projection of n different points P. Thus the projection is not proper if $n > 1$.

The projection may, however, be proper when the vertex S_h

$(h \leqslant r-k-2)$ is specially situated with respect to V_k^n. If S_h meets V_k^n, the projection U_k will be of order less than n. For example, the projection of V_k^n from a point of itself is in general a U_k^{n-1}.

4.52. *Normal manifolds*

DEFINITION. A manifold V_k^n is *normal* in S_r if it cannot be obtained as a proper projection of a manifold of the same order in S_t $(t > r)$. V_k^n is said to be *supernormal* if it cannot be obtained by projecting a manifold U_k from a vertex meeting U_k in only a finite number of simple points.

§ 5. Algebraic correspondence

5. Definition. An algebraic correspondence between two irreducible manifolds U, V is a set of relations enabling the coordinate-ratios of a variable point of either manifold to be expressed as algebraic functions of the coordinate-ratios of a point of the other.

The correspondence is *uni-rational* (from U to V), if to a generic point of U there corresponds a unique point of V. In this case, if the coordinates of a point of U are $x_0, x_1,..., x_r$, and if those of a point of V are $y_0, y_1,..., y_s$, the equations of the correspondence can be written in the form

$$\rho y_i = F_i(x_0, x_1,..., x_r) \quad (i = 0, 1,..., s), \tag{1}$$

where the F_i are homogeneous polynomials of the same degree.

The correspondence is *birational* if to a generic point of either manifold there corresponds a unique point of the other. In this case the uni-rational equations (1) can be rationally reversed to give

$$\rho x_j = G_j(y_0, y_1,..., y_s) \quad (j = 0, 1,..., r), \tag{2}$$

where the G_j are likewise homogeneous polynomials of the same degree.

Transformations such as the collineations and proper projections we have already considered are simple types of birational transformation. Another special class in this category is the class of the so-called Cremona transformations, defined as follows.

DEFINITION. A Cremona transformation is a birational transformation of a linear space S_r into another (or the same) linear space S_r'.

The general idea of birational correspondence, as above defined, is fundamental. It is, namely, the root idea of what we call *invariantive algebraic geometry*, in which we try to regard all manifolds which are birationally transformable into each other as a single abstract algebraic manifold, and in which, therefore, we are only interested in those properties of manifolds which are birationally invariant.

5.1. *Rational manifolds*

DEFINITION. A rational manifold V_k is one which can be birationally transformed into a space S_k.

It is a simple matter to state various types of sufficient condition for the rationality of a manifold. Thus for curves we give

THEOREM XXI. *A curve V_1^n is necessarily rational if it lies in S_n, but not in any space of lower dimension than n.*

To prove this, we construct the S_{n-2} through $n-1$ fixed points of general position on V_1^n, and we consider the pencil of primes through this S_{n-2}, no one of which, by hypothesis, contains the curve entirely. Each of these primes meets V_1^n in one variable point; and conversely, through a generic point of V_1^n there passes one prime of the pencil. The points of V_1^n are therefore in birational correspondence with the primes of the pencil; and these in turn may be made to correspond birationally with the points of a line. This proves the theorem.

COROLLARY. *A curve V_1^n of S_n cannot have a multiple point.*

For if V_1^n had a double point P, the prime containing P and $n-1$ other points of V_1^n would meet the curve in $n+1$ points in all and hence contain it entirely.

For surfaces there is a similar result:

THEOREM XXII. *A surface V_2^n of S_r is necessarily rational if an S_{r-3} can be found which contains so many points or curves of V_2^n that a generic secundum through it meets the surface in only one further point.*

For in this case the points of V_2^n are in birational correspondence with the secunda through an S_{r-3}, and therefore also with the points of a plane.

EXAMPLES

1. Show that a primal V_{r-1}^n with an $(n-1)$-fold point is rational. Also, that a V_k^n with an $(n-1)$-fold point must lie in an S_{k+1}.

2. Prove that, for any V_k^n of S_r (and not in any S_{r-1}), $r \leqslant n+k-1$.

3. Show that the projection of a surface from a simple point of itself contains a line corresponding to the neighbourhood of the vertex of the projection.

NOTES AND EXAMPLES ON CHAPTER I

1. *Some incidence properties of lines.* It is a familiar fact that in ordinary space a single transversal can be drawn from a given point to two given lines. This result can easily be extended to any space $[2r+1]$ of an odd number of dimensions. Thus, if in $[2r+1]$ we take two spaces α, β of dimension r, we can draw through a given point P the two spaces (P,α) and (P,β), each of dimension $r+1$; these meet in a line which meets both α and β.

Ex. Show that in $[2r]$ a unique transversal can in general be drawn from a given point to two given spaces $[r-1]$ and $[r]$.

2. In ordinary space, two transversals can in general be drawn to four given lines. In [4], three given lines have in general a single transversal; for the prime or solid containing two of the lines meets the third line in a point from which the transversal in question can be drawn.

The former result can be suggested by the specialization principle already used in § 3.2. Let l_1, l_2, l_3, l_4 be the given lines and assume that l_1, l_2 intersect; the number of transversals then remains finite, one of them being the unique transversal to l_3, l_4 from the common point of l_1, l_2, and the other lying in the common plane of l_1, l_2.

In the four-dimensional problem, however, we observe that the method of specialization fails; for if we suppose two of the three lines to intersect, the number of transversals becomes infinite.

3. *Incidence relations for spaces.* In § 1.2 we saw that the freedom of spaces $[k]$ in $[r]$ is $(r-k)(k+1)$. We consider now the number of conditions imposed on these spaces by various incidence relations of simple type.

(i) In general $r-k$ primes intersect in a $[k]$; and we may consider the number of conditions required to make a larger number h of primes meet in a $[k]$. Clearly we must make each of $h-r+k$ of the primes contain $k+1$ points of the $[k]$ defined by the rest; and the required number of conditions is therefore $(k+1)(h-r+k)$.

(ii) The number of conditions required to make two spaces $[h]$ and $[k]$ intersect in a $[t]$, where $t > h+k-r$, is $(t+1)(t+r-h-k)$. For by (i) the sets of $r-h$ and $r-k$ primes defining $[h]$ and $[k]$ must be subjected to $(t+1)(2r-h-k-r+t)$ conditions in all.

(iii) An important particular case of (ii) occurs when $t = 0$; thus the number of conditions for a *point* intersection is $r-h-k$.

(iv) The freedom of spaces $[k]$ which contain a given space $[t]$ is $(r-k)(k-t)$. For $t+1$ of the $k+1$ points which define $[k]$ may be taken to lie in $[t]$, whence the freedom of the spaces $[k]$ is

$$(r-k)(k+1)-(r-k)(t+1) = (r-k)(k-t).$$

4. *Harmonic inversion.* We have shown in Ex. 1 that a unique transversal l can be drawn from an arbitrary point P of $[2r+1]$ to two given spaces α, β of dimension r. Suppose now that l meets α, β in A, B respectively; and let P' be the harmonic conjugate of P with respect to A, B. The transformation (P, P') is called a *harmonic inversion* with respect to

α and β; we now show that it is a collineation (of a special kind). Suppose then that the equations of α, β are $x_i = 0$ $(i = 0, 1, ..., r)$ and $x_j = 0$ $(j = r+1, r+2, ..., 2r+1)$ respectively. If P is the point $(y_0, y_1, ..., y_{2r+1})$, it is clear that any point on l has coordinates $(y_0, y_1, ..., y_r, \lambda y_{r+1}, ..., \lambda y_{2r+1})$. The point P' is obtained by taking $\lambda = -1$; and the transformation is therefore a collineation in which every point of α, β is invariant.

Ex. Obtain the equations of a harmonic inversion in $[2r]$ based on spaces $[r-1]$ and $[r]$.

5. Show that in S_r, if P' is the projection of P from a vertex H on to a space Π, then the coordinates of P' are linear functions of those of P; but that the transformation is not a collineation because its matrix is singular.

6. *Quadric and cubic primals in* [4]. In later chapters we shall be specially concerned with two primals—the quadric and the cubic—of [4]; at this stage the following introductory remarks may be noted.

(i) A quadric primal V_3^2 of [4] contains ∞^3 lines; for the lines of [4] form a system of freedom 6, and three conditions are imposed on a line if it is to lie on a given quadric.

(ii) Conversely, any primal of [4] which contains ∞^3 lines, and which is not generated by ∞^1 planes, must be a quadric. For through an arbitrary point of the given primal V, there must pass ∞^1 lines, lying on the primal and forming a cone Γ; and the section of V by a prime Π containing a generator of Γ is a ruled surface v, since every prime section of V must be ruled. Now, if V is not a quadric, v is not a quadric surface; and it has therefore only one generator through a generic point of itself. Hence Π cannot contain a second generator of Γ; and Γ must therefore be a pencil of lines whose plane lies on V.

This result is due to Severi.†

(iii) In general V_3^2 does not contain any plane; if, however, it does contain a plane ϖ, the pencil of primes through ϖ meet the primal again in a pencil of planes, all of which pass through a point O of ϖ. Evidently O is a double point of the primal, which is therefore a cone with vertex O. Thus V_3^2 contains in general two systems of planes, all passing through O; any two planes of the same system meet only at O, while any two of opposite systems meet in a line, the prime containing them being tangent to V_3^2.

The latter result follows also from the generation of the cone by lines joining the points of a quadric directrix surface to the vertex O.

(iv) If, however, V_3^2 is a line-cone (§ 4.4), the two systems of generating planes coalesce. In this case an arbitrary prime section of V_3^2 is a conical surface V_2^2, while a section through the vertex consists of a pair of planes.

7. The general cubic primal V_3^3 of [4] contains an ∞^2 system (or congruence) of lines. Since a generic prime section of V_3^3 is a general cubic surface of ordinary space, it follows that 27 lines of the system in question lie in an arbitrary prime (Ch. VII, § 2.2). To find how many of the lines pass through an arbitrary point P of V_3^3, we observe that such lines must

† *Rend. Palermo*, **15** (1901), 33.

form the complete intersection of the primal with the first and second polars of P; and since the former is a quadric cone and the latter is a prime, there are 6 lines of the system which pass through P.

Since the second polar of P is actually the tangent prime there, its section of V_3^3 is a cubic surface having a node at P. We have thus verified incidentally that a nodal cubic surface contains 6 lines passing through the node.

BOOKS RECOMMENDED FOR FURTHER READING

BAKER, *Principles of geometry*, i.
BERTINI, *Geometria proiettiva degli iperspazi*, chs. i–ix.
HODGE and PEDOE, *Methods of algebraic geometry*, i.
TODD, *Projective and analytical geometry*.
VAN DER WAERDEN, *Algebraische Geometrie*.

CHAPTER II

PLANE CURVES

§ 1. ELEMENTARY PROPERTIES

1. A PLANE curve C^n of order n is represented by an equation $f(x,y,z) = 0$, in which f is a polynomial of degree n, homogeneous in x, y, z. Since the number of terms in f is $\frac{1}{2}(n+1)(n+2)$, the curve depends on the ratios of that number of coefficients, and therefore, effectively, on the values of $\frac{1}{2}n(n+3)$ arbitrary parameters; and since to make C^n pass through a given point imposes a linear condition on the coefficients, it follows that there exists a C^n (in general unique) through $\frac{1}{2}n(n+3)$ given points. We say therefore that curves of order n have *freedom* $\frac{1}{2}n(n+3)$.

1.1. *Multiple points.* In the previous chapter we defined the term 'multiple point' for algebraic manifolds in general. Before continuing, it will be convenient to repeat and amplify this definition for the case of plane curves.

DEFINITION. A multiple point of order k (or k-fold point, $k > 1$) of C^n is a point P of the curve such that a generic line through P meets the curve in only $n-k$ further points.

To investigate more closely the behaviour of a curve at a multiple point, let us take the point under consideration to be the vertex Z of the triangle of reference, and let us write the equation of the general C^n in the form

$$f(x,y,z) \equiv u_0 z^n + u_1 z^{n-1} + \ldots + u_n = 0, \tag{1}$$

where $u_i \equiv u_i(x,y)$ is homogeneous of degree i in x and y. A generic line through Z has an equation of the form $x/\xi = y/\eta$; and the point of this line whose coordinates are $(\rho\xi, \rho\eta, 1)$ lies on C^n if ρ is any one of the roots of the equation

$$u_0 + \rho u_1(\xi,\eta) + \rho^2 u_2(\xi,\eta) + \ldots + \rho^n u_n(\xi,\eta) = 0. \tag{2}$$

To make C^n have a k-fold point at Z, we have to make equation (2) have k zero roots for every value of the ratio ξ/η: this requires that $u_0, u_1, \ldots, u_{k-1}$ should vanish identically. For the equation then to have an additional zero root, the ratio ξ/η would have to take one of the particular values given by the equation $u_k(\xi,\eta) = 0$; and such a value would correspond to a line having at least $k+1$

of its intersections with C^n coincident in Z. Such a line we call a *nodal tangent* to C^n at Z. Thus, if Z is a k-fold point of C^n, the equation of C^n is of the form

$$z^{n-k}u_k(x,y)+z^{n-k-1}u_{k+1}(x,y)+...+u_n(x,y) = 0, \tag{3}$$

where $u_k(x,y) = 0$ is the equation of the k nodal tangents, distinct or otherwise, of C^n at Z.

A multiple point of a curve is said to be *ordinary* if the curve has all its nodal tangents distinct at that point.

From (3) it appears that the imposition of a k-fold point on C^n at Z implies the vanishing of $\frac{1}{2}k(k+1)$ coefficients; and clearly the same number of linear conditions will be implied by the imposition of a k-fold point on C^n at any given point of the plane. This leads to

THEOREM I. *If all the linear conditions involved are independent, the curves C^n which contain a set of assigned points $K_1, K_2,..., K_s$, to assigned multiplicities $k_1, k_2,..., k_s$ respectively, form a system of freedom r, given by*

$$r = \tfrac{1}{2}n(n+3)-\tfrac{1}{2}\sum_{1}^{s} k_i(k_i+1). \tag{4}$$

The word 'freedom' is used here, as elsewhere, to denote the number of points of general position which determine a unique curve of the system passing through them.

In special cases, the actual (or effective) freedom may be greater than that indicated above. Even then, however, the expression on the r.h.s. of (4) is significant as a known lower bound for the effective freedom r: we refer to it therefore as the *virtual freedom* of the system.

1.2. *Number of intersections of two curves.* According to the simplest case of Bézout's Theorem (Ch. I, § 3.2), a C^m and a C^n in the plane will meet altogether in mn points; but as these intersections need not be all distinct, the result just stated has little meaning until some rule has been formulated and justified which will enable us to assign to each common point of C^m and C^n its proper multiplicity as an intersection. We postpone till later (Ch. IV, § 4.1) a formal consideration of this problem; but in anticipation of their subsequent justification we shall apply now a few intuitively plausible rules for estimating intersection multiplicity.

Thus, for example, a point of simple contact of C^m and C^n will count twice in their intersection group; and a simple contact of the rth order (as usually defined) will count $(r+1)$-fold. Of special importance, however, is the following rule:

If a point P is h-fold on C^m and k-fold on C^n, and if i is the multiplicity of P in the group of mn intersections of the two curves, then always $i \geqslant hk$; and if the nodal tangents of C^m at P are distinct from those of C^n, then $i = hk$.

Thus in general, then, if C^m and C^n meet altogether in points O_i $(i = 1, 2, ..., s)$, whose multiplicities on C^m are h_i and whose multiplicities on C^n are k_i, and if, at each of these points, the nodal (or ordinary) tangents of C^m are all distinct from those of C^n, then

$$mn = \sum_1^s h_i k_i.$$

1.3. *Polar curves.* As in Ch. I, § 3.1, if P_0 is the point (x_0, y_0, z_0), the curves obtained by equating to zero the coefficients of $\lambda, \lambda^2, ..., \lambda^{n-1}$ in the expansion of $f(x+\lambda x_0, y+\lambda y_0, z+\lambda z_0)$ are called the polar curves of P_0 with respect to C^n. If we write f for $f(x, y, z)$, and f_0 for $f(x_0, y_0, z_0)$, and if we use the operators

$$\Delta = x_0\frac{\partial}{\partial x}+y_0\frac{\partial}{\partial y}+z_0\frac{\partial}{\partial z},$$

$$\Delta_0 = x\frac{\partial}{\partial x_0}+y\frac{\partial}{\partial y_0}+z\frac{\partial}{\partial z_0},$$

we obtain successive polar curves $\Gamma^{n-1}, \Gamma^{n-2}, ..., \Gamma^2, \Gamma^1$, whose equations may be written either in the forms

$$\Delta f = 0, \quad \Delta^2 f = 0, \quad ..., \quad \Delta^{n-1} f = 0$$

respectively, or in the alternative forms

$$\Delta_0^{n-1} f_0 = 0, \quad \Delta_0^{n-2} f_0 = 0, \quad ..., \quad \Delta_0 f_0 = 0.$$

These curves are respectively the first, second,..., $(n-1)$th polars of P_0 with respect to C^n; or, alternatively, they may be called the polar $(n-1)$ic, the polar $(n-2)$ic,..., the polar line, of P_0 with respect to C^n. If the n intersections of the line joining an arbitrary point P to P_0 are expressed in the form $P+\lambda_i P_0$ $(i = 1, ..., n)$, then Γ^r is the locus of those points P for which vanishes the sum of products of the λ_i, taken r at a time, and any such relation of P and P_0 to the n points $P+\lambda_i P_0$ is invariant. Thus all the polar curves are what we may call geometrical *covariants* of P_0 and C^n: that is to say, they are loci dependent only on P_0 and C^n, and independent therefore of any particular system of homogeneous coordinates in terms of which their equations may happen to be expressed.

By the argument of Ch. I, § 3.1, if P_0 lies on C^n, then its polar line is the tangent at that point; and for arbitrary position of P_0, the points of contact of tangents from P_0 to C^n lie at the intersections of C^n with the first polar of P_0.

Concerning the behaviour of the various polar curves of P_0 at a multiple point of C^n, we have the result (cf. Ch. I, § 3.11):

THEOREM II. *A k-fold point of C^n is $(k-r)$-fold (at least) on the r-th polar of every point of the plane $(k \geqslant r)$.*

1.4. *Class of a curve.* The *class m* of a curve C^n is the number of proper tangents to C^n which pass through a generic point of the plane. (The word 'proper' is used here only to exclude lines through multiple points, which we do not regard as tangents, though some of their intersections with the curve are coincident.)

To evaluate m, we use the result stated in § 1.3, namely, that the points of contact of tangents from the generic point P_0 to C^n lie at the intersections of C^n with the first polar of P_0; only we now have to exclude those of these intersections which fall at multiple points of C^n, since the joins of such points to P_0 are not proper tangents. To find how many of the $n(n-1)$ intersections of the two curves fall at a k-fold point of C^n, we suppose this point to be the vertex Z of the triangle of reference; so that, by § 1.1, the equation of C^n is of the form

$$f(x,y,z) \equiv z^{n-k}u_k+z^{n-k-1}u_{k+1}+\ldots+u_n = 0.$$

In this case, the first polar of P_0 has equation

$$\Delta f \equiv z^{n-k}v_{k-1}+z^{n-k-1}v_k+\ldots+v_{n-1} = 0,$$

where, in particular, $v_{k-1} = x_0\dfrac{\partial u_k}{\partial x}+y_0\dfrac{\partial u_k}{\partial y}$. If the factors of u_k are all distinct, so that the multiple point Z is *ordinary*, then no one of them coincides in general with any factor of v_{k-1}: whence, by § 1.2, precisely $k(k-1)$ intersections of the two curves fall at Z. There follows from this, then, the result:

THEOREM III. *If the multiple points of C^n are all ordinary and of multiplicities $k_1, k_2, \ldots, k_s$ respectively, then the class m of C^n is given by*

$$m = n(n-1) - \sum_{1}^{s} k_i(k_i-1). \tag{5}$$

1.5. *Inflexions. The Hessian of C^n.* If $P(x,y,z)$ is any point of C^n, the equation of the tangent at P (in current coordinates X, Y, Z) is

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z} = 0.$$

If this tangent is *inflexional* at P, then, in addition to the above, any point (X, Y, Z) on it must satisfy the further condition

$$\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)^2 f = 0,$$

which is the equation of the polar conic of P; and this means that *the polar conic of an inflexion breaks up into the inflexional tangent and another line.*

To find the inflexions of C^n, we consider therefore, to begin with, the locus of all points $P(x,y,z)$ whose polar conics are degenerate. If the polar conic of P is a line-pair whose vertex is (X, Y, Z), then

$$Xf_{xx}+Yf_{xy}+Zf_{xz} = 0,$$
$$Xf_{yx}+Yf_{yy}+Zf_{yz} = 0,$$
$$Xf_{zx}+Yf_{zy}+Zf_{zz} = 0.$$

If we eliminate X, Y, Z from these equations, we obtain the condition

$$\frac{\partial(f_x, f_y, f_z)}{\partial(x,y,z)} = 0;$$

this is the equation of a curve of order $3(n-2)$ which we call the *Hessian* of C^n. It is the locus of points whose polar conics are degenerate.

On the other hand, if we multiply the equations by x, y, z and add, then, by Euler's Theorem on homogeneous functions, we obtain

$$Xf_x+Yf_y+Zf_z = 0.$$

Hence, if P lies on C^n, so that all its polars touch C^n at P, either (X, Y, Z) lies on the tangent at P, in which case P must be an inflexion, or f_x, f_y, f_z all vanish at P, in which case P is a multiple point of C^n. Hence:

The inflexions of C^n are those of the intersections of C^n with its Hessian which do not fall at multiple points of C^n.

Thus if C^n is non-singular, it has $3n(n-2)$ inflexions; but if C^n has multiple points, the behaviour of the Hessian at these is rather complicated.† In Ch. IV we succeed in by-passing this difficulty.

EXAMPLES

1. Prove that if P lies on the rth polar of Q, then Q lies on the $(n-r)$th polar of P.

2. Prove that from an ordinary k-fold point of C^n there can be drawn $m-2k$ tangents to touch the curve elsewhere.

3. Show that the curve $(x+y+z)^3 = 27xyz$ has a node at $(1, 1, 1)$, and find the equation of the nodal tangents.

4. A *cusp* is a node (double point) at which the nodal tangents are coincident. (A stricter definition will be given in Ch. III.) Show that the equation of a cubic curve with a cusp can be reduced to the form $y^2z+f_3(x, y) = 0$.

§ 2. SUB-ADJOINT CURVES. APPARENT GENUS OF A CURVE

2. Definition. A curve C' is said to be *sub-adjoint* to a given curve C if it has a point of multiplicity $k-1$ (at least) at every multiple point of order k of C. If all the multiple points of C are ordinary, C' is simply *adjoint*‡ to C.

It follows from (4) that the virtual freedom of sub-adjoints C^{n-3} to a curve C^n is $D-1$, where

$$D = \tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\sum k_i(k_i-1). \tag{6}$$

The number D so defined is called the *apparent genus* or *deficiency* of C^n; when the multiple points of C^n are all ordinary, it is identical with the genus p, to be defined later in Ch. III, § 3.3. We now prove

THEOREM IV. *For an irreducible curve, $D \geqslant 0$.*

Consider the sub-adjoints C^{n-1} to a given curve C^n. Plainly such curves exist; for the first polar of any point of the plane is of order $n-1$ and it satisfies the conditions of sub-adjointness. Also, by (4), they can be made to pass through a set of arbitrary points of C^n numbering at least

$$\tfrac{1}{2}(n-1)(n+2)-\tfrac{1}{2}\sum k_i(k_i-1) = D+2(n-1). \tag{7}$$

† At an ordinary k-fold point of C^n, the Hessian has in general a $(3k-4)$-fold point, and its nodal tangents are the k nodal tangents of C^n together with the $2k-4$ lines which form the Hessian group of those k lines. For this and other special properties cf. *Encyk. der math. Wiss.* III. C 4, 339–42.

‡ If C has multiple points which are not ordinary, then it may have implicit (infinitely near) multiple points as well as explicit multiple points (cf. Ch. III, § 3.1); and C' is *adjoint* to C if it has a $(k-1)$-fold point at every k-fold point, explicit or implicit, of C.

If C^n is irreducible, this number plainly cannot exceed the number of intersections, remote from the multiple points, of a sub-adjoint C^{n-1} with C^n. Now this number, by the rule given in § 1.2, cannot exceed

$$n(n-1)-\sum k_i(k_i-1) = 2D+2(n-1),$$

and it follows therefore that

$$D+2(n-1) \leqslant 2D+2(n-1),$$

i.e. $D \geqslant 0$.

COROLLARY. *An irreducible curve C^n cannot have more than $\frac{1}{2}(n-1)(n-2)$ double points.*

For if the only singularities of C^n are d double points, the theorem and equation (6) give $\frac{1}{2}(n-1)(n-2)-d \geqslant 0$.

EXAMPLES

1. An irreducible C^3 can have at most one node which may be a cusp. An irreducible quartic may have at most three nodes, some or all of which may be cusps: if the three nodes lie at the vertices of the triangle of reference, the equation of the curve is

$$ay^2z^2+bz^2x^2+cx^2y^2+2xyz(fx+gy+hz) = 0;$$

and if $f^2 = bc$, $g^2 = ca$, $h^2 = ab$, the three nodes are cusps.

2. The apparent genus D may be defined as *the virtual number of linearly independent sub-adjoints of order $n-3$ possessed by C^n.* Thus if a quartic has two nodes A, B, then the line AB is the only sub-adjoint of order **1**, so that $D = 1$.

2.1. The following table shows the virtual freedom r_m of the sub-adjoints of order m, for different values of m; and it gives also the maximum number i_m of their *free* intersections with C^n, where by free intersections we mean such as do not occur at the multiple points K_i of C^n.

m	r_m	i_m
$n-1$	$2(n-1)+D$	$2(n-1)+2D$
$n-2$	$n-2+D$	$n-2+2D$
$n-3$	$D-1$	$2(D-1)$
$n-4$	$D-n+1$	$2D-n+2$

It thus appears that

(1) when $D = 0$, there may be no sub-adjoints C^{n-3}, but the sub-adjoints C^{n-2} always exist for $n > 2$; moreover, if C^n is irreducible, their effective freedom is r_{n-2} since, if it exceeded this, we could make a curve C^{n-2} meet C^n in more than i_{n-2} assigned points;

(2) when $D = 1$, there may be one sub-adjoint C^{n-3} but no more, while the sub-adjoints C^{n-2} always exist;

(3) when $D = 2$, there is a pencil of sub-adjoints C^{n-3}, each of them meeting C^n in at most two free points.

§ 3. Rational curves

3. Sufficient condition for rationality. A curve is *rational* if its points can be placed in algebraic (1, 1) correspondence (or birational correspondence) with the points of a line. For plane curves we now establish the following *sufficient* condition for rationality.

THEOREM V. *If $D = 0$, then C^n is rational.*

To prove this, we consider sub-adjoints C^{n-2}, observing that, by § 2.1, if $D = 0$, these curves have freedom $n-2$ and meet C^n in $n-2$ free points. Those of them which pass through $n-3$ generically chosen fixed points of C^n will therefore form a *pencil*, given by an equation of the form

$$U+\lambda V = 0, \tag{8}$$

such that each curve of this pencil meets C^n in only one variable point. The curves of this pencil, and hence also the values of the parameter λ, are clearly in birational correspondence with the points of C; and since λ may be regarded as the coordinate of a variable point on a given line, it follows that C^n is rational.

It is of interest to note how the actual parametric representation of C^n would be obtained from the above. Let C^n have the equation

$$f(x,y,z) = 0, \tag{9}$$

and let

$$px+qy+rz = 0 \tag{10}$$

be an arbitrary line. The resultant of (8), (9), and (10) expresses the condition that the line (10) should pass through one of the points common to (8) and (9). By the theory of elimination, the resultant will be of degree n in λ, and of degree $n(n-2)$ in p, q, r. It must, then, be resoluble into the product of $n(n-2)$ factors, linear in p, q, r, but not all distinct; and since only one of the intersections of (8) and (9) is variable, only one of the factors will contain λ. Let this be

$$A(\lambda)p+B(\lambda)q+C(\lambda)r.$$

Identifying this with (10), we obtain

$$x:y:z = A(\lambda):B(\lambda):C(\lambda), \tag{11}$$

where A, B, C are polynomials of degree n in λ.

Finally, it should be added that the condition $D = 0$ is not in fact a necessary one for the rationality of a curve; a condition which is both necessary and sufficient will be obtained in a subsequent chapter in the form $p = 0$, where p is the genus of C^n.

3.1. *The rational normal curve of order n.* The definition of rationality implies in the first place that a rational curve C^n admits a parametric representation in the form

$$x:y:z = A(\lambda):B(\lambda):C(\lambda), \tag{11}$$

where A, B, C are polynomials; it implies also, however, that this representation can be so chosen that, to a generic point of the curve, there corresponds only one value† of the parameter λ. Supposing then that this condition is satisfied by (11), it is clear that A, B, C must be of degree $\leqslant n$, except possibly for a common factor which may be discarded; for the equation obtained by making the substitution (11) in the equation of an arbitrary line must be of degree n in λ. Hence:

The parametric equations of any rational C^n can be written in the form (11), *where A, B, C are polynomials of degree n in λ.*

We now prove

THEOREM VI. *Any rational C^n in the plane is a projection of the projectively unique rational normal curve Γ^n of order n in S_n.*

We assume that C^n is represented parametrically in the simple form just explained; only we now write ϕ_0, ϕ_1, ϕ_2 for A, B, C respectively. Let us choose, then, as is always possible, $n-2$ further polynomials $\phi_3, \phi_4,..., \phi_n$, also of degree n, such that all the polynomials $\phi_0, \phi_1,..., \phi_n$ are linearly independent. The equations

$$x_0:x_1:...:x_n = \phi_0:\phi_1:...:\phi_n \tag{12}$$

define a curve Γ^n lying in space S_n but not in any space of lower dimension (whence, by Ch. I, § 5.1, Γ^n is rational). The curve C^n can

† The significance of this condition is sufficiently illustrated by the parametric equations $x:y:z = \lambda^4:\lambda^2:1$, in which each point of the conic $xz = y^2$ corresponds to two values $\pm\lambda$ of the parameter.

be obtained from Γ^n by projecting the latter from the S_{n-3} whose equations are $x_0 = x_1 = x_2 = 0$ on to the plane

$$x_3 = x_4 = ... = x_n = 0,$$

and by identifying x_0, x_1, x_2 with x, y, z. Also the projection is a proper one, since the points of both C^n and Γ^n are in birational correspondence with the values of the parameter λ.

To show that the curve Γ^n is projectively unique, we show that its equations can always be reduced, by a change of the frame of reference in S_n, to the fixed canonical form

$$X_0 : X_1 : ... : X_n = \lambda^n : \lambda^{n-1} : ... : \lambda : 1. \tag{13}$$

We solve the equations $\rho x_i = \phi_i$ $(i = 0, 1,..., n)$, for the ratios

$$\lambda^n : \lambda^{n-1} : ... : \lambda : 1 : \rho,$$

noting that the solution is unique, with $\rho \neq 0$, since the coefficient-matrix of the ϕ_i is non-singular; and this yields immediately equations of the form (13), in which the X_i are $n+1$ linearly independent linear expressions in the x_i. By taking these latter to be proportional to new coordinates in a different system of reference, the result stated follows at once.

The curve Γ^n whose equations, in canonical form, are (13), is called the *rational normal curve of order n in* S_n.

EXAMPLES

1. Prove directly, from the parametric representation (11), that the *class* of the general rational C^n is $2(n-1)$, and that the number of its *inflexions* is $3(n-2)$.

2. For the curve $zu_{n-1}(x,y)+u_n(x,y) = 0$, the sub-adjoints C^{n-2} are all reducible, consisting each of $n-2$ lines of the pencil $y = \lambda x$. The parametric representation is obtained by putting $y = \lambda x$ in the equation.

3. Show that any nodal cubic can be reduced to the form $x^3+y^3 = axyz$, and hence find a parametric representation.

4. The quartic $y^2z^2+z^2x^2 = 2x^2y^2$ has nodes at the vertices of the triangle of reference, so that, having $D = 0$, it is rational. To obtain its parametric representation by the method of § 3, we use the pencil of conics

$$yz-xy = \lambda(zx-xy),$$

whose base points are the three nodes and the fixed point $(1, 1, 1)$ of the curve. Show that the parametric equations of the quartic are

$$x:y:z = (\lambda^2+1)(1+2\lambda-\lambda^2):(\lambda^2+1)(\lambda^2+2\lambda-1):\lambda^4-6\lambda^2+1.$$

3.2. *Nodes of a rational C^n.* We have seen in § 2 that a curve C^n cannot have more than $\frac{1}{2}(n-1)(n-2)$ nodes; we now show that this limit is attained for the rational curve C^n given by

$$x:y:z = A(\lambda):B(\lambda):C(\lambda), \tag{14}$$

when A, B, C are three general polynomials of degree n in λ.

A node of the curve defined by the above equations is a point to which there correspond, in general, two distinct values, λ and μ say, of the parameter instead of only one. Such pairs of values must satisfy the relations

$$\frac{A(\lambda)}{C(\lambda)} = \frac{A(\mu)}{C(\mu)}, \qquad \frac{B(\lambda)}{C(\lambda)} = \frac{B(\mu)}{C(\mu)} \quad (\lambda \neq \mu), \tag{15}$$

which have, in general, a finite number of solutions.

We assume in the first place, without incurring thereby any loss of generality, that the particular line $z = 0$ meets the curve in n distinct points: the equation $C(\lambda) = 0$ has then n distinct roots, $\gamma_1, \gamma_2,..., \gamma_n$ say. Resolving $A(\lambda)/C(\lambda)$ and $B(\lambda)/C(\lambda)$ into partial fractions, we obtain

$$\frac{A(\lambda)}{C(\lambda)} = A_0 + \sum_{r=1}^{n} \frac{A_r}{\lambda-\gamma_r}, \qquad \frac{B(\lambda)}{C(\lambda)} = B_0 + \sum_{r=1}^{n} \frac{B_r}{\lambda-\gamma_r};$$

so that (15) are equivalent to

$$\sum \frac{A_r}{\lambda-\gamma_r} = \sum \frac{A_r}{\mu-\gamma_r}, \qquad \sum \frac{B_r}{\lambda-\gamma_r} = \sum \frac{B_r}{\mu-\gamma_r}, \tag{16}$$

and hence to two equations for $\lambda\mu$ and $\lambda+\mu$, namely,

$$\sum \frac{A_r}{\lambda\mu-\gamma_r(\lambda+\mu)+\gamma_r^2} = 0, \qquad \sum \frac{B_r}{\lambda\mu-\gamma_r(\lambda+\mu)+\gamma_r^2} = 0. \tag{17}$$

There are $\frac{1}{2}n(n-1)$ solutions of these equations, with $\lambda = \gamma_r$, $\mu = \gamma_s$ $(r \neq s)$, which are to be rejected. To find the others, we take $\lambda\mu$ and $\lambda+\mu$ to be coordinates X and Y in the plane, so that the equations (17) represent two curves of order $n-1$ intersecting in $(n-1)^2$ distinct points. Of these, $\frac{1}{2}n(n-1)$ correspond to the rejected solutions, so that the number of required solutions is

$$(n-1)^2 - \tfrac{1}{2}n(n-1) = \tfrac{1}{2}(n-1)(n-2).$$

EXAMPLES

1. Find the parameter pairs corresponding to the nodes of the quartic in Ex. 4 of the previous set (p. 34).

2. Show that, for the curve given by (14) to have a node at the point $(0, 0, 1)$, the functions A, B must have the special forms

$$A(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)A_1, \qquad B(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)B_1;$$

and that, if $\lambda_1 = \lambda_2$, the node is a cusp.

3. Show that the parametric equations of a quartic having cusps at the vertices of the triangle of reference and passing through (1, 1, 1) are

$$x:y:z = (\lambda-b)^2(\lambda-c)^2:(\lambda-c)^2(\lambda-a)^2:(\lambda-a)^2(\lambda-b)^2.$$

§ 4. Curves for which $D = 1$

4. Transformation to a plane cubic. In the preceding section we saw that every curve† for which $D = 0$ could be birationally transformed into a straight line. In the same way, we now discuss curves having $D > 0$, in order to find out what are the curves of lowest degree into which they may be transformed; and we begin with those for which $D = 1$.

If a C^n has unit deficiency, it follows from § 2.1 that its sub-adjoints C^{n-2} have virtual freedom $n-1$, and that they meet C^n in sets of at most n free points. If these sub-adjoints have freedom greater than $n-1$, or if they meet C^n in fewer than n free points, then the reasoning employed in § 3 shows that C^n is rational. Excluding this case, we suppose that the C^{n-2} in question have effective freedom $n-1$ and meet C^n in sets of n free points.

We choose now $n-3$ fixed points of general position on C^n, and we consider those of the sub-adjoints C^{n-2} which pass through them. In the first place, the freedom of these curves is 2; so that the system formed by them is a *net*, whose equation is of the form

$$\lambda U+\mu V+\nu W = 0. \tag{18}$$

And in the second place, the curves of this net meet C^n in sets of only three variable points, all their remaining intersections with C^n being at its multiple points K_i and at the fixed points just imposed.

Consider now another plane, in which the coordinates of a general point are x', y', z'; and let us denote by Γ the curve described in this plane by the point $P'(x',y',z')$ whose coordinates are proportional to the functions U, V, W of a variable point $P(x,y,z)$ of C^n. The equations

$$\frac{x'}{U(x,y,z)} = \frac{y'}{V(x,y,z)} = \frac{z'}{W(x,y,z)}, \tag{19}$$

together with the equation $f(x,y,z) = 0$ of C^n, may evidently be regarded as a parametric representation of Γ, employing the ratios $x:y:z$ as two related parameters, instead of one independent parameter. Since U, V, W are polynomials, any given point P of C^n

† We suppose here, and in the sequel, that the curves we consider are irreducible.

has a unique corresponding point P' on Γ; we wish now to show that, conversely, a generic point P' of Γ corresponds to only one point of C^n.

To show this, we suppose that, if possible, at least two points $P(x,y,z)$ and $P^*(x^*,y^*,z^*)$ of C^n always correspond to the same point P' of Γ. In this case, clearly

$$\frac{U(x,y,z)}{U(x^*,y^*,z^*)} = \frac{V(x,y,z)}{V(x^*,y^*,z^*)} = \frac{W(x,y,z)}{W(x^*,y^*,z^*)},$$

so that those curves of the net $\lambda U+\mu V+\nu W = 0$ which pass through any point P of C^n must all pass in consequence through a further point P^* of C^n. Such a situation will certainly not occur *in general* in the type of transformation here employed. In the present instance it is impossible in view of the fact that a unique curve of the net can be drawn through two arbitrary points P, Q of C^n, and that this curve would then have to pass through two further points P^*, Q^* of C^n, whereas the generic curve of the net has actually only three variable intersections with C^n.

Thus Γ is in birational correspondence with C^n.

The order of Γ is the number of its points which lie on an arbitrary line $\lambda x'+\mu y'+\nu z' = 0$; and, by equations (19), this is the number of variable points in which C^n is met by a generic curve of the net given by (18), which is three. Hence Γ is a cubic curve, general or nodal; and this proves

THEOREM VII. *Any curve for which $D = 1$ is birationally transformable into a plane cubic.*

It can be shown (cf. Ex. 9 below) that the coordinates of a variable point of a plane non-singular cubic are expressible rationally in terms of elliptic functions of a single parameter; and it follows therefore that also the coordinates of a variable point of any curve transformable birationally into such a curve are likewise expressible rationally in terms of elliptic functions of a parameter. For this reason all such curves are called *elliptic*.

Here again it is worth while recalling that $D = 1$ is only a sufficient condition for a curve to be either elliptic or rational.

4.1. *Elliptic curves in higher space.* There is no immediate analogue for elliptic curves of the derivation of all rational C^n by projection from a unique curve in $[n]$. This is due primarily to the fact that two plane non-singular cubics are not in general birationally transformable into each other. For them to be so, a certain *modulus*, defined in Ex. 15 below, must have the same value for both curves; so that whereas from the invariantive point of view all rational curves are identical, from the same point of view there exists a single-parameter family of elliptic curves.

We may, however, note the following fairly obvious result.

THEOREM VIII. *Any curve of order n in $[n-1]$ (and not contained in any space of lower dimension) is elliptic or rational.*

For if Γ^n is such a curve, its projection from a point of itself is a curve Γ^{n-1} in $[n-2]$; similarly the projection of the latter from a point of itself is a Γ^{n-2} in $[n-3]$; and so on, until we obtain finally a plane cubic Γ^3.

4.2. *Reduction of C^n for the values 2, 3, 4, of D.* By methods very similar to that employed in the preceding case, we can reduce curves for which the deficiency has other low values besides 0 and 1.

If $D = 2$, the sub-adjoints C^{n-2} of C^n have virtual freedom n and meet C^n in sets of at most $n+2$ free points; and here again we may suppose these numbers to be effective, since, in the contrary case, the previous methods show that C^n is elliptic or rational. Those of these curves which pass through $n-2$ fixed points of C^n form, by § 2.1, a net, $\lambda U+\mu V+\nu W = 0$, whose curves meet C^n in sets of four free points, and the equations

$$x':y':z' = U:V:W$$

transform C^n as before into a curve Γ which, for generic choice of the fixed points on C^n, is in birational correspondence with C^n, and which is now of order 4. Thus these curves are transformable into plane quartics.

(Actually, though it does not emerge easily from the above method, the plane quartics so arising are *uni-nodal*, being themselves therefore of deficiency 2, as one would expect.)

If $D = 3$, C^n has, by § 2.1, a net of sub-adjoints C^{n-3} which meet it in sets of 4 free points. Using this, in the preceding manner, as the basis of a transformation of C^n, we find that in this case also, if C^n is not rational or elliptic, it is transformable into a plane quartic.

(In this case the plane quartics which arise are normally non-singular, being themselves of deficiency 3; they may, however, have one, two, or three nodes, the last two cases occurring if C^n is elliptic or rational.)

If $D = 4$, C^n has sub-adjoints C^{n-3} of freedom 4, meeting the curve in sets of 6 free points. They form a system—called a *web*—whose equation is of the form

$$\lambda U+\mu V+\nu W+\rho T = 0.$$

If we apply to C^n the transformation

$$x':y':z':t' = U:V:W:T,$$

we obtain, as birational transform of C^n, a curve of order 6 in S_3.

Collecting these results, we have

THEOREM IX. *Curves of deficiencies 2 and 3 can be birationally transformed into curves of order 4 at most, and those of deficiency 4 into sextic curves in ordinary space.*

NOTES AND EXAMPLES ON CHAPTER II

1. *The standard rational cubic.* The reduced equation $x^3+y^3 = axyz$ of a nodal cubic admits the parametric representation

$$x:y:z = at:at^2:1+t^3.$$

Establish the following deductions:

(i) If t_1, t_2, t_3 are (the parameters of) three collinear points of the curve, then $t_1 t_2 t_3 = -1$. Deduce that from a given point of the curve there can be drawn two tangents to touch the curve elsewhere, and that the curve has three inflexions which are collinear.

(ii) If the curve is met by a conic in the points t_i $(i = 1,..., 6)$, then in general $t_1 t_2 ... t_6 = 1$. Deduce that, if the conic touches the curve at t_1, t_2, t_3, then $t_1 t_2 t_3 = 1$; also that there exist three proper *sextactic* conics (having 6-point contact with the curve).

(iii) If t_1, t_2, t_3 are the roots of the equation $t^3+At^2+Bt-1 = 0$, prove that the tritangent conic whose points of contact are t_1, t_2, t_3 has equation

$$a^2z^2+2aAyz+(A^2+2B)y^2+2(AB-2)xy+(B^2-2A)x^2 = 0.$$

2. *General parametric cubic.* For the general rational cubic whose equations are

$$x:y:z = a_0 t^3+a_1 t^2+a_2 t+a_3:b_0 t^3+...+b_3:c_0 t^3+...+c_3,$$

prove that the condition for collinear points t_1, t_2, t_3 is

$$p_0+p_1\sum t_i+p_2\sum t_i t_j+p_3 t_1 t_2 t_3 = 0,$$

where p_0, p_1, p_2, p_3 are the third-order determinants extracted from the matrix whose rows are $(a_i b_i c_i)$ for $i = 0, 1, 2, 3$. Deduce an equation giving the parameters of the three inflexions and verify that these are collinear.

Show also that the node of the cubic corresponds to the two values of the parameter given by

$$(p_0+p_1 t)(p_2+p_3 t) = (p_1+p_2 t)^2.$$

(If t, t^* are the two parameters required, then the points t, t^* are collinear with an arbitrary point of the curve.)

3. Show that the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}} = z^{\frac{2}{3}}$ may be represented parametrically by the equations

$$x:y:z = (1-t^2)^3:8t^3:(1+t^2)^3,$$

and find its multiple points.

4. Obtain parametric representations for the curves whose equations in polar coordinates are

(i) $r = a(1+\cos\theta)$, (ii) $r^{\frac{1}{3}}\cos\frac{1}{3}\theta = a^{\frac{1}{3}}$, (iii) $r^2 = a^2\cos 2\theta$.

Prove that, in case (iii), four tangents can be drawn to the curve from a point of itself and that their points of contact are collinear.

(*Solutions.* The curve (i) is the inverse in the circle $x^2+y^2 = a^2$ of the parabola $y^2+a(2x-a) = 0$. The curve (ii) is the nodal cubic

$$27ay^2 = (8a+x)^2(a-x);$$

put $y = t(8a+x)$. The curve (iii) is the quartic $(x^2+y^2)^2 = a^2(x^2-y^2)$, nodal at the origin and at the circular points; a suitable rationalizing pencil of sub-adjoint conics is $x^2+y^2 = ta(x+y)$, giving

$$x:y:z = at(t^2+1):at(t^2-1):t^4+1;$$

and the four points t_1, t_2, t_3, t_4 are collinear if $t_1 t_2 t_3 t_4 = 1$ and $\sum t_i t_j = 0$.)

5. Show that the curve $(x^2+y^2-zx)^2 = z^2(x^2+y^2)$ has cusps at $(0,0,1)$ and $(1,\pm i,0)$. By means of the pencil of conics $x^2+y^2 = tyz$, obtain a parametric representation of the curve.

6. The quartic $y^2z^2 = x^4-x^3z+x^2z^2$ has an ordinary node and a tacnode (point of self-contact); and the conics which touch it at the latter and meet it in the node and in the point $(1,1,1)$ form the pencil $x(x-z) = t(x^2-yz)$. Show that the curve is met residually by the generic conic of this pencil in the point for which

$$x:y:z = (t^2-1)(2t-1):(t^3+1)(t-1):(2t-1)^2.$$

Verify that the pairs of values of the parameter corresponding to the node and tacnode are $(1,-1)$ and $(\frac{1}{2},\infty)$ respectively.

7. If P is any point of a curve C, prove that the first polar (and therefore, by induction, all the polars) of P with respect to C touch C at P.

If C is the special type of trinodal quartic whose equation, referred to the nodal triangle, is of the form $a/x^2+b/y^2+c/z^2 = 0$, prove that the polar cubic with respect to C of any point on C breaks up into a conic and a line.

(The curve (iii) in Ex. 4 reduces to the above form by writing $X = x+iy$, $Y = x-iy$; hence its geometrical property there noted.)

8. *Rational normal* C^n. Prove that the rational normal C^n in $[n]$ may be defined as the locus of the point of intersection of corresponding primes of n related pencils; also that the bases $[n-2]$ of the pencils are each $(n-1)$-secant to C^n.

9. *General plane cubic; parametric representation.* In order to obtain the simplest parametric representation of the general plane cubic C^3, we must first reduce its equation to one of the simplest possible forms.

We choose our triangle of reference so that the vertex $(0, 1, 0)$ is one of the points of inflexion of C^3—by § 1.5, the curve has nine of these—and such that the line $z = 0$ is the corresponding inflexional tangent. This reduces the equation of C^3 to the form

$$zy^2+2(px+qz)yz = f_3(x,z),$$

where f_3 is a cubic polynomial, homogeneous in x and z. Writing

$$y+px+qz = Y,$$

we obtain the modified form

$$zY^2 = u_3(x,z),$$

which in turn reduces, after a further simple alteration, to the form

$$Y^2Z = 4X^3-g_2XZ^2-g_3Z^3 \quad \text{or} \quad (Y/Z)^2 = 4(X/Z)^3-g_2(X/Z)-g_3,$$

where g_2 and g_3 are constants. This is the *first canonical form* of the equation of the general plane cubic.

The above gives a parametric representation of C^3, in terms of Weierstrassian elliptic functions, by putting

$$X:Y:Z = \wp u:\wp' u:1.$$

We denote the periods of $\wp u$ by 2ω and $2\omega'$.

10. *Intersections of C^3 with a C^m.* Many geometrical properties of the cubic are easily deducible from the above representation. Thus, consider the intersections of C^3 with a curve C^m whose equation is $F(X,Y,Z) = 0$; the values $u_1, u_2,..., u_{3m}$ of the parameter u at these intersections are the zeros of the function $F(\wp u, \wp' u, 1)$. Since F is a polynomial, its only poles are at the points $u \equiv 0 \pmod{2\omega, 2\omega'}$, so that the sum of $u_1, u_2,...$ must likewise be congruent to zero. Thus

$$u_1+u_2+...+u_{3m} \equiv 0 \pmod{2\omega, 2\omega'}.$$

Hence: *The last point of intersection of a C^m with a plane non-singular cubic is uniquely determined by the remaining $3m-1$.*

11. *Geometrical properties of C^3.* From the preceding result we may deduce the following properties of C^3:

(i) From a generic point of C^3 there can be drawn four tangents to touch the curve elsewhere.

(ii) The tangents to C^3 at three collinear points meet the curve again in three collinear points.

(iii) If a conic passes through four fixed points of C^3, the chord joining the two remaining intersections meets the curve in a fixed point. (A similar result for the intersections of C^3 with curves of any order should be established.)

(iv) If, of the $3(m+n)$ intersections of C^3 with a curve C^{m+n}, $3m$ lie on a curve C^m, the remainder lie on a curve C^n.

(v) There exist three systems of proper conics touching C^3 at three distinct points, and the six points of contact of any two conics of the same system lie on a conic.

(vi) The cubic has nine inflexions which lie by threes on twelve lines; the points of contact of the three tangents that can be drawn to C^3 from any inflexion are collinear.

(vii) There exist 27 proper conics having six-point (sextactic) contact with C^3, and their points of contact are those of the tangents drawn from the inflexions.

12. *Second canonical equation of C^3.* In accordance with (vi) above, we may choose a triangle of reference whose sides YZ, ZX, XY contain by threes all nine inflexions of C^3. Remembering always that any join of a pair of inflexions contains a third, we may select one line meeting YZ, ZX, XY in inflexions and take it to have equation $x+y+z = 0$; and it can then be shown that the remaining six inflexions lie at the intersections of YZ, ZX, XY with the lines

$$x+\epsilon y+\epsilon^2 z = 0, \qquad x+\epsilon^2 y+\epsilon z = 0,$$

where ϵ is an imaginary cube root of unity. The equation of C^3 is therefore of the form

$$(x+y+z)(x+\epsilon y+\epsilon^2 z)(x+\epsilon^2 y+\epsilon z) = kxyz,$$

or, say,

$$x^3+y^3+z^3 = 3mxyz.$$

This last is the *second canonical form* of the equation of a general plane cubic.

Deduce that the Hessian of C^3 is another cubic H^3 with the same nine points of inflexion; that the pencil of cubics determined by C^3 and H^3 contains the Hessian of each of its curves; and that every cubic is the Hessian of three others.

13. *Geometry of a net of conics.* By inspecting the three partial derivatives of a general cubic polynomial in x, y, z, we find that three quadratic polynomials $(a_i, b_i, c_i, f_i, g_i, h_i)(x, y, z)^2$ $(i = 1, 2, 3)$, are the derivatives of a cubic polynomial if and only if the following eight conditions are satisfied: $b_1 = h_2$, $c_1 = g_3$, $a_2 = h_1$, $c_2 = f_3$, $a_3 = g_1$, $b_3 = f_2$, $f_1 = g_2 = h_3$. Given any net of conics having equation $pU+qV+rW = 0$, we try to choose three combinations $S_i = p_i U+q_i V+r_i W$ $(i = 1, 2, 3)$, to satisfy all eight of the above relations; the unknowns p_i, q_i, r_i are thereby subjected to eight linear conditions which are sufficient in general to determine their ratios uniquely. We conclude therefore:

The general net of conics is the net of polar conics of a unique cubic curve.

If we take this cubic to be $x^3+y^3+z^3 = 3mxyz$, its net of polar conics is $p(x^2-myz)+q(y^2-mzx)+r(z^2-mxy) = 0$, and this is therefore the *canonical equation of a general net of conics*. The *Jacobian* curve of the net (cf. Ch. VI) is the Hessian of the above cubic.

14. *Nine associated points.* Cubic curves which pass through eight given points of the plane form a pencil whose equation is of the form $U+kV = 0$; and they therefore pass through a ninth common point, namely, that which unites with the given eight to form the complete set of intersections of U and V. For this reason the nine intersections of a pair of cubic curves are said to form a set of *nine associated points*, being such that all cubics through any eight of them pass through the ninth.

Most of the properties given in Ex. 11 can be proved, without the aid of elliptic functions, by applying the above property to various special and limiting groups of nine associated points.

15. *Modulus of a plane cubic.* In virtue of the fundamental theorem of algebra, every algebraic function† which is not a constant must take every assigned value at least once; and, conversely, if it can be shown that a given algebraic function can never take a given particular value, then it will follow that this function is a constant. On this is based a proof (due to Zeuthen‡) of the following theorem:

The four tangents which can be drawn to a general plane cubic from a point of itself form a pencil of constant cross-ratio. (Salmon's Theorem.)

For the four tangents drawn from a variable point P of C^3 are always distinct from each other, even in the limiting case when P is an inflexion. Hence their cross-ratio can never take any one of the three values 0, 1, ∞, which characterize coincidence. It is therefore a constant.

In the usual notation, if e_1, e_2, e_3 are the zeros of the cubic $4x^3-g_2x-g_3$, it may be shown that the cross-ratio is $(e_3-e_2)/(e_1-e_2)$.

This is called the *modulus* (in its irrational form) of the curve.

16. Show that the curve whose parametric equations are

$$x:y:z = \wp^2u:\wp'u:1$$

is a binodal quartic; while the curve given by

$$x:y:z = \wp u:\wp'u:\wp^2u$$

is a tacnodal quartic. Show also that these curves can be transformed into plane cubics.

† If $F(w, z)$ is a polynomial in z and w, then the equation $F(w, z) = 0$ defines w as a (many valued) *algebraic function* of z. The definition of an algebraic function of several variables is on similar lines.

‡ *Abzählenden Methoden der Geometrie*, pp. 83–5.

BOOKS RECOMMENDED FOR FURTHER READING

Coolidge, *Algebraic plane curves*, Book I.

Bertini, *Complementi di geometria proiettiva*, § 5.

CHAPTER III

THE QUADRATIC TRANSFORMATION

§ 1. Rational transformations

1. The reader will be familiar already with the concept of a plane collineation (Ch. I, § 2), a transformation in which to any point P of the plane there is made to correspond a point Q whose coordinates are proportional to assigned linear homogeneous functions of those of P. Such transformations leave the projective properties, and in particular the order and singularities, of curves unaltered. In advancing beyond such properties, our first step is to introduce other transformations of the plane into itself, which establish relations between curves of different orders and possessing different sets of singularities.

The most general *rational transformation* of the plane, which makes correspond to a generic point $P(x,y,z)$ a unique point $Q(x',y',z')$, is defined by equations of the form

$$\rho x' = \phi_1(x,y,z), \quad \rho y' = \phi_2(x,y,z), \quad \rho z' = \phi_3(x,y,z), \tag{1}$$

where ϕ_1, ϕ_2, ϕ_3 are homogeneous polynomials of degree n in x, y, z. We assume that ϕ_1, ϕ_2, ϕ_3 are linearly independent, and we denote the transformation by $\mathbf{T}$. Thus we may write

$$Q = \mathbf{T}(P).$$

Consider now the net of curves Φ defined by the equation

$$\lambda\phi_1+\mu\phi_2+\nu\phi_3 = 0, \tag{2}$$

where λ, μ, ν are arbitrary parameters. As P describes any given curve Φ, the corresponding point Q evidently describes a line. Thus the curves of the net are correlated by the transformation with the lines of the plane; and conversely, given any net of curves such as Φ, a linear representation of the curves of this net on the lines of the plane is equivalent to a rational transformation of the plane.

The curves Φ may have *base points* O_i common to them all. Since each such point is a common zero of ϕ_1, ϕ_2, ϕ_3, the equations (1) to determine its corresponding point $\mathbf{T}(O_i)$ are illusory; and conversely, any point, termed a *singular point*, which renders equations (1) illusory is a base point of Φ. Hence:

THEOREM I. *The singular points of any rational transformation are the base points (if any) of the associated net of curves* Φ.

Any two general curves Φ_1, Φ_2 of Φ define a pencil of curves $\Phi_1+\lambda\Phi_2$ of the net. Let us denote by N the number of *free* intersections of Φ_1, Φ_2, not occurring at the base points O_i; and let us denote by G^N the group of such intersections, $P_1,\ldots, P_N$ say. The number N we shall term the *grade* of Φ.

To curves of the arbitrary pencil $\Phi_1+\lambda\Phi_2$ of Φ, there correspond, by the transformation, lines of a pencil $L_1+\lambda L_2$; and if the vertex of the latter pencil is Q, then, clearly, every point P_i of G^N corresponds to the same point Q. Conversely, it is easy to see that if any two points of the plane have the same transform Q, then they belong to the same free intersection-set G^N of some pencil in Φ. Thus:

THEOREM II. *In any rational transformation of the plane into itself, whose generating net Φ is of grade N, an arbitrary point Q is the transform of N points $P_1,\ldots, P_N$, which together form the group of free base points (excluding those of Φ) of a pencil of curves of Φ.*

Symbolically, if $\mathbf{T}^{-1}$ is the reverse of the transformation $\mathbf{T}$, we write

$$\mathbf{T}^{-1}(Q) = G^N.$$

1.1. ***Cremona transformations.*** The general rational transformation of the plane is, as we have just seen, an $(N, 1)$ correspondence between the points P and Q; and this means that, when the ratios of x', y', z' are given, the equations (1) have in general N distinct solutions for the ratios of x, y, z. If $N = 1$, however, so that these equations have only one solution, the ratios $x:y:z$ will be rational functions of $x':y':z'$. In this case the equations of the reverse transformation will be of the form

$$\rho x = \psi_1(x',y',z'), \quad \rho y = \psi_2(x',y',z'), \quad \rho z = \psi_3(x',y',z'), \qquad (3)$$

where ψ_1, ψ_2, ψ_3 are polynomials of degree n', say.

DEFINITION. A Cremona transformation is a rational transformation whose reverse is also rational.

Thus if $\mathbf{T}$ is a Cremona transformation, so is $\mathbf{T}^{-1}$.

DEFINITION. A homaloidal net of curves in the plane is one whose grade is unity.

Equations (1) define a Cremona transformation if and only if the net given by (2) is homaloidal. On the other hand, from any given homaloidal net we can derive many Cremona transformations;

for if $\bar{\phi}_1$, $\bar{\phi}_2$, $\bar{\phi}_3$ are any three independent linear combinations of ϕ_1, ϕ_2, ϕ_3, the net given by (2) can also be expressed in the form

$$p\bar{\phi}_1+q\bar{\phi}_2+r\bar{\phi}_3 = 0,$$

and the transformation defined by

$$\rho x'' = \bar{\phi}_1, \qquad \rho y'' = \bar{\phi}_2, \qquad \rho z'' = \bar{\phi}_3$$

is based on the same net. It is easily seen, however, that x'', y'', z'' are fixed linear functions of x', y', z'; and we may express this result by saying that the second transformation is obtained by superimposing a collineation on the former. We enunciate therefore

THEOREM III. *With any Cremona transformation there is associated a homaloidal net of curves* Φ; *and conversely, any homaloidal net of curves generates an infinity of Cremona transformations, any one of which is the product of any other by a plane collineation.*

A collineation is itself, of course, the simplest kind of Cremona transformation whose homaloidal net is composed of the lines of the plane; it is, however, of a type which we agree to regard in this connexion as trivial. Two Cremona transformations will be considered as being essentially the same if one of them is the product of the other by a collineation.

DEFINITION. The order of a Cremona transformation is the order n of the curves of its generating homaloidal net Φ.

Thus also, the order n' of the curves of the reverse homaloidal net Ψ is the order of the reverse transformation $\mathbf{T}^{-1}$. If the number of intersections of two curves A and B is denoted by AB, and if L and L' are lines, then

$$n = L\Phi = \mathbf{T}(L)\mathbf{T}(\Phi) = \Psi L' = n'.$$

This proves

THEOREM IV. *A Cremona transformation and its reverse transformation have the same order.*

1.2. *Quadratic transformations.* A Cremona transformation of order 2 must be based on a homaloidal net of conics; and such a net must consist evidently of conics through three fixed points, for these form a system of freedom 2 and grade 1. Let the three fixed points be distinct, and let us take them as points of reference X, Y, Z. The equation of the net then takes the form

$$\Phi = \lambda yz+\mu zx+\nu xy = 0,$$

and the simplest Cremona transformation derivable from this net is that whose equations are

$$\rho x' = yz, \qquad \rho y' = zx, \qquad \rho z' = xy. \tag{4}$$

These equations may also be written in the form

$$x':y':z' = 1/x:1/y:1/z; \tag{4'}$$

and hence this transformation—which is evidently self-inverse—is called the *reciprocal transformation*. If we denote it by $\mathbf{T}_0$, and if we denote any other transformation based on Φ by $\mathbf{T}$, then $\mathbf{T} = \mathbf{A}\mathbf{T}_0$, where $\mathbf{A}$ is a collineation.

DEFINITION. Any transformation of the plane whose equations can be reduced by suitable choice of the reference system to the form (4) will be called a *standard quadratic transformation* (s.q.t.).

Such, in particular, is any transformation of the form

$$x':y':z' = ayz:bzx:cxy;$$

for this reduces to the reciprocal form when the coordinates are multiplied by $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ respectively.

For the s.q.t., the reverse homaloidal net Ψ is identical with Φ, and their three base points A, B, C are called the *base points* or *fundamental points* of the transformation. The transformation carries conics through A, B, C into lines, and carries lines into conics through A, B, C.

EXAMPLES

1. Prove that the transformation which makes correspond to any point P the intersection of its polars with respect to two fixed conics is, in general, a standard quadratic transformation.

2. A quadratic transformation which transforms conics through A, B, C into lines, and which transforms lines into conics through A, B, C, is either a standard quadratic transformation with A, B, C as base points, or it is the product of such a transformation by a collineation which interchanges two, or all three, of A, B, C.

(Hint: If $\mathbf{A}$ is either of the collineations

$$x':y':z' = x:z:y \quad \text{or} \quad x':y':z' = y:z:x,$$

and if $\mathbf{T}$ is a s.q.t. having its base points at the vertices A, B, C of the triangle of reference, then the transformation $\mathbf{AT}$ has the property in question.)

3. Prove that if I, J are the circular points at infinity, then ordinary inversion with respect to a point O is the product of a s.q.t. whose base points are O, I, J with a collineation which leaves O invariant and interchanges I and J.

§ 2. Properties of the standard quadratic transformation

2. For a more detailed discussion of the s.q.t., we take its equations in the form (4), the reverse equations then being

$$\rho x = y'z', \qquad \rho y = z'x', \qquad \rho z = x'y'. \tag{5}$$

Its base points are X, Y, Z, and conics through these will be referred to as Φ-conics.

As a point P describes an irreducible curve C, its corresponding point Q in general describes an irreducible curve D which we call the *proper transform* of C, writing $D = \mathbf{T}(C)$.

Exceptionally, however, we note that the transform of any point on a line joining two base points, Y, Z say, is fixed at the third base point X, unless indeed the point coincides with one of Y, Z which have no transforms. The lines YZ, ZX, XY transform therefore into the points X, Y, Z respectively; and for this reason they are called the *fundamental lines* of the transformation.

A line through X has an equation of the form $my+nz = 0$; using (5) and rejecting the factor $x' = 0$, we find that its proper transform has the equation $mz'+ny' = 0$. Hence:

A line through a base point (other than a fundamental line) transforms to another line through the same base point.

2.1. *Neighbourhoods of base points.* A base point has itself no transform; but we may consider the behaviour of $Q = \mathbf{T}(P)$ as P approaches the base point X, for example, along a path $\mathscr{L}$. The point Q is the vertex of a pencil of lines, $L_1+\lambda L_2$, which corresponds to a pencil of conics, $\Phi_1+\lambda\Phi_2$, whose base points are X, Y, Z, and P. As P tends to X along $\mathscr{L}$, the pencil $\Phi_1+\lambda\Phi_2$ tends to the pencil of conics passing through X, Y, Z and touching the tangent, t say, to $\mathscr{L}$ at X. Q tends simultaneously, therefore, to the vertex Q_t of the pencil of lines corresponding to this limiting pencil of conics.

Now the tangent at X to the conic whose equation is

$$\lambda yz+\mu zx+\nu xy = 0$$

is $\mu z+\nu y = 0$; and if we fix the ratio $\mu:\nu$, the corresponding line $\lambda x'+\mu y'+\nu z' = 0$ passes through the point $(0, \nu, -\mu)$. Hence, as P approaches X in an assigned direction t, its transform Q approaches a definite point Q of YZ. In particular a pair of corresponding lines through X meet YZ in points corresponding inversely to the directions of the lines at X. In conclusion then:

To every direction t at X there corresponds a definite point O of YZ, such that if a curve C has a branch issuing from X in direction t, then the proper transform C' of C passes through O. Conversely, also, if O is any point of YZ, other than Y or Z, *then any curve passing through O transforms to a curve which has a branch issuing from X in the direction corresponding to O.*

2.2. *Transforms of curves.* For convenience we introduce the following notation:

(*a*) We use the symbol $C^n(X^\alpha, Y^\beta, Z^\gamma)$ to denote a curve of order n, having points of multiplicities α, β, γ respectively at X, Y, Z.

(*b*) We use the symbol (CD) to denote the set of points, *other than* X, Y, Z, in which two curves C and D intersect.

(*c*) We use the symbol CD to denote the total number of points in the set (CD).

Consider now a curve $\mathscr{C} = C^n(X^\alpha, Y^\beta, Z^\gamma)$, and let us investigate its proper transform $\mathscr{C}'$ which we may suppose to be of the type $C^{n'}(X^{\alpha'}, Y^{\beta'}, Z^{\gamma'})$. We suppose throughout that no branch of $\mathscr{C}$ at X or Y or Z has a fundamental line as its tangent there, so that YZ, for example, meets $\mathscr{C}$ in $n-\beta-\gamma$ points (distinct or otherwise) remote from Y and Z.

The order n' is the number of intersections of $\mathscr{C}'$ with an arbitrary line L. Hence

$$n' = L\mathscr{C}' = \mathbf{T}(L)\mathbf{T}(\mathscr{C}') = \Phi\mathscr{C} = 2n-\alpha-\beta-\gamma,$$

since a general Φ-conic meets $\mathscr{C}$ in $2n$ points of which α fall at X, β at Y, and γ at Z.

The index α' is the number of intersections absorbed in X when $\mathscr{C}'$ is met by a line L_1 through X. Hence

$$\alpha' = n'-L_1\mathscr{C}' = n'-\mathrm{T}(L_1)\mathscr{C} = n'-L_1'\mathscr{C} = n'-(n-\alpha)$$
$$= n-\beta-\gamma,$$

for the transform of L_1 is another line L_1' through X. This proves

THEOREM V. *The transform of a curve of the type $C^n(X^\alpha, Y^\beta, Z^\gamma)$ by a standard quadratic transformation whose base points are X, Y, Z is a curve of the type*

$$C^{2n-\alpha-\beta-\gamma}(X^{n-\beta-\gamma}, Y^{n-\gamma-\alpha}, Z^{n-\alpha-\beta}).$$

We may verify at once that, by applying the same rule to the transformed type, we arrive back at the original.

We note further that at X, for example, the $n-\beta-\gamma$ nodal tangents to $\mathscr{C}'$ arise from the $n-\beta-\gamma$ intersections of $\mathscr{C}$ with YZ at points remote from Y and Z. If $\mathscr{C}$ touches YZ at a point O, then $\mathscr{C}'$ has a *cuspidal branch* at X whose cuspidal tangent is the transform of OX. On the other hand, if $\mathscr{C}$ has an ordinary double point at O, then $\mathscr{C}'$ has two distinct branches issuing from X in the same direction—a type of singularity which we term a *tacnode*. This insight into the difference between a cusp and a tacnode is a first example of the use of the s.q.t. in the analysis of singularities.

The following are some examples of the application of Theorem V:

Original curve	*Transform*
Line	Conic through X, Y, Z
Line through X	Line through X
General conic	Quartic with nodes at X, Y, Z
Conic inscribed in XYZ	Quartic with cusps at X, Y, Z
Conic through X	Cubic through Y, Z, nodal at X
Cubic through Y, Z	Quartic through X, nodal at Y and Z
Quartic through X, Y, Z	Quintic with nodes at X, Y, Z.

As a last example, we consider a curve $\mathscr{C}$ which has an inflexion at X. The inflexional tangent, XO say, meets $\mathscr{C}$ in two points consecutive to X; hence, if O' is the point of YZ corresponding to the inflexional direction at X, the transformed curve $\mathscr{C}'$ touches XO' at O'.

More generally, if two curves $\mathscr{C}$ and $\mathscr{D}$ have r-point contact at X, then their transforms have $(r-1)$-point contact at a point of YZ.

§ 3. The resolution of singularities

3. One of the most fascinating branches of algebraic geometry is that which deals with the analysis of the various types of singularity which an algebraic curve or higher manifold can possess. In this section we deal only with the case of plane curves, and we attempt no more than an elementary introduction to the ideas involved, together with an elementary proof of one important theorem.

3.1. ***The standard resolution.*** We have already explained in the preceding chapter the distinction between multiple points of a curve which are ordinary and those which are not ordinary. Actually a curve with only ordinary multiple points presents, in many connexions, scarcely greater difficulty than a curve with no singularities at all. It is our purpose presently to show that any plane curve can be transformed, by a series of quadratic transformations, into one whose singularities are all ordinary.

Consider now the case of an arbitrary s-fold point O of a given curve $\mathscr{C}$ of order n; and let us choose a standard quadratic transformation of which one base point lies at O, while the other two, A, B say, satisfy the following conditions:

(i) A, B do not lie on $\mathscr{C}$,

(ii) OA and OB each meet $\mathscr{C}$ in $n-s$ distinct points, remote from O,

(iii) AB meets $\mathscr{C}$ in n distinct points.

A s.q.t., applied to a multiple point of a curve, and satisfying all the above conditions, will be called a single standard *resolution* of the point.

The above resolution transforms $\mathscr{C}$ into a curve $\mathscr{C}'$, of order $2n-s$, with *ordinary* multiple points of orders n, $n-s$, $n-s$, respectively, at O, A, B; and $\mathscr{C}'$ will meet AB in s points, distinct or otherwise, remote from A and B. Suppose that σ_1 of these s intersections coincide at O_1, σ_2 coincide at O_2,..., σ_k coincide at O_k, k being the number of distinct nodal tangents to $\mathscr{C}$ at O. Then at any one of these points, say O_i, $\mathscr{C}'$ may have, for example, a simple point and σ_i-point contact with AB at O_i, or it may have a σ_i-fold point at O_i, or it may have a lower singularity with one or more branches touching AB at O_i.

Only if all the points O_i, corresponding to directions of nodal tangents to $\mathscr{C}$ at O, are simple for $\mathscr{C}'$, do we say that the singularity of $\mathscr{C}$ at O is completely *resolved.*

We may regard the points O_i as corresponding to fictitious points of $\mathscr{C}$, 'infinitely near' to O, in the directions of the k nodal tangents; we denote these fictitious points by the same symbols O_i as their explicit 'transforms', and we say that they are in the *first neighbourhood* of O, since they are made actual by a single resolution. If the explicit points O_i have multiplicities s_i on $\mathscr{C}'$, then clearly $\sum_1^k s_i \leqslant s$; whence the following theorem:

THEOREM VI. *If a curve $\mathscr{C}$ has an s-fold point at O, then it possesses a number k of (fictitious) points, $O_1,\ldots, O_k$, in the first neighbourhood of O $(1 \leqslant k \leqslant s)$; and these points have, on $\mathscr{C}$, definite multiplicities $s_1,\ldots, s_k$ (defined by a resolution of O), such that* $1 \leqslant \sum_1^k s_i \leqslant s$.

Thus, for example, $\mathscr{C}$ has two simple points in the first neighbourhood of an ordinary double point; it has only one simple point in the first neighbourhood of a cusp; and it has an ordinary double point in the first neighbourhood of a tac-node.

When we have resolved the s-fold point O of $\mathscr{C}$ into the set of explicit points O_i of $\mathscr{C}'$, we may proceed to resolve any one of these latter in the same way into a set of explicit points O_{ij} $(j = 1,..., k_i)$ on a curve $\mathscr{C}''$, and these points may be regarded as representing points of $\mathscr{C}$ in the first neighbourhood of O_i, and therefore in the *second neighbourhood* of O. Continuing in this way, we can describe, evidently, the sets of multiple points which $\mathscr{C}$ possesses in successive neighbourhoods, first, second, third,..., of O. The crucial question is whether the process always comes to an end with the final complete resolution of the original singularity into a set of simple points. The answer which follows is one of many famous theorems due to Noether.

3.2. *Noether's transformation.* The fundamental theorem we now prove may be stated as follows:

THEOREM VII. *Any irreducible plane algebraic curve can be transformed, by a finite succession of standard quadratic transformations, into a curve whose multiple points are all ordinary.*

Let the curve under consideration be $\mathscr{C}$, of order n; and among the multiple points of $\mathscr{C}$ which are not ordinary let O be that of highest, or equal highest, multiplicity s. Also let $r_1,..., r_\mu$ be the multiplicities of all the remaining explicit multiple points of $\mathscr{C}$.

As in the preceding section, we apply to O a standard resolution, with base points O, A, B, satisfying the conditions there stated. Any multiple point of $\mathscr{C}$ other than O gives rise to a similar multiple point of the transformed curve $\mathscr{C}'$; but $\mathscr{C}'$ has new ordinary multiple points at O, A, B, of multiplicities n, $n-s$, $n-s$ respectively, and points $O_1,..., O_k$ of multiplicities $s_1,..., s_k$, on AB, remote from A and B. These points O_i, representing points of $\mathscr{C}$ infinitely near to O, need not be ordinary on $\mathscr{C}'$; but we now prove that unless every one of them is actually simple on $\mathscr{C}$, the deficiency D' of $\mathscr{C}'$ is less than the deficiency D of $\mathscr{C}$. Thus:

$$D = \tfrac{1}{2}(n-1)(n-2) - \tfrac{1}{2}s(s-1) - \tfrac{1}{2}\sum_{1}^{\mu} r_i(r_i-1),$$

and

$$D' = \tfrac{1}{2}(2n-s-1)(2n-s-2)-\tfrac{1}{2}\sum_{1}^{\mu} r_i(r_i-1)-\tfrac{1}{2}n(n-1)-$$
$$-2.\tfrac{1}{2}(n-s)(n-s-1)-K,$$

where $K = \frac{1}{2}\sum_{1}^{k} s_i(s_i-1)$. On reduction we obtain

$$D' = \tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}s(s-1)-\tfrac{1}{2}\sum_{1}^{\mu} r_i(r_i-1)-K,$$

so that
$$D' = D-K.$$

Hence, unless $s_i = 1$ $(i = 1,..., k)$, we have $K > 0$, and $D' < D$.

Thus if the multiple point O is not completely resolved by the transformation, the deficiency of $\mathscr{C}$ is diminished by it. On the other hand, the deficiency of any irreducible curve cannot be negative (Ch. II, § 2). It follows therefore that only a *finite* sequence of resolutions will be required to transform $\mathscr{C}$ into a curve with only ordinary multiple points; and this proves the theorem. We note the following corollaries incidental to the proof:

COROLLARY 1. *Any singularity of an irreducible plane algebraic curve can be completely resolved, by a succession of s.q.t., into a set of simple points of a transform of the curve. The process of resolution defines a decomposition of the singularity into one explicit multiple point and a number of simple or multiple implicit (fictitious) points of which the latter lie in various successive neighbourhoods of the former and of each other.*

COROLLARY 2. *By each resolution of a singular point O of $\mathscr{C}$, the deficiency of $\mathscr{C}$ is diminished by $\frac{1}{2}\sum_{1}^{k} s_i(s_i-1)$, where $s_1,..., s_k$ are the multiplicities of the points of $\mathscr{C}$ in the first neighbourhood of O.*

3.3. *The genus of a curve.* From the theorem just proved and its corollaries it appears at once that if a curve $\mathscr{C}$ is transformed into $\mathscr{C}^*$ (with only ordinary multiple points) by a Noether transformation T, then the deficiencies D and D^* of $\mathscr{C}$ and $\mathscr{C}^*$ are connected by the relation

$$D^* = D-\tfrac{1}{2}\sum s(s-1), \tag{6}$$

where the summation extends to every implicit multiple point (of multiplicity s) which may be assigned to the neighbourhood of an explicit multiple point of $\mathscr{C}$ by any one of the simple resolutions of which T is composed. This implies that D^* is equal to the number p given by

$$p = \tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\sum r_i(r_i-1), \tag{7}$$

where the summation now extends to every multiple point (of multiplicity r_i), explicit or implicit, of $\mathscr{C}$.

In the next chapter (Ch. IV, § 5) we shall prove that any two birationally equivalent plane curves, such as, for example, any two different Noether transforms of $\mathscr{C}$, have the same deficiency if their multiple points are all ordinary; and this means that the number p defined by (7) has a unique value, i.e. its value is independent of the particular Noether transformation implicit in its definition.

The number p is therefore a well-defined character of $\mathscr{C}$ and we call it the *genus* of the curve. It is a refinement of the cruder deficiency hitherto employed.

For a curve with only ordinary multiple points, the genus and the deficiency are equal; and we shall refer to their common value henceforth as the *genus* p of the curve. A curve with multiple points which are not ordinary may also have its genus equal to its deficiency, provided that the points in question, as, for example, if they are simple cusps, have no implicit multiple points in their neighbourhoods.

3.4. *Intersection multiplicity.* For a satisfactory rounding off of our remarks on multiple points it would be necessary to prove, as is in fact the case, that the implicit multiple points in the neighbourhood of an arbitrary singular point of a curve $\mathscr{C}$, and the multiplicities of these points, are independent of the sequence of resolutions used to define them. In simple cases, such as that of a tacnode, this is sufficiently obvious; but a complete proof of the general result† presents considerable difficulty and we shall not attempt it here. It depends on the fact (Theorem of Puiseux) that if any singular point of $\mathscr{C}$ is taken as the origin O of non-homogeneous coordinates x, y, then the points of $\mathscr{C}$ in a neighbourhood of O belong to a finite number of branches which each admit a parametric representation of the form

$$x = t^{\nu}, \qquad y = a_0 t^{\mu_0}+a_1 t^{\mu_1}+\dots,$$

where the indices ν and $\mu_0 < \mu_1 < \dots$ are positive integers; and

† See the references to Enriques–Chisini and van der Waerden at the end of this chapter.

the result can be proved by a direct algebraic estimation of the intersection multiplicity of $\mathscr{C}$ at O with any other curve $\mathscr{C}'$ which has one or more branches originating in O.

We may, however, indicate very briefly how the method of resolution may be applied, in simple cases, to the problem of estimating the number of intersections of a C^m and a C^n which coincide in a point O which is s-fold on C^m and t-fold on C^n (cf. also Ch. IV, § 4.1).

We write $mn = d+\delta$, where d is the total number of intersections of the curves remote from O, and δ is the required intersection multiplicity at O.

On applying a resolution to O, as in § 3.1, the $(2m-s)(2n-t)$ intersections of the transformed curves fall into the following groups: (i) intersections at O, A, B, totalling $mn+2(m-s)(n-t)$, which we regard as irrelevant, since they do not correspond to intersections of C^m and C^n; (ii) intersections, remote from O and from the line AB, which we assume to correspond to the d intersections of C^m and C^n remote from O; and (iii) a possible total number δ_1 of intersections *on* AB, remote from A and B, occurring when the nodal tangents to C^m at O are not all distinct from those of C^n. Thus we may write

$$d+\delta_1 = (2m-s)(2n-t)-mn-2(m-s)(n-t) = mn-st;$$

whence

$$\delta = st+\delta_1.$$

Thus δ_1 represents the excess of δ over what the intersection multiplicity at O would be if the nodal tangents of C^m were distinct from those of C^n. If the δ_1 intersections on AB are simple to enumerate, then δ is found; in the contrary case, further resolutions will eventually evaluate δ in the form

$$\delta = st+\sum s_i t_i+\sum\sum s_{ij} t_{ij}+\ldots,$$

where the summations extend to all the *common* implicit points of C^m and C^n in the first, second,... neighbourhoods of O, and the s- and t-symbols represent the relevant multiplicities at these points.

Thus, for example, if C^m has a cusp at O, while C^n passes simply through O and touches the cuspidal tangent, then $\delta_1 = 1$, and $\delta = 3$.

Similarly, if C^m has a tacnode at O and C^n touches the tacnodal tangent, then C^n passes simply through two *consecutive* nodes of C^m, and $\delta = 4$.

NOTES AND EXAMPLES ON CHAPTER III

1. *The trinodal quartic.* By using a s.q.t., prove that the class of a trinodal quartic is 6; show also that the curve has 4 double tangents and 6 points of inflexion.

If two of the nodes are at the circular points I, J, prove that the curve has two systems of bitangent circles and four foci.

2. *The tricuspidal quartic.* In the same way, prove that a tricuspidal quartic has one double tangent, that it has no inflexions, and that its class is 3. Prove also that the cuspidal tangents concur in the harmonic pole of the double tangent with respect to the triangle of cusps.

If two of the cusps are at I and J, prove that the curve has one system of bitangent circles and one focus.

3. Deduce in the same way the class and number of inflexions of a nodal or cuspidal cubic.

4. *Binodal quartic.* By transforming it into a cubic, prove that from either node of a binodal quartic a tetrad of tangents can be drawn to touch the curve elsewhere; also that the two tetrads so arising have the same cross-ratio. (Use Ch. II, Ex. 15.)

Show that a bicircular quartic (nodal at I and J) has 16 foci which belong by fours, as limiting point-circles, to four distinct systems of bitangent circles of the curve. (Use Ch. II, Ex. 10.)

5. *A special quadratic transformation.* Discuss the properties of the transformation whose equations are $x':y':z' = x^2:yz:xz$, or, in non-homogeneous coordinates, $x' = x$, $y' = y/x$.

If Φ is a net of conics which have one ordinary base point and one fixed point of contact, prove that any transformation based on Φ is the product of a transformation such as the above by a collineation.

6. Form the equations of a quadratic transformation based on a net of conics which have a fixed three-point contact.

7. Supposing a curve to have only ordinary singularities, prove that its deficiency (genus) is unaltered by any standard quadratic transformation which does not introduce not-ordinary singularities of the transformed curve.

8. If C_0 has an ordinary cusp at O, and C is a variable curve constrained to have only a simple point at O (and not to coincide with C_0), prove that C cannot be made to have more than three coincident intersections with C_0 at O.

9. *The special quadratic transformation as instrument of resolution.* When a given curve has a special singularity which we wish to analyse, it is natural to take the point at which the singularity occurs as origin O of non-homogeneous coordinates x, y, and to take OX to be the nodal tangent, or one of the nodal tangents, at O. This last assumption, however, excludes the use of the reciprocal transformation $x' = 1/x$, $y' = 1/y$ as an instrument of resolution; for this transformation has OX as fundamental line.

In this case we may conveniently resort to the special quadratic transformation $x' = x$, $y' = y/x$ (Ex. 5) as a means of resolving the neighbourhood of O, this having OY, but *not* OX, as a fundamental line. Its relevant properties are as follows:

(i) It transforms directions through O (points of the first neighbourhood of O) into the points of OY;

(ii) It transforms the particular direction OX through O into the point O of OY;

(iii) It transforms every point of OX (except O) into itself. Only the fundamental line OY must not be a nodal tangent at O.

Thus, for example, if C is the curve $y^2 = x^3$, its transform C' (obtained by writing xy for y) is $y^2 = x$, and it touches OY at O; thus C has a *cusp* at O.

Again, if C has equation $(y-ax^2)(y-bx^2) = x^5$, C' has equation

$$(y-ax)(y-bx) = x^3$$

and, if $a \neq b$, it has an ordinary node at O; thus C has a *tacnode* at O.

Analyse in this way the singularities at the origin of the curves whose equations are

$$\text{(i)} \quad (y-ax^2)^2 = x^5, \qquad \text{(ii)} \quad y^3 = x^4, \qquad \text{(iii)} \quad y^2(y-ax) = x^4,$$

$$\text{(iv)} \quad (y-ax^2)(y-bx^2)(y-cx^2) = x^7.$$

BOOKS RECOMMENDED FOR FURTHER READING

BERTINI, *Complementi*, §§ 5, 6, 8.

COOLIDGE, *Algebraic plane curves*, Book II.

ENRIQUES–CHISINI, *Teoria geometrica* . . ., vol. ii, Book IV.

SEVERI–LÖFFLER, *Vorlesungen über algebraische Geometrie*, ch. ii.

VAN DER WAERDEN, *Algebraische Geometrie*, ch. ix.

CHAPTER IV

RATIONAL CORRESPONDENCES

§ 1. Correspondences on a line

1. The general (m, n) correspondence. If $F(x, y)$ is a polynomial of degree m in x and of degree n in y, then the equation

$$F(x, y) = 0 \tag{1}$$

determines an (m, n) correspondence between the variables x and y, such that to any x there correspond n values, $y_1, \ldots, y_n$, of y, while to any y there correspond similarly m values, $x_1, \ldots, x_m$, of x. If we think of x, y as parameters fixing the positions of points P, Q on a line l, then the equation establishes an (m, n) correspondence between P and Q on l.

Instead of points on a line, the parameters x, y may, of course, fix the positions of points on a rational curve, lines of a plane pencil, curves of a linear pencil, or generally the elements of any simply infinite rational field; our theory therefore includes correspondences in every such field.

A *coincidence* of a point P of l with one of its corresponding points $Q_1, \ldots, Q_n$ occurs when one of the numbers y_i is equal to the x from which it arises. Such coincidences are given therefore by the equation

$$F(x, x) = 0, \tag{2}$$

which is evidently of degree $m+n$ in general. Thus:

A rational (m, n) correspondence between points P, Q of a line has, in general, $m+n$ united points $U_1, \ldots, U_{m+n}$.

We may denote the correspondence from P to Q by $\mathbf{T}$, and that from Q to P by $\mathbf{T}^{-1}$, so that

$$\mathbf{T}(P) = \sum_1^n Q_i, \qquad \mathbf{T}^{-1}(Q) = \sum_1^m P_i.$$

A *branch-point* P^* of $\mathbf{T}$ is a point such that two of the n points Q_i corresponding to it coincide; and similarly a branch-point Q^* of $\mathbf{T}^{-1}$ is a point such that two of the m points P_i corresponding to it coincide. If we write (1) in the form

$$F \equiv X_0 y^n + X_1 y^{n-1} + \ldots + X_n = 0, \tag{3}$$

where X_i $(i = 0, 1,..., n)$ is a polynomial of degree m in x, then this equation in y has two coincident roots if

$$\frac{\partial F}{\partial y} \equiv nX_0 y^{n-1} + ... + X_{n-1} = 0. \quad (4)$$

Thus the parameters x of the branch-points P^* are obtained by eliminating y between (3) and (4). We may exclude the irrelevant power of X_0 from the eliminant by replacing (3) by the equation

$$nF - y\frac{\partial F}{\partial y} \equiv X_1 y^{n-1} + 2X_2 y^{n-2} + ... + nX_n = 0; \quad (5)$$

the eliminant of (4) and (5) is then of degree $n-1$ in the coefficients of each equation and therefore of degree $2m(n-1)$ in x. Thus:

In a general rational (m, n) correspondence between points P, Q of a line, there are $2m(n-1)$ points P^* which are branch-points of the forward correspondence **T** *and there are $2n(m-1)$ branch-points Q^* of the backward correspondence* $\mathbf{T}^{-1}$.

1.1. ***The image curve of the correspondence.*** If we denote by C the plane curve whose equation, in Cartesian coordinates, is $F(x, y) = 0$, then C is called the *image curve* of the correspondence **T**. Any point (x, y) of C represents a pair of points (P, Q) which correspond in **T**; and accordingly C gives a picture of the totality of all such corresponding pairs. By using this image, the results of the preceding section may be interpreted as follows.

In the first place, C is in general of order $m+n$; and by augmenting it, if necessary, by a convenient multiple of the line at infinity, it may be considered to be always of this order.

A line parallel to the y-axis OY represents a definite position of P and it meets C in n points corresponding to the n pairs (P, Q_i) of the correspondence; similarly a line parallel to OX meets C in m points corresponding to pairs (P_i, Q). Thus C has the point at infinity on OY as m-fold point and the point at infinity on OX as n-fold point.

The united points U_i correspond to the $m+n$ intersections of C with the line $x = y$.

The branch-points P^* are associated with tangents to C which are parallel to OY; or, more precisely, with lines drawn through the m-fold point of C to meet the curve in two coincident points in the finite plane. Now the class of C, taking account of its two multiple points at infinity, has ordinarily the value

$$(m+n)(m+n-1)-m(m-1)-n(n-1) = 2mn;$$

and of the $2mn$ tangents from the m-fold point along OY, $2m$ are absorbed by the m nodal tangents there. Thus there remain $2m(n-1)$ tangents parallel to OY, to which correspond a similar number of branch-points P^* as already found.

1.11. *Representation by an (n, m) curve on a quadric.* Another elegant representation of all the pairs of the correspondence is obtained by regarding x and y as parameters of generators of opposite systems on a quadric surface S (cf. Ch. VII, § 1). Any pair of points (P, Q) will then be associated with the intersection of the x-generator corresponding to P with the y-generator corresponding to Q; and the curve, D say, whose points represent the pairs of the (m, n) correspondence will be an (n, m) curve on S, meeting generators of the two systems in n and m points respectively. Conversely, any (n, m) curve on S defines an (m, n) correspondence between x and y.

The relation $x = y$ defines a particular plane section of S, meeting D in the $m+n$ points corresponding to united points.

The branch-points P^* and Q^* are associated with generators of the two systems which touch D. From our previous result there follows the corollary that, in general, an (α, β) curve on a quadric is touched by $2\alpha(\beta-1)$ generators of the first system and by $2\beta(\alpha-1)$ generators of the second system.

Example

A general (m, n) correspondence has $\frac{1}{2}(m^2+n^2-m-n)$ *involutory* pairs (P, Q), such that P belongs to the set $\mathbf{T}(Q)$ and also Q belongs to the set $\mathbf{T}(P)$.

Solution: The involutory pairs will have parameters satisfying the two equations $F(x, y) = 0$ and $F(y, x) = 0$, representing C and its reflection C' in the line $x = y$. In general these two curves meet in a number

$$(m+n)^2-2mn = m^2+n^2$$

of points not at infinity; and these will include the $m+n$ united points. The remainder correspond, in symmetrical pairs, to the stated number of involutory pairs of the correspondence.

1.2. *Involutions.* Among correspondences with equal indices, we may distinguish two important special types as follows.

A correspondence $\mathbf{T}$ is said to be *symmetrical* if the correspondence in $\mathbf{T}$ of any pair (P, P') implies the correspondence in

$\mathbf{T}$ of the reversed pair (P', P). If the equation of $\mathbf{T}$ is $F(x, x') = 0$, then the polynomial $F(x, x')$ must remain unaltered, save possibly in sign, by the interchange of x and x'.

A correspondence $\mathbf{T}$ may also be *cyclically symmetrical*. To explain this very special type, we suppose $\mathbf{T}$ to be an $(n-1, n-1)$ correspondence, and we denote by G^n a typical group of n points $(P_1, \ldots, P_n)$ formed by adding to any point P_1 the $n-1$ points $P_2, \ldots, P_n$ which correspond to it by $\mathbf{T}$. Then $\mathbf{T}$ is cyclically symmetrical if every such group G^n arises symmetrically from each of its points; in other words, all the points of G^n must correspond mutually in $\mathbf{T}$.

Thus, for example, any symmetrical (2, 2) correspondence on a conic k is generated by chords PQ of k which touch another conic k'; but if and only if k is triangularly circumscribed to k', is the correspondence cyclically symmetrical. The groups G^3 in this case are the triads of points of k whose joins touch k'.

A cyclically symmetrical $(n-1, n-1)$ correspondence on a line l clearly separates the points of l into ∞^1 mutually exclusive groups of n points; this leads to the idea of an involution of order n on a line, which we define as follows:

DEFINITION. A series of sets G^n of n points on a line l is said to be an *involution of order n* if (i) the points of a generic set G^n of the series are all distinct and all variable, and (ii) an arbitrary point of l belongs to precisely one set of the series.

We now prove the following fundamental theorem:

THEOREM I. *The sets of any involution on a line form a linear series, given by an equation of the form $U + \lambda V = 0$, where U and V are polynomials in the coordinate x of a point of the line, and λ is a variable parameter.*

To prove this, we consider first the cyclically symmetrical $(n-1, n-1)$ correspondence on the line, which is determined by the condition that two points correspond if they belong to the same group of the involution; and we suppose the equation of this correspondence to be $F(x, x') = 0$, where F is a polynomial of degree $n-1$ in x and in x'.

The equation $$(x - x')F(x, x') = 0 \tag{6}$$
defines a set G^n of the involution if either x' or x is regarded as fixed and the equation solved for x or x'. We may write it in the form
$$a_0(x')x^n + a_1(x')x^{n-1} + \ldots + a_n(x') = 0, \tag{7}$$

where each coefficient $a_i(x')$ is a polynomial in x' of degree not exceeding n; and we observe then that, since all the roots of (7) vary with x', the coefficient $a_0(x')$ is not identically zero (for this would give a fixed infinite root), nor can all the ratios

$$\frac{a_1}{a_0},\quad \frac{a_2}{a_0},\quad \dots,\quad \frac{a_n}{a_0},$$

be independent of x' (for this would fix all the roots).

Choose now any one of the above ratios which really does vary with x', and let this be, for example, $a_1(x'):a_0(x')$. If $x_1,\dots, x_n$ are the n roots of (7), one of them, say x_1, being x' itself, then this equation by hypothesis remains unaltered (as an equation for x) if x' is replaced by any one of $x_2,\dots, x_n$. Thus

$$\frac{a_1(x_1)}{a_0(x_1)} = \frac{a_1(x_2)}{a_0(x_2)} = \dots = \frac{a_1(x_n)}{a_0(x_n)} = -\sum_1^n x_i.$$

Hence $x_1,\dots, x_n$ are roots of an equation

$$a_1(x)-\lambda a_0(x) = 0 \tag{8}$$

for some value of the parameter λ; and they are in fact the only roots of this equation since they are in general distinct, and a_0, a_1 have degrees not exceeding n. This proves the theorem.

A point of the line which counts twice in the set G^n to which it belongs is called a *double point* of the involution; and the set of all such double points is called the *Jacobian set* of the involution. This set coincides evidently with the set of united points of the associated $(n-1, n-1)$ correspondence; and it follows therefore that an involution of order n has in general $2n-2$ double points.

1.3. *Lüroth's Theorem*. The theorem of the preceding section leads at once to the following important result.

THEOREM II. *Any curve which is capable of a rational parametric representation is rational.*

We need only consider the case of a plane curve C, as the proof for a curve in space of any dimension is exactly similar. Suppose then that C has a parametric representation

$$x:y:z = f_1(t):f_2(t):f_3(t),$$

where f_1, f_2, f_3 are polynomials in the parameter t. We have to show that the points of C are in (1, 1) correspondence with the points of a line, or say, with the values of another (or the same) parameter λ.

To any value of t there corresponds certainly a unique point of C; but it may well happen (as, for example, with the parametric equations $x:y:z = t^4:t^2:1$) that every point of C is given by more than one value of t, so that the correspondence between points of C and values of t is not one-one but one-many.

Suppose then that a general point of C is given always by each of σ values of t (in general all distinct), forming a group G^σ, so that the points of C are in $(1,1)$ correspondence with the ∞^1 groups G^σ so arising. These groups G^σ clearly form an involution; and hence, by Theorem I, they are given, for various values of the parameter λ, by an equation of the form

$$u(t)-\lambda v(t) = 0.$$

The groups G^σ, and hence also the points of C, are in $(1,1)$ correspondence with the values of this new parameter $\lambda = u(t)/v(t)$; and this proves that C is rational.

The new (uniformizing) parameter λ is evidently a rational function of t, which takes the same value at every point of a set G^σ.

Note. It is an interesting fact that the above theorem has an analogue for surfaces, but not for threefolds or for manifolds of higher dimension. This means, effectively, that although every involution on a line or in a plane is necessarily rational, yet there exist spatial involutions whose point-sets cannot be put in $(1,1)$ correspondence with the points of another space.

§ 2. Correspondences in a Plane

2. The fundamental form. In regard to pairs of associated points (P,Q) in a plane, we adopt, for greater generality, a slightly different point of view. The theory of a correspondence on a line can be regarded as the geometry of ∞^1 ordered point-pairs on the line; and the immediate analogue of this in the plane would be the geometry of a simply or doubly infinite system of ordered point-pairs in the plane. With a plane point-pair, however, there is associated in general a joining line; and a more complete theory is evolved if we take this formally into consideration.

We introduce, therefore, as the basis of our plane theory, a *fundamental form*, consisting of a line l and an ordered pair of points (P,Q) on l; and we denote this combination by the symbol $(P,Q;l)$. If P, Q coincide, the form is not defined till the line l through the point of coincidence is assigned.

We shall consider below the geometry of (i) a simple infinity, (ii) a double infinity, of fundamental forms $(P, Q; l)$, embracing thus the theory of plane correspondences of a simple infinity or of a double infinity of pairs of points.

2.1. ***One-dimensional correspondence in a two-dimensional field.*** Consider a simply infinite (algebraic) system Σ_1 of fundamental forms $(P, Q; l)$. Such a system has evidently three characteristic indices, namely:

(*a*) the number α_1 of forms of Σ_1 in which P lies on a given line,

(*b*) the number α_2 of forms of Σ_1 in which Q lies on a given line,

(*c*) the number μ of forms of Σ_1 in which l passes through a given point.

A form becomes *special* if, l remaining definite, P and Q coincide; we denote by ξ the number of such special forms—briefly called *coincidences*—which occur in Σ_1, and we proceed to prove

THEOREM III. *A one-dimensional correspondence whose indices, in a two-dimensional projective field, are* $(\alpha_1, \alpha_2; \mu)$ *has a number of coincidences* ξ *given by*

$$\xi = \alpha_1 + \alpha_2 - \mu. \tag{9}$$

To prove this, we establish a correspondence between rays u, v through a fixed point O, by the condition that u, v contain, respectively, the point P and the point Q of a fundamental form of Σ_1. The indices of this correspondence are clearly (α_1, α_2); and a united ray arises either from a coincidence of P with Q, or from a form of Σ_1 whose axis l passes through O. Hence $\xi + \mu = \alpha_1 + \alpha_2$, which proves the theorem.

Generalization. The above theorem and its proof can be generalized at once to apply to a one-dimensional correspondence in an r-dimensional field. The indices α_1, α_2 are now the numbers of forms $(P, Q; l)$ of Σ_1, in which P and Q respectively lie in an arbitrary $[r-1]$; and μ is the number of those whose axes l meet an arbitrary $[r-2]$. The number ξ of coincidences is still given by (9).

2.11. *The symmetrical case.* The expression of the above results must, for convenience, be modified when the correspondence between P and Q is symmetrical, i.e. when, for every form $(P, Q; l)$ belonging to Σ_1, the associated form $(Q, P; l)$ likewise belongs to

Σ_1. In this case $\alpha_1 = \alpha_2 = \alpha$ say; but we must distinguish between the index μ of Σ_1 and the number, ν say, of joins PQ which pass through a given point; for each such join is associated with two forms of Σ_1, so that $\mu = 2\nu$.

If $[P, Q; l]$ is used to denote a modified fundamental form consisting of a line l and an *unordered* pair of points P, Q on l, then any simply infinite family Σ_1' of these modified forms defines a symmetrical correspondence of the type under consideration; and the number ξ of coincidences (forms of Σ_1' in which P, Q coincide) is given by

$$\xi = 2(\alpha - \nu), \tag{9'}$$

where α is the number of forms of Σ_1' in which one of the points P, Q lies on a fixed line, while ν is the number of forms in which l passes through a fixed point.

This result admits of the same generalization as Theorem III.

Examples

1. In any (α, α') correspondence between the points of two curves C, C', of orders n, n' respectively, the class μ of the envelope of joins of corresponding points is given by

$$\mu = \alpha n' + \alpha' n - i,$$

where i is the number of intersections of C and C' which correspond to themselves.

2. In any unsymmetrical (α, β) correspondence between points P, Q of a curve C of order n, the class μ of the envelope of PQ is given by

$$\mu = n(\alpha + \beta) - \delta,$$

where δ is the number of the united points of the correspondence on C.

For a symmetrical (α, α) correspondence, the formula is

$$\mu = n\alpha - \tfrac{1}{2}\delta.$$

3. If S_1, S_2, S_3 are three general conics, discuss the correspondence of the ∞^1 pairs of points which are conjugate with respect to all three conics.

4. If curves C, C', of orders n, n' respectively, lie on a ruled surface of order k and meet the generators in α, α' points respectively, prove that they intersect in i points, where

$$i = n\alpha' + n'\alpha - k\alpha\alpha'.$$

Verify this result independently for curves on a quadric. [Use the extension of Theorem III to point-pairs of space.]

5. Show, by means of a correspondence, that the locus of points of intersection of corresponding curves of two related pencils of orders n_1 and n_2 respectively, is a curve of order $n_1 + n_2$.

Show, more generally, that if there is an (r_1, r_2) correspondence between the curves of two simply infinite systems, of orders n_1, n_2 and *indices* μ_1, μ_2

respectively, then the locus of intersections of corresponding curves is of order $\mu_2 r_1 n_1 + \mu_1 r_2 n_2$. (The *index* μ of a simply infinite system is the number of curves of the system which pass through an arbitrary point.)

6. If C, C' are two plane curves of classes m, m', prove that the locus of the point of intersection of tangents to C and C' which make a fixed angle with each other (in a fixed sense) is in general of order $2mm'$. If C' is identical with C, prove that the corresponding locus for C is of order $2m(m-1)$.

7. A plane curve C is of order n and class m, and PQ is a variable chord of it which passes through a fixed point. Prove that, in general, the order of the curve described by the point of intersection of the tangents at P and Q is $\frac{1}{2}m(2n-3)$. How must this result be modified for a curve possessing cusps?

8. A correspondence of ∞^1 pairs of points (P_1, P_2) in the plane is such that (i) to arbitrary points P_1, P_2 there correspond ν_2 points P_2 and ν_1 points P_1 respectively, (ii) an arbitrary line contains n_1 points P_2 and n_2 points P_1, (iii) the number of pairs (P_1, P_2) whose joining line passes through an arbitrary point is s. Show that the number of coincidences of P_1 with P_2 is

$$n_1\nu_1 + n_2\nu_2 - s.$$

State the corresponding result for ∞^1 point-pairs in space.

9. A plane curve C has order n and class m. Show that, in general, it is touched by $n\nu + m\mu$ curves of any system of indices μ and ν (i.e. such that μ curves of the system pass through a point and ν touch any line).

10. Find the number of lines of a regulus which touch a surface of order n.

2.2. *Two-dimensional correspondence in a two-dimensional field.* We consider next a doubly infinite system Σ_2 of forms $(P, Q; l)$ in the plane. It will normally happen that every point of the plane is the point P of one or more forms of Σ_2, and similarly for Q. Thus the geometry of Σ_2 is the geometry of a correspondence which normally extends over the whole plane. As indices of Σ_2 we take the following:

(*a*) the number α_1 of forms of Σ_2 in which Q is fixed,
(*b*) the number α_2 of forms of Σ_2 in which P is fixed,
(*c*) the number α_{12} of forms of Σ_2 in which P lies on one given line and Q lies on another given line,
(*d*) the number μ of forms of Σ_2 in which l is a given line.

We expect a simple infinity of forms of Σ_2 to have P coincident with Q, and we refer to the locus of such points of coincidence of P with Q as the *coincidence-locus* of Σ_2. Similarly we refer to the total envelope of coincidence-lines l (associated with coincidence-points) as the *coincidence-envelope* of Σ_2. It should be remarked, however, that Σ_2 may possess a finite number of isolated *complete coincidences* U, such that if l is an arbitrary line through U, then $(U, U; l)$ is a form of Σ_2. Such points enter as component curves of order zero into the coincidence-locus, and give plane pencils as components of the coincidence-envelope.

We now prove the following theorem:

THEOREM IV. *A two-dimensional correspondence, of indices* $(\alpha_1, \alpha_2, \alpha_{12}; \mu)$, *in a two-dimensional projective field has a coincidence-locus of order* ξ, *and a total coincidence-envelope of class* ζ, *where*

$$\text{(i)} \quad \xi = \alpha_{12} - \mu; \qquad \text{(ii)} \quad \zeta = \alpha_1 + \alpha_2 + \mu.$$

Let the form $(P, Q; l)$ vary, in Σ_2, so that P describes a line L. Then Q describes an associated curve-locus K and l describes an envelope Λ. The order of K—the number of its intersections with a line L'—is equal evidently to α_{12}. But K meets L itself in μ points Q belonging to the μ forms of Σ_2 for which $l \equiv L$, and in the ξ points in which L meets the coincidence curve. Hence $\xi + \mu = \alpha_{12}$, which proves (i).

Furthermore, the class of Λ, which we may denote by λ, is evidently the number of forms of Σ_2 whose point P lies on L and whose line l passes through an arbitrary point; by taking this arbitrary point to lie on L, we obtain at once

$$\lambda = \alpha_2 + \mu.$$

Now consider the correspondence between rays u, v, through a fixed point O of the plane, such that there is a form of Σ_2 for which P lies on u, Q lies on v, and l passes through a second fixed point O'. If u is fixed, the number of corresponding rays v is clearly $\lambda = \alpha_2 + \mu$; and similarly, if v is fixed, the number of corresponding rays u is $\alpha_1 + \mu$. Coincidences arise from (i) the ζ coincidence-lines through O', and (ii) the ray OO', which evidently counts as a μ-fold coincidence. Thus

$$\alpha_2 + \mu + \alpha_1 + \mu = \zeta + \mu;$$

i.e.

$$\zeta = \alpha_1 + \alpha_2 + \mu.$$

This completes the proof of the theorem, which, like the preceding one, can be extended (with slight modifications) to systems Σ_2 in ordinary or higher space (cf. Ex. 3 below).

EXAMPLES

1. In the most general plane collineation, $\alpha_1 = \alpha_2 = \alpha_{12} = \mu = 1$. By Theorem IV, then, $\xi = 0$, $\zeta = 3$. Thus there are three complete isolated coincidences.

2. In *any* plane collineation, $\alpha_1 = \alpha_2 = \alpha_{12} = 1$. Then $\xi = 1-\mu$, so that either $\mu = 1$, $\xi = 0$, as above; or $\mu = 0$, $\xi = 1$. In the latter case $\xi = 1$, $\zeta = 1$; and the collineation is an homology in which the coincidence curve is a line and there is one complete coincidence in addition.

3. For a system Σ_2 in [3], if the coincidence curve is of order ξ and the coincidence-lines form a ruled surface of order ζ, then formulae analogous to those of Theorem IV are

$$\text{(i)}\quad \xi = \alpha_{12}-\mu, \qquad \text{(ii)}\quad \zeta = \alpha_1+\alpha_2+\mu-\nu,$$

where α_1 is the number of forms of Σ_2 in which Q lies in an arbitrary line, α_2 has a similar definition, α_{12} is the number of forms in which P and Q lie in assigned planes, μ is the number in which l lies in a given plane, and ν is the number in which l passes through an arbitrary point.

4. An interesting special case is that in which there are only ∞^1 points P. Suppose, namely, that P always lies on a fixed curve c of order α, that to any point P of c there correspond all the points Q of a curve q of order α', and that as P varies on c the curve q describes a system of index β. Then

$$\alpha_1 = \beta, \quad \alpha_2 = 0, \quad \alpha_{12} = \mu = \alpha\alpha', \quad \xi = 0, \quad \zeta = \beta+\alpha\alpha'.$$

The coincidence envelope consists in this case of the $\beta+\alpha\alpha'$ points P which lie on their corresponding curves q; and if q also is a fixed curve, the envelope consists of the $\alpha\alpha'$ intersections of c with q.

These results admit of immediate generalization to space of arbitrary dimension, and we may deduce, in particular, the following useful result:

If, in space of any dimension $r \geqslant 2$, the points of a curve c of order α are in birational correspondence with the primes of a developable of class β, then the number of points of c which lie on their corresponding primes is $\alpha+\beta$.

§ 3. ZEUTHEN'S RULE

3. Multiplicities of the coincidences. The chief difficulty in applying the correspondence principle to the solution of geometrical problems lies undoubtedly in reckoning the multiplicities of the coincidences, particularly when these separate themselves, in any given correspondence, into distinct groups occurring in quite different ways. In the geometric representation of § 1.2, the coincidences of an (m, n) correspondence on a line were represented by the intersections of the curve C, whose equation is $F(x,y) = 0$, with the line $x = y$. In any given case, the curve C may have multiple intersections with $x = y$, arising either from tangencies of C with this line, or from multiple points of C on it, or from combinations of these two possibilities. Unless therefore we have some rule enabling us, in any case of doubt, to estimate the precise multiplicity of any given coincidence, it is often impossible to arrive at an exact conclusion. In the previous work, we have implicitly assumed the possibility of verifying that the multiplicities of coincidences were what they appeared to be.

Suppose then that the point given by $x = y = a$ is a coincidence of the correspondence represented by $F(x,y) = 0$, and consider the correspondence (x,y) in the neighbourhood of a. To every x near a, there will correspond n values of y of which one or more, say $y_1,\ldots,y_s$, will likewise be near to a. To calculate the multiplicity of a in the group of $m+n$ united points, we may apply the following theorem discovered by Zeuthen:

THEOREM V (*Zeuthen's Rule*). *The multiplicity of any united point a in the group of united points of a correspondence (x,y) on a line is equal to the sum of the orders of the infinitesimals*

$$y_1-x, \quad y_2-x, \quad y_3-x, \quad \ldots, \quad y_s-x$$

as compared with $x-a$ (as unitary infinitesimal), where $y_1,\ldots,y_s$ are those of the values of y, corresponding to a given x, which tend to a with x.

Before proving this formally, we indicate briefly how it operates. What is wanted evidently is the number of intersections of the line $x = y$ with the curve C at the point (a,a); but what is available usually, from the geometrical constructions setting up the correspondence, is a knowledge only of its algebraicity, its indices m, n, and the way in which coincidences of different types arise.

It is possible usually, in the case of any coincidence a, to see how many points of the set $\mathbf{T}(x)$ tend to a as $x \to a$; but this may be different from the number of points of the set $\mathbf{T}^{-1}(y)$ which tend to a as $y \to a$; and it is important to realize that *either or both these numbers may be quite different from the required coincidence-multiplicity at a.*

Consider, for example, the following diagrams showing what may happen in the geometrical representation:

Fig. 1 represents an ordinary simple coincidence in which one y tends to a as $x \to a$.

Fig. 2 simple coincidence; as $x \to a$, two values of y tend to a.

Fig. 3 double coincidence; as $x \to a$, one value of y tends to a, and as $y \to a$, one value of x tends to a.

Fig. 4 double coincidence; as $x \to a$, three values of y tend to a, and as $y \to a$, three values of x tend to a.

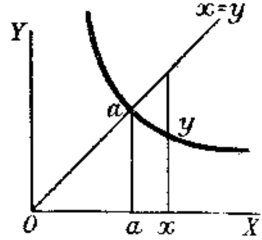


FIG. 1.

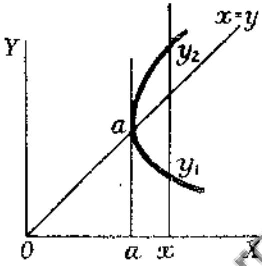


FIG. 2.

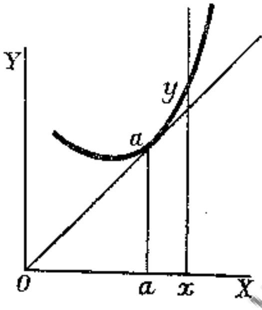


FIG. 3.

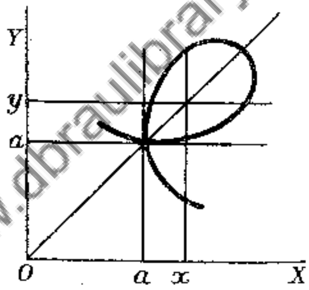


FIG. 4.

Thus the number of y which tend to a as x tends to a is no certain indication of the multiplicity of a; for any one of these y may count, as it were, as a multiple, or as a fraction, of a coincidence. The necessity for a closer inspection, on the lines of Zeuthen's rule is apparent.

Returning now to the proof of Zeuthen's rule, we prefix the following lemma.

LEMMA. *If $\Phi(X, Y) = 0$ is the equation of an algebraic curve passing through the origin, and not containing the line $X = 0$ as component, and if, as $X \to 0$, a set $Y_1,\ldots, Y_s$ of corresponding values of Y tend to zero, then the multiplicity of the intersection of the curve with $Y = 0$ at the origin is equal to the sum of the orders of the infinitesimals $Y_1,\ldots, Y_s$, relative to X.*

To prove this, we write the equation of the curve in the form

$$\Phi(X, Y) \equiv A_0 + A_1 Y + \ldots + A_n Y^n = 0, \tag{1}$$

where the A_i are polynomials in X. By saying that Φ has an intersection of multiplicity ρ with $Y = 0$ at the origin, we mean simply that the polynomial $\Phi(X, 0)$ has X as ρ-fold factor. Thus ρ is such that $$A_0 = X^\rho A \quad (\rho \geqslant 1, A(0) \neq 0).$$

Let $Y_1,..., Y_n$ be the roots of (1); and suppose that, as $X \to 0$, $Y_1,..., Y_s$ tend to zero, $Y_{s+1},..., Y_\sigma$ remain finite, while $Y_{\sigma+1},..., Y_n$ tend to infinity. Since X is not a factor of Φ, the equation $\Phi(0, Y) = 0$ is not illusory and its roots are the limiting values of $Y_1,..., Y_n$. It follows from this that X is a factor of A_n, $A_{n-1},...,$ $A_{\sigma+1}$, but not of A_σ. Now

$$A_\sigma = \pm A_0 \sum (1/p_\sigma),$$

where p_σ is a typical product of any σ of the n roots. As $X \to 0$, the particular term $1/(Y_1 ... Y_\sigma)$ in the summation is evidently of a greater order of magnitude than any other; and hence, since A_σ does not vanish with X,

$$Y_1 ... Y_\sigma = KA_0 = K'X^\rho,$$

where neither K nor K' vanishes with X. Finally, since $Y_{s+1},..., Y_\sigma$ have finite non-zero limits as $X \to 0$, it follows that

$$Y_1 ... Y_s = K''X^\rho,$$

where K'' is finite non-zero at $X = 0$; and this proves the lemma.

Zeuthen's rule now follows from the lemma by a simple change of axes. Thus, to evaluate the multiplicity of intersection of the curve $F(x, y) = 0$ with the line $x = y$ at the point (a, a), we write

$$X = x-a, \qquad Y = y-x,$$

and the problem reduces to that of the lemma. If $y_1,..., y_s$ are those values of y, corresponding to given x, which tend to a with x, then the required multiplicity is equal to the sum of the orders of $Y_i = y_i - x$ $(i = 1,..., s)$, relative to $X = x-a$; and this proves the theorem.

COROLLARY. *In a correspondence between rays u, v of a pencil, the multiplicity of any coincidence ray a is equal to the sum of the orders, relative to the angle $\widehat{ua}$ as unitary infinitesimal, of the infinitesimal angles*

$$\widehat{uv_1}, \quad \widehat{uv_2}, \quad ..., \quad \widehat{uv_s},$$

where $v_1,..., v_s$ are the rays v, corresponding to u, which tend to a with u.

This follows at once by taking a section of the pencil by a generic line of the plane; if this line is p and if the vertex of the pencil is O, then any small segment of p subtends a *comparable*† infinitesimal angle at O.

If the vertex of the pencil is at infinity, Zeuthen's rule will be applied, of course, directly to a section.

3.1. ***Modification for a one-dimensional correspondence in the plane.*** In § 2.1 we considered a system Σ_1 of ∞^1 forms $(P, Q; l)$; and by joining P, Q by rays u, v to a fixed point O and considering the correspondence (u, v), we found the number ξ of coincidences of P with Q. We wish now to extend Zeuthen's rule to enable us to calculate the multiplicity of any coincidence U in such a case.

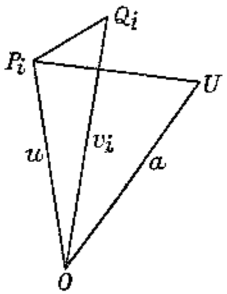


FIG. 5.

We choose O so that the coincidence ray $a \equiv OU$ satisfies the two following conditions: (i) a contains no coincidence besides U, (ii) a is not the joining line l, limiting or otherwise, of any form of Σ_1. In virtue of these conditions, the multiplicity of U, as a coincidence in Σ_1, is equal to that of a as a coincidence ray in the correspondence (u, v).

Of all the points P, belonging to forms of Σ_1, which lie on u, a certain sub-set $P_1, \ldots, P_\sigma$ will approach U as $u \to a$; and of the companions $Q_1, \ldots, Q_\sigma$ of these points, a certain sub-set, say $Q_1, \ldots, Q_s$ $(1 \leqslant s \leqslant \sigma)$, will likewise approach U; and in virtue of (ii) no companion point Q of any other point P of u can tend to U. Thus, of all the rays v corresponding to u, the set $OQ_1, \ldots, OQ_s$, and these alone, tend to a as $u \to a$. By Th. V, Cor. 1, the multiplicity of a is the sum of the orders of the angles $\widehat{uv_i}$ relative to $\widehat{ua}$. Now if δ is the perpendicular distance of U from u, δ and $\widehat{ua}$ are like infinitesimals (OU being finite); and if Q_i corresponds to P_i, their join, by (ii), tends to a limit distinct from a, so that the length P_iQ_i and the angle $\widehat{uv_i}$ are like infinitesimals. Hence the multiplicity of U—equal to that of a—is the sum of the orders of the lengths of such joins, relative to δ. This proves the following rule:

THEOREM VI. *The multiplicity of a united point U of any correspondence of ∞^1 point-pairs in the plane is to be evaluated as follows. We take a line L, passing close to U, at distance δ from it, and not neighbour to any join (limiting or otherwise) of a pair of the corre-*

† We speak of two infinitesimals as being *comparable* or *like* if their ultimate or limiting ratio is finite and non-zero.

spondence; and we consider pairs (P, Q) of the correspondence, close to the united point U, such that P lies on L. Then the multiplicity of U is equal to the sum of the orders of the infinitesimal distances PQ between the points of every such pair, relative to δ as unit infinitesimal.

Instead of δ we may take, as unit infinitesimal, any one of the distances PU, provided that L is not neighbour to the limiting join of P and U.

§ 4. Applications of Zeuthen's Rule

4. Zeuthen's rule opens the way to a series of elegant and rigorous applications of the correspondence principle to the solution of fundamental enumerative problems. Foremost among these applications is the following proof of Bézout's Theorem on the number of intersections of two curves of orders m and n (Ch. I, § 3.2), and the interpretation of the theorem in the case when the intersections in question are not all distinct.

4.1. THEOREM VII (*Bézout's Theorem*). *Two irreducible plane curves of orders m and n respectively meet in mn points, distinct or coincident.*

Let the curves be C^m and C^n, and let O be an arbitrary point not on either curve. Also let Σ_1 be the system of fundamental forms $(P, Q; l)$ such that P lies on C^m, Q lies on C^n, while l passes through O. There are m points P_i on an arbitrary line; and to these there correspond mn points Q_{ij} on the rays OP_i. Thus the index α_2 of the system (cf. § 2.1) is mn; and similarly α_1 has the same value. Moreover, any ray through O contains mn pairs PQ, and one such ray passes through an arbitrary point; so that the third index μ of Σ_1 is likewise mn. Hence Σ_1 has ξ coincidences where

$$\xi = \alpha_1+\alpha_2-\mu = mn+mn-mn = mn.$$

Since any coincidence U is necessarily an intersection of C^m and C^n, the theorem is proved.

Of more importance than the above proof is the fact that Zeuthen's rule now enables us to assign a definite intersection multiplicity to various types of multiple intersection of C^m and C^n. We employ the rule in the form given in Theorem VI. Thus:

(i) If U is an *ordinary* intersection, at which C^m and C^n have simple points and distinct tangents, then only one pair of corresponding points (P, Q) tends to U; the lengths PQ and PU are comparable infinitesimals (cf. Fig. 6); U counts *simply* therefore as a coincidence of P with Q, i.e. as an intersection of C^m and C^n.

(ii) If U is a point of *simple contact* of C^m and C^n (cf. Fig. 7), then only one pair (P, Q), with P on L, tends to U; but the length PQ is an infinitesimal of order 2 relative to PU. Thus U counts

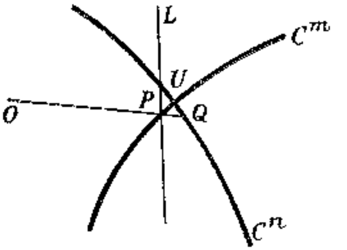


FIG. 6.

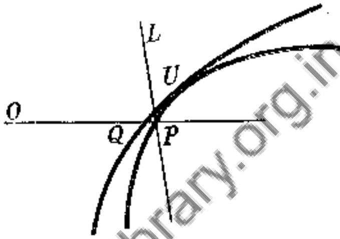


FIG. 7.

twice as an intersection of C^m and C^n. More generally, if U is an ordinary contact of order $r-1$, then PQ is of order r relative to PU, so that U is an r-fold intersection.

(iii) If U is an i-fold point of C^m and a j-fold point of C^n, and if the nodal tangents of C^m are all distinct from those of C^n at U, then L contains i points P close to U, to each of which correspond j points Q close to U; and all the lengths PQ and PU are like infinitesimals. Thus U counts ij-fold in the group of intersections of C^m and C^n.

In the same way we could evaluate the intersection multiplicity in more complicated cases. The method reduces the problem in each case to a local estimation of relative orders of magnitude of certain infinitesimal segments.

4.2 *Plücker's equations.* A plane curve may in general be regarded both as a point-locus, given by a point equation $F(x, y, z) = 0$, and as an envelope, given by a tangential equation $\Phi(l, m, n) = 0$; and in many cases, by shifting from one of these aspects to the other, we may usefully apply the principle of duality to derive new formulae from such as we may have already discovered. Before attempting to do this, however, we must recognize and overcome one difficulty, namely that, as regards the point and tangential *equations* involved, a general point-locus is a special type of envelope, and a general envelope is a special type of point-locus.

Consider a point-locus C represented in homogeneous coordinates by $F(x,y,z) = 0$; and consider also the envelope C', represented by $F(l,m,n) = 0$, which is dual to C. To a double point of C there will correspond a double tangent of C'; and if the double point becomes a cusp (by coincidence of the nodal tangents), the double tangent becomes an inflexional tangent (by coincidence of the points of contact).

Now a point-locus, given by a general equation of order n in x, y, z, has no double points or cusps, but has in general some double tangents and inflexions; and dually, a general envelope has double points and cusps, but no double tangents or inflexions. Hence in order to apply the principle of duality, we must allow the curves under consideration to have both classes of singularities, point and tangential. For simplicity, we suppose the typical curve C which we propose to study to have no higher singularities than those we have just mentioned; that is to say, we suppose that the enumerative properties of C are sufficiently described, for our present purposes, by the following six characters:

- n the order of C,
- m the class of C,
- δ the number of double points of C,
- τ the number of double tangents of C,
- κ the number of cusps of C,
- ι the number of inflexions of C.

These six characters are called the *Plücker numbers* of C. We look for relations among them, remembering that, from every relation we find, we may derive a dual relation by the substitution

$$\begin{pmatrix} n & m & \delta & \tau & \kappa & \iota \\ m & n & \tau & \delta & \iota & \kappa \end{pmatrix}.$$

4.21. *The class m.* To determine the class of C, we consider the simply infinite system of point-pairs (P,Q) of C, whose joins PQ pass through a fixed point O; this system is the analogue for a single curve of that depending on two distinct curves in the proof of Bézout's Theorem. The indices α_1, α_2, μ of the system are each equal to $n(n-1)$; and the number of coincidences is therefore $n(n-1)$. A coincidence is either (i) a point of contact of a proper tangent to C from O, or (ii) a double-point of C, or (iii) a cusp of C.

A point of contact T counts once as a coincidence; for on applying Theorem V, with the line L as shown in Fig. 8, one pair (P, Q) tends to T, and the distances PQ, PT are like infinitesimals.

A double point D counts twice as a coincidence (cf. Fig. 9); for two pairs (P_1, Q_1) and (P_2, Q_2) tend to D, and the lengths concerned are comparable as infinitesimals.

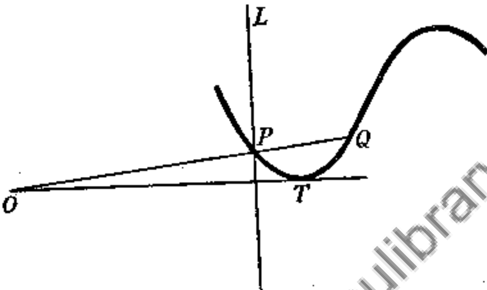


FIG. 8.

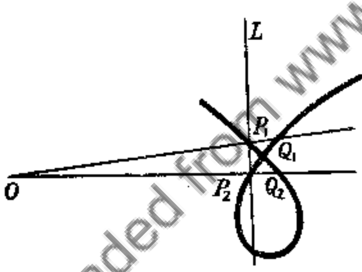


FIG. 9.

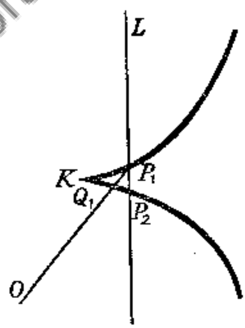


FIG. 10.

A cusp K counts thrice as a coincidence; for again two pairs (P_1, Q_1) and (P_2, Q_2) tend to K (cf. Fig. 10), but each of $P_1 Q_1$ and $P_2 Q_2$ is of order 3/2 as compared with $P_1 K$ or $P_2 K$, as can be verified immediately by taking the local equation of the curve in the form $y = ax^{3/2}+\dots$. Thus

$$n(n-1) = m+2\delta+3\kappa. \tag{1}$$

This, and the corresponding dual equation, can be written in the forms

$$m = n(n-1)-2\delta-3\kappa, \tag{2}$$

$$n = m(m-1)-2\tau-3\iota. \tag{3}$$

4.22. *A further relation.* To find a further relation, we consider another simply infinite system of point-pairs of C, defined by the condition that Q is any one of the $n-2$ further points in which the tangent at P meets C. For this system, $\alpha_1 = n(n-2)$; and since $m-2$ tangents can be drawn from a point Q of the curve to touch it elsewhere, $\alpha_2 = n(m-2)$. Finally, $\mu = m(n-2)$; for every one of the m tangents to C through an arbitrary point

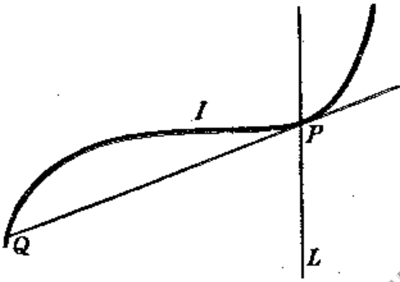


Fig. 11.

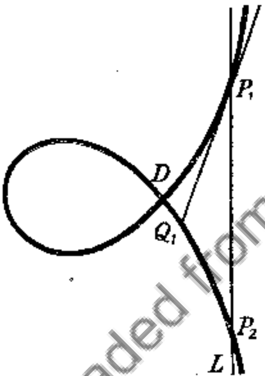


Fig. 12.

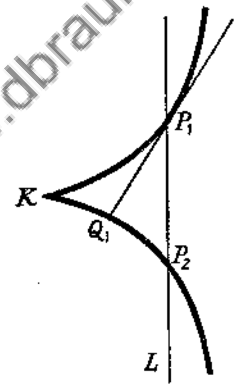


Fig. 13.

contains $n-2$ pairs (P, Q). Thus the number ξ of coincidences is given by

$$\xi = n^2-4n+2m = 3n(n-2)-4\delta-6\kappa.$$

The coincidences occur evidently at inflexions, double points, and cusps of C.

An inflexion I is a simple coincidence; for, in applying Theorem V, one pair (P, Q) tends to I, and PI, PQ are like infinitesimals whose ultimate ratio is actually $\frac{1}{2}$ (cf. Fig. 11).

A double point D counts twice (cf. Fig. 12); for two pairs (P_1, Q_1) and (P_2, Q_2) tend to D, and each of the lengths P_1Q_1 and P_2Q_2 is comparable with P_1D or P_2D.

A cusp K also counts twice (cf. Fig. 13); for the application of the rule leads to the same state of affairs as in the case of a double point. Thus $\xi = \iota+2\delta+2\kappa$, and this gives the formula

$$3n(n-2) = \iota+6\delta+8\kappa, \tag{4}$$

to which we add the dual relation

$$3m(m-2) = \kappa+6\tau+8\iota. \tag{5}$$

4.23. The relations (2), (3), (4), (5) are not independent, as appears by multiplying the first pair by 3 and adding; but any three of them are independent and suffice, for example, to express m, τ, ι in terms of n, δ, κ. In fact all possible relations between the six Plücker numbers are consequences of (2), (3), and (4). Among such relations we note in particular

$$3(m-n) = \iota-\kappa; \tag{6}$$

and if we introduce the genus p of C (cf. Ch. III, § 3.3), given by

$$p = \tfrac{1}{2}(n-1)(n-2)-\delta-\kappa, \tag{7}$$

we can derive a dual formula for the genus, namely

$$p = \tfrac{1}{2}(m-1)(m-2)-\tau-\iota. \tag{8}$$

Finally the reader may derive the particularly simple formulae

$$m = 2n+2p-2-\kappa, \tag{9}$$

$$n = 2m+2p-2-\iota. \tag{10}$$

4.24. *Plane cubics.* By means of the Plücker equations obtained above, we may characterize the three different types of plane cubic as follows:

	n	m	δ	κ	τ	ι	p
Non-singular cubic .	3	6	0	0	0	9	1
Nodal cubic . .	3	4	1	0	0	3	0
Cuspidal cubic . .	3	3	0	1	0	1	0

It may be noted that the dual concept—the curve of class three (or class-cubic)—is in general of order six with nine cusps. Only the cuspidal cubic is self-dual, with order and class both equal to three.

Examples

1. Find by the above methods the class and number of inflexions of a curve of order n possessing only ordinary multiple points O_i of multiplicities λ_i.

2. Obtain from Plücker's equations a proof that the generic rational curve of order n has $\frac{1}{2}(n-1)(n-2)$ double points.

3. Enumerate all the quartic curves which have only Plücker singularities, and find their Plücker numbers. Show that one such type is self-dual.

§ 5. The genus of a curve

5. Invariance of the genus. In this section we propose to give an elementary discussion of the fundamental invariant of any irreducible algebraic curve, namely its genus p, already mentioned in Ch. III, § 3.3. A quite independent and more far-reaching discussion will be given later in Ch. XII.

We recall that when a plane curve C, of order n, has only ordinary multiple points, the genus p of C is to be identified with the deficiency; and we note the useful fact that in this case the class m of C may be expressed in terms of n and p by the formula

$$m = 2n+2p-2. \tag{1}$$

We begin by proving

THEOREM VIII. *If the points of two irreducible plane curves, with only ordinary multiple points, can be put in* $(1,1)$ *algebraic correspondence, then the genera of the curves are equal.*

Let the curves be C, C', of orders n, n' and genera p, p', and let the variable points P on C and P' on C' be in algebraic $(1,1)$ correspondence. Plainly the genus of either curve will not be altered by applying to it an arbitrary collineation of the plane, or indeed any standard quadratic transformation which does not introduce multiple points which are not ordinary; so we may suppose, without loss of generality, that the multiple points of C are distinct from, and not specially related† to, those of C'. Our proof employs an intermediate curve Γ which we construct as follows.

We choose two points of the plane, namely A and B, which are generally situated with respect to C and C' and also with respect to the correspondence (P, P'); and, in particular, we suppose that (i) A, B do not lie on either curve, (ii) the join AB meets C in n distinct points and C' in n' distinct points, (iii) AB does not contain any pair of corresponding points P, P', (iv) no pair of

† We may assume, in particular, that to the i positions of P at any i-fold point of C there correspond i simple points P' of C'.

rays through A and B respectively touch C and C' at corresponding points P and P'. We then define Γ as the locus of the point of intersection Q of corresponding rays AP, BP'.

We observe first that Γ, being generated by an (n', n) correspondence between rays $a = AP$ and $b = BP'$ of the pencils (A) and (B), is of order $n+n'$; for this is the number of coincidences of the correspondence determined by a, b on an arbitrary line. Secondly, Γ has an *ordinary* n'-fold point at A and an ordinary n-fold point at B; for to the ray BA through B there correspond n' rays through A (distinct in virtue of (ii) and (iii)) which are tangents to n' branches of Γ at A; and similarly for B.

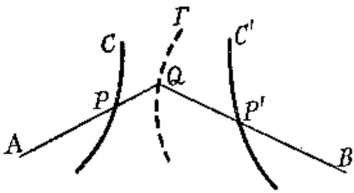


FIG. 14.

Since A and B are ordinary multiple points of Γ, the numbers μ_1, μ_2 of proper tangents from each of them, drawn to touch the curve at simple points elsewhere, are given by

$$\mu_1 = \mu - 2n', \qquad \mu_2 = \mu - 2n,$$

where μ is the class of Γ, each nodal tangent at either multiple point counting twice in the limiting set of μ tangents from that point. Our proof will consist then in showing that the sets of proper tangents from A to C and to Γ are identical, as also those from B to C' and to Γ; for this, by (1), would give us the relations

$$\mu - 2n' = 2n + 2p - 2,$$

$$\mu - 2n = 2n' + 2p' - 2,$$

since the multiple points of C, C' are ordinary; and these imply that $p' = p$.

Consider then proper tangents from A to C and to Γ. Any ray a of the first pencil meets C in n points P_i $(i = 1,..., n)$, to which there correspond successively n points P'_i of C', n rays BP'_i of the second pencil, and eventually n points Q_i of Γ on a. If a is a proper tangent to C, two of the points P_i are simple consecutive points of C. The pair of points P'_i corresponding to these will in general be simple consecutive points of C'; their join will not pass through B (condition (iv)); the ray joining one of them to B will not in general contain any other point P'_i corresponding to a different original point P_i on a; and hence they project from B

into simple consecutive points of Γ on a. Conversely also, it is easy to verify that, in general (using the full degree of generality available), a pair of simple consecutive points Q_i of Γ on a can only arise from a similar pair on C' and eventually from a pair of simple consecutive points of C on a. Thus the proper tangents from A to C are identical with those from A to Γ; and, by what we have already said, this proves the theorem.

We observe now that the above theorem, as pointed out in Ch. III, § 3.3, justifies us in defining the genus of any irreducible plane curve C as that of any Noether transform (with ordinary multiple points) of C. Also, with this, we can remove immediately the restriction that the curves C, C' referred to in the theorem should have only ordinary multiple points; for any pair of Noether transforms of C and C' will be in (1, 1) correspondence if C and C' are, and their genera, equal to those of C and C', must therefore be equal.

It should be noted that the above theorem provides only a necessary condition for the birational equivalence of two curves. A curve of genus $p > 0$ has, in fact, other birational invariants besides its genus, namely, certain invariant parameters called *moduli*. Thus, for example, it can be shown that two non-singular plane cubics are birationally transformable into each other if and only if a certain modulus has the same value for both curves, this modulus being a function of the constant cross-ratio of the tetrad of tangents drawn from a point of the curve to touch it elsewhere (see Ch. II, Ex. 15).

5.1. *Genus of a curve in space of any dimension.* Any irreducible curve C in r-dimensional space ($r > 2$) can be projected in many ways into a birationally equivalent curve C_1 in the plane; but all such projections, being birationally equivalent to each other, have the same genus. We therefore define the genus of C to be that of any plane birational projection of itself; and this leads at once to the following extension of the preceding theorem:

THEOREM IX. *The genus of any curve, defined where necessary as that of any plane birational projection of itself, is invariant over birational transformation.*

We note in conclusion that every simply infinite and irreducible algebraic family of entities has a genus, which is that namely of any curve whose points are in (1, 1) correspondence with the entities of the family. Thus, for example,

(a) the tangents to any plane curve of genus p form an envelope of genus p (cf. equation (8), § 4.23);

(b) any ruled surface, regarded as a simply infinite family of lines, has a genus, equal to that of the prime sections of the surface;

(c) any simply infinite family of planes in ordinary space has a genus, equal to that of the envelope of lines in which the planes meet a fixed plane.

EXAMPLE

Prove that the genus of a generic (α, β) curve on a quadric surface is $(\alpha-1)(\beta-1)$.

§ 6. CORRESPONDENCES BETWEEN CURVES

6. Zeuthen's formula. As an immediate extension of Theorem VIII, we may investigate on the same lines the relation between the genera p, p' of two curves C, C' whose points are in algebraic (α, α') correspondence. Such a correspondence will have in general a number δ of branch-points D on C, to each of which there corresponds a set of α' points of C' of which two coincide in a coincidence-point I' of C'; and it will have likewise a number δ' of branch-points D' on C', each associated with a coincidence-point I on C. The theorem of Zeuthen which we now propose to prove is as follows:

THEOREM X. *If there exists between the points of two curves C, C', of genera p, p', an (α, α') correspondence having δ branch-points on C and δ' branch-points on C', then*

$$\alpha'(2p-2)+\delta = \alpha(2p'-2)+\delta'. \tag{2}$$

Since we are evidently dealing with properties of curves which are invariant over birational transformation, we may suppose that C, C' are plane curves of orders n, n', with only ordinary multiple points; and since any particular s-fold point can always be resolved into s distinct points, we may legitimately assume that none of the branch-points or coincidence-points of the correspondence lie at the multiple points of either curve.

Proceeding then as in § 5, and keeping to the same notation, we construct as before the curve Γ which is the *image of all pairs of corresponding points (P, P') of the (α, α') correspondence.*

To this end, we choose A, B as before, but subject to the additional restriction that their join must contain no branch-point or coincidence-point; and we define Γ as the locus of the point of intersection Q of corresponding rays $a = AP$ and $b = BP'$ through A and B. Any ray a meets C in n points P_i ($i = 1,..., n$), to which correspond $n\alpha'$ points P'_{ij} ($j = 1,..., \alpha'$); and the joins of these to B are $n\alpha'$ rays b_{ij} which meet a in $n\alpha'$ points Q_{ij} of Γ. Thus the correspondence between a and b has indices $n'\alpha$, $n\alpha'$; and Γ is therefore a curve of order $n'\alpha+n\alpha'$, having ordinary multiple points, of multiplicities $n'\alpha$, $n\alpha'$, at A and B respectively.

Consider then the proper tangents from A to Γ. If two of the points Q_{ij} on a are simple consecutive points of Γ, then they may be assumed, for general choice of A, B, to arise from simple consecutive points of C' in the set P'_{ij} referred to above. Now two such points of C' may arise either from consecutive points of C belonging to the set P_i on a, or from the same point of this set. In the former case a is a proper tangent to C and the set P'_{ij} contains α' coincidences at points corresponding to the point of contact of a; and in the latter case a contains one of the δ branch-points D and the set P'_{ij} contains a single coincidence at the point I' associated with D. Thus, of the proper tangents from A to Γ, α' arise from each proper tangent to C from A, and one arises from each join of A to a branch-point D. If μ is the class of Γ, the number of proper tangents from A to Γ is $\mu-2n'\alpha$; and it follows therefore that

$$\mu-2n'\alpha = \delta+(2n+2p-2)\alpha'.$$

In the same way we deduce that

$$\mu-2n\alpha' = \delta'+(2n'+2p'-2)\alpha;$$

whence on eliminating μ we obtain the relation

$$(2p-2)\alpha'+\delta = (2p'-2)\alpha+\delta'.$$

The invariantive theory of curves provides an alternative proof of the above theorem, based on ideas of a different order (cf. Ch. XII, § 3.3); and it gives also general properties of the sets of united points in correspondences on an irrational curve (of arbitrary genus). The scope of the present chapter is limited because the only instruments we have as yet at our disposal are correspondences in a *rational* field.

EXAMPLE 1. If a plane non-singular cubic curve C^3 is projected from a point O of its plane on to a line l, then there is a $(3, 1)$ correspondence between the points of C^3 and l. Here $p = 1$, $p' = 0$, $\delta = 0$; so that, by the formula, $\delta' = 6$. This agrees with the fact that six tangents to C^3 pass through O.

If O lies on C^3, the resulting $(2, 1)$ correspondence has four branch-points on l.

EXAMPLE 2. If $\phi_0,..., \phi_n$ are $n+1$ linearly independent polynomials of order n in a parameter t, the general plane parametric curve C whose equations are $x_0:x_1:x_2 = \phi_0:\phi_1:\phi_2$ can be considered as the projection of the rational normal curve Γ^n whose equations are $\rho x_i = \phi_i$ $(i = 0,..., n)$, from the $[n-3]$ joining the points of reference $X_3,..., X_n$. If the projection is not birational, suppose that σ points of Γ^n project into a single point of C and that C is of genus p. For the $(\sigma, 1)$ correspondence between Γ^n and C, we have $\delta = 0$, and hence

$$\delta' = -2-(2p-2)\sigma.$$

Since $\delta' \geqslant 0$, this gives $p = 0$. Thus C is rational (cf. § 1.3).

§ 7. CHARACTERS OF SPACE CURVES

7. The general non-singular curve in S_3. In space S_r of any dimension r, the two characters of a curve which we regard as most important are its order and its genus, a curve of order n and genus p being denoted frequently by a symbol of the form ${}^pC^n$. When a ${}^pC^n$ is projected from a vertex V in S_r into a ${}^{p_1}C^{n_1}$ in S_{r_1}, then the sets of free points in which it is met by primes through V project into the sets of points in which the projected curve is met by primes of S_{r_1}; so that $n_1 = n-d$, where d is the number of points of ${}^pC^n$ which lie in V. On the other hand, $p = p_1$ by definition, provided only that the projection is birational.

Consider now a non-singular curve C, of order n and genus p, in ordinary space S_3; and let C' be its projection from a generic point O of space on to a plane Π. Then C' is also of order n and genus p.

The *rank* r of C we define to be the number of tangents of C which meet an arbitrary line. This number is equal to the class of C'; for tangents to C' which pass through a point O' of Π are projections of tangents to C which meet OO'; and hence, by equation (1) of § 5,

$$r = 2n+2p-2. \qquad (1)$$

The *number of apparent double points of C*, which we denote by h, is the number of chords of C which pass through an arbitrary point. Locating this arbitrary point at O, the number in question must be the number of double points of C'; and hence

$$h = \tfrac{1}{2}(n-1)(n-2)-p. \tag{2}$$

The *class* n' of C we define to be the number of osculating planes of C which pass through an arbitrary point. An osculating plane of C which passes through O clearly projects into an inflexional line of C'; and hence, by equation (4) of § 4.22, in which we write $\delta = h$ and $\kappa = 0$, we obtain the relation

$$n' = 3n(n-2)-6h = 3(n-2)+6p. \tag{3}$$

Finally the number of *bitangent-planes* of C which pass through an arbitrary point, a character which we denote† by τ, is equal to the number of bitangents of C'; whence, by equation (3) of § 4.21,

$$n = r(r-1)-2\tau-3n',$$

or

$$\tau = 2(n-2)(n-3)+2p(2n+p-7). \tag{4}$$

7.1. *The developable of planes.* A simply infinite algebraic system Δ of planes in ordinary space—dual of a curve—is called a developable of planes. For such a system it will be convenient to introduce certain terms to represent notions derived by duality from a curve C. Thus:

A *focal line* of Δ is a line of intersection of two consecutive planes of the system; it is dual to a tangent line of C.

A *focal point* of Δ is a point of intersection of three consecutive planes of the system; it is the dual of an osculating plane of C.

An *axis* of Δ is a line of intersection of any pair of planes of the system; it is the dual of a chord of C.

Any curve C defines an *osculating developable*, generated by its osculating planes; and dually, any developable Δ has a *cuspidal edge*, locus of its focal points. (The significance of the latter name will be apparent in the sequel.)

A developable has a genus p, identical, for example, with that of its cuspidal edge, whose points are in (1, 1) correspondence with the planes of the system. Its most important characters besides this are its *class* n'—the number of its planes which pass through an arbitrary point—and its *rank* r—the number of its focal lines which meet an arbitrary line.

If a developable is non-singular—in the appropriate dual sense—then four relations between its characters are provided by the

† In the systematic notation of § 7.2, this character is renamed ν'.

duals of equations (1),..., (4) above. Thus, for example, its rank is given by

$$r = 2n'+2p-2. \tag{5}$$

7.2 *The Cayley-Plücker equations.* For a curve C in space, it is well known that the osculating planes at simple consecutive points P, P' of C meet in the tangent to C at P; also that the osculating planes at consecutive points P, P', P'' meet in P. Hence

The cuspidal edge of the osculating developable of a curve C is C itself; and, dually, the osculating developable of the cuspidal edge of any developable Δ is Δ itself.

This means evidently that a space curve can be regarded, when we find it convenient to do so, as the cuspidal edge of its own osculating developable; and this forms the basis of the extension to space curves of the duality method employed in obtaining the complete set of Plücker equations of a plane curve.

As in the case of plane curves, however, a non-singular point-locus C has not in general a non-singular osculating developable. In fact C will have in general a finite number of *stationary osculating planes*, each of which meets the curve in four consecutive points; and each of these will be a *stationary double plane* (dual of a cusp) of the osculating developable, counting twice in any set of n' concurrent osculating planes to which it belongs. Dually, a non-singular developable will have in general a finite number of *stationary focal points* which will be *cusps of the cuspidal edge*. In order to be able to apply duality in the desired manner, we must allow the curve C which is to be the subject of our discussion to be possessed of a finite number κ of simple cusps; and for simplicity we suppose that it has no other multiple points of any kind.

We define therefore, for the curve C and for its osculating developable Δ, the following sets of dual characters:

Curve-locus C	*Osculating developable*
n = order	n' = class
r = rank	r = rank
h = number of chords through an arbitrary point	h' = number of axes in an arbitrary plane
ν = number of points in a plane through which pass two tangents	ν' = number of planes through a point which contain two tangents
κ = number of cusps	κ' = number of stationary osculating planes

We denote by T the ruled surface generated by tangents to C, noting that its order is r. This surface has a double curve—locus of intersections of pairs of distinct tangents to C—whose order is evidently ν.

To obtain relations between the nine characters defined above, we employ two processes which are dual to each other; namely, projection from a point O, and section by a plane Π. To lines of the cone V which projects C from O there correspond dually the lines of the envelope W in which Π is met by osculating planes of C. The point-locus enveloped by W we denote by w. To tangent planes of V, which join O to tangents of C, there correspond dually points of w; and hence w is the section by Π of the tangent surface T. To osculating planes of C through O—which are evidently *inflexional* tangent planes to V, since each contains three consecutive generators—there correspond by duality the intersections of C with Π, which are therefore *cusps* on the curve w. Thus:

The section of the tangent surface T by an arbitrary plane has cusps at the intersections of this plane with C. In other words, C is a cuspidal curve on its own tangent surface.

This explains the name *cuspidal edge* for the locus of focal points of a developable.

Continuing the argument, we observe that V has h double generators, which are, namely, the chords of C which pass through O; and hence, dually, w has h' double tangents. Also V has ν' double tangent planes to which correspond by duality ν double points of w. Finally V has κ cuspidal generators to which correspond by duality κ' inflexional lines of w.

We can now write down all the usual Plücker numbers of a generic plane section of V—which is a plane projection of C—and of a generic plane section w of T. Thus:

	Order	*Class*	*Double points*	*Cusps*	*Inflexions*	*Double tangents*
Projection of C . .	n	r	h	κ	n'	ν'
Section of T . .	r	n'	ν	n	κ'	h'

By applying Plücker's equations to each of these curves we obtain six independent relations between the nine assumed characters of C. For convenience, however, we adjoin the genus p of the curve as an additional variable; and we may then take a fundamental set of seven relations to be as follows:

$$p = \tfrac{1}{2}(n-1)(n-2)-h-\kappa, \tag{6}$$

$$p = \tfrac{1}{2}(r-1)(r-2)-n'-\nu', \tag{7}$$

$$p = \tfrac{1}{2}(r-1)(r-2)-\nu-n, \tag{8}$$

$$p = \tfrac{1}{2}(n'-1)(n'-2)-\kappa'-h', \tag{9}$$

$$r = 2n+2p-2-\kappa, \tag{10}$$

$$r = 2n'+2p-2-\kappa', \tag{11}$$

$$n+n' = 2r+2p-2. \tag{12}$$

Either (7) or (8) may evidently be replaced by

$$n-n' = \nu'-\nu. \tag{13}$$

The modification of these results which become necessary when C possesses actual double points, inflexions, double osculating planes, or double tangents presents no special difficulty (cf. Ex. 4 below).

Examples

1. For a twisted cubic $n = 3$ and $p = 0$; and $\nu, \nu', \kappa, \kappa'$ are all necessarily zero. The formulae give $n' = 3$, $r = 4$, and $h = h' = 1$.

2. If C is the curve of intersection of two general quadrics, then $n = 4$, $h = 2$, and $\kappa = 0$. The formulae give

$$p = 1, \quad r = 8, \quad n' = 12, \quad \nu = 16, \quad \nu' = 8, \quad \kappa' = 8.$$

The tangent surface is an octavic ruled surface having C as cuspidal curve and having a further ordinary double curve of order 16.

3. Find all the Cayley-Plücker characters of a general rational quartic curve in space.

4. If a curve C in space has δ ordinary double points, δ' double osculating planes, σ double tangent lines, and ι inflexional lines, then these give rise to the following *additional* special points or tangents of the projection and section:

(*a*) Projection of C: δ double points, σ double tangents, ι inflexions.

(*b*) Section of T: δ' double tangents, σ double points, ι cusps.

Calculate the values of the characters of a curve of intersection of two quadrics which has (*a*) a double point, (*b*) a cusp.

§ 8. Space curves defined as intersections

8. The most straightforward way to define a curve in space is to define it as a complete or partial intersection of two surfaces; that is to say, we describe the curve as the complete intersection of two given surfaces, or we describe it as the residual curve of intersection of two surfaces which pass through one or more assigned curves. In either case there arises the problem of determining the characters of a curve so defined in terms of the orders of the surfaces and the characters of the assigned curve or curves. In what follows we apply the Correspondence Principle to the solution of this problem.

We shall find it most convenient to apply the Correspondence Principle in the form of a theorem which is the space-dual of the space-generalization of Theorem III, to which we have already referred in § 2.1; this theorem may be stated as follows:

If $(\alpha_1, \alpha_2; \mu)$ is the set of indices of any simply infinite family of pairs of corresponding planes (p, q) of space, then the number of coincidences of p with q is $\alpha_1+\alpha_2-\mu$.

Strictly speaking, the theorem refers to a simply infinite family of forms $(p, q; L)$, each consisting of a pair of planes p, q intersecting in a line L (limiting or otherwise); the index α_1 is the number of forms of the family in which p passes through a given point; α_2 is the corresponding number associated with q; and μ is the number of forms in which L meets a given line.

8.1. *Complete intersection of two surfaces.* If two surfaces F_1, F_2, of orders N_1, N_2, meet in a curve C, then this curve, by the theorem of Bézout, will be of order $n = N_1 N_2$. Supposing C to be free from multiple points, we propose to calculate its rank r, genus p, and number h of apparent double points.

We consider the pair of tangent planes p_1, p_2 to F_1, F_2 respectively at any point P of C, these planes intersecting in the tangent line L to C at P. As P describes C, the fundamental form $(p_1, p_2; L)$ describes a simply infinite family Σ_1 of such forms. The index α_1 of Σ_1 is the number of tangent planes to F_1 at points of C which pass through an arbitrary point O; but this is the same as the number of points in which the first polar of O with respect to F_1 meets C. Hence $\alpha_1 = n(N_1-1)$; and similarly $\alpha_2 = n(N_2-1)$. The third index μ of Σ_1 is the number of tangent lines L of C which meet an arbitrary line; so that $\mu = r$. Now since C has no multiple points, p_1 never coincides with p_2; and hence, by the theorem quoted above,

$$0 = n(N_1-1)+n(N_2-1)-r,$$

or

$$r = n(N_1+N_2-2) = N_1 N_2(N_1+N_2-2). \tag{1}$$

From the formula $r = 2n+2p-2$, we deduce that

$$2p-2 = N_1 N_2(N_1+N_2-4); \tag{2}$$

and by projecting C on to a plane we find that

$$h = \tfrac{1}{2}N_1 N_2(N_1-1)(N_2-1). \tag{3}$$

If F_1 and F_2 touch at any point U, then their curve of intersection has in general a double point at U, and their common tangent plane there is a true coincidence of p_1 with p_2; but this coincidence occurs in two distinct forms of Σ_1, in which the lines L are respectively the two nodal tangents to C at U. A simple application of Zeuthen's rule shows that such a coincidence counts twice. Hence if C has δ ordinary double points, the formula for r becomes

$$r = N_1 N_2(N_1+N_2-2)-2\delta, \tag{4}$$

and there are consequent modifications in the formulae for p and h.

8.2. ***Intersection of two surfaces in two curves.*** We suppose now that F_1, F_2 are surfaces of assigned orders N_1, N_2, which are made to pass through a given non-singular curve C of order n, rank r, and genus p. The surfaces will in general intersect residually in a curve C' which will meet C in some points. Assuming that C' is also non-singular and that the intersections of C with C' are all distinct, we propose to calculate the number i of these intersections and the order n', rank r', and genus p' of C'.

The value of n' is given by the equation

$$n+n' = N_1 N_2. \tag{5}$$

To evaluate i, we consider the correspondence between the tangent planes to F_1, F_2 at a variable point of C; and we find, as in the preceding case, that the indices of this correspondence have the values given by

$$\alpha_1 = n(N_1-1), \qquad \alpha_2 = n(N_2-1), \qquad \mu = r.$$

The only coincidences of the correspondence arise from the contacts of F_1, F_2 at the intersections of C with C'; and the coincidence arising from each such contact is a simple one. Thus

$$i = n(N_1+N_2-2)-r,$$

or

$$r+i = n(N_1+N_2-2). \tag{6}$$

Similarly

$$r'+i = n'(N_1+N_2-2). \tag{7}$$

From (6), writing $r = 2n+2p-2$, we obtain

$$i = n(N_1+N_2-4)-(2p-2), \tag{8}$$

which determines i; and from (6) and (7) we derive the relations

$$r'-r = (n'-n)(N_1+N_2-2), \tag{9}$$

and

$$2(p'-p) = (n'-n)(N_1+N_2-4), \tag{10}$$

which determine r' and p' as required.

From (6) and (7) we obtain by addition

$$r+r'+2i = N_1 N_2(N_1+N_2-2), \tag{11}$$

and this agrees with (4). It suggests also, by comparison with (1), that the system of tangents of the composite curve $C+C'$ may be taken to consist of (*a*) tangents of C, (*b*) tangents of C', (*c*) i plane pencils, each counted twice, at the intersections of C with C'.

8.3. ***Numerical characters of a composite curve.*** The method of the preceding section may be employed, with only trivial modifications, in the general case in which F_1, F_2 intersect in $s+1$ non-singular curves C_α ($\alpha = 1,..., s+1$), which meet each other only in distinct points. If we denote the order, rank, and genus of C_α by n_α, r_α, p_α, and the number of intersections of C_α, C_β by $i_{\alpha\beta}$, then the correspondence of tangent planes to F_1, F_2 along C_α gives the relation

$$r_\alpha + \sum_{\lambda\neq\alpha} i_{\alpha\lambda} = n_\alpha(N_1+N_2-2); \tag{12}$$

and the $s+1$ equations of this type suffice to determine the rank of C_{s+1} and the numbers $i_{s+1,\lambda}$ ($\lambda = 1,..., s$), all the other numbers being supposed to be known.

We may also use the above equations to investigate the way in which an arbitrary composite curve C behaves, when it functions as a partial intersection of a pair of surfaces. To this end we take C to be the curve composed of the s curves $C_1,..., C_s$, residual to C_{s+1}; so that it is a curve of order $n = \sum_1^s n_\alpha$. We write i for the total number of mutual intersections of components of C; we write I for the total number of points in which C meets C_{s+1}; and we introduce the number r which is defined by the equation

$$r = \sum_1^s r_\alpha + 2i. \tag{13}$$

The result of adding all but the last of the $s+1$ equations such as (12) may be written in the form

$$r+I = n(N_1+N_2-2),$$

and the last of these equations is

$$r_{s+1}+I = n_{s+1}(N_1+N_2-2).$$

Comparing these with (6) and (7), it appears that when C functions as a partial intersection, it behaves enumeratively like a curve of rank r. The number r defined by (13) will therefore be called the *numerical rank* of C. Writing $r = 2n+2p-2$, we find for the numerical genus p of C the formula

$$2p-2 = \sum_1^s (2p_\alpha-2)+2i, \tag{14}$$

or

$$p = \sum_1^s p_\alpha+i-s+1. \tag{15}$$

These results may be recorded as follows:

THEOREM XI. *A composite curve C, composed of s distinct non-singular curves C_α which intersect each other in i distinct points, behaves, as a partial intersection of two surfaces, like a single (non-singular) curve of rank r and genus p given by*

$$r = \sum_1^s r_\alpha+2i, \qquad p = \sum_1^s p_\alpha+i-s+1.$$

This theorem applies of course to disconnected as well as to connected composite curves; and it is quite independent of the possibility or impossibility of generating C as the limit of a variable irreducible curve. If C is such a limit, however, the formula for r admits of a simple limiting interpretation.

The numerical genus of s skew lines is $1-s$, indicating that for a composite curve p may be negative. Conversely, if a residual intersection curve is found by calculation to have a negative genus, then it must be composite.

8.4. For purposes of illustration we add a list of all non-singular space curves of orders 3, 4, and 5. A curve which is the residual intersection of surfaces of orders m and n passing through assigned curves C, C',..., will be represented symbolically in the form $F^m . F^n-(C+C'...)$; the curves C, C',..., are non-intersecting, and the symbols l, l' represent lines.

(1) Twisted cubic $C^3 \equiv F^2.F^2-(l)$.

(2) Rational quartic ${}^0C^4 \equiv F^2.F^3-(l+l')$.

(3) Elliptic quartic ${}^1C^4 \equiv F^2.F^2$.

(4) Rational quintic ${}^0C^5 \equiv F^3.F^3-(l+{}^0C^3)$.

(5) Elliptic quintic ${}^1C^5 \equiv F^3.F^3-({}^0C^4)$.

(6) Quintic of genus 2 ${}^2C^5 \equiv \begin{cases} F^2.F^3-(l), \\ F^3.F^3-({}^1C^4). \end{cases}$

There exist non-singular sextics of genera 0, 1, 2, 3, 4. If $p = 4$, the curve is the complete intersection of a quadric and a cubic surface; while if $p = 3$, *either* ${}^3C^6 \equiv F^3.F^3-({}^0C^3)$, *or* ${}^3C^6$ is a (2, 4) curve on a quadric.

EXAMPLES

1. If surfaces of orders N_1, N_2, N_3 pass through a non-singular curve of order n and genus p, prove that the number (supposed finite) of their residual points of intersection is $N_1 N_2 N_3 - E$, where

$$E = n(N_1+N_2+N_3-4)-2p+2.$$

(E is called the *equivalence* of the curve for the surfaces concerned.)

2. If surfaces of order N are passed through a non-singular curve C (possibly composite) of (numerical) order n and genus p, and if these surfaces do not possess any further common curve, prove that the genus π of the residual intersection curve of two of them and the number γ of residual intersections of any three of them are given by

$$\pi = p+(N-2)(N^2-2n),$$
$$\gamma = N^3-n(3N-4)+2p-2.$$

(π and γ are called the *linear genus* and *grade* of the system.)

3. Exhibit, after the manner of § 8.4, the sextic curves of genera 1, 2, 3, 4 which are partial intersections of two cubic surfaces.

4. Show that a skew polygon of n sides has numerical genus 1 or 0 according as it is closed or open.

5. If quartic surfaces are drawn through generic non-singular curves ${}^0C^3$ and ${}^0C^4$, prove that two of these surfaces meet residually in a ${}^3C^9$, and find the numbers of points in which this meets ${}^0C^3$ and ${}^0C^4$. Show also that three of the surfaces meet residually in 12 points.

BOOKS RECOMMENDED FOR FURTHER READING

ZEUTHEN, *Lehrbuch der abzählenden Methoden in der Geometrie.*
ENRIQUES-CHISINI, *Teoria geometrica delle equazioni*, ii, Book III.
BAKER, *Principles of geometry*, v, ch. viii, and vi, ch. ii.
SCHUBERT, *Kalkül der abzählenden Geometrie.*
BERTINI, *Complementi*, § 7.
COOLIDGE, *Algebraic plane curves*, Book II, ch. iv.

CHAPTER V

SYSTEMS OF PLANE CURVES

§ 1. Noether's Theorem

1. In special cases we have already had occasion to assume that any curve through all the n^2 intersections of two curves of the same order n necessarily belongs to the linear pencil of curves of order n defined by the two curves. This result, as it happens, is itself only a particular case of a general theorem, of fundamental importance for the whole of algebraic geometry—the so-called $Af+B\phi$ theorem of Noether. It is our object in this chapter to introduce the reader to this theorem by proving it in the simplest case, by indicating the method by which the theorem may be extended to the so-called 'simple case', and by using it to prove some other results of geometrical interest. For complete and detailed discussions of the theorem we refer the reader to Severi-Löffler, *Vorlesungen über algebraische Geometrie*, ch. v, and to van der Waerden, *Einführung in die algebraische Geometrie*, ch. viii.

The simplest case, which we now prove, may be stated as follows:

THEOREM I. *If a C^m and a C^n, whose equations are $U = 0$ and $V = 0$ respectively, intersect in mn distinct points, then any C^r which passes through all the intersections of the C^m and the C^n has an equation of the form $AU+BV = 0$, where A and B are polynomials of degrees $r-m$ and $r-n$ respectively.*

We prove the theorem in three stages: (i) for sufficiently large r, (ii) for $r \geqslant m+n$, and (iii) for $r < m+n$.

(i) In the first place, for sufficiently large values of r, exactly mn conditions are imposed on C^r by its passage through the points (UV). To show this, it is sufficient to observe that $mn-1$ lines, each containing only one of these points, constitute a curve of order $mn-1$ which passes through $mn-1$ of the points (UV) without necessarily containing the remaining one; whence, if $r > mn-1$, the independence of the conditions is certainly assured. The equation of C^r must therefore contain $\frac{1}{2}(r+1)(r+2)-mn$ arbitrary coefficients.

As against this, we have to consider the freedom of curves whose equations can be written in the form $AU+BV = 0$. The arbitrary coefficients in the expression $AU+BV$ are those occurring in the polynomials A and B, which are of orders $r-m$ and $r-n$ respectively; but this expression is associated with the same curve as is associated with any expression of the form

$$(A-VX)U+(B+UX)V,$$

in which X is an arbitrary polynomial of degree $r-m-n$. Allowing for this, the number of arbitrary coefficients in the equation of a C^r which can be expressed in the required form is

$$\tfrac{1}{2}(r-m+1)(r-m+2)+\tfrac{1}{2}(r-n+1)(r-n+2)-$$
$$-\tfrac{1}{2}(r-m-n+1)(r-m-n+2) = \tfrac{1}{2}(r+1)(r+2)-mn.$$

This proves the theorem for values of r exceeding $mn-1$.

(ii) Next, supposing that $r \geqslant m+n$, we shall show that the theorem is true for curves of order $r-1$ if it holds for curves of order r.

Let F (or $F = 0$) represent a curve of order $r-1$, passing through the points (UV) and let L (or $L = 0$) represent an arbitrary line. Then LF is a curve of order r which satisfies the conditions of our inductive hypothesis, so that

$$LF \equiv AU+BV \equiv (A-VX)U+(B+UX)V,$$

where X is the arbitrary polynomial of degree $r-m-n$ already referred to and contains $\tfrac{1}{2}(r-m-n+1)(r-m-n+2)$ arbitrary coefficients, a number which is at least equal to $r-m-n+1$.

The line L meets U in m points not lying on V; so that, by the above equation, they must lie on B and therefore on $B+UX$. Similarly L meets V in n points lying on A and therefore on $A-VX$. Now choose X so that $A-VX$ passes through $r-m-n+1$ further points of L and thus, in all, through $r-m+1$ points of this line. The curve $A-VX$ will then contain L as part, so that we may write

$$A-VX \equiv LA',$$

where A' is of order $r-m-1$. It follows now that

$$B+UX \equiv LB',$$

where B' is of order $r-n-1$. Hence

$$F \equiv A'U+B'V.$$

The induction process thus established proves Noether's Theorem for $r \geqslant m+n-1$.

(iii) Lastly suppose that $r = m+n-\gamma$, where $\gamma > 0$ and r is not less than m or n.

Let C^r have equation $F = 0$, and let $F' = 0$ be the equation of any curve C^γ not passing through any of the points (UV). Applying the previous result to the curve FF', we may write

$$FF' \equiv AU+BV \equiv (A-kV)U+(B+kU)V,$$

where A and B are of orders n and m respectively and k is an arbitrary constant.

The $m\gamma$ points (UC^γ) must all lie on B and therefore on $B+kU$; hence, if we choose k so that the latter curve meets C^γ in one further point, it must break up into C^γ and a curve B' of order $m-\gamma$. It follows then that $A-kV$ must break up into C^γ and a curve A' of order $n-\gamma$, so that

$$F \equiv A'U+B'V.$$

This completes the proof of what we have called the simplest case of Noether's Theorem. The so-called 'simple case' may be stated as follows:

THEOREM I*a*. *If $U = 0$ and $V = 0$ are curves of orders m and n, and if these curves are such that at any point P which is common to both and which is h-fold on U and k-fold on V the branch tangents of U are all distinct from those of V, and if C^r is any curve which has an $(h+k-1)$-fold point, at least, at every point such as P, then the equation of C^r may be written in the form $AU+BV = 0$, where A and B are polynomials, of degrees $r-m$ and $r-n$, such that the curves $A = 0$ and $B = 0$ have points of multiplicities $k-1$ and $h-1$ at least at every point P.*

The proof of this result can be constructed on almost exactly the same lines as that already given, the main differences being in the more complicated estimations of freedom due to the assigned multiple points of C^r, and also (in this case) the less trivial verification of the fact that for sufficiently large r the conditions imposed by these assigned multiple points are independent.

We state, finally, the most general form of the theorem for plane curves, based on the concept of infinitely near intersections of curves:

THEOREM I*b*. *The conclusion of Theorem I a remains valid for any algebraic curves U and V, intersecting in any arbitrary manner,* *provided only that U and V have no common part and that C^r has an $(h+k-1)$-fold point at least at every common point, whether explicit or implicit, which is h-fold for U and k-fold for V.*

EXAMPLES

1. If a C^m and a C^n meet in mn points of which nr lie on a C^r, prove that the remaining $(m-r)n$ lie on a C^{m-r}.

2. Deduce Pascal's theorem for a conic from this result.

3. If a variable cubic is drawn through seven fixed points of a fixed cubic C_1, show that the line joining the two remaining intersections passes through a fixed point of C_1.

(Solution: Let C_2 and C_3 be two of the cubics through the seven fixed points, and let L_1 be the join of their two remaining intersections, with similar meanings for L_2 and L_3. By Noether's Theorem we have $L_1 C_1 \equiv AC_2+BC_3$ and we identify A and B as L_2 and L_3. Hence the three points $(L_3 C_1)$ lie either on C_2 or L_2. Since L_3, C_1, and C_2 meet only in two points, it follows that L_2 and L_3 meet on C_1.)

1.1. ***The Cayley–Bacharach Theorem.*** Our first applications of Noether's Theorem are to problems on *tied systems of points*, i.e. sets of points such that the passage of curves of a certain order through some of them implies the passage of these curves through the rest.

From the last part of the proof of Noether's Theorem it appears that, when $r < m+n$, the curve C^r can in general be made to contain all the points (UV) by imposing on it fewer than mn conditions. For, if $r = m+n-\gamma$, the number of arbitrary coefficients in the form $AU+BV$ is

$$\tfrac{1}{2}(r-m+1)(r-m+2)+\tfrac{1}{2}(r-n+1)(r-n+2)$$
$$= \tfrac{1}{2}(r+1)(r+2)-mn+\tfrac{1}{2}(\gamma-1)(\gamma-2).$$

Thus when $\gamma > 2$, the number of conditions imposed on C^r is less than mn. In particular, taking $\gamma = 3$, we may prove

THEOREM II. *Every curve C^{m+n-3} which passes through $mn-1$ of the points (UV), assumed distinct, passes through them all.*

Suppose that the curve in question has equation $W = 0$, and let $L = 0$ be any line through the last of the points (UV). Then, by Noether's Theorem,

$$LW \equiv AU+BV.$$

Since L meets V in n points, of which one only lies on U, $n-1$ must lie on A which is a curve of order $n-2$. Hence A, and therefore also B, contains L as part. Writing $A = LA'$ and $B = LB'$, we have

$$W \equiv A'U + B'V,$$

which proves the result stated.

1.2. Consider next the case $r = m+n-\gamma$, where $\gamma > 3$ but is so restricted that r is not less than m or n. We prove then

THEOREM III. *Every $C^{m+n-\gamma}$ $(\gamma > 3)$ which passes through $mn - \frac{1}{2}(\gamma-1)(\gamma-2)$ of the points (UV) passes through the remainder, except when these remaining $\frac{1}{2}(\gamma-1)(\gamma-2)$ points lie on a $C^{\gamma-3}$.*

Denote by $P_1, P_2, \ldots$ the $\frac{1}{2}(\gamma-1)(\gamma-2)$ points (UV) through which the $C^{m+n-\gamma}$ is not assumed to pass. Through all but one of these, say P_1, we can draw a curve $C^{\gamma-3}$ which, with $C^{m+n-\gamma}$, constitutes a C^{m+n-3} passing through all but one of (UV); hence, by Theorem II, it will contain P_1.

There are now two possibilities: if P_1 lies on $C^{\gamma-3}$, we have the exceptional case mentioned in the theorem; but if not, P_1 must lie on $C^{m+n-\gamma}$.

We now repeat the argument, taking P_2 instead of P_1; and again, if $P_1, P_2, \ldots$ do not lie on a $C^{\gamma-3}$, it follows that P_2 also must lie on $C^{m+n-\gamma}$. And so for $P_3, P_4, \ldots$.

The above two theorems were originally enunciated by Cayley in a single statement which took no account of the exceptional case. The necessary modification is due to Bacharach (*Math. Ann.* **26**, 275).

It may be noticed that both the theorems in question admit immediate extensions by use of Theorem I *a* in place of Theorem I. Thus, for example, we have

THEOREM II *a*. *If some of the intersections of U and V are multiple and others simple, each multiple intersection, h-fold on U and k-fold on V, being such that the branch tangents of U are distinct from those of V, then every C^{m+n-3} which passes $(h+k-1)$-fold through each multiple intersection and also through all but one of the simple intersections passes also through the last simple intersection.*

EXAMPLES

1. All curves C^n passing through $\frac{1}{2}n(n+3)-1$ general points pass through $\frac{1}{2}(n-1)(n-2)$ further points.

2. If two quartics meet in 16 distinct points, any other quartic which passes through 13 of them will contain the remaining 3, unless these happen to be collinear. In this exceptional case, prove that quartics through the 13 points form a net (instead of a pencil), that any pair of them meet

residually in three collinear points, and that the line of collinearity may be any line of the plane.

3. Show that, if more than $\frac{1}{2}(\gamma-1)(\gamma-2)$ of the points (UV) lie on a $C^{\gamma-3}$, the remaining points have the property that the curves $C^{m+n-\gamma}$ which pass through all but one pass through all.

4. If two quartics, nodal at O, meet in 12 further distinct points, prove that these impose only 11 conditions on quintics which have an assigned triple point at O.

§ 2. Serret's Theorem

2. The Cayley–Bacharach Theorem, and other properties of tied point sets, can be established by an analytical method, due to Serret, which is based on the following result:

THEOREM IV. *The necessary and sufficient condition that any curve C^r, of given order r, which passes through $q-1$ of a set of q given points should pass through them all, is that there should exist a linear relation (or syzygy) connecting the rth powers of the tangential equations of the q points.*

Let the given points be (a_s, b_s, c_s) $(s = 1, 2, ..., q)$, and let the equation of C^r be
$$\alpha x^r + \beta x^{r-1}y + ... = 0.$$
If C^r passes through the first $q-1$ points of the set, we have
$$\alpha a_s^r + \beta a_s^{r-1}b_s + ... = 0 \quad (s = 1, 2, ..., q-1). \tag{1}$$
Then if, by virtue of (1), the equation
$$\alpha a_q^r + \beta a_q^{r-1}b + ... = 0$$
holds for all values of $\alpha, \beta, ...,$ we must have an identity of the form
$$\sum_{s=1}^{q-1} \lambda_s \{\alpha a_s^r + \beta a_s^{r-1}b_s + ...\} \equiv \lambda_q(\alpha a_q^r + \beta a_q^{r-1}b_q + ...) \tag{2}$$
for all values of $\alpha, \beta, ...,$ where $\lambda_1, \lambda_2, ..., \lambda_q$ are constants and $\lambda_q \neq 0$. Equating coefficients of $\alpha, \beta, ...,$ we have
$$\lambda_q a_q^r = \sum \lambda_s a_s^r, \qquad \lambda_q a_q^{r-1}b_q = \sum \lambda_s a_s^{r-1}b_s,$$
and so on.

These equations can be combined into a single identity. For if
$$a_s u + b_s v + c_s w = 0$$
is the tangential equation of the point (a_s, b_s, c_s), we have the syzygy
$$\sum_{s=1}^{q} k_s(a_s u + b_s v + c_s w)^r \equiv 0 \tag{3}$$
—an identity in u, v, w—in which the constants $k_1, k_2, ...$ are the same as $\lambda_1, \lambda_2, ...,$ except that the sign of the last is reversed.

The necessity of Serret's condition is thus established; to prove its sufficiency, starting from (3), we have merely to reverse the steps of the argument.

It is clear from the nature of the proof that the theorem extends at once to primals in any number of dimensions.

EXAMPLES

1. If $q = \frac{1}{2}(r+1)(r+2)$, Serret's Theorem evidently expresses the condition that the q given points should lie on a C^r.

2. If $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ are the parameters of six points on the conic $(\theta^2, \theta, 1)$, and if $u_\alpha = 0$, $u_\beta = 0$,... are the equations of the points, prove that

$$\sum u_\alpha^2/(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)(\alpha-\eta) = 0.$$

3. Prove that, if $F(x,y,z)$ is any homogeneous polynomial of degree r in x, y, z, then

$$F\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right)[(au+bv+cw)^r] = r!\,F(a,b,c).$$

Deduce the sufficiency part of Serret's Theorem.

2.1. It was first pointed out by Richmond† that the Cayley-Bacharach Theorem is a simple consequence of Theorem IV.

Let (a_s, b_s, c_s) be any one of the points (UV). If we can establish the existence of a syzygy

$$\sum_{s=1}^{mn} k_s(a_s u+b_s v+c_s w)^{m+n-3} \equiv 0, \tag{4}$$

then Theorem II will follow. Actually the required relation (4) was obtained by Jacobi,‡ who showed that

$$k_s = a_s \Big/ \frac{\partial(U,V)}{\partial(b_s,c_s)} = b_s \Big/ \frac{\partial(U,V)}{\partial(c_s,a_s)} = c_s \Big/ \frac{\partial(U,V)}{\partial(a_s,b_s)}.$$

Again, since (4) is an identity, it can be differentiated repeatedly with respect to u, v, or w. Then, if $F(\partial/\partial u, \partial/\partial v, \partial/\partial w)$ is a homogeneous polynomial of degree ν in the operators, we evidently have

$$\sum_{s=1}^{mn} k_s F(a_s, b_s, c_s)(a_s u+b_s v+c_s w)^{m+n-3-\nu} = 0.$$

In particular, taking $\nu = \gamma-3$,

$$\sum_{s=1}^{mn} k_s F(a_s, b_s, c_s)(a_s u+b_s v+c_s w)^{m+n-\gamma} = 0. \tag{5}$$

Hence, if $F(x,y,z) = 0$ is the equation of a $C^{\gamma-3}$ passing through certain of the points (UV), $F(a_s, b_s, c_s)$ vanishes at these points, so that (5) reduces to a syzygy among the remaining points.

† *Proc. Lond. Math. Soc.* (2) **19** (1921), xxix. ‡ *Werke*, iii. 285, 610.

A curve $C^{m+n-\gamma}$ which passes through all but one of the latter will therefore pass through all. This proves Theorem III.

2.2. The following extension of Serret's Theorem is due to Clifford.†

THEOREM V. *Let α and β be the quotient and remainder when r is divided by n, and let*

$$N = \tfrac{1}{2}\alpha n(n+3)+\tfrac{1}{2}(\beta+1)(\beta+2).$$

Then, in order that N points should lie on a C^n, it is sufficient that a syzygy $\sum_{s=1}^{N} k_s(a_s u+b_s v+c_s w)^r \equiv 0$ should hold between them.

For if $F(x,y,z) = 0$ is the equation of a C^β passing through $\frac{1}{2}\beta(\beta+3)$ of the points, and we operate on the given identity with $F(\partial/\partial u, \partial/\partial v, \partial/\partial w)$, we are left with a syzygy of order αn between $\frac{1}{2}\alpha n(n+3)$ points. Operating on this $\alpha-1$ times by curves C^n each of which passes through $\frac{1}{2}n(n+3)$ of these points, we reduce the syzygy to one of order n between $\frac{1}{2}(n+1)(n+2)$ points, whence, by Serret's Theorem, these lie on a C^n. But since these $\frac{1}{2}(n+1)(n+2)$ points may be any of the N given points, it follows that all the latter lie on a C^n.

EXAMPLE. Taking $n = 3$, $r = 4$, we have $\alpha = \beta = 1$ and $N = 12$. Then if a syzygy $\sum k_s(a_s u+b_s v+c_s w)^4 \equiv 0$ holds between 12 given points, these will lie on a cubic.

EXAMPLES ON CHAPTER V

1. (i) If six of the intersections of a cubic and a quartic lie on a conic, prove that the remaining six (assumed distinct) lie on another conic; and that the four remaining intersections of the two conics with the quartic are collinear.

(ii) If two of the quartics through twelve given points meet in four further points of which three are collinear, prove that every pair of the quartics has the same property, and that the fourth remaining intersection lies on all the curves.

2. A line L meets a curve C^n in n distinct points, the tangents at which are $T_1,\ldots, T_n$. Prove that every curve of order n which touches C^n at its intersections with L has an equation of the form

$$L^2A+T_1T_2\ldots T_n = 0.$$

Deduce that the $n(n-2)$ simple intersections of the T_i with C^n lie on a curve of order $n-2$.

3. A binodal quartic is met by a cubic through the nodes in eight further points of which four lie on a conic through the nodes. Prove that the remaining four lie on another conic through the nodes.

† *Math. Papers*, 119.

CHAPTER VI

LINEAR SYSTEMS OF CURVES AND THEIR PROJECTIVE MODELS

§ 1. The general linear system

1. In this chapter we propose to discuss some general characteristics of linear systems of plane curves, but more particularly those which relate directly to the construction and properties of the rational surfaces which serve as projective models of such systems. In Chapter VII the results here obtained will form the basis for a discussion in some detail of all the more important types of well-known rational surfaces, as also for our remarks on plane Cremona transformations and involutions. The reader may well find it advantageous to study some of the applications in conjunction with the generalities here discussed.

1.1. *Definitions.* We begin with a formal definition of a linear system and a series of further definitions of qualifying terms.

If $f_0 = 0$, $f_1 = 0, \ldots, f_r = 0$ are the equations of $r+1$ linearly independent curves C^n, then the curves given by

$$\lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_r f_r = 0, \tag{1}$$

for all possible sets of values of the parameters $\lambda_0, \lambda_1, \ldots, \lambda_r$, are said to form a *linear system of freedom* r, and we denote this system by (f).

We say that (f) is *reducible* if and only if its generic curve is reducible. This will be the case if every curve of (f) breaks up into a fixed part and a variable part, or into a number of variable parts.

We say that (f) is *simple* if it does not possess the peculiarity that all those of its curves which pass through a generic point of the plane pass in consequence through one or more other variable points of the plane.

In the contrary case, if the points of the plane arrange themselves, relative to (f), into sets of k points $(k > 1)$ such that curves of (f) through any point P contain the whole set to which P belongs, then (f) is said to be *compound*, and the sets form a *plane involution* of order k.

The *actual base* of (f) consists of all the points (or fixed comves) common to all curves of (f), together with a n of their multiplicities for a generic curve of the

hat (f) is *complete* if it is not contained in any more r system of curves of the same order and with the same

results of Chs. II and III we know that all the curves n order n which contain a set of assigned base points, rily distinct, to assigned multiplicities form a linear This system is necessarily complete; but its *assigned* less than the actual. It may be found, indeed, that of (f) possess additional base points, or even fixed , which were not assigned, or they may have greater than the assigned at some of the assigned base points. s notation, we shall use two forms which are more han the general notation $C^n(k_i)$. Thus:

ymbol $C^n(O_1^{k_1}O_2^{k_2}...O_s^{k_s})$ will represent, primarily, a actual or assigned multiplicities $k_1,..., k_s$ at $O_1,..., O_s$; lso be used to represent a linear system with actual $_s^{k_s}$), or the complete linear system with $(O_1^{k_1}...O_s^{k_s})$ as e, as the context will indicate.

mbol $C^n[k_1^{s_1}, k_2^{s_2},..., k_\nu^{s_\nu}]$ will represent, in the same way, order n with s_1 specified k_1-fold points, s_2 specified s, and so on; or it will represent—as the context —a linear system with an actual or assigned base of

ni's Theorem. One of the first questions which lf is whether the generic curve of a linear system *ariable* multiple point. The answer—due originally to s follows, and it represents an essential simplification uent theory.

. *If the generic curve of a linear system has a multiple st be a base point of the system.*

ve give is due to Picard. Consider a pencil of curves the system, and let the equation of this pencil be he coordinates being non-homogeneous. If the curve multiple point at $P(x,y)$, then

$$\phi_x+\lambda_0\psi_x = 0, \qquad \phi_y+\lambda_0\psi_y = 0.$$

If P is not a base point of the pencil, so that ϕ and ψ do not both vanish there, we can eliminate λ_0 between these two equations and $\phi+\lambda_0\psi = 0$, obtaining

$$\psi\phi_x-\phi\psi_x = 0, \qquad \psi\phi_y-\phi\psi_y = 0,$$

that is
$$\frac{\partial}{\partial x}(\phi/\psi) = 0, \qquad \frac{\partial}{\partial y}(\phi/\psi) = 0.$$

Thus ϕ/ψ, regarded as a function of the independent variables x, y, is a constant, so that P could only vary, if at all, along a curve of the pencil. It follows that if the generic curve of the pencil has a multiple point, this must be a base point of the pencil, at which ϕ and ψ both vanish; and the corresponding result for the linear system follows at once.

1.3. *Characters of a linear system.* The primary characters of a linear system (f) are its *freedom* r (already defined), its *genus* p which is the genus of a generic curve of (f), and its *grade* ν which is the number of free intersections of two generic curves of (f).

When we attempt to calculate these numbers for the complete system $C^n(O_1^{k_1}O_2^{k_2}...O_s^{k_s})$ defined by an assigned base, we are held up by two considerations: (i) the actual base may differ from the assigned, and (ii) the linear conditions apparently imposed by the assigned base may not be independent. We may, however, write down formulae for *virtual* characters defined as follows:

The *virtual genus* p' and the *virtual grade* ν' of the complete system $C^n(O_1^{k_1}...O_s^{k_s})$ are defined by the formulae

$$p' = \tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\sum k_i(k_i-1), \tag{2}$$
$$\nu' = n^2-\sum k_i^2; \tag{3}$$

so that $p' = p$ and $\nu' = \nu$ when the actual base is the same as the assigned.

The *virtual freedom* r' of the same system is defined by

$$r' = \tfrac{1}{2}n(n+3)-\tfrac{1}{2}\sum k_i(k_i-1); \tag{4}$$

so that $r' = r$ if the conditions imposed by the base points are all independent. It is important to note that always

$$r \geqslant r'. \tag{5}$$

If $r = r'$, the system is said to be *regular*.

If $r > r'$, the system is said to be *superabundant*, and the excess $r-r'$ is called its *superabundance* with respect to the assigned base.

EXAMPLES

1. The system $C^3(O_1^2 O_2^2)$ is reducible, being composed of the conics $C^2(O_1 O_2)$ each augmented by the fixed line $O_1 O_2$. We have $r = 3$, $\nu = 1$, $p = -1$, in agreement with (5); while for the conics alone we have $\nu = 2$, $p = 0$. The system $C^3(O^3)$ is also reducible in a different way, being composed of triads of lines through O.

2. Any system $C^3[1^7]$ is compound since all curves of it which pass an eighth arbitrary point pass also through the associated ninth. More generally, any linear net of curves is compound if its grade exceeds unity. The really interesting compound systems, however, are those—comparatively rare—which are of freedom $r > 2$. Such, for example, is $C^6[2^8]$, a system of freedom three, for which the generic set of free intersections of a pair of curves of the system consists of two *tied* point-pairs.

3. Show that there exist superabundant pencils of 9-nodal sextics.

§ 2. PLANE REPRESENTATION OF A RATIONAL SURFACE

2. Projective model of a simple linear system. If a linear system of curves is (*a*) simple, (*b*) irreducible, and (*c*) of effective freedom $r > 2$, then we can represent its curves birationally on the prime sections of a rational surface in the following manner.

We suppose that the system (f) given by (1) has the properties in question; and we consider the set of equations

$$x_0 : x_1 : \ldots : x_r = f_0 : f_1 : \ldots : f_r. \tag{6}$$

Since, by hypothesis, $f_0, f_1, \ldots, f_r$ are linearly independent, so also are $x_0, x_1, \ldots, x_r$; hence we may take these to be homogeneous coordinates of a point in S_r. To any point $P(x, y, z)$ of the plane there corresponds in this way a point $Q(x_0, x_1, \ldots, x_r)$ of S_r; and as P describes the whole plane, Q will describe a surface F of which (6) are the parametric equations.

If Q describes the section of F by the prime whose equation is

$$a_0 x_0 + a_1 x_1 + \ldots + a_r x_r = 0, \tag{7}$$

then P describes the curve of (f) whose equation is

$$a_0 f_0 + a_1 f_1 + \ldots + a_r f_r = 0. \tag{8}$$

To a generic point P of the plane there corresponds clearly a unique point Q of F. Suppose now that, if possible, to a generic point Q of F there correspond several points P, P',... of the plane; by (6), these points would each give the same set of values to the ratios $f_0 : f_1 : \ldots : f_r$, and they would therefore impose the same condition on curves of (f) to contain them. This, however, is expressly excluded by the condition that (f) is a simple system;

hence to a generic point of F there corresponds a unique point of the plane.

It follows that F is rational. The curves of (f) are therefore in birational correspondence with the prime sections of F and have the same genus p.

The order of F—the number of points in which it is met by an arbitrary secundum S_{r-2}, represented by two equations such as (7)—is evidently the grade ν of (f).

These results may be summarized as follows:

THEOREM II. *If (f) is simple, irreducible, and of freedom $r > 2$, with genus p and grade ν, its curves may be represented birationally on the prime sections of a rational surface which is situated in S_r, is of order ν, and has section genus p.*

We shall denote the surface by ${}^pF^\nu$ when we wish to indicate its order and section genus, or by ${}^pF^\nu[r]$ when we wish also to indicate the space in which it is situated. The surface may be described as the *projective model* (or *image*) of the given linear system.

2.1. *Curves on a rational surface.* Given the system of plane curves (f) which represents the prime sections of a rational surface F, we can determine all the other curves lying on F, the number of their mutual intersections, and other geometrical properties of the surface. To this end we suppose that the curves of (f) are of the type $C^n(O_1^{k_1}O_2^{k_2}...)$ which specifies all their base points. Then to any curve $C^{n'}(O_1^{k'_1}O_2^{k'_2}...)$ of the plane there corresponds on F a curve Γ^μ, say, whose order μ is the number of its intersections with an arbitrary prime section of F. It follows that μ is equal to the number of variable (or free) intersections of $C^{n'}(O_1^{k'_1}O_2^{k'_2}...)$ with a curve $C^n(O_1^{k_1}O_2^{k_2}...)$, whence

$$\mu = nn' - \sum_{i=1}^{s} k_i k'_i. \tag{9}$$

In the same way we see that any other curve $\Gamma^{\mu'}$, which is represented by $C^{n''}(O_1^{k''_1}O_2^{k''_2}...)$, meets Γ^μ in N points, where

$$N = n'n'' - \sum_{1}^{s} k'_i k''_i. \tag{10}$$

From this formula, by putting $n'' = n'$, $k''_i = k'_i$, we may deduce the (virtual) grade of the curves Γ^μ, assuming that they form a linear system of freedom one at least.

Again, the curve in which F is met by a primal of order ρ is represented evidently by a curve of the type $C^{\rho n}(O_1^{\rho k_1} O_2^{\rho k_2}\ldots)$; for the primal may break up into ρ primes, in which case the curve in question is a set of ρ prime sections.

It remains, finally, to examine the images on F of the neighbourhoods of the base points O_i. To the point O_i itself, of course, there corresponds no definite point of F; for all the polynomials f_i vanish at O_i, so that the equations (6) become indeterminate. However, to the ∞^1 points of the first neighbourhood of O_i, which is a convenient way† of referring to the ∞^1 directions in which a curvilinear branch may pass through O_i, there will correspond the points of a curve Ω on F. Now, in the first place, the points of Ω are in birational correspondence with the lines of the pencil whose vertex is O_i; so that Ω is a *rational* curve. Next, if we suppose that the generic curve of (f) has k_i distinct nodal tangents at O_i, which are all variable, the corresponding generic prime section of F will meet Ω in k_i points; so that Ω is of order k_i. Whence

THEOREM III. *To the neighbourhood of a k_i-fold base point of (f) there will correspond in general a rational curve of order k_i on F.*

The same argument shows that if a number t of the nodal tangents to the generic curve of (f) at O_i are fixed, then the order of Ω is reduced to $k_i - t$. If all the nodal tangents are fixed, then the first neighbourhood of O_i is represented on F by a single point.

2.2. *Fundamental curves.* The existence of exceptional points O_i of the plane which transform, in a certain sense, into curves of F suggests the possibility of the reverse phenomenon, namely, the existence of certain exceptional curves of the plane which transform into points of F. If a curve Θ of the plane is to transform into a point E of the surface, then every point of Θ must present the same linear condition to curves of (f) required to contain it; in other words, Θ must be a fundamental curve of (f) according to the following definition:

A *fundamental curve* of (f) is a curve which presents but a single condition on curves of (f) required to contain it as component. Such curves may also be characterized by the obvious fact that

† For the concept of the neighbourhood of a point and the meaning of its resolution into an explicit curve, we refer the reader to the analysis of Ch. III.

they must have no variable intersections with curves of (f). The simplest example is provided by the system of conics through two fixed points A, B, for which the line AB is a fundamental line.

Supposing then that Θ is a fundamental curve of (f), we may ask whether the corresponding point E is simple or multiple on F. This may be answered in two ways.

In the first place, sections of F by primes through E are represented, effectively, by curves of the system (f_1) residual to Θ with respect to (f). Hence, if (f_1) has grade ν_1, the generic secundum through E will meet the surface in only ν_1 further points, so that E is a $(\nu-\nu_1)$-fold point of F.

Alternatively we may argue that, if E is a j-fold point of F, then the section of F by a generic prime through E will have a j-fold point at E with, in general, j distinct nodal tangents. Since Θ is the image of the neighbourhood of E, this means that j must be equal to the number of variable points in which Θ is met by a generic curve of (f_1). Hence:

THEOREM IV. *To any fundamental curve Θ of (f) there corresponds a single point E whose multiplicity for F is the excess of the grade of (f) over that of the system (f_1) residual to Θ with respect to (f). This again is equal in general to the number of variable intersections of curves of (f_1) with Θ.*

2.3. *Projection of a rational surface.* Consider now the linear sub-system $(f)_1$ of (f), obtained by imposing on the parameters λ_i which occur in (1) the fixed linear condition

$$a_0\lambda_0+a_1\lambda_1+...+a_r\lambda_r = 0. \tag{11}$$

To the curves of $(f)_1$ there will clearly correspond the ∞^{r-1} sections of F by primes through the fixed point A of S_r whose coordinates are $(a_0, a_1,..., a_r)$. Now if $r > 3$, the surface F may be projected from A into a surface F' in a prime S_{r-1}, sections of F by primes through A projecting into prime sections of F'. Hence, assuming the projection to be a proper one,† the curves of the sub-system $(f)_1$ represent birationally the prime sections of F'; and conversely, any surface F' which is a proper projection of F from a point of S_r is represented on the plane by means of a sub-system of (f) such as $(f)_1$.

If A does not lie on F, then F' will have the same order ν and the same section genus p as F; this corresponds to the fact that

† This is equivalent to the assumption that $(f)_1$ is itself a simple system.

the grade and genus of $(f)_1$ are the same as those of (f). If, however, A is a generic simple point of F, the section genus of F' will still be p, but its order will be $\nu-1$; in this case (11) is the special type of linear condition which is equivalent to imposing an additional simple base point A_1 on (f), so that $(f)_1$ is of grade $\nu-1$. By Theorem III, the neighbourhood of A_1 is represented on F' by a line; and this is none other than the projection of the tangent plane to F at A which contains the first neighbourhood of A on F. These results may be summarized as follows.

THEOREM V. *Any proper projection of F from an external point on to a prime S_{r-1} $(r > 3)$ is represented on the plane by a linear sub-system of curves of (f), having freedom $r-1$ and grade ν. A proper projection from an ordinary point of F is represented by the curves of (f) which have an additional simple base point; and it contains a line, arising from the tangent plane at the point of projection, which represents the neighbourhood of the new base point.*

Provided that r is sufficiently large, the process of projection may be continued, each projection imposing one further linear condition on (f).

2.31. *Rational normal surfaces.* In accordance with the definition of Ch. I, § 4.52, a surface which belongs to S_r, and not to any space of lower dimension, is said to be *normal* in S_r if it is not a proper projection of any surface of the same order in a higher space. The surface F referred to in the preceding theorem will certainly not be normal unless the linear system (f) is complete. However, if (f) is complete, then it is certainly not a linear subsystem of any more ample system of the same grade; and F is therefore normal in S_r.

From this and the equations of § 1.3, we deduce

THEOREM VI. *A rational surface is normal if and only if the linear system representing its prime sections is complete. If this linear system is also regular† then the resulting surface ${}^pF^\nu$ is normal in $[\nu-p+1]$.*

Surfaces represented by regular systems are termed *non-special*; if ${}^pF^\nu$ is normal in space $[\nu-p+1+\delta]$, where $\delta > 0$, it is called *special*.

2.4. *Rational surfaces in S_4 and S_3.* If F is situated in S_r $(r > 4)$, then it will not in general possess any multiple points;

† We are supposing here, as we may, that the actual base is assigned.

but if it is situated (though not necessarily normal) in S_4 or in S_3, then it will in general possess such singularities as are indicated in the following theorem.

THEOREM VII. *If F lies in S_4, then it possesses in general only a finite number of improper double points. If it lies in S_3, then it possesses in general a double curve which has a number of triple points triple also for the surface.*

In the case $r = 4$, the parametric equations of F are of the form

$$\rho x_i = f_i(x, y, z) \quad (i = 0, \ldots, 4).$$

As in Ch. II, § 3.2, we observe that a double point of F will correspond in general to a pair of points, (x, y, z) and (x', y', z'), such that

$$\frac{f_i(x, y, z)}{f_0(x, y, z)} = \frac{f_i(x', y', z')}{f_0(x', y', z')} \quad (i = 1, 2, 3, 4).$$

These are four equations for the ratios $x:y:z$ and $x':y':z'$; and they will have in general a finite number of solutions, other than $x:y:z = x':y':z'$, leading to a finite number of double points of F.

A double point of F, of the general type under consideration, corresponds to a pair of points of the plane, which has evidently the property that curves of (f) through either of the points also pass through the other; such a pair of points is said to be *neutral* with respect to (f). A curve of (f) acquires no additional multiple point by passing through the points of a neutral pair, so that the section of F by a prime through the corresponding double point has the same genus as a generic prime section of the surface. For this reason such double points are said to be *improper* or *accidental*. In special cases (f) may possess fundamental curves to which there may correspond *proper* multiple points of F, such as diminish the genus of prime sections passing through them.

In the case $r = 3$, we find similarly that (f) possesses ∞^1 neutral pairs and in addition a finite number of *neutral triads* (P, Q, R), such that curves of (f) which contain any one of P, Q, R contain also the other two. The neutral pairs will correspond to the points of a double curve D of F; and the neutral triads will correspond to triple points of F which will evidently also be triple points of D, as stated in the theorem.

2.5. *Freedom of primals through F.* If $r \geq 4$, we may wish to find the freedom, ρ_l say, of primals M^l, of any given order l, through a given rational surface F. This amounts to finding the *postulation* of F for primals of order l—a number, ϕ_l say, which is defined as the number of conditions to which primals M^l are subjected by being made to contain F. Evidently

$$\rho_l = \binom{l+r}{r} - 1 - \phi_l. \tag{12}$$

Primals of order l will meet F in a system of curves C_l whose freedom, $r_{(l)}$ say, is that of the system C'_l which images C_l in ϖ (the plane of the representation). This means that a primal M^l can be made to pass through $r_{(l)}$ generic points of F without containing the surface; but that $r_{(l)}+1$ generic points of F present to M^l the same condition as the whole surface. Hence

$$\phi_l = r_{(l)} + 1. \tag{13}$$

By § 2.1, and retaining the notation there used, C'_l is composed of curves of the system $(f_l) \equiv C^{ln}(O_1^{lk_1} O_2^{lk_2} \ldots)$; but we have no guarantee that C'_l is the *complete* system (f_l). If this latter has freedom r_l, we must write

$$r_{(l)} = r_l - \Delta_l \quad (\Delta_l \geqslant 0),$$

where Δ_l is the *deficiency* from completeness of C'_l and therefore also of C_l.

If the base points O_i are all of general position in ϖ, then we may suppose that (f_l) is a regular system and r_l is given by the formula

$$r_l = \tfrac{1}{2}ln(ln+3) - \tfrac{1}{2}\sum lk_i(lk_i+1),$$

which reduces, in virtue of (2) and (3), to the form

$$r_l = \binom{l+1}{2}\nu - l(p-1).$$

Hence
$$\phi_l = \binom{l+1}{2}\nu - l(p-1) - \Delta_l + 1. \tag{14}$$

For $r > 4$, all we can safely assert about Δ_l, especially for low values of l, is that $\Delta_l \geqslant 0$; beyond that, each case has to be investigated on its own.

For $r = 4$, F possesses in general a number δ of improper double points represented by neutral point-pairs in ϖ; these will evidently be neutral for C'_l, but not for the complete system (f_l) if $l \geqslant 2$; and hence $\Delta_l \geqslant \delta$.

Combining these results with equations (12), (13), and (14), we arrive at the following conclusions:

THEOREM VIII. *If ${}^pF^\nu$ is a rational surface in S_r, representable on a plane ϖ by means of a linear system of curves whose base points are of general position† in ϖ, then the freedom ρ_l of primals of order l through the surface is at least equal to*

$$\binom{l+r}{r}-\binom{l+1}{2}\nu+l(p-1)-2 \tag{15}$$

if $r > 4$, and is at least equal to

$$\binom{l+r}{r}-\binom{l+1}{2}\nu+l(p-1)-2+\delta \tag{16}$$

if $r = 4$, δ being then the number of improper double points of ${}^pF^\nu$.

In either case, the lower limit given in the theorem is called the *virtual freedom* of primals of order l through the surface.

§ 3. BIRATIONAL EQUIVALENCE OF LINEAR SYSTEMS

3. Birationally equivalent systems. When the linear system (f) of § 2 is taken to be a homaloidal net, i.e. a (simple) linear system of freedom 2 and grade 1, the transformation which arises is a Cremona transformation of ϖ into another plane ϖ' (cf. Ch. III, § 1). We shall postpone till later (Ch. VII, § 7) any detailed consideration of such transformations; but we must point out here the important role they may be made to play in the classification of linear systems of curves generally.

We note in the first place that Cremona transformations of a plane into itself form a group; for all inverses and products of such transformations are of the same type. Also it is evident, by simple substitution, that every linear system of freedom $r \geqslant 1$ is transformed by any Cremona transformation into another linear system of the same freedom, grade, and genus.

We may therefore group together as *birationally equivalent* all linear systems which are transformable into each other by Cremona transformation of the plane, the relation so defined being reflexive, symmetrical, and transitive.

† If the base points in the plane representation are not of general position in ϖ, then (f_l) must be supposed to have superabundance $\epsilon_l \geqslant 0$; in this case

$$\phi_l = \binom{l+1}{2}\nu-l(p-1)+\epsilon_l-\Delta_l+1,$$

from which it is no longer possible to infer any lower bound for ρ_l.

3.1. *Invariant projective models.* If F, in the notation of § 2, is a projective model of (f), then any projective transform of F is again a projective model of (f); for it is easy to see that any general collineation of the $[r]$ in which F lies is equivalent, in the plane, to choosing a new set of $r+1$ linearly independent reference curves for (f), in place of the original set $f_0, f_1,..., f_r$. Hence:

THEOREM IX. *The model of any simple linear system, of freedom $r \geqslant 2$, is a projectively invariant rational surface in $[r]$.*

Suppose now, on the other hand, that the rational surface F is given, and let us consider all possible representations of F on the plane ϖ. Clearly any one such representation, by means of a linear system (f) say, will be transformed, by any Cremona transformation of the plane, into another representation by means of a birationally equivalent linear system (f'); and any two representations of F must be related in this way. Hence:

THEOREM X. *Every rational surface is the image of a complete class of birationally equivalent linear systems of curves in the plane.*

Thus every property of a rational surface represents an invariant property of linear systems of curves in the plane.

§ 4. COMPOUND SYSTEMS AND INVOLUTIONS

4. Characteristic sets. We find it convenient at this stage to introduce another term of general usage in connexion with linear systems of curves:

A *characteristic set* of any linear system (f) is the set of free intersections of two curves of (f).

If (f) is a complete system defined by an assigned base, it is sometimes necessary to distinguish between *effective* characteristic sets, whose points are all variable, and *virtual* characteristic sets, which may contain unassigned fixed points.

Consider now a compound system (f) of freedom $r \geqslant 2$, such that all curves of (f) through a generic point P_1 of the plane pass in consequence through $k-1$ further points $P_2,..., P_k$ $(k \geqslant 2)$. With respect to such a system, the points of the plane evidently arrange themselves into mutually exclusive sets $G^k \equiv (P_1,..., P_k)$, each presenting but a single condition to curves of (f) required to contain it; and these sets G^k form a plane involution of order k.

Clearly any characteristic set of (f) must consist of a number ν_1 of complete sets G^k; so that if (f) is of grade ν we shall have

$$\nu = k\nu_1. \tag{17}$$

For the obvious type of compound system in which (f) is a linear net of grade $\nu > 1$, we shall have $\nu_1 = 1$, $\nu = k$.

A double point of the involution is a point which counts twice in the set G^k to which it belongs, and the locus of such double points is called the *coincidence curve* of the involution.

4.1. *Image surface of the involution.* Equations (6) still define a surface F which is a rational transform of the plane; but now all the points of any set G^k are seen to be represented by a single point of F, so that ϖ is in $(k, 1)$ correspondence with F. The sets G^k, however, are in birational correspondence with the points of F, so that F is the *projective image* of the involution.

The order of F is evidently ν_1; for this is the number of sets G^k which make up a characteristic set of (f). Thus, for example, if (f) is a net, then F is a plane.

The *branch curve* of F—a term which originates in the concept of F as a k-sheeted surface of total order ν—is the locus of points of F which represent double points of the involution, or rather the sets G^k containing these double points. The points of this curve will be in birational correspondence with the points of the coincidence curve of the involution.

It can be shown, though we are not at this stage in a position to do so, that every surface such as F is rational; or, in other words, that every plane involution is rational.

§ 5. Jacobian systems

5. We shall conclude this chapter by investigating certain other properties of linear systems, of a kind very different from those so far considered, but which have nevertheless very important applications to the theory of rational surfaces in general. These concern in part the more immediate problem of determining the representation in the plane of the double curve of a rational surface in [3]; but they will also open the way later for the direct calculation, from the plane representation, of the numerical characters of any rational surface (cf. Ch. IX).

5.1. *Jacobian curves*

DEFINITION. The locus of double points of curves of a net is called the *Jacobian curve* of the net.

If the equation of the net is

$$\lambda f+\mu g+\nu h = 0, \tag{18}$$

where f, g, and h are of degree n, the Jacobian is evidently given by the equation

$$\begin{vmatrix} f_x & g_x & h_x \\ f_y & g_y & h_y \\ f_z & g_z & h_z \end{vmatrix} = 0. \tag{19}$$

This curve, which we denote by $J(f,g,h)$, is of order $3(n-1)$.

We observe that

The Jacobian is also the locus of contacts of curves of the net. For, at a point where two curves of the net touch, there is a pencil of curves of the net in contact, and, of these, one has a node.

For a precise algebraical definition of the Jacobian, equation (19) is sufficient; for the curve defined by it is plainly a *covariant* of the net—independent of the choice of the coordinate system and independent also of the choice of the three curves f, g, h in terms of which the equation of the net is expressed.

5.11. *Singularities.* We have now to determine the nature and position of the multiple points of the Jacobian. For this purpose let us suppose that the curve f has a node at $(0,0,1)$, its equation being

$$f = (ax^2+2bxy+cy^2)z^{n-2}+... = 0.$$

Let $x = 0$ be the common tangent to the ∞^1 curves of the net which pass through $(0,0,1)$, so that g has an equation of the form

$$g = a'xz^{n-1}+(b'x^2+2c'xy+d'y^2)z^{n-2}+... = 0.$$

In general, the curve h will not pass through the point in question, so we may write

$$h = a''z^n+(b''x+c''y)z^{n-1}+... = 0.$$

We then find that

$$J(f,g,h) = 2na'a''(bx+cy)z^{3n-4}+....$$

Thus, *the tangent to the Jacobian is harmonically separated from the common tangent $x = 0$ by the nodal tangents of the curve f.* It will be defined except when

(i) $a'' = 0$, i.e. when $(0,0,1)$ is a base point of the system;

(ii) $a' = 0$, i.e. when all curves of the net passing through $(0,0,1)$ have a node there;

(iii) $b = c = 0$, i.e. when one curve of the net has either a cusp at $(0,0,1)$, with $x = 0$ as cuspidal tangent, or a singularity of order 3 at least at this point.

Whence:

THEOREM XI. *A point will be a singularity of the Jacobian if*

(i) *it is a base point of the net,*

(ii) *it is at least a double point for all curves of the net passing through it,*

(iii) *it is a cuspidal point for one of the curves, and all curves through it touch the cuspidal tangent, or it is a singularity of higher order of a curve of the net.*

The behaviour of the Jacobian at a base point is determined by

THEOREM XII. *At an ordinary i-fold base point of the net, the Jacobian has a $(3i-1)$-fold point.*

Let $(0,0,1)$ be the base point in question, and let

$$f = z^{n-i}f'+\dots,\quad g = z^{n-i}g'+\dots,\quad h = z^{n-i}h'+\dots,$$

where it is assumed that f', g', and h' have no repeated or common factors.

In forming the Jacobian we need take account only of the terms of lowest degree in x and y. Thus:

$$J(f,g,h) \equiv (n-i)\begin{vmatrix} z^{n-i}f'_x+\dots, & z^{n-i}g'_x+\dots, & z^{n-i}h'_x+\dots \\ z^{n-i}f'_y+\dots, & z^{n-i}g'_y+\dots, & z^{n-i}h'_y+\dots \\ z^{n-i-1}f'+\dots, & z^{n-i-1}g'+\dots, & z^{n-i-1}h'+\dots \end{vmatrix}.$$

The terms of lowest degree in x and y are apparently to be found in the determinant $|f'_x, f'_y, f'|$, of degree $3i-2$; but this is identically zero. Thus in general the required terms are of degree $3i-1$, and it is easily seen that these do not vanish.

5.12. *Reducible net.* We consider next the case when all curves of the net are reducible, with a fixed part which is counted a certain number of times. We have first

THEOREM XIII. *A fixed k-fold component of the net counts $3k$ times in the Jacobian.*

Let the fixed part have equation $F = 0$, so that

$$f = F^k f',\qquad g = F^k g',\qquad h = F^k h'.$$

Then

$$J(f,g,h) \equiv |F^k f'_x + kF^{k-1}F_x f',\ F^k g'_x + kF^{k-1}F_x g',\ F^k h'_x + kF^{k-1}F_x h'|.$$

This determinant may be expressed as the sum of eight others of the same order, of which four vanish identically since they have at least two identical columns. Thus:

$$J(f,g,h) \equiv F^{3k}J(f',g',h') + kF^{3k-1}\sum_{f,g,h}|f'_x,\ g'_x,\ F_x|h'.$$

Now

$$\sum |f'_x,\ g'_x,\ F_x|h' \equiv \sum \begin{vmatrix} f', & g', & h' \\ f'_y, & g'_y, & h'_y \\ f'_z, & g'_z, & h'_z \end{vmatrix} F_x \equiv \frac{1}{\alpha}J(f',g',h')\sum_{x,y,z} xF_x$$

by Euler's Theorem, where α is the degree of f', g', h'; and by a second application of Euler's Theorem it follows that this expression contains F as a factor. Thus $J(f,g,h)$ is divisible by F^{3k}, and the theorem is established.

COROLLARY 1. *If, further, F is l-fold for a pencil of the curves, and m-fold for one curve of the net ($k < l \leqslant m$), it counts $k+l+m-1$ times in the Jacobian.*

This is proved in a similar manner by writing $f = F^k f'$, $g = F^l g'$, $h = F^m h'$.

COROLLARY 2. *A fundamental curve of the net is, in general, a simple component of the Jacobian.*

For in this case $k = 0$, $l = m = 1$.

Note. For certain purposes it is desirable to introduce the concept of the Jacobian of any three curves f, g, h, not necessarily of the same order. In this case we define the Jacobian as the locus of points whose polar lines with respect to the three curves are concurrent, thereby obtaining the equation (19) above.

EXAMPLE. Prove that if the curves have a common point, this lies on the Jacobian.

5.2. *Jacobian systems*

DEFINITION. Let (f) be the general linear system, of freedom $r \geqslant 3$, defined by (1), and let J_{ijk} denote the Jacobian curve of f_i, f_j, f_k ($i,j,k = 0,\ldots,r$). Then the Jacobian curve of every net contained in (f) belongs to the linear system

$$\sum_{i,j,k}\lambda_{ijk}J_{ijk} = 0, \qquad (20)$$

and this linear system is called the *Jacobian system* (f_J) of (f).

In general (f_j) only *contains* all the Jacobian curves,† and it is, plainly, the minimum linear system with this property; but for $r = 3$, it is easy to verify that every curve of (f_j) is the Jacobian curve of some net of (f), so that in this case (f_j) consists entirely of Jacobian curves.

If (f) is a system $C^n(k_i)$, then by the results of the last section (f_j) is a system $C^{3n-3}(3k_i-1)$; but we cannot assert, of course, that (f_j) is complete, even if (f) is complete.

This result has an important application to rational surfaces. Suppose that F is a surface situated in S_r; and let Σ be the net of primes passing through a fixed S_{r-3}. Then ∞^1 of these primes will touch F, i.e. meet it in curves having a double point at a simple point of the surface. The locus of these points of contact is called the *curve of contact* of tangent primes of Σ, and its order is called the *rank* of F.

If F is rational, its prime sections corresponding to a system $C^n(k_i)$ in ϖ, the curve of contact corresponds to the Jacobian of the net of curves which represent sections by the primes of Σ. Hence:

THEOREM XIV. *The curves of contact on F are represented in the plane by curves of the Jacobian system of $C^n(k_i)$.*

5.21. *Double curve of a rational surface.* If F is in S_3, then its normal singularities, as already noted in § 2.4, are a double curve D together with a number of triple points at triple points of D. The image of D in ϖ will be a curve Δ, locus of pairs of points, neutral for $C^n(k_i)$, representing the points of D; and the triple points will be represented by triads of points, neutral for $C^n(k_i)$, which are all double points of Δ. This last fact, not hitherto mentioned, is not difficult to establish, and it will be made evident by the considerations of Ch. IX, § 2.12.

To determine the character of Δ, we observe that the curve of contact of tangent planes to F from an arbitrary point P, together with D, make up the complete intersection of F with the first polar of P. Hence, if F is of order ν, the curve Δ is in general of the form

$$C^{(\nu-1)n}\{(\nu-1)k_i\}-C^{3n-3}(3k_i-1)$$

† The freedom of nets contained in (f)—which is the freedom of planes in $[r]$—is $3(r-2)$; while the freedom of (f_j) is $\binom{r+1}{3}-1$. These numbers are equal if $r = 3$.

so far as its behaviour at the base points is concerned: and it will also possess $3t$ further double points, arising, as indicated above, from the t triple points of F which are also triple points of D. Hence:

THEOREM XV. *If the rational surface F in [3] has only normal singularities, then its double curve is represented in ϖ by a curve*

$$C^{(\nu-4)n+3}\{(\nu-4)k_i+1, 2^{3t}\}.$$

This result requires modification if, for example, F has an ordinary isolated i-fold point represented by a fundamental curve Ω of $C^n(k_i)$. In this case the true curve Δ representing the double curve of F is obtained by subtracting $(i-2)\Omega$ from that given in the theorem; for the curve representing the section of F by the first polar of P will have Ω as $(i-1)$-fold component, while the Jacobian of a net of $C^n(k_i)$ will have Ω as simple component.

5.3. *Jacobian sets*

DEFINITION. Among the curves of any pencil there will be, in general, a certain number of curves having double points not common to all curves of the pencil. The set of all such double points is called the *Jacobian set* of the pencil.

If $f+\lambda g = 0$ is the equation of a pencil of curves of order n, then the points of the Jacobian set must satisfy the equations

$$f_x+\lambda g_x = 0, \qquad f_y+\lambda g_y = 0, \qquad f_z+\lambda g_z = 0, \tag{21}$$

and they are therefore the points of intersection of the curves

$$\phi \equiv \begin{vmatrix} f_x & g_x \\ f_z & g_z \end{vmatrix} = 0, \qquad \psi \equiv \begin{vmatrix} f_y & g_y \\ f_z & g_z \end{vmatrix} = 0 \tag{22}$$

other than those common to the curves $f_z = 0$, $g_z = 0$.

Since the curves ϕ and ψ are of order $2(n-1)$, it follows that, if f and g have no common multiple point, the number of points in the Jacobian set is

$$\delta = 4(n-1)^2-(n-1)^2 = 3(n-1)^2. \tag{23}$$

Suppose now that f and g, being irreducible, have a common k-fold point of general character, at which the tangents to the two curves are distinct; and let us estimate the reduction in δ due to this point.

We take the point, as we evidently may, to be $(0, 0, 1)$, so that the polynomials f and g are of the forms

$$f \equiv z^{n-k}u+\dots, \qquad g \equiv z^{n-k}v+\dots,$$

where u and v are polynomials, of degree k in x and y, with no common factor.

The leading terms in the equations of ϕ and ψ reduce to

$$\frac{n-k}{k}y(u_x v_y - v_x u_y)z^{2n-2k-1} \quad \text{and} \quad \frac{n-k}{k}x(u_x v_y - v_x u_y)z^{2n-2k-1}$$

respectively, i.e. ϕ and ψ have $(2k-1)$-fold points at $(0,0,1)$ with $2k-2$ nodal tangents in common; and it may be verified that the contacts of the branches of ϕ and ψ are in fact simple. Thus the total intersection multiplicity of these two curves at $(0,0,1)$, and in the first neighbourhood of this point, is

$$(2k-1)^2+(2k-2) = 4k^2-2k-1.$$

On the other hand, the curves $f_z = 0$, and $g_z = 0$ have k-fold points at $(0,0,1)$, with distinct nodal tangents there.

Thus we obtain for δ the formula

$$\delta = 4(n-1)^2-(4k^2-2k-1)-\{(n-1)^2-k^2\},$$

i.e.

$$\delta = 3(n-1)^2-(k-1)(3k+1). \tag{24}$$

Hence

THEOREM XVI. *The reduction in δ due to a k-fold base point of general type is $(k-1)(3k+1)$.*

REFERENCES AND EXAMPLES

(See end of Chapter VII.)

CHAPTER VII

SPECIAL RATIONAL SURFACES AND PLANE CREMONA TRANSFORMATIONS

In this chapter we propose to apply the general theory of the preceding chapter to an investigation, *seriatim*, of all the more important types of rational surfaces in ordinary and higher space; and this will be followed by a brief account, on the same lines, of some general and special properties of plane Cremona transformations and involutions.

We begin, as is natural, with the general quadric surface F^2 in ordinary space, which well illustrates the general theory.

§ 1. Plane representation of the quadric surface

1. To obtain a birational representation of F^2, we have merely to project it from a point O of itself on to a plane ϖ, so that to a generic point P of the surface there corresponds the point P' in which OP meets ϖ. In this correspondence, however, there are exceptional elements in both directions; there are, namely, the two generators λ_0, μ_0 of the surface which pass through O and project into points A, B of ϖ; and there is the line AB in ϖ, projection of the tangent plane to F^2 at O, whose points are the projections of the points of the neighbourhood of O on F^2.

The generic plane section of F^2 is a conic which meets λ_0 and μ_0 and projects into a conic through A and B; so that the representation of F^2 on ϖ is by means of the system of conics $C^2(A^1B^1)$. In accordance with general theory, this system has freedom 3 and grade 2, and it has AB as fundamental line (cf. Ch. VI, § 2.2).

Sections of F^2 by planes through O project into the lines of ϖ, these lines being the variable components of curves of $C^2(A^1B^1)$ which have been made to contain the fundamental line AB as component.

The λ-generators of F^2 (of the same system as λ_0) all meet μ_0 and project therefore into the lines of the pencil vertex B; similarly the μ-generators project into the lines of the pencil vertex A.

A generic (α, β) curve of F^2, which meets λ-generators in α points and μ-generators in β points, projects into a curve $C^{\alpha+\beta}(A^{\alpha}B^{\beta})$. It will be observed that the latter curve has no *free* intersections with AB in accordance with the fact that it represents a curve of F^2 which does not pass through O. A 'general' curve $C^n(A^\alpha B^\beta)$ of the plane represents a 'special' curve of F^2; for the former has $n-\alpha-\beta$ free intersections with AB, and hence the latter has an $(n-\alpha-\beta)$-fold point at O.

If the (α, β) curve above referred to has no multiple points, its genus, equal to that of $C^{\alpha+\beta}(A^\alpha B^\beta)$, is

$$\tfrac{1}{2}(\alpha+\beta-1)(\alpha+\beta-2)-\tfrac{1}{2}\alpha(\alpha-1)-\tfrac{1}{2}\beta(\beta-1) = (\alpha-1)(\beta-1).$$

Also the number of intersections of an (α, β) curve with an (α', β') curve, equal to the number of free intersections of the projected curves, is given by

$$(\alpha+\beta)(\alpha'+\beta')-\alpha\alpha'-\beta\beta' = \alpha\beta'+\alpha'\beta.$$

The curve in which F^2 is met by a surface F^α is an (α, α) curve represented by $C^{2\alpha}(A^\alpha B^\alpha)$. Conversely, by using the parametric equations of F^2 in the form

$$x_0 : x_1 : x_2 : x_3 = xy : xz : yz : z^2,$$

it may readily be shown that any (α, α) curve is the complete intersection of F^2 with a surface F^α.

Suppose now that $\alpha > \beta$; then the curve $C^{\alpha+\beta}(A^\alpha B^\beta)$, together with $\alpha-\beta$ lines through B, forms a curve of the type $C^{2\alpha}(A^\alpha B^\alpha)$. Hence

An (α, β) curve for which $\alpha > \beta$ is the residual intersection of F^2 with a surface F^α passing through $\alpha-\beta$ generators of the second system.

EXAMPLES

1. Examine the systems of cubic and quartic curves which lie on F^2 and show how they may be obtained as systems of complete or partial intersections of F^2 with other surfaces.

2. Show that there are two types of quintics and three types of sextics on F^2, and determine their genera.

3. Show that an (α, β) curve can be drawn through $\alpha\beta+\alpha+\beta$ given points of F^2.

4. Show that a quadric cone may be represented on the plane by means of the system of conics which have a fixed point of contact.

§ 2. THE GENERAL CUBIC SURFACE IN S_3

2. Plane representation. The general cubic surface, like the quadric, can be represented birationally on a plane; but this representation is something much less obvious—and much more striking—than the simple projection which sufficed in the case of the quadric. We derive the representation from the following lemma:

The general homogeneous equation of the third degree in $x_0,\ldots, x_3$ can be written in the form

$$\Delta \equiv \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0,$$

where $u_1,\ldots, w_3$ are linear homogeneous functions of $x_0,\ldots, x_3$.

We omit, for reasons of space, a rigorous proof† of this lemma; but we may show, by 'counting constants', that the result is what we should expect, and the procedure is interesting as being applicable generally to determinantal manifolds. The equation $\Delta = 0$ apparently contains 35 disposable constants; however, its generality is that of the equation $\Delta' = 0$, where Δ' is the product, by rows, of Δ with an arbitrary third-order determinant A whose elements are constants. Again, the generality of $\Delta' = 0$ is that of $\Delta'' = 0$, where Δ'' is the product by columns, of Δ' with a second arbitrary determinant B. Since each of A, B contains 8 disposable constants, it would appear that the equation $\Delta = 0$ depends in reality on 19 disposable constants, and this is precisely the number associated with the general equation of the third degree.

We suppose then that F^3 has equation $\Delta = 0$, and we observe that this equation is the result of eliminating λ, μ, ν from the equations

$$\left.\begin{aligned} \lambda u_1 + \mu v_1 + \nu w_1 &= 0 \\ \lambda u_2 + \mu v_2 + \nu w_2 &= 0 \\ \lambda u_3 + \mu v_3 + \nu w_3 &= 0 \end{aligned}\right\}. \tag{1}$$

† Perhaps the simplest proof of the lemma is that which is based on the reduction of the equation of the general cubic surface to the *Cayley–Salmon* form

$$pqr + p'q'r' \equiv \begin{vmatrix} 0 & p & p' \\ q' & 0 & q \\ r & r' & 0 \end{vmatrix} = 0,$$

where $p,\ldots, r'$ are linear functions of $x_0,\ldots, x_3$; and this form is itself derived by establishing the existence of at least one line on the surface, and by showing that through this line there pass five planes which meet the surface residually in pairs of lines. It may be added that there is one and only one *special* type of cubic surface which does not admit of the required type of representation. For a general account of these matters and an extensive bibliography of the cubic surface, we may refer the reader to Pascal's *Repertorium*, ii_2, ch. xxxiv; and for an alternative derivation of the lines of the surface, to Ex. I at the end of this chapter.

In other words, there exist values of λ, μ, ν (not all zero) satisfying these equations if and only if x_0, x_1, x_2, x_3 are the coordinates of a point P of F^3. If we interpret λ, μ, ν as homogeneous coordinates of a point P' of a plane ϖ, the resulting correspondence between P and P' is clearly birational.

To obtain the explicit equations of the representation, we solve the equations (1) for the ratios of x_0, x_1, x_2, x_3, the resulting solution being of the form

$$x_0:x_1:x_2:x_3 = L_0:L_1:L_2:L_3, \tag{2}$$

where L_0, L_1, L_2, L_3 are cubic polynomials in λ, μ, ν. Hence if P describes a plane section of F^3, P' describes a cubic curve in the plane, and this will be non-singular since the generic plane section of F^3 is not rational.

Since the system of cubics (L) whose equation is

$$aL_0+bL_1+cL_2+dL_3 = 0 \tag{3}$$

represents plane sections of F^3, it must have grade 3. Thus, in the most general case, (L) is the system of cubics passing through six distinct base points $O_1,\ldots, O_6$ of general position in ϖ; and conversely, any such system in ϖ certainly does represent plane sections of a cubic surface in space S_3. In conclusion then:

The general cubic surface in S_3 can be represented birationally on a plane by means of a six-base-point system of cubics $C^3(O_1,\ldots, O_6)$.

2.1. *Properties of the representation.* As we shall frequently have occasion to make use of the representation just obtained, we begin by summarizing its general properties in some detail as follows.

In the first place, three points of the surface are collinear if and only if their images in the plane form with $O_1,\ldots, O_6$ a set of nine associated points.

Next, the representing system $(L) = C^3(O_1,\ldots, O_6)$ has no fundamental curves, such as would transform exceptionally into points of F^3; but to the neighbourhoods of its six base points $O_1,\ldots, O_6$ there correspond, by Ch. VI, § 2.1, six lines on F^3 which we denote by $a_1,\ldots, a_6$. Clearly no two of these lines intersect.

An arbitrary non-singular curve Γ on F^3 may be supposed to be represented in ϖ by a curve $C^m(O_1^{k_1}, O_2^{k_2},\ldots, O_6^{k_6})$, in which case its order n and its genus p are given by

$$n = 3m-\sum_1^6 k_i, \qquad p = \tfrac{1}{2}(m-1)(m-2)-\tfrac{1}{2}\sum_1^6 k_i(k_i-1). \tag{4}$$

The freedom r of the system to which it belongs is given by

$$r = \tfrac{1}{2}m(m+3)-\tfrac{1}{2}\sum k_i(k_i+1). \tag{5}$$

Also the number i of its intersections with any other curve Γ', represented by $C^{m'}(O_1^{k'_1}\dots O_6^{k'_6})$, is given by

$$i = mm'-\sum k_i k'_i. \tag{6}$$

The section of F^3 by a surface of order k will be represented by a curve $C^{3k}(O_1^k\dots O_6^k)$.

2.2. *Lines on F^3.* If a line lies entirely on F^3, then it is common to a pencil of plane sections; and it must be represented in the plane therefore, either by a base point (as already mentioned) or by a line or a conic at most; if it is represented by a line, then this line must be a join of two base points so that it may have only one free intersection with curves of (L); and, for the same reason, if it is represented by a conic, this must pass through five of the base points. There exist, therefore, in all, twenty-seven lines on F^3, represented as follows:

6 lines a_i $(i = 1,\dots,6)$ represented by neighbourhoods of $O_1,\dots, O_6$,

15 lines c_{ij} $(i,j = 1,\dots,6, i\neq j)$ represented by lines $O_i O_j$,

6 lines b_j $(j = 1,\dots,6)$ represented by conics $C^2(O_2\dots O_6)$, etc.

The six lines a_i do not intersect one another; and similarly the lines b_j are all mutually skew, since their representative conics meet only in tetrads of the points O_i; but clearly a_i will meet b_j, save only when $i = j$. These two sets of six lines—intersecting only as stated—are said to form a *double-six*.

The line c_{ij}, represented by $O_i O_j$, will clearly meet a_i and a_j; also it will meet b_i and b_j, for the conics representing these lines have each one free intersection with $O_i O_j$; and finally, since two joins $O_i O_j$ and $O_h O_k$ meet in a free point only if O_i, O_j, O_h, O_k are all different, c_{ij} will meet c_{hk} if and only if $i, j \neq h, k$.

All the intersections of the twenty-seven lines may be summarized therefore as follows:

a_i meets b_j if $i \neq j$,

c_{ij} meets a_k or b_k if $k = i, j$,

c_{ij} meets c_{hk} if $i, j \neq h, k$.

Corresponding to the twenty-seven lines of F^3, we have twenty-seven pencils of conics lying in planes through the lines. These are represented in the plane by pencils of cubics, conics, or lines such as, for example, $C^3(O_1^2 O_2 \dots O_6)$, $C^2(O_1 O_2 O_3 O_4)$, $C^1(O_1)$.

2.3. *Symmetry of the lines.* The lines of the surface, as we have obtained them, appear to form three distinct groups a_i, b_j, c_{ij}; but this grouping is in fact a feature only of the method of representation and not an essential property of the configuration as a whole. To make this clear, we may consider the effect of transforming ϖ birationally into itself by means of a quadratic transformation whose fundamental points are three of the points O_i, say O_1, O_2, O_3. Such a transformation carries (L) into a system $(L') \equiv C^3(O_1 O_2 O_3 O_4' O_5' O_6')$, where O_4', O_5', O_6' are the transforms of O_4, O_5, O_6; and this gives at once a new representation of F^3 on ϖ by means of the system (L'). For this new representation, however, the partition of the twenty-seven lines into sets a_i', b_j', c_{ij}' is different from the original. The three lines a_1, a_2, a_3 become c_{23}', c_{31}', c_{12}'; conversely, c_{23}, c_{31}, c_{12} become a_1', a_2', a_3'; also b_4, b_5, b_6 become c_{56}', c_{64}', c_{45}', and so on.

Thus, by a suitable transformation, we may arrange for any given line of F^3 to be represented in any of the three possible ways, and the distinction between the three groups of lines disappears. Actually, the twenty-seven lines form a perfectly symmetrical configuration; and this is abundantly illustrated in the many detailed properties of the lines which are given in examples at the end of the chapter.

2.4. *Other curves on the surface.* After lines and conics, we may inquire how many systems of twisted cubics lie on F^3; and we find no less than 72 distinct systems—each a net of grade unity—represented by (*a*) the lines of the plane, (*b*) the 20 nets of conics through three base points, (*c*) the 30 nets of cubics which have a node at one base point and pass through four others, (*d*) the 20 nets of quartics which have nodes at three base points and pass through the remaining three, (*e*) the net of quintics which have nodes at all six base points. These 72 systems fall into 36 complementary pairs, such that quadrics through curves of either system of a pair meet F^3 residually in the curves of the other; thus, for example, the system represented by lines of the plane is complementary to that represented by $C^5(O_1^2 \dots O_6^2)$ because any line and any quintic of the type in question unite to form a $C^6(O_1^2 \dots O_6^2)$ representing a quadric section of F^3. For this particular pair of systems also, the curves of the first have the six lines b_j as chords, those of the second have the lines a_i as chords, while curves of both systems have the lines c_{ij} as unisecants; and generally, each of the 36 pairs of complementary systems is associated in this way with a particular double-six of lines on the surface.

In exactly the same way, and supposing we had any reason to do so, we could enumerate the systems of curves of any given order and genus on F^3; but we pass on now to give some illustrations of the way in which the representation may be used to give complete and rapid solutions of particular intersection problems.

Thus, for example, to find the residual intersection of a quadric F^2 and a cubic surface F^3 which pass through a given conic C^2, we may suppose F^3 represented as above on ϖ in such a way that C^2 is represented by a line through O_1. Since the complete intersection is represented by $C^6(O_1^2 \dots O_6^2)$, the residual intersection is represented by $C^5(O_1\, O_2^2 \dots O_6^2)$, and it is therefore an *elliptic quartic* ${}^1C^4$ which meets C^2 in $5-1 = 4$ points. In the same way, if F^2 and F^3 meet in a line l, represented, we may suppose, by $C^2(O_1 \dots O_5)$, then their residual intersection, represented by $C^4(O_1 \dots O_5\, O_6^2)$, is a ${}^2C^5$ which meets l in $8-5 = 3$ points. Alternatively, of course, we might have supposed l to be represented by the neighbourhood of O_1 and the image of ${}^2C^5$ would then have been $C^6(O_1^3\, O_2^2 \dots O_6^2)$. This illustrates a point which is often found puzzling at first, namely, that if a curve is made to contain the neighbourhood of a base point O_i (as a component of order zero), then its multiplicity at O_i is *increased* by unity.

As a last illustration we may find the residual intersection of two cubic surfaces F^3, G^3 which pass through a twisted cubic C^3 and through one or both of two lines l, l' which meet neither C^3 nor each other. The complete intersection of G^3 with F^3 will be represented by $C^9(O_1^3 \dots O_6^3)$; and if we take C^3 to be represented by $C^5(O_1^2 \dots O_6^2)$, as we may, there remains a curve $C^4(O_1 \dots O_6)$. The lines l, l', since they do not meet C^3 or each other, must then be supposed to be represented by conics, say $C^2(O_1 \dots O_5)$ and $C^2(O_2 \dots O_6)$. If F^3, G^3 contain only C^3 and l, their residual intersection, represented by $C^2(O_6)$, is a *rational quintic* ${}^0C^5$, meeting C^3 in eight point and l in two points. If F^3, G^3 contain l' in addition, then $C^2(O_6)$ must break up into $C^2(O_2 \dots O_6)$ together with the neighbourhoods of O_2, O_3, O_4, O_5; so that ${}^0C^5$ breaks up into l' and four other skew lines which are chords of C^3 and transversals of l and l'. This degeneration could have been deduced from the fact that there do exist four chords of C^3 which meet l and l'.

Many further examples of the use of the above representation will occur, both in the examples at the end of this chapter and incidentally in other chapters.

§ 3. The Veronese surface and its projections

3. We propose next to discuss together all the surfaces which can be represented birationally on a plane by means of systems of *conics*, a group which includes the quadric surface already discussed separately in § 1. The parent surface of this group, from which all the others may be derived by projection, is a remarkable surface in five-dimensional space—named after its discoverer Veronese—whose prime sections image the totality of conics in the plane. This surface—the first ever to be constructed in higher space—is notable for its habit of being an exception to theorems about surfaces in general; in what follows we shall try to explain some of its peculiar characteristic properties.

3.1. ***The Veronese surface.*** Since all the conics in a plane ϖ form a linear system, of grade 4 and freedom 5, whose equation may be written in the form

$$\lambda_0 x^2+\lambda_1 y^2+\lambda_2 z^2+\lambda_3 yz+\lambda_4 zx+\lambda_5 xy = 0, \tag{1}$$

the Veronese surface of which this system is the projective model is a quartic surface F^4, in five-dimensional space S_5, and its parametric equations are

$$\frac{x_0}{x^2}=\frac{x_1}{y^2}=\frac{x_2}{z^2}=\frac{x_3}{yz}=\frac{x_4}{zx}=\frac{x_5}{xy}. \tag{2}$$

The explicit equations of the surface take the simple form

$$\left\| \begin{matrix} x_0 & x_5 & x_4 \\ x_5 & x_1 & x_3 \\ x_4 & x_3 & x_2 \end{matrix} \right\|_2 = 0. \tag{3}$$

The $(1, 1)$ correspondence between the points of F^4 and those of ϖ is unexceptional in both directions, and it carries every curve of order n in the plane into a curve of order $2n$ on the surface. Hence F^4 *contains only curves of even order.*

3.11. *The conics* λ *and chord locus* Ω *of* F^4. To the lines of ϖ there correspond on F^4 the conics of a doubly infinite system λ. By the incidence properties of the former, any two of the conics λ meet in a unique point of F^4 and define a pencil of these conics through the same point; also through any two points of F^4 there passes a unique conic of the system. The planes of the conics λ are called the *conic planes* of F^4, and they generate a primal which we denote by Ω.

The ∞^4 chords of F^4 all lie in the conic planes of the surface; for any chord PQ lies in the plane of the unique conic λ through P and Q. Thus, instead of filling the whole of the ambient space, as the chords of a surface in S_5 normally do, the chords of F^4 generate only the primal Ω; i.e.

F^4 has no chord through a generic point† *of* S_5.

If a point lies on one chord of F^4, then it lies on a pencil of such chords in the conic plane which passes through it.

The equation of Ω may be obtained as follows. Let

$$lx+my+nz = 0$$

be the equation of any line of ϖ; then, by multiplying this equation in turn by x, y, z, we see that the conic λ corresponding to the line is the section of F^4 by the plane whose equations are

$$lx_0+mx_5+nx_4 = 0,$$

$$lx_5+mx_1+nx_3 = 0,$$

$$lx_4+mx_3+nx_2 = 0;$$

whence Ω, which is the locus of such conic planes, has equation

$$\begin{vmatrix} x_0 & x_5 & x_4 \\ x_5 & x_1 & x_3 \\ x_4 & x_3 & x_2 \end{vmatrix} = 0. \tag{4}$$

The first member of this equation is a symmetrical cubic determi-

† Cf. Severi, *Rend. Palermo*, **15** (1901), 33, where it is shown that F^4 is the only non-singular surface in S_5 which has the property in question. Compare also Chapter IX, § 2.11.

nant whose first derivatives plainly vanish at every point of F^4; hence:

The primal Ω which is the locus of conic planes (and therefore also the chord-locus) of F^4 is a cubic symmetroid with F^4 as double surface.

3.12. *Tangential properties of F^4.* Among the chords of F^4 we must include the ∞^3 tangent lines to the surface, lying by pencils in its ∞^2 tangent planes. Hence:

The primal Ω contains all the tangent planes to F^4 as a second system of generating planes.

Consider next then the ∞^4 tangent primes of F^4, of which there is a double infinity through each tangent plane. If a prime touches a surface, it meets it in a curve which has a double point. Now prime sections of F^4 are the family of rational quartics ${}^0C^4$ represented by conics in ϖ; and the only nodal ones among them are those which break up into pairs of the conics λ, represented by line-pairs of the plane. Hence:

The tangent primes of F^4 are those which meet it in pairs of conics.

This property of having tangent prime sections which are all reducible is another notable peculiarity† of F^4.

We note finally that among the conics of the plane, representing prime sections of F^4, there are ∞^2 repeated lines representing prime sections which are repeated conics; so that, corresponding to each conic of the surface, there exists a prime which touches F^4 all along it. Such primes are termed the *contact primes* of F^4. The coordinates $(\lambda_0,\ldots,\lambda_5)$ of any contact prime are such that

$$\lambda_0 x^2+\lambda_1 y^2+\lambda_2 z^2+\lambda_3 yz+\lambda_4 zx+\lambda_5 xy \equiv (lx+my+nz)^2;$$

and hence, regarding l, m, n as homogeneous parameters, the parametric equations of the envelope of contact primes are

$$\frac{\lambda_0}{l^2}=\frac{\lambda_1}{m^2}=\ldots=\frac{\lambda_5}{2lm}.$$

† It can be shown that any surface which contains ∞^2 conics is a Veronese surface or one of its projections. More generally, any surface of S_3 having ∞^2 reducible plane sections is either the projection of a Veronese surface or a scroll. This result is known as the Kronecker–Castelnuovo theorem; cf. Castelnuovo, *Rend. Lincei*, (5), **3** (1894), 22.

Comparing these with (2), it follows that the envelope of contact primes of F^4 is a *Veronese envelope* Φ^4, dual of a Veronese surface.

3.13. *Representation of conics of the plane on points of* S_5. Instead of associating, as above, the general conic-locus $k(\lambda_i)$ given by (1) with the *prime* $\Pi(\lambda_i)$ of S_5, we may equally well associate it with the *point* $P(\lambda_i)$ of S_5 whose coordinates are $(\lambda_0,...,\lambda_5)$. By doing this we obtain a representation, intuitively more acceptable, of the ∞^5 conic-loci of the plane on the ∞^5 points of S_5; and in this representation the repeated lines of ϖ, originally associated with the primes of the Veronese envelope Φ^4, are now imaged, by duality, by the points of a Veronese surface F^4. It may readily be verified then that line-pairs generally in ϖ are imaged by points of the chord-locus Ω of F^4. Any pencil of conic-loci in ϖ is imaged by a line of S_5; and similarly, linear systems of freedoms 2, 3, and 4 are imaged by planes, solids, and primes of S_5.

Furthermore, if $K(\Lambda_i)$ is the conic-envelope in ϖ whose equation is

$$\Lambda_0 l^2+\Lambda_1 m^2+...+\Lambda_5 lm = 0,$$

the above representation of the conic-loci of ϖ on the points of S_5 is intimately associated with a certain analogous representation of the conic-envelopes $K(\Lambda_i)$ on the primes of S_5. We observe in fact that any given linear condition on the coefficients λ_i of k may be written in the form

$$\lambda_0\Lambda_0+\lambda_1\Lambda_1+\lambda_2\Lambda_2+\tfrac{1}{2}\lambda_3\Lambda_3+\tfrac{1}{2}\lambda_4\Lambda_4+\tfrac{1}{2}\lambda_5\Lambda_5 = 0,$$

which is the condition that k should be apolar to a certain conic-envelope $K(\Lambda_i)$. Hence, by associating any conic-envelope $K(\Lambda_i)$ with the prime Π of S_5 whose coordinates are

$$\Lambda_0,\ \Lambda_1,\ \Lambda_2,\ \tfrac{1}{2}\Lambda_3,\ \tfrac{1}{2}\Lambda_4,\ \tfrac{1}{2}\Lambda_5,$$

we ensure that a conic-locus $k(\lambda_i)$ is apolar to a conic-envelope $K(\Lambda_i)$ in ϖ if and only if, in S_5, the image point $P(\lambda_i)$ of the former lies in the image prime Π of the latter. It may readily be verified then, for example, that in this associated representation the (Veronese) envelope of primes Π which represent repeated points of ϖ is simply the envelope of contact primes of F^4, while the more ample (cubic symmetroidal) envelope of primes Π which represent point-pairs of ϖ is simply the envelope of tangent primes of F^4; and the representation has many other interesting properties.

3.2. ***Projections of the Veronese surface.*** Returning now to rational surfaces represented by systems of conics, we observe that any such surface in S_4 will be represented on ϖ by means of an ∞^4 system, and any such surface in S_3 by means of an ∞^3 system of conics. Any linear ∞^4 system consists of all the conics of ϖ whose coefficients λ_i are subjected to a single linear condition, this condition being tantamount to requiring that the conics shall all be apolar to a fixed conic-envelope K (cf. § 3.13); and similarly any linear ∞^3 system consists of conics which are apolar to each of two conic-envelopes K, K' (and therefore also to every conic-envelope of the tangential pencil $K+\lambda K'$). If F^4 is the Veronese surface defined in § 3.1, conics of ϖ which are apolar to K represent sections of F^4 by primes through a fixed point O of S_5, and their projective model is the projection of F^4 from O into S_4; and similarly, conics apolar to K and K' represent sections of F^4 by primes through two fixed points O, O' of S_5, and their projective model is the surface in S_3 obtained by projecting F^4 from the line OO'. Different cases arise according as the envelopes K, K' are proper or degenerate; or, in other words, according as the points O, O' are generally or specially situated with respect to F^4.

3.21. *The surfaces* $F^4[4]$ *and* $F^4_{(1)}[4]$. The conic K referred to above may be either (1) a proper conic-envelope, or (2) a point-pair U, V, or (3) a repeated point A^2; conics apolar to K are accordingly either a general linear ∞^4 system, or the system of conics which have an assigned pair of conjugate points U, V, or the system of conics which pass through a simple base point A; and the projective models of these systems are projections of F^4 from a vertex O which is, in the three respective cases, a general point of S_5, a general point lying in a conic-plane of F^4, a point of F^4 itself.

In each of the first two cases, in which O is not on F^4, the surface obtained is of order 4 and is usually called a *projected Veronese surface*; we denote it by $F^4[4]$ or by $F^4_{(1)}[4]$ according as it is a generic projection of F^4 or a projection from a point of a conic-plane.

The generic projection $F^4[4]$ is represented then by means of conics apolar to a proper conic-envelope K; and it has no double points.

The special projection $F^4_{(1)}[4]$ has a *double line, d* say, projection of the conic of F^4 in whose plane the vertex O is taken to lie; its plane representation, as indicated, is by means of the system of conics which have a given pair of conjugate points U, V, and we observe that this system has in fact ∞^1 neutral pairs of points on UV, namely pairs of harmonic conjugates with respect to U and V, which represent the individual (improper) double points of the surface on d. The points U, V themselves, being coincident neutral pairs, represent what are called *pinch-points* of the surface, such being the name given by Cayley to points of any double curve of a surface at which the two tangent planes to the surface coincide; in the present case they might also be described as *improper cusps*.

3.22. *The rational normal cubic scroll.* The surface obtained by projecting F^4 from a point O of itself is a cubic scroll R^3, for the conics of F^4 which pass through O project into lines on R^3. Its plane representation, as already noted, is by means of conics with a simple base point A.

The lines of ϖ which pass through A represent evidently the generators of R^3; but the line, d say, of R^3 which represents the neighbourhood of A (and arises by projection from the neighbourhood of O) is not a generator. In fact it evidently meets every generator and is therefore a simple *directrix line* of R^3. The generators meet any conic of R^3, represented by a line of ϖ, in a range related to the range they cut on d; hence

The rational normal cubic scroll is the locus of joins of corresponding points of related ranges on a line and on a conic of S_4.

Any surface so generated, provided the line and the conic are of general position in S_4, is a cubic scroll; and its plane representation, by means of conics $C^2(A)$, may be obtained by projecting it (birationally) from a generator.

3.23. *The Steiner surface.* Coming now to surfaces in S_3, we consider first the projection of F^4 from a generic line l of S_5; the quartic surface so obtained, which we denote by $F^4[3]$, is called a Steiner surface, and it is represented on ϖ by the linear ∞^3 system of conics, (k) say, apolar to the conic-envelopes of a generic tangential pencil $K+\lambda K'$.

The conics (k) are apolar, in particular, to the three point-pairs of the pencil, say (P, P'), (Q, Q'), (R, R'); and it follows therefore, exactly as in § 3.21, that the joins of these pairs, which are, namely, the sides of the common self-polar triangle of the pencil, represent double lines d_1, d_2, d_3 of the surface. The point-pairs themselves, being the united points of the involutions of neutral point-pairs on their joins, represent pairs of pinch-points on d_1, d_2, d_3. The three double lines arise by projection from the three conics of F^4 whose planes meet l, agreeing with the fact that the locus Ω of all such conic-planes is of order three.

In addition, however, to its three involutions of neutral pairs, the system (k) possesses a neutral triad, which is namely the self-conjugate triad, (X, Y, Z) say, of $K+\lambda K'$; for the pairs (Y, Z), (Z, X), (X, Y), being harmonic with respect to (P, P'), (Q, Q'), (R, R') respectively, are neutral for (k), so that conics of (k) through any one of the three points X, Y, Z contain the remaining two. Hence (X, Y, Z) represents a triple point T of the surface, projection of a unique trisecant plane of F^4 through l; and the three double lines d_1, d_2, d_3 clearly concur in T. Thus:

The Steiner surface $F^4[3]$ possesses three double lines (each endowed with two pinch-points) which concur in a triple point of the surface.

We record also the not unimportant corollary:

A unique trisecant plane of a Veronese surface passes through a generic line of S_5; and, consequently, a unique trisecant line of a projected Veronese surface passes through a generic point of S_4.

If we take $K+\lambda K'$ to have equation

$$(a+\lambda)l^2+(b+\lambda)m^2+(c+\lambda)n^2 = 0,$$

then the equation of (k) is of the form

$$\lambda_0 \sum (b-c)x^2+\lambda_1 yz+\lambda_2 zx+\lambda_3 xy = 0;$$

the parametric equations of the Steiner surface are

$$\frac{x_0}{\sum (b-c)x^2} = \frac{x_1}{yz} = \frac{x_2}{zx} = \frac{x_3}{xy},$$

and its explicit equation is

$$(b-c)x_2^2 x_3^2+(c-a)x_3^2 x_1^2+(a-b)x_1^2 x_2^2 = x_0 x_1 x_2 x_3.$$

The triple point is the vertex of reference X_0, and the double lines are the joins of this point to X_1, X_2, X_3. By a convenient choice of the unit point, the equation of any quartic surface having

X_0X_1, X_0X_2, X_0X_3 as double lines may be reduced to the form (given by Kummer)

$$x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2 = x_0x_1x_2x_3.$$

3.24. *The cubic scroll in S_3.* If we project the rational normal cubic scroll R^3 from a generic point O of S_4, we obtain a cubic scroll in ordinary space which we denote by $R^3[3]$. This surface will be represented on ϖ by means of the ∞^3 system of conics (k) which pass through a fixed point A and are apolar to a fixed conic envelope K. Besides the repeated point A^2, the tangential pencil $A^2+\lambda K$ will contain one other point-pair, (U, V) say; and hence (k) may be defined as the system of conics which pass through A and have (U, V) as a given pair of conjugate points.

The projected surface, like the original, has a *simple directrix line*, which we denote by e, represented by the neighbourhood of A; but it has in addition a *double directrix line*, d say, which is the projection of a conic of R^3 whose plane passes through O. The points of this double-line image ∞^1 pairs of points of the line UV, which, being harmonically conjugate with respect to U and V, are neutral for (k); hence, through any point P of d there pass two generators of the surface which are distinct except when P is one of the two pinch-points on this line.

If P and Q denote the points in which a generator of the surface meets d and e respectively, then there is a $(1, 2)$ correspondence between P and Q whose branch-points are the pinch-points. Conversely, it is not difficult to verify that any general $(1, 2)$ correspondence between the points of two skew lines defines a cubic scroll $R^3[3]$.

The reduced equation of $R^3[3]$, obtained as in the preceding section, is

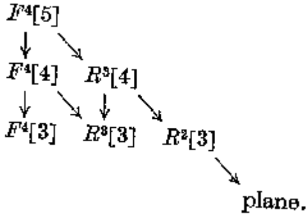
$$x_0x_3^2 = x_1x_2^2.$$

Here the double directrix line is X_0X_1, the simple directrix line is X_2X_3, and the pinch-points are X_0 and X_1.

By projecting R^3 from a point which lies in one of the planes which join the directrix to a generator, a special type of $R^3[3]$ is obtained, on which d and e coincide.

3.25. *Summary.* To complete the list of projections of the Veronese surface, we project R^3 from a point of itself, obtaining a quadric surface R^2, represented by $C^2(AB)$ as in § 1; and finally, we may project R^2 from a point of itself, thereby obtaining a plane which is birationally representable on ϖ by $C^2(ABC)$.

The various projections of the Veronese surface may now be arranged in a table as follows:



§ 4. Rational scrolls

4. Plane representation. In regard to rational scrolls, we propose to prove first the following general result:

Any rational scroll of order n may be represented on the plane by means of a system of curves of order n with one $(n-1)$-fold base point and $n-1$ further simple base points.

If the scroll is in higher space, then a linear system of curves by means of which it may be represented on the plane will be unaltered in essentials, save for a diminution of freedom, by any general projection of the scroll into space of lower dimension; and it will be sufficient therefore to prove the above result for rational scrolls in ordinary space S_3.

We suppose then, for convenience, that we are dealing with a scroll R of order n in ordinary space, possessing a double curve D which is the locus of points in which generators of R happen to intersect. We shall show first that *for any such scroll, whether rational or not, every generator is met by $n-2$ others*; or, in other words, every generator meets D in $n-2$ points. Consider then the curve C, of order $n-1$, in which R is met residually by any plane π through a generator g_0. Through the generic point P of C there will pass a unique generator g, the correspondence between P and g being birational; so that, as g approaches g_0, P must tend to a definite point P_0 which will be one of the intersections of C with g_0. But C meets g_0 in $n-1$ points, say $P_0, P_1,..., P_{n-2}$, through which there must pass $n-1$ generators of R of which only one is g_0; hence the remaining $n-2$ must be generators of R which happen to meet g_0; and $P_1,..., P_{n-2}$ are the intersections of g_0 with D. It follows, furthermore, that if two planes π, π' through g_0

meet R residually in curves C, C', then these curves meet g_0 in $n-2$ common points $P_1,..., P_{n-2}$; and hence R is the locus of joins of birationally corresponding points P, P' of C and C', such that P comes to coincidence with P' in each of the $n-2$ positions $P_1,..., P_{n-2}$.

When R is a rational scroll, whose generators are in birational correspondence with the values of a parameter λ, then C and C'—loci of points $P(\lambda)$ and $P'(\lambda)$—have parametric equations of the form

$$\rho x_i = u_i(\lambda) \quad (i = 0,..., 3),$$

and

$$\rho x_i = v_i(\lambda) \quad (i = 0,..., 3),$$

where the polynomials u_i and v_i are all of order $n-1$; and there exist $n-2$ values of λ, say $\lambda_1,..., \lambda_{n-2}$, for which $P(\lambda)$ and $P'(\lambda)$ coincide. The resulting parametric equations of R itself are then

$$\rho x_i = u_i(\lambda)+\mu v_i(\lambda) \quad (i = 0,..., 3);$$

and if we interpret (λ, μ) as non-homogeneous coordinates of a point in a plane ϖ, then R is birationally represented on ϖ by means of the system of curves (f) whose equation is

$$\sum_0^3 k_i\{u_i(\lambda)+\mu v_i(\lambda)\} = 0.$$

The curves of (f) are of order n; they have an $(n-1)$-fold base point at $(0,\infty)$ and a simple base point at $(\infty, 0)$; and if we write

$$\frac{u_0(\lambda_i)}{v_0(\lambda_i)} = \frac{u_1(\lambda_i)}{v_1(\lambda_i)} = ... = \frac{u_3(\lambda_i)}{v_3(\lambda_i)} = -\mu_i \quad (i = 1,..., n-2),$$

it is clear that the points (λ_i, μ_i) are $n-2$ further simple base points of the system. Hence R is represented on ϖ by means of a system of the type $C^n[(n-1)^1, 1^{n-1}]$, as was required to be proved; and conversely, any system of this type, of freedom $r \geqslant 3$, is evidently of grade n and represents a rational scroll whose generators correspond to lines of ϖ through the $(n-1)$-fold base point.

4.1. *Normal rational scrolls.* A complete linear system of the type $C^n[(n-1)^1, 1^{n-1}]$ represents a normal rational scroll R^n which belongs therefore to space of dimension r given by

$$r = \tfrac{1}{2}n(n+3)-\tfrac{1}{2}n(n-1)-(n-1) = n+1.$$

Hence: *A rational scroll of order n is normal in $[n+1]$.*

Conversely it can be shown that, with the single exception of the Veronese surface F^4 in [5], the rational normal scrolls are the only surfaces of order n in $[n+1]$ and not in any lower space; or, in other words, the only surfaces having prime sections which are rational normal curves. Also, since any $F^n[n+1]$ projects from a point of itself into a $F^{n-1}[n]$, a succession of such birational projections will lead down to a quadric F^2 in [3], and thence to a plane ϖ on which the projections of the prime sections of the original surface will be curves of order n, and those of its generators, if it is ruled, the lines of a pencil. This means that the plane representation of the normal scroll $R^n[n+1]$ by means of a system $C^n[(n-1)^1, 1^{n-1}]$ is obtainable by direct projection of the surface from an $[n-2]$ which meets it in $n-1$ general points; these lie on $n-1$ generators which project into the simple base points, and they must all lie also on a simple directrix C^{n-1} which projects into the $(n-1)$-fold base point.

The representation just obtained is not in general the simplest available. In fact, if O is the $(n-1)$-fold base point and A, B are simple base points, then a quadratic transformation with O, A, B as fundamental points reduces $C^n[(n-1)^1, 1^{n-1}]$ to $C^{n-1}[(n-2)^1, 1^{n-3}]$; and in the most general case this reduction process can be continued till the number of simple base points is diminished to 0 or 1. In special cases, however, and in particular when all or most of the simple base points lie on a line, the reduction process may break down owing to the occurrence of infinitely near base points in the transformed systems. If no such stoppage occurs, $C^n[(n-1)^1, 1^{n-1}]$ reduces ultimately to one or other of the systems

$$C^m[(m-1)^1] \quad \text{or} \quad C^{m+1}[m^1, 1^1],$$

in which $m = \frac{1}{2}(n+1)$ and $m = \frac{1}{2}n$ respectively; and these systems represent scrolls $R^{2m-1}[2m]$ and $R^{2m}[2m+1]$ which are, in one sense at least, the *general* rational normal scrolls in $[2m]$ and $[2m+1]$.

From a different point of view, one can assert simply that there exist different species of $R^n[n+1]$ which are distinguished from one another by the order of their *minimum directrix curve or curves.* Indeed if Δ^λ and $\Delta^{n-\lambda}$ are any pair of rational curves in birational correspondence, then the locus of joins of corresponding points is in general a rational R^n which possesses, for $\lambda = 1, 2,\ldots$, a directrix line, a directrix conic,...; and if Δ^λ and $\Delta^{n-\lambda}$ are normal and of general position in space of sufficiently high dimension, then the least space containing them is an $[n+1]$, so that R^n is also normal.

For the scroll $R^{2m-1}[2m]$ represented by $C^m[(m-1)^1]$, there is a unique minimum directrix curve Δ^{m-1} represented by the neighbourhood of the $(m-1)$-fold base point; and the directrix curves of next lowest order are Δ^m (not meeting Δ^{m-1}) represented by the lines of the plane. The surface is the locus of joins of corresponding points of related rational normal curves of orders $m-1$ and m respectively.

For the scroll $R^{2m}[2m+1]$, represented by $C^{m+1}[m^1, 1^1]$, there is a pencil of (non-intersecting) minimum directrix curves Δ^m, represented by lines through the simple base point; and the surface is the locus of joins of corresponding points of two related rational normal curves of order m.

Finally, the type of scroll $R^n[n+1]$ which possesses a minimum directrix curve Δ^λ, of order $\lambda < \frac{1}{2}(n-1)$, will admit—in the first instance—of plane representation by means of a system $C^n[(n-1)^1, 1^{n-1}]$ in which $n-\lambda$ of the simple base points lie on a line representing C^λ; but this system will admit of some reduction by quadratic transformations, though not usually to the same extent as when λ has one or other of the values $\frac{1}{2}(n-1)$ or $\frac{1}{2}n$.

EXAMPLES

1. Any system $C^3[2^1, 1^2]$ in which the three base points are distinct is reducible by quadratic transformation to $C^2[1^1]$; hence, as we have already seen in § 3.22, there is but one species of $R^3[4]$ which is not a cone.

2. There are two species of rational normal quartic scroll $R^4[5]$. The first of these—and in one sense the more general—is the locus of joins of related points of two general conics in [5]; and its plane representation is by means of the system $C^3[2^1, 1^1]$. The second, which is the locus of joins of related points of a line and twisted cubic in [5], is represented in the first instance by a system $C^4[3^1, 1^3]$ in which the three simple base points are collinear; but this system may be reduced to a system $C^3[2^1(1^1)]$ in which the simple base point is in the first neighbourhood of the double base point.

3. Prove that the general rational quartic scroll in [4] has a unique improper double point. (If the surface is generated by related conics C, C', then the two generators which intersect lie in a certain plane through the point of intersection of the planes of C and C'.)

4. Prove that the general rational quartic scroll in [3] has a double curve which is a twisted cubic; also that its generators cut this curve in pairs of a general symmetrical (2, 2) correspondence.

§ 5. DEL PEZZO SURFACES

5. The plane representation. Having considered, in the two preceding sections, surfaces of order n which are normal in $[n+1]$, it is natural to ask next what surfaces of order n (other than cones) are normal in $[n]$. To this question Del Pezzo found the surprising answer† that for $n > 9$ there are no such surfaces; while, for $n \leqslant 9$, the surfaces which arise form—with only one exception—a single simple series $F^9, F^8, \ldots, F^3$, such that F^n $(3 \leqslant n \leqslant 8)$ is always the projection of F^{n+1} from a point of itself. All these surfaces, including the exceptional one which is a second octavic surface F^8_1, have been named *Del Pezzo surfaces* in honour of their discoverer.

The starting-point of Del Pezzo's analysis is the obvious fact that any surface $F^n[n]$ projects from a point of itself into a surface $F^{n-1}[n-1]$, and hence, by a generic series of such projections, into a cubic surface (possibly a cone) in ordinary space. We may not reproduce his arguments here; but the conclusion reached is that all the surfaces concerned are rational, that with one exception they are the surfaces represented on the plane by means of systems of non-singular cubics $C^3[1^{9-n}]$ $(n = 9, 8, \ldots, 3)$, and that the exception is the surface F^8_1, which is represented on the plane by the system $C^4[2^2]$. The parent surface of the main sequence is $F^9[9]$, the *Veronesean* of plane cubics; this projects from a point of itself into $F^8[8]$, and so on. On the other hand, $F^8_1[8]$ projects from a point of itself into the surface $F^7[7]$ of the main sequence; for the projection in question is representable by means of a system $C^4[2^2, 1^1]$, and this is transformable into a system $C^3[1^2]$ by a quadratic transformation of the plane. Thus we have the scheme

$$\begin{array}{l} F^9 \to F^8 \to F^7 \to F^6 \to F^5 \to F^4 \to F^3. \\ \qquad\;\; F^8_1 \nearrow \end{array}$$

The numbers of lines and of pencils of conics which exist on each of the Del Pezzo surfaces are shown in the table:

	F^9	F^8	F^8_1	F^7	F^6	F^5	F^4	F^3
Lines . . .	0	1	0	3	6	10	16	27
Pencils of conics .	0	1	2	2	3	5	10	27

The reader should work out these numbers for himself from the plane representations, and he should investigate also the configurations of the various sets of lines, showing, for example, that the six lines of F^6 form a skew hexagon which is a prime section of the surface.

† *Rend. Palermo*, **1** (1887), 241.

It may be noted, finally, that F_1^8 is the surface which is represented on a quadric surface Q by means of the complete system of sections of Q by other quadrics; for all quadric sections of Q are represented, in the ordinary plane representation of Q, by curves of a system $C^4[2^2]$. Indeed in this connexion the six surfaces

$$F_1^8,\ F^7,\ F^6,\ F^5,\ F^4,\ F^3$$

form a different straight sequence, being all representable on Q by means of quadric sections through 0, 1,..., 5 points of this surface.

5.1. *Segre's surface F^4.* The quartic Del Pezzo surface F^4 is also known as Segre's surface, and it is particularly important in that it is the general surface of intersection of two quadrics in [4]. To see this, we observe first that, if we substitute $r = 4, l = 2$, $\nu = 4, p = 1$ in Theorem VIII, Ch. VI, we find that the number of linearly independent quadrics through F^4 is at least two; and it obviously cannot be more than two. Conversely also, if G^4 is any general quartic surface of intersection of two quadrics Ω, Ω' in [4], then G^4 certainly contains some lines, as may be seen by projecting it into a cubic surface from a point of itself; also, if l is any one of these lines, then any plane through l meets G^4 in only one residual point, namely the point of intersection of the residual lines in which it meets Ω, Ω', so that G^4 projects birationally from l on to a plane ϖ; and finally prime sections of G^4, since they are elliptic quartic curves which meet l, project from l into non-singular cubics in ϖ, and these cubics must form a linear system $C^3[1^5]$, of grade 4 and freedom 4, so that G^4 is a Del Pezzo F^4 as stated. The five base points $[1^5]$ are projections of five lines of G^4 (from the total of sixteen) which meet the chosen line l.

The historic association of this surface with Segre dates from an epoch-making memoir† of his about *cyclides*—quartic surfaces (in ordinary space) passing doubly through the absolute conic—in which he demonstrated that all the remarkable properties which these surfaces were known to possess were essentially properties of pencils of quadrics in four-dimensional space, any cyclide being a projection of the base surface of such a pencil.

5.2. *The Del Pezzo double plane and the Geiser involution.* In addition to the proper Del Pezzo surfaces already listed, there exists another improper but interesting member of the family,

† *Math. Ann.* **24** (1884), 313.

namely the *double plane* F^2 obtained by projecting F^3 from a point of itself on to a plane. This surface is to be regarded as a pair of superimposed planes π_1, π_2 which cross into each other all along a *branch curve* Γ, projection of the curve of contact, T say, of the proper tangent cone to F^3 from the vertex of projection O. Clearly T is the sextic curve, nodal at O, in which F^3 is met by the polar quadric of O; and Γ is a plane non-singular quartic. The general plane section of F^2 is evidently a double line with four branch points.

The surface F^2 must be birationally representable, in the usual way, on a (simple) plane ϖ by means of a system $C^3[1^7]$, and this representation will evidently be such that each pair of superimposed points P_1, P_2 of F^2 is represented in ϖ by a pair of points Q_1, Q_2—in general distinct—which form with $[1^7]$ a set of nine associated points. The totality of pairs of points such as Q_1, Q_2 constitutes an involution of pairs of points of ϖ, and this particular involution—generated by cubic curves through seven fixed points—is known as the *Geiser involution*. Each pair of points Q_1, Q_2 represents the pair of points in which F^3 is met by a line through O; and the *coincidence-curve* of the involution is therefore a sextic $K = C^6[2^7]$, image of T on F^3 and of Γ on F^2 (Ch. VI, § 5.1).

The surface F^2 contains in all 56 lines which lie, in 28 superimposed pairs, along the 28 bitangents of Γ. Each of these pairs, with one exception, is such that one of its lines is the projection of a line of F^3, while the other is the projection of the residual conic in the plane joining this line to O. The lines of the last remaining pair represent respectively the neighbourhood of O and the nodal cubic in which F^3 is met by the tangent plane at O. The reader should investigate for himself the representation in ϖ of all these lines. (Compare also Ex. 1 below.)

§ 6. Surfaces with multiple points

6. We have already referred, in Ch. VI, § 2.4, to *improper* multiple points of rational surfaces, i.e. points which correspond to neutral point-sets of the representing system (f); and we have also discussed, in Ch. VI, § 2.2, *proper* multiple points such as are represented by fundamental curves of (f), including possibly neighbourhoods of base points (curves of order zero) at which the tangent or (nodal) tangents are fixed for all curves of (f). We propose now to give some examples of these possibilities.

6.1. *A normal surface with an improper node.* It is easy enough to see that any surface F in [4] which is a general projection of a surface F_0 in [5] may be expected to possess a finite number of improper nodes arising from chords of F_0 which pass through the vertex of projection. Thus, for example, one chord of a rational normal quartic scroll $R^4[5]$ passes through an arbitrary point, so that the general rational quartic scroll in [4] has one improper node. It may be of interest, however, to give an example of a normal surface in [4] which possesses an easily detected improper node.

The complete system $C^5[2^2, 1^{10}]$, which we write here in the form $C^5[2^1, 2^1, 1^{10}]$, has freedom 4 and grade 7; and it represents a surface ${}^4F^7$ which is normal in [4]. Now the two curves $C^4[2^1, 1^1, 1^{10}]$ and $C^4[1^1, 2^1, 1^{10}]$ are uniquely defined; they have only two free intersections, U, V say; and by the extended Cayley–Bacharach Theorem (Ch. V, Th. II a) all the curves of the system $C^5[2^1, 2^1, 1^{10}]$ which pass through either of U, V also pass through the other. Thus U, V represent an improper node D of ${}^4F^7$.

It may be verified incidentally that ${}^4F^7$ is the general residual surface of intersection of two cubic primals through a pair of planes, the improper node being at the intersection of the planes; also ${}^4F^7$ contains eleven skew lines and a double-eleven of isolated conics.

6.2. *Nodal cubic surfaces in* [3]. For examples of surfaces possessing proper double points, we pass over the ordinary quadric cone, and we turn to the nodal F^3 in [3].

There are three possible ways in which a system $C^3[1^6]$, representing a surface F^3, can be so specialized as to possess a fundamental curve Ω representing the neighbourhood of a proper double point of F^3; thus (i) the six base points may lie on a conic Ω^2, (ii) three of them may lie on a line Ω^1, or (iii) two of the base points may be consecutive, so that the cubics touch a fixed line at a fixed point A, the neighbourhood of A being then a fundamental curve Ω^0 of order zero. In each of these cases, the curves (f_1), residual in (f) to Ω, have two free intersections with Ω, so that plane sections through the corresponding point O of F^3 are all nodal at O. Case (i) is that which arises when a nodal cubic surface is birationally represented on the plane by direct projection from the node.

There is, it may be remarked, no essential difference between the above three methods of representing a nodal F^3; for any system $C^3[1^6]$ which is specialized in any one of the three ways in question can be transformed, by a quadratic transformation of the plane, into another system $C^3[1^6]$ which is specialized in either of the other two possible ways. By taking the fundamental curve to be a conic, however, it becomes more immediately obvious that a uni-nodal F^3 possesses only twenty-one lines; for in this case the six lines normally represented by conics no longer exist.

6.21. *The four-nodal cubic surface.* By combining specializations of the above types, we can obtain special systems $C^3[1^6]$ which represent, in various alternative ways, cubic surfaces possessing two, three, or four nodes. By far the most interesting of these surfaces is the four-nodal F^3; and the most convenient representation of this surface is by means of *cubic curves through the vertices of a complete quadrilateral* whose sides are then the four fundamental lines representing the nodes. This surface has nine lines represented by the vertices and diagonals of the quadrilateral, and it has many other interesting properties of which some are indicated in Ex. 12, 13 at the end of this chapter.

§ 7. Plane Cremona transformations

7. Homaloidal nets. If $\phi = C^n(k_i)$ is a homaloidal net of curves (of freedom 2 and grade 1), then any three linearly independent curves of ϕ define a Cremona transformation

$$x':y':z' = \phi_1:\phi_2:\phi_3.$$

This carries curves of ϕ into lines, and it carries lines into the curves of a second homaloidal system $\psi = C^n(k_i')$ associated with the reverse transformation (cf. Ch. III, § 1.1).

Since ϕ has grade 1, and since its curves are necessarily rational, we have

$$n^2 - \sum k_i^2 = 1, \tag{1}$$

$$(n-1)(n-2) - \sum k_i(k_i-1) = 0; \tag{2}$$

and these imply the equivalent relations

$$3(n-1) = \sum k_i, \tag{3}$$

$$\tfrac{1}{2}n(n+3) - \tfrac{1}{2}\sum k_i(k_i+1) = 2. \tag{4}$$

From this last relation, and by observing that no linear system of grade 1 can have freedom greater than 2, we deduce

THEOREM I. *Every homaloidal net is a complete system of curves, and the conditions imposed by its base points are all independent.*

To construct a table of homaloidal systems (for increasing n) we have only to find solutions of (1) and (3) which correspond to systems of *irreducible* curves. If we suppose that the base points are all distinct and that their multiplicities are arranged in descending order

$$k_1 \geqslant k_2 \geqslant k_3 \geqslant \ldots,$$

the elimination of reducible systems is aided by the inequalities

$$k_1+k_2 \leqslant n, \quad k_1+k_2+\ldots+k_5 \leqslant 2n, \quad k_1+k_2+\ldots+k_9 \leqslant 3n, \quad \text{etc.} \tag{5}$$

expressing that no line, conic, cubic,..., should have too many intersections with a generic curve of the system. For $n \leqslant 6$ there exist the following types of irreducible homaloidal net:

$n = 1$ lines of the plane
$n = 2$ $C^2[1^3]$
$n = 3$ $C^3[2^1, 1^4]$
$n = 4$ $C^4[3^1, 1^6]$, $C^4[2^3, 1^3]$
$n = 5$ $C^5[4^1, 1^8]$, $C^5[3^1, 2^3, 1^3]$, $C^5[2^6]$
$n = 6$ $C^6[5^1, 1^{10}]$, $C^6[4^1, 2^4, 1^3]$, $C^6[3^3, 2^1, 1^4]$, $C^6[3^2, 2^4, 1^1]$.

The irreducibility of any one of these systems is easily established by transforming it, by a series of quadratic transformations, into the lines of the plane, and we propose next to examine the question as to whether this test can always be applied.

7.1. We prove first the fundamental result:

THEOREM II (*Noether's Inequality*). *If k_1, k_2, k_3 are the three highest or equal highest base-point multiplicities of a homaloidal net of curves of order $n \geqslant 2$, then*

$$k_1+k_2+k_3 > n.$$

We suppose all the base-point multiplicities k_i arranged in descending order. If $n \geqslant 2$, the total number s of base points (distinct or otherwise) is at least three; for if there were only two we should have

$$k_1+k_2 \leqslant n, \qquad 3n-3 = k_1+k_2,$$

and these give $3n-3 \leqslant n$, i.e. $n = 1$.

Thus for $n \geqslant 2$, we may multiply (3) by k_3 and subtract from (1), and this gives

$$k_1(k_1-k_3)+k_2(k_2-k_3)-\sum_4^s k_i(k_3-k_i) = n^2-1-3k_3(n-1).$$

This may be written in the form

$$(n-1)\{k_1+k_2+k_3-(n+1)\}$$
$$= (k_1-k_3)(n-1-k_1)+(k_2-k_3)(n-1-k_2)+\sum_4^s k_i(k_3-k_i);$$

whence, since $n > 1$, $n-1 \geqslant k_i$, and $k_i \geqslant k_{i+1}$, it follows that

$$k_1+k_2+k_3 \geqslant n+1,$$

which proves the theorem.

This theorem implies (i) that the three base points of highest multiplicity of a homaloidal net ϕ are not collinear, and (ii) that if these three points can be taken as the base points of a quadratic transformation, then ϕ is reduced by this transformation to a homaloidal net of lower degree (cf. Ch. III, Th. V). In particular, if the base points of ϕ are all distinct and of general position, then ϕ can be reduced by a succession of standard quadratic transformations (s.q.t.) to the lines of the plane; and this implies further that any Cremona transformation representing ϕ directly on the lines of the plane can be regarded as the product of a finite number of s.q.t. and a collineation. We express this result by saying that such a Cremona transformation is *factorable*.

We have shown therefore that any Cremona transformation whose fundamental points are distinct and of general position is factorable. One of the most justly famous of all geometrical theorems carries this result to the stage of complete generality as follows:

THEOREM III (*Noether–Castelnuovo Theorem*). *Every Cremona transformation of the plane can be represented as the product of a finite number of standard quadratic transformations together with a collineation.*

The proof of this theorem, which involves rather delicate consideration of clustering singularities, may not be given here. The original statement and proof were given by Noether in 1871; and they remained unchallenged till 1901, when C. Segre pointed out a flaw in the reasoning which arose from Noether's incomplete knowledge of infinitely near points on algebroid branches. The first complete proof, valid for every possible disposition of base points, was given by Castelnuovo in 1901. The theorem has no analogue in ordinary or higher space.

EXAMPLE. Show how to factorize (into s.q.t.) a special quadratic transformation in which two or three of the fundamental points are consecutive.

[Effectively we have to transform the special homaloidal net into the lines of the plane by s.q.t. If the base points O_1, O_2 are consecutive, while O_3 is distinct, we apply first a generic s.q.t. with O_1, O_3 as two of its base points; this reduces the special net to a standard net with three distinct base points. A similar procedure can be applied when O_1, O_2, O_3 are consecutive.]

7.2. *Base points and fundamental curves.* We have already seen in Ch. III, Th. IV that if ϕ, ψ are the homaloidal nets generating a Cremona transformation and its inverse, then ϕ and ψ are of the same order n; we now inquire how the base of ψ may be determined from a knowledge of ϕ. We assume for simplicity that the base points concerned are all distinct.

To the neighbourhood of any ordinary ν-fold base point of either net there corresponds a rational fundamental curve of order ν of the other; and each fundamental curve so arising will meet its residual curves in the net in one free point (cf. Ch. VI, § 2.2). Furthermore, neither ϕ nor ψ can have any other fundamental curves than those arising in this way. In order, therefore, to find the base of ψ, we have only to discover the fundamental curves $C^\nu(\kappa_i)$ of ϕ, and we know that these must satisfy the conditions

$$n\nu - \sum k_i \kappa_i = 0,$$

$$(\nu-1)(\nu-2) - \sum \kappa_i(\kappa_i - 1) = 0.$$

Thus, for example, if ϕ is the system $C^6[4^1, 2^4, 1^3]$, or, say,

$$C^6(A^4 B_1^2 B_2^2 B_3^2 B_4^2 C_1 C_2 C_3),$$

then its fundamental curves are (i) the four lines AB_i, (ii) the conic $C^2(AB_1 B_2 B_3 B_4)$, (iii) the three cubics $C^3(A^2 B_1 B_2 B_3 B_4 C_i C_j)$; and ψ is therefore a system of the type $C^6[3^3, 2^1, 1^4]$.

Another example, of very much greater importance, is the *De Jonquières transformation* which is the only general type of Cremona transformation which exists for arbitrary order n. The net ϕ which generates it is composed of curves of order n which have an $(n-1)$-fold base point O together with $2n-2$ further simple base points A_i $(i = 1, 2, \ldots, 2n-2)$, i.e. it is of the type

$C^n[(n-1)^1, 1^{2n-2}]$. The fundamental curves of this net are (i) the $2n-2$ lines OA_i, and (ii) the unique curve $C^{n-1}[(n-2)^1, 1^{2n-2}]$; and the reverse transformation is therefore of the same type as the original.

It may readily be verified that when the base points are all of general position, a De Jonquières net is reducible to the lines of the plane by a succession of n standard quadratic transformations; and conversely, the product of n generic s.q.t. which have one fundamental point in common is a De Jonquières transformation of order n.

It will be clear to the reader by now that the enumeration of Cremona transformations would be little more than a mechanical operation; it is indeed largely a problem of Diophantine arithmetic based on equations (1) and (3), and the complete solution for all values of $n \leqslant 16$ has been given by H. P. Hudson in her treatise on the subject. There exists, however, a very extensive literature dealing with general properties of the configuration of base points and fundamental curves, and with groups of transformations; but these developments, as far as the rest of geometry is concerned, tend inevitably to be overshadowed and isolated by the subject's own fundamental theorem.

§ 8. Plane involutions of order two

8. In Ch. VI, § 4, we have already discussed, in general terms, involutions I_k of sets of k points in the plane, and the surfaces which serve as projective models for such involutions. Naturally the first examples of plane involutions to attract attention have been those of order two; and indeed these have a particular interest in that every I_2 obviously generates a *symmetrical* Cremona transformation of the plane into itself; and conversely. We propose, therefore, to give some account of several well-known I_2 which occupy key positions among the whole totality of their kind. The simplest I_2 of all, the plane harmonic homology, will be assumed to be sufficiently well known to require no special mention here.

8.1. *The Geiser involution.* This involution, which is generated by cubic curves through seven fixed points, has already been dealt with in some detail in § 5.2 where it was related to the involution of pairs of points of a cubic surface F^3 which are collinear with a fixed point O of F^3. To discover the Cremona transformation which carries each point Q_1 of the plane into its mate Q_2, we observe that, when Q_1 describes a line, its image point R_1 on F^3 describes a twisted cubic curve Γ, the image point R_2 of Q_2 describes the residual sextic curve in which F^3 is met by the cubic cone projecting Γ from O, and the curve described by Q_2 is therefore an octavic curve with triple points at the seven base points of the net of cubics. Thus the homaloidal net associated with the Geiser involution is of the type $C^8[3^7]$.

8.2. *The Bertini involution.* Another and even more remarkable I_2 is the so-called Bertini involution which derives from the fact that a general system of curves of the type $C^6[2^8]$ is not simple; in fact such a system, which is of freedom 3 and grade 4, has the property that each of its characteristic sets consists of two pairs of a fixed involution I_2. To see this we may observe (i) that the system $C^6[2^8]$ may be represented on a cubic surface F^3 by the system of curves in which F^3 is met by quadrics which touch F^3 at two fixed points A, B; (ii) that if α, β are the tangent planes to F^3 at A, B, then *conics* touching α, β at A, B plainly meet F^3 in pairs of points of an involution, and (iii) that any pair of quadrics touching α, β at A, B intersect in a pair of the conics in question, so that their sections of F^3 intersect in two pairs of the involution.

It may be shown without much difficulty that the Cremona transformation generated by this involution has the system $C^{17}[6^8]$ as its homaloidal net, and the coincidence curve of the involution is a $C^9[3^8]$. The image surface of the involution is a quadric.

It may be remarked that there exist alternative approaches, both to the Geiser and to the Bertini involutions, by way of involutions of pairs of points of space. Thus six fixed points of space determine an involution of pairs of points which augment the six to sets of eight associated points; and this space involution determines on any quadric through the six points a subordinate involution which projects from any one of these points into a Geiser involution in the plane. In the same way, seven fixed points of space determine another space involution of which any pair consists of the two residual intersections of an elliptic quartic *curve* through the seven points with a quartic *surface* passing doubly through the same points; and here again the space involution determines a subordinate involution on any quadric through the seven points, which projects, from any one of the seven, into a Bertini involution in the plane.†

8.3. *De Jonquières involutions.* Besides the two important involutions just described, there exists a whole series of I_2, named after De Jonquières, which generalize the projective equivalent of ordinary plane inversion with respect to a circle.

A De Jonquières $I_2(n)$ is determined completely, for any value of $n \geqslant 2$, by a fixed point O and a fixed curve Γ^n which possesses an $(n-2)$-fold point at O; a pair of the involution is then any pair of points P, Q such that P and Q lie on a line through O and separate harmonically the two residual intersections of this line with Γ^n.

Clearly the coincidence curve of the involution is Γ^n.

To find the homaloidal net of curves ϕ which characterizes the Cremona transformation defined by the involution, we consider the general curve ϕ described by Q when P describes a line λ. Clearly ϕ meets any line through O in only one point other than O; the line λ meets the first polar of O with respect to Γ^n in $n-1$ points which all have O as mate (as do all points of this curve), so that ϕ has an $(n-1)$-fold point at O and is therefore of order n; and finally ϕ obviously passes through the $2n-2$ points of contact of tangents from O to Γ^n. Hence the required homaloidal net is of the type $C^n[(n-1)^1, 1^{2n-2}]$, and the Cremona transformation which permutes P, Q is a De Jonquières transformation of order n.

8.4. *The general result.* We may add in conclusion that the whole subject of the present section has been rounded off by a remarkable theorem due, in the first instance, to Bertini. This asserts that every plane involution of pairs of points is derivable by Cremona transformation of the plane from one or other of four types:

(i) the plane harmonic homology,
(ii) the De Jonquières involution,
(iii) the Geiser involution,
(iv) the Bertini involution.

Bertini's original proof of this theorem was subjected to criticism; but the result was later established rigorously by other writers.

† For further details of this method, and of the two involutions concerned, the reader is referred to the account in Baker, vi, ch. iii.

NOTES AND EXAMPLES ON CHAPTERS VI AND VII

The general cubic surface F^3

1. Show that the *apparent contour* of a general F^3 from a point P of itself, i.e. the generic plane section of the proper tangent cone from P, is a non-singular quartic curve k^4 (cf. § 5.2).

Show also that, of the 28 bitangents of k^4 (Ch. IV, § 4.2), all but one are projections of lines of F^3, while the last is the projection of the tangent plane at P which touches the tangent cone along the two inflexional tangents at P.

2. Prove that the general F^3 has 120 tritangent planes which meet it in triads of lines such as (a_1, b_2, c_{12}) or (c_{12}, c_{34}, c_{56}).

3. A *Steiner trihedral* is a set of three tritangent planes of F^3 whose point of concurrence does not lie on the surface. Show that the rows and columns of the array

$$\begin{matrix} c_{23} & b_3 & a_2 \\ a_3 & c_{31} & b_1 \\ b_2 & a_1 & c_{12} \end{matrix}$$

are the traces on F^3 of *two associated Steiner trihedrals* which intersect each other in these nine lines.

Show that every Steiner trihedral is associated in this way with a second Steiner trihedral. (Cf. Pascal, *Repertorium*, ii. 789, where also the 120 pairs of associated Steiner trihedrals are exhibited.)

4. If the planes of two associated Steiner trihedrals have equations $u_i = 0$, $v_i = 0$ $(i = 1, 2, 3)$, prove that the equation of F^3 may be written in the form $u_1 u_2 u_3 = \lambda v_1 v_2 v_3$. Deduce that the equation of F^3 may be reduced (in 120 distinct ways) to the form

$$X_1 X_2 X_3 = Y_1 Y_2 Y_3,$$

where the X_i, Y_i are linear functions of the coordinates and connected by two linear identities, of which one may be taken to be

$$X_1 + X_2 + X_3 \equiv Y_1 + Y_2 + Y_3.$$

By writing $X_1 = \xi_2 + \xi_3$, $Y_1 = -(\xi_5 + \xi_6)$, etc., show that the equation of F^3 becomes

$$\xi_1^3 + \xi_2^3 + \ldots + \xi_6^3 = 0,$$

where the ξ_i are connected by two identities of which one is $\sum_1^6 \xi_i \equiv 0$; and write down the equations of 15 lines on the surface. Find also the equations of the remaining 12 lines by using the other identity, say $\sum_1^6 a_i \xi_i \equiv 0$, connecting the ξ_i.

5. Show from first principles that through any line of a general F^3 there pass five tritangent planes. If α, β are any two of these planes, construct a pair of associated Steiner trihedrals which contain α, β respectively.

6. Prove that the general cubic surface may be generated as

(i) the locus of conics in which the planes of a pencil are met by their corresponding quadrics in a related pencil of quadrics;

(ii) the locus of poles of a given plane with respect to the quadrics of a given net.

7. Show that the general F^3 may be birationally represented on the plane by drawing through any point P of the surface the unique transversal to two given skew lines on the surface.

Show also that, by this method, the plane sections of F^3 are represented in the plane by quartics $C^4[2^2, 1^5]$; and deduce from this the ordinary representation by means of $C^3[1^6]$.

8. A *double-six configuration of lines*, such as was referred to in § 2.2, is completely determined by any set of five skew lines $b_1,..., b_5$ which have one and only one transversal a_6. For, on the one hand, $a_1,..., a_5$ are uniquely determined as second transversals of sets of four of $b_1,..., b_5$; and on the other hand, we see, by counting constants, that there exists a (unique) F^3 which contains the original six lines, and this necessarily contains $a_1,..., a_5$ and a sixth line b_6 meeting these five.

Among the lines of F^3 there are, in all, 36 double-sixes.

9. Show, by means of a plane representation of either surface, that a quadric F^2 and a cubic surface F^3 intersect in general in a sextic curve ${}^4C^6$ of genus 4, and prove that this can break up in any one of the following ways: (i) into a rational quartic ${}^0C^4$ and two of its trisecants, (ii) into a ${}^2C^5$ and one of its trisecants, (iii) into two twisted cubics meeting in five points, (iv) into an elliptic quartic ${}^1C^4$ and one of its quadrisecant conics, (v) into two triads of generators of opposite systems of F^2.

10. Show that, in the plane representation of the cubic surface F^3, the sections of F^3 by other cubic surfaces are represented by the *complete* system of curves $C^9[3^6]$.

If (F) is the complete system of cubic surfaces through a non-singular curve C of order n and genus p, and if surfaces of (F) intersect residually in irreducible curves of order $9-n$, prove that the genus π of these curves, the number t of points in which they meet C, and the grade γ of (F) are given by

$$\pi = 9-2n+p, \qquad t = 2n-2p+2, \qquad \gamma = 25-5n+2p.$$

Show also how to find the freedom ρ of (F).

Discuss the cases when C is an elliptic quintic, a rational quintic, and an elliptic sextic, and explain why π is negative in the last two cases.

11. Find from the plane representation the freedom of cubic surfaces through (i) a ${}^1C^4$, (ii) a ${}^0C^4$, and (iii) a ${}^3C^6$.

Deduce that cubic surfaces through a ${}^3C^6$ generate a Cremona transformation of space.

The four-nodal cubic surface

12. Prove that the system of curves $C^3[1^6]$, in which the base points are the intersections of four lines, represents a four-nodal cubic surface, and that this contains nine lines of which six are the joins of the nodes in pairs.

Show that the equation of the surface may be taken to be $\sum_{0}^{3} (1/x_i) = 0$, and hence obtain the equations of the lines. Show also that the reciprocal of the surface is Steiner's surface.

Prove that, in the reciprocal transformation $\rho x_i = 1/x_i'$ $(i = 0, 1, 2, 3)$ between two spaces S_3 and S_3', planes of each space correspond to cubic surfaces having four common nodes, and lines to twisted cubics passing through the nodes.

13. (i) Show that the equation of any four-nodal cubic surface can be written in the form $|u_{rs}| = 0$, where $|u_{rs}|$ is a symmetrical determinant of the third order whose elements are linear functions of the coordinates.

(ii) Deduce that, if the points of a space S_3 are projectively related to the conics of a web (k) situated in a plane ϖ, the line-pairs of the web (and therefore the vertices of these line-pairs) are in correspondence with the points of a four-nodal cubic surface F^3; and that the nodes of F^3 correspond to the four coincident line-pairs of (k).

(iii) Show that the conics of the web which touch any line l in ϖ correspond to the points of a quadric cone having its vertex on F^3 and touching F^3 along a twisted cubic which passes through the nodes. Prove also that any two such cubics form the complete intersection of F^3 with a quadric.

(iv) By means of the same representation, prove that the tangent cone to F^3 from any point of itself breaks up into a pair of quadric cones.

The Veronese surface and its projections

14. By using the equations of the Veronese surface (§ 3.1), write down the equation of the linear ∞^5 system of quadrics through the surface.

Show that these quadrics form a homaloidal system, i.e. that any five of them intersect in only one free point. (This follows from the fact that the rational transformation of S_5 which they define is rationally reversible.)

15. Given a conic Ω in a plane ϖ, prove that ϖ may be represented birationally on a projected Veronese surface $F^4[4]$ in such a way that all the triads of points of ϖ which are self-polar for Ω are represented on F^4 by the triads of intersections of this surface with its ∞^3 trisecants.

16. Show that any non-singular elliptic sextic curve in [4] which possesses more than two trisecants possesses an infinity of trisecants which generate a cubic scroll. Deduce, by the method of Ex. 15, that if a plane cubic is drawn through the vertices of three triangles which are self-polar for a conic, then it contains an infinity of such inscribed triangles which are self-polar for the conic.

17. Show that a projected Veronese surface $F^4[4]$ contains a (unique) *inflexional rational normal quartic curve* K—corresponding to the conic Ω in Ex. 15—such that (i) every trisecant through a point P of K touches F^4 at P, and (ii) every tangent to K is an inflexional trisecant of F^4.

18. Show that any plane which meets a projected Veronese surface F^4 in a conic k meets the surface residually in a single point P, that the trisecants of the surface which pass through P constitute the pencil, vertex P, in the plane of k, and that the relation between P and k may be regarded as that of pole and polar with respect to the inflexional curve K of the surface.

19. Show that the parametric equations of a cubic scroll Γ in [4] may be taken to be

$$x_0:x_1:x_2:x_3:x_4 = x^2:y^2:yz:zx:xy.$$

Deduce that the surface lies on a net of quadric cones.

20. Prove that in [4] any quadric which contains a plane is a cone with vertex in the plane; and that two such cones with a plane in common intersect residually in a cubic scroll.

21. Show that the general cubic scroll in [3] may be generated by setting up a (1, 2) correspondence between the points of two skew lines and joining corresponding points; and obtain its reduced equation in this way.

Rational scrolls and Del Pezzo surfaces

22. Prove that the parametric equations of the most general rational normal quartic scroll ${}^0R^4[5]$ may be taken to be

$$x_0:x_1:x_2:x_3:x_4:x_5 = x^2y:xyz:yz^2:x^2z:z^3:xz^2.$$

Show that there are ∞^5 quadrics through the surface.

23. Show that the equations

$$x_0:x_1:x_2:x_3:x_4 = x^2y:xyz:zx(x+y+z):z^2(x+y+z):z^3$$

represent a scroll ${}^0R^4[4]$ which is a projection of ${}^0R^4[5]$, and determine the vertex and prime of the projection.

Prove that the projected scroll is the intersection of the quadric cone $x_0x_3 = x_1x_2$ with the cubic primal $x_2x_4(x_0+x_1) = x_3x_1(x_2-x_1)$, residual to a pair of planes.

Prove also that the scroll has an improper node which corresponds, in the plane representation, to the pair of points $(0, 1, -1)$ and $(1, 0, -1)$.

24. Show that there are two distinct types of ${}^0R^4[5]$, of which one is the locus of joins of corresponding points of two projectively related conics, while the other is the locus of joins of corresponding points of a line and a twisted cubic which are projectively related. Show that the first is represented by a general system $C^3[2^1, 1^1]$; and that the second may be represented either by a system of cubics which have one fixed node and one infinitely near simple base point, or by a system $C^4[3^1, 1^3]$ in which the three simple base points are collinear.

25. Obtain, as in the last example, two types of ${}^0R^5[6]$ and three types of ${}^0R^6[7]$, and discuss the plane representations of these scrolls.

26. Show that, on the quintic Del Pezzo surface F^5, the ten lines may be arranged as follows:

(i) six lines forming a skew hexagon,

(ii) three transversals of pairs of opposite sides of the hexagon,

(iii) a line meeting these transversals.

Show that the surface contains five pencils of conics such that any two conics of different pencils have one common point, that it contains also five nets of twisted cubics, and that the surface lies on five linearly independent quadrics of [5].

Show also that the equations

$$x_0:x_1:x_2:x_3:x_4 = x^2(y-z):y^2(z-x):z^2(x-y):yz(y-z):zx(z-x)$$

represent a projected Del Pezzo quintic surface in [4].

Prove that this surface has an improper node which corresponds to the pair of points $(\omega,\omega^2,1)$ and $(\omega^2,\omega,1)$, where ω is a complex cube root of unity.

27. Prove that the equations of the sextic Del Pezzo surface ${}^1F^6[6]$ may be written in the parametric form

$$x_0:x_1:x_1':x_2:x_2':x_3:x_3' = xyz:y^2z:yz^2:z^2x:zx^2:x^2y:xy^2,$$

and that its chord locus is the primal

$$\begin{vmatrix} x_0 & x_1 & x_1' \\ x_2' & x_0 & x_2 \\ x_3 & x_3' & x_0 \end{vmatrix} = 0,$$

whose two systems of generating solids are those which meet the surface in twisted cubic curves.

Show that the surface projects *doubly* from the plane

$$x_0 = x_2+x_3' = x_3+x_1' = x_1+x_2' = 0$$

into a four-nodal cubic surface.

28. If all plane cubics are represented by the points of S_9, prove that a threefold line is represented by a point P of a certain Del Pezzo ${}^1F^9$, that all cubics which have the same line as double component are represented by the points of the tangent plane to ${}^1F^9$ at P, and that all cubics which have the same line as simple component are represented by the points of the osculating [5] of ${}^1F^9$ at P.

29. Prove the following results:

(i) A curve of order n in $[n]$ is necessarily rational, normal, and without multiple points.

(ii) A surface of order n in $[n+1]$ is necessarily rational with rational normal prime sections.

(iii) A surface of order n in $[n]$ is either rational or a cone, and its prime sections are either rational or elliptic.

(iv) A surface with normal prime sections is necessarily normal.

Surfaces of section genus 2

30. Surfaces of section genus 2 were first discussed systematically by Castelnuovo† as part of a wider problem. He showed that those of them which are not ruled are rational and representable on the plane by systems of quartics $C^4[2^1, 1^r]$ $(r = 0, 1,..., 8)$. The parent or *supernormal* surface of

† *Rend. Palermo*, **4** (1890), 73.

this type is evidently ${}^2F^{12}[11]$, and from it we derive by projection a series of surfaces ${}^2F^n$ $(4 \leqslant n \leqslant 11)$ normal in $[n-1]$. The most noteworthy are:

(i) ${}^2F^5[4] \equiv C^4[2^1, 1^7]$, residual intersection of a quadric cone and a cubic primal which have a plane in common. The surface contains 14 lines, a pencil of conics, and a pencil of plane cubics residual to the conics.

(ii) ${}^2F^4[3] \equiv C^4[2^1, 1^8]$. This surface has a double line whose points correspond to pairs of points of the cubic $C^3[1^1, 1^8]$. Planes through the double line meet the surface residually in conics of which eight break up into line pairs.

Surfaces of section genus 3

31. The rational normal surfaces of section genus 3 have likewise been classified by Castelnuovo† who has shown that they fall into the following three classes:

$$(a)\quad {}^3\phi^n[n-2] \equiv C^4[1^{16-n}] \quad (5 \leqslant n \leqslant 16),$$
$$(b)\quad {}^3\psi^n[n-2] \equiv C^6[2^7, 1^{8-n}] \quad (5 \leqslant n \leqslant 8),$$
$$(c)\quad {}^3\chi^n[n-2] \equiv C^5[3^1, 1^{16-n}] \quad (5 \leqslant n \leqslant 16).$$

The sextic surface of the first class, namely, ${}^3\phi^6[4] \equiv C^4[1^{10}]$, is called the *Bordiga surface*, its properties having been first investigated in detail by Bordiga‡ and later by F. P. White.§ It is easy to see that the locus of the point of intersection of corresponding primes of four collinear nets in [4] is a surface of this type; and conversely. The surface contains in general 10 lines and 10 plane cubic curves, each member of either set meeting all but one of the other; and the existence of the surface therefore implies the existence in [4] of *double-ten configurations of lines and planes* analogous to double-six configurations in [3].

32. Show that, by specializing the 10 base points, a Bordiga ${}^3\phi^6$ may be made to have (*a*) a double line, or (*b*) five distinct double points.

33. Prove that a general surface of the type ${}^3\phi^5$ possesses a double twisted cubic curve; also, conversely, that a general quintic surface in [3] which has a double twisted cubic curve is a rational surface of the type ${}^3\phi^5$.

34. Show that each of the surfaces ${}^3\psi^n$ contains a net of elliptic quartic curves of which any pair meet in two points; also that ${}^3\psi^7$ contains a pencil of plane cubics with only one base point.

Show also that the normal surface ${}^3\psi^6$ possesses a double line, and that ${}^3\psi^5$ possesses three concurrent double lines.

Show also that all the surfaces ${}^3\psi^n$ are birationally representable on the Del Pezzo double plane ${}^1F^2$ by quadric sections with r simple base points $(r = 0, 1, 2, 3)$.

35. Prove that each of the surfaces ${}^3\chi^n$ contains a pencil of conics, and that ${}^3\chi^5$ possesses a triple line which corresponds in the plane representation to the curve $C^4[2^1, 1^{11}]$.

† *Atti Acc. Torino*, **25** (1890), 695.
‡ *Mem. Acc. Lincei*, (4), **4** (1887), 182.
§ *Proc. Camb. Phil. Soc.* **21** (1923), 221.

Plane Cremona transformations

36. Find the fundamental curves and reverse systems of all the homaloidal nets listed in § 7.

37. Discuss homaloidal nets of order n which possess an $(n-2)$-fold base point.

38. Show that the Jacobian of a homaloidal net is composed of the fundamental curves of the net.

39. Show that if the maximum base point multiplicity of a homaloidal net is 2, then the order of the net does not exceed 5.

40. A homaloidal net is said to be *symmetric* if its base point multiplicities are all equal. Prove that the only symmetric homaloidal nets, excluding the lines of the plane, are

$$C^2[1^3], \quad C^5[2^6], \quad C^8[3^7], \quad C^{17}[6^8].$$

Discuss the fundamental curves of these nets, and prove that each of the nets has fewer base points than any other homaloidal net of the same order. (Use the identity $\sum k_i^2 \sum 1-(\sum k_i)^2 = \frac{1}{2}\sum_{i,j}(k_i-k_j)^2$.)

NOTE ON A CHARACTERISTIC PROPERTY OF THE VERONESE SURFACE

We have already remarked that a surface which contains ∞^2 conics is the Veronese surface or one of its projections. This property may be established as follows.

If a surface F contains an irreducible system Σ of ∞^2 conics, then a finite number of conics of Σ pass through two given points of F. If we suppose, in the first place, that any two conics of Σ meet in a single variable point, then only one of the conics will pass through two given points of F. Moreover, in this case, the ∞^1 conics passing through a point P of F form a rational pencil (P), for they can be put in birational correspondence with the directions in the tangent plane at P.

Consider now, in a plane ϖ, two pencils of lines having centres at Q_1 and Q_2 respectively; these we denote by (Q_1) and (Q_2). We relate (Q_1) and (Q_2) projectively to two pencils of conics (P_1) and (P_2)—defined by two points of F—in such a way that the conic common to (P_1) and (P_2) corresponds to the line common to (Q_1) and (Q_2). To a generic point P of F, through which passes a single conic of each pencil, there corresponds a point Q of ϖ, intersection of the corresponding lines of (Q_1) and (Q_2); and conversely, to a generic point Q of ϖ there corresponds a unique point P of F. Furthermore, if Q describes a generic line of ϖ, then P describes a conic of Σ; and if P describes a generic prime section of F, Q describes a curve of order 2, i.e. a conic, of ϖ. Thus F is rational, and representable on ϖ by a system of conics; and it is therefore a Veronese surface or one of its projections.

Supposing, in the second place, that the conics of Σ are not unisecant to each other, we observe that they must be bisecant; for if they met each other in more than two points, they would all be coplanar. Two conics of Σ lie therefore in a space [3]; and this, moreover, contains all the conics of Σ, for these meet the [3] in four points. Hence a quadric surface of [3] which contains two conics of F and one further point of F will contain F entirely. Thus F is a quadric.

ADDITIONAL READING AND BOOKS OF REFERENCE ON THE SUBJECTS OF CHAPTERS VI AND VII

Coolidge, *Algebraic plane curves*, Book IV.

Baker, *Principles of geometry*, iii–vi.

Pascal, *Repertorium der höheren Mathematik*, ii. 2.

Bertini, *Complementi*, §§ 9–13.

——, *Introduzione*, chs. xiv, xv, xvi.

Telling, *The Rational quartic curve in space of three and four dimensions.* [In this work the general projected Veronese surface in [4] is derived from its unique rational normal inflexional ${}^0C^4$.]

Conforto, *Superficie razionali.*

Hudson, *Cremona transformations.*

Edge, *Ruled surfaces.*

CHAPTER VIII

LINEAR SYSTEMS OF SURFACES, RATIONAL MANIFOLDS, AND HIGHER CREMONA TRANSFORMATIONS

§ 1. Linear systems of surfaces

1. In passing from the consideration of linear systems of plane curves and their projective models to the analogous theory for linear systems of surfaces in [3] or of primals in $[r]$, there is much that admits of immediate generalization—at least in theory; but there is one essential difference, namely, in the existence—for these higher linear systems—of *base manifolds*, as distinct from mere isolated base points, common to all members of a system. The essential novelty of these becomes particularly apparent when we proceed to construct the projective models of such systems, or to consider, for example, how a multiple curve of a surface in [3] may be 'resolved'. What is essentially new is quite sufficiently illustrated by the case of linear systems of surfaces in [3], and we shall therefore confine our attention largely to this case. The reader should have no serious difficulty in extending the results to higher space as required.

1.1. ***Linear systems of surfaces.*** If $F_0,..., F_r$ are linearly independent homogeneous polynomials, of the same degree n in x, y, z, t, then the equation

$$\lambda_0 F_0+\lambda_1 F_1+...+\lambda_r F_r = 0, \tag{1}$$

which we suppose to be irreducible, defines a linear system of surfaces (F), of freedom r in an ordinary space S_3. These surfaces may have *base points*, simple or multiple, and *base curves*, simple or multiple.

A *characteristic curve* (or F-curve) of the system is a free intersection curve—residual to the base curves—of a pair of F-surfaces; and the genus of the generic characteristic curve is called the *curvilinear genus* of (F).

A *characteristic set* of (F) is the set of free intersections—residual to the base points and base curves—of three F-surfaces; and the number of points in a generic set of this kind is called the *grade* of (F).

The system (F) is *simple* if all the F-surfaces which pass through a generic point of S_3 do not pass in consequence through any further point of S_3.

If $r \geqslant 3$, the equations

$$\frac{X_0}{F_0} = \frac{X_1}{F_1} = \dots = \frac{X_r}{F_r} \tag{2}$$

make correspond to any general point $P(x,y,z,t)$ of S_3 a point $Q(X_0, X_1,\dots, X_r)$ of $[r]$; and as P describes S_3, Q will describe, in general, a threefold V in $[r]$, which is the projective model of the system (F).

If $r = 3$, then (F) is called a *web*, and three possibilities arise:

(i) If (F) has grade $N > 1$, then it is complex (not simple), and its characteristic sets G^N form an involution of sets of N points of S_3. The projective model V is another space S_3' whose points represent the sets G^N of S_3.

(ii) If (F) has grade 1, then it is simple, and equations (2) transform S_3 birationally into another space S_3'. Such a system (F), of grade 1 and freedom 3, is called a *homaloidal web*.

(iii) If (F) has grade 0, then F-surfaces through an arbitrary point have necessarily a curve in common; this is the case, for example, with quadrics through two skew lines. The projective model in such cases is a surface whose points represent the curves of a congruence (doubly infinite system) generated by (F).

If $r > 3$, then (F) is usually simple, and V is a rational threefold which is birationally represented on S_3. It may happen, however, that (F) is complex, generating an involution of sets of points of S_3; if this be so, then V is the projective image of the sets of the involution; and if V is rational the space involution is said to be rational.

1.2. *Rational threefolds.* Suppose now that (F) is simple so that the correspondence between V and S_3 is birational. To F-surfaces there correspond prime sections of V; to F-curves there correspond sections of V by secunda, or, as we shall say, *curvilinear sections* of V; and to characteristic sets of (F) there correspond the sets of points in which V is met by spaces $[r-3]$. Hence

THEOREM I. *If (F) is a simple linear system of surfaces of freedom r, grade N, and curvilinear genus p, then the surfaces of (F) represent birationally the prime sections of a threefold V of order N and curvilinear section genus p in r-dimensional space.*

Those properties of rational threefolds which are fairly obvious extensions of analogous properties of rational surfaces will now be stated, the proofs being left to the reader.

(1) The manifold V is *normal* (not a projection of any manifold of the same order and dimension) if (F) is *complete* (not contained in any more ample linear system with the same base).

(2) If Q is a simple point of V representing a point P of S_3, then the projection V_1 of V from P is represented on S_3 by means of the sub-system (F_1) of (F) which has P as a simple base point.

The tangent solid to V at Q gives rise, on projection, to a plane lying on V_1; this plane represents the neighbourhood of Q on V.

(3) If V is projected birationally from a $[k]$, T say, which meets V in a simple curve C, then the projected manifold V_1 in $[r-k-1]$ is represented on S_3 by means of the sub-system (F_1) of (F) which has a new base curve C_1 corresponding to C.

The tangent solid to V at any point Q of C meets T in the tangent line to C at Q, and it projects therefore into a line lying on V_1. Thus V_1 contains a scroll of such lines, each representing the whole first neighbourhood of a point of C.

(4) To any curve C_1 of S_3 there corresponds in general a curve C on V whose order is the number of free intersections of C_1 with a generic F-surface.

To the lines of S_3 there correspond ∞^4 rational C^n on V.

(5) To any surface ϕ_1 in S_3 there corresponds on V a surface ϕ whose order is the number of free intersections of ϕ_1 with a generic F-curve.

Prime sections of ϕ correspond to the free curves in which ϕ_1 is met by F-surfaces; and from this, when ϕ_1 is rational, there can be deduced the plane representation of ϕ.

To the planes of S_3 there correspond on V the surfaces of a homaloidal web, each represented on a plane of S_3 by the system of curves of order n in which this plane meets (F).

1.21. *Limit points of V.* The equations (2) which determine the point Q of V corresponding to P in S_3 break down if P is a base point of (F); but there will then be limit points of V associated with approaches to P in S_3. To discuss these our point of view must be that the correspondence between the surfaces of (F) and the *primes* of $[r]$ is fundamental. From this aspect, a general point of $[r]$ corresponds to a *general linear condition* on (F); while a point of V corresponds to a *special linear condition* which requires F-surfaces to pass through a fixed point. To every limiting special linear condition on (F) there will correspond a limit point of V; and conversely, any limit point of V represents a limiting special linear condition on (F).

1.22. *Neighbourhood of a simple isolated base point.* If O is a simple isolated base point of (F), the condition on F-surfaces to touch at O a given line through O is a limiting special linear condition; and to this condition there will correspond therefore a definite point of V. Thus we may say that to every direction t through O there corresponds a point $Q(t)$ of V. As t varies about O, $Q(t)$ describes a surface E which is in fact a plane; for a secundum K meets it in the unique point $Q(t)$ which corresponds to the direction at O of the F-curve associated with K. Further, to any line of E, intersection of E with a prime Π, there corresponds a plane pencil of directions at O, lying in the tangent plane at O to the F-surface corresponding to Π; and it follows that the correspondence between the star (O) and the plane E is homographic. Hence

THEOREM II. *To the neighbourhood of a simple isolated base point O of (F) there corresponds a plane E on V, the correspondence between directions through O and points of E being homographic.*

Similar considerations show that if O is a λ-fold base point of (F) through which pass α linear branches of simple base curves, then *to the neighbourhood of O there corresponds, in general, a rational surface of order $\lambda^2-\alpha$ on V.*

1.23. *Neighbourhood of a simple non-singular base curve.* Let C be a simple isolated non-singular base curve of (F), and let τ be the tangent to C at any point P. Then all directions t at P (other than τ) which lie in a plane α through τ impose the *same* limiting special linear condition on (F), namely, that of touching α at P; and to this linear condition there corresponds therefore a point $Q(\alpha)$ of V. As α varies about τ, $Q(\alpha)$ must describe a curve on V; and this is in fact a line r, for a prime meets it only at the point corresponding to the tangent plane at P to the associated F-surface. Also, as P describes C, r describes a scroll R lying on V; this scroll represents the whole neighbourhood of C in S_3; its order—the number of its intersections with a secundum—is given by the number of points in which a general F-curve meets C. Hence

THEOREM III. *To the whole neighbourhood of a simple isolated non-singular base curve C of (F) there corresponds on V a scroll R. Each generator r of R represents the neighbourhood of a point P of C, and each point of r represents a pencil of directions through P, lying in a plane touching C at P. The order of R is the number of intersections of C with an F-curve.*

If C is an isolated λ-fold base curve of (F), a similar argument shows that to the neighbourhood of each point of C there corresponds a rational curve of order λ on V, and that this generates a surface whose order is again equal to the number of intersections of C with an F-curve.

1.24. *Algebraic verification.* Taking non-homogeneous coordinates (x, y, z) for a point of S_3, we may suppose the equations of V to be

$$\rho X_i = F_i(x, y, z) \quad (i = 0, 1, \dots, r). \tag{3}$$

If the origin O is a simple base point of (F), we may write

$$F_i(x, y, z) = a_i x + b_i y + c_i z + G_i,$$

where every term G_i is of degree 2 at least. Then to a point infinitely near O, in direction $\lambda:\mu:\nu$, there corresponds the limit point of V given by

$$\rho X_i = a_i \lambda + b_i \mu + c_i \nu \quad (i = 0, 1, \dots, r). \tag{4}$$

For varying values of $\lambda:\mu:\nu$, equations (4) represent a plane whose points are in homographic correspondence with rays through O.

Again, if OZ is a simple base line of (F), we may write

$$F_i(x, y, z) = xU_i + yV_i, \tag{5}$$

where U_i, V_i are polynomials in x, y, z. If x and y are small, equations (3) become, to a first approximation,

$$\rho X_i = x\overline{U}_i + y\overline{V}_i, \tag{6}$$

where $\overline{U}_i$ and $\overline{V}_i$ are polynomials in z only. For any assigned value of z, these equations represent a line whose points correspond to values of the ratio $x:y$; and for varying z, equations (6) represent a ruled surface on V corresponding to the whole neighbourhood of OZ.

The general case of a simple base curve of (F) reduces to the above by introducing curvilinear coordinates u, v, w such that the curve in question is given by $u = v = 0$.

1.3. ***Fundamental surfaces and curves.*** A surface or curve is *fundamental* for (F) if it has no free intersection with F-surfaces, so that it imposes only one condition on these surfaces to contain it entirely. To any such surface or curve there will correspond a single point of V.

If O is the point of V corresponding to a fundamental surface E of (F), and if the multiplicity of O on V is $\alpha \geqslant 1$, then the curve in which V is met by a secundum through O will in general have α branches passing through O; and hence the surfaces residual to E in (F) will intersect each other in free curves which each meet E in α free points.

Again, if a fundamental curve L (not lying on a fundamental surface) transforms to a β-fold point O of V, then the section of V by a secundum through O corresponds to a curve T in S which is the intersection of two F-surfaces through L, residual to L and the base curves; and T must in general meet L in β points. Hence

THEOREM IV. *To any fundamental surface E whose residual surfaces in (F) intersect in free curves having α free intersections with E, there corresponds an α-fold point of V; and to any fundamental curve L (not lying on a fundamental surface) whose complementary curves, in the system of F-curves, meet L in β points, there corresponds a β-fold point of V.*

It should be noted that a fundamental curve of the above type may be *isolated*, or it may belong to a simply infinite family of such curves, representing a simple or multiple curve on V.

§ 2. Some important rational threefolds

2. By way of illustration and application of the preceding theory we proceed to describe briefly some of the more important rational threefolds with known space representations.

We shall use the notation ${}^{p}V_3^n$ to indicate the order n and curvilinear genus p of the threefolds concerned.

Nearly all the manifolds we discuss are representable by means of systems of quadric surfaces (of freedom $r \geqslant 4$), and they fall into three groups corresponding to systems of quadrics which possess (a) a base conic, (b) a base line, or (c) only base points at most. We consider these groups in turn.

2.1. *The general quadric primal in* [4]. If a general quadric primal Q of [4] is projected from a point of itself, say O, on to a solid S_3, then its prime sections project into quadric surfaces F passing through a conic k which is the projection of the quadric cone of lines of Q through O. The representation is in every way analogous to that of the quadric surface (Ch. VII, § 1). Conversely,

Quadric surfaces of S_3 which pass through a fixed conic k represent prime sections of a quadric primal Q of [4].

The F-curves are conics meeting k in two points, and the characteristic sets are the unrestricted point-pairs of S_3. The plane of k is fundamental for (F) and represents the neighbourhood of O on Q, and the neighbourhoods of points of k represent the lines on Q which pass through O.

If k is a pair of intersecting lines, Q is a point-cone; and if k is a repeated line (of contact), Q is a line-cone.

2.2. *The planar* $^0V_3^4$ *of* [6]. Consider next the system (F) of all quadrics through a fixed line u of S_3. This has freedom 6 and grade 4; and the F-curves are twisted cubics meeting u twice. The projective model is therefore a $^0V_3^4$ of [6].

To any plane π of S which passes through u there corresponds a plane lying on V; for the F-quadrics meet π residually in the lines of this plane. Thus V is generated by a simply infinite family of planes corresponding to the planes of the pencil (π). The whole neighbourhood of u is represented on V by a quadric surface ϕ (cf. Th. III), the generators of one system of ϕ representing the neighbourhoods of points of u, while those of the other system represent sections of the neighbourhood of u by the planes π. Any line of S_3 represents a conic on V which meets every generating plane in one point; and hence, plainly, V may be defined as the locus of a plane which joins a variable point of any one of these conics to a corresponding generator of the second system on ϕ. Hence

Quadrics through a line u represent prime sections of a planar $^0V_3^4$ *of* [6], *which is generated by a projective correspondence between the points of a conic and the lines of a regulus.*

The reader should verify that whereas the general prime section of V is a rational normal quartic scroll, a prime through a generating plane meets V residually in a rational cubic scroll, and a prime through two generating planes meets V residually in the unique surface ϕ.

The birational representation of V on S_3 is equivalent to a direct projection of the manifold from any one of its generating planes.

2.21. *The projectively generated* ${}^0V_3^3$ *in* [5]. Of far greater importance, in many ways, than the manifold just discussed is the planar ${}^0V_3^3$ in [5] which is the projection of ${}^0V_3^4$ from a generic point of itself. This new manifold V is represented on S_3 by quadrics F through a line u and a point P; and, besides the generating planes α represented by planes through u, it contains ∞^2 directrix lines λ represented by lines through P. The planes through u cut related ranges on lines through P, and the latter meet the former in collinear fields. It follows then that V is (i) the locus of planes joining corresponding points of three related ranges, and (ii) the locus of lines joining corresponding points of two collinear plane fields. Hence

Quadrics through a line and a point represent prime sections of a ${}^0V_3^3$ *of* [5], *locus of planes which join corresponding points of three related ranges.*

The generic prime section of V is evidently a cubic scroll in [4].

If we take u to be the line XY and P to be the vertex Z of the tetrahedron of reference, then the equation of (F) is

$$\lambda_0 t^2+\lambda_1 xt+\lambda_2 yt+\lambda_3 zt+\lambda_4 zx+\lambda_5 yz = 0,$$

and the parametric equations of ${}^0V_3^3$ are

$$\frac{X_0}{t^2} = \frac{X_1}{xt} = \frac{X_2}{yt} = \frac{X_3}{zt} = \frac{X_4}{zx} = \frac{X_5}{yz}.$$

The explicit equations of the manifold are

$$\frac{X_0}{X_3} = \frac{X_1}{X_4} = \frac{X_2}{X_5},$$

and these lead at once to the two projective generations referred to above.

The representation of V on S_3 is equivalent to a projection of the manifold from a line p in one of its generating planes. The isolated base point P is the projection of this plane, and the points of u are the projections of the directrix lines through the points of p.

2.22. *The planar ${}^0V_3^3$ of* [4]. The planar ${}^0V_3^3$ which we have just described contains a net of quadric surfaces ψ, namely, those which are represented in S_3 by planes through the isolated base point P. Each of these quadrics ψ meets every plane of ${}^0V_3^3$ in a line, its other system of generators being composed of ∞^1 directrix lines of ${}^0V_3^3$; and the solids which contain the quadrics, being given by

$$\lambda X_0+\mu X_1+\nu X_2 = 0 = \lambda X_3+\mu X_4+\nu X_5,$$

are such that one and only one of them passes through a generic point T of [5].

Now project ${}^0V_3^3$ from T into a planar cubic threefold V_1 of [4]. The unique quadric surface ψ whose containing solid passes through T projects into a double plane δ of V_1, its generating lines of both systems projecting into the tangents to a conic k in δ; and each plane of ${}^0V_3^3$, since it meets ψ in a line, projects into a plane of V_1 which meets δ in a line touching k. Hence

The general planar ${}^0V_3^3$ of [4] *possesses a double plane which is met by all the generating planes of the manifold in the tangents to a conic.*

2.3. *The Del Pezzo threefolds.* We consider next the set of rational threefolds which are represented by systems of quadrics with at most a finite number of base points. The first of these—the common ancestor of all threefolds representable by quadrics—is the octavic manifold ${}^1V_3^8$ in [9] which is the projective model of all quadrics in S_3. This manifold, by analogy with the Veronese surface, is called the *Veronesean* of quadrics in S_3. Its prime sections, being representable on quadric surfaces by all quadric sections of these surfaces, are octavic Del Pezzo surfaces of the second species (Ch. VII, § 5), and its curvilinear sections are elliptic. To the planes of S_3 there correspond on ${}^1V_3^8$ ∞^3 Veronese surfaces intersecting by pairs in ∞^4 conics.

From ${}^1V_3^8$ we obtain by successive projection a series of threefolds ${}^1V_3^{8-r}$ $(1 \leqslant r \leqslant 5)$, situated in $[9-r]$ and represented by systems (F) of quadrics through r base points A_i $(i = 1, 2,..., r)$. For each r, ${}^1V_3^{8-r}$ contains r planes corresponding to the neighbourhoods of the points A_i, and r doubly infinite line systems corresponding to stars of lines with vertices at A_i. If a plane joins three base points A_i, then it represents an additional plane on ${}^1V_3^{8-r}$; so that this manifold contains $r+\binom{r}{3}$ planes in all. Finally, each join $A_i A_j$ of two base points is a *fundamental line* of (F), whose complementary curves are twisted cubics bisecant to $A_i A_j$; hence, by Theorem IV, each such join represents a double point of ${}^1V_3^{8-r}$, which therefore possesses $\binom{r}{2}$ quadric nodes.

These results are summarized in the following table:

Manifold	*Normal space*	*Nodes*	*Planes*	*Line-systems*	*Remarks*
${}^1V_3^8$	[9]	0	0	0	Veronesean of quadrics.
${}^1V_3^7$	[8]	0	1	1	
${}^1V_3^6$	[7]	1	2	2	The two planes meet at the node.
${}^1V_3^5$	[6]	3	4	3	One plane met by the other three in lines.
${}^1V_3^4$	[5]	6	8	4	Double-four of planes.
${}^1V_3^3$	[4]	10	15	6†	Segre cubic primal.

We add notes on the last two of the series.

2.31. *The tetrahedral quartic threefold.* Let the quadrics which represent prime sections of ${}^1V_3^4$ have base points at the vertices of the tetrahedron of reference in S_3, so that (F) has equation

$$\lambda_0 yz+\lambda_1 zx+\lambda_2 xy+\lambda_3 xt+\lambda_4 yt+\lambda_5 zt = 0.$$

Then the parametric equations of ${}^1V_3^4$ are

$$\frac{X_0}{yz}=\frac{X_1}{zx}=\frac{X_2}{xy}=\frac{X_3}{xt}=\frac{X_4}{yt}=\frac{X_5}{zt},$$

from which we obtain

$$X_0 X_3 = X_1 X_4 = X_2 X_5.$$

The prime sections of this threefold, which is normal in [5], are Segre quartic surfaces (Ch. VII, § 5.1). Four of its planes $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ correspond to the neighbourhoods of the points X, Y, Z, T, while the other four $\beta_1, \beta_2, \beta_3, \beta_4$ correspond to the faces of the fundamental tetrahedron; and these eight planes form a *double-four*

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{matrix}$$

† The sixth line-system represents the system of twisted cubics through the five base points, as explained in the sequel.

such that every α_i meets β_j $(i \neq j)$ in a line. The planes α_i meet in pairs at the six nodes, as do also the planes β_i.

This ${}^1V_3^4$ is the base of a pencil of quadrics in [5], given by

$$aX_0X_3+bX_1X_4+cX_2X_5 = 0,$$

where $$a+b+c = 0.$$

If one of these quadrics is regarded as the quadric Ω whose points represent the lines of [3], then ${}^1V_3^4$ represents on Ω a tetrahedral complex of lines of S_3.

2.32. *The Segre cubic primal.*† The primal ${}^1V_3^3$ of [4] which is represented on S_3 by quadrics through five fixed points A_i is called the Segre cubic primal; and it possesses many remarkable properties. It has 10 nodes, and a *symmetrical* system of 15 planes of which 5 correspond to points A_i and 10 to the planes $A_iA_jA_k$. The symmetry of the planes appears by observing that each of them contains 4 of the nodes; for instance, the plane corresponding to A_1 contains the nodes corresponding to A_1A_2, A_1A_3, A_1A_4, A_1A_5, while that corresponding to $A_1A_2A_3$ contains the nodes represented by A_1A_2, A_2A_3, A_3A_1, A_4A_5. Also every plane is met in lines by six others, namely, by the planes corresponding to $A_1A_iA_j$ $(i,j \neq 1)$ for the first-mentioned plane, and by the planes corresponding to A_1, A_2, A_3, $A_1A_4A_5$, $A_2A_4A_5$, $A_3A_4A_5$ for the second.

Besides the 5 line-systems represented by the stars (A_i), ${}^1V_3^3$ contains a *sixth* system represented by twisted cubics through $A_1,\ldots, A_5$; for any such cubic has one free intersection with an arbitrary F-surface. This sixth system is exactly on a par with the other five; for if we apply to S_3 a reciprocal transformation

$$x':y':z':t' = 1/x:1/y:1/z:1/z,$$

based on $A_1A_2A_3A_4$ and leaving A_5 invariant, the system (F) is left unaltered, and the twisted cubics transform to the lines of the star (A_5). The same transformation carries the neighbourhood of each vertex of reference into the opposite face, and vice versa.

Taking the line-system represented by the cubics as typical, we see that the lines of this system all meet five skew planes of ${}^1V_3^3$, namely, those represented by the points $A_1,\ldots, A_5$; and, in fact, any line in [4] which meets four of these planes must lie on the unique

† *Mem. Acc. Torino* (2) **39** (1888), 3.

cubic primal ${}^1V_3^3$ containing them, and meet the fifth plane of the set. Thus:

A Segre cubic primal contains six pentads of associated *planes, such that all lines which meet four planes of a pentad meet the fifth, and it is generated simply by the doubly infinite system of transversal lines of each of the pentads.*

2.33. *Double solid with Kummer quartic branch surface.* By a procedure exactly analogous to that by which the rational Del Pezzo double plane was derived from the cubic surface (Ch. VII, § 5.2), we can derive from the Segre ${}^1V_3^3$ a *rational Del Pezzo double solid* ${}^1V_3^2$ which is at least as interesting as any of the ordinary threefolds of the series. We project ${}^1V_3^3$, namely, from a simple point O of itself on to a solid Π, the resulting projection being regarded as a manifold ${}^1V_3^2$ consisting of two superimposed solids Π_1, Π_2 which cross into each other all along a *branch-surface* ψ, projection of the *apparent contour* of ${}^1V_3^3$ from O. The prime sections of ${}^1V_3^2$ are, of course, Del Pezzo double planes.

The cone of proper tangents from O is of order 4 (as appears by considering a plane section through O); furthermore, it has 16 *double lines* which are, namely, the lines joining O to the ten nodes of ${}^1V_3^3$, and the six lines of this primal (one from each generating system) which pass through O. Thus ψ, which is the section of this cone by Π, is a 16-nodal quartic surface, of a type known as *Kummer's surface*. The six line-systems on ${}^1V_3^3$ project into six distinct systems of bitangent lines of ψ.

In accordance with the general theory, the double solid ${}^1V_3^2$ is birationally representable on a simple solid S_3, being in fact the projective model of a system (F) of quadrics through 6 points A_i. This means that characteristic sets of (F) are pairs of points (P_1, P_2) of an involution I_2 of S_3, these pairs being such, namely, as form with the six points A_i sets of eight associated points; and each pair (P_1, P_2) represents a pair of *superimposed* points of ${}^1V_3^2$. The branch-surface ψ represents the locus of coincident pairs (P, P) of I_2. Thus

The ∞^3 *quadrics through* 6 *base points of* S_3 *represent prime sections of a Del Pezzo double solid* ${}^1V_3^2$ *whose branch-surface is a* 16-*nodal Kummer quartic surface.*

The following are some of the more interesting properties of ${}^1V_3^2$:

(i) Let T denote the involutory correspondence between the points of a characteristic set (P_1, P_2) of (F). Then if A is any locus in S_3, the corresponding locus $T(A)$ is said to be *conjugate* to A in the involution I_2, and the images of A and $T(A)$ are superimposed (but in general distinct) loci on ${}^1V_3^2$. If A and $T(A)$ coincide, however, A is said to be *compounded of* I_2, and the image of A is a double (i.e. two-sheeted) locus on ${}^1V_3^2$. Clearly F-surfaces and F-curves are compounded of I_2.

(ii) If U is any united point of I_2, the F-quadrics through U all touch a certain line at U, and hence one of them is a cone. Thus the coincidence locus of I_2 is also that of vertices of cones of (F), i.e. the Jacobian surface

$$\frac{\partial(F_0, F_1, F_2, F_3)}{\partial(x,\ y,\ z,\ t\ )} = 0,$$

where F_0, etc., are linearly independent quadrics of (F). This locus is a quartic surface W known as the *Weddle surface*, having nodes at the 6 points A_i and containing the 10 lines of intersection of pairs of planes through these points. To the points of W there correspond birationally the points of the Kummer surface ψ on ${}^1V_3^2$.

(iii) ${}^1V_3^2$ possesses

16 *nodes*, corresponding to the joins $A_i A_j$ and the twisted cubic $A_1 A_2 \dots A_6$,

32 *planes*, forming 16 overlapping pairs, of which 10 pairs are represented in S_3 by plane-pairs such as $A_1 A_2 A_3$, $A_4 A_5 A_6$, while each of the remaining 6 pairs is represented by a point A_i and the quadric cone which has its vertex at A_i and passes through the other base points,

12 *line-systems*, forming 6 overlapping pairs, of which any pair is represented by lines through a point A_i and twisted cubics through the other base points.

The complete symmetry of these systems follows as usual by applying reciprocal transformations to S_3 with four of the points A_i as fundamental points.

The 16 nodes of ${}^1V_3^2$ are, of course, the nodes of ψ. The plane of each of the 16 overlapping pairs contains 6 nodes of ψ lying on a conic; for if such a pair is represented in S_3 by the planes $A_1 A_2 A_3$, $A_4 A_5 A_6$, the line of intersection of these represents a conic on ψ which passes through the nodes represented by the sides of the triangles $A_1 A_2 A_3$ and $A_4 A_5 A_6$; and a similar result holds, by symmetry, in all the other cases. Each of the 16 planes in question must therefore touch ψ along a conic, i.e. it is a *trope* of ψ, dual of a conic node of the surface.

Finally, the 6 systems of overlapping line-pairs of ${}^1V_3^2$ are located in 6 systems of bitangent lines of ψ (distinct from lines lying in the tropes); for any such line-pair is represented by an F-curve which consists of a line through one base point and its conjugate twisted cubic through the other five; and these meet the coincidence surface W in the same two points, which shows that the corresponding (double) line of ${}^1V_3^2$ is a bitangent of ψ. In conclusion, then,

The Kummer surface ψ *has* 16 *planes which touch it along* 16 *conics, each of which contains* 6 *of the* 16 *nodes. The* 32 *planes of* ${}^1V_3^2$ *overlap in pairs along the* 16 *planes in question; and the* 12 *line-systems overlap in pairs along* 6 *systems of bitangents to* ψ.

2.4. *The general intersection of two quadric primals in* [5]. Having already encountered, in § 2.31, a special case of the intersection of two quadrics in [5], namely, the tetrahedral threefold, we now turn our attention to the most general threefold of this type. The intersection of two general quadrics Ω, Ω' in [5] is a ${}^1V_3^4$ whose prime sections are Segre quartic surfaces; and it contains therefore a doubly infinite system of lines, of which 16 lie in any prime section. If g is any one of these lines, then the manifold projects birationally from g on to a space S_3; for any plane through g meets Ω, Ω' in residual lines which intersect in a unique point of ${}^1V_3^4$. The same argument, incidentally, proves the rationality of the intersection of two quadric primals in space of any dimension $r \geqslant 4$.

In the projection from g, the prime sections of ${}^1V_3^4$—since each of them meets g in one point—project into a system (F) of cubic surfaces in S_3, and these will have a base curve Γ whose points are projections of lines of ${}^1V_3^4$ which meet g. Now two of these F-surfaces will meet, residually to Γ, in an elliptic quartic curve, projection of a curvilinear section of ${}^1V_3^4$; and it follows therefore (by Ch. IV, § 8.4) that Γ is a ${}^2C^5$ which is met by F-curves in 8 points. Hence

The complete intersection ${}^1V_3^4$ *of two general quadrics of* [5] *projects birationally from a line of itself into a solid, its prime sections being represented in this solid by cubic surfaces through a quintic curve of genus* 2.

By Theorem III the whole neighbourhood of Γ represents an octavic scroll of lines of ${}^1V_3^4$ which meet g.

The curve Γ is a (3, 2) curve on a unique quadric surface containing it; and it therefore possesses a regulus of trisecants which are evidently fundamental lines of (F). To these trisecants there correspond the points of g; and conversely, the tangent solids to ${}^1V_3^4$ at points of g project into the trisecants of Γ.

Since the prime sections of ${}^1V_3^4$ are normal in [4], it follows that the manifold itself is normal in [5].

The special importance of the above representation is in its application to line-geometry. For if we take the quadric Ω to represent the lines of ordinary space, then ${}^1V_3^4$ represents on Ω a *general quadratic line-complex*; and our representation is therefore equivalent to a birational representation of the lines of such a complex on S_3 (cf. Ch. X, § 2).

Other examples of rational manifolds and their representations will be found in the examples at the end of this chapter.

§ 3. Resolution of a curve

3. As a further application of the methods of the present chapter we propose to consider the general *resolution of a curve* in S_3, i.e. the transformation of its neighbourhood into an explicit ruled surface on a suitable transform of S_3; and we apply the method to derive some properties of double curves on surfaces which will be of use in the next chapter.

Let C be an irreducible non-singular curve of S_3, and let (F) consist of all surfaces, of some sufficiently high order n, which pass simply through C. Then the projective model of (F) is a threefold V in higher space on which the neighbourhood of C is resolved into a scroll R whose generators represent neighbourhoods of points of C.

We saw in § 1.23 that if t is the tangent to C at any point P, and if r is the generator of R corresponding to P, then every point Q of r represents a pencil of directions at P in a plane π through t. In other words, the section of the neighbourhood of P by π corresponds to the whole neighbourhood of Q in V. We now show further that

To the individual directions (other than t itself) which issue from a point P of C and lie in a plane π through the tangent t to C at P, there correspond, in V, sections of the neighbourhood of the point Q *of R associated with π by planes through the generator r associated with P.*

This means, in effect, that if P_1 approaches P in a generic assigned direction u at P, then its corresponding point Q_1 approaches the point Q associated with the plane (tu) in some direction lying in a definite plane through r.

To prove this we note first that any F-surface which has a conic node at P transforms to the section of V by a prime through r; for its nodal cone contains, besides t, one free direction in every plane through t. Now all F-surfaces which are nodal at P and whose nodal cones contain an assigned direction u at P intersect each other (unless their nodal cones coincide) in curves of which one branch at P is in direction u; and the corresponding prime sections of V therefore intersect each other in r and in one branch passing through the point Q corresponding to the plane (tu) at P. Hence these prime sections must touch a fixed plane through r at Q, lying in the tangent solid to V at this point; and this plane therefore corresponds to the direction u at P.

Thus to the ∞^2 individual directions u at P (excluding t itself) there correspond ∞^2 planes τ through r, each lying in the tangent solid to V at some point of r. A prime through r contains, as we have seen, ∞^1 of these planes, associated with directions u at P forming a quadric cone containing t; hence the planes τ meet a secundum in points of a surface represented on the star (P) by quadric cones through a line, and therefore on a plane by means of conics through a point. Thus the planes τ form a line-cone, vertex r, whose base is a cubic scroll; the generating solids of this cone are tangent to V at points of r; and its directrix solid—corresponding exceptionally to t—is the solid joining r to its consecutive generator on R. This gives

THEOREM V. *The tangent solids to V along r generate a cubic-scroll line-cone whose vertex is r and whose directrix solid is that which contains the tangent planes to R at all points of r.*

Clearly any simple directrix curve of R may be regarded as corresponding to a simple curve C' *infinitely near* to C in S_3; and this provides a basis for the discussion of curvilinear contact and curvilinear singularities generally in S_3.

3.1. *Application to surfaces with a double curve.* Consider now a surface ϕ of S_3 which has C as double curve. At a generic point of C, the surface is assumed to have two sheets with distinct tangent planes; and its image on V is a surface ϕ' which meets R in a curve Γ intersecting the generators of R in two points. The double curve C of ϕ is thus resolved into the simple curve Γ of ϕ'.

At a finite number of points of C (called *pinch-points*) the two tangent planes coincide; these correspond evidently to generators of R which touch Γ.

Let P be a pinch-point on C, having a *pinch-plane* π passing through the tangent to C at P; then Γ touches the generator r corresponding to P at that point Q of r which corresponds to π. Hence also ϕ' touches r at Q; and its tangent plane at this point contains r and corresponds, by what we have said above, to a particular direction γ at P in π. This direction is called the *cotangent* to ϕ at P.

If ϕ is of order N, an arbitrary curve $\mathscr{C}$ of S_3, of order ν, meets ϕ in $N\nu$ points; and in general ϕ' meets the image $\mathscr{C}'$ of $\mathscr{C}$ on V in the same number of points. If $\mathscr{C}$ passes through a point of C, however, two of its intersections with ϕ are absorbed there, and $\mathscr{C}'$ has only $N\nu-2$ intersections with ϕ' in V. If $\mathscr{C}$ meets C at the pinch-point P and touches the pinch-plane π at P, then $\mathscr{C}'$ passes through Q and meets ϕ' in $N\nu-3$ points elsewhere. Lastly, if and only if $\mathscr{C}$ touches the cotangent at P, $\mathscr{C}'$ *touches* ϕ' at Q and meets ϕ' in $N\nu-4$ points elsewhere. Hence

THEOREM VI. *Any curve which passes simply through a pinch-point P of ϕ and touches the pinch-plane π at P has three coincident intersections with ϕ at P, unless it touches the cotangent in π, when the number of intersections which coincide at P is four.*

Note. The restriction that C should be non-singular is obviously irrelevant, provided that P itself is a simple point of C. In the next chapter we shall deal with surfaces ϕ whose double curves ordinarily possess triple points T which are triple also for ϕ. If we resolve any such curve C, the F-surfaces passing simply through it are all nodal at the points T, with nodal cones containing the three branch tangents of C at each of these points; and the transform of the neighbourhood of C consists in all of a scroll R augmented by a number of planes corresponding to the neighbourhoods of the points T. But this does not affect the above conclusions, provided that P is remote from every T.

§ 4. Cremona transformations of space

4. We propose next to give a short account of the Cremona transformations which transform a space S_3 into another space S'_3. Such transformations are certainly not without interest for their own sake, and they have various theoretical applications, as, for example, to the resolution of singularities of space curves and surfaces. Perhaps, however, their most obvious significance is that —like their analogues in the plane—they classify linear systems of surfaces into classes of birationally equivalent systems, all the systems of any one class having the same projective model. It should be noted, however, that there is no analogue for space of the Noether–Castelnuovo reduction theorem for the plane, and the development of the subject tends accordingly to be an enumeration of cases of special interest or importance.

4.1. *Generalities.* If T is any Cremona transformation of S_3 into a space S'_3, then T is generated by a homaloidal web of surfaces Φ in S_3, representing planes in S'_3; and T^{-1} is generated by a homaloidal web of surfaces Ψ in S'_3, representing planes of S_3. To Φ-curves in S_3 there correspond lines of S'_3, and to Ψ-curves in S'_3 there correspond lines of S_3. Hence follows

THEOREM VII. *In any space Cremona transformation with generating systems Φ, Ψ, the order of Φ-surfaces is equal to that of Ψ-curves, and the order of Ψ-surfaces is equal to that of Φ-curves.*

Thus, when Φ has been chosen, the orders of the Ψ-surfaces and Ψ-curves are determined; our knowledge of the base elements of Ψ is then to be derived, in simple cases, from the fundamental surfaces and curves of Φ. In particular:

(*a*) To any fundamental surface A of Φ there corresponds a base point O of Ψ; and if Φ_1 is the system of surfaces residual to A in Φ (representing planes through O in S'_3), then the multiplicity of O for Ψ is equal to the difference between the orders of the Φ-curves and the Φ_1-curves.

(*b*) To any simply infinite family of fundamental curves $\mathscr{L}$ of order λ of Φ, there corresponds a λ-fold base curve of Ψ; the order of this curve is equal to the number of curves $\mathscr{L}$ which lie on a generic Φ-surface.

Lastly, it is clear that any Ψ-surface is represented on a plane π of S_3 by means of the system of curves in which π is met by Φ-surfaces.

4.2. *Construction of homaloidal systems.* For the systematic construction of Φ-systems in S_3, two standard methods are available:

(i) by use of postulation and equivalence formulae,

(ii) by the method (due to Cremona) of using a presupposed Φ-surface.

In (i) the procedure is to impose on surfaces of any given order n sufficient base curves and base points to reduce their freedom to 3 and their grade to 1. We thus require *postulation formulae*, giving the diminution of freedom imposed on surfaces F^n by various types of base condition, and *equivalence formulae*, estimating corresponding reductions of grade in systems of F^n. The construction of such formulae is in general a matter of considerable difficulty and delicacy, which we do not attempt here; so that this method is available only in the simplest cases.

In (ii) the essential consideration is that the generic (rational) surface Φ of a homaloidal web is met by the other surfaces of the web in a homaloidal net of curves to which correspond, in the plane representation of Φ, the curves of a plane homaloidal net. We begin then by choosing some rational surface Φ_0^n which is, if possible, to be a generic Φ-surface. In the plane π in which Φ_0 is birationally represented, we construct first the system of curves (L) which represent the free curves of intersection of Φ_0 with surfaces of order n which have the same multiple points and curves as Φ_0; and we then reduce (L) to a homaloidal net (L^*) by imposing fixed components A^* and base points O_i^* on (L). It will then be the case that surfaces of order n which have the same multiple points and curves as Φ_0, which have further simple base curves A on Φ_0 corresponding to A^* on π, and further base points or base contacts at points O_i of Φ_0 corresponding to O_i^*, will form a linear system Φ which meets Φ_0 in the homaloidal net of curves corresponding to L^*. As Φ_0 is supposedly a generic member of the system, it follows that Φ is a homaloidal web.

It may be mentioned that Cremona transformations of S_3 may also be generated by

(iii) direct geometrical construction,

(iv) representing a rational threefold in two different ways (e.g. by two different projections) on spaces S_3 and S_3'.

4.3. *Special transformations.* The following table shows some of the simplest and most useful Cremona space transformations; each of these is denoted by a symbol **T** with appropriate suffixes, showing the order of the surfaces Φ and Ψ.

	Φ-*surfaces*	*Base of* Φ	Φ-*curves*	Ψ-*surfaces*	*Base of* Ψ	Ψ-*curves*
$\mathbf{T}_{22}$	Quadrics.	Conic k, point O.	Conics through O meeting k twice.	Quadrics.	Conic k', point O'.	Conics through O' meeting k' twice.
$\mathbf{T}_{23}$	Quadrics.	Line l, points A, B, C.	Twisted cubics through A, B, C having l as chord.	Cubic scrolls.	Double line l', lines a, b, c which meet l'.	Conics meeting a, b, c, l'.
$\mathbf{T}_{24}$	Quadrics.	Point of contact T, points A, B, C.	Quartics through A, B, C, nodal at T.	Steiner surfaces.	Concurrent double lines a, b, c, conic k meeting a, b, c.	Conics meeting a, b, c, k.
$\mathbf{T}^*_{33}$	4-nodal cubic surfaces.	Nodes A, B, C, D, six joins of these nodes.	Twisted cubics through A, B, C, D.	4-nodal cubic surfaces.	Nodes A', B', C', D', six joins of these nodes.	Twisted cubics through A', B', C', D'.
$\mathbf{T}_{33}$	General cubic surfaces.	Sextic curve c of genus 3.	Twisted cubics meeting c in 8 points.	General cubic surfaces.	Sextic curve c' of genus 3.	Twisted cubics meeting c' in 8 points.

The homaloidal character of all the systems concerned can be verified directly by the method (ii) outlined above, for the plane representations of all the surfaces concerned have already been given. We now comment briefly on each transformation in turn.

$\mathbf{T}_{22}$. This transformation is symmetrical. The fundamental elements for Φ are (*a*) the plane of k, and (*b*) the lines joining O to points of k; and to these there correspond respectively O' and the points of k'. If O is the point $(0,0,0,1)$ and the equations of k are $t = 0 = (x,y,z)_2$, then

$$\Phi \equiv \lambda_0 xt + \lambda_1 yt + \lambda_2 zt + \lambda_3 (x,y,z)_2 = 0,$$

and the equations of **T** (similar to those of $\mathbf{T}^{-1}$) are

$$x':y':z':t' = xt:yt:zt:(x,y,z)_2.$$

The transformation may be generated by projecting a quadric in [4] from two distinct points of itself into spaces S_3 and S'_3. Ordinary inversion with respect to a sphere is an example of $\mathbf{T}_{22}$. A most important special case is that whose equations, in non-homogeneous coordinates, are

$$x' = x, \qquad y' = y/x, \qquad z' = z/x,$$

a transformation used in the resolution of singularities of space curves.

T_{23}. Here Φ has three pencils of fundamental lines, with vertices at A, B, C respectively, and one pencil of fundamental conics through A, B, C and the point of l coplanar with these three; they give rise to the three simple and one double base lines of Ψ.

T_{24}. Here Φ has three pencils of fundamental conics touching the assigned contact plane π at T and passing through two of A, B, C; and it has also a pencil of fundamental lines through T, lying in π, of which two lie on every Φ-quadric. These systems give the three double lines and the base conic of Ψ.

T^*_{33}. This is the reciprocal transformation

$$x':y':z':t' = 1/x:1/y:1/z:1/t$$

already referred to in § 2.32. Here Φ consists of cubic surfaces with nodes at X, Y, Z, T which contain in consequence the six joins of these points; it has four fundamental planes—the planes of reference—which transform into the base nodes X', Y', Z', T' of Ψ; but in this case the neighbourhoods of the six base lines of Φ transform abnormally into the neighbourhoods of the corresponding base lines of Ψ. It should be noted that *this transformation carries all the lines of space into twisted cubics through four fixed points; and conversely.* In fact, if P describes the line whose parametric equations are

$$x:y:z:t = (a_1+\lambda b_1):(a_2+\lambda b_2):(a_3+\lambda b_3):(a_4+\lambda b_4),$$

then P' describes the twisted cubic whose equations are

$$x':y':z':t' = (a_1+\lambda b_1)^{-1}:(a_2+\lambda b_2)^{-1}:(a_3+\lambda b_3)^{-1}:(a_4+\lambda b_4)^{-1}.$$

T_{33}. This is the *standard cubo-cubic transformation* of space, the most interesting and useful perhaps, with its large variety of special cases, of all Cremona transformations of S_3. It is symmetrical, each homaloidal system consisting of cubic surfaces through a base ${}^3C^6$. To establish the existence of the transformation we use the representation of a general cubic surface Φ on a plane ϖ by a system $C^3[1^6]$; and we observe that any sextic curve c of genus 3 on Φ may be supposed to be represented on ϖ by a curve $C^8[3^6]$. Clearly then cubic surfaces through c meet Φ in the homaloidal net of twisted cubics represented by the lines of ϖ; and it follows that they form a homaloidal system. From this also we see at once that the Φ-curves are twisted cubics meeting c in 8 points; and further that each Φ-surface contains 6 trisecants of c, namely, those corresponding to the base points in its plane representation.

The trisecants of c are the only fundamental elements of Φ, and they transform into the points of the base curve c' of Ψ. Also since the Ψ-curves meet c' in 8 points, the trisecants of c must form a scroll R of order 8; and the trisecants of c' a similar scroll R'. Further, since c projects from any point of itself into a plane trinodal quintic, it follows that 3 trisecants of c pass through each point of the curve; and this means that c is a triple curve on R. Hence *to the whole neighbourhood of either base curve there corresponds the trisecant scroll of the other, this scroll being of order 8 and passing triply through the base curve concerned.*

The transformation also admits of the following definition:

The general $\mathbf{T}_{33}$ *is the resultant of three correlations between* S_3 *and* S_3', *in which, to any point* P *of* S_3, *there corresponds the intersection* P' *of the three corresponding planes of* S_3'.

In fact, lines and planes of S_3 are obviously carried by such a transformation into twisted cubics and cubic surfaces of S_3'; and the base curve c in S_3 is the locus of points P whose corresponding planes intersect in a line instead of in a point. The transformation admits evidently of immediate generalization, giving a $\mathbf{T}_{rr}$ $(r \geqslant 2)$ which is the resultant of r correlations between S_r and S_r'.

If we make S_3' coincide with S_3, and take the correlations to be polarities with respect to three quadrics S_1, S_2, S_3, we obtain the particular result:

If P' *is the intersection of the polar planes of any point* P *with respect to three given quadrics* S_1, S_2, S_3, *then the correspondence between* P *and* P' *is an involutory transformation* $\mathbf{T}_{33}$ *of space.*

The point-pairs (P, P') so arising are the pairs of points which are conjugate for all the quadrics of the linear net defined by S_1, S_2, S_3, and the use of the transformation is of cardinal importance in investigating the properties of such a net. In particular, the base curve c of the transformation is the Jacobian curve (locus of vertices of cones) of the net; and the trisecants of c are the *exceptional lines* corresponding to the points of this curve.†

§ 5. A quadro-cubic transformation of S_4

5. As regards Cremona transformations of higher space, we may say that the main interest tends, even more than for S_3, to be confined to special cases of some particular simplicity or importance.

† For further details the reader may consult Reye, *Geometrie der Lage*, iii, chs. 17, 18.

Thus, for example, T_{22} of S_3 generalizes immediately to a T_{22} of S_r (embracing inversion with respect to a hypersphere) whose generating Φ-system consists of quadrics V^2_{r-1} through a quadric V^2_{r-2} and a point; likewise T_{23} of S_3 admits of easy generalization; and T_{33} generalizes, as already stated, to the T_{rr} of S_r which is the resultant of r correlations of S_r.

Instead of enlarging further on such general types, we propose to limit ourselves in this section to describing one particularly simple transformation of S_4, namely, a quadro-cubic transformation in which the Φ-primals are quadrics through an elliptic quintic curve ${}^1C^5$, while the Ψ-primals are all Segre cubic primals through an elliptic quintic scroll ${}^1R^5$ It will be seen that there is much interesting geometry associated with this transformation, and this is equally true of several at least of its specialized forms.

5.1. ***The general transformation of S_4.*** By straightforward generalization of previous results, we note that any Cremona transformation of S_4 will transform a homaloidal system of primals Φ in S_4 into the solids of S'_4, and another such system Ψ in S'_4 into the solids of S_4; also Φ-primals, Φ-surfaces, and Φ-curves (corresponding to solids, planes, and lines of S'_4) will have orders equal respectively to those of Ψ-curves, Ψ-surfaces, and Ψ-primals (corresponding to lines, planes, and solids of S_4).

Any simple base point O of Φ may be either (i) isolated, or (ii) a point of a base curve b, or (iii) a point of a base surface β; and its neighbourhood transforms in these three cases into a fundamental solid, plane, or line of Ψ respectively; also the points of this fundamental locus correspond homographically to lines (linear directions) through O, planes through the tangent line to b at O, or solids through the tangent plane to β at O, as the case may be. To the whole neighbourhood of a curve such as b there corresponds a planar threefold in S'_4, and to the whole neighbourhood of a base surface such as β there corresponds a threefold locus of ∞^2 lines.

We now proceed to apply these results to our particular case.

5.2. ***Quadrics through a ${}^1C^5$ in S_4.*** We have already remarked (Ch. VII, § 5.1) that two general quadrics of S_4 intersect in a Segre quartic surface ${}^1F^4$, representable on a plane π by curves $C^3[1^5]$; and it follows from this that any third quadric Φ meets ${}^1F^4$ in an octavic curve, base of a net of quadrics, which is represented in π

by a curve $C^6[2^5]$. Now plainly this octavic curve can break up into an elliptic quintic Γ, represented by a $C^5[2^5]$, and a twisted cubic ${}^0C^3$, represented by a line of π, which meets Γ in five points. If we now fix Γ, all the quadrics Φ which pass through it will evidently meet ${}^1F^4$ in a homaloidal net of twisted cubics, a net of the quadrics passing through each cubic; and they will therefore form a system of freedom 4 and grade 1, i.e. a homaloidal system. Thus *the quadrics of* S_4 *which pass through an elliptic quintic curve* Γ (*normal in* S_4) *generate a Cremona transformation of* S_4 *into a space* S_4'.

5.3. *The reverse transformation.* For the system Φ just described, the Φ-surfaces are Segre ${}^1F^4$ through Γ, and the Φ-curves were seen to be twisted cubics meeting Γ in five points. Hence, by § 5.1, the Ψ-primals are of order 3, the Ψ-surfaces are of order 4, and the Ψ-curves are conics. Hence Ψ must have a base surface, R say, of order 5.

Each Ψ-primal is represented on a solid Π of S_4 by the system of quadric surfaces in which Π is met by the primals Φ, i.e. by quadric surfaces through the 5 points in which Π meets Γ. Hence, by § 2.32, *the primals* Ψ *are Segre* 10-*nodal cubic primals.*

Again, each Ψ-surface is represented on a plane π of S_4 by the ∞^4 conics in which the Φ-primals meet this plane; so that *the* Ψ-*surfaces are projected Veronese quartic surfaces.*

Consider next, then, the ∞^2 chords of Γ, which are evidently fundamental lines of Φ, and correspond therefore to the *points* of the base surface R of Ψ.

By projecting Γ (i) from a generic line of S_4 into a plane 5-nodal quintic, and (ii) from a unisecant of Γ into a binodal quartic, it appears that *the chord-locus of* Γ *is a quintic threefold* M_3^5 *which passes triply through* Γ.

Also, since an elliptic quintic curve in [3] lies on a quadric surface if and only if it possesses a node, it follows that a point of S_4 is vertex of a quadric cone through Γ if and only if it lies on a chord of the curve. *Hence* M_3^5 *is the locus of vertices of quadric cones through* Γ. From any point A of Γ itself, Γ projects into a ${}^1C^4$ base of a pencil of quadric surfaces; and this means that A is the vertex of a pencil of quadric cones through Γ, and that through A there pass in particular four lines which are vertices of line-cones of the pencil, and from each of which Γ projects into a repeated conic. Each of the lines in question must therefore be met by infinitely many chords of Γ, forming a rational normal cubic scroll ρ with the line as directrix; and hence Γ *possesses an infinity of chordal cubic scrolls* ρ *whose directrices—unisecants of* Γ *and vertices of line-cones of* Φ*—concur by fours in the points of* Γ.

Returning now to the correspondence between chords of Γ and points of R, we remark that the order of the curve r of R which corresponds to a chordal scroll ρ is given by the number of chords in which ρ is met by a quadric Φ; and this is plainly 1, since ρ and Φ both contain Γ. Hence:

The base surface R *of* Ψ *is a quintic scroll whose generators represent the cubic chordal scrolls* ρ *of* Γ.

Finally, to the neighbourhood of each point A of Γ there corresponds, by § 5.1, a fundamental plane α of Ψ, whose points correspond homographically to the planes through the tangent, t say, to Γ at A. Now a point of α lies on R if and only if its associated plane through t contains a chord AP of Γ; whence, since all such planes form the elliptic cubic cone projecting Γ from t, it follows that α meets R in an elliptic cubic curve homographically related to the cone in question. Thus:

The scroll R *is elliptic, possessing* ∞^1 *elliptic cubic directrix curves whose planes represent the neighbourhoods of points of* Γ.

The locus of the planes α is a quintic threefold Ω_3^5: for any line of S'_4 meets the five planes α which correspond to the five intersections of the corresponding Φ-curve with Γ. Plainly, also, any two planes α intersect in a point of R; so that Ω_3^5 has R as its double surface. The scroll R and the planar threefold Ω_3^5 are, in fact, four-dimensional duals of each other.

In conclusion then we have

THEOREM VIII. *There exists a quadro-cubic Cremona transformation of* S_4 *with the following properties*: (i) Φ *consists of quadrics through a* ${}^1C^5$ *which we denote by* Γ, Φ*-surfaces are Segre quartics through* Γ, *and* Φ*-curves are* ${}^0C^3$ *quintisecant to* Γ; (ii) Ψ*-primals are Segre* 10*-nodal cubic primals through a scroll* ${}^1R^5$, *or say* R, Ψ*-surfaces are projected Veronese quartics, and* Ψ*-curves are quintisecant conics of* R; (iii) Φ *has a fundamental primal* M_3^5, *chord-locus of* Γ, *which transforms to* R, *and* Ψ *has a fundamental primal* Ω_3^5, *locus of planes of* ∞^1 *directrix cubic curves of* R, *which represents the whole neighbourhood of* Γ.

EXAMPLES

1. Show that 5 coprime points of Γ correspond to 5 associated planes of Ω_3^5.

2. Show that an arbitrary line meets 5 associated planes of Ω_3^5, and an arbitrary plane meets 5 associated lines of R.

3. If u, v are the parameters of P, Q in a suitable elliptic parametric representation of Γ, show that the chord PQ describes one of the chordal scrolls ρ if and only if $u+v \equiv k$, where k is a constant.

4. Show that each Ψ-surface meets R in a ${}^6C^{10}$.

5. Discuss the special cases of the above transformation in which Γ breaks up into (i) a rational normal ${}^0C^4$ and a chord, (ii) a twisted cubic and a conic, which meet in two points.

(For further details of this transformation the reader is referred to the account in *Phil. Trans. Royal Soc.* A **228** (1929), 331–76.)

EXAMPLES ON CHAPTER VIII

Rational Manifolds

1. Show that the projective model of the system of cubic surfaces through a rational quartic curve ${}^0C^4$ in S_3 is a ${}^1V_3^5$, normal in [6]; also that, if the ${}^0C^4$ degenerates into three skew lines and a transversal, then ${}^1V_3^5$ becomes identical with the Del Pezzo quintic threefold of § 2.3.

Show that the ∞^1 trisecants of ${}^0C^4$ represent the points of a conic k, from which ${}^1V_3^5$ projects birationally into S_3; and that the points of ${}^0C^4$ arise from a scroll of lines R^{10} which has k as triple curve.

2. Show that cubic surfaces through three skew lines represent a ${}^1V_3^6$ in [7], and that this threefold represents *unexceptionally* all the triads of planes through three given lines, or, dually, all the triads of points of three given lines.

3. Prove that the general quadric V_3^2 of [4] can be represented on S_3 by means of cubic surfaces passing through a curve ${}^1C^5$, the trisecants of this curve corresponding to the points of an elliptic quintic curve on the quadric.

[For other examples of threefolds representable by systems of cubic surfaces, including several nodal cubic primals of [4], see Semple, *Proc. Camb. Phil. Soc.* **25** (1929), 145.]

4. *The determinantal quartic primal of* [4]. Prove that if four primes in [4] describe related point-stars, then their point of intersection describes a quartic primal ${}^2V_3^4$ which is rational, being represented on S_3 by means of the quartic surfaces which pass through a ${}^{11}C^{10}$. Assuming, as may be shown to be the case, that this curve possesses 20 quadrisecants, show that these correspond to 20 nodes of the primal.

5. *Rational planar threefolds.* Show that any irreducible V_3^{r-2} of $[r]$ is a rational threefold which is either the locus of ∞^1 planes, or, if $r = 6$, a cone projecting a Veronese surface from an external point. Show by projection that in the first case the threefold is representable on S_3 by

means of scrolls R^{r-2} which have a common $(r-3)$-fold line and $r-3$ simple base lines meeting it. Show also that the Veronese cone is representable by means of quadrics touching a fixed plane at a fixed point.

6. *The* ${}^1V_4^6$ *of* [8]. Show that the projective model of the system of quadrics through two fixed lines of S_4 is a rational ${}^1V_4^6$ of [8] which possesses two distinct doubly infinite systems of generating planes; also that the above representation of this manifold on S_4 is obtainable by direct projection from a [3] which meets ${}^1V_4^6$ in a quadric surface. Show that ${}^1V_4^6$ is image, without exception, of the pairs of points (P, P') of two planes π, π'; and identify the surfaces of ${}^1V_4^6$ which represent collineations of π on π'. (See Segre, 'Sulle varietà che rappresentano le coppie di punti di due piani o spazi', *Rend. Circ. Mat. Palermo*, **5** (1891), 192.)

7. *The Perazzo primal in* [5]. Show that the cubic primal in [5] whose equation is

$$x_1 x_2 x_3 + x_1' x_2' x_3' = 0$$

is representable on [4] by means of quadrics which pass through a fixed skew quadrilateral and one further fixed point. Show that this primal has nine double lines which form the sides and diagonals of a skew hexagon; also that it is met by any tangent prime in a Segre V_3^3.

Cremona Transformations

8. *Special cases of* $\mathbf{T}_{33}$. For some purposes, and in particular for the reduction of surfaces with multiple curves, the special cases of a Cremona transformation—in which the base curves are composite—are at least as important as the original. To obtain such cases of $\mathbf{T}_{33}$ for example (§ 4.3), the obvious course is to use the plane representation of the cubic surface and to allow the plane image of the base sextic c to degenerate in different ways. It may be verified, then, that the following important cases arise:

	Components of C	*Intersections of Φ-curves with components of C*	*Reverse transformation*
$\mathbf{T}_{33}^{(1)}$	${}^0C^5$, quadrisecant line l	8 with ${}^0C^5$, 0 with l	Similar
$\mathbf{T}_{33}^{(2)}$	${}^1C^5$, trisecant l	7 with ${}^1C^5$, 1 with l	$\mathbf{T}_{33}^{(5)}$
$\mathbf{T}_{33}^{(3)}$	${}^2C^5$, chord l	6 with ${}^2C^5$, 2 with l	Similar
$\mathbf{T}_{33}^{(4)}$	${}^0C^4$, 4-secant conic k	6 with ${}^0C^4$, 2 with k	Similar
$\mathbf{T}_{33}^{(5)}$	${}^1C^4$, two skew chords l, l'	4 with ${}^1C^4$, 2 each with l, l'	Similar
$\mathbf{T}_{33}^{(6)}$	${}^1C^4$, trisecant conic k	5 with ${}^1C^4$, 3 with k	$\mathbf{T}_{33}^{(2)}$
$\mathbf{T}_{33}^{(7)}$	${}^0C^3$, ${}^0D^3$ (four intersections)	4 with each curve	Similar
$\mathbf{T}_{33}^{(8)}$	Four skew lines l_i, two transversals m_j	2 with each l_i, 0 with each m_j	Similar

9. Show that by use of $\mathbf{T}^*_{33}$ any quintic surface with 4 triple points may be transformed into a cubic surface, so that such a surface is in general rational; and obtain its plane representation.

Show, on the other hand, that a quintic surface with 5 triple points is in general irrational; also that it can be rational if it possesses a double twisted cubic through the triple points.

10. *Surfaces with multiple curves*. By means of the transformations listed in § 4.3 and in the preceding examples, establish the following examples of reduction of surfaces with multiple curves to well-known simple surfaces:

$\mathbf{T}_{22}$: $F^4[k^2, O] \rightarrow$ cubic surface.
$\mathbf{T}_{22}$: $F^4[l^3, A, B, C] \rightarrow$ cubic scroll.
$\mathbf{T}_{33}$: F^6 with double ${}^3C^6 \rightarrow$ quadric.
$\mathbf{T}^{(2)}_{33}$: $F^6[l, ({}^1C^5)^2] \rightarrow$ cubic surface.
$\mathbf{T}^{(6)}_{33}$: $F^5[l^2, l'^2, ({}^1C^4)^1] \rightarrow$ cubic surface.
$\mathbf{T}^{(6)}_{33}$: $F^5[k, ({}^1C^4)^2] \rightarrow$ cubic surface.
$\mathbf{T}^{(7)}_{33}$: $F^5[({}^0C^3)^2, {}^0D^3] \rightarrow$ cubic surface.

Deduce the plane representation of the surfaces concerned.

11. *Enumerative problems for twisted cubics.*† A twisted cubic in ordinary space has twelve degrees of freedom; and we would therefore expect a finite number of these curves to satisfy six conditions of weight 2, such as, for example, the condition P that the curve should pass through an assigned point or the condition L that the curve should have an assigned line as chord. We consider therefore the problems represented by the compound conditions $P^\alpha L^\beta$ $(\alpha+\beta = 6)$ which express that the cubic should pass through α assigned points and have β assigned lines as chords.

To solve the seven problems of this group we may apply three of the transformations already described, namely, $\mathbf{T}^*_{33}$, $\mathbf{T}_{23}$, and $\mathbf{T}^{(8)}_{33}$, of which the last is generally known as *Wakeford's transformation*.‡

In the first place, $\mathbf{T}^*_{33}$ transforms all the cubic curves through 4 fixed points of S_3 into the lines of S'_3, and the lines of S_3 into cubics through 4 fixed points of S'_3. Thus the problem P^6 transforms to that of drawing a line through 2 points of S'_3; and P^5L^1 transforms to that of drawing a chord of a twisted cubic through a given point. Hence $P^6 = P^5L^1 = 1$.

Next, $\mathbf{T}_{23}$ transforms cubics satisfying a condition P^3L into lines of S'_3, and lines of S_3 into a system of conics in S'_3. As before, we deduce that $P^5L^1 = 1$; but the problem P^4L^2 transforms to that of drawing a line through a given point A to intersect a given conic k in 2 points, and this is evidently *poristic* with no solution or an infinity of solutions according as A lies off or on the plane of k. The problem P^3L^3 transforms to that of drawing a common chord of two conics; whence, obviously, $P^3L^3 = 1$.

Finally, $\mathbf{T}^{(8)}_{33}$ transforms all the twisted cubics which have four given lines $l_1,\ldots, l_4$ as chords into the lines of S'_3, and the lines of S_3 into cubics of S'_3 which have four lines $l'_1,\ldots, l'_4$ as chords. Hence P^2L^4 is reduced to

† For the general problem of twisted cubics satisfying 12 assigned conditions, see Todd, *Proc. Royal Soc.* A, **131** (1931), 286.

‡ *Proc. Lond. Math. Soc.* (2), **21** (1922), 98.

the operation of joining two points of S_3'; P^1L^5 is reduced to that of drawing a chord from a given point to a given twisted cubic; and L^6 is reduced to that of finding the number of *additional* common chords of two twisted cubics which have $l_1', \ldots, l_4'$ as *assigned* common chords. Now two twisted cubics have in general 10 common chords, as may be proved either (*a*) by a correspondence argument, or (*b*) by allowing each curve to degenerate and using the principle of Conservation of Number (cf. Ch. XI), or (*c*) by the result of Theorem XX of Chapter X. It follows therefore that $P^2L^4 = 1$, $P^1L^5 = 1$, and $L^6 = 6$.

[The problem for elliptic quartic curves analogous to that here considered for twisted cubics has been discussed in detail by Welchman, *Proc. Camb. Phil. Soc.* **27** (1931), 20; see also Todd, ibid. p. 538, for further enumerative results about these curves.]

12. *Monoids and monoidal transformations.* A *monoid* in ordinary space S_3 is any irreducible surface of order n which possesses an $(n-1)$-fold point; and a *monoidal transformation* is a Cremona transformation in which all the Φ-surfaces are monoids of order n with a fixed $(n-1)$-fold point.

If Φ is a generic monoid of order n with O as its $(n-1)$-fold point, then Φ has a nodal cone K, of order $n-1$, which meets Φ in $n(n-1)$ lines; and, by projection from O, the surface is representable on a plane π by a system of curves $C^n[1^{n(n-1)}]$ for which all the base points lie on an $(n-1)$-ic curve k, projection of K. If Φ' is another monoid of order n with the same multiple point, then Φ' meets Φ in a curve of the system whose image in π is $C^{2n-1}[1^{n(n-1)}]$, and this curve has a point of multiplicity $(n-1)^2$ at O; but if Φ and Φ' have the same nodal cone at O, then the image in π of their intersection curve sheds off k as component and is reduced in consequence to a free n-ic curve of π; and since this curve has $n(n-1)$ intersections with k, the curve of intersection of Φ' with Φ has in this case a point of multiplicity $n(n-1)$ at O.

The two simplest monoidal transformations which arise from the above are obtained by reducing the image systems $C^{2n-1}[1^{n(n-1)}]$ and C^n to the lines of the plane π; in the first case, when the nodal cone at O is variable, there is a simple base curve of order n^2-n with an $(n-1)(n-2)$-fold point at O; in the second, when the nodal cone at O is fixed, there is a simple base curve of order n^2-n with a point of multiplicity $(n-1)^2$ at O; and in both cases the Φ-curves are plane curves of order n with an $(n-1)$-fold point at O.

Ex. Discuss the *monoidal involution* in S_3 (analogue of the De Jonquières involution (Ch. VII, § 8.3)) defined by any surface of order n with an $(n-2)$-fold point.

13. *The general* $\mathbf{T}_{44}$ *of* S_4. Show that the four-dimensional Cremona transformation which is the resultant of 4 correlations of S_4 on S_4' is a quarto-quartic transformation $\mathbf{T}_{44}$ whose homaloidal systems Φ and Ψ each consist of determinantal quartic primals through a fixed projectively generated ${}^{11}F^{10}$, locus of points of intersection of corresponding primes of 5 related point-stars.

Show also that the Φ-surfaces and Ψ-surfaces are Bordiga ${}^3F^6$ (cf. Ch. VII, Ex. **31**), and that the Φ-curves and Ψ-curves are normal ${}^0C^4$ which meet the base surface in 15 points. (Todd, *Proc. Camb. Phil. Soc.* **26** (1930), 323.)

14. *A* $\mathbf{T}_{22}$ *of* S_5 *and a* $\mathbf{T}_{24}$ *of* S_4. If a, b, c, f, g, h are the coordinates of a general point of S_5 and if A, B, C, F, G, H are their cofactors in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

prove that the transformation

$$\frac{a'}{A} = \frac{b'}{B} = \dots = \frac{h'}{H}$$

has the inverse transformation

$$\frac{a}{A'} = \frac{b}{B'} = \dots = \frac{h}{H'}.$$

Hence deduce the result: *Quadrics through a Veronese surface generate a symmetrical Cremona transformation* $\mathbf{T}_{22}$ *of* S_5.

If any one of the above quadrics, representing a prime S_4 say, is projected from a point of itself on to a space S_4', show that the resulting representation of S_4 on S_4' may be characterized as follows: *The quadrics of* S_4 *which pass through a normal* ${}^0C^4$ *and a point generate a quadro-quartic transformation* $\mathbf{T}_{24}$ *of* S_4 *whose reverse system consists of quartic primals passing doubly through a quadric surface and simply through a projected Veronese surface, these two surfaces meeting in the section of the latter by the prime containing the former.*

BOOKS RECOMMENDED FOR FURTHER READING

Baker, *Principles of geometry*, iv, vi.

Bertini, *Introduzione*, Appendice, ch. ii.

Conforto, *Le superficie razionali*, Book II.

Reye, *Geometrie der Lage*, iii.

Sturm, *Die Lehre von den geometrischen Verwandtschaften*, iv.

Hudson, *Cremona transformations.*

CHAPTER IX

PROJECTIVE CHARACTERS OF CURVES AND SURFACES

§ 1. CURVES

1. Elementary projective characters. In the previous discussion of plane and space curves we had occasion to introduce various characters of a curve (in connexion, for instance, with the Plücker equations). Among these there are two, the *order* and the *rank*, which are of fundamental importance; for, as will appear, the definitions given of these characters extend immediately to curves in any space [r] and, moreover, the characters are invariant under projection. We shall call them the *elementary projective characters* of a curve.

For surfaces in higher space we shall obtain a set of four characters which are likewise invariant under projection. The position is somewhat obscured if we begin, as the analogy with space curves might suggest, by considering surfaces in ordinary space: first, because proper projection of such surfaces is impossible and, secondly, because one of the four characters of a surface situated in [r] ($r > 3$) has no precise analogue for a surface in [3]. These characters, which we shall call the elementary projective characters of a surface, will be described in § 2.

The elementary projective characters of a curve or surface are not of course the sole projective characters of the manifold in question. But they have a twofold importance in the theory; for it will appear in this chapter that (i) a wide group of problems require for their solution a knowledge of these characters alone, and (ii) certain birational invariants of a manifold (as, for example, the genus of a curve) are expressible in terms of the elementary projective characters. The case of surfaces is considered from this point of view in the concluding sections of this chapter.

1.1. ***Curves in*** [r]. The definition of rank of a space curve, given in Chapter IV, extends immediately to a curve C of [r] ($r > 3$). Thus, of the ∞^1 tangents to C, a finite number μ_1 will meet a generic secundum [$r-2$]; this number is called the *rank* of C. The *order* μ_0 of C is of course the number of its points which lie in a generic prime [$r-1$].

It follows, then, that if C is projected on a prime Π from a generic point O (not on the curve), its projection C' will have order μ_0 and rank μ_1; for the number of tangents to C' which meet a given $[r-3]$ in Π is clearly the number of tangents to C which meet the $[r-2]$ which joins the $[r-3]$ to O, while the order of C is preserved by the projection (see Ch. I, § 5).

If C is without singularities so also is C'; for a multiple point of C' could arise only if a chord or multisecant of C passed through O, and this will not happen when O is in general position.

By successive projections from point vertices, we finally obtain a curve in [3], of order μ_0 and rank μ_1, which, if C is non-singular, is of general character. It follows from Ch. IV, § 7, that the genus p of C is given by the formula

$$\mu_1 = 2(\mu_0+p-1). \tag{1}$$

We may also, as in Ch. IV, introduce the character h, representing the *number of apparent nodes* of C, i.e. the number of chords which meet a given $[r-3]$. As before, then, we have the relations

$$h = \tfrac{1}{2}(\mu_0-1)(\mu_0-2)-p, \tag{2}$$

$$h = \tfrac{1}{2}\mu_0(\mu_0-1)-\tfrac{1}{2}\mu_1. \tag{3}$$

1.2. *Intersection problems for curves.* In space $[r]$, $r-1$ irreducible primals, generically situated with respect to each other, will meet in an irreducible curve; and $r-1$ primals which are drawn through a given curve will in general meet in a second curve. The problems (already solved in the case $r = 3$) of determining the characters of these curves will now be considered in the general case.†

Let $f_i = 0$ $(i = 1, 2,..., r-1)$ be the equations of $r-1$ given primals, in general position, having orders n_i $(i = 1, 2,..., r-1)$. The curve C common to them all has order

$$\mu_0 = n_1 n_2 ... n_{r-1}. \tag{1}$$

The tangent at any point P of C is the line common to the $r-1$ tangent primes to the primals at P; and of such tangents, a finite number μ_1 will meet a given secundum Π.

† The results which follow are due to Veronese, *Math. Ann.* **19** (1882), 161.

To determine the character μ_1 we suppose that Π is the intersection of the primes A, B whose respective equations are

$$A \equiv \sum_{i=0}^{r} a_i x_i = 0, \qquad B \equiv \sum_{i=0}^{r} b_i x_i = 0, \tag{2}$$

and we consider the Jacobian primal $J \equiv J(f_1, f_2, \ldots, f_{r-1}, A, B)$ whose equation is

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_0}, & \ldots, & \frac{\partial f_{r-1}}{\partial x_0}, & a_0, & b_0 \\ \frac{\partial f_1}{\partial x_1}, & \ldots, & \frac{\partial f_{r-1}}{\partial x_1}, & a_1, & b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0. \tag{3}$$

This is evidently the locus of a point P such that the $r-1$ polar primes of P with respect to f_i are concurrent with the primes A and B. If P lies on C, this means that the tangent to C at P must meet the secundum $\Pi = A \, . \, B$. The rank μ_1 is therefore equal to the number of intersections of J with C. And since J is obviously of order $\sum_{i=1}^{r-1} (n_i - 1)$, we obtain the result

$$\mu_1 = n_1 n_2 \ldots n_{r-1} \{\sum (n_i - 1)\}. \tag{4}$$

1.3. We pass now to the second problem. Suppose that the $r-1$ primals considered above are drawn through a non-singular curve C of order μ_0 and rank μ_1. In general their residual intersection will be an irreducible curve C' of order μ_0' and rank μ_1'. The curves C and C' will intersect at a certain number ϵ of points; at each of these the $r-1$ tangent primes to the primals f_i have in common the plane of the tangents to C and C' there.

The order of C' is given by the equation

$$\mu_0 + \mu_0' = n_1 n_2 \ldots n_{r-1}. \tag{5}$$

Consider now, as in § 1.2, the Jacobian of f_i and an arbitrary secundum $\Pi = A \, . \, B$. This meets C in the points P such that the tangent to C at P meets Π; and it also passes through the ϵ intersections of C with C' since, at each of these points, the tangent primes to all the f_i have a plane in common, and this plane meets Π.

Since the Jacobian is of order $\sum_{i=1}^{r-1} (n_i - 1)$, it follows that

$$\mu_1 + \epsilon = \mu_0 \{\sum (n_i - 1)\}. \tag{6}$$

In a similar manner we deduce that

$$\mu_1'+\epsilon = \mu_0'\{\sum (n_i-1)\}. \tag{7}$$

These equations determine the numbers μ_1' and ϵ; from them we derive the result

$$\mu_1'-\mu_1 = (\mu_0'-\mu_0)\{\sum (n_i-1)\}. \tag{8}$$

From this equation it follows that, if $\mu_0' = \mu_0$, then $\mu_1' = \mu_1$.

1.31. The generalization of the previous results is now obvious. Suppose that, in $[r]$, the $r-1$ primals f_1 meet in a set of s non-singular curves $C^{(1)}, C^{(2)},..., C^{(s)}$ having orders $\mu_0^{(i)}$ and ranks $\mu_1^{(i)}$ $(i = 1, 2,..., s)$, and that the curves $C^{(i)}$, $C^{(j)}$ meet in ϵ_{ij} points $(i = 1, 2,..., s;\ j = 1, 2..., s;\ i \neq j)$.

Then we shall have

$$\sum \mu_0^{(i)} = n_1 n_2 ... n_{r-1}, \tag{9}$$

$$\mu_1^{(l)}+\sum_j \epsilon_{lj} = \mu_0^{(l)}\{\sum (n_i-1)\} \quad (l = 1, 2,..., s). \tag{10}$$

From these equations it follows that, if the orders, ranks, and mutual intersections of $s-1$ of the curves are given, then the order and rank of the remaining curve, and the numbers of its intersections with the other curves, are completely determined. These are extensions of the results obtained in Ch. IV.

1.4. *Equivalence of a curve.* In general, r given primals in $[r]$, of orders n_i $(i = 1, 2,..., r)$, meet in $n_1 n_2 ... n_r$ points; but if they have in common a curve C, they will meet in a certain number N of points external to C. The difference $n_1 n_2 ... n_r - N$ is called the *equivalence* of C for the primals in question.

The value of N is immediately deducible from (6); for, with the previous notation, the points common to the r primals, external to C, are those in which the rth primal meets C', other than those lying on C itself.

Hence,
$$N = \mu_0' n_r - \epsilon.$$
That is,
$$N = n_1 n_2 ... n_r - \mu_0\{\sum (n_i-1)\} + \mu_1 - \mu_0. \tag{11}$$

EXAMPLES

1. Show that three quadrics of [4] in general meet in a ${}^5C^8$; but that, if they have a common line, their residual intersection is a ${}^3C^7$ meeting the line in three points.

2. Determine the residual intersection of three quadrics of [4] which have in common (*a*) a conic, (*b*) two skew lines, (*c*) a line and a conic, skew or intersecting.

3. Find the equivalence of (a) a line, (b) a rational cubic, (c) an elliptic quartic for four quadrics of [4].

4. Two surfaces of [3], of orders m and n respectively, have as partial intersection an irreducible curve of order μ_0 and genus p. Prove that

$$2p \leqslant (mn-2\mu_0')(m+n-4)+(\mu_0'-1)(\mu_0'-2),$$

where $\mu_0' = mn-\mu_0$.

5. $C_1, C_2, C_3,\ldots$ are a set of irreducible curves, without singular points, of genera $p_1, p_2, p_3,\ldots$. The curves intersect only in pairs, the total number of such intersections being N. Show that the composite curve $C_1+C_2+C_3\ldots$ may be formally regarded as a curve of genus p, where

$$p-1 = \sum (p_i-1)+N.$$

§ 2. SURFACES

2. The elementary projective characters of a surface. Consider a surface F situated in $[r]$ $(r > 3)$. This is met by a generic secundum $[r-2]$ in a certain number μ_0 of points, where μ_0 is the *order* of F; thus the section of F by a generic prime is a curve of order μ_0; the rank μ_1 of this curve is defined as the *rank* of F.

Let us suppose that F is irreducible, and let f denote its projection on space [3] from a general vertex $[r-4]$. At a general point of itself f possesses a unique tangent plane; and, since f is likewise irreducible, its tangent planes form an irreducible aggregate. Assuming, then, as we shall do in the sequel, that f is not developable, it follows that at a general point of F the surface possesses a unique tangent plane, and that the tangent planes of F form an irreducible aggregate of dimension 2. Hence a finite number of tangent planes to F can be found to satisfy two given conditions. In particular

(1) A number μ_2 of such planes can be found to meet a given $[r-2]$ in a line. This number is called the *class* of F.

The class may be given another interpretation. Consider any prime through a tangent plane of F; its section of F is a curve having a node at the point of contact of the tangent plane. We shall call any such prime a *tangent prime* to F; then we may say that μ_2 is equal to the number of tangent primes of F which belong to a general pencil.

(2) A number ν_2 of tangent planes will meet a given $[r-4]$; this character is called the *type*† of F.

† In Italian, *ceto* (i.e. rank).

The numbers μ_0, μ_1, μ_2, ν_2 constitute the *elementary projective characters* of F. In terms of these, many other important characters of the surface may be expressed.

In particular, for a surface in [4], μ_2 is the number of tangent planes of F which meet a given plane in a line; and ν_2 is the number of tangent planes (*or* tangent lines) which pass through a given point.

If we attempt to formulate similar characters for a surface in ordinary space, we shall obtain the order, rank (of a plane section), and class, i.e. the number of tangent planes through a given line. But the previous definition of the type will not extend to the present case; we shall, however, see shortly how to define a suitable character to take its place.

2.01. *The curve* Δ. In order that a tangent plane to F should meet a given space $[r-3]$, a single condition must be fulfilled; hence there are ∞^1 such planes, and ∞^1 corresponding points of contact, lying on a curve Δ. We shall now prove that

The points of contact of tangent planes which meet a generic space $[r-3]$ *lie on a curve* Δ *of order* μ_1. *Also any two curves* Δ *corresponding to generic spaces* $[r-3]$ *meet in* $\mu_2+\nu_2$ *points.*

Let Π be a given space $[r-3]$ and Δ the corresponding curve of contact. Then the order of Δ is the number of tangent planes of F which meet Π and have their points of contact on a given prime Ω. Since any prime Ω will meet Δ in the same number of points, we may suppose that Ω *contains* Π: in this case the number of tangent planes satisfying the required conditions will be equal to the number of tangent lines to the section of F by Ω which meet Π; and this, by definition, is μ_1.

Again, let Π, Π' be given spaces $[r-3]$, and let Δ, Δ' be the corresponding curves of contact. The number of points common to Δ and Δ' is equal to the number of tangent planes to F which meet Π and Π'. In order to calculate this number we shall assume† that it remains unchanged when, by varying Π' continuously, we make it come to lie in a space $[r-2]$ passing through Π. The tangent planes which now meet Π and Π' consist of those which meet the $[r-4]$ common to Π and Π', and those which meet $[r-2]$ in a line; the required number is thus $\mu_2+\nu_2$.

† For remarks upon this type of argument see Chapter XI.

EXAMPLES

1. Show that the ∞^1 tangent planes to F at points of a prime section generate a manifold V_3 of order μ_1. What is the order of the V_3 generated by tangent planes which meet a given $[r-3]$?

2. Show that, if F is situated in $[r]$, where $r > 4$, its projection on $[r-1]$ from an external point has the same characters $\mu_0, \mu_1, \mu_2, \nu_2$ as F. (Hence the origin of the term *projective character*.)

2.1. *Generic surfaces in* $[r]$. In all that follows in this chapter, we must for the sake of simplicity confine ourselves to surfaces which are free from what we may call *proper* singularities; and to make this restriction precise, we look for a basis on which to decide what *accidental* or *improper* singularities a surface must be assumed to possess in order to secure a reasonable measure of generality. This will lead to the concept of a surface possessing only *normal singularities*, a phrase which is usually abbreviated to *generic surface* (the term generic here being used in a special sense).

If $r \geqslant 5$, we define a generic surface to be one which has no multiple points of any kind. This is based on the consideration that, in $[r]$ $(r \geqslant 5)$, a surface, regarded as a two-parameter point-locus, cannot in general be expected to intersect itself; or, in other words, that there will not in general be any sets of solutions u_1, v_1, u_2, v_2 of the equations

$$\frac{x_0(u_1, v_1)}{x_0(u_2, v_2)} = \frac{x_1(u_1, v_1)}{x_1(u_2, v_2)} = \dots = \frac{x_r(u_1, v_1)}{x_r(u_2, v_2)}$$

other than those for which $u_1 = u_2$, $v_1 = v_2$ (cf. Ch. VI, § 2.4). The assumption in question is supported, as it happens, by the fact that any algebraic surface is known to be transformable birationally into a non-singular surface of S_r $(r \geqslant 5)$.

If $r < 5$, we take for normal singularities those which are in general possessed by surfaces obtained by *general* projection of non-singular surfaces in [5]; and this convention will apply also to such surfaces as are not themselves obtainable by projection in this way. It is easy to verify that this principle does in fact lead to the same results as would arise from considering the surfaces, in [3] or [4], from the point of view of their parametric representations.

2.11. *Generic surfaces in* [4]. *A generic surface in* [4] *is one which has at most a finite number of improper (accidental) double points.* For if F^* is a non-singular surface of [5], the ∞^4 chords of the surface in general fill the ambient space,† so that a finite number of them pass through any given point O. If F^* is projected from O into a surface F of [4], any one of these chords—OP_1P_2, say—will project into a double point Q of F. Clearly F will in general possess two tangent planes at Q—projections of those at P_1 and P_2—which meet only at Q.‡ And the section of F by a generic prime through Q will be a curve of the same genus as the generic prime section of F. A node of F having this last property will be defined as an *improper* node of the surface.

2.12. *Chord-cone and chord-curve.* Consider next the chords of a generic surface F in [4] which pass through a general point T. There will be a simple infinity of such chords TQ_1Q_2 forming a chord-cone K; and the locus of Q_1, Q_2 is a curve—intersection of K with F—which we may call the chord-curve $\mathscr{K}$.

Among the chords from T, there will in general be a finite number of trisecants $TQ_1Q_2Q_3$; for a surface in [4] ordinarily possesses ∞^3 trisecant lines, and these will usually fill the ambient space. Any such trisecant is a triple generator of K; in fact it will be approached along three separate sheets of this cone by three variable chords

$$TQ_1'Q_2', \qquad TQ_2''Q_3', \qquad TQ_1''Q_3'';$$

and since we have two distinct points Q_1', Q_1'' approaching Q_1 (and lying on different sheets of the chord-cone) it follows that *the chord-curve has nodes at each of the three intersections of F with any trisecant from T* (cf. Ch. VI, § 5.21).

Finally, the chord-cone will evidently contain any line TD joining T to any improper node D of F. The chord TQ_1Q_2 which tends to TD will have one of the end points Q_1 on one sheet of F at D, and the other end point Q_2 on the other sheet. Hence *the chord-curve on F has a node at each improper node of F.*

2.13. *Formula for the improper nodes.* We shall now obtain a formula expressing the number of improper nodes of F in terms of the elementary projective characters.

With the notation of the previous sections, the chord-cone K

† As already remarked in Chapter VII, the Veronese surface is an exception in that its chords generate only a V_4.

‡ For special positions of O, the tangent planes at Q may be cospatial or even coincident. When the tangent planes meet only at Q, Q is called an improper node of the *first species*; in the other cases we say that it is of the second species.

will be of order h (cf. § 1.1). Then, taking a section of F by a prime through T, we see that the chord curve $\mathscr{K}$ is of order $2h$.

Now fix an arbitrary plane ϖ, and consider the symmetrical correspondence between primes Π_1, Π_2 through ϖ which contain the end points Q_1, Q_2 respectively, of some chord TQ_1Q_2 through T.

If Π_1 is given, Q_1 has $2h$ possible positions; there are $2h$ positions of the associated point Q_2, and therefore $2h$ positions for Π_2. Thus there is a $(2h, 2h)$ correspondence between Π_1 and Π_2.

The $4h$ coincidences of Π_1 with Π_2 arise as follows:

(i) If Π_1 passes through the point of contact U of one of the ν_2 tangent lines from T to F, then we can take $U \equiv Q_1 \equiv Q_2$, and one of the primes corresponding to Π_1 coincides with Π_1. Each such coincidence is simple.

(ii) If Π_1 passes through an improper node D of F, then D, as we have seen, is a double point of $\mathscr{K}$, and therefore counts twice among the set of points Q_1 in which Π_1 meets $\mathscr{K}$; furthermore, if $Q_1 \equiv D$, then $Q_2 \equiv D$, and hence two of the primes corresponding to Π_1 coincide with Π_1. Each such coincidence is therefore double.

(iii) If Π_1 contains one of the h chords of F which pass through T and meet ϖ, say the chord TAB, then $\Pi_2 \equiv \Pi_1$ if Q is taken to be either A or B. Thus every coincidence so arising is double.

The application of Zeuthen's rule in checking the multiplicities (i)–(iii) is immediate. Hence, if d is the number of improper nodes of F, we have the formula

$$\nu_2+2d+2h = 4h,$$

or, by § 1.1,

$$\nu_2+2d = \mu_0(\mu_0-1)-\mu_1.$$

Examples

1. If F is a non-ruled surface situated in [4], show that a tangent plane to F in general meets the surface in μ_0-4 points other than the point of contact, and hence that μ_0-4 tangents at that point meet F elsewhere.

2. With the same hypotheses about F, prove that the trisecants of F, which pass through a simple point T of the surface generate a *trisecant-cone* K (analogue of the chord-cone for T not on F) whose order h' is given by

$$2h' = (\mu_0-1)(\mu_0-2)-(\mu_1-2).$$

Prove also that the *trisecant-curve* $\mathscr{K}$ in which K meets F has a multiple point of order μ_0-4 at T, and that $\mathscr{K}$ is of order $2h'+(\mu_0-4)$.

3. Applying the correspondence method of § 2.13, and using the above results, prove that ν_2-4 tangents can be drawn to F from T.

4. Deduce that, *if a generic, non-ruled surface in* [r], *where* $r > 4$, *is projected from a point of itself, its projection will be a surface of order* μ_0-1, *rank* μ_1-2, *class* μ_2, *and type* ν_2-4, *where* μ_0, *etc., denote the characters of the given surface.*

2.14. *Generic surfaces in* [3]. We now apply § 2.12 again to the problem of characterizing *normal singularities* for a surface in [3].

When the generic surface F in [4] is projected from T into a surface f in [3], the chords TQ_1Q_2 of F evidently project into the points of a *double curve* Γ of f, the pairs of points (Q_1, Q_2) of the chord-curve $\mathscr{K}$ being in (1, 1) correspondence with the points of Γ. If TQ_1Q_2 projects into the point R of Γ, then the tangent plane to K along the generator TQ_1Q_2 projects into the tangent to Γ at R; and the two tangent planes to F at Q_1, Q_2, since they contain the tangent lines to $\mathscr{K}$ at Q_1, Q_2 respectively, project into the two tangent planes of f—in general distinct—which intersect in the tangent to Γ at R.

There will, however, be ν_2 tangents TQ_1Q_1 of F which pass through T; these will give rise to *pinch-points* of f on Γ, i.e. points of the double curve at which the two tangent planes to the surface coincide (cf. Ch. VIII, § 3.1). The tangent planes at these points are called *pinch-planes*.

Next we observe that the improper nodes of F project into ordinary points of Γ. Finally, the trisecants of F, in number t say, which pass through T, project into triple points of f which are also triple on Γ. For, in the notation of § 2.12, the variable point-pairs $(Q_1'Q_2')$, $(Q_2''Q_3')$, $(Q_1''Q_3'')$ project into three points of Γ which approach the triple point of f along separate branches of Γ as the three chords such as $TQ_1'Q_2'$ tend to the trisecant $TQ_1Q_2Q_3$. At the triple point f possesses three distinct tangent planes which are the joins in pairs of the three nodal tangents to Γ at this point.

Our conclusion then is that

A generic surface in [3] *is one whose only singularities are* (i) *a double curve* Γ, (ii) *a finite number* ν_2 *of pinch-points on* Γ, *and* (iii) *a finite number* t *of triple points which are also triple for* Γ.

Clearly the surface f has the same order μ_0, rank μ_1, and class μ_2 as F; the type ν_2 of F passes into the number of pinch-points on the double curve of f.

§ 3. The Cayley-Zeuthen Equations

3. Given any irreducible surface f in [3], with normal singularities and known characters μ_0, μ_1, μ_2, ν_2, we shall show that other important characteristics of f, such as the rank of its double curve, the number of its triple points, and so on, are thereby determined. The formulae which give these numbers are particular cases of more general results, first obtained empirically by Salmon, and later established by Cayley and Zeuthen.

Notation. The order and rank of the double curve Γ are denoted by ϵ_0 and ϵ_1 respectively; t is the number of triple points. Of the ∞^1 tangent planes to f at points of Γ, a finite number ρ will pass through a given point; this number is called the *class of immersion* of Γ in f.

The tangent cone drawn to f from an arbitrary point P touches f along a proper curve of contact Δ, which is of order μ_1 (§ 2.01). Among the tangents from P, there are some which have *inflexional contact* with f, and others which have *double contact* with f. We denote the numbers of these tangents by κ and δ respectively.

3.1. Since the first polar of P passes simply through Γ and meets f residually in Δ, we have

$$\mu_0(\mu_0-1) = 2\epsilon_0+\mu_1. \tag{1}$$

Consider now the intersections of Γ with the second polar of P. A point Q of f lies on both the first and second polars of P if, and only if, PQ has *three* of its intersections with f coincident at Q; and if Q lies on Γ it satisfies this condition in general only if one of the two tangent planes to f at Q passes through P, or again if Q is one of the triple points of f, which are all simple on the second polar of P. Hence

$$\epsilon_0(\mu_0-2) = \rho+3t. \tag{2}$$

Consider next the intersections of the second polar with Δ. These include, evidently, the κ points of contact of the inflexional tangents from P; but it is clear that Δ will also pass through the ρ points mentioned above, which are the points of contact of tangent planes at points of Γ which pass through P. Hence these are included in the required intersections. Thus

$$\mu_1(\mu_0-2) = \kappa+\rho. \tag{3}$$

3.2. *Tangent planes at points of* Γ. The next equation is derived from correspondence principles. Consider the ∞^1 pairs of tangent planes to f at points of Γ, and apply the method of Ch. IV, §§ 2.1, 2.11, to find the number of coincidences of the planes of a pair, i.e. the number ν_2 of pinch-points of f.

Since the correspondence between the planes is symmetrical, the appropriate formula is (9′) of Ch. IV, § 2.11, generalized (and dualized) in the manner laid down in § 2.1 for correspondence in space. This gives

$$\nu_2 = \xi = 2(\alpha-\nu),$$

where α, the number of plane-pairs in which one plane passes through an assigned point, is evidently equal to ρ, while ν, the number of plane-pairs for which the line of intersection of the planes meets a fixed line, is evidently ϵ_1. Hence

$$\nu_2 = 2\rho-2\epsilon_1. \tag{4}$$

3.3. In order to establish the remaining equations we have first to examine more closely the behaviour of the first polars and the contact curves Δ at the singularities of f. We begin by proving the following

LEMMA. *Let P be a generic point, and Q a point of Γ. Then the tangent plane at Q to the first polar of P with respect to f is the harmonic conjugate, in regard to the tangent planes to f at Q, of the plane joining P to the tangent to Γ at Q.*

This may be proved geometrically by regarding PQ as the limit of a chord PQ_1Q_2 which intersects the two sheets of f at Q in Q_1, Q_2, and by applying the original definition of the first polar. The following simple analytical proof may also be given.

Using non-homogeneous coordinates, let Q be the origin and P the point (x_0, y_0, z_0). Let the tangent planes to f at Q have equations $x = 0$, $y = 0$; then the equation of f is of the form

$$2xy+f_3(x,y,z)+\ldots = 0,$$

where $f_3(x,y,z)$ is a homogeneous cubic polynomial in the three coordinates, and the omitted terms are all of higher degree.

The tangent plane at Q to the first polar of P is represented by the equation

$$y_0x+x_0y = 0.$$

The harmonic conjugate of this plane, with respect to the planes $x = 0$, $y = 0$, is the plane $y_0x-x_0y = 0$; and this establishes the lemma.

We now apply this result to find all the intersections of Γ with the curve of contact Δ of tangent planes from P. If Q is any such intersection, the first polar of P—since it contains Γ and Δ—must touch one sheet of f at Q; and this, by the lemma, is only possible if either (i) one of the tangent planes of f at Q passes through P, or (ii) the two tangent planes of f at Q coincide. The points arising from (i) are among those (already considered in § 3.1) which lie on the second polar of P; while from (ii) we deduce the result that

All the contact curves Δ pass through the pinch-points of f.

3.4. *Formula for the bitangents.* Consider now the cone K of proper tangents to f from a generic point P. This cone, of order μ_1, meets f in Δ counted twice, and in a residual curve Δ^*, of order $\mu_1(\mu_0-2)$, which meets each generator of K in μ_0-2 points.

Let U, V be any pair of points of Δ^* which lie on the same generator of K; and let us calculate the number of coincidences of the symmetrical correspondence between U and V by the method of Ch. IV, §§ 2.1, 2.11. This is the number ξ given by

$$\xi = 2(\alpha-\nu),$$

where α is the number of (unordered) pairs (U, V) for which U lies on a fixed plane, so that

$$\alpha = \mu_1(\mu_0-2)(\mu_0-3),$$

while ν is the number of pairs for which UV meets a fixed line; thus

$$\nu = \mu_1 . \tfrac{1}{2}(\mu_0-2)(\mu_0-3).$$

Hence

$$\xi = \mu_1(\mu_0-2)(\mu_0-3).$$

The coincidences occur in two ways, as follows:

(i) Any one of the δ bitangents which can be drawn to f from P is evidently an intersection of two distinct sheets of the cone K. If we denote such a bitangent by b, and its points of contact by M, N, then one branch of Δ^* (lying in one sheet of K) will touch b at N, while another (lying in the other sheet of K) will touch b at M. Thus each bitangent furnishes two coincidences.

(ii) If X is any one of the intersections of K with Γ which are *not* intersections of Δ with Γ, then X is evidently a double point of Δ^* and a coincidence of the correspondence; also, by an argument exactly similar to that given in Ch. IV, § 4.21, it appears that each such coincidence counts twice. Since K meets Γ in the

ν_2 pinch-points and touches Γ at the ρ other points where Δ meets Γ, the number of its residual intersections with Γ is $\mu_1 \epsilon_0 - \nu_2 - 2\rho$.

Hence we have

$$\mu_1(\mu_0-2)(\mu_0-3) = 2\delta+2(\mu_1\epsilon_0-2\rho-\nu_2). \tag{5}$$

3.5. *Formula for the class.* Consider finally the curve C which is the intersection, residual to Γ, of the first polars of two general points P, P'. Its order is $(\mu_0-1)^2-\epsilon_0$, and its intersections with Γ fall into the following groups:

(i) The t triple points. Each of these is simple on C; for each first polar has a node at these points and, of the four generators common to their nodal cones at a triple point, three are tangents to Γ.

(ii) The ϵ_1 points, Q say, of Γ at which the tangents to Γ meet PP'. For at any such point Q the first polars touch (by the lemma of § 3.3), since the planes joining PP' to the tangent at Q coincide. Thus the common curve of the first polars has a node at Q, one branch being Γ and the other C.

(iii) The ν_2 pinch-points. Each of these counts for three intersections of C with f. This result has been established in Ch. VIII, § 3.1, by resolving Γ by a system of surfaces passing through it; there it was seen that a linear branch approaching a pinch-point in the pinch-plane has three intersections there with f, except when it approaches in one definite direction (that of the cotangent).

The remaining intersections of C with f are evidently the μ_2 points of contact of the tangent planes to f which pass through PP'. We thus obtain the equation

$$\mu_2 = \mu_0\{(\mu_0-1)^2-\epsilon_0\}-3t-2\epsilon_1-3\nu_2. \tag{6}$$

This formula shows the reduction in the class of f due to the double curve, its pinch-points, and triple points.

3.6. *Summary of results.* The formulae (1)–(6), which constitute the Cayley–Zeuthen equations, may be used to determine the six characters ϵ_0, ϵ_1, t, ρ, κ, δ in terms of the elementary projective characters μ_0, μ_1, μ_2, ν_2. In practice, however, it is generally the case that a surface with normal singularities is specified by the characters μ_0, ϵ_0, ϵ_1, and t; and so it is convenient to express the remaining characters in terms of these. The solution of the equations will then proceed as follows:

From (1), $$\mu_1 = \mu_0(\mu_0-1)-2\epsilon_0. \tag{7}$$

From (2), $$\rho = \epsilon_0(\mu_0-2)-3t. \tag{8}$$

Substituting in (4) from (8), we have

$$\nu_2 = 2\epsilon_0(\mu_0-2)-6t-2\epsilon_1. \tag{9}$$

Substituting in (6) from (9), we have

$$\mu_2 = \mu_0(\mu_0-1)^2-(7\mu_0-12)\epsilon_0+4\epsilon_1+15t. \tag{10}$$

From (3) and (8),

$$\kappa = \mu_0(\mu_0-1)(\mu_0-2)-3(\mu_0-2)\epsilon_0+3t. \tag{11}$$

From (5), (8), and (9),

$$\delta = \tfrac{1}{2}\mu_0(\mu_0-1)(\mu_0-2)(\mu_0-3)-2\epsilon_0(\mu_0-2)(\mu_0-3)- -2\epsilon_1-12t+2\epsilon_0(\epsilon_0-1). \tag{12}$$

Other formulae deduced from the equations are given in Ex. 1 below.

EXAMPLES

1. Deduce from the Cayley-Zeuthen equations the following formulae:

$$\kappa = \mu_2+2\nu_2-\mu_1; \qquad t = \binom{\mu_0}{3}-\tfrac{1}{2}\mu_1\mu_0+\tfrac{1}{3}(2\mu_1+2\nu_2+\mu_2);$$

$$\epsilon_1 = \mu_1\mu_0-\mu_1-\tfrac{5}{2}\nu_2-\mu_2; \qquad \rho = \mu_1\mu_0-\mu_1-2\nu_2-\mu_2.$$

2. Prove that the curve Δ has rank $R = \kappa+\mu_2$ and hence that $R = 2\mu_2+2\nu_2-\mu_1$. (If P is the vertex of the cone corresponding to Δ, consider the tangents to Δ which meet any line through P.)

3. Show from first principles that, for a surface of order μ_0, without singularities, $\kappa = \mu_0(\mu_0-1)(\mu_0-2)$, $\delta = \frac{1}{2}\mu_0(\mu_0-1)(\mu_0-2)(\mu_0-3)$.

4. Show that, for a cubic scroll in [3], $\mu_1 = 4$, $\nu_2 = 2$, $\kappa = 3$, and that the curve Δ is a rational quartic. Deduce also that a cubic scroll in [4] has no improper nodes.

5. Show that, for a quartic surface with a double conic, $\nu_2 = 4$, $\mu_2 = 12$, $\rho = 4$, $\kappa = 12$, $\delta = 4$. Deduce that the quartic surface, which is the complete intersection of two general quadrics in [4], has no improper nodes.

6. For Steiner's surface ($\mu_0 = 4$, $\epsilon_0 = 3$, $\epsilon_1 = 0$, $t = 1$) show that $\nu_2 = 6$, $\mu_2 = 3$, $\rho = 3$, $\kappa = 9$, $\delta = 0$; and that Δ is an elliptic sextic. Deduce the characters of Veronese's surface in [5] (see Ch. VII, § 3).

In Exx. 4–6, obtain the image of Δ in the plane representation of the surface.

7. Using the method outlined in the examples following § 2.13, prove that, if a non-ruled surface of [4] is projected from one of its improper nodes, then, with the usual notation, the projection has order μ_0-2, rank μ_1-4, class μ_2, and that the double curve contains ν_2-8 pinch-points.

8. By considering the (1, 2) correspondence between the points of a chord-curve of a surface situated in [4] and those of the double curve on its projection, prove that $2q-2 = 2(2p-2)+\nu_2$, where p, q are the respective genera of the double curve and the chord-curve.

9. From the fact that the Del Pezzo quintic can be projected into the surface of Ex. 5, deduce that its characters are $\mu_0 = 5$, $\mu_1 = 10$, $\mu_2 = 12$, $\nu_2 = 8$. Proceed similarly to obtain the projective characters of the other Del Pezzo surfaces (see Ch. VII, § 5).

§ 4. Some properties of Scrolls

4. In Ch. VII we encountered various types of rational scrolls and discussed certain of their properties. We shall now see that some of these properties are true of scrolls in general.

Let R denote a scroll, i.e. a non-developable ruled surface. Through a generic point P of R there passes a unique generator lying in the tangent plane to R at P. As P moves along the generator, the tangent plane will in general vary homographically with P; and if R is situated in $[r]$, where $r \geqslant 4$, the tangent planes to R at points of a generator will form a pencil in the solid joining the generator to its consecutive generator.

As regards the singularities of R, we make the same conventions as for non-ruled surfaces: if R is situated in $[r]$, where $r > 4$, we suppose that it is free from singular points. If R is situated in [4], it possesses in general a finite number of improper nodes, given by the formula of § 2.13. Through each such node there pass two generators, in general distinct. And if R is situated in [3], we assume it to possess normal singularities in the sense already defined.

4.1. Suppose now that R is a scroll in [3], of order μ_0 and rank μ_1, having normal singularities. Through a generic point of the double curve on R there pass two distinct generators, each defining a tangent plane there. At a pinch-point, however, the two generators coincide in a *torsal generator*† of R, along which the tangent plane to the surface does not vary. Through each triple point of the double curve there pass three generators, in general distinct from one another.

The following properties of R are particularly important:

I. *Each generator is met by $\mu_0 - 2$ others, at points of the double curve.*

This result has already been established in Ch. VII, § 4.

II. *The class of R is equal to its order, i.e. $\mu_2 = \mu_0$.*

For the tangent planes to R which pass through a given line l each contain a generator which meets l; so that the number μ_2

† The name *torsal generator* was given by Cayley to any generator of a scroll which meets its consecutive generator (necessarily in a pinch-point).

of such tangent planes is equal to the number of generators which meet l, i.e. μ_0.

III. *The number of pinch-points is* $2(\mu_1-\mu_0)$.

Let ϖ, ϖ' be any two planes through a given line l. On l we define a correspondence in the following manner. Through a point P of l there pass μ_1 tangents to the curve in which ϖ meets R; the generator of R through a point Q of contact meets ϖ' in a point Q', and the tangent there to the section of R by ϖ' will meet l in a point P'. The correspondence between P and P' is symmetrical, with indices μ_1; the coincidences arise (i) from the ν_2 generators at the pinch-points (for if QQ' is such a generator, the tangent plane is the same at all points of it); and (ii) from the points in which R meets l. When P is at one of these points, two points Q, and therefore also two points P', coincide with P. Hence $\nu_2 = 2\mu_1-2\mu_0$, as stated above.

4.2. *Projective characters of scrolls.* Next suppose that R is a generic scroll of $[r]$, where $r > 3$. It will project from a general vertex into a scroll of [3] having normal singularities, and to this the results of § 4.1 will apply. Whence

IV. *For a generic scroll situated in* $[r]$ $(r > 3)$, $\mu_2 = \mu_0$ *and* $\nu_2 = 2(\mu_1-\mu_0)$.

It follows that a generic scroll situated in any space is specified, as regards its elementary projective characters, by the order μ_0 and rank μ_1 alone.

4.3. *Representation on a quadric of* [5]. It is interesting to obtain these results from the representation of the lines of [3] by the points of a quadric Ω of [5] (see Ch. X). Suppose that R is a scroll of [3], possessing normal singularities; then to the generators of R correspond the points of a curve C lying on Ω; C is of order μ_0 and rank μ_1, and, moreover, is free from multiple points.

We first recall that, if two lines of [3] intersect, their representative points are such that their join lies entirely on Ω. To determine, then, how many generators of R meet a given generator g, we require to know how many generators of Ω can be drawn through the image point G of g to meet C elsewhere. All such generators of Ω lie in the tangent prime to Ω at G; and this prime, which is tangent to C also, therefore meets C in μ_0-2 points elsewhere.

Again, the pairs of consecutive generators of R at the pinch-points obviously correspond to tangents to C lying entirely on Ω.

Now, since C has rank μ_1, its tangents generate a scroll Σ of order μ_1, on which C is cuspidal; and since such tangents already meet Ω in two points, the residual intersection of Σ and Ω must consist of tangents to C which lie on Ω. Since the order of this residual intersection is $2\mu_1-2\mu_0$ it follows that $\nu_2 = 2(\mu_1-\mu_0)$.

Examples

1. If a scroll R is of genus p (this being also the genus of a prime section) show that $\nu_2 = 2(\mu_0+2p-2)$.

2. If R is a generic scroll of [4], show that the number d of improper nodes is $\frac{1}{2}\mu_0(\mu_0-5)-3(p-1)$.

Show also that at a general point of R there are μ_0-3 tangents which meet R elsewhere.

3. In the previous question, show that the trisecant curve based on a general point P of R is of order $2h-\mu_0+1$, and that it has a multiple point of order μ_0-3 at P.

Using the correspondence method of § 2.13, deduce that the projection of R from P possesses ν_2-2 pinch-points on the double curve.

4. Prove that, if a generic scroll of $[r]$ ($r > 4$), having characters μ_0, μ_1, μ_2, ν_2, is projected from a point of itself, the characters of the projection are μ_0-1, μ_1-2, μ_2-1, ν_2-2.

5. Show that the rational quartic scroll has characters $\mu_0 = \mu_2 = 4$, $\mu_1 = 6$, $\nu_2 = 4$, and that the rational quintic scroll has characters

$$\mu_0 = \mu_2 = 5, \quad \mu_1 = 8, \quad \nu_2 = 6.$$

4.4. *Curves on a scroll.* It has been shown in Ch. IV how to obtain the number of intersections of any two given curves on a scroll R. We denote by the symbol (α,β) a curve of order α on R which meets each generator in β points. If then (α',β') is any other curve on R we find that it meets the first curve in i points, where

$$i = \alpha\beta'+\alpha'\beta-\mu_0\beta\beta'. \tag{1}$$

We may use this formula to find the number of pinch-points (or the type) of R. Consider the curves Δ of contact introduced in § 2.01; there it was shown that any two such curves meet in $\mu_2+\nu_2$ points. Now Δ is a curve with the characters $\alpha = \mu_1$, $\beta = 1$; whence, from (1), $\mu_2+\nu_2 = 2\mu_1-\mu_0$, or

$$\nu_2 = 2(\mu_1-\mu_0). \tag{2}$$

Next we obtain a general formula for the genus ρ of a curve (α,β). For, considering the correspondence $(1,\beta)$ between the points of the curve and the generators of R, we have (Ch. IV, § 6)

$$2(\rho-1) = 2\beta(p-1)+\delta,$$

where δ is the number of generators which touch the curve and p is the genus of R. To find δ we consider the involutory correspondence $(\beta-1, \beta-1)$ between points of the curve lying on the same generator: as above, we derive the result

$$\delta = 2\alpha(\beta-1)-\mu_0\beta(\beta-1). \quad (3)$$

It follows that

$$2(\rho-1) = 2\beta(p-1)+(2\alpha-\mu_0\beta)(\beta-1). \quad (4)$$

EXAMPLE. Verify the formulae (1), (3), and (4) for curves lying on the rational scroll which is represented on the plane by the system of curves $C^n(O^{n-1})$.

§ 5. INTERSECTION PROBLEMS FOR SURFACES

5. The complete intersection of $r-2$ primals† $(r > 3)$. Let F denote the surface of intersection of $r-2$ primals of $[r]$ $(r > 3)$ whose equations are $f_i = 0$ $(i = 1, 2,..., r-2)$ and whose orders are $n_1, n_2,..., n_{r-2}$ respectively. We suppose that at each point of F the primals possess tangent primes and that these are linearly independent; then F will be irreducible and without singular points.

The order μ_0 and rank μ_1 of F are immediately found by applying §1.3 to a general prime section of the whole figure. Thus we have

$$\mu_0 = n_1 n_2 ... n_{r-2}, \qquad \mu_1 = n_1 n_2 ... n_{r-2}\left\{\sum_1^{r-2}(n_i-1)\right\}. \quad (1)$$

5.01. To determine the class μ_2 of F, we take two general primes A, B whose equations are

$$A \equiv \sum_0^r a_i x_i = 0, \qquad B \equiv \sum_0^r b_i x_i = 0,$$

and we consider the Jacobian manifold $J \equiv J(f_1, f_2, ..., f_{r-2}, A, B)$, of dimension $r-2$, whose equations are

$$\left\| \begin{matrix} \dfrac{\partial f_1}{\partial x_0} & . & . & . & \dfrac{\partial f_1}{\partial x_r} \\ \dfrac{\partial f_2}{\partial x_0} & . & . & . & \dfrac{\partial f_2}{\partial x_r} \\ . & . & . & . & . \\ a_0 & . & . & . & a_r \\ b_0 & . & . & . & b_r \end{matrix} \right\| = 0. \quad (2)$$

† The results of this and the following sections are due to Severi, *Mem. Accad. Torino*, (2), 52 (1903), 61.

This is evidently the locus of a point P such that the $r-2$ polar primes of P with respect to the primals f_i, together with A and B, form a linearly dependent set of r primes, i.e. meet in a line. If now P lies on F, this means that the tangent plane to F at P meets the secundum $A.B$ in a line. The class μ_2 of F is therefore equal to the number of intersections of J with F.

We write for convenience $m_i = n_i-1$ $(i = 1, 2,..., r-2)$, and we denote the order of J by $\phi_r(m_1, m_2,..., m_{r-2})$. In order to calculate this number we observe, first, that in the matrix (2) of r rows and $r+1$ columns, all the elements in the ith row $(i \leqslant r-2)$ are polynomials of degree m_i in the coordinates, while the elements of the last two rows are constants.

Extracting the two rth order determinants which have in common the first $r-1$ columns of (2), we see that

$$\phi_r(m_1, m_2,..., m_{r-2}) = (\sum m_i)^2 - \psi_r(m_1, m_2,..., m_{r-2}), \tag{3}$$

where $\psi_r(m_1, m_2,..., m_{r-2})$ denotes the order of the V_{r-2} represented by

$$\left\| \begin{matrix} \dfrac{\partial f_1}{\partial x_0} & \dfrac{\partial f_2}{\partial x_0} & . & . & . & \dfrac{\partial f_{r-2}}{\partial x_0} & a_0 & b_0 \\ . & . & . & . & . & . & . & . \\ \dfrac{\partial f_1}{\partial x_{r-2}} & \dfrac{\partial f_2}{\partial x_{r-2}} & . & . & . & \dfrac{\partial f_{r-2}}{\partial x_{r-2}} & a_{r-2} & b_{r-2} \end{matrix} \right\| = 0. \tag{4}$$

Dealing in a similar way with this matrix, we obtain

$$\psi_r(m_1, m_2,..., m_{r-2}) = (m_1+m_2+...+m_{r-3})(m_1+m_2+...+m_{r-4}+m_{r-2}) - \phi_{r-2}(m_1, m_2,..., m_{r-4}).$$

We thus have the relation

$$\phi_r(m_1, m_2,..., m_{r-2}) = (m_1+m_2+...+m_{r-2})^2 - (m_1+m_2+...+m_{r-3})(m_1+m_2+...+m_{r-4}+m_{r-2}) + \phi_{r-2}(m_1, m_2,..., m_{r-4}). \tag{5}$$

Now $\phi_3(m_1) = m_1^2$, and

$$\phi_4(m_1, m_2) = (m_1+m_2)^2 - m_1 m_2 = m_1^2+m_2^2+m_1 m_2.$$

Using these particular cases, we deduce from the recurrence relation (5) that

$$\phi_r(m_1, m_2,..., m_{r-2}) = \sum m_i^2 + \sum m_i m_j. \tag{6}$$

Hence, from (3),

$$\psi_r(m_1, m_2,..., m_{r-2}) = \sum m_i m_j. \tag{7}$$

It now follows that the class of F, i.e. the number of intersections of F with J, is, by Bézout's Theorem,

$$\mu_2 = n_1 n_2 ... n_{r-2}\{\sum (n_i-1)^2+\sum (n_i-1)(n_j-1)\}. \tag{8}$$

5.02. To determine the type ν_2 of F we have recourse to the curve Δ, already considered in § 2, which is the locus of points of contact of tangent planes to F which meet a given space α of dimension $r-3$. Consider now, in conjunction with Δ, the Jacobian of the $r-2$ primals f_i and three primes passing through a generic space β, of dimension $r-3$. The Jacobian, which is a primal of order $\sum (n_i-1)$, meets Δ in those points at which the tangent planes to F meet α and β.

Since, by § 2.01, the number of such planes is $\mu_2+\nu_2$, it follows that

$$\mu_2+\nu_2 = \mu_1\{\sum (n_i-1)\}. \tag{9}$$

Hence, by (1) and (8), we obtain the formula

$$\nu_2 = n_1 n_2 ... n_{r-2}\{\sum (n_i-1)(n_j-1)\}. \tag{10}$$

5.1. ***The intersection of $r-2$ primals in two surfaces*** ($r > 4$). Let F be a generic (non-singular) surface of $[r]$, where $r > 4$, having characters μ_0, μ_1, μ_2, ν_2; and let $f_i = 0$ ($i = 1, 2,..., r-2$) be the equations of $r-2$ primals of orders n_i which pass through F. These primals will in general have for residual intersection a surface F' whose characters μ_0', μ_1', μ_2', ν_2' we wish to determine.

By hypothesis, F is free from singular points, so that the primals f_i are not constrained to possess any singular points on F. If we suppose them to be of general character, the surface F' will also be without singularities. By taking a prime section of the whole figure, and applying § 1.3, we obtain for the order μ_0' and rank μ_1' of F' the expressions

$$\mu_0' = n_1 n_2 ... n_{r-2}-\mu_0, \tag{1}$$

$$\mu_1' = \mu_1+(n_1 n_2 ... n_{r-2}-2\mu_0)\sum (n_i-1). \tag{2}$$

It follows also from § 1.3 that F and F' intersect in a curve C whose order ϵ_0 is given by

$$\epsilon_0 = \mu_0 \sum (n_i-1)-\mu_1. \tag{3}$$

At any point P of C, the tangent planes to F and F' have in common the tangent to C, and the space [3] containing them is common to the $r-2$ tangent primes to f_i at P. We shall now show that F and F' have no common points other than those of C. For, supposing that F and F' meet at a point O not on C, consider a generic prime section of the figure through O; then the corresponding curve sections of F and F' will meet in only ϵ_0-1 points elsewhere; whence O must lie on C.

5.11. We proceed to find the remaining characters of F'; the formulae obtained involve certain auxiliary characters connected with the curve C. These are:

The rank ϵ_1 of C.

The order X of the V_4 generated by the ∞^1 spaces [3] common to the tangent primes to f_i at points of C.

The orders z, z' of the manifolds V_3 generated by the tangent planes to F and F' respectively at points of C. These characters are called the *classes of immersion* of C in F and F'.

Consider first the Jacobian of f_i and four primes A, B, C, D passing through a given space $[r-4]$. This is the V_{r-2} defined by

$$\left\| \begin{matrix} \dfrac{\partial f_1}{\partial x_0} & \dfrac{\partial f_2}{\partial x_0} & . & . & . & \dfrac{\partial f_{r-2}}{\partial x_0} & a_0 & b_0 & c_0 & d_0 \\ . & . & . & . & . & . & . & . & . & . \\ \dfrac{\partial f_1}{\partial x_r} & \dfrac{\partial f_2}{\partial x_r} & . & . & . & \dfrac{\partial f_{r-2}}{\partial x_r} & a_r & b_r & c_r & d_r \end{matrix} \right\| = 0. \qquad (4)$$

It is clear from § 5.01 that V_{r-2} is of order $\psi_{r+2}(m_1, m_2, \ldots, m_{r-2}, 0, 0)$, i.e. $\sum (n_i-1)(n_j-1)$. It meets F in the ν_2 points of contact of the tangent planes to F which meet $[r-4]$, and in the X points of contact of the spaces [3] above defined which meet $[r-4]$. Thus

$$\nu_2+X = \mu_0 \sum (n_i-1)(n_j-1). \qquad (5)$$

Similarly, we obtain

$$\nu_2'+X = \mu_0' \sum (n_i-1)(n_j-1). \qquad (6)$$

Hence, by subtracting (5) from (6),

$$\nu_2' = \nu_2+(n_1 n_2 \ldots n_{r-2}-2\mu_0) \sum (n_i-1)(n_j-1). \qquad (7)$$

5.12. Consider now the curve Δ and the Jacobian J already defined in § 5.02 in association with the spaces α and β of dimension $r-3$. J is of order $\sum (n_i-1)$, and contains the curve C, since at each point of C the tangent primes to f_i meet in a [3] instead of in a plane. Evidently J meets Δ in the $\mu_2+\nu_2$ points of contact of tangent planes to F which meet α and β, and in the z points common to C and Δ. Hence

$$\mu_2+\nu_2+z = \mu_1 \sum (n_i-1). \tag{8}$$

Similarly, $$\mu_2'+\nu_2'+z' = \mu_1' \sum (n_i-1). \tag{9}$$

5.13. Next, consider the ∞^1 point-pairs P, P' in which the tangent planes to F and F' with a common point of contact on C meet a fixed space Π of dimension $r-2$. Taking a fixed $[r-4]$, ϖ, in this space, we define a correspondence between the pencil of primes in Π which contain ϖ by associating with the prime (P,ϖ) the prime (P',ϖ). The indices of the correspondence are z and z'; the coincidences arise from the X spaces [3] which meet ϖ, and the ϵ_1 tangents to C which meet Π. Hence

$$X+\epsilon_1 = z+z'. \tag{10}$$

5.14. Finally, we apply a similar method to the tangent planes to F and the tangent primes to J along the common curve C. In the fixed space Π of dimension $r-2$, we associate the points P in which these tangent planes meet Π, with the spaces Σ, of dimension $r-3$, in which the tangent primes to J meet it. The former describe a curve of order z, and the latter a developable of class ζ, where ζ is the *class of immersion* of C in J, i.e. the number of tangent primes to J, at points of C, which pass through a given point Q. Since this is equal to the number of intersections of C with the first polar of Q with respect to J, we have

$$\zeta = \epsilon_0\{\sum (n_i-1)-1\}. \tag{11}$$

The points P and the spaces Σ are said to correspond when they are the respective traces on Π of a tangent plane to F and a tangent prime to J having the same point of contact. Thus in Π we have a curve of order z, whose points are in (1, 1) correspondence with the spaces Σ of a developable of class ζ. Hence, by Ch. IV, § 2.2, Ex. 4, it follows that the number of points P which lie in their corresponding spaces Σ is $z+\zeta$. The coincidences arise in two ways:

(i) from the ϵ_1 tangents to C which meet Π;

(ii) from the points of C at which the tangent planes to F lie in the corresponding tangent primes to J. Clearly these are the points of contact, in number z, of tangent planes to F, at points of C, which meet the space β. It follows that

$$z+\zeta = \epsilon_1+z.$$

Hence, from (11),

$$\epsilon_1 = \epsilon_0\{\sum n_i - r + 1\}. \tag{12}$$

5.15. *Solution of the equations.* From the above results we can obtain explicit formulae for all the characters of F' and C. The characters μ_0', μ_1', ν_2', ϵ_0 have already been given by equations (1), (2), (7), and (3) respectively. Next, to obtain ϵ_1 we substitute for ϵ_0 in (12), whence

$$\epsilon_1 = \{\mu_0 \sum (n_i - 1) - \mu_1\}(\sum n_i - r + 1). \tag{13}$$

It therefore remains to determine μ_2'. To this end we first add (8) and (9) and then substitute for $z+z'$ from (10). Hence

$$\mu_2' + \nu_2' = (\mu_1 + \mu_1') \sum (n_i - 1) - (\mu_2 + \nu_2) - X - \epsilon_1. \tag{14}$$

We now calculate X and ϵ_1 from (5) and (13); and since μ_1' and ν_2' have already been obtained, the value of μ_2' follows. The method of procedure is illustrated below.

EXAMPLES

1. Using the equations of § 5, show that two quadrics of [4], in general position, meet in a surface whose characters are

$$\mu_0 = 4, \quad \mu_1 = 8, \quad \mu_2 = 12, \quad \nu_2 = 4.$$

2. Calculate the characters of the surface common to a quadric and cubic primal of [4], in general position.

3. Show that three quadrics of [5], in general position, intersect in a surface whose characters are $\mu_0 = 8$, $\mu_1 = 24$, $\nu_2 = 24$, $\mu_2 = 48$.

4. Suppose now that the quadrics have a common plane F; the characters of F are $\mu_0 = 1$, $\mu_1 = \mu_2 = \nu_2 = 0$. From (1)–(3) above we have at once $\mu_0' = 7$, $\mu_1' = 18$, $\epsilon_0 = 3$. Thus C is a plane cubic; from (13) we obtain $\epsilon_1 = 6$, so that C is non-singular. From (7) it follows that $\nu_2' = 18$. And finally, in order to determine μ_2' we first use (5), which gives $X = 3$, and then substitute for X, ϵ_1, and ν_2' in (14), obtaining $\mu_2' = 27$.

5. Show that three quadrics of [5] which have a quadric surface in common meet residually in a sextic surface whose characters are $\mu_0 = 6$, $\mu_1 = 14$, $\mu_2 = 20$, $\nu_2 = 12$, and which meets the quadric surface in an elliptic quartic.

6. In the examples following § 3.6 it was shown that a Veronese surface of [5] has characters $\mu_0 = 4$, $\mu_1 = 6$, $\nu_2 = 6$, $\mu_2 = 3$. From the plane representation of the surface (see Ch. VII, § 3) prove that three quadrics of [5] can certainly be found to contain the surface, and that the residual intersection has the same characters.

7. Using the results of § 4, show that a rational quartic scroll in [5] has characters $\mu_0 = 4$, $\mu_1 = 6$, $\mu_2 = 4$, $\nu_2 = 4$. Deduce from the plane representation that three quadrics can be found to contain the surface, and find the characters of the residual intersection.

5.2. ***The case*** $r = 4$. Discussion of the case $r = 4$ proceeds on somewhat different lines: first, because a surface F of [4] possesses in general a finite number of improper nodes and, secondly, because —as we shall see—a primal containing it also possesses in general a finite number of nodes, not only at the nodes of F but elsewhere on F.

If f is any primal through F, then f will clearly have nodes at the d improper nodes of F; for if, as we shall in future assume, these nodes are of the first species (see § 2.11), the two tangent planes to F at such a node do not lie in a prime. We must, however, also suppose—and it will in general be the case†—that f possesses a number δ of additional nodes at *simple* points of F. For two conditions only must be satisfied if f is to have a node at an assigned simple point of F; and it is therefore to be suspected that these conditions will in fact be fulfilled at a finite number of points of F.

Suppose that f_1, f_2 are two primals, of orders n_1, n_2, passing through F and meeting residually in a surface F' which meets F in a curve C. At a node D of F, the complete intersection $F+F'$ will have a fourfold point, with quartic nodal cone $\mathscr{K}$ which is the intersection of the quadric nodal cones K_1, K_2 of f_1, f_2, at D. Plainly $\mathscr{K}$ will consist of the two tangent planes of F (these intersecting only at D) and of two further planes (likewise intersecting only at D) which form with the first two the quadrilateral of planes common to K_1, K_2. Hence F' has an improper node at D, with the second pair of planes as its tangent planes; and the curve C has four branches at D, with the edges of the quadrilateral as tangents, along which the two sheets of F' meet the two sheets of F.

At a general point of the curve C, the tangent planes to F and F' lie in a prime [3]. This means that f_1 and f_2 *touch along* C. We may thus expect the primals to have a number of nodes on C, at simple points of the curve; and we denote the numbers of such nodes for f_1 and f_2 by δ and δ' respectively. As in § 5.1, we can show that F and F' have no intersections external to C; and similarly we can show that if f_1 or f_2 has a node on F, it is either an improper node of F or a simple point of C.

† The simplest example is that of quadrics through a rational normal cubic scroll, which are all cones; and the reader will also recall, for example, that cubic primals through an elliptic quintic scroll ${}^1R^5$ have each ten nodes on the surface (cf. Ch. VIII, § 5.3).

5.21. Using the notation of the previous sections, we may now quote a number of results already obtained.

In the first place, the number d of improper nodes of F is given by

$$2d = \mu_0(\mu_0-1)-\mu_1-\nu_2. \tag{1}$$

Next, the order μ_0' and rank μ_1' of F' are given by

$$\mu_0' = n_1 n_2-\mu_0, \qquad \mu_1' = (n_1 n_2-2\mu_0)(n_1+n_2-2)+\mu_1. \tag{2}$$

The order ϵ_0 of C is given by

$$\epsilon_0 = \mu_0(n_1+n_2-2)-\mu_1. \tag{3}$$

We also have the equations

$$X+\epsilon_1 = z+z', \tag{4}$$

$$\left.\begin{aligned}\mu_2+\nu_2+z &= \mu_1(n_1+n_2-2)\\ \mu_2'+\nu_2'+z' &= \mu_1'(n_1+n_2-2)\end{aligned}\right\}. \tag{5}$$

Since F' has the same number of improper nodes as F, it follows from (1) that

$$\nu_2' = \nu_2+(n_1 n_2-2\mu_0)(n_1-1)(n_2-1). \tag{6}$$

5.22. Consider next the intersections of F with the first polars of a generic point P with regard to f_1 and f_2; of these, ν_2 are the points of contact of tangent planes to F which pass through P; X are points of contact of common tangent primes to f_1 and f_2 at points of C; and $2d$ lie at the improper nodes of F. Thus

$$\nu_2+X+2d = \mu_0(n_1-1)(n_2-1). \tag{7}$$

Hence, by (1), (2), and (3), we derive the formula†

$$X = \mu_0\mu_0'-\epsilon_0. \tag{8}$$

5.23. We consider now the intersections of C with the first polar of P with respect to f_1. Of these, X are points of contact of common tangent primes to f_1 and f_2; $4d$ lie at the improper nodes of F (since these are simple on the first polar and quadruple on C); and the remainder δ are at the nodes of f_1 which lie at simple points of C. Thus

$$X+4d+\delta = \epsilon_0(n_1-1). \tag{9}$$

For the final equation we have to introduce the character λ representing *the number of lines through a fixed point P which meet C and meet F again elsewhere.* If F is projected from P on [3], the

† This is a particular case of a formula, due to Pieri, for the number of intersections of two surfaces of [4] with a common curve; the number in question is $\mu_0\mu_0'-\epsilon_0-X$. See Baker, vi. 251.

double curve of the projected surface meets the projection of C in $\lambda+4d$ points so that, considering the first polar of an arbitrary point in [3], we have

$$z+\lambda+4d = \epsilon_0(\mu_0-1). \tag{10}$$

Similarly, the corresponding character λ' of F' is given by

$$z'+\lambda'+4d = \epsilon_0(\mu_0'-1). \tag{11}$$

Now consider the cone projecting C from a general point P; this meets f_1 in C and in a residual curve C' of order $\epsilon_0(n_1-1)$; and any intersection of C' and f_2, which does not lie on C, furnishes a line through P which meets C and F or F' again elsewhere.

The curves C and C' have the following intersections:

(i) at the δ nodes of f_1 which are simple for F;

(ii) at the d improper nodes of F, which are quadruple for C and double for f_1. These are evidently quadruple for C', and thus absorb eight of the intersections of C' and f_2;

(iii) at the X points of contact of common tangent primes to f_1 and f_2 which pass through P. These intersections count twice since C' touches f_2 there.

We thus obtain the relation

$$\lambda+\lambda'+\delta+8d+2X = \epsilon_0(n_1-1)n_2, \tag{12}$$

and this gives, by (9),

$$\lambda+\lambda'+X+4d = \epsilon_0(n_1-1)(n_2-1). \tag{13}$$

Hence, by (4), (10), (11), and (13), the value of ϵ_1 is given by

$$\epsilon_1 = \mu_0(n_1+n_2-2)(n_1+n_2-3)-2\mu_0(\mu_0-1)-\mu_1(n_1+n_2-5)+2\nu_2. \tag{14}$$

5.24. Knowing X and ϵ_1 we can now, by means of (5), express μ_2' in terms of the given characters.

From (9) we derive a formula for *the number of nodes which f_1 must have at simple points of F*, namely,

$$\delta = \mu_0\{n_1(n_1-2)-\mu_0\}-\mu_1 n_1+2(\mu_0+\mu_1+\nu_2). \tag{15}$$

Thus, in conclusion, the characters of F' and C, the immersion characters of C, and the number of nodes of f_1 (and, similarly, of f_2) are all determined by the above equations.

It is clear that the formula (15) is of general application in [4]; it gives the number of nodes which a primal of any order n_1 acquires, at simple points of a surface F, by being made to contain F.

EXAMPLES

1. Prove that, if the primals f_1 and f_2 have a common plane, their residual intersection has characters

$$\mu_0 = n_1 n_2 - 1, \quad \mu_1 = (n_1 n_2 - 2)(n_1 + n_2 - 2), \quad \nu_2 = (n_1 n_2 - 2)(n_1 - 1)(n_2 - 1),$$
$$\mu_2 = n_1 n_2 \{(n_1 - 1)^2 + (n_2 - 1)^2 + (n_1 - 1)(n_2 - 1) + 1\} - 3(n_1 + n_2 - 2)^2 - 1.$$

2. It follows from (1) that the intersection of two quadrics of [4], residual to a plane, is a cubic scroll. The quadrics, which are necessarily cones, each contain two systems of planes; one system meets the scroll in generators, and the other in conics.

3. Show that the intersection of a quadric and cubic primal of [4], residual to a plane, is a quintic surface having characters $\mu_0 = 5$, $\mu_1 = 12$, $\nu_2 = 8$, $\mu_2 = 20$. Show also that one system of planes of the quadric meets the surface in conics, and the other in cubics; and find the number of nodes possessed by the cubic primal.

4. If the quadric and cubic have in common two skew planes, show that the residual intersection is a rational quartic scroll, meeting all the planes of one system of the quadric in lines, and the planes of the other system in cubics with a node at the intersection of the two planes. Find the number of nodes on the cubic primal.

5. If through this scroll a pair of cubic primals are drawn, show that the residual intersection has characters $\mu_0 = 5$, $\mu_1 = 10$, $\nu_2 = 8$, $\mu_2 = 12$. This is a projected Del Pezzo quintic; it meets the scroll in a curve of order $\epsilon_0 = 10$, and rank $\epsilon_1 = 26$, having a quadruple point at the common node of the two surfaces. Each cubic primal has 7 nodes on this curve.

5.3. *Intersection of a surface and a primal.* There are two problems to consider in this connexion: (i) the intersection of a primal f with a surface F of $[r]$, when f and F are generally situated; and (ii) the intersection of f and F residual to a common curve C. Since the results for case (i) can be deduced from those for (ii) we shall only treat the latter.

Suppose that F is a generic surface of order μ_0 and rank μ_1, and that C is a non-singular curve, simple on f, of order ϵ_0 and rank ϵ_1. Let the residual intersection C' be of order ϵ_0' and rank ϵ_1', meeting C in i points. We have further to assume that the class x of immersion of C in F is given; its class of immersion in f—which we denote by ξ—is found at once by considering the intersection of C with a first polar of f. Thus, if f is of order n, we have

$$\xi = \epsilon_0(n-1). \tag{1}$$

The order of C' is given by the equation

$$\epsilon_0 + \epsilon_0' = \mu_0 n. \tag{2}$$

Also, if x', ξ' are the classes of immersion of C' in F, f respectively, we obtain, by the method of § 5.14, the relations

$$x+\xi = \epsilon_1+i, \qquad x'+\xi' = \epsilon_1'+i. \tag{3}$$

Next, we consider the curve Δ of contact of tangent planes to F which meet a given space $[r-3]$. This curve, which is of order μ_1, meets f in the points of contact of the tangent planes to F, at points of C and C', which meet the $[r-3]$. Hence

$$x+x' = \mu_1 n. \tag{4}$$

In a similar manner, considering the curve of contact of tangent primes to f which pass through a given point, we obtain the result

$$\xi+\xi' = \mu_0 n(n-1). \tag{5}$$

From (2) we now deduce ϵ_0'; from (1) and (3) we find i; and from (3), (4), and (5) the characters ϵ_1', x', ξ' may be deduced.

5.4. *The intersection of $r-1$ primals of $[r]$ with a common surface.* The preceding results may be applied to determine the curve of residual intersection of $r-1$ primals of $[r]$ which have a common surface F. If, for simplicity, we suppose that F is without nodes, even in the case $r = 4$, the results obtained will apply for all values of r (> 3).

Let $f_1, f_2,..., f_{r-1}$ be the primals in question; and let F' be the surface, residual to F, in which $f_1, f_2,..., f_{r-2}$ intersect. In the notation of § 5.1, we see that the required curve C' is the intersection of f_{r-1} with F', residual to the curve C common to F and F'. Now the projective characters ϵ_0 and ϵ_1 of C are known, and its class z' of immersion in F' may be found from § 5.12; its class of immersion in f_{r-1} is of course $\epsilon_0(n_{r-1}-1)$. Thus, from the previous section, ϵ_0' and ϵ_1' may be calculated.

Finally, we may calculate the *equivalence* of F for r primals $f_1, f_2,..., f_r$ which contain it. For, if the curves C, C' meet in i points, the equivalence N in question is given by the formula

$$N = \epsilon_0' n_r - i,$$

where the number i of points common to C, C' is found as in § 5.3.

EXAMPLES

1. Find the character x for a rational normal C^4 on a Segre surface.

2. A quadric primal intersects the rational normal scroll, represented by the plane curves $C^n(O^{n-1})$, in a generator; find the characters of the residual intersection.

3. Verify the above formulae for the case of three quadrics of [4] which have a common plane. What is the equivalence of this plane for four quadrics containing it?

4. Show that three cubic primals of [4] which have a Segre surface in common meet elsewhere in a septimic curve of genus 3. Hence find the equivalence of the surface for four cubic primals containing it.

5.5. *Table of surfaces.* We conclude this section by giving a list of generic surfaces in higher space, up to and including those of the sixth order, with their projective characters and other information. The following notation is employed:

${}^pF^{\mu_0}$ and ${}^pR^{\mu_0}$ denote surfaces and scrolls respectively, of order μ_0 and section genus p; r is the dimension of their normal space, and d the number of improper nodes possessed by the surface, or its general projection on [4]. If the latter surface can be obtained as the intersection of two primals, of orders m and n, residual to a surface ${}^{p'}F^{\mu_0'}$, we write ${}^pF^{\mu_0} = (m, n) - {}^{p'}F^{\mu_0'}$. For this purpose we denote a plane by F^1.

The results are shown in the accompanying table.

	μ_1	μ_2	ν_2	r	d	
${}^0R^3$	4	3	2	4	0	$(2, 2) - F^1$
${}^0R^4$	6	4	4	5	1	$(2, 3) - 2F^1$
${}^0F^4$	6	3	6	5	0	$(3, 3) - {}^1R^5$
${}^1F^4$	8	12	4	4	0	$(2, 2)$
${}^0R^5$	8	5	6	6	3	$(3, 3) - {}^0R^3 - F^1$
${}^1R^5$	10	5	10	4	0	$(3, 3) - {}^0F^4$
${}^1F^5$	10	12	8	5	1	$(3, 3) - {}^0R^4$
${}^2F^5$	12	20	8	4	0	$(2, 3) - F^1$
${}^0R^6$	10	6	8	7	6	$(3, 4) - {}^0R^6$
${}^1R^6$	12	6	12	5	3	$(3, 3) - 3F^1$
${}^1F^6$	12	12	12	6	3	$(3, 4) - {}^1R^6$
${}^2F^6$	14	20	12	5	2	$(3, 3) - {}^0F^2 - F^1$
${}^3F^6$	16	27	14	4	0	$(3, 3) - {}^0R^3$
${}^3F^6_1$	16	28	12	4	1	$(2, 4) - 2F^1$
${}^4F^6$	18	42	12	4	0	$(2, 3)$

We add a brief description of these surfaces.

(i) ${}^0R^{\mu_0}$ and ${}^1R^{\mu_0}$ are respectively rational and elliptic scrolls.

(ii) ${}^0F^4$ is Veronese's surface.

(iii) ${}^1F^{\mu_0}$ are Del Pezzo surfaces.

(iv) ${}^2F^{\mu_0}$ are of the species known as Castelnuovo surfaces,† which are represented on a plane by systems of nodal quartics; the two surfaces shown are given by $C^4(2, 1^7)$ and $C^4(2, 1^6)$ respectively.

† *Rend. Palermo*, **4** (1890); cf. Ch. VII, Ex. 30.

(v) ${}^3F^6$ is the Bordiga surface,† represented on a plane by the quartics $C^4(1^{10})$. It may also be defined by the system of equations

$$\left\| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & . & . & a_{24} \\ a_{31} & . & . & a_{34} \end{matrix} \right\| = 0,$$

in which a_{ij} are linear functions of the coordinates in [4].

(vi) ${}^3F_1^6$ is a rational surface, represented on the plane by the curves $C^5(3, 1^{10})$. It is the surface of lowest order which is normal in [4] and which possesses an improper node.

§ 6. The projective characters of a rational surface

6. Plane representation. We consider now, as in Ch. VI, the general rational surface F whose prime sections correspond to the curves of a given simple linear system (f) in the plane, and we propose to express the projective characters of F as characters of (f). We specify (f) as a system $C^n(k_i)$; but it is to be understood that (f) is not necessarily the complete system that would be so specified.

Let us suppose then, in the first place, that F is a generic surface of $[r]$, where $r > 3$. Its order μ_0 is given by

$$\mu_0 = n^2 - \sum k_i^2; \tag{1}$$

and, from the formula $\mu_1 = 2\mu_0 + 2p - 2$, we deduce that

$$\mu_1 = 3n(n-1) - \sum k_i(3k_i - 1). \tag{2}$$

This last formula may also be interpreted directly; for μ_1 is the order of a contact curve Δ, and equal therefore to the number of free intersections of a curve $C^n(k_i)$ with the Jacobian curve J of a net of curves of (f); and, by Ch. VI, § 5.1, J is a curve of the type $C^{3(n-1)}(3k_i - 1)$. The system (f) may, of course, possess fundamental curves, which would, by Ch. VI, § 5.12, be simple components of J; but since these have no free intersections with curves of (f), they do not affect the formula.

In what follows we shall assume, for the sake of simplicity, that the base points of (f) are all distinct; and we shall assume, furthermore, that the fundamental curves of (f), if any such exist, are all irreducible, and correspond only to *simple* points of F. Any fundamental curve of this simple type is called an *exceptional curve*

† *Mem. Accad. Lincei*, (4), 4 (1887); cf. Ch. VII, Ex. 31.

in the plane, just as any one of the curves on F which represents the neighbourhood of a base point of (f) is called an exceptional curve of F relative to the representation.

Suppose then that $\Omega \equiv C^{\alpha}(\beta_i)$ is an exceptional fundamental curve of the plane whose points represent (birationally) the points of the first neighbourhood of a simple point P on F. We have, plainly,

$$n\alpha - \sum k_i \beta_i = 0. \tag{3}$$

Also, since the section of F by a prime through P is represented by a curve $C^{n-\alpha}(k_i - \beta_i)$ and meets the neighbourhood of P in one point, it follows that

$$(n-\alpha)\alpha - \sum (k_i - \beta_i)\beta_i = 1, \tag{4}$$

whence, by (3),

$$\alpha^2 - \sum \beta_i^2 = -1. \tag{5}$$

Hence: *The virtual grade of an exceptional fundamental curve of (f) is -1.*

Since Ω is necessarily rational, we have

$$(\alpha-1)(\alpha-2) - \sum \beta_i(\beta_i - 1) = 0 \tag{6}$$

whence, by subtracting (5) from (6), we have

$$3\alpha - \sum \beta_i = 0. \tag{7}$$

Finally, if $C^{\alpha'}(\beta_i')$ is a second exceptional curve, we must have

$$\alpha\alpha' - \sum \beta_i \beta_i' = 0, \tag{8}$$

which expresses that the curves represent distinct points P, P' of F.

6.1. *Class and type of F.* We now proceed to find the remaining characters of F.

To find the class μ_2, we have to consider evidently the Jacobian set of a generic pencil of curves of (f), i.e. the set of (free) nodes of nodal curves of the pencil; for these will represent in general the points of contact with F of tangent primes in a given pencil. Now it was shown in Ch. VI, § 5.3 that the number δ of points in the Jacobian set is given by

$$\delta = 3(n-1)^2 - \sum (k_i - 1)(3k_i + 1); \tag{9}$$

and hence, if (f) possesses no fundamental curves, we have $\mu_2 = \delta$. However, if (f) possesses an exceptional fundamental curve $C^{\alpha}(\beta_i)$, this curve, together with a residual $C^{n-\alpha}(k_i - \beta_i)$, forms one of the δ nodal curves of the pencil; but this reducible curve will not in general represent the section of F by a tangent prime. Hence, if (f) possesses e such exceptional fundamental curves, we have

$$\mu_2 = 3(n-1)^2 - \sum (k_i-1)(3k_i+1) - e. \tag{10}$$

In order to determine the type ν_2 we observe, as in § 2.01, that two curves of contact, Δ, Δ', meet in $\mu_2+\nu_2$ points; we have thus to find the number of free intersections of the corresponding Jacobian curves, from which, however, the exceptional curves—fixed components of all Jacobian curves—must first be removed. If the e exceptional curves are $C^{\alpha^{(1)}}(\beta_i^{(1)})$, $C^{\alpha^{(2)}}(\beta_i^{(2)})$,..., we have

$$\mu_2+\nu_2 = \{3(n-1) - \sum \alpha^{(j)}\}^2 - \sum_i \Big(3k_i - 1 - \sum_j \beta_i^{(j)}\Big)^2$$

$$= 9(n-1)^2 - \sum (3k_i-1)^2 - 2\Big\{3(n-1)\sum \alpha^{(j)} - \sum_i (3k_i-1)\sum_j \beta_i^{(j)}\Big\} +$$

$$+ (\sum \alpha^{(j)})^2 - \sum_i \Big(\sum_j \beta_i^{(j)}\Big)^2.$$

Now, by (3) and (7),

$$3(n-1)\sum \alpha^{(j)} - \sum_i (3k_i-1)\sum_j \beta_i^{(j)} = -3\sum \alpha^{(j)} + \sum \sum_j \beta_i^{(j)}$$
$$= -e.$$

Also, by (5) and (8),

$$\sum (\alpha^{(j)})^2 - \sum_i (\sum \beta_i^{(j)})^2 = -e.$$

Thus
$$\mu_2+\nu_2 = 9(n-1)^2 - \sum (3k_i-1)^2 + e,$$

whence, by (10), we deduce the formula

$$\nu_2 = 6(n-1)^2 - 2\sum \{(3k_i+1)(k_i-1)+2\} + 2e. \tag{11}$$

6.2. *Rational surfaces in* [3]. We now assume that F is situated in [3], and that the surface possesses only normal singularities. In this case analogous reasoning will lead to the same formulae for μ_0, μ_1, and μ_2; while (11) now gives the number of pinch-points on the double curve. As regards the double curve Γ itself, we have seen in Ch. VI, § 5.21, that this is represented in the plane by the pairs of points of a curve γ of order $(\mu_0-4)n+3$, having a multiple point of order $(\mu_0-4)k_i+1$ at each k_i-fold base point of $C^n(k_i)$. We have also seen that each of the t triple points of F corresponds to a triad of double points on γ, such that any curve $C^n(k_i)$ passing through any one point of the triad passes through all three.

EXAMPLES

1. From the plane representation of Veronese's surface we deduce the characters $\mu_0 = 4$, $\mu_1 = 6$, $\mu_2 = 3$, $\nu_2 = 6$; whence, by the Cayley–Zeuthen equations, $t = 1$. The double curve of the projection on [3] (Steiner's surface) is represented by the sides of a triangle, whose vertices correspond to the triple point.

2. For the cubic scroll $C^2(O^1)$ we have $\mu_0 = \mu_2 = 3$, $\mu_1 = 4$, $\nu_2 = 2$. More generally, for the rational scroll represented by $C^n(O^{n-1})$ we have

$$\mu_0 = 2n-1 = \mu_2, \quad \mu_1 = 2(\mu_0-1), \quad \nu_2 = 2(\mu_1-\mu_0),$$

verifying the results of § 4.

3. For the Del Pezzo surfaces represented by $C^3(1^{9-n})$, show that $\mu_0 = n$, $\mu_1 = 2n$, $\mu_2 = 12$, $\nu_2 = 4(n-3)$. Show also that the octavic surface of the second species, which is represented by the system $C^4(2^2)$, has the same characters as the surface represented by $C^3(1)$.

4. Show that, for the projection of Segre's surface on [3], the double curve corresponds to a curve $C^3(1^5)$. Show also that, for the projection of the Del Pezzo quintic on [3], the double curve is an elliptic quintic with one triple point, corresponding in the plane representation to a curve $C^6(2^4)$ with a further triad of nodes.

5. Determine the characters of the Castelnuovo surfaces, corresponding to the systems of curves $C^4(2, 1^{8-n})$ $(0 \leqslant n \leqslant 8)$. Show that, for $n = 0$, the surface is a quartic, situated in [3], and possessing a double line.

6. Determine the characters of the Bordiga surfaces, corresponding to the systems $C^4(1^{11-n})$ $(0 \leqslant n \leqslant 11)$. Find the order and rank of the double curve in the case $n = 11$, and the curve corresponding to it in the plane.

7. Prove that, for a surface F of [3], the genera p and q of Γ and γ are connected by the relation $\nu_2 = 2(q-1)-4(p-1)$.

§ 7. THE NUMERICAL INVARIANTS OF A SURFACE

7. Arithmetic genus of a surface. In Chs. III—IV we defined the genus of a curve and established its invariance under birational transformation. If now we look for characters of a surface analogous to the genus of a curve, it is natural to begin by seeking to extend the definition of genus already given. At present, however, we can do no more than introduce such characters and illustrate some of their properties, leaving further discussion of them to Ch. XIII.

We first recall that the genus p of a plane curve C^n having ordinary k_i-fold points is defined by the equation

$$p = \tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\sum k_i(k_i-1). \tag{1}$$

We see from (1) that the number $p-1$ is the *virtual freedom* of the curves of order $n-3$ which are adjoint to C^n.

It is possible to define the genus of a surface in a similar way. If F^n is a surface of [3], it is found that the virtual freedom of the adjoint surfaces of order $n-4$, suitably defined, is a birational invariant; this is denoted by p_a-1, where p_a is called the *arithmetic* (or numerical) *genus* of F^n. For a surface having normal singularities it may be shown that, with the usual notation,

$$12(p_a+1) = 12\mu_0-8\mu_1+2\mu_2+\nu_2. \tag{2}$$

7.1. *The Zeuthen–Segre invariant.* A different line of approach is afforded by the definition of the rank of a curve. If C is a plane curve of order n and genus p, its rank r is given by the formula

$$r = 2(n+p-1). \tag{3}$$

Here r is the number of coincidences among the series of ∞^1 sets of points cut out on C by a pencil of lines with a generic vertex; in other words, r is the number of points in the Jacobian set of the series.

Now let F be a generic surface, situated in $[r]$ ($r \geqslant 3$), and consider on F a pencil of prime sections. The curves of the pencil have μ_0 base points, and μ_2 of the curves possess a node. We then define the character I by the equation

$$I = \mu_2-\mu_0-4p, \tag{4}$$

where p is the genus of a prime section. Since $2(p-1) = \mu_1-2\mu_0$, it follows that

$$I = \mu_2-2\mu_1+3\mu_0-4. \tag{5}$$

More generally, we may consider the pencil of curves cut out on F by any pencil $U-\lambda V = 0$ of primals of a given order, which may possibly have base elements on F. Assuming, for simplicity, that no curve of the pencil has a multiple part, we may compute the expression

$$\delta-\sigma-4p,$$

where δ is the number of nodal members of the pencil, σ the number of base points, and p the genus of a generic curve. It can be shown that this number is equal to I, and hence that it is invariant for all pencils of curves on F which satisfy the condition stated. I is called the *Zeuthen–Segre invariant* of F.

From the invariant property of I we might expect that, when F is transformed birationally into a surface F', the values of I, calculated for pencils of curves on F and F', would be the same. That this is not always true is seen by comparing the values, given below, of I for various rational surfaces, e.g. the plane, quadric, and cubic surfaces. A reason for the discrepancy is suggested if we calculate I for any generic rational surface F' and its plane representation F. Using the results of § 6 we have

$$I' = 3(n-1)^2 - \sum (k_i-1)(3k_i+1) - e - 6n(n-1) + $$
$$+2\sum k_i(3k_i-1) + 3(n^2 - \sum k_i^2) - 4 = -1+e'-e,$$

where e and e' are the numbers of exceptional curves on the two surfaces. It may be shown that a similar relation holds between any two surfaces in birational correspondence: that is, if I, I' denote the Zeuthen–Segre invariants of the surfaces F and F', and e, e' the numbers of exceptional curves on F, F', then

$$I-e = I'-e'. \tag{6}$$

Thus I is a *relative invariant* for birational transformation.

7.2. *The Castelnuovo–Enriques invariant.* We may obtain a third invariant in a manner which is not, however, suggested by analogy with the theory of curves. Consider the net of curves cut on F by primes passing through a given $[r-3]$. By § 3.6, Ex. 2, the curve Δ of contact of tangent primes belonging to the net has rank $R = 2\mu_2+2\nu_2-\mu_1$ and, since Δ is of order μ_1, its genus π is given by

$$2(\pi-1) = R-2\mu_1 = 2\mu_2+2\nu_2-3\mu_1. \tag{7}$$

We now define the character ω by the equation

$$\omega = 1+(\pi-1)-9(p-1), \tag{8}$$

where, as before, p is the genus of a generic curve of the net. We thus find that

$$\omega = 1+\mu_2+\nu_2-6\mu_1+9\mu_0. \tag{9}$$

More generally, for any net of irreducible curves on F, with analogous characters p, π, and with β base points, it can be shown that the expression

$$\omega = 1+(\pi-1)-9(p-1)+\beta \tag{10}$$

has a value independent of the net. It is generally known as the Castelnuovo–Enriques invariant of F, or the *virtual linear genus* of F.

By calculating the value ω' of this invariant for a rational surface F' we find, from § 6, that $\omega' = 10+e-e'$. And since, by (9), the value of ω for a plane is 10, this result may be expressed in the form

$$\omega'+e' = \omega+e. \tag{11}$$

Thus ω is a relative invariant for the rational surfaces considered.

In fact it may be shown that ω is a relative invariant for all birational transformations of a surface.

It follows from (6) and (11) that the expression $I+\omega$ is an *absolute* invariant for birational transformation. Moreover, from (2), (5) and (9) we obtain the identity

$$I+\omega = 12p_a+9. \tag{12}$$

EXAMPLES

1. Using equation (2), calculate the invariant p_a for the plane, quadric, and cubic surfaces. By means of § 6, calculate p_a for the generic rational surface. (These results suggest that $p_a = 0$ for all rational surfaces; this is actually so.)

2. Using the results of § 4, show that, for a generic scroll of genus p, $p_a = -p$.

3. Show that, for a non-singular surface F^n in [3], the invariant I is equal to $(n-2)(n^2-2n+2)$, and hence that for the plane, quadric, and cubic, I takes the values -1, 0, and 5 respectively.

4. Prove that, if I is calculated for a pencil of plane curves of order n, having an ordinary k-fold base point, the value -1 is obtained.

5. Show that, for a generic scroll of genus p, $I = -4p$.

6. Prove that, for a non-singular surface F^n in [3], $\omega-1 = n(n-4)^2$. (Thus $\omega-1$ is equal to the grade of the system of curves cut on F^n by surfaces of order $n-4$. A similar property of this invariant may be established for surfaces in general.)

7. If the net of curves is defined as the intersection with F^n of the surfaces, of order m, given by the equation $U+\lambda V+\mu W = 0$, then $p-1 = \frac{1}{2}mn(m+n-4)$. The Jacobian curve is the intersection of F^n with the Jacobian surface of F^n, U, V, and W. Hence

$$\pi-1 = n(3m+n-4)(3m+2n-8).$$

With these values it is found, as before, that $\omega-1 = n(n-4)^2$.

8. Show that, for a generic scroll of genus p, $\omega-1 = -8(p-1)$.

9. Calculate the invariants p_a, I, and ω for the surfaces given in § 5.5.

EXAMPLES ON CHAPTER IX

1. *Scrolls which are generated by secants to curves.* In [3], the lines which meet three general lines generate a quadric surface; or, what is the same thing, there are two lines which meet four given lines in general position. These facts may be used to determine the orders of certain scrolls, as the following examples show.

Ex. 1. Consider a curve C of [3], of order n, with h apparent nodes, and let l, l' be two lines in general position: it is required to find the order $N(C, l, l')$ of the scroll generated by the secants of C, l, and l'.

Evidently the order in question is the number of such secants which meet a third generic line l''; that is to say, the number of points in which C meets the scroll of secants to l, l', and l''; or, symbolically,

$$N(C,l,l') = nN(l'',l,l').$$

Since $N(l'',l,l') = 2$, we have $N(C,l,l') = 2n$.

It is clear that C is a simple directrix curve on the scroll. To find the multiplicity of, say l, on the scroll, we have to determine how many secants of C and l' pass through a general point P of l. Now the plane (P,l') meets C in n points, whence the required multiplicity is n.

Ex. 2. With a similar notation, consider two curves C, C' and a line l, all in general position. Then the order $N(C,C',l)$ of the scroll generated by secants of C, C', and l is the number of such lines which meet a generic line l'. Thus

$$N(C,C',l) = n'N(C,l',l)$$
$$= 2nn', \quad \text{by Ex. 1.}$$

Evidently the multiplicity of C on this scroll is n', while that of l is nn'.

Ex. 3. With a similar notation, show that $N(C,C',C'') = 2nn'n''$ and find the multiplicity of each curve on the scroll.

Ex. 4. Consider next the scroll generated by chords of C which meet a given line l; denote its order by $N(C,C,l)$. We first observe that the multiplicity of l on this scroll is h; now, taking any plane ϖ through l, we see that precisely $\binom{n}{2}$ generators lie in it. Hence, since ϖ meets the scroll in a composite curve of order $h+\binom{n}{2}$, we have

$$N(C,C,l) = h+\binom{n}{2}.$$

The multiplicity of C on the scroll is clearly $n-1$.

Ex. 5. Find the order $N(C,C,C')$ of the scroll of chords of C which meet C'; and determine the multiplicities of C and C' on it.

Ex. 6. Deduce that the number of chords of C which meet C' and C'', when the curves are in general position, is $n'n''\left[h+\binom{n}{2}\right]$.

Ex. 7. Find the number of lines which meet four given curves in general position. (Use Ex. 3.)

Ex. 8. Given a curve C of order n and with h apparent nodes, situated in $[r]$, show that the chords of C which meet a given space $[r-2]$ generate a scroll of order $h+\binom{n}{2}$. By considering the $(1,2)$ correspondence between C and the curve in which this scroll meets $[r-2]$, find the genus of the latter curve.

2. *Some formulae for surfaces in* [3]. Let f be a surface of [3], having normal singularities, for which we employ the notation of § 3. Let f' be a surface of order l passing simply through the double curve Γ of f; this implies that f' is constrained to have ordinary nodes at the t triple points of f, but is otherwise supposed to be free from singularities. Let C denote

the residual intersection of f and f'; suppose that it has order ϵ_0' and rank ϵ_1' and that it meets Γ in i points; at these points f and f' touch.

Evidently we have $\epsilon_0' = l\mu_0 - 2\epsilon_0$. In order to find ϵ_1' and i we consider, first, a correspondence (P, P') of points on a generic line g such that the tangent plane to f at a point of C passes through P while the tangent plane to f' at the same point passes through P'. The indices of the correspondence are the classes of immersion of C in f and f' respectively, i.e. $\epsilon_0'(\mu_0 - 1) - i$, and $\epsilon_0'(l-1)$. The coincidences of the correspondence arise from the ϵ_1' tangents to C which meet g, and the i common tangent planes to f and f'. Thus

$$\epsilon_0'(\mu_0 + l - 2) - i = \epsilon_1' + i.$$

Consider next an analogous correspondence defined by the tangent planes to f and f' at points of Γ. The classes of immersion of Γ in f and f' are ρ and $\epsilon_0(l-1) - 3t$ respectively, and the indices of the correspondence are therefore ρ and $2\epsilon_0(l-1) - 6t$. Of the coincidences, $2\epsilon_1$ arise from the tangents to Γ which meet g, and i from the common tangent planes to f and f'.

Hence

$$\rho + 2\epsilon_0(l-1) - 6t = 2\epsilon_1 + i.$$

Now, by the Cayley–Zeuthen equations (§ 3), we have

$$\rho = \epsilon_1 + \tfrac{1}{2}\nu_2, \quad \text{and} \quad 3t = \epsilon_0(\mu_0 - 2) - \rho.$$

Inserting these values, we obtain the results, due to Severi,†

$$\epsilon_1' = \mu_0 l(\mu_0 + l - 2) - 2\epsilon_0(3l - \mu_0) - 2\epsilon_1 - 3\nu_2,$$

$$i = 2\epsilon_0(l - \mu_0 + 1) + \epsilon_1 + \tfrac{3}{2}\nu_2.$$

The grade of the system of curves $|C|$, i.e. the number of intersections, not on Γ, of C with a surface f', is then

$$\nu = l\epsilon_0' - i = \mu_0 l^2 - 2\epsilon_0(2l - \mu_0 + 1) - \epsilon_1 - \tfrac{3}{2}\nu_2.$$

The genus p' of C is given by

$$2(p'-1) = \epsilon_1' - 2\epsilon_0'.$$

Ex. 1. Show that, when $l = \mu_0 - 3$, the characters of the system $|C|$, expressed in terms of the characters μ_0, μ_1, μ_2, ν_2 of f, are given by

$$\nu = 4(\mu_0 - \mu_1) + \mu_2 + \nu_2, \qquad 2(p'-1) = 10\mu_0 - 9\mu_1 + 2(\mu_2 + \nu_2).$$

Ex. 2. Show that, when $l = \mu_0 - 4$, p' is the Castelnuovo–Enriques invariant ω (§ 7.2), and that $\nu = \omega - 1$.

(*Note.* When the surfaces f' of order $\mu_0 - 3$ or $\mu_0 - 4$ do not exist, the numerical results just obtained may be interpreted as the characters of a *virtual* system of curves traced on the surface f.)

3. *Intersection of surfaces with a common multiple curve.* By similar methods we may find the characters of the intersection of two surfaces in [3], residual to a common multiple curve. Suppose that, with the previous notation, the curve Γ is without multiple points, and that it is r_1-fold on a surface of order n_1 and r_2-fold on a surface of order n_2. We

† Severi, op. cit. (§ 5).

then obtain for the characters of the residual intersection C the formulae

$$\epsilon_0' = n_1 n_2 - r_1 r_2 \epsilon_0,$$
$$i = \epsilon_0(r_1 n_2 + r_2 n_1 - 2r_1 r_2) - r_1 r_2 \epsilon_1,$$
$$\epsilon_1' = n_1 n_2(n_1+n_2-2) - \epsilon_0 r_1 r_2(n_1+n_2-2) + 2r_1 r_2 \epsilon_1 - -2\epsilon_0(r_1 n_2 + r_2 n_1 - 2r_1 r_2).$$

From these results it follows that three surfaces f_i of orders n_i $(i = 1, 2, 3)$ passing through Γ with multiplicities r_i $(i = 1, 2, 3)$ meet residually in a finite number N of points given by

$$N = n_3 \epsilon_0' - r_3 i$$
$$= n_1 n_2 n_3 - \epsilon_0(\sum n_i r_j r_k - 2r_1 r_2 r_3) + \epsilon_1 r_1 r_2 r_3.$$

4. *Degeneration methods for intersection problems.* We may obtain the latter formula more readily by a degeneration method due to Salmon and afterwards used by James.† We shall assume that the number N will be unaltered by any particular choice of the surfaces f_i, provided always that it remains finite. Let us suppose then that each surface f_i breaks up into r_i surfaces F_i of order α_i, each passing simply through the given curve, together with $n_i - r_i \alpha_i$ arbitrary planes ϖ_i. The number N is then given by the symbolical expression

$$N = \sum F_i F_j F_k + \sum F_i F_j \varpi_k + \sum F_i \varpi_j \varpi_k + \sum \varpi_i \varpi_j \varpi_k.$$

Now the number $F_1 F_2 F_3$ has already been obtained (cf. Ch. IV, § 8.4, Ex. 1), and the remaining terms are easily evaluated. When all the results are added together it will be found that the numbers α_i disappear, and the previous value for N is reobtained.

In view of the formal character of the work it can be argued that the surfaces F_i need not effectively exist; they may instead be considered as virtual entities.

Ex. Using the degeneration method, show that, in $[k]$, k primals of orders n_i, which contain an r_i-fold curve Γ $(i = 1, 2,..., k)$ meet elsewhere in N points, where

$$N = n_1 n_2 ... n_k - \epsilon_0(\sum n_1 r_2 ... r_k - \overline{k-1}\, r_1 r_2 ... r_k) + \epsilon_1 r_1 r_2 ... r_k.$$

Hence show that the residual curve of intersection of the first $k-1$ primals meets Γ in i points, where

$$i = \epsilon_0(\sum n_1 r_2 ... r_{k-1} - \overline{k-1}\, r_1 r_2 ... r_{k-1}) - \epsilon_1 r_1 r_2 ... r_{k-1}.$$

Deduce the rank ϵ_1' of the residual intersection.

5. *Composite surfaces in* $[r]$. Consider, in $[r]$ $(r > 4)$, a set of $k-1$ irreducible surfaces F^i $(i = 1, 2,..., k-1)$, each free from multiple points. Let F^i, F^j meet in a curve C^{ij} of order ϵ_0^{ij} and rank ϵ_1^{ij}, whose class of immersion in F^i is ${}^i z^{ij}$. Let X^{ij} denote the class of the developable of common tangent spaces [3] to F^i, F^j at points of C^{ij}. Suppose also that F^i, F^j have no isolated intersections, i.e. common points not on C^{ij}, but that C^{ij}, C^{jk}, C^{ki} meet in ξ^{ijk} points.

† *Proc. Camb. Phil. Soc.* **21** (1923), 435. For an account of the assumptions involved see Ch. XI.

Through this set of surfaces are drawn $r-2$ primals of orders n_i $(i = 1, 2, ..., r-2)$, meeting residually in an irreducible surface F^k.

Ex. 1. Denoting the characters of F^i by μ_0^i, etc., establish the following results:

$$\sum \mu_0^i = N, \qquad \mu_1^\alpha + \sum_i \epsilon_0^{\alpha i} = \mu_0^\alpha S,$$

$$\nu_2^\alpha + \sum_i X^{\alpha i} = \mu_0^\alpha T, \qquad X^{\alpha\beta} + \epsilon_1^{\alpha\beta} = {}^\alpha z^{\alpha\beta} + {}^\beta z^{\alpha\beta},$$

$$\mu_2^\alpha + \nu_2^\alpha + \sum_i {}^\alpha z^{\alpha i} = \mu_1^\alpha S,$$

$$\epsilon_1^{\alpha\beta} + \sum_i \xi^{\alpha\beta i} = (S-1)\epsilon_0^{\alpha\beta},$$

where $\alpha, \beta = 1, 2, ..., k$, and

$$N = n_1 n_2 ... n_{r-2}, \quad S = \sum (n_i - 1), \quad T = \sum (n_i - 1)(n_j - 1).$$

Ex. 2. By comparing these equations with those obtained in § 5.1 for the intersection of $r-2$ primals in two surfaces, show that the composite surface $F^1 + F^2 + ... + F^{k-1}$ may be regarded as a surface with virtual characters μ_0, μ_1, ν_2, μ_2 given by the formulae

$$\mu_0 = \sum_1^{k-1} \mu_0^i,$$

$$\mu_1 = \sum_1^{k-1} \mu_1^i + 2 \sum_{i,j=1}^{k-1} \epsilon_0^{ij},$$

$$\nu_2 = \sum_1^{k-1} \nu_2^i + 2 \sum_{i,j=1}^{k-1} X^{ij},$$

$$\mu_2 = \sum_1^{k-1} \mu_2^i + 3 \sum_{i,j=1}^{k-1} \epsilon_1^{ij} + 2 \sum_{i,j=1}^{k-1} \epsilon_0^{ij} - \sum_{i,j=1}^{k-1} X^{ij} + 6 \sum_{i,j,m=1}^{k-1} \xi^{ijm}.$$

What interpretation of the systems of tangent lines and planes to a limiting composite surface is suggested by those results?

Ex. 3. These results are due to Semple,† who has also obtained analogous formulae in the more difficult case $r = 4$. Here it must be supposed that each surface F^i has in general a finite number of improper nodes which may be nodes of other surfaces of the set. To make the situation clear consider two surfaces F^1, F^2 of [4], meeting in a curve C^{12} of known characters and in a number i^{12} of points not on C^{12}, which are simple for both surfaces. If two primals of orders n_1 and n_2 are drawn through the surfaces, their residual intersection F^3 will meet F^1, F^2 in curves C^{13}, C^{23}, and will have improper nodes at the i^{12} common points of F^1 and F^2; these will be double points of C^{13} and C^{23}. With regard to the other nodes, in order to obtain a completely symmetrical scheme, we assume they are distributed as follows:

On F^1: d^{12} common to F^1 and F^2; d^{13} isolated; i^{23} at simple points of F^2.

On F^2: d^{12} common to F^1 and F^2; d^{23} isolated; i^{13} at simple points of F^1.

It follows that we shall have

On F^3: d^{13} nodes common with F^1; d^{23} common with F^2; i^{12} lying at simple points of F^1 and F^2.

† *Proc. Roy. Irish Acad.* **41** (1933), 70.

The curve C^{12} will have d^{12} 4-fold points at the common nodes of F^1 and F^2, and $i^{13}+i^{23}$ double points at nodes of either of these surfaces which are simple on the other. Similar results hold for C^{13} and C^{23}.

Finally, the three curves, and hence also the three surfaces, meet in a certain number ξ of points which are simple on all of them.

There are seven sets of equations between the various characters, of which the following are typical. (The notation is as before.)

$$\mu_0^1+\mu_0^2+\mu_0^3 = n_1 n_2,$$
$$\mu_1^1+\epsilon_0^{12}+\epsilon_0^{13} = \mu_0^1(n_1+n_2-2),$$
$$X^{23}+i^{23} = \mu_0^2\mu_0^3-\epsilon_0^{23},$$
$$\nu_2^1+2(d^{12}+d^{13}+i^{23}) = \mu_0^1(\mu_0^1-1)-\mu_1^1,$$
$$X^{23}+\epsilon_1^{23} = {}^2z^{23}+{}^3z^{23},$$
$$\mu_2^1+\nu_2^1+{}^1z^{12}+{}^1z^{13} = \mu_1^1(n_1+n_2-2);$$
$$(n_1+n_2-3)\epsilon_0^{23}-4d^{23}-2i^{12}-2i^{13} = \epsilon_1^{23}+\xi.$$

Ex. 4. Deduce from the previous equations that the composite surface $F^1+F^2+F^3$ may be regarded as having the virtual characters μ_0, μ_1, ν_2, and μ_2, where

$$\mu_0 = \mu_0^1+\mu_0^2+\mu_0^3,$$
$$\mu_1 = \mu_1^1+\mu_1^2+\mu_1^3+2(\epsilon_0^{23}+\epsilon_0^{31}+\epsilon_0^{12}),$$
$$\nu_2 = \sum\nu_2^i+4\sum(d^{ij}+i^{ij})+2\sum X^{ij},$$
$$\mu_2 = \sum\mu_2^i+2\sum\epsilon_0^{ij}+3\sum\epsilon_1^{ij}-\sum X^{ij}+6\xi+4(\sum d^{ij}+\sum i^{ij}).$$

Ex. 5. These results admit of an obvious extension to the case of a composite surface in [4] having $k-1$ components; and again, they may be applied to find the virtual characters of a surface consisting entirely of planes, provided they have the kind of intersections contemplated above. For further details Semple's paper may be consulted.

Another problem that arises in this connexion is the following. Suppose that an irreducible surface F tends to a limiting form F^1+F^2, where F^1 and F^2 are irreducible surfaces: then it is required to determine the limiting form of the system of tangent planes to F. This problem has been solved† in the simple case where F is a generic surface in $[r]$ ($r > 3$) and where F^1 and F^2 intersect in a non-singular curve C. In the previous notation, it may be shown that the limiting envelope consists of

(i) the tangent planes to F^1 and F^2;

(ii) the tangent planes to C which are cospatial with the tangent planes to F^1 and F^2, these planes being counted twice;

(iii) a number $N = 2\epsilon_0+\epsilon_1-X$ of solid point-stars whose centres lie at N distinct points of C, the position of these points depending upon the particular passage to the limit.

Roth has also‡ discussed the limiting envelope of a non-singular surface of [3] which breaks up into a set of planes in general position.

Analogous problems concerning manifolds of higher dimension are treated in a paper by B. Segre.§

† Roth, *Proc. Camb. Phil. Soc.* **29** (1933), 88. ‡ Ibid. **28** (1932), 45.
§ *Proc. Lond. Math. Soc.* (2) **47** (1942), 351.

6. *The apparent nodes of a surface in* [5]. A generic surface F situated in [5] has a finite number of apparent nodes, due to the chords of F which pass through a general point. If this number is zero, the ∞^4 chords of F will generate a primal V_4 which meets a generic line l in a finite number ν of points P_i; and since no chord of F passes through a generic point of l, it follows that through each of the points P_i there passes a cone of chords. It may be proved† that this cone is in fact a plane pencil; intuitively the result is clear since, if the cone were of order $\mu > 1$, any prime through l would meet F in a curve having the property that, of its $\mu\nu$ chords which meet l, μ pass through each of the points P_i. Hence V_4 is generated by ∞^3 planes, each of which meets F in a curve; and it may be shown that this property is characteristic of the Veronese surface (see Ch. VII, § 3.11).

The above result is due to Severi,‡ who has also determined the types of surface for which $d = 1$. These are (a) the rational quartic scroll, and (b) the Del Pezzo quintic surface.

7. *The apparent triple points of a surface in* [4]. As remarked in § 2, a surface F of [4] has in general ∞^3 trisecants, of which a finite number t will pass through a general point. Unless F is a cubic scroll or Segre surface (in which cases it lies on an infinity of quadrics) it will certainly possess trisecants; if, then, $t = 0$, it follows from a theorem of Severi (see the Examples on Ch. I) that the trisecants must generate a primal which is either a quadric or locus of ∞^1 planes, and that in the latter case the planes will meet F in curves of order 3 at least.

Conversely, it is obvious that if F lies on a quadric, $t = 0$. Since surfaces of all orders can be found to lie on a quadric, it follows that there are surfaces of all orders having $t = 0$. Some familiar examples of such surfaces are:

(i) ${}^0R^4$; in this case the trisecants lie in one system of planes of a quadric cone, and the planes meet the scroll in cubics having nodes at the improper node of ${}^0R^4$. This follows from the definition of the scroll as the partial intersection of a quadric and cubic primal (§ 5.5), or from the representation on the plane by curves $C^3(2, 1)$;

(ii) the elliptic scroll ${}^1R^5$; the trisecants lie in ∞^1 planes which generate a V_3^5, normal in [4], the planes meeting the scroll in cubics;

(iii) the Castelnuovo surface ${}^2F^5$, which is the partial intersection of a cubic primal and a quadric cone; the planes of one system of the cone meet the surface in cubics, and those of the other in conics.

The surfaces for which $t = 1$ have been determined by Ascione§ and Severi (op. cit.). There are four distinct types:

(i) the projection of Veronese's surface,
(ii) the projection of the Del Pezzo quintic,
(iii) the projection of the rational quintic scroll,
(iv) the Bordiga sextic surface, represented on the plane by the curves $C^4(1^{10})$.

† See Bertini, *Introduzione*, ch. xv. ‡ *Rend. Palermo*, **15** (1901), 33.
§ *Rend. Lincei*, (5) **6** (1897)$_1$, 162.

8. *Some properties of scrolls.* By means of the table given in § 5.5 and the Cayley–Zeuthen equations, establish the following results:

Ex. 1. The projection of ${}^0R^4$ on [3] has for double curve a twisted cubic.

Ex. 2. The scroll ${}^1R^5$ in [4] is the double surface on a V_3^5 with elliptic curve sections, which is the locus of ∞^1 planes, the planes meeting ${}^1R^5$ in cubic curves. The projection of ${}^1R^5$ on [3] has for double curve an elliptic quintic, on which there are 10 pinch-points.

Ex. 3. The projection of ${}^0R^5$ on [4] possesses three improper nodes, the plane of the nodes meeting the scroll in a trinodal quartic, which is a directrix of the scroll. The projection of ${}^0R^5$ on [3] has for double curve an elliptic sextic, on which there are one triple point and six pinch-points.

Ex. 4. The projection of ${}^1R^6$ on [4] possesses three improper nodes and contains three binodal plane quartics whose nodes are situated at those of the scroll. ${}^1R^6$ may be obtained as the intersection of three quadric primals of [5], residual to two skew planes. These planes meet ${}^1R^6$ in directrix plane cubics; and, conversely, ${}^1R^6$ may be defined as the locus of joins of corresponding points of two generically situated birationally related plane cubic curves in [5].

9. *Matrix curves and surfaces.*

Ex. 1. Show that the curve represented by the equation

$$\left\| \begin{matrix} a_m & a_{m+h} & a_{m+k} \\ a_n & a_{n+h} & a_{n+k} \end{matrix} \right\| = 0,$$

where a_r is a polynomial of degree r in the coordinates (x_0, x_1, x_2, x_3), is of order

$$N = m^2+mn+n^2+(h+k)(m+n)+hk,$$

and that it possesses H apparent nodes, where

$$H = \tfrac{1}{2}mn(m-1)(n-1)+\tfrac{1}{2}(N-mn)(m+n+h-1)(m+n+k-1).$$

Determine, for $N \leqslant 10$, what non-singular curves can be represented in this way.

Ex. 2. Show that the surface F_n represented by the vanishing of all minors of order n in a matrix of n rows and $n+1$ columns, whose elements are linear functions of coordinates in [4], has projective characters

$$\mu_0 = \tfrac{1}{2}n(n+1), \quad \mu_1 = \tfrac{2}{3}n(n-1)(n+1), \quad d = 0,$$
$$\nu_2 = \tfrac{1}{12}n(n^2-1)(3n-2), \quad \mu_2 = \tfrac{1}{3}n(n^2-1)(5n-6).$$

Show also that F_2 is a cubic scroll, F_3 a Bordiga sextic, and that the Jacobian of four quadric primals is an F_4.

Prove that each surface F_n is residual to a surface F_{n-1} with respect to two primals of order n.

RECOMMENDED FOR FURTHER READING

Baker, *Principles of geometry*, vi.

CHAPTER X

THE GEOMETRY OF LINE SYSTEMS

Introduction. Up to the present stage in this book we have confined our attention almost exclusively to the study of algebraic point-systems (loci) and to the dual concept of algebraic prime-systems (envelopes). There is, however, a rich field of investigation which deals with systems of subordinate spaces S_k of any projective space S_r. More particularly, there arises the question as to whether there exists for these systems any fundamental coordinate representation which would enable us to define linear systems of such S_k, analogous to the primes, secunda, and other linear systems of points of S_r.

In the present chapter we give some preliminary account of these matters, beginning, naturally enough, with the simplest case, namely, that of line geometry in ordinary space. We first introduce, and then develop briefly, the concept of line coordinates of S_3. In most elementary treatises on coordinate geometry of three dimensions some account is given of these coordinates and their chief properties; so our treatment, though logically self-contained, omits many of the detailed explanations and illustrations which would otherwise be required.

§ 1. LINE GEOMETRY IN S_3

1. The coordinates of a line. If $(x) = (x_0, x_1, x_2, x_3)$ and $(y) = (y_0, y_1, y_2, y_3)$ are two distinct points of S_3, their join is a determinate line p. Consider now the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}$$

of the coordinates of the two points; it is clear that, if (x) and (y) are replaced by $(x') = a(x)+b(y)$ and $(y') = c(x)+d(y)$ respectively, then each determinant $|x'\,y'|_{ij}$ of the new matrix is equal to the product of the corresponding determinant $|x\,y|_{ij}$ by the constant $ad-bc$. It follows that the ratios of the determinants of the matrix are unaltered if the points (x), (y) of p are replaced by any other two points of p: that is to say, they are functions of the line p itself.

If we write $|x\,y|_{ij} = \delta_{ij}$, and take

$$p_1 = \delta_{01}, \quad p_2 = \delta_{02}, \quad p_3 = \delta_{03}, \quad p_4 = \delta_{23}, \quad p_5 = \delta_{31}, \quad p_6 = \delta_{12}, \tag{1}$$

then the six quantities p_i may be called the *homogeneous coordinates* of the line p relative to the usual frame of reference.

Since the lines of S_3 have only freedom four, the six coordinates p_i cannot be independent; in fact they are connected by the *fundamental homogeneous quadratic identity*

$$\Omega_{pp} \equiv p_1 p_4 + p_2 p_5 + p_3 p_6 = 0. \tag{2}$$

This follows at once from Laplace's expansion, in terms of minors of the first two rows, of the vanishing determinant

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{vmatrix}.$$

Thus the coordinate representation of the lines of S_3 is by means of such homogeneous six-vectors (p_i) as satisfy the relation (2). This result corresponds to the essentially *non-linear* character of the totality of these lines; for whereas two generic points of space define a unique line which is their 'join', two generic lines of S_3 define no corresponding simply-infinite family of lines which could in any sense be described as their 'join'.

It may now readily be verified that any non-null vector (p_i) which satisfies (2) is indeed the vector of a unique line p; the four points in which p meets the faces of the tetrahedron of reference have coordinates

$$\begin{array}{cccc} (\ 0, & p_1, & p_2, & p_3), \\ (-p_1, & 0, & p_6, & -p_5), \\ (-p_2, & -p_6, & 0, & p_4), \\ (-p_3, & p_5, & -p_4, & 0\ ). \end{array} \tag{3}$$

Also the four planes joining p to the vertices of this tetrahedron have coordinates

$$\begin{array}{cccc} (\ 0, & p_4, & p_5, & p_6), \\ (-p_4, & 0, & p_3, & -p_2), \\ (-p_5, & -p_3, & 0, & p_1), \\ (-p_6, & p_2, & -p_1, & 0\ ). \end{array} \tag{4}$$

The *dual coordinates* π_i of the line p, defined as the determinants constructed from the coordinates $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $(\beta_0, \beta_1, \beta_2, \beta_3)$ of any two planes through p, are connected with the ordinary coordinates by the relations

$$\frac{\pi_1}{p_4} = \frac{\pi_2}{p_5} = \frac{\pi_3}{p_6} = \frac{\pi_4}{p_1} = \frac{\pi_5}{p_2} = \frac{\pi_6}{p_3}. \tag{5}$$

This follows at once by identifying the points (3) or the planes (4) with the analogous points or planes which would arise in the dual procedure. Thus the two processes lead essentially to the same coordinate representation of the lines of S_3.

1.1. *Intersecting lines.* If (p_i) and (q_i) are the coordinates defined by the point-pairs (x), (y) and (X), (Y) respectively, then the equation

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ X_0 & X_1 & X_2 & X_3 \\ Y_0 & Y_1 & Y_2 & Y_3 \end{vmatrix} = 0$$

is evidently the necessary and sufficient condition that the two lines should intersect; if the determinant is expanded by Laplace's method, this condition clearly takes the form

$$\Omega_{pq} \equiv \sum (p_1 q_4 + q_1 p_4) = 0. \tag{6}$$

This is the polarized form of the fundamental relation $\Omega_{pp} = 0$. The expression Ω_{pq} on the left is called the *mutual invariant* of p and q. We thus have

THEOREM I. *Two lines p, q intersect if and only if their mutual invariant Ω_{pq} vanishes.*

It will be noticed that the condition $\Omega_{pq} = 0$ is linear in the coordinates of each of the lines p and q.

1.2. *Linearly related lines.* If (p) and (q) are two arbitrary homogeneous vectors of the type $(p_1,\ldots,p_6)$, then the vector $\lambda(p)+\mu(q)$ satisfies the fundamental relation (2) if and only if

$$\lambda^2\Omega_{pp} + 2\lambda\mu\Omega_{pq} + \mu^2\Omega_{qq} = 0. \tag{7}$$

This is true for *all* values of the ratio $\lambda:\mu$ if and only if

$$\Omega_{pp} = \Omega_{qq} = \Omega_{pq} = 0, \tag{8}$$

i.e. if (p) and (q) represent actual intersecting lines p and q. When these conditions are satisfied, all the lines $\lambda(p)+\mu(q)$ generate the *plane pencil* defined by p, q; for each of them clearly meets every line incident to p and q.

Again, the condition that $\lambda(p)+\mu(q)+\nu(r)$ should represent a line is plainly

$$(\Omega_{pp}, \Omega_{qq}, \Omega_{rr}, \Omega_{qr}, \Omega_{rp}, \Omega_{pq})(\lambda, \mu, \nu)^2 = 0. \tag{9}$$

This condition is true identically if and only if (p), (q), (r) represent three actual lines p, q, r which all intersect each other, i.e. concur in a point P or lie in a plane π. If such is the case, every line $\lambda(p)+\mu(q)+\nu(r)$ meets every line incident to p, q, and r; this means that the system of lines either forms a *point-star*, with vertex P, or else fills the *plane* π.

If the vectors (p), (q), (r) represent three actual lines p, q, r which do not intersect each other, then (9) reduces to

$$\Omega_{qr}\mu\nu+\Omega_{rp}\nu\lambda+\Omega_{pq}\lambda\mu = 0.$$

In this case, therefore, there are only ∞^1 lines which are linearly dependent on p, q, r, and these generate the *regulus* which contains p, q, r, since each of them meets every line which meets p, q and r. Hence we have

THEOREM II. *If p, q are two intersecting lines, then any linear combination of their coordinate vectors represents a line of the pencil defined by them; and if p, q, r are three lines which meet in a point P or which lie in a plane π, then every linear combination of their coordinate vectors represents a line through P or a line of π respectively. If p, q, r do not intersect each other, there exist only ∞^1 lines which are represented by linear combinations of their coordinate vectors, and these generate a regulus to which p, q, r belong.*

We may add that systems of lines which are linearly dependent on k given lines ($k = 4, 5$) are best considered in relation to the linear conditions thereby imposed on their coordinates.

Any line is linearly dependent on six given lines of general position.

1.3. *The linear complex.* A *complex* of lines in S_3 is an algebraic system of ∞^3 lines whose coordinates satisfy a single equation $F(p_i) = 0$, which is independent of the fundamental identity. If F is of degree n, the complex is of *order* n; if $n = 1$, the complex is said to be *linear*.

The equation of any linear complex $\mathscr{L}$ may be written in the form

$$\sum (\alpha_4 p_1 + \alpha_1 p_4) = 0,$$

where $\alpha_1,\ldots,\alpha_6$ are constants. In the particular case where $\Omega_{\alpha\alpha} = 0$, the coefficients $\alpha_1,\ldots,\alpha_6$ are clearly the coordinates of a certain line α, and this is met by every line of $\mathscr{L}$. In this case, then, $\mathscr{L}$ consists of all the lines which meet a fixed line; the linear complex is then called *special*, and the fixed line is its *axis*.

In the general case, the lines of $\mathscr{L}$ which pass through any given point $P(x_i)$ form a pencil in the plane whose equation is

$$\sum \{\alpha_4(x_0 X_1 - x_1 X_0) + \alpha_1(x_2 X_3 - x_3 X_2)\} = 0,$$

i.e. the plane whose coordinate vector $\mathbf{u} = (u_0, u_1, u_2, u_3)$ is given by $\tilde{\mathbf{u}} = \mathbf{T}\mathbf{x}$, where $\tilde{\mathbf{u}}$ is the transpose of $\mathbf{u}$ and $\mathbf{T}$ is the skew-symmetric matrix†

$$\begin{pmatrix} 0 & \alpha_4 & \alpha_5 & \alpha_6 \\ -\alpha_4 & 0 & \alpha_3 & -\alpha_2 \\ -\alpha_5 & -\alpha_3 & 0 & \alpha_1 \\ -\alpha_6 & \alpha_2 & -\alpha_1 & 0 \end{pmatrix}.$$

This plane is called the *polar plane* of P with respect to the complex $\mathscr{L}$. Dually, the lines of $\mathscr{L}$ which lie in a given plane $\pi(u_i)$ pass through the *pole* P of the plane, given by $\mathbf{x} = \mathbf{T}^{-1}\tilde{\mathbf{u}}$. Hence

THEOREM III. *The general linear complex generates a skew-polarity (P, π) of space; and, conversely, the* self-polar *lines of a skew-polarity of space constitute a linear complex.*

The linear complexes of S_3, being representable by the totality of homogeneous coordinate vectors (α_i), form an ∞^5 system. It follows that there is a unique linear complex which contains five general lines of S_3.

1.31. *The general complex.* If $F(p_i) = 0$ is the equation of a general complex Γ of order n, then the lines of Γ which pass through a point $P(x_i)$ form a cone of order n; its equation is derived by replacing the quantities p_i in the equation of Γ by the appropriate determinants of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ X_0 & X_1 & X_2 & X_3 \end{pmatrix}.$$

This cone is called the *complex cone* of Γ.

† This matrix is non-singular if $\mathscr{L}$ is not special.

Dually, the lines of Γ which lie in a plane envelop a curve of class n, which is called a *complex curve* of Γ.

1.4. *Line congruences.* A congruence of lines of S_3 is an algebraic system of ∞^2 lines; of these, a finite number m, say, pass through a generic point, and a finite number n lie in a generic plane. We denote such a congruence by the symbol $K^{(m,n)}$, and call m, n the *indices* of the system: m is also called the *order*, and n the *class*, of K.

If K consists of all the lines common to two complexes, of orders μ and ν respectively, then evidently $m = n = \mu\nu$; such a congruence is called a *complete intersection*. It will appear from the discussion in § 2.3 that in general congruences are not complete intersections.

There are three congruences of particularly simple types:

(i) The *point-star*, with indices $(1, 0)$;
(ii) The *ruled plane*, with indices $(0, 1)$;
(iii) The *linear congruence* $(1, 1)$, which is the complete intersection of two linear complexes.

Suppose that $K^{(1,1)}$ is the intersection of the linear complexes whose equations are

$$\sum (\alpha_4 p_1 + \alpha_1 p_4) = 0, \qquad \sum (\beta_4 p_1 + \beta_1 p_4) = 0.$$

Then $K^{(1,1)}$ is contained in every complex of the pencil

$$\sum \{(\alpha_4 + \lambda\beta_4)p_1 + (\alpha_1 + \lambda\beta_1)p_4\} = 0.$$

We observe that in general two complexes of this pencil are special, namely, those given by the roots of the equation

$$\sum (\alpha_1 + \lambda\beta_1)(\alpha_4 + \lambda\beta_4) = 0.$$

Hence, if these roots are unequal, $K^{(1,1)}$ consists of all the lines meeting the axes of the two special complexes in question. This gives

THEOREM IV. *A linear congruence $K^{(1,1)}$ consists, in general, of all the lines which meet two given lines.*

These two fixed lines are called the *directrices* of $K^{(1,1)}$. In particular cases, of course, they may coincide.

§ 2. THE REPRESENTATION ON S_5

2. The quadric Ω. We next proceed to develop for our line geometry the kind of representation which is found to be almost indispensable to S_k-geometry in general: that is to say, its reduction to the ordinary geometry of points, with their familiar terminology and properties.

To this end we regard the coordinates (p_i) of a line p of S_3 as the homogeneous coordinates of a point P of S_5. Then, in virtue of the fundamental identity $\Omega = 0$, it follows that P always lies on the quadric primal of S_5 whose equation is

$$p_1 p_4 + p_2 p_5 + p_3 p_6 = 0.$$

We denote this quadric by M_4^2 or by the symbol Ω itself. Hence

THEOREM V. *The lines of* [3] *may be represented birationally, and without exception, on the points of a quadric* Ω *of* [5].

The quadric Ω arising in this way is in fact a quite general quadric primal of [5]; for its equation may be reduced, by the linear transformation

$$p_1 = X_1 + iX_4, \quad p_4 = X_1 - iX_4, \text{ etc.},$$

to the canonical form $$\sum_1^6 X_i^2 = 0.$$

By this representation we may reduce the consideration of complexes, congruences, and scrolls of lines in [3] to the study of threefolds, surfaces, and curves which lie on the quadric Ω in [5]. This, in the following sections, we propose to accomplish; it will also appear that the representation throws a reflected light on the geometry of Ω itself.

2.1. *Preliminary properties of* Ω. We require first to note the main facts concerning polarity with respect to Ω; these are such immediate generalizations of familiar results in two and three dimensions that we may record them without comment:

(i) Every point P has a *polar prime* Π in regard to Ω; and Π meets Ω in the locus of points of contact of tangent primes from P.

(ii) Every line t has a *polar solid* T, and vice versa; t is the join of the points of contact of the two tangent primes which may be drawn to Ω through T; and T meets Ω in the locus of points of contact of tangent primes through t.

(iii) Every plane π has a polar plane π'; each of these planes meets Ω in a conic whose points are the points of contact of tangent primes through the other.

In each case, of course, every point of either of the two spaces concerned is *conjugate* to every point of the other. The condition of conjugacy for two points $P = (p_i)$ and $Q = (q_i)$ is

$$\Omega_{pq} \equiv \sum (p_1 q_4 + q_1 p_4) = 0.$$

2.11. *Lines of* Ω. For a pair of points P, Q which lie on Ω—and which therefore represent lines p, q of S_3—the condition of conjugacy implies that the tangent prime at P passes through Q; thus the line PQ, which has two intersections with Ω at P and a third intersection at Q, must lie entirely on Ω. Conversely, if P, Q are any two points of a line which lies on Ω, the tangent prime at P must contain Q; hence the necessary and sufficient condition for conjugacy is that PQ should lie on Ω. By § 1.1, the corresponding algebraic condition $\Omega_{pq} = 0$ is precisely the condition that p and q should intersect; and, by § 1.2, the pencil of lines then defined by p, q will be represented by the line PQ on Ω. Hence

THEOREM VI. *Two points of* Ω *represent intersecting lines of* [3] *if and only if their join lies entirely on* Ω; *and the points of this join then represent the lines of the pencil defined by the two lines.*

We shall call a line which lies on Ω a *generator* of Ω.

2.12. *Planes of* Ω. It also follows from § 1.2 that the points of Ω which represent the lines of a point-star form an aggregate such that the join of any two lies on Ω: hence the point-star corresponds to a plane α, say, lying entirely on Ω. Similarly, a ruled plane of lines is represented by a plane β lying entirely on Ω. Clearly the ∞^3 planes α, corresponding to the point-stars of [3], are a distinct system from that formed by the ∞^3 planes β, corresponding to the ruled planes. Hence

THEOREM VII. *The quadric* Ω *possesses two distinct triply-infinite systems of generating planes, corresponding respectively to the point-stars and ruled planes of* [3].

We shall call these the α-planes and β-planes, or planes of the first and second systems of Ω. We now determine their incidence relations.

In the first place, any two α-planes, representing point-stars (P_1) and (P_2), always meet in a point corresponding to the common ray $P_1 P_2$. Similarly, any two β-planes always meet in a point.

An α-plane and a β-plane, representing a star (P) and a ruled plane (π) respectively, will in general not meet at all; but if P lies in π, so that (P) and (π) have a pencil in common, the corresponding planes α and β meet in a line. Hence

THEOREM VIII. *Two planes of the same system on Ω always meet in a point. Two planes of opposite systems in general do not meet, but if they intersect, they have a line in common.*

2.13. *Conics of* Ω. Any three points P, Q, R of Ω define a plane section k of Ω; in general this is an irreducible conic which, by Theorem II, represents the regulus $\mathscr{R}$ of [3] containing the lines corresponding to P, Q, R. If π is the plane of k, then the polar plane π' of π meets Ω in a conic k' with the property that every point of k is conjugate to every point of k'; i.e. the join of any point of k and any point of k' lies on Ω. This means that every line of the regulus $\mathscr{R}'$ corresponding to k' meets every line of $\mathscr{R}$, whence $\mathscr{R}$ and $\mathscr{R}'$ are complementary reguli, lying on the same quadric surface. Hence

THEOREM IX. *Every conic which is the section of Ω by a plane π represents a regulus of [3]; and the complementary regulus is represented by the section of Ω by the polar plane π' of π in regard to Ω.*

2.14. *Other space sections of* Ω. It is clear from the definition of a linear complex that this system will be represented on Ω by a prime section. The section of Ω by a generic prime Π is a general quadric threefold V_3^2; if, however, Π touches Ω at P, the section V_3^2 is a cone, with vertex P, and every point of the cone is conjugate to P. It follows that in this case the corresponding linear complex is special, consisting of all the lines which meet the image p of P. Hence

THEOREM X. *The prime sections of Ω represent the linear complexes of [3], the tangent prime sections representing special linear complexes.*

Consider the lines of a special linear complex whose axis is p; they are distributed into ∞^1 point-stars whose vertices lie on p, and ∞^1 ruled planes whose planes pass through p. We thus see that, on the cone V_3^2 which is the corresponding tangent prime section of Ω, there are ∞^1 planes of each system, all passing through the vertex P of V_3^2. We therefore have the following results:

THEOREM XI. *Through each point of Ω there pass ∞^1 α-planes and ∞^1 β-planes, lying in the tangent prime section at that point. Through any line of Ω there passes one plane of each system.*

In conclusion, we note that a linear congruence $K^{(1,1)}$, intersection of two linear complexes, will be represented by the quadric surface V_2^2 in which Ω is met by a solid T. The polar line t of T meets Ω in two points P, Q which are conjugate to every point of V_2^2; these two points therefore represent the *directrices* of $K^{(1,1)}$ (§ 1.4).

It may, however, happen that T is tangent to Ω; in that case its section V_2^2 will in general be a cone with vertex R, say. The directrices of the congruence then coincide with the line r corresponding to R.

Finally, if T touches Ω along a line, the cone V_2^2 is a plane-pair (α, β), and the congruence $(1, 1)$ is reducible, consisting of two congruences $(1, 0)$ and $(0, 1)$.

Examples

1. Prove that the lines of a linear complex which meet a given line also meet a second line. (This is called the *polar line* of the first in regard to the given complex.)

Verify the result algebraically by assuming the equation of the complex to be $\Omega_{ap} = 0$, the coordinates of the given line to be (q_i), and by observing that any line, not (q_i), with coordinates $(a_i + \lambda q_i)$ satisfies the conditions for the second line of the problem.

2. If the tetrahedron of reference is $A_0 A_1 A_2 A_3$, and if $A_0 A_2$, $A_0 A_3$, $A_1 A_2$, $A_1 A_3$ belong to a given linear complex, show that the equation of the complex must be of the form $ap_1 + bp_4 = 0$. Deduce that the equation of any linear complex can be reduced to this form.

Determine, for such a complex, the coordinates of the polar line of a given line (q_i).

3. Two linear complexes are said to be *conjugate* if their corresponding primes in [5] are conjugate with respect to Ω. Prove that the polar line, in regard to one of the complexes, of any line belonging to the other, likewise belongs to the other.

4. Show that the equations of any linear congruence with distinct directrices may be put in the form $p_1 = 0$, $p_4 = 0$.

Show that, in the case where the directrices coincide, the equations may be written as $p_1 = 0$, $p_2 + kp_5 = 0$. Show that the lines of the congruence touch a quadric at points of the directrix.

5. Show that the equations of any regulus may be reduced to the form

$$p_1 = k_1 p_4, \qquad p_2 = k_2 p_5, \qquad p_3 = k_3 p_6,$$

and that the equations of the complementary regulus are obtained by changing k_i into $-k_i$.

6. Prove that, if $X_i = 0$ $(i = 1, 2, \ldots, 6)$ are the equations of six linear complexes which are mutually conjugate, the equation of Ω may be written as $\sum_1^6 X_i^2 = 0$ (that is, the corresponding primes in [5] form a self-polar simplex with regard to Ω).

7. Taking X_i instead of p_i for line coordinates, obtain (i) the condition that two given lines (X_i), (Y_i) should intersect, (ii) the condition that a given linear complex $\sum a_i X_i = 0$ should be special.

8. Deduce from the representation on [5] that four given lines in [3] have in general two common transversals. What is the condition that these transversals should coincide?

9. Prove that the expression Ω_{pq} is a relative invariant for linear transformations of the system of line coordinates; and that, if p, q, r, s are given lines, the expression $\dfrac{\Omega_{pq}\Omega_{rs}}{\Omega_{ps}\Omega_{rq}}$ is an absolute invariant for such transformations.

Hence show that, if the four lines have a single common transversal,

$$\pm\sqrt{(\Omega_{pq}\Omega_{rs})}\pm\sqrt{(\Omega_{qr}\Omega_{ps})}\pm\sqrt{(\Omega_{rp}\Omega_{qs})} = 0.$$

10. Show that the polar line (q_i), in regard to the quadric surface $\sum_1^4 a_i x_i^2 = 0$, of a given line (p_i) has coordinates

$$q_1 = a_3 a_4 p_4, \qquad q_4 = a_1 a_2 p_1, \quad \text{etc.}$$

Prove that, if any line p meets the polar of p' in regard to a quadric surface, then p' meets the polar of p in regard to the quadric.

11. Prove that, if any line p meets a line p_0 and its polar line q_0 in regard to a quadric F, then the polar line q of p also meets p_0 and q_0.

The tangent cone from any fixed point O to F meets a fixed plane π in the conic σ; and any line p of space is associated with the ordered pair of points (P, P') of π in which π is met by the transversals from O to the two pairs of non-intersecting generators of F which meet p. Prove that (i) a line and its polar line are represented by the same ordered point-pair (P, P'); (ii) two intersecting pairs of polar lines (forming a skew quadrilateral) are represented by pairs (P, P'), (P_1, P_1') for which P, P_1, and also P', P_1', are conjugate in regard to σ.

Formulate the theorem about pairs of polar lines for F which corresponds to the theorem that two triangles PQR, STU which are reciprocal for σ are in perspective.

By making F degenerate tangentially into a disc-quadric, deduce a proof of the Petersen–Morley Theorem (cf. § 3.5, Ex. 6).

2.2. *The general line complex.* The line systems which we have just examined are those corresponding to the manifolds of lowest orders on Ω. We now proceed to investigate the line systems which are represented by more general manifolds of various dimensions, beginning with the line complex.

We defined a complex Γ in § 1.31 as the system of lines whose coordinates satisfy a relation of the form

$$F(p_i) = 0, \tag{1}$$

where F is a polynomial of order n, homogeneous in p_i. We say that Γ is of order n, and that it is *general* if $F(p_i)$ is general of its order.

In the representation on [5], the equation (1) is that of a primal V_4^n; this intersects Ω in a threefold V_3^{2n} whose points represent the lines of Γ. Any line which lies on Ω will meet V_4^n, and thus V_3^{2n}, in n points: that is to say, a general pencil contains n lines of Γ. This gives a geometrical proof of the fact, already noted in § 1.31, that the lines of Γ which pass through a given point lie on a cone of order n, and those which lie in a given plane envelop a curve of class n.

The complex Γ, as defined above, is represented by a threefold on Ω which is the complete intersection of Ω with a primal. At this point it is natural to inquire whether there exist line systems of dimension three whose representative threefolds cannot be obtained as complete intersections of primals with Ω. It will be proved later (see § 3, Ex. 10) that in fact every threefold on Ω is the complete intersection of Ω with a primal of some order; hence the concept of line complex, as already formulated, is the widest possible, at least in space S_3.

2.21. *Singular points and planes. The singular congruence.* Consider any point P of V_3^{2n} representing a line p of Γ, and let Π, Π' be the tangent primes to Ω and V_4^n at P. Then the tangent solid T to V_3^{2n} at P is the intersection of Π and Π'.

The section of Ω by Π is a cone V_3^2 which is generated by ∞^1 α-planes and by ∞^1 β-planes; these meet V_3^{2n} in curves of order n, passing simply through P, which represent the complex cones of Γ at points of p and the complex curves of Γ in planes through p respectively. The section of this cone V_3^2 by T is a cone V_2^2 whose generators are the tangents at P to the sets of curves in question. Hence, by § 2.14,

Of all the plane pencils containing a given line p of Γ, those for which p counts twice in the set of n lines common to Γ and the pencil generate a linear congruence whose directrices coincide in p.

Consider now the case in which the cone V_2^2, section of Ω by T, breaks up into a pair of planes α and β, representing a particular point K of p and a particular plane κ through p respectively. Since α and β lie in T, they will now meet V_3^{2n} in curves having double points at P; and this means that p is a double line of the complex cone with vertex K, while p is also a double tangent of the complex curve in κ. In this case we call p a *singular line* of Γ; and we call K and κ the *singular point* and the *singular plane* associated with the line in question.

The condition that P should represent a singular line of Γ can also be expressed very simply in another way. For if T meets Ω in a pair of planes, then every prime through T—and therefore Π' in particular—is tangent to Ω; and conversely, if Π' touches Ω at some point Q, then PQ lies on Ω and T meets Ω in a pair of planes through PQ. Hence:

A point P of V_3^{2n} represents a singular line of Γ if and only if the tangent prime to V_4^n at P touches Ω at some point Q.

There will evidently be ∞^2 singular lines of Γ, forming a congruence; this we call the *singular congruence* of Γ. We now prove

THEOREM XII. *The singular congruence of Γ is represented on Ω by a surface of order $4n(n-1)$, which meets each plane of Ω in $2n(n-1)$ points.*

Let us take the equation of Ω in the canonical form $\sum x_i^2 = 0$; and suppose that the equation of V_4^n is $f(x_i) = 0$. Then the tangent prime to V_4^n at the point (x_i) has equation $\sum \frac{\partial f}{\partial x_i} y_i = 0$. The condition that this prime should touch Ω is $\sum \left(\frac{\partial f}{\partial x_i}\right)^2 = 0$. It follows that the singular congruence is represented by the surface which is the complete intersection of the primals whose equations are

$$\sum x_i^2 = 0, \qquad f(x_i) = 0, \qquad \sum \left(\frac{\partial f}{\partial x_i}\right)^2 = 0. \tag{2}$$

This surface is of order $4n(n-1)$, and so it meets any solid section of Ω in $4n(n-1)$ points. Hence if the solid intersects Ω in a plane-pair, each of the planes will meet the surface in $2n(n-1)$ points.

2.22. *The singular surface.* We now consider the locus of the points K and the envelope of the planes κ; this brings us to

THEOREM XIII. *The locus of singular points of Γ is a surface of order $2n(n-1)^2$, and the envelope of singular planes is a surface of class $2n(n-1)^2$; also these surfaces are identical.*

The order of the locus in question is the number of singular points which lie on a given line l; and this is equal to the number of α-planes of Ω which touch V_4^n and pass through a given point L of Ω. Similarly, the class of the envelope is equal to the number of β-planes which touch V_4^n and pass through L. Thus the order and class are equal.

Since the planes of the two systems are generators of the cone V_3^2 in which the tangent prime S_4 at L meets Ω, it follows that we require the number of planes, of either system, which touch the section V_3^n of V_4^n by S_4. If we project from L on to a solid S_3 of S_4, the two systems of planes project into complementary reguli of a quadric surface ϕ of S_3, while the tangent planes of V_3^n which pass through L project into the tangent lines of a surface ψ, the apparent contour of V_3^n from L. These tangent lines will form a complex Γ' whose order—equal to that of its complex cones—will be the order of the cone of tangent planes to V_3^n which contain a given line through L; and this is plainly $n(n-1)^2$. Hence, in S_3, Γ' has $2n(n-1)^2$ lines in common with either regulus of ϕ; and it follows then that $2n(n-1)^2$ planes of either system of Ω pass through L and touch V_4^n. This proves the first part of the theorem.

To prove the second part, it will suffice to show that if K is the singular point associated with any given singular line p, and if κ is the corresponding singular plane through p, then κ is the tangent plane at K to the locus of singular points. We do this, in effect, by showing that any line which passes through K and lies in κ has two coincident intersections with the locus in question at K.

Suppose then, in the notation of the preceding section, that PQ is the line on Ω which represents the pencil of lines (K, κ); we have to show that the plane α through PQ counts twice in the set of $2n(n-1)^2$ α-planes which pass through any given point L of PQ and touch V_4^n. Keeping to the notation of the first part of this proof, we observe that the two planes α and β through PQ touch V_4^n at the same point P and project therefore into two lines a, b of S_3 which touch ψ at the same point P_1. Plainly, then, the quadric ϕ, which contains these lines, touches ψ at P_1; and from this it follows at once (for example, from the plane representation of ϕ) that each of a, b counts twice in the set of generators of the same regulus of ϕ which touch ψ. Thus α counts twice in the set of α-planes through L which touch V_4^n: and this proves the second part of the theorem.

The locus of singular points, which is also the envelope of singular planes, is called the *singular surface* of the complex Γ.

2.23. *The quadratic complex.* The above results have an interesting application to the theory of the quadratic complex. This is represented on Ω by the threefold in which Ω is met by a quadric primal V_4^2. In general V_4^2 and Ω have a common self-polar simplex; if this is taken as the simplex of reference, the equations of the threefold will be of the form

$$\sum x_i^2 = 0, \qquad \sum a_i x_i^2 = 0.$$

Then the singular congruence is represented by the surface common to the primals

$$\sum x_i^2 = 0, \qquad \sum a_i x_i^2 = 0, \qquad \sum a_i^2 x_i^2 = 0.$$

The singular surface of the complex is, by Theorem XIII, a surface of order four and class four. It may be shown that this is the well-known Kummer surface which has been discussed in another connexion in Ch. VIII, § 2.33; it possesses 16 nodes and 16 singular tangent planes, or tropes, forming a $(16_6, 16_6)$ configuration. For the relevant details we may refer the reader to the existing literature of the subject.†

Examples

1. Show that the general complex of order n depends upon

$$\tfrac{1}{12}(n+1)(n+2)^2(n+3)$$

coefficients.

2. Show that, for the general quadratic complex, the singular surface is the locus of points for which the complex cones are plane-pairs, and also the envelope of planes for which the complex curves are point-pairs.

3. Prove that the quartic threefold common to two general quadric primals of [5] contains ∞^2 lines; and that, of these, 4 pass through a generic point, and 16 lie in a generic prime section, of the threefold. (*Note.* The prime section is a Segre quartic surface (Ch. VII, § 5.1).)

Give the interpretation of these results in the theory of the quadratic complex.

4. Defining the quadratic complex, as in § 2.23, by the equations $\sum x_i^2 = 0$, $\sum a_i x_i^2 = 0$, prove that any plane of Ω which touches the quadric $\sum a_i x_i^2 = 0$ also touches every quadric of the system $\sum x_i^2/(a_i+\lambda) = 0$. (We may call this, by analogy, a confocal system of quadrics.)

Interpret this result in the geometry of line complexes.

5. Prove that the lines belonging to the *tetrahedral complex* defined by the equation $ap_1p_4+bp_2p_5+cp_3p_6 = 0$ meet the faces of the tetrahedron of reference in ranges of constant cross-ratio.

Prove also that the singular surface of the complex consists of the four faces of the tetrahedron, and that any line through one of its vertices, or in one of its planes, belongs to the complex.

Discuss the representation of the complex on Ω.

6. Two pencils of lines are in homographic correspondence; prove that

† e.g. Jessop, ch. vi, and Baker, iv, ch. vii.

the lines which meet corresponding lines of the pencils generate a tetrahedral complex.

7. Show that, in space S_4, there are ∞^3 planes which meet three fixed lines, and that these planes intersect any fixed prime in lines of a tetrahedral complex.

8. Prove that the tangents to the quadric surface $\sum_0^3 a_i x_i^2 = 0$ belong to the quadratic complex $a_0 a_1 p_1^2 + a_2 a_3 p_4^2 + ... = 0$. What is the singular surface in this case? Discuss the representation of the complex on Ω.

9. Show that the complex of lines which meet the plane curve $f(x_1, x_2, x_3) = 0, x_0 = 0$ is given by the equation $f(p_1, p_2, p_3) = 0$. Examine the representation on Ω of the complex of lines which meet a given conic.

10. More generally, show that any space curve may be specified by the complex of its secants; and find the equation of the complex of secants to the twisted cubic

$$x_0 : x_1 : x_2 : x_3 = 1 : t : t^2 : t^3.$$

11. Using the same parametric equations, obtain the line coordinates of a chord of the twisted cubic. Hence show that any quadratic complex which contains the chords of the curve must be tetrahedral.

2.3. *The general congruence: its order and class.* In § 1.4 we defined a line congruence K as an aggregate of ∞^2 lines (or *rays*); we shall now develop its properties by using the representation on Ω.

Evidently K will correspond to a surface F of order μ_0, say; this is met by any solid, in particular one containing a plane-pair of Ω, in μ_0 points. If α and α' are any planes of the first system on Ω, there is an infinity of planes of the second system meeting each of these in a line. Let β be any one such plane, and suppose it meets F in n points; then since each solid (α, β), (α', β) meets F in μ_0 points, it follows that α and α' each meet F in $\mu_0 - n = m$ points. Thus every α-plane meets F in m points, and every β-plane meets F in n points, where $m + n = \mu_0$.

Accordingly, in [3], a finite number m of lines of K pass through a generic point, and a finite number n lie in a generic plane. As already stated in § 1.4, we call m the order and n the class of K, and denote the congruence by $K^{(m,n)}$.

2.31. *The rank of K.* In addition to its order μ_0, F possesses three elementary projective characters μ_1, μ_2, and ν_2 (Ch. IX, § 2); these represent the rank, class, and type of F. Since, however, the single character μ_0 must be replaced by the pair (m, n), it would appear that, unlike the generic surface of the preceding chapter, *five*, not four, elementary projective characters are required to specify F. That the five characters are actually independent will be shown later (see Ex. 6, § 2.35).

As a first step towards describing F geometrically we prove

THEOREM XIV. *Of the chords of F, a finite number $r = mn - \frac{1}{2}\mu_1$ pass through a given point of Ω: that is, r pairs of rays of K are concurrent on and coplanar with a given line of* [3].

This character r is called the *rank* of K.

Let L be any point of Ω, not on F; then the chords of F which pass through L must be generators of Ω, lying in the tangent prime S_4 at L: thus they are chords of the section C of F by S_4. Now C lies on the cone, vertex L, in which S_4 meets Ω; and if g is any generator of this cone which passes through L, the chords of C meeting g must either lie in one of the two planes of Ω through g, or pass through L. Since the number of chords of C incident to g is equal to h, where

$$h = \binom{m+n}{2} - \tfrac{1}{2}\mu_1,$$

while the number of chords lying in one or other of the two planes is $\binom{m}{2} + \binom{n}{2}$, it follows that the number of chords passing through L is

$$r = \binom{m+n}{2} - \tfrac{1}{2}\mu_1 - \left\{\binom{m}{2} + \binom{n}{2}\right\} = mn - \tfrac{1}{2}\mu_1. \tag{1}$$

If, in [3], the line l corresponds to the point L, for each of these r chords we have a pair of rays which lie in a pencil containing l.

2.32. *The double rays. Improper nodes of F.* In Ch. IX, § 2 we defined the generic surface of [5] as one without singular points; and we showed that if such a surface, having characters μ_0, μ_1, μ_2, ν_2, is projected from a general point, the surface so obtained in [4] has a number d of improper nodes, where

$$d = \binom{\mu_0}{2} - \tfrac{1}{2}\mu_1 - \tfrac{1}{2}\nu_2. \tag{2}$$

These nodes arise from the d chords of F which pass through the vertex of projection.

If, then, we wish to define a congruence K of general character, at first sight it would seem natural to say that its image F on Ω is to be a generic surface of [5]. But it is easily seen that such a definition would be too restrictive; for it is the case that *a surface*

F which is required to lie on Ω must be expected to possess a finite number of improper nodes.

The justification of this remark follows from the same argument by which (Ch. IX, § 2) we showed that a surface of [4] must be expected to possess nodes. A point of Ω, like a point of S_4, is determined by four parameters, or coordinates; and a surface on Ω will be obtained by making each of these coordinates a function of two independent variables. Hence the surface may be expected to intersect itself at a finite number of points; and, as in the previous chapter, we see that these points are in fact improper nodes of the surface.

Accordingly, we shall define a *generic surface* of Ω as one whose only singularities are a *finite number of improper nodes.* This number is readily obtained from equation (2) as follows.

Suppose that the surface F, with the above-mentioned projective characters, is projected on [4] from a general point L of Ω. The nodes of the projected surface arise from (i) the improper nodes of F, or (ii) the r chords of F which pass through L. Also the total number of nodes is given by (2): hence, if δ denotes the number of nodes on F, we have

$$d = r + \delta, \tag{3}$$

where the value of r is given by equation (1).

To each improper node of F there corresponds a *double ray* l of K, which counts twice among the lines of K lying in a plane through l or passing through a point of l. We thus have

THEOREM XV. *A congruence which is represented by a generic surface F of Ω possesses a finite number δ of double rays, given by the formula*

$$\delta = \binom{m}{2} + \binom{n}{2} - \tfrac{1}{2}\nu_2.$$

These correspond to the improper nodes of F.

2.33. *Singularities of a congruence. General congruences.* The double rays of K may be termed accidental singularities of the congruence; in contrast to these, there may be essential (i.e. non-accidental) singularities of a different kind. Thus, suppose that an α-plane of Ω meets F in a curve; then, through the corresponding point of [3] there will pass a cone of lines of K. Similarly, corresponding to a β-plane which meets F in a curve, there will be a plane containing an infinity of lines of K. Points and planes such as these are called *singular* points and planes of the congruence.

A congruence, as will appear shortly, may possess a finite or infinite number of singular points and planes; in view of this circumstance it is convenient to restrict still further our concept of a general congruence. Accordingly, we frame the following

DEFINITION. A congruence K is *general* if

(i) its only singular lines are double rays, corresponding to improper nodes of the representative surface F,

and if

(ii) it possesses at most a finite number of singular points and planes.

It follows from our definition that the surface on Ω which represents a general congruence is necessarily generic in the above sense, but that not every generic surface on Ω is the image of a general congruence.

Any congruence which is not general will be called *special*. We may note here that, if the congruence K is general, then F cannot be a *ruled* surface (see Ex. 10, § 2.35).

An important type of special congruence is that of the chords of a given space curve C. If C is of order n, and has h apparent nodes, the congruence is of order h and class $\binom{n}{2}$. Through each point of C there passes a cone of chords; thus the congruence possesses a curve of singular points. In general there are also ∞^1 triple lines, corresponding to the trisecants of C, and a finite number of sextuple lines, corresponding to the quadrisecants of C. These multiple lines will be represented by singular points of the surface which maps the congruence on Ω.

2.34. *Focal points and planes.* Let K be a general congruence, and F its image on Ω. Consider the tangent plane to F at a generic point L; since this is also tangent to Ω, it meets Ω in a pair of generators g_1, g_2 through L. These represent pencils in [3], each containing the corresponding line l and one line of K consecutive to l. Hence,

Any ray of K is met by two consecutive rays.

The points of intersection of these lines with l are called the *focal points* (or *foci*) of l; they correspond to the α-planes of Ω which can be drawn through g_1 and g_2. Similarly, the planes containing l and one of the consecutive lines are called the *focal planes* of l; they correspond to the β-planes which can be drawn through g_1 and g_2. We shall call g_1 and g_2 the *focal generators* relative to L. We now prove

THEOREM XVI. *The locus of focal points of a general congruence is a surface of order $\mu_1 - 2n$, and the envelope of focal planes is a surface of class $\mu_1 - 2m$.*

Evidently the order of the locus in question is equal to the number of α-planes which pass through a given point P of Ω and which contain a focal generator. All such planes are contained in one system of planes of the cone in which the tangent prime S_4 at P meets Ω; and all such focal generators are tangent to the curve C in which S_4 meets F. Consider, then, any β-plane of the cone with vertex P; this meets C in n points. Since C is, by hypothesis, of rank μ_1, the number of tangents to C which meet β and have their points of contact external to β is $\mu_1 - 2n$; and each of these tangents furnishes one α-plane containing a focal generator. In the same manner we may show that there are $\mu_1 - 2m$ β-planes through P which contain a focal generator of F.

It should be noted that a special, in contrast to a general, congruence, need not possess a surface-locus of focal points or a surface-envelope of focal planes; for example, the foci of the congruence formed by the chords of a curve all lie on the curve itself.

2.35. *The focal surface.* Let l be any ray of the congruence, P_1, P_2 its focal points, and π_1, π_2 the corresponding focal planes. If l is represented by the point L of F, and if g_1, g_2 are the focal generators relative to L, then P_1, P_2 correspond to the planes α_1, α_2 on Ω which pass through g_1, g_2 respectively, and π_1, π_2 correspond to the planes β_1, β_2 which pass through g_1, g_2. We now establish

THEOREM XVII. *The focal planes π_1, π_2 touch the locus of focal points at P_2 and P_1 respectively.*

We have to prove that, if m is any line in π_1, passing through P_2, then two of the intersections of m with the focal locus are at P_2.

Consider the corresponding figure in S_5: to m there corresponds a point M of Ω, common to β_1 and α_2. The tangent prime S_4 to Ω at M contains β_1 and α_2, and therefore g_1, g_2 also; hence it must contain the tangent plane to F at L. Thus its section C of F will have a node at L, and therefore the rank of C is μ_1-2. Reasoning now as in the proof of the last theorem, we see that the number of tangents to C, at ordinary points of the curve, which meet any β-plane through M, and have their points of contact outside that plane, is equal to μ_1-2-2n: which proves the result.

From this follows

THEOREM XVIII. *The locus of focal points of a general congruence is identical with the envelope of focal planes; and the rays of the congruence are bitangents to this locus.*

Consider the focal generators g_1, g_2 for the surface F on Ω; there are ∞^2 such lines, and their locus is a manifold V_3 whose order is at once seen to be $2\nu_2$, where ν_2 is the type of F. For, by definition, the number of tangents to F which meet a given line is equal to ν_2; and if the line lies on Ω, these tangents must be focal generators. It follows that V_3 is of order $2\nu_2$.

For a general congruence K, the two systems of planes of Ω passing through the focal generators are each irreducible. But suppose that the congruence is special, that, for instance, it has a singular curve: then there are ∞^1 α-planes which meet F in curves, and the ∞^2 tangents to these curves form an irreducible manifold V'_3 which is a component of V_3. The identity of the focal locus with the focal envelope no longer holds.

For a general congruence, however, we may speak of the *focal surface* ϕ, regarding it either as the locus of the focal points or the envelope of the focal planes. We see, from the proof of Theorem XVII, that all rays of the congruence are bitangents of the focal surface. But not all bitangents of the surface belong to the congruence; for in general ϕ will possess ∞^1 tritangents and a finite number of quadritangents, and the congruence containing them will consequently possess multiple rays of types which we have excluded from the present discussion.†

EXAMPLES

1. Show that the rays of K which meet a given line l generate a scroll of order $m+n$ and rank μ_1. Show also that there is a double curve on the scroll, and that its order is $r+\binom{n}{2}$.

† For other results concerning congruences, see L. Roth, *Proc. Camb. Phil. Soc.* 27 (1931), 190.

2. By considering the case where l is a ray of K, show that any ray of K is met by two consecutive rays.

3. Prove that the character ν_2 of F is equal to the number of focal planes which pass through a given point and have their foci on a given plane.

4. Prove that the character μ_2 of F is equal to the rank of the focal surface ϕ.

(By definition, μ_2 is equal to the number of primes of a given pencil which touch F. Suppose that the primes all pass through the solid containing two planes α, β of Ω; any such prime must touch Ω at a point P of the line g common to α, β. If the prime touches F at Q, the line p corresponding to P touches ϕ at a focus of the ray corresponding to Q.)

5. Show that the congruence which is the complete intersection of two general complexes of orders k and k' has characters

$$m = n = kk', \quad \mu_1 = 2kk'(k+k'-1), \quad \mu_2 = 2kk'\{(k+k'-1)^2-kk'+1\},$$
$$\nu_2 = 2kk'(kk'-1).$$

Show also that the congruence is general, that its focal surface is of order and class $2kk'(k+k'-2)$, and that it has no double rays.

6. To show that the five characters $m, n, \mu_1, \mu_2, \nu_2$ of a general congruence are independent.

We observe, first, that these characters are *additive*, i.e., if a surface F'', with characters m'', n'', etc., is composed of two irreducible surfaces F, F' having characters $m, n,\ldots$, and $m', n',\ldots$, then

$$m'' = m+m', \qquad n'' = n+n', \quad \text{etc.}$$

Suppose, then, that there exists an identical relation $f(m,n,\mu_1,\mu_2,\nu_2) = 0$ between the five characters. It follows that the function f must satisfy the equation $f(m+m',n+n',\ldots) = f(m,n,\ldots)+f(m',n',\ldots)$. Now the general solution of this functional equation is

$$f(m,n,\ldots) = Am+Bn+C\mu_1+D\mu_2+E\nu_2,$$

where the capital letters denote constants. Thus the identical relation must be

$$Am+Bn+C\mu_1+D\mu_2+E\nu_2 = 0.$$

In particular, it must hold for the congruences which are complete intersections of two complexes. Inserting the values of m, n,... obtained in Ex. 5, we deduce at once that $C = D = E = 0$, $A+B = 0$; and, from the last result, we have $A = B = 0$.

7. Prove from first principles that the number of double rays of a general congruence K is $\binom{m}{2}+\binom{n}{2}-\frac{1}{2}\nu_2$. (Let π, π' be two fixed planes, and consider the correspondence between points A, B of π such that a ray through A and a ray through B meet on π'; then apply Theorem IV, Ch. IV.)

8. Show that the number of line-pencils, each containing 3 rays of K, which have a line in common with a given generic line-pencil is

$$t_0 = t-\binom{m}{3}-\binom{n}{3},$$

where t, the number of trisecant planes of F which pass through a given line of S_5, is given by the formula (Ch. IX, § 3.6, Ex. 1)

$$t = \binom{m+n}{3} - \tfrac{1}{2}\mu_1(m+n) + \tfrac{1}{3}(2\mu_1 + 2\nu_2 + \mu_2).$$

9. Discuss the representation on Ω of the congruence formed by the chords of a non-singular space curve of order n and genus p; and determine the singularities of the corresponding surface F.

Show that the congruence of chords of a twisted cubic is represented by a Veronese surface on Ω, and that the surface contains ∞^2 conics, among which there are ∞^1 conics whose planes lie on Ω.

Discuss also the congruence of chords of (i) an elliptic quartic, (ii) a non-singular rational quartic.

10. If F is a generic scroll on Ω, show that the corresponding congruence K contains ∞^1 pencils of rays, whose vertices and planes give rise to singular curves and developables of K. Obtain the focal locus and focal envelope of the congruence.

11. Determine the characters of the singular congruence of a general complex of order n (§ 2.21).

12. (i) Prove that, for the general quadratic congruence $(2, n)$, the focal surface ϕ must be of order four, and therefore of class $2n$; also that the representative surface F has elliptic sections, and possesses $\frac{1}{2}(n-2)(n-3)$ improper nodes.

(ii) Show that the congruence $(2, 2)$ is represented on Ω by a Segre quartic surface (see Ch. VII, § 5.1), and hence prove that the congruence is the complete intersection of a linear and a quadratic complex.

Prove that, to the 16 lines on the Segre surface there correspond 16 pencils of rays of the congruence, and obtain the 16_6 configuration of singular points and planes. Prove also that ϕ is a Kummer surface having the singular points and planes for nodes and tropes.

(iii) Show that the congruence $(2, 3)$ is represented on Ω by a Del Pezzo quintic surface F (Ch. VII, § 5), and that the 10 lines of F correspond to singular pencils of the congruence. By considering the focal surface ϕ, prove that the congruence possesses in addition 5 singular points, through each of which there passes a quadric cone of rays.

13. If the equation of Ω is taken in the form $\sum x_i^2 = 0$, prove that the six quadratic congruences $x_i = 0$, $\sum a_i x_i^2 = 0$ have the same Kummer surface for focal surface.

2.4. ***Ruled surfaces. Scrolls and developables.*** In Ch. IX, § 4.3, we have already used the representation on Ω in order to obtain various enumerative properties of scrolls, i.e. non-developable ruled surfaces. The methods employed there were all based on the following obvious fact: *A scroll of order μ_0 and genus p, situated in* [3], *is represented on Ω by a curve of order μ_0 and genus p.*

It is clear, moreover, that if the scroll is free from multiple generators, the curve in question is without singularities.

Let R be the scroll and C the corresponding curve. Suppose that, with the terminology and notation of Ch. IX, R is endowed with normal singularities: that is to say, a double curve of order ϵ_0 (locus of intersection of pairs of distinct generators), on which lie ν_2 pinch-points (intersections of pairs of consecutive generators) and t triple points (intersections of three distinct, non-coplanar generators).

It is interesting to observe how these singularities of R are reflected in the representation. To the points of the double curve there will correspond ∞^1 α-planes of Ω which are bisecant to C; among these are ν_2 generators which touch C. To the t triple points on the double curve there correspond t α-planes of Ω which meet C in three distinct points. Since R is self-dual,† there will also be t β-planes which are trisecant to C; these correspond to tritangent planes of R, each containing three distinct generators.

Consider next the representation of a developable surface D, of order μ_0 and genus p. Its generators will likewise correspond to the points of a curve C, of order μ_0 and genus p. Since, however, any two consecutive generators of D intersect, it follows that the join of any two consecutive points of C must lie on Ω. Thus

A developable of order μ_0 and genus p, situated in [3], *is represented on Ω by a curve of order μ_0 and genus p, all of whose tangents lie on Ω.*

Evidently the α-planes which can be drawn through these tangents correspond to the points of the edge of regression E on D; the β-planes containing these tangents correspond to the osculating planes of E.

If D is of general character it will possess, in addition to the cuspidal edge E, a double curve containing triple points and pinch-points; their representation on Ω is the same as for a scroll R.

EXAMPLES

1. Show that any scroll in [3] which possesses two linear directrices is represented on Ω by a curve on a quadric surface. Hence establish the existence of (i) a cubic scroll with one simple and one double linear directrix, (ii) an elliptic quartic scroll with two double linear directrices, (iii) a rational quartic scroll with one simple and one triple directrix.

† The self-duality of R clearly imposes severe restrictions upon the nature of C; thus a generic curve on Ω will not in general represent a generic scroll of S_3.

2. Show that any scroll in [3] with a linear directrix is represented on Ω by a curve lying in a tangent prime to Ω. Hence establish the existence of rational quartic scrolls with a simple linear directrix.

3. Taking the equation of Ω in the form $\sum x_i^2 = 0$, show that the parametric equations

$$\rho x_i = a_i t^3 + b_i t^2 + c_i t + d_i \quad (i = 1, 2, ..., 6)$$

represent a cubic scroll, provided that a_i, etc., satisfy certain relations. Deduce that the scroll has in general two cuspidal generators. Hence, by assuming that these are given by the values $t = 0, \infty$ of the parameter, reduce the parametric equations of the scroll to a standard form.

4. Obtain the image, on Ω, of the developable of tangents to (i) a twisted cubic, (ii) an elliptic quartic.

5. Show that three general line complexes of orders k, k', k'' intersect in a scroll of order $2kk'k''$ and rank $2kk'k''(k+k'+k''-2)$. Show also from first principles that the scroll possesses $4kk'k''(k+k'+k''-3)$ cuspidal generators.

6. A congruence is represented on Ω by a generic surface of order μ_0 and rank μ_1; show that the scroll which is common to the congruence and to a general complex of order k has order $\mu_0 k$ and rank $\mu_0 k(k-1)+\mu_1 k$.

§ 3. Other representations of the lines of S_3

3. First representation on S_4. Rationality of the line system. So far we have discussed the line geometry of [3] by referring the aggregate of lines to the points of a quadric Ω of [5]. But this is only one of many representations of the aggregate in question; and in the present section we shall examine some alternatives. Generally speaking, the representation which we choose in a given problem is determined by its appropriateness to the situation. Each of the methods to be described has its own special advantages.

The most obvious alternative to the representation we have hitherto used is obtained by projection of Ω. Let O be any point of Ω, S_4 a given prime, and consider the projection of Ω from O on S_4. This gives a birational representation of the points of Ω on those of S_4. The exceptional elements of the correspondence arise from the cone of generators of Ω which pass through O; this projects into a quadric surface ω, lying in the solid S_3 in which S_4 meets the tangent prime to Ω at O; each point of ω corresponds to a generator of this cone. Generators of Ω which do not pass through O project into secants of ω.

As regards the planes of Ω, those which pass through O project into the two reguli of generators of ω; the rest project into planes, each containing one generator of ω; and the projections of the

α-planes, for instance, each contain a generator of the same regulus on ω, which is itself the projection of a β-plane.

It follows also that a general linear complex, corresponding to an arbitrary prime section of Ω, will be represented by a quadric V_3^2 through ω; a general linear congruence will be represented by a quadric surface containing a conic of ω; and a regulus by a conic which meets ω in two points. In conclusion, then, we have

THEOREM XIX. *The lines of* [3] *may be represented birationally by the points of a space* S_4, *so that the system of linear complexes corresponds to that of the quadric primals passing through a fixed quadric surface.*

We have thus shown, incidentally, that the lines of [3] form a *rational* aggregate. We note also that there are ∞^5 quadric primals of S_4 which contain a given quadric surface ω; for there are ∞^{14} quadric primals in S_4, and nine conditions are imposed on any quadric primal which is to contain ω. Thus the freedom of these primals is equal to that of the corresponding linear complexes.

This representation of the lines of [3] was first proposed by Schumacher† in connexion with the theory of congruences; his account of the matter, which is incomplete, was revised by James.‡

3.1. ***Halphen's Theorem.*** We shall now apply the representation to prove the following result, due to Halphen:

THEOREM XX. *The number of rays common to two congruences* (m, n) *and* (m', n') *in general position is* $mm'+nn'$.

Denote the congruences by K and K' respectively; then, by what has been said, they will correspond in S_4 to two surfaces F, F', of orders $m+n$ and $m'+n'$. The surfaces will meet each generator of ω of one system in m, m' points respectively, and each generator of the other system in n, n' points; hence F intersects ω in a curve C which we may denote by the symbol (m, n), and F' intersects ω in a curve C' having the corresponding symbol (m', n').

Since, by hypothesis, K and K' are in general position, so also are F and F'; they will therefore meet in $(m+n)(m'+n')$ points in all. But among these are the points common to C and C', in number $mn'+m'n$, which do not correspond to common rays of

† Schumacher, *Math. Ann.* **37** (1900), 100.
‡ James, *Messenger of Math.* **55** (1925), 9.

K and K'. Deducting these, we obtain for the number of common rays the expression

$$(m+n)(m'+n')-(mn'+m'n) = mm'+nn'.$$

COROLLARY. *The number of chords common to two space curves of orders n, n' and ranks r, r' is in general*

$$\left\{\binom{n}{2}-\tfrac{1}{2}r\right\}\left\{\binom{n'}{2}-\tfrac{1}{2}r'\right\}+\binom{n}{2}\binom{n'}{2}.$$

For the chords of the first curve form a congruence of order h and class $\binom{n}{2}$, where h, the number of apparent nodes, is given by the formula $r = n(n-1)-2h$; and similarly for the second curve.

EXAMPLES

1. In the representation of the lines on S_4, obtain the image of (i) a special linear complex, (ii) a linear complex containing the line corresponding to the vertex O of projection, (iii) a linear congruence containing that line.

2. Show that a general quadratic complex is represented by a quartic primal in S_4 on which ω is a double surface, and that a general quadratic congruence (2, 2) corresponds to a Segre surface meeting ω in an elliptic quartic curve.

3. Interpreting the quadric primals of S_4 which contain ω as a system of hyperspheres, prove that the linear complexes of S_3 may be represented on the hyperspheres of S_4, in such a way that two conjugate complexes correspond to two orthogonal hyperspheres.

4. Taking the section of the figure in S_4 by a prime in that space, deduce Lie's correspondence between the lines and spheres of [3].

Let Π be the prime in question: then, if P is any point of Ω, corresponding to a line p of [3], the tangent prime to Ω at P meets Ω in a quadric threefold which we project from O on to S_4. Its section by Π is a quadric surface which, for all positions of P, passes through a fixed conic, namely, the section of ω by Π. Thus any line p can be made to correspond to a sphere of Π. Conversely, however, to any sphere of Π there correspond two points of Ω, and therefore two lines of [3], which are polars of one another in a certain linear complex. (The proof of the latter statement is left to the reader, who should also show that, if two lines of [3] intersect, the corresponding spheres will touch; and conversely.)

5. Show that, if a general congruence K is represented on S_4 by a surface F^*, then, with the notation of § 2.32, F^* will have r improper nodes on ω, and δ improper nodes external to ω.

Describe the nature of the congruence for which the representative surface meets ω in a finite number of points.

6. Prove that the number of α-planes of Ω which pass through a point O of Ω and touch the image F of a congruence K is equal to the number

of generators of the same system on ω which are tangent to F^*. Deduce that the focal surface of K is of order $2n(m-1)-2r$.

7. Prove that K possesses a scroll of rays with coincident foci which are flecnodal tangents to the focal surface ϕ, and show that the scroll is of order $2(\mu_1-m-n)$. (Let $S(l)$ denote the scroll of rays which meet a given line l; this is of order $m+n$, and touches ϕ along a curve of order μ_1. Consider the involutory correspondence between the foci on this curve.)

8. Show that, for a non-singular surface in [3], the inflexional tangents form a congruence of order $n(n-1)(n-2)$ and class $3n(n-2)$, and that the bitangents form a congruence of order $\frac{1}{2}n(n-1)(n-2)(n-3)$ and class $\frac{1}{2}n(n-2)(n^2-9)$.

Examine for what values of n, if any, either of these congruences is general.

9. By means of Halphen's Theorem, obtain the number of (i) inflexional tangents, (ii) bitangents common to two non-singular surfaces, of orders n and n', in general position in [3].

10. To prove that *any threefold on Ω is the complete intersection with Ω of a primal.*

Let V_3 be the threefold in question; its section by a tangent solid to Ω at any point O, not on V_3, will be a curve lying on the quadric cone in which the solid intersects Ω. The generators of the cone all meet the curve in the same number (n, say) of points; hence V_3 is of order $2n$, and meets each plane of Ω through O in a curve of order n.

Projecting from O, we obtain in S_4 a manifold V_3' of order $2n$ on which ω is an n-fold surface. Suppose that ω is the intersection of the quadric threefold whose equation is $A = 0$ with the solid whose equation is $B = 0$; then the equation of V_3' must be of the form

$$a_0 A^n + a_1 A^{n-1}B + \dots + a_n B^n = 0,$$

where a_i is a homogeneous polynomial of degree i in the coordinates $(x_0,\dots,x_4)$ in S_4. If we assume that the equation of the prime S_4 itself is $x_5 = 0$, and that O is the opposite vertex of the fundamental simplex, it follows that the equation of Ω is $A + Bx_5 = 0$. Hence V_3 is the complete intersection of Ω with the primal whose equation is

$$a_0 x_5^n - a_1 x_5^{n-1} + \dots + (-1)^n a_n = 0,$$

and the proposition is established.

3.2. *Second representation (by quadrics through a ${}^0C^4$).*

In addition to the straightforward projection just considered, there exists a quite different representation of Ω on S_4; and since this possesses certain advantages in regard to some important problems, we propose to give a short account of it here. The representation, like its predecessor, is by means of a system of quadrics Φ; but now, instead of a base surface, the system Φ has only a base curve Γ which is a rational normal quartic of S_4.

The idea is a very simple one: namely, that in the equations

$$\frac{p_1}{x_0y_3-y_0x_3} = \frac{p_2}{x_1y_3-y_1x_3} = \dots = \frac{p_6}{x_0y_1-y_0x_1} \qquad (1)$$

which define the coordinates of a line, we take the x_i and the y_i to be fixed linear homogeneous functions of the coordinates of a point of S_4, so that these equations become forthwith a parametric representation of Ω. The primals of S_4 which thereby represent prime sections of Ω form the system Φ of quadrics whose equation is

$$\lambda_1(x_0y_3-y_0x_3)+\dots+\lambda_6(x_0y_1-y_0x_1) = 0, \qquad (2)$$

and this is the fivefold system of quadrics which contain the rational normal ${}^0C^4$ whose equations are

$$\left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{matrix} \right\| = 0. \qquad (3)$$

Conversely, if Φ is the system of quadrics through a given ${}^0C^4$, say Γ, then the freedom of Φ is plainly 5; its generic Φ-surface is a Segre ${}^1F^4$, represented by a system $C^3(A_1,\dots,A_5)$, and it contains the curve Γ which is represented, say, by a curve $C^3(A_1^2, A_2, A_3, A_4)$; its generic Φ-curve is a ${}^0C^4$ (residual to Γ), represented by $C^3(A_2, A_3, A_4, A_5^2)$, which evidently meets Γ in 6 points; and it follows at once that Φ has grade 2. Hence

THEOREM XXI. *The quadric Ω may be represented birationally on S_4 in such a way that its prime sections correspond to the quadrics which pass through a fixed rational normal quartic† curve Γ of S_4. Solid sections and plane sections of Ω correspond respectively to Segre ${}^1F^4$ through Γ and to rational normal ${}^0C^4$ which meet Γ in 6 points.*

3.21. *Details of the correspondence.*‡ We now proceed to set out some of the more important properties of the representation, prefixing these for convenience by the following simple lemma:

The transform of a 4-nodal cubic surface ${}^1F^3$ by quadrics through the 4 nodes is a Veronese quartic surface in [5].

This is a simple deduction from the known plane representation of ${}^1F^3$; but it also follows immediately from the fact that a

† If Γ is taken to have parametric representation $\{\xi_0,\dots,\xi_4\} = \{\theta^4, \theta^3, \theta^2, \theta, 1\}$, then the explicit parametric representation of Ω is obtained by writing

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{pmatrix}$$

in equations (1).

‡ For a more detailed account, see Semple, *Proc. Lond. Math. Soc.* (2), **35** (1932), 294.

reciprocal transformation based on the 4 nodes carries the linear ∞^5 system of quadrics through the nodes into itself, and the ${}^1F^3$ into a plane.

We note then the following facts which are all fairly obvious applications of the ideas set out in Ch. VIII.

(i) *The chords of* Γ, *being fundamental lines of* Φ, *represent the points of a Veronese surface* σ *on* Ω. For these chords generate a cubic primal passing doubly through Γ, so that they are in birational correspondence with the points of the 4-nodal cubic surface ${}^1F^3$ in which this primal is met by a solid; and the result stated then follows from the lemma given above, since prime sections of σ represent sections of ${}^1F^3$ by the primals Φ.

(ii) *To the neighbourhoods of points of* Γ *there correspond* ∞^1 *planes lying on* Ω; *and these generate a sextic threefold* W *which has* σ *as double surface.* They are plainly the conic-planes of σ which lie entirely on Ω.

(iii) *To the solids of* S_4 *there correspond* ∞^4 *tetrahedral threefolds on* Ω (*cf.* VIII, § 2.31); *and these form the complete system of quartic threefolds* Ψ *in which* Ω *is met by quadrics through* σ. For, in the first place, each of the threefolds in question is represented on a solid of S_4 by means of the quadric surfaces—through 4 fixed points—in which the solid is met by Φ-primals; secondly, the threefold passes through σ because the solid meets every chord of Γ; and, finally, if a quadric section of Ω contains σ, then its homologue in S_4, which is a quartic primal passing doubly through Γ, must break up into the cubic chord-locus of Γ and a solid.

(iv) *To the planes of* S_4 *there correspond* ∞^6 *Veronese surfaces on* Ω; but among these there are some—corresponding to the various types of secant planes of Γ—which reduce effectively to cubic scrolls, quadric surfaces, or planes. As to these last, we have

(v) *To the* ∞^3 *trisecant planes of* Γ *there correspond the* ∞^3 *planes of one generating system, say* α-*planes, of* Ω. To discover the surfaces of S_4 which represent β-planes, we note that if a Φ-surface consists in part of a trisecant plane of Γ, then the remainder is a cubic scroll which passes through Γ and has a chord of this curve for its directrix. Hence

(vi) *There are* ∞^3 *cubic scrolls* ρ *through* Γ, *each having a chord of this curve for its directrix, and these represent the* ∞^3 *planes* β *of the second generating system of* Ω. It should be noted, however, that the planes of W are a special system of β-planes which represent neighbourhoods of points of Γ.

We may note finally the special significance of the representation in terms of line-geometry of S_3. Plainly σ, being a $(3, 1)$ surface on Ω, represents a congruence $(3, 1)$ in S_3, i.e. the congruence of axes of a cubic envelope (dual of the congruence of chords of a twisted cubic); and the threefolds Ψ, which form a homaloidal system on Ω, represent tetrahedral complexes through the congruence in question. Hence

THEOREM XXII. *All the quadratic complexes which contain a given* $(3, 1)$ *or* $(1, 3)$ *congruence are tetrahedral, and any five of them have one and only one further line in common.*

EXAMPLES

1. Show that the chords of a rational normal quartic generate a cubic primal of S_4 on which the quartic is a double curve.

2. Show that, in the second representation of the lines of [3] on S_4, a special linear complex corresponds to a quadric cone containing Γ. Deduce that the trisecant planes of a rational normal quartic which pass through a given point generate a quadric cone.

Establish this result also by taking the equations of the quartic in the form

$$x_0:x_1:x_2:x_3:x_4 = 1:t:t^2:t^3:t^4.$$

3. By considering the system of solids in S_4, prove that the quadric primals of S_5 which pass through a given Veronese surface define a homaloidal system.

3.3. *Associated triads of quartic curves.* The above representation leads incidentally to an interesting property of rational normal quartic curves, originally due to James,† which we may mention here. The result to be proved is

THEOREM XXIII. *If C_1 and C_2 are two rational normal quartic curves which meet in six points, then all trisecant planes of C_1 which meet C_2 meet a third associated quartic C_3 passing through the same six points.*

Our proof consists in taking C_1 to be the curve Γ of the preceding section, and C_2 to be a Φ-curve C representing the conic k in which Ω is met by a plane π, and in then showing that the required curve

† *Proc. Camb. Phil. Soc.* **21** (1923), 673. See also Semple, *Journal London Math. Soc.* **7** (1932), 266; Babbage, *Proc. Camb. Phil. Soc.* **28** (1932), 421; Welchman, ibid. pp. 275 and 416; Telling, ibid. p. 403; and Todd, *Proc. London Math. Soc.* (2), **33** (1932), 328.

C_3 is that represented by the conic k' in which Ω is met by the polar plane π' of π.

To show first then that any trisecant plane of Γ which meets C meets the curve C' corresponding to k', we have to show that any α-plane which meets k likewise meets k'. This is, in fact, obvious; for if the α-plane meets k in P, then it lies in the tangent prime at P, which is a prime through π', and it therefore meets π' in a point which lies on k'.

The same argument, with β-planes, shows that k' meets the same six planes of W as k; so that C and C' meet Γ in the same six points, and this completes the proof of the theorem.

The theorem has an interesting extension† which states that the relation between the three curves C_1, C_2, C_3 is entirely symmetrical, and also that three general cubic primals with six common nodes intersect, residually to the fifteen joins of the nodes, in a general triad of associated quartics such as we have described.

3.31. *The fifth associated line.* We shall show next that the above theorem, when C_1 degenerates into three skew lines l_1, l_2, l_3 and their unique transversal line λ, yields the fundamental theorem of four-dimensional line-geometry, namely, the theorem of the fifth associated line.

To show this we observe first that in the argument of § 3.2 the degenerate form above suggested is an admissible form for Γ; for if $C^3(A_1^2, A_2, A_3, A_4)$ breaks up, as it may, into the lines A_1A_2, A_1A_3, A_1A_4 and the neighbourhood of A_1, then Γ breaks up as suggested; and the Φ-curves meet each of l_1, l_2, l_3 in two points, but do not meet λ. Thus *the quadric Ω can also be represented on S_4 by quadrics Φ through the degenerate‡ form (l_1, l_2, l_3, λ) of Γ, and in this case its plane sections are represented by ${}^0C^4$ having l_1, l_2, l_3 as chords.*

With this modification, the surface σ degenerates into three α-planes, α_1, α_2, α_3 say, representing the three linear congruences of fundamental lines meeting pairs of l_1, l_2, l_3, together with the (passive) β-plane, β_0 say, which meets these three planes in lines; and *the general α-plane corresponds to a plane which meets l_1, l_2, l_3,*

† See, for example, Semple, loc. cit., and Bronowski, *Quart. J. of Math.* (Oxford), **14** (1943), 5.

‡ By suitable choice of coordinates (ξ_i) in S_4, this parametric representation of Ω may be taken in the form

$$\frac{p_1}{\xi_3\,\xi_4} = \frac{p_2}{\xi_4\,\xi_2} = \frac{p_3}{\xi_2\,\xi_3} = \frac{p_4}{\xi_1\,\xi_2} = \frac{p_5}{-\xi_3\,(\xi_0+\xi_1)} = \frac{p_6}{\xi_0\,\xi_4}.$$

If the conic k, in the notation of the preceding section, meets $\alpha_1, \alpha_2, \alpha_3$, then the curve C is reduced effectively to a line l_4, the rest of it (*qua* Φ-curve) consisting of the three lines which meet l_4 and a pair of l_1, l_2, l_3; and any general line of S_4 can be so regarded as l_4. But if k meets $\alpha_1, \alpha_2, \alpha_3$, so likewise must k'; and hence C' is reduced effectively, like C, to a line l_5 of S_4. Thus the fact that all α-planes which meet k also meet k' means in this case that all the transversal planes of l_1, l_2, l_3 which meet l_4 also meet l_5. This gives

THEOREM XXIV. *With any four general lines of S_4 there is associated a determinate fifth line such that all planes which meet the first four lines meet also the fifth.*

The last line is called the *fifth associated line* relative to the given tetrad.

Since, in S_4, the line and the plane are dual to one another, it follows that the lines of S_4 which meet four given planes meet also a fifth associated plane.

EXAMPLES

1. Prove that the lines of [4] which meet four given planes (and therefore the fifth associated plane) generate a cubic primal containing the five planes and having nodes at the ten points at which the planes meet in pairs.

(Let $\pi_1, \pi_2, \pi_3, \pi_4$ denote the given planes; then it is required to find how many lines incident to π_i meet a given line l. Suppose that l is taken to meet π_4 at a point P, in which case it lies in a prime Π through π_4. Then there is one line through P which meets π_1, π_2, π_3, and there are two lines meeting l and the three lines in which Π intersects π_1, π_2, π_3.)

2. Prove that the trisecant planes of a rational normal quartic which meet a given line all meet a second such quartic having six points in common with the first.

3.4. *The representation of congruences.* Returning now to the subject of line geometry in [3], we show how the present representation of Ω on S_4 may be used in the discussion of certain properties of line systems.

We have seen that a line complex in [3] may be represented by a threefold which is the complete intersection of Ω with a primal of some order; from this point of view its properties may be examined. So also may those of a congruence which is the complete intersection of two complexes. However, for a general congruence $K^{(m,n)}$, where $m \neq n$, the representative surface F on Ω is not obtainable as a complete intersection: in fact the primary problem in this connexion is that of establishing the *existence* of a congruence with given indices. For such a purpose the representation of the lines of [3] on Ω, and the projection of Ω on S_4 (§ 3) are of little use in themselves. As we have seen, the surface F on Ω is usually a special case of a surface having the same elementary projective characters, possessing improper nodes, and other particular features; this will be further demonstrated by the examples which follow. If instead of the direct representation on Ω we wished to use the projection on S_4, we should have to prove that a surface of given order in S_4 can be drawn to meet a given quadric surface in a certain curve, on which it must have a prescribed number of improper nodes; and even for congruences of low order and class this is difficult to accomplish.

In the following set of examples the methods of representing the simplest types of congruences are outlined. For this purpose we require to note how the singular points and double rays of a congruence are mapped on S_4 when we use the second representation (§ 3.2).

Suppose that the congruence K corresponds to a surface F on Ω, which in turn corresponds to a surface F' in S_4. Then a singular point of K, corresponding to an α-plane which meets F in a curve, will be represented in one of two ways:

(i) by a trisecant plane of Γ which meets F' in a curve.

(ii) by a point of intersection of F' with Γ.

A double ray of K, corresponding to an improper node of F, will be represented by

(i) an improper node of F', or

(ii) a chord of Γ which meets F' in two points not on Γ.

The singular planes of K will evidently correspond to cubic scrolls through Γ which meet F' in curves residual to Γ; but these are obviously more difficult to identify.

For further information concerning the following results, as well as for the classification of quadratic and cubic congruences in general, we refer the reader to the paper already quoted on p. 261.

EXAMPLES

1. Prove that an irreducible (1, 3) congruence is necessarily a congruence of chords of a twisted cubic.

2. Show that the quadric surfaces of S_4 which meet Γ in $6-n$ points ($2 \leqslant n \leqslant 6$) represent general quadratic congruences $(2, n)$. Prove that

the corresponding surfaces on Ω are Del Pezzo surfaces which are of general character only when $n = 2, 3$.

Show that each of these congruences is contained in at least one tetrahedral complex.

Show that the congruence (2, 6) of this type is represented by a Del Pezzo octavic surface of the *second* kind (cf. Ch. VII, § 5).

3. Denoting the latter surface by F, and its image on S_4 by Q, show that four lines of each regulus of Q lie in trisecant planes of Γ, and that these represent conics on F. Hence show that the corresponding congruence $K_2^{(2,6)}$ has four singular points of order four and eight of order two, so that it may be indicated by the symbol $K_2^{(2,6)}[4^4, 2^8]$.

4. Discuss similarly the remaining congruences of Ex. 2, showing that their singular points are given by the following scheme:

$$K^{(2,5)}[4^1, 3^3, 2^6, 1^8], \qquad K^{(2,3)}[2^5, 1^{10}],$$
$$K^{(2,4)}[3^2, 2^6, 1^6], \qquad K^{(2,2)}[1^{16}].$$

5. Prove that the normals to a central quadric surface form a congruence (6, 2) which is contained in a tetrahedral complex, and that the normals at points of a generator lie on a hyperbolic paraboloid.

Prove that there are eight generators of the quadric for which this paraboloid degenerates tangentially into a parabola.

Show that the congruence is dual to the type $K_2^{(2,6)}$ described above.

6. Show that, if the Veronese surface σ degenerates into a cubic scroll R and an α-plane containing a generator of R, then Ω is represented on S_4 by means of quadrics passing through a twisted cubic C^3 and a line l meeting C^3. Show also that the ϕ-curves meet l twice and C^3 four times; and that the solid containing C^3 corresponds to an α-plane.

7. Using this representation, show that a cubic scroll through l corresponds to a congruence $K_1^{(2,6)}[5^1, 3^6, 2^4, 1^1]$ whose image on Ω is a Del Pezzo octavic surface of the first kind.

8. Show also that a projected Veronese surface having l as double line represents a congruence $K^{(2,7)}[6^1, 3^{10}]$.

9. Using the representation by quadrics through Γ, show that a general cubic scroll of S_4 represents a congruence $K^{(3,9)}[2^6]$ of section genus 2, having 21 double rays.

10. Show that a general cubic surface, lying in a solid of S_4, represents a congruence $K^{(3,9)}[6^4]$ of section genus 4, having 6 triple rays; and that, if the surface meets Γ in four points, it corresponds to a congruence $K^{(3,5)}[3^4, 1^4]$ without multiple rays.

3.5. ***Study's dual representation.*** We now describe a third representation of the lines of [3]; this is due to Study† and, being of a metrical character, finds its natural applications in the field of real Euclidean geometry.

† See Blaschke, *Differentialgeometrie*, i, ch. ix, for a detailed account.

For the definition of our line coordinates we shall suppose that the frame of reference adopted in §1 consists of the plane at infinity $x_0 = 0$, and three mutually perpendicular planes $x_i = 0$ $(i = 1, 2, 3)$. As before, we assume that a given line p is specified by two of its points (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3); but we modify slightly the definitions of p_1, p_2, p_3, writing

$$l = (x_0 y_1 - x_1 y_0)/d, \quad m = (x_0 y_2 - x_2 y_0)/d, \quad n = (x_0 y_3 - x_3 y_0)/d, \tag{1}$$

where d is the distance between the points in question. Thus (l, m, n) are the direction cosines of p, and satisfy the relation

$$l^2 + m^2 + n^2 = 1. \tag{2}$$

The remaining coordinates of p are denoted by λ, μ, ν, where

$$\lambda = nx_2 - mx_3, \qquad \mu = lx_3 - nx_1, \qquad \nu = mx_1 - lx_2. \tag{3}$$

Clearly we have the identical relation

$$l\lambda + m\mu + n\nu = 0 \tag{4}$$

as before. Also the condition of intersection of two lines, with coordinates $(l, m, n, \lambda, \mu, \nu)$ and $(l', m', n', \lambda', \mu', \nu')$, is

$$\sum (l\lambda' + l'\lambda) = 0. \tag{5}$$

The condition that the lines should be at right angles is

$$ll' + mm' + nn' = 0. \tag{6}$$

It thus appears that the 6-vector (p_i), which has served hitherto to specify the line p, is now replaced by the pair of 3-vectors defined by (1) and (3). We shall write

$$(l, m, n) = \mathbf{p}_1, \qquad (\lambda, \mu, \nu) = \mathbf{p}_2. \tag{7}$$

In virtue of (2) and (4), the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ satisfy the relations

$$\mathbf{p}_1^2 = 1, \qquad \mathbf{p}_1 \cdot \mathbf{p}_2 = 0. \tag{8}$$

We now require a suitable calculus by which to treat real line-geometry from this new point of view. This is provided by the concept of *dual numbers* which we proceed to explain.

We define a *dual number* $a + \epsilon b$ as a symbol in which a, b are real and ϵ is an element such that $\epsilon^2 = 0$; we call a and b the real and dual parts of the number. The sum and product of two such numbers $a + \epsilon b$, $a' + \epsilon b'$ are then defined by the equations

$$(a + \epsilon b) + (a' + \epsilon b') = (a + a') + \epsilon(b + b'), \tag{9}$$

$$(a + \epsilon b)(a' + \epsilon b') = aa' + \epsilon(ab' + ba'). \tag{10}$$

It is obvious that addition and multiplication when so defined are commutative. We shall write the dual zero $0+\epsilon 0$ as 0, for brevity.

We next apply this concept to vector analysis. Combining the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of (7) we write

$$\mathbf{P} = \mathbf{p}_1+\epsilon\mathbf{p}_2 \tag{11}$$

and call $\mathbf{P}$ the *dual vector* representing the given line p. Clearly any real line specifies a unique vector $\mathbf{P}$ such as (11), whose real and dual parts satisfy the conditions (8): conversely, any such vector specifies a unique line.

The definitions of the sum, scalar product, and vector product of two dual vectors $\mathbf{P} = \mathbf{p}_1+\epsilon\mathbf{p}_2$, $\mathbf{Q} = \mathbf{q}_1+\epsilon\mathbf{q}_2$ are as follows:

$$\mathbf{P}+\mathbf{Q} = (\mathbf{p}_1+\mathbf{q}_1)+\epsilon(\mathbf{p}_2+\mathbf{q}_2), \tag{12}$$

$$\mathbf{P}.\mathbf{Q} = \mathbf{p}_1.\mathbf{q}_1+\epsilon(\mathbf{p}_1.\mathbf{q}_2+\mathbf{p}_2.\mathbf{q}_1), \tag{13}$$

$$\mathbf{P}\wedge\mathbf{Q} = \mathbf{p}_1\wedge\mathbf{q}_1+\epsilon(\mathbf{p}_1\wedge\mathbf{q}_2+\mathbf{p}_2\wedge\mathbf{q}_1). \tag{14}$$

We may then verify that the dual vectors satisfy the laws of ordinary vector algebra. Further, the triple scalar product $\mathbf{P}.\mathbf{Q}\wedge\mathbf{R}$ of three dual vectors may be evaluated by the above rules.

We observe that, by (8) and (13),

$$\mathbf{P}^2 = \mathbf{p}_1^2+2\epsilon\mathbf{p}_1.\mathbf{p}_2 = 1.$$

Hence *the vector* $\mathbf{P}$ *may be represented as a point of a unit dual sphere.*

We now prove that

The necessary and sufficient condition that two lines should intersect at right angles is that the scalar product of their corresponding dual vectors should be zero.

Let $\mathbf{P}$, $\mathbf{Q}$ correspond to the lines p, q; then, if p, q intersect at right angles, we have $\mathbf{p}_1.\mathbf{q}_1 = 0$, $\mathbf{p}_1.\mathbf{q}_2+\mathbf{p}_2.\mathbf{q}_1 = 0$, by (4) and (5). Hence, by (13), $\mathbf{P}.\mathbf{Q} = 0$. Conversely, if $\mathbf{P}.\mathbf{Q} = 0$, it follows from (13) that $\mathbf{p}_1.\mathbf{q}_1 = 0$, and $\mathbf{p}_1.\mathbf{q}_2+\mathbf{p}_2.\mathbf{q}_1 = 0$. These conditions express the fact that p and q intersect at right angles.

These last two results may be combined in the following proposition:

THEOREM XXV. *The lines of real three-dimensional space can be represented by unit dual vectors in such a way that two lines intersect at right angles if and only if the scalar product of their corresponding dual vectors vanishes.*

So far as the application of dual vectors to algebraic geometry is concerned, the above result is the most important; its use is illustrated in the examples which follow.

EXAMPLES

1. Verify that the dual vectors $\mathbf{P}$, $\mathbf{Q}$ satisfy the relations

$$\mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P}, \qquad \mathbf{P}\wedge\mathbf{Q} = -\mathbf{Q}\wedge\mathbf{P}, \qquad \mathbf{P}\wedge\mathbf{P} = 0.$$

2. Prove that two lines p, q are parallel if and only if $\mathbf{P}\wedge\mathbf{Q} = 0$.

3. Deduce from Theorem XXV that, if p, q are non-parallel lines, the dual vector $\mathbf{P}\wedge\mathbf{Q}$ represents the common perpendicular transversal to p and q. (In general $\mathbf{P}\wedge\mathbf{Q}$ will be non-normalized, i.e. its square will not be unity, but for most purposes this is immaterial.)

4. Prove that, if the lines p, q, r possess a common perpendicular transversal, then $\mathbf{P}\cdot\mathbf{Q}\wedge\mathbf{R} = 0$. What conclusion can be drawn from the condition $\mathbf{P}\cdot\mathbf{Q}\wedge\mathbf{R} = 0$?

5. Establish the identity

$$\mathbf{P}\wedge(\mathbf{Q}\wedge\mathbf{R}) = (\mathbf{P}\cdot\mathbf{R})\mathbf{Q}-(\mathbf{P}\cdot\mathbf{Q})\mathbf{R}.$$

6. *The Petersen–Morley Theorem. If p, q', r, p', q, r' are six lines in space forming a skew right-angled hexagon, then the three lines drawn to meet respectively p, p'; q, q'; r, r' at right angles have a common perpendicular transversal.*

The following proof is due to Todd.† Denote by $\mathbf{P}$, $\mathbf{Q}'$,... the dual vectors corresponding to p, q',.... Let a, b, c be the three lines in question, with corresponding vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. Since, by hypothesis, p' is a common perpendicular transversal of q and r, we may take $\mathbf{P}' = \mathbf{Q}\wedge\mathbf{R}$, to within a scalar multiple, and similarly we may write $\mathbf{Q}' = \mathbf{R}\wedge\mathbf{P}$, $\mathbf{R}' = \mathbf{P}\wedge\mathbf{Q}$.

Since a meets p and p' at right angles, we may take $\mathbf{A} = \mathbf{P}\wedge\mathbf{P}'$. Hence, by Ex. 5, $\mathbf{A} = \mathbf{P}\wedge(\mathbf{Q}\wedge\mathbf{R}) = (\mathbf{P}\cdot\mathbf{R})\mathbf{Q}-(\mathbf{P}\cdot\mathbf{Q})\mathbf{R}$, with analogous formulae for $\mathbf{B}$ and $\mathbf{C}$. Adding these results we deduce that $\mathbf{A}+\mathbf{B}+\mathbf{C} = 0$, so that $\mathbf{A}\cdot\mathbf{B}\wedge\mathbf{C} = -(\mathbf{B}+\mathbf{C})\cdot\mathbf{B}\wedge\mathbf{C} = 0$. Hence either b is parallel to c, or a, b, c have a common perpendicular transversal.

7. Assuming that the dual vector $\mathbf{P}$ is a function of a scalar parameter t, show that the line p generates a ruled surface. Show also that the surface is a right conoid with axis a if, for all values of t, $\mathbf{A}\cdot\mathbf{P} = 0$.

8. If $\mathbf{P}(t)$ represents a generator of a non-developable ruled surface, show that the vector $\mathbf{R} = \mathbf{P}\wedge\mathbf{Q}$, where $\mathbf{Q} = \mathbf{P}'/\sqrt{\mathbf{P}'^2}$, represents the common perpendicular transversal to consecutive generators p and p_1. (The point common to p and r is the *central point* relative to the generator p, and its locus is the *line of striction* of the surface.)

Prove that the line q is normal to the surface at the corresponding central point.

Establish the formulae

$$\mathbf{Q}' = -\lambda\mathbf{P}+\mu\mathbf{R}, \quad \mathbf{R}' = -\mu\mathbf{Q}, \text{ where } \lambda = \sqrt{\mathbf{P}'^2} \text{ and } \mu = \mathbf{P}\cdot\mathbf{P}'\wedge\mathbf{P}''/\mathbf{P}'^2.$$

† *Math. Gazette* **20** (1936), 184; for other proofs see Ex. 11, p. 243, and Lyons and Frith, *Proc. Camb. Phil. Soc.* **30** (1934), 192.

§ 4. Line geometry in S_4

4. Coordinates of a line: the fundamental identities. In S_4 the lines form an aggregate of dimension six; hence it is at once obvious that the line geometry in this space will be both richer in content and more complicated in structure than the system we have so far studied. Also, as will appear, certain phenomena which present themselves have no analogue in the line geometry of S_3, and therefore serve to broaden our view of the problem of line systems in general. It is clear, then, that our account of the subject will perforce be of a summary nature, and that the reader must be referred to the literature for more detailed information.

We begin by defining the coordinates of a line in S_4; this is effected precisely as in § 1. Thus, if $(x) = (x_0, x_1, x_2, x_3, x_4)$ and $(y) = (y_0, y_1, y_2, y_3, y_4)$ are distinct points of S_4, we see that the ratios of the determinants

$$p_{ij} = |x\ \ y|_{ij} \quad (i, j = 0, 1, 2, 3, 4;\ i \neq j),$$

extracted from the matrix

$$\begin{pmatrix} x_0 & . & . & . & x_4 \\ y_0 & . & . & . & y_4 \end{pmatrix}$$

will be unaltered if (x) and (y) are replaced by any two points of the line p joining (x) and (y). We now define the ten quantities

$$p_{01},\quad p_{02},\quad p_{03},\quad p_{04},\quad p_{12},\quad p_{13},\quad p_{14},\quad p_{23},\quad p_{24},\quad p_{34}$$

as the homogeneous coordinates of p, and we shall frequently make use of the fact that $p_{ji} = -p_{ij}$ $(i, j = 0, ..., 4)$; clearly there is a (1, 1) correspondence, without exceptional elements, between the lines p and the sets of ten coordinates p_{ij} which, taken in a prescribed order, form the components of the coordinate vector of p.

Since the totality of lines (p) has freedom six, it follows that these coordinates cannot be independent; in fact, by the method adopted in § 1, we obtain five quadratic identities between them, namely,

$$\left.\begin{aligned} \Omega_{pp}^{(0)} &\equiv p_{12}p_{34} + p_{23}p_{14} + p_{31}p_{24} = 0 \\ \Omega_{pp}^{(1)} &\equiv p_{02}p_{34} + p_{23}p_{04} + p_{30}p_{24} = 0 \\ \Omega_{pp}^{(2)} &\equiv p_{01}p_{34} + p_{13}p_{04} + p_{30}p_{14} = 0 \\ \Omega_{pp}^{(3)} &\equiv p_{01}p_{24} + p_{12}p_{04} + p_{20}p_{14} = 0 \\ \Omega_{pp}^{(4)} &\equiv p_{01}p_{23} + p_{12}p_{03} + p_{20}p_{13} = 0 \end{aligned}\right\}. \qquad (1)$$

Here we encounter the first complication of the problem; for the relations (1), though linearly independent, must in fact be equivalent to three conditions only, giving seven independent homogeneous parameters with which to define a line p. The geometrical interpretation of this fact will be given in § 5.

4.1. *Intersecting lines.* The necessary and sufficient conditions that two lines p, q, defined by the coordinates (p_{ij}) and (q_{ij}), should intersect, are that all the four-rowed determinants of the matrix

$$\begin{pmatrix} x_0 & . & . & . & x_4 \\ y_1 & . & . & . & y_4 \\ X_0 & . & . & . & X_4 \\ Y_0 & . & . & . & Y_4 \end{pmatrix}$$

in which $|x\,y|_{ij} = p_{ij}$, $|X\,Y|_{ij} = q_{ij}$, should vanish. We thus obtain five conditions which, in virtue of (1), are effectively equivalent to two. The first of these is

$$\Omega_{pq}^{(0)} \equiv \sum (p_{12}\,q_{34} + p_{34}\,q_{12}) = 0,$$

which is the polarized form of $\Omega_{pp}^{(0)}$; the remainder are similarly obtained by polarizing $\Omega_{pp}^{(1)}, \ldots, \Omega_{pp}^{(4)}$. Denoting the set of relations by the symbol $\Omega_{pq}^{(i)} = 0$, we have

THEOREM XXVI. *Two lines p, q of S_4 intersect if and only if their coordinates satisfy the relations $\Omega_{pq}^{(i)} = 0$ $(i = 0, 1, 2, 3, 4)$.*

4.11. *Linearly related lines.* Taking, as in § 1.2, a linear combination of two or more coordinate vectors, we inquire under what circumstances every vector so obtained represents a line; using the relations (1) and Theorem XXVI, we see that this will be the case if and only if the lines of the system are mutually intersecting. We thus obtain the following systems, of dimension 1, 2 and 3 respectively:

(i) the *pencil*: there are evidently ∞^8 such systems, each defined by a pair of intersecting lines;

(ii) the *ruled plane*: there are ∞^6 such systems, each defined by a triad of coplanar non-concurrent lines;

(iii) the *solid point-star*: there are ∞^7 such systems, each defined by a prime (solid) of S_4, and a point of the prime; or by a triad of concurrent, non-coplanar lines;

(iv) the *point-star*: this consists of all the lines through a given point. There are ∞^4 such systems, and each is defined by a tetrad of concurrent lines. Hence

THEOREM XXVII. *In S_4 there exist the following systems of linearly related lines:*

(i) *the pencil, defined by a pair of intersecting lines;*

(ii) *the ruled plane, defined by three coplanar lines;*

(iii) *the solid point-star, defined by three concurrent, non-coplanar lines;*

(iv) *the point-star, defined by four concurrent lines not lying in a prime.*

These systems have freedom 8, 6, 7, 4 *respectively.*

4.2. *The linear complex.* We define a *general linear complex* $\mathscr{L}$ as the totality of ∞^5 lines (p) whose coordinates satisfy a relation of the form

$$\sum a_{ij} p_{ij} = 0, \tag{2}$$

where the ten coefficients a_{ij} are arbitrary and we may take $a_{ij} = -a_{ji}$. We say, instead, that the complex is *special* if the a_{ij} satisfy the conditions

$$\Omega^{(i)}_{aa} = 0 \quad (i = 0, 1, ..., 4). \tag{3}$$

We note first that ∞^3 lines of $\mathscr{L}$ lie in a given prime S_3, forming a linear complex in that space. Next, we observe that ∞^2 lines of $\mathscr{L}$ pass through a generic point $P(x_i)$; evidently they lie in the prime through P whose equation is

$$\sum a_{ij}(x_i X_j - x_j X_i) = 0. \tag{4}$$

The coordinate vector $\mathbf{u} \equiv (u_0, ..., u_4)$ of this prime is given by the equation

$$\tilde{\mathbf{u}} = \mathbf{a}\mathbf{x}, \tag{5}$$

where $\tilde{\mathbf{u}}$ is the transposed matrix of $\mathbf{u}$, and $\mathbf{a}$ is the skew-symmetric matrix (a_{ij}). Now the determinant $\Delta \equiv |a_{ij}|$, being skew-symmetric† and of odd order, vanishes identically; furthermore, the cofactors A_{ij} of a_{ij} in Δ are found to have the values

$$A_{ii} = (a_{\delta\alpha} a_{\beta\gamma} + a_{\delta\beta} a_{\gamma\alpha} + a_{\delta\gamma} a_{\alpha\beta})^2, \tag{6}$$

where $\alpha, \beta, \gamma, \delta$ are the subscripts other than i, and

$$A_{ij} = -(a_{i\alpha} a_{\beta\gamma} + a_{i\beta} a_{\gamma\alpha} + a_{i\gamma} a_{\alpha\beta})(a_{j\alpha} a_{\beta\gamma} + a_{j\beta} a_{\gamma\alpha} + a_{j\gamma} a_{\alpha\beta}), \tag{7}$$

where $i \neq j$, and α, β, γ are the subscripts other than i, j.

† See Cullis, *Matrices and determinoids*, vol. ii (Cambridge, 1918), 521.

Since $\Delta = 0$, the five linear forms given by

$$u_\alpha \equiv a_{\alpha i} x_i \quad (\alpha = 0,\ldots, 4) \tag{8}$$

are linearly dependent, and they all vanish simultaneously at the point O whose coordinates ξ_i are given by

$$\xi_0/\phi_{1234} = -\xi_1/\phi_{0234} = \xi_2/\phi_{0134} = -\xi_3/\phi_{0124} = \xi_5/\phi_{0123}, \tag{9}$$

where $\phi_{\alpha\beta\gamma\delta} = a_{\alpha\beta} a_{\gamma\delta} + a_{\alpha\gamma} a_{\delta\beta} + a_{\alpha\delta} a_{\beta\gamma}$, and where, it will be noted, any interchange of two suffixes of $\phi_{\alpha\beta\gamma\delta}$ changes its sign.†

It is clear, then, that the skew-polarity defined by (5) is such that the prime (4) corresponding to any point P always passes through‡ the fixed point O and all points on OP have the same polar prime; this point is called the *centre* of the skew-polarity, and of the associated linear complex. Thus we have

THEOREM XXVIII. *The lines of a general linear complex $\mathscr{L}$ which pass through a generic point lie in a prime; but there is one point, uniquely determined by $\mathscr{L}$, such that every line through it belongs to $\mathscr{L}$.*

COROLLARY 1. *The lines of $\mathscr{L}$ which lie in a given plane form a pencil.*

COROLLARY 2. *The lines of $\mathscr{L}$ which lie in a given prime form a special linear complex if and only if the prime contains the centre of $\mathscr{L}$; and in that case the axis of the complex passes through the centre.*

Consider next the case of a *special* linear complex, for which the coefficients a_{ij} in (2) satisfy the relations (3). In this case all the first minors of Δ vanish: hence, by a property of skew-symmetric determinants, all the second minors of Δ likewise vanish, i.e. the five primes with the equations $a_{\alpha,i} x_i = 0$ have a common *plane*. Thus the centre of the complex is no longer a point but a plane; hence

THEOREM XXIX. *The lines of a special linear complex consist of all the lines which meet a given plane.*

This plane is called the *axial plane* of the complex.

Clearly there are ∞^9 general linear complexes but only ∞^6 special linear complexes in S_4.

† If i is the subscript other than $\alpha, \beta, \gamma, \delta$, then $\phi_{\alpha\beta\gamma\delta} = \pm\Omega_{\alpha\alpha}^{(i)}$.

‡ For $u_\alpha \xi_\alpha = a_{\alpha i} x_i \xi_\alpha = -(a_{i\alpha} \xi_\alpha) x_i = 0$.

EXAMPLES

1. Show that any line p through the centre O of a skew-polarity of S_4 is such that all points of p have the same polar prime Π through p, and that p and Π meet any prime Σ, not through O, in pole and polar plane of a skew-polarity in Σ.

2. Show also that every line r, not through O, has a polar plane π through O, and that all lines r in a plane ρ through O have the same polar plane π.

3. Deduce from Ex. 2 that the lines of a linear complex which meet a fixed line r of general position meet the polar plane of r with respect to the associated skew-polarity.

4.3. *Pencils of linear complexes.* Let $\mathscr{L}^{(1)}$, $\mathscr{L}^{(2)}$ be general linear complexes, in general position; and consider the system K_4 of lines common to the pencil $\mathscr{L}^{(1)}+\lambda\mathscr{L}^{(2)}$. Clearly ∞^1 lines of K_4 will pass through a given point, forming a pencil, and one line of the system will lie in a given plane, namely, the line common to the pencils of lines belonging to $\mathscr{L}^{(1)}$ and $\mathscr{L}^{(2)}$ which lie in this plane.

Consider now the locus of centres of the system of complexes; if the equations of $\mathscr{L}^{(1)}$ and $\mathscr{L}^{(2)}$ are $\sum a_{ij}^{(1)}p_{ij} = 0$, $\sum a_{ij}^{(2)}p_{ij} = 0$ respectively, the centre of $\mathscr{L}^{(1)}+\lambda\mathscr{L}^{(2)}$ is obtained by writing $a_{ij} = a_{ij}^{(1)}+\lambda a_{ij}^{(2)}$ in (9). We then see that the polynomials† $\phi_{\alpha\beta\gamma\delta}$ are quadratics in λ, so that the locus of centres is a conic; and K_4 contains every line lying in the plane of this conic. Hence

THEOREM XXX. *The general linear ∞^4 system of lines in S_4, base of a pencil of linear complexes, contains one ruled plane π, and the locus of centres of the complexes is a conic in π.*

4.4. *Nets of linear complexes. The linear congruence.* Consider next the system K_3 of lines common to the net

$$\lambda\mathscr{L}^{(1)}+\mu\mathscr{L}^{(2)}+\nu\mathscr{L}^{(3)}$$

formed by three general linear complexes. Evidently one line of K_3 will pass through a generic point of S_4; the system is therefore called a *linear congruence*.

A *singular point* of K_3 is defined as one through which there passes more than one (and thus an infinity) of its lines. It is clear that such lines will form a pencil. Since a unique complex of the net can be made to contain any two general lines, in particular two *arbitrary* lines through a singular point, we deduce that the

† It will appear from § 4.5 that these polynomials have no common zero.

latter is the centre of a complex of the net. Thus, *the locus of singular points of a linear congruence is also the locus of centres of the linear complexes which define it.* We call this the *singular surface* of K_3. We now prove

THEOREM XXXI. *The singular surface of a linear congruence is a projected Veronese surface, of which the lines of the congruence are trisecants.*

For the skew-polarity associated with the general complex of the net has a matrix of the form $\lambda\mathbf{a}^{(1)}+\mu\mathbf{a}^{(2)}+\nu\mathbf{a}^{(3)}$, of which the typical element is $a_{ij} = \lambda a_{ij}^{(1)}+\mu a_{ij}^{(2)}+\nu a_{ij}^{(3)}$. Thus, in (9), the functions $\phi_{\alpha\beta\gamma\delta}$ are linearly independent homogeneous quadratic forms in λ, μ, ν, and they have no common zero;† whence, by Ch. VII, § 3, the locus of centres of complexes of the net is a projected Veronese surface F^4. The ∞^2 pencils contained in the net of complexes have the ∞^2 conics of F^4 as their loci of centres; and any two of the pencils have one complex in common, corresponding to the intersection of the two conics on F^4.

Now any trisecant of F^4 belongs to the congruence since it contains three centres of (linearly independent) complexes of the net. And since we know that a unique trisecant of F^4 passes through a generic point of S_4, we may identify the lines of the congruence with the trisecants of F^4.

4.5. *Webs of linear complexes. The Segre cubic primal.* The lines common to a general web of linear complexes form an ∞^2 system K_2, and thus generate a ruled primal V_3 which may be regarded as a natural analogue of the regulus in ordinary space. We shall show that V_3 is actually the ten-nodal cubic primal already discussed in Ch. VIII, § 2.32; more precisely we shall prove

THEOREM XXXII. *The locus of centres of a web of linear complexes is a Segre ten-nodal cubic primal containing the lines common to the complexes of the web; and all the lines are incident to a set of five associated planes lying on the cubic primal.*

If we denote the web by $\lambda\mathscr{L}^{(1)}+\mu\mathscr{L}^{(2)}+\nu\mathscr{L}^{(3)}+\rho\mathscr{L}^{(4)}$, the locus of centres of complexes of the system is again given parametrically by (9), in which the a_{ij} are linear homogeneous functions of λ, μ, ν, ρ. These equations are equivalent to a representation of the

† This also will appear from § 4.5.

locus V_3, on the space S_3 of the coordinates $(\lambda, \mu, \nu, \rho)$, by means of the system of quadrics

$$\Phi \equiv \theta_0 \phi_{1234} + \theta_1 \phi_{0234} + \ldots + \theta_4 \phi_{0123} = 0.$$

Now any base point of this system must satisfy the conditions

$$\begin{aligned} \phi_{0234} &\equiv a_{02} a_{34} + a_{03} a_{42} + a_{04} a_{23} = 0, \\ \phi_{1234} &\equiv a_{12} a_{34} + a_{13} a_{42} + a_{14} a_{23} = 0, \\ -\phi_{0134} &\equiv (a_{03} a_{14} - a_{04} a_{13}) - a_{01} a_{34} = 0, \\ \phi_{0124} &\equiv (a_{04} a_{12} - a_{02} a_{14}) - a_{01} a_{42} = 0, \\ -\phi_{0123} &\equiv (a_{02} a_{13} - a_{03} a_{12}) - a_{01} a_{23} = 0. \end{aligned} \tag{10}$$

By substituting the values of a_{34}, a_{42}, a_{23} given by the last three of these equations in the first two, we see that those of the eight intersections of the last three quadrics ϕ which do not lie on the plane $a_{01} = 0$ lie on the first two; and since $a_{01} = 0$ plainly meets the last three surfaces at its intersections with the twisted cubic whose equations are

$$\frac{a_{02}}{a_{12}} = \frac{a_{03}}{a_{13}} = \frac{a_{04}}{a_{14}},$$

it follows that Φ is an ∞^4 system of quadrics with five base points. Hence, by Ch. VIII, § 2.32, the *locus of centres of complexes of the web is a Segre ten-nodal cubic primal* V_3^3.

Denoting the five base points by $O_1, O_2, \ldots, O_5$, we recall that the neighbourhood of O_i corresponds to a plane on V_3^3. If we now compare the equations (10) with the conditions (3) for a linear complex to be special, we see that the coordinates of a base point O_i are precisely a set of values of $(\lambda, \mu, \nu, \rho)$ which render the corresponding complex of the web special. Hence the five planes on V_3^3 which correspond to the neighbourhoods of the points O_i are *the axial planes of five special linear complexes contained in the web*; and, moreover, these are the only special complexes in the system.

Since the lines of K_2 are common to these special complexes, they must meet all five of the corresponding axial planes. Hence they lie on V_3^3 and form one system of generating lines on the primal, namely, the one corresponding in the space representation to the system of twisted cubics through (O_i). Conversely, any line which meets four of the axial planes belongs to the system K_2 and must obviously lie on V_3^3; we may then identify it with one of the generating lines represented by twisted cubics through

(O_i), whence it follows that the line meets the fifth axial plane; that is to say, the planes form an associated set.

Incidentally we have obtained the following result:

COROLLARY. *A general web of linear complexes contains five special complexes.*

4.6. *The elliptic quintic scroll.* From the above results we have

THEOREM XXXIII. *The lines common to five linear complexes generate an elliptic quintic scroll; and the axial planes of the ∞^1 special complexes contained in the linear system defined by the given five meet the scroll in cubic curves.*

This follows from the fact that the lines in question are those common to two Segre cubic primals through a projected Veronese surface; by the formulae of Ch. IX, § 5.2, these primals meet residually in a quintic surface whose sections are elliptic. Since any axial plane of a special complex contained in the system meets the cubic primals in a cubic curve common to both, it follows that this curve lies on the scroll.

Conversely, any plane which meets the scroll in a cubic curve is the axial plane of a special complex contained in the system; for it contains a directrix curve of the scroll and so is met by all the generators.

Since five generators of this scroll meet a generic plane, we have

THEOREM XXXIV. *Six linear complexes in general have five common lines.*

These lines form an associated set; for four of the lines in general define a linear ∞^5 system of complexes which contain them, i.e. the system defined by the six given complexes; and hence any plane which meets four of the lines is an axis of one of the ∞^2 special complexes contained in the system, and so meets all five lines.

COROLLARY. *Six planes of S_4 are in general met by five associated lines.*

This follows from the duality of line and plane in S_4.

For further information concerning the linear complexes in S_4 reference may be made to Castelnuovo's original paper.†

† *Atti Ist. Veneto,* (7) 2 (1891), 855.

EXAMPLES

1. A *null system* is defined by the correlation $u_i = \sum_j a_{ij} x_j$ $(i = 0,..., 4)$, where $\sum u_i x_i = 0$. Two points P, P' are said to be conjugate with regard to the system if P' lies in the prime (u_i) corresponding to the point $P(x_i)$. Prove that the totality of lines PP' generate a linear complex.

2. Show that any special linear complex can be represented by the equation $p_{01} = 0$, and that the equation of any general linear complex can be written in the form $ap_{01} + bp_{23} = 0$. What is the centre of the latter?

3. Prove that the lines of a system K_4 which meet a given line generate a quadric point cone, and determine the vertex of the cone.

4. Taking the parametric equations of a rational normal quartic curve in the form $x_0:x_1:x_2:x_3:x_4 = 1:t:t^2:t^3:t^4$, show that the tangents to the curve are contained in a net of linear complexes.

Prove also that any general net of linear complexes is the system of complexes containing the tangents to a rational normal quartic.

5. Prove that the lines of a linear congruence which meet a given line l generate a cubic scroll having l for directrix.

By supposing l to belong to the congruence, deduce that all the lines of the congruence are trisecants of the singular surface. Also, by considering the lines of the congruence which lie in a given S_3, show that the singular surface is cut by S_3 in a quartic $(3, 1)$ curve on a quadric surface.

Hence prove that the surface is a projected Veronese surface.

6. Interpret the results of Examples 15–17 on Ch. VII in relation to the geometry of a linear congruence of S_4.

7. Prove that a unique Segre cubic primal can be drawn to contain a given projected Veronese surface F and an arbitrary plane. Deduce that there is an ∞^6 linear system of such primals through F.

If this system is represented upon the primes of S_6, show that a general point of S_4, which determines a unique trisecant of F, corresponds to a unique point of S_6, but that the latter corresponds to all the points of the trisecant.

8. Show that two projected Veronese surfaces have in general five common trisecants.

9. Given that the locus of centres of the linear complexes of a web is the Segre primal V_3^3 of which one system of generating lines is that common to the complexes of the web, determine what pencils of complexes in the web correspond to the remaining five systems of generating lines on V_3^3.

10. Discuss the system of lines which meet three given planes of S_4. In what respects does it differ from the general linear congruence?

Prove that the lines which meet four given planes generate a Segre cubic primal, and deduce that six given planes are in general met by five lines.

§ 5. THE REPRESENTATION ON S_9

5. The manifold M_6^5. Proceeding as in § 2 we next obtain a representation of our line system by the points of a manifold; as before, this is effected by taking the coordinates p_{ij} of a line p to be homogeneous coordinates of a point which, in the present instance, will be situated in S_9. Since the aggregate of lines in S_4 has dimension six, the resulting manifold M will also have dimension six. Its prime sections will correspond to linear complexes; and, generally, its S_k sections $(3 \leqslant k \leqslant 8)$ will represent line systems common to $9-k$ linear complexes. Thus the order of M, i.e. the number of points in which it is met by a generic S_3, is equal to the number of lines common to six general linear complexes; and this, by Theorem XXXIV, is five. We shall denote the manifold by M_6^5; the section genus of M_6^5 is equal to that of the scroll of lines common to five general linear complexes, so that the [4]-sections are elliptic quintic curves. Hence

THEOREM XXXV. *The lines of S_4 can be represented birationally, and without exception, upon the points of a manifold M_6^5 of S_9, whose curve sections are elliptic.*

It is interesting to deduce this result independently from the equations of § 4. Each of the equations (1), interpreted geometrically, represents a quadric primal of S_9; and, since the equation of each is expressed in terms of six coordinates only, the primals in question are [3]-cones.

Consider now the three cones whose equations are $\Omega^{(2)} = 0$, $\Omega^{(3)} = 0$, $\Omega^{(4)} = 0$; these intersect in a manifold V_6^8. Writing the equations in the form

$$\begin{aligned} p_{01}p_{34} &= p_{03}p_{14} - p_{04}p_{13}, \\ p_{01}p_{24} &= p_{02}p_{14} - p_{04}p_{12}, \\ p_{01}p_{23} &= p_{02}p_{13} - p_{03}p_{12}, \end{aligned} \tag{1}$$

we see that V_6^8 contains the V_6^3 whose equations are

$$p_{01} = 0, \qquad p_{02}/p_{12} = p_{03}/p_{13} = p_{04}/p_{14}. \tag{2}$$

We now show that the residual component M_6^5 lies on the cones

$$\Omega^{(0)} = 0, \qquad \Omega^{(1)} = 0.$$

Multiplying equations (1) by p_{12}, p_{31}, p_{14} in order and adding, we obtain

$$p_{01}(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23}) = 0.$$

Rejecting the component $p_{01} = 0$, which does not contain M_6^5, there remains $\Omega^{(0)} = 0$. Similarly we may show that M_6^5 lies on $\Omega^{(1)} = 0$.

Next, to show that the manifold M_6^5 has elliptic curve sections we consider a generic S_4-section of the cones $\Omega^{(i)}$ ($i = 2, 3, 4$); we thus obtain three general quadric primals in S_4 passing through the cubic curve in which S_4 meets the manifold V_6^3 represented by (2). This curve is rational; whence, by the formulae of Ch. IX, § 1, we deduce that the residual intersection of the three quadrics is an elliptic quintic.

As an immediate consequence of Theorem XXXV, we have the

COROLLARY. *A scroll of order n in S_4 is represented on M_6^5 by a curve of order n.*

For the number of generators of the scroll which meet a plane π is equal to the number of points in which the corresponding curve of M_6^5 is met by the prime which corresponds to the special linear complex defined by π.

5.1. *The linear spaces on M_6^5.* In this and the following sections we examine briefly the chief properties of M_6^5. Here the contrast between line geometry in S_3 and S_4 is marked by the fact that, instead of using M_6^5 as a means to discuss the line systems in S_4, it is at first more practicable to reverse the procedure.

The only linear spaces of various dimensions which the manifold M_6^5 can contain are those corresponding to the systems of linearly related lines in S_4. From what has been said in § 4.11, there will be two main types, namely,

(i) ∞^6 *planes* ρ, representing ruled planes, and

(ii) ∞^4 *solids* Σ, corresponding to point-stars.

Included in (ii) there are ∞^8 *lines*, representing the pencils of S_4, and ∞^7 *planes* σ, corresponding to solid point-stars.

Consider first the system of lines. From § 4.1 we see that two lines p, q of S_4 intersect if and only if the corresponding points P, Q are conjugate with regard to the cones $\Omega^{(i)}$ containing M_6^5; this means that the line PQ lies on all these cones and therefore on M_6^5.

The totality of lines q which meet p, and the pencils which such lines form with p, correspond respectively to the points and generators of a conical manifold V_4 with vertex P, lying on M_6^5. The order of V_4 is equal to the number of lines which meet p and which lie in a general web of linear complexes. By § 4.5, this number is three; also all the generating lines must lie in the tangent space S_6 to M_6^5 at P; hence

THEOREM XXXVI. *The aggregate of lines which meet a given line p of S_4 is represented by a conical manifold V_4^3, with vertex P, which is the section of M_6^5 by the tangent space S_6 at P.*

Evidently V_4^3 contains the ∞^1 Σ-solids corresponding to the point-stars whose centres are on p; it also contains the ∞^2 ρ-planes corresponding to the ruled planes passing through p. These solids and planes are the joins of P to the generating planes and directrices respectively of a rational normal V_3^3 of S_5 (Ch. VIII, § 2.21) in the tangent S_6 at P.

Consider next the system of ρ-planes; since a ruled plane is uniquely determined by a pencil of lines, it follows that a single ρ-plane can be drawn through a given line of M_6^5. And, as we have already remarked, through a given point of M_6^5 there pass ∞^2 ρ-planes. In general two ruled planes have no common line but, exceptionally, they may meet in a line. Hence

THEOREM XXXVII. *Through any line of M_6^5 there passes a single ρ-plane, and through any point of M_6^5 there pass ∞^2 such planes. In general two ρ-planes do not meet; but, if they intersect, they meet only in a point.*

The σ-planes are less significant in the theory since clearly they consist of all the planes lying in the Σ-solids: we therefore confine our attention to the latter. Their chief properties are expressed by

THEOREM XXXVIII. *Through any point of M_6^5 there pass ∞^1 Σ-solids, and through any line or σ-plane of M_6^5 there passes one such solid. Any two Σ-solids meet in a point.*

The first of these properties has already been noted above. As regards the others, any line or σ-plane of M_6^5 corresponds to a pencil or solid point-star, whose centre defines a unique point-star containing the lines of the given system. Also any two point-stars in S_4 have a unique line in common, namely, the join of their centres.

EXAMPLES

1. Prove that M_6^5 is the base manifold of a linear ∞^4 system of quadric [3]-cones, whose vertices are its Σ-solids.

2. Show that, if two ρ-planes do not meet, there is a unique Σ-solid which meets both planes in lines.

3. Show that two σ-planes do not in general meet; but that they may intersect in a point or in a line.

4. Show that a Σ-solid and a ρ-plane do not in general meet; but that, if they intersect, they have a common line.

5. Prove that, if a ρ-plane and a σ-plane intersect, they may do so either in a point or in a line.

6. Show that the ∞^4 lines lying in a solid of S_4 correspond on M_6^5 to a quadric M_4^2 of S_5; and that one such quadric passes through two generic points of M_6^5.

Show also that two such quadrics intersect in a ρ-plane, and that three of the quadrics meet in a point.

7. Prove that the ρ-planes which meet a given σ-plane in lines of a given pencil generate a quadric cone V_3^2 with vertex at the vertex of the pencil.

5.2. *Prime sections of M_6^5.* We consider next the prime sections M_5^5 of M_6^5. These correspond to linear complexes of S_4; and there will be two types, those which correspond to general and to special linear complexes respectively.

As regards the first, we have seen that a general complex has a unique centre (§ 4.2), such that every line through this point belongs to the complex. Hence:

The generic prime section of M_6^5 contains a unique Σ-solid.

In the second case, we are concerned with a special complex whose lines meet a fixed axial plane π. If T is the corresponding prime of S_9, and if ρ_0 is the plane of M_6^5 corresponding to π, then T contains the tangent [6] to M_6^5 at every point of ρ_0; for, by Theorem XXXVI, the section of M_6^5 by any such tangent [6] corresponds to the aggregate of lines of S_4 which meet a line of π. Also T contains in all ∞^2 solids Σ (all meeting ρ_0 in lines) which correspond to point-stars of S_4 with vertices on π. We now note the following significant property of M_6^5:

Every tangent prime of M_6^5 touches this manifold at every point of a ρ-plane.

For to any prime containing the tangent [6] to M_6^5 at a point P there corresponds a linear complex of S_4 which contains all the lines meeting the line p corresponding to P; and since, by § 4.2, a linear complex cannot have a central line without being special (having a central plane), it follows that the prime in question touches M_6^5 at all points of a ρ-plane. Hence:

THEOREM XXXIX. *The section of M_6^5 by a general tangent prime whose plane of contact is ρ contains ∞^2 Σ-solids, and it corresponds to the special linear complex of S_4 whose axial plane is that corresponding to ρ.*

5.21. *Self-duality of M_6^5.* We observe now that M_6^5 has only ∞^6 tangent primes T and that these correspond biunivocally with the ∞^6 planes of S_4. This leads us to

THEOREM XL. *The tangent primes to M_6^5 form a developable Δ_6^5 (system of ∞^6 primes, of class 5) which is dual to M_6^5, and this duality corresponds to that between lines and planes of S_4.*

To prove this we consider the plane π of S_4 which is the intersection of primes $(l_0,..., l_4)$ and $(m_0,..., m_4)$; and we define the coordinates π_{ij} of π by writing

$$\pi_{ij} = \begin{vmatrix} l_i & l_j \\ m_i & m_j \end{vmatrix}$$

in the same way as we defined the coordinates of a line. The line p which joins the points $(x_0,..., x_4)$ and $(y_0,..., y_4)$ has coordinates p_{ij} and it meets π if and only if

$$\begin{vmatrix} l_0 x_0+...+l_4 x_4 & l_0 y_0+...+l_4 y_4 \\ m_0 x_0+...+m_4 x_4 & m_0 y_0+...+m_4 y_4 \end{vmatrix} = 0.$$

This gives
$$\sum \pi_{ij} p_{ij} = 0,$$

and this means that the coordinates π_{ij} of π in S_4 are identical with the coordinates of the tangent prime of M_6^5 which corresponds to the special complex with π as axial plane. This proves the theorem.

By associating any plane of S_4 with its polar line with respect to the quadric $\sum_0^4 x_i^2 = 0$, plane and line having then the same coordinates, we can interpret the self-duality of M_6^5 as a definite (1, 1) correspondence between its points and tangent primes. In this correspondence we find easily, for example, that to a ρ-plane of M_6^5 there corresponds a tangent [6], and to a Σ-solid of M_6^5 there corresponds a [5] which meets M_6^5 in a quadric M_4^2.

5.3. ***The representation of M_6^5 on S_6.*** To discuss sections of M_6^5 by spaces of lower dimension than a prime, it is convenient to make use of a birational projection of the manifold on to a space S_6, this giving a representation of the lines of S_4 on the points of S_6, analogous to the Schumacher representation of the lines of S_3 on the points of S_4. The projection in the present case will be from a ρ-plane of M_6^5, say ρ_0; and we proceed now to show that M_6^5 projects birationally from ρ_0.

If we take ρ_0, as we may, to be the plane corresponding to the (ruled) plane π of S_4 whose equations are $x_0 = x_1 = 0$, then ρ_0 is the plane joining the reference points P_{34}, P_{24}, P_{23} in S_9, and we may take the S_6 of projection to be that whose equations are $p_{34} = p_{24} = p_{23} = 0$. Consider now the equations of M_6^5 as discussed in § 5. We saw there that M_6^5 is the intersection, other than a V_6^3 in the prime $p_{01} = 0$, of the three quadric cones given by equations (1); and we now observe that, for $p_{01} \neq 0$, these three equations define p_{34}, p_{24}, p_{23} rationally in terms of the remaining seven coordinates. In other words, a generic point Q of S_6 is the projection from ρ_0 of a unique point P of M_6^5, i.e. the projection of M_6^5 from ρ_0 is birational.

From the same equations also, we see that prime sections of M_6^5 project into quadrics of the linear ∞^9 system Φ whose equation in S_6 is

$$\Phi \equiv p_{01}\,U+\lambda(p_{03}\,p_{14}-p_{04}\,p_{13})+\mu(p_{02}\,p_{14}-p_{04}\,p_{12})+ +\nu(p_{02}\,p_{13}-p_{03}\,p_{12}) = 0, \qquad (1)$$

where U is a linear form, with arbitrary coefficients, in the seven coordinates of a point of S_6. Plainly these quadrics all pass through the planar V_3^3 (described in Ch. VIII, § 2.21) whose equations are

$$p_{01} = 0, \qquad p_{02}/p_{12} = p_{03}/p_{13} = p_{04}/p_{14}, \qquad (2)$$

and they are the complete system of quadrics through this V_3^3. It follows therefore that M_6^5 is represented birationally on S_6 by means of all quadrics through a planar V_3^3 which lies in a prime, Π say, of S_6.

Consider now the prime Π and the base locus V_3^3 from the point of view of the projection. Plainly Π is the projection of the prime T_0, with equation $p_{01} = 0$, which touches M_6^5 at every point of ρ_0, and V_3^3 is the projection of the section of M_6^5 by T_0. This section corresponds to the special linear complex of lines of S_4 meeting π; it contains ∞^2 Σ-solids, representing stars of S_4 with vertices on π, which meet ρ_0 in lines and project into the ∞^2 directrix lines of V_3^3; and it contains ∞^1 quadrics M_4^2 through ρ_0, representing solids through π, which project into generating planes of V_3^3. Each point of V_3^3 is the projection of a σ-plane, lying in one of the Σ-solids and in one of the M_4^2, which meets ρ_0 in a line.

From the viewpoint of the representation of M_6^5 by means of the quadrics Φ, we note that V_3^3 contains ∞^2 quadric surfaces lying in solids which fill Π simply and completely; and these solids, being fundamental for Φ, represent the points of ρ_0, each one of them being the projection of the tangent [6] to M_6^5 at one such point. The ∞^2 quadrics V_4^2 of Π which pass through V_3^3 are all line-cones,† each of them projecting V_3^3 from a directrix line; and they represent the lines of ρ_0.

The ∞^4 Σ-solids of M_6^5, other than those which meet ρ_0 in lines, project into the solids through generating planes of V_3^3; and the ∞^6 ρ-planes, other than ρ_0 itself and those‡ which meet ρ_0, project into planes through directrix lines of V_3^3.

In conclusion, then, we have

THEOREM XLI. *The manifold M_6^5 projects birationally from any ρ-plane of itself (representing a ruled plane of [4]) on to S_6, and the resulting representation on S_6 is by means of quadrics through a rational normal planar V_3^3, lying in a prime Π of S_6. The whole prime Π corresponds exceptionally to the plane vertex of projection ρ; and the points, directrix lines, and generating planes of V_3^3 correspond exceptionally to σ-planes which meet ρ in lines, Σ-solids which meet ρ in lines, and quadrics M_4^2 on M_6^5 which contain ρ, respectively.*

We pass on now to space sections of M_6^5. A generic prime section of M_6^5 contains an infinity of ρ-planes; and, by Theorem XXX, a generic secundum section contains a single ρ-plane. Hence, taking sections of the previous figure, we obtain the following results:

COROLLARY 1. *The generic prime section M_5^5 of M_6^5 is rational, and is representable on S_5 by quadrics through a cubic scroll.*

COROLLARY 2. *The generic secundum section M_4^5 of M_6^5 is rational, and is representable on S_4 by quadrics through a twisted cubic curve.*

The above method is inapplicable to the section of M_6^5 by a generic space S_3, since this section does not contain a ρ-plane.

For many further details concerning M_6^5, the reader may consult a paper by Todd, *Proc. Lond. Math. Soc.* (2) **30** (1930), 513. For a different method of deriving the above and other representations of M_6^5 and its projections, see Semple, ibid. 500.

EXAMPLES

1. Interpret, in terms of M_6^5, the condition that a given line and plane of S_4 should intersect.

† The fact that these V_4^2 are all line-cones means that a Φ-quadric cannot have a double point outside Π without acquiring a double plane, i.e. every prime which touches M_6^5 at a point touches it at every point of a plane.

‡ These project into lines lying in generating planes of V_3^3.

2. Extend the duality of § 5.21 to a complete table embracing all the important spaces related to M_6^5.

3. Show, by the method of duality or otherwise, that a generic [6] which contains a plane ρ of M_6^5 lies in three tangent primes. Deduce that, if a linear congruence in [4] contains a ruled plane, then it consists of all the lines which meet three fixed planes.

4. Establish the converse of the result given in the last example, and deduce that the trisecant planes of M_6^5 are dual to the spaces [6] which contain a plane ρ.

5. Prove that the M_3^5 in which M_6^5 is met by a generic [6] through a plane ρ is a Del Pezzo threefold, representable on S_3 by means of quadrics through three points. Deduce that it contains three planes σ meeting the plane ρ in lines.

6. Prove that the generic [5] through a plane ρ meets M_6^5 residually in a Veronese surface intersecting ρ in a conic; also that this surface represents the system of lines in [4] meeting four fixed planes which possess a common transversal plane (meeting them all in lines), the lines of this plane being excluded.

7. Six lines in [4] have a common transversal line λ: prove that there is a unique plane, not through λ, which meets the six lines. (First formulate the dual theorem in [4].)

8. Prove that, in general, a [5]-section of M_6^5 residual to a plane σ is a rational quartic scroll meeting σ in a conic.

9. Show that the generic [5]-section of M_6^5 is a Del Pezzo quintic surface, representable on the plane by curves $C^3[1^4]$.

Show that this surface, and therefore also M_6^5, projects birationally from any plane which meets it in a conic.

10. Show, by the method of the last example, that the generic [6]-section of M_6^5 is a rational M_3^5, and that this is representable on S_3 by cubic surfaces through a ${}^0C^4$.

Obtain the same result by associating the trisecants of a projected Veronese surface in [4] with the points in which they meet a fixed S_3.

§ 6. General line systems in S_4

6. The line systems we have hitherto discussed are those which correspond to space sections of M_6^5; in the following paragraphs we shall describe very briefly the general line systems of various dimensions in S_4. First, we consider the main characteristics of these systems in their purely geometrical aspect; then we examine the appropriate numerical characters for their specification; and, lastly, we state formulae for the intersection of two or more line systems in S_4.

As we have seen, there are, in all, systems of five different dimensions to be considered, represented on M_6^5 by manifolds V_k $(1 \leqslant k \leqslant 5)$; these we denote by the symbol $\{k\}$, for brevity.

6.1. *Systems* $\{5\}$*: the general complex.* The system corresponding to a V_5 is called a *complex*; it may be proved† that any such manifold must be the complete intersection of M_6^5 with a primal of some order m. Thus the equation of the complex, in terms of the coordinates p_{ij}, will be of the form $F(p_{ij}) = 0$, where F is a homogeneous polynomial of degree m in p_{ij}. Clearly the ∞^2 lines of the complex which pass through a given point lie on a cone of order m, and those which lie in a given plane envelop a curve of class m. If the complex is of general character, the sole number m suffices to describe it completely.

6.2. *Systems* $\{4\}$. Corresponding to a manifold V_4 we have a line system $\{4\}$. Through a given point of S_4 there will in general‡ pass ∞^1 lines of the system, lying on a conical surface of order m; and in a given plane there will be a finite number n of lines of the system. We call m the *order* and n the *class* of $\{4\}$. In the representation on M_6^5, m and n are the respective numbers of points in which V_4 is met by a σ-plane and a ρ-plane. The ∞^2 lines of $\{4\}$ which lie in a given prime form a congruence of indices (m, n).

The order of V_4 is equal to the number of points in which it is met by a general surface section of M_6^5; it will be shown in § 6.3 that this number is $3m+2n$.

The various projective characters of V_4 will of course give rise to other numerical characters of $\{4\}$, besides m and n, but no detailed study of these has yet been made.

6.3. *Systems* $\{3\}$. *General and special congruences.* The systems of dimension three are in certain respects analogous to line congruences of ordinary space; we shall therefore call them congruences. Through a generic point of S_4 there will pass a finite number m of lines of the congruence; and in a generic prime there will lie ∞^1 lines of the system, forming a scroll of order n. We call m the *order* and n the *class* of the system.

In order to define a general congruence we first exclude those ∞^3 line systems for which $m = 0$. Each of these generates a primal;

† See Note 5 (iv), p. 295.

‡ The exceptional case is the system of lines lying in a prime.

and it has been proved (see the Examples on Chapter I) that they fall into two groups:

(i) the general quadric V_3^2: through each point of this primal there passes a conical quadric surface of lines;

(ii) the planar primals:† in this case the lines are distributed in ∞^1 planes, one of which passes through a generic point of the primal.

We call these *restricted* congruences, since no lines of such systems pass through a generic point; the other type we call *unrestricted*.

Consider an unrestricted congruence K, represented on M_6^5 by a manifold V_3, and let p be a generic ray of K, corresponding to the point P of V_3. At P there are ∞^2 tangent lines to V_3, lying in the tangent solid S_3 at P; these give rays consecutive to p. Now we have seen in § 5.1 that there is a cone V_4^3 of lines through P, lying in the section of M_6^5 by the tangent space S_6 at P. Since this cone will be met by S_3 in three lines, we have

THEOREM XLII. *Any ray of an unrestricted congruence is met by three consecutive rays.*

This may be compared with the result established for line congruences in S_3 (§ 2.34). The three points of intersection of the given ray with its consecutive rays are called foci of the congruence.

We may expect that the locus of the foci of the ∞^3 rays will in general be a primal. It may, however, happen that the locus is a surface or curve. Consider, for example, the congruence of trisecants to a surface in S_4; here the locus of foci is the surface itself—in this respect the congruence is analogous to that formed by the chords of a curve in S_3.

If there exists a focal primal, we say that the congruence is *general*; in that case we may prove

THEOREM XLIII. *The rays of a general congruence are tritangent to the focal primal.*

The demonstration of this result, which is due to Segre,‡ is outlined in the Examples below.

In the representation on M_6^5, the congruence corresponds to a threefold V_3 which meets each Σ-solid in m points and each quadric M_4^2 in a curve of order n. The order of V_3 is equal to the number

† For these see Roth, *Proc. Lond. Math. Soc.* (2) **33** (1932), 115, and Welchman, *Proc. Camb. Phil. Soc.* **29** (1933), 103.

‡ *Rend. Palermo*, **2** (1888), 148.

of rays which lie in a linear congruence generally situated with respect to V_3. We now show that this number is $m+2n$. The proof depends on two simple results:

(i) the number of rays of the congruence which meet a given plane and a given line is $m+n$; this is obtained by assuming the line and plane to intersect;

(ii) the number of rays of the congruence which meet three given planes is $m+2n$; this follows from (i) by supposing that two of the given planes lie in a prime.

If we now assume that the linear congruence in question is the intersection of three special linear complexes, we require the number of rays which meet all three of their axial planes, and this is $m+2n$, by (ii).

Returning to the manifold V_4 of § 6.2 which represents a system {4} of characters (m, n), we first observe that a general linear complex meets the system in a congruence of order m and class $m+n$. By what we have proved, the threefold which represents this congruence is of order $m+2(m+n)$; that is to say, V_4 is of order $3m+2n$.

6.4. *Systems* {2}. *Ruled primals.* We come now to systems of ∞^2 lines; assuming that these are not sub-systems of the restricted congruences considered in § 6.3, we see that a finite number of the lines will meet a given line in S_4. Thus the lines generate a primal which we shall call a *ruled primal*. In certain respects it is analogous to a ruled surface in S_3; there may, however, be essential differences between the two types of manifold. Thus the primal may be *multiply ruled*, i.e. through each of its points there may pass a number μ (> 1) of generating lines: we call μ the *index* of the system. These may form an irreducible system, as on the general cubic primal, for which $\mu = 6$; or they may form two or more such systems, with different values of μ: for example, on the 6-nodal cubic primal there are three systems, two with index 1 and the third with index 4; and on the Segre primal there are six systems, each with index unity. It is clear that in all cases $\mu \leqslant 6$, since the number of generating lines (if finite) through an ordinary point of the primal cannot exceed the number of 4-point tangents there, and these are the intersection of the tangent prime with the polar quadric and polar cubic of the point in question.

We now define the order m and class n of the ruled system as the respective numbers of lines which meet a given line, and which lie in a given prime. For a system of index μ it is plain that the order of the primal V generated by the lines is m/μ.

We assume henceforth that V is *simply ruled*, i.e. that $\mu = 1$; this is the case which corresponds exactly to that of a scroll of order greater than two. In general V will possess a double surface F, locus of intersection of pairs of generators; on this there will be a curve C, triple for V and for F, locus of intersection of triads of generators; and C will contain a finite number of quadruple points, at which concur four generators; these points are quadruple for V and sextuple for F. A ruled primal with no other multiple points than these is said to be endowed with *normal singularities*; for such manifolds there holds the following theorem, due to James:†

THEOREM XLIV. *If V is a simply ruled primal of order m, with normal singularities, then each generator of V is met by $m-3$ others, at points of the double surface.*

This result corresponds to the property of ruled surfaces established in Ch. VII, § 4; the proof is indicated below.

The generators of V will be represented on M_6^5 by the points of a surface whose order is the number of generators of V which lie in the system K_4 common to two linear complexes. If, as before, we assume that these are special, with axial planes π_1 and π_2, we see that the number of generators which meet π_1 and π_2 is $m+n$. This result is valid for both multiply and simply ruled primals.

6.5. *Ruled surfaces in S_4.* As already remarked, a ruled surface of order m will be represented by a curve C^m on M_6^5; the genus p of C^m will be that of the ruled surface. If the latter is a generic scroll its only singularities will be a finite number of improper nodes, at which concur a pair of generators; corresponding to these we shall have a finite number of chords of C^m which lie on M_6^5: C^m itself will be free from multiple points.

On the other hand, if the given surface is developable, its image C is characterized by the fact that all the tangents to the curve lie on M_6^5. These results are similar to those obtained in § 2 for ruled surfaces in S_3.

† *Proc. Lond. Math. Soc.* (2) **28** (1928), 161. For other results concerning ruled primals, see Roth, ibid. (2) **35** (1933), 380.

6.6. *The generalized Halphen Theorem.* Any two systems $\{i\}$, $\{j\}$ of S_4 will in general meet in a third system $\{i+j-6\}$ $(i+j \geqslant 6)$; in particular, if $i+j = 6$, they will in general have a finite number of common lines. This number is deducible from a formula analogous to that established in § 3.1.

We have already defined the characters (m, n) in the cases $i = 4$, 3, 2; now, in order to state the formula in general terms, we adopt the convention that a complex of order m is also of class m, and that a scroll of order m has class zero. We may then prove

THEOREM XLV. *Two systems $\{i\}$ and $\{j\}$ $(i+j = 6)$ in S_4, with characters (m, n) and (m', n') respectively, have in general $mm'+nn'$ common lines.*

The result is obvious in the case $i = 5$, $j = 1$; for, in general, a primal of order m meets a curve of order m' in mm' points. For the proof of the remaining cases we refer the reader to James's paper on the intersection of line systems which is quoted at the end of the chapter. The theorem may also be applied to determine the order and class of the system $\{i+j-6\}$ when $i+j > 6$; examples of this application will be found below.

EXAMPLES

1. Determine the order and class of the system of lines in S_4 which are
(*a*) incident to a given surface,
(*b*) incident to a given curve,
(*c*) incident to two given lines, or to a line and a conic,
(*d*) bisecant to a given surface or curve.
Describe the singular elements in each case.

2. Show that a congruence (m, n) in S_4 projects into a complex of order n in S_3, and that a ruled system (m, n) projects into a congruence (m, n) in S_3.

3. Show that a curve C^m on M_6^5 corresponds in general, by the representation of § 5.3, to a curve C_1^m meeting the base V_3^3 in m points.
Deduce that a scroll of order m and genus p in S_4 has in general $\frac{1}{2}(m-2)(m-3)-3p$ improper nodes.
(This mode of derivation is due to Todd, *Journal Lond. Math. Soc.* **7** (1932), 194.)

4. Discuss the properties of the rational cubic and quartic scrolls of S_4 from the point of view of their representation on M_6^5.

5. Examine the representation on M_6^5 of (i) a planar primal, (ii) a simply ruled primal, (iii) a multiply ruled primal.

6. Prove that the congruence of lines on a general quadric primal of S_4 is represented on M_6^5 by the V_3^8 which is the Veronesean of quadrics of S_3.

7. Show that a surface F on M_6^5 corresponds in general, in the representation on S_6, to a surface F' of the same order which meets the base V_3^3 in a simple or multiple curve of F'.

8. Determine the characters of the line system common to

(i) two complexes of given orders,

(ii) a {5} and a {4} with given characters,

(iii) a complex and a ruled system {2},

(iv) three given complexes.

9. Prove that the rays of a general congruence $K(m,n)$ which meet a given line generate a scroll R of order $m+n$. What curve corresponds to R on the manifold V_3 which represents K on M_6^5?

Given that V_3 is free from singularities and that R has rank r, prove that the focal primal of K is of order $r-2n$ (cf. § 2.35). (The order of the focal primal ϕ is equal to the number of Σ-solids passing through a given point L of M_6^5 and containing a tangent to V_3 which also lies entirely on M_6^5. Now all Σ-solids through L lie on a rational cone V_4^3 (§ 5.1); this intersects V_3 in a curve C^{m+n}, each of the Σ-solid generators meeting C^{m+n} in m points. Considering the $(1, m)$ correspondence between the points of C^{m+n} and the generators of V_4^3, we deduce from the theorem of Ch. IV, § 6 that the number of tangent Σ-solids is $2(p-1)+2m$, where $2(p-1) = r-2(m+n)$.)

10. If l is any ray of the congruence K, P_1, P_2, P_3 the corresponding foci, and l_1, l_2, l_3 the relative consecutive rays, prove that the solid through l, l_1, l_2 touches the focal primal ϕ at P_3. Deduce that l is tritangent to ϕ. (The proof follows the method of § 2.35, using the preceding example.)

11. *James's Theorem.* If V is a simply ruled primal of order m, with normal singularities, then each generator of V is met by $m-3$ others. (We use here the correspondence theorem of Ch. IV, § 2.2. Let g be the generator in question, and Π, Π' any two primes meeting in a plane ϖ. Let Π' cut g in Q; and suppose that α is any plane through g, meeting Π' in a line a through Q and Π in a line b which cuts V in $m-1$ points L external to g. The generators of V through the points L meet Π' in $m-1$ points A which we associate with the lines a through Q. Let QA meet ϖ in A', and let α (or a) meet ϖ in B'. Consider the correspondence $(m-1, m-1)$ between the points A', B'. If n is the class of the ruled system, the number of pairs of corresponding points A', B' which lie on a given line of ϖ is $n-1$. The coincidences of the correspondence arise either from the generators which meet g, or from the envelope of lines joining coincident points in ϖ, which is of class $m+n$.)

12. Prove that the double surface on V is of order $R-\frac{1}{2}(2n-m)(m-1)$, where R is the rank of the congruence obtained by projecting the ruled system on to a generic prime.

13. Prove that, if the ruled system on V is projected from a generic point O on to a given prime, the focal surface of the congruence so obtained is the projection of the (proper) surface of contact of tangents drawn to V from O.

Show that the latter surface has order μ_1, rank μ_2, and class μ_3, where μ_1 and μ_2 are the rank and class of a prime section of V, and μ_3 is the class of V. Deduce that $\mu_3 = \mu_1 + 2(n-m)$, where (m,n) are the indices of the congruence.

14. Given that the ruled system $\{2\}$ is represented on M_8^5 by a surface F with elementary projective characters $\mu_0\,(=m+n)$, μ_1, μ_2, ν_2, obtain the connexion between these characters and those of $\{2\}$.

NOTES AND EXAMPLES ON CHAPTER X

1. *Self-collineations of* Ω. If the coordinates of a line in [3] are taken in the form d_{ij} $(i,j = 0,\ldots,3,\ d_{ij}+d_{ji} = 0)$, show that the most general direct and opposite self-collineations of the quadric Ω in S_5 are given by equations of the form

$$\mathbf{d}' = \mathbf{a}\mathbf{d}\tilde{\mathbf{a}} \quad \text{and} \quad \overline{\mathbf{d}}' = \mathbf{a}\mathbf{d}\tilde{\mathbf{a}},$$

where $\mathbf{d}$, $\mathbf{d}'$ are the skew-symmetric matrices (d_{ij}) and (d'_{ij}), $\overline{\mathbf{d}}$ is the matrix obtained from $\mathbf{d}$ by interchange of d_{01}, d_{02}, d_{03} with d_{23}, d_{31}, d_{12} respectively, and $\mathbf{a}$, $\tilde{\mathbf{a}}$ are respectively an arbitrary non-singular 4×4 matrix and its transposed.

2. *Results for line geometry in* S_4. For the further study of line systems in S_4 the following works may be consulted:

Results mainly of an enumerative character; James, *Proc. Lond. Math. Soc.* (2) **28** (1928), 161.

The general quadratic complex in S_4: B. Segre, *Atti Ist. Veneto*, **88** (1929)$_2$, 595.

Congruences of the first and second orders: Marletta, *Rend. Palermo*, **28** (1909), 353; *Giorn. di Mat.* **50** (1912), 17; also Bordiga, *Atti Ist. Veneto*, (6) **6** (1888), 919.

3. *Line geometry in* S_r. The method by which we have defined line coordinates in S_3 and S_4 extends at once to any space S_r. We thus obtain a set of $\frac{1}{2}r(r+1)$ coordinates p_{ij} satisfying a number of quadratic relations $\Omega_{pp}^{(i)} = 0$ which effectively reduce the number of independent coordinates to the freedom $2r-2$ of lines in S_r. Taking p_{ij} to be homogeneous coordinates in a space S_σ, where $\sigma = \frac{1}{2}r(r+1)-1$, we obtain a representation, without exceptional elements, of the lines on a manifold G of dimension $2r-2$ and order $\dfrac{1}{r-1}\dbinom{2r-2}{r}$. This is called the *Grassmannian* of the system. The chief properties of G, such as its rationality and the nature of its space representation,† are particular instances of those of the more general Grassmannian which we consider below.

Ex. Show that the general linear complex in S_r defines a skew-polarity which is singular if r is even.

4. *Results for line geometry in* S_r. Two line systems in S_r, of complementary dimensions i and $2r-2-i$, will in general have a finite number of common lines. The definition of a set of characters for such systems, and the deduction of a formula, analogous to Halphen's, for the number of common lines, have been given by James, *Proc. Lond. Math. Soc.* (2) **24** (1925), 359.

† For this see Semple, *Proc. Lond. Math. Soc.* (2) **30** (1930), 500.

Among particular results may be mentioned a study by Todd, *Proc. Lond. Math. Soc.* (2) **33** (1932), 328 of doubly infinite line systems, including a detailed account of those represented on the appropriate Grassmannian by Del Pezzo surfaces.

5. *S_k-coordinates. The Grassmannian $G(k,r)$.* In S_r the spaces S_k form an aggregate of freedom $R=(r-k)(k+1)$, and any one of these spaces is determined by a set of $k+1$ generic points. From the matrix of their coordinates we may extract $\binom{r+1}{k+1}$ determinants of order $k+1$, which we call the Grassmann coordinates of S_k. Taking these to be homogeneous coordinates of a point in S_σ, where $\sigma=\binom{r+1}{k+1}-1$, we obtain a representation, without exceptional elements, of the spaces S_k on the points of a manifold $G(k,r)$. This manifold has been studied by Severi, *Annali di Mat.* (3) **24** (1915), 89; its chief properties are as follows:

(i) The order of $G(k,r)$ is $1!\,2!\ldots k!\,R!/(r-k)!\,(r-k+1)!\ldots r!$.

(ii) $G(k,r)$ is the manifold of least order on which the S_k of S_r can be represented birationally without exceptional elements.

(iii) The manifold is rational.

(iv) Any manifold of dimension $R-1$ lying on $G(k,r)$ is the complete intersection of the Grassmannian with a primal.

The problem of representing the Grassmannian on a space S_R has been studied by Semple, *Proc. Lond. Math. Soc.* (2) **32** (1931), 200, who has shown that the representation may be obtained by projecting $G(k,r)$ on S_R from a space of dimension $\binom{r+1}{k+1}-(r-k)(k+1)-2$ contained in it; this is the generalization of the method of § 5.3. In this way prime sections of $G(k,r)$ are made to correspond to primals of order $k+1$ of S_R.

6. *S_k-geometry.* For the study of linear S_k-systems reference may be made to Kantor, *Journal für Math.* **118** (1897), 74, where the representation of the lines of S_r is also considered.

Systems of ∞^2 planes in [4] have been considered by Segre, *Rend. Lincei*, (5) **30** (1921), 67, and by Welchman, *Proc. Camb. Phil. Soc.* **28** (1932) 275, 416.

The planes of S_5, and their Grassmannian M_9^{42}, have been discussed in detail by Segre, *Annali di Mat.* (3) 27 (1918), 75.

BOOKS RECOMMENDED FOR READING AND CONSULTATION

BAKER, *Principles of geometry*, iv.

BERTINI, *Complementi*, §§ 14–16, and *Introduzione*, ch. v.

JESSOP, *Treatise on the line complex.*

ROOM, *The geometry of determinantal manifolds*, ch. x and ch. xiii.

EDGE, *Ruled surfaces.*

BLASCHKE, *Differentialgeometrie*, i.

CHAPTER XI

SOME PROBLEMS OF ENUMERATIVE GEOMETRY

§ 1. Introduction

1. The typical problem of enumerative geometry. Broadly speaking, any geometrical investigation whose purpose is to obtain a formula for a certain number may be called an enumerative problem. Thus, the entire contents of Chapter IX, and much of Chapter IV, may be brought under this heading. Important enumerative problems of types not included in those chapters are, for example:

Ex. 1. To find the number of lines which meet four given lines in [3].

Ex. 2. To find the number of intersections of two curves of given character lying on a given rational surface.

Ex. 3. To find the number of conic-loci of a plane which pass through three given points and touch two given lines.

Such problems, although differing superficially to a great extent, may for the most part be characterized in general terms; and before broaching the particular questions to which this chapter is devoted, it will be helpful to describe the general type of problem with which the subject deals.

Consider an algebraic system Σ of geometrical forms or entities (F): for instance, the conics of a plane, the lines of [3], or the quadric primals of [4]. If the system has dimension d, a finite number (possibly zero) of its members can be found to satisfy a set of d given independent algebraic conditions Γ. *The problem of determining how many members of Σ satisfy Γ is the typical problem of enumerative geometry.*

Thus in any problem there are two main elements:

I. The system Σ of *geometrical forms* (F).

II. The *condition-figure*, or assemblage of given points, lines, curves,... with which the forms of Σ are to have assigned relations. We may denote this figure equally well by the symbol Γ; and no confusion will arise thereby.

For instance, in Ex. 1 above, the condition-figure is a set of four given lines; while in Ex. 3, it is a set of three given points and two given lines.

1.1. *The two methods of calculation.* The theoretical approach to a solution of the typical problem would consist in

(*a*) expressing Σ in terms of suitably chosen parameters (λ_i)

(whose number would generally exceed d; so that there would be a set of relations between them);

(*b*) writing down the conditions which Γ would impose on them; and hence

(*c*) determining the number of sets of (λ_i) which satisfy the equations so imposed, as well as the initial relations.

But, as a rule, even in comparatively simple problems, this method is so complicated as to be unworkable. In fact it was in order to avoid these complications that the special technique of enumerative geometry was evolved. This technique is based on the application of one or, more usually, both of two characteristic methods:

I. *Specialization of the condition-figure.*

II. *The use of degenerate forms.*

We illustrate these in turn.

Ex. 1. Given four lines a, b, c, d in [3], to find how many lines meet all four.

The system Σ of the lines of [3] has dimension 4; and in order that a line of Σ should meet a, a single condition must be fulfilled. When a, b, c, d are in general position, we may thus expect a finite number of solutions to the problem.

Suppose now that b varies continuously until it comes to lie in a plane π through a; and let a, b meet at P. Then the number of solutions to the problem is obviously two: for a single transversal can be drawn from P to c, d; and, if c, d meet π in C, D respectively, the line CD meets a and b as well as c and d.

We see, then, that when the condition-figure is specialized in this way, there are precisely two solutions to the problem. What we now wish to assert is, that when a, b are *not* coplanar, there are still two solutions. In fact we wish to affirm that, since with our chosen specialization of the condition-figure the number of solutions remains finite, this must be the number of solutions in the most general case.

Here the justification of the method is easily obtained by referring to the representation of the lines of [3] by the points of the quadric primal Ω of [5] (Ch. X, § 2). Suppose that a, b, c, d are represented by the points A, B, C, D of Ω; then any line p which meets a, say, corresponds to a point P of Ω such that A and P are conjugate with respect to Ω. Thus the images P of the solutions to our problem are among the points common to the four tangent primes to Ω at A, B, C, D; and, for general positions of these points, the primes will meet in a line having two intersections with Ω. In our specialization of the condition-figure the points A and B are conjugate with respect to Ω; and clearly this does not affect the conclusion, for there are still two intersections with Ω of the line common to the tangent primes at A, B, C, D.

Ex. 2. To find the number of twisted cubics lying on a quadric surface and passing through five given points of the quadric.

Here the system Σ is that of all twisted cubics in [3], while the condition-figure is a quadric and five general points of the surface. Since Σ is of dimension 12, and 7 conditions must be fulfilled if a space cubic is to lie on a given quadric, we expect a finite number of solutions to the problem.

We now specialize the figure by supposing that four of the given points are coplanar. Then all the cubics satisfying the given conditions must break up into the plane section of the quadric through the four points in question, and one of the generators through the fifth point. In this case, then, there are two solutions to the problem; and we wish to assert that there are two in general.

It should be noted that the specialization method does not invariably lead to valid results; every time it is used a verification similar to that of Ex. 1 should be carried out. We shall return to this question in §6.

1.2. *Calculation of multiplicities.* In the kind of argument illustrated above there is usually a characteristic difficulty which did not, as it happened, present itself there. Generally speaking, when we attempt to calculate a number by using degenerate forms, we shall be faced by one or more multiple solutions; the problem is then to estimate the correct multiplicity of each such solution.

Ex. 3. In the enumerative plane geometry of conics it is often required to consider the degenerate form consisting of a pair of distinct lines. Regarded as a point locus this is clearly of order two; but it has no envelope, properly speaking. If, however, we regard the line-pair as the limit of a non-degenerate conic, by forming the tangential equation of the latter and proceeding to the limit we obtain for the envelope of the line-pair the pencil of lines through its vertex, each of the lines being counted twice.

1.3. *The calculus of conditions.* In enumerative geometry the methods of specialization and degeneration already described are used in conjunction with a *symbolic calculus of conditions.*† Its use is justified, as we shall show, by the circumstance that the various conditions Γ which are imposed on a system Σ may be regarded as symbols which to a certain extent obey the laws of algebra.

Consider now a system Σ of forms (F) having dimension d; in connexion with this we can envisage various condition-figures Γ, Γ', Γ'',..., each of which imposes a certain set of conditions on Σ. We denote the complete set of conditions imposed by Γ by the symbol c; thus, if Σ is regarded as fixed, c is defined relative to

† This, like much of the technique of the subject, is in great measure due to H. Schubert, whose work (see the references at the end of the chapter) should be consulted for further information.

a prescribed condition-figure. We speak of c as a *condition* or condition-symbol.

(1) With any condition c there is associated a *dimension*: we say that c has dimension k (or is a k-fold condition) if among the ∞^d forms of Σ there are ∞^{d-k} which satisfy c.

Two conditions c, c' are said to be *independent* if the two sets of parameters which define the corresponding condition-figures Γ, Γ' are independent of one another.† With this concept in mind, we shall often find it convenient to use c in a second sense; we shall denote by c the *number* of forms F which satisfy c and $d-k$ other given conditions independent of c. No confusion arises from the double function assigned to the symbol. But it is evident that, when c is so used, its value will in general vary with the nature of these other $d-k$ conditions.

(2) We now define the *sum* $c+c'$ of two conditions of equal dimension k as the condition that Σ should satisfy c or c'. Evidently we have $c+c' = c'+c$. We may also use these symbols in the second sense as explained above. By an obvious extension we may define the sum $c+c'+c''+...$ of three or more conditions of equal dimensions.

(3) We define the *product* cc' of any two independent conditions, of dimensions k, k' respectively, as the condition that Σ should satisfy both c and c'; or again, as the number of forms satisfying c, c' and $d-k-k'$ other given independent conditions. It is clear that we have $cc' = c'c$. We may then proceed to define the product $cc'c''...$ of three or more independent conditions.

(4) Two conditions c, c', which are not necessarily identical, are said to be *equivalent* when they are of the same dimension k and when, if c'' is any other condition, independent of c and c', of dimension $d-k$, the number cc'' is equal always to the number $c'c''$. The corresponding condition-numbers c and c' are then equal, and we write $c = c'$.

(5) We can now introduce the concepts of *integral multiple* and *integral power* of a condition. Thus, if we have the set of $n-1$ equations

$$c = c' = c'' = ... = c^{(n-1)},$$

† A strict algebraic definition of independence is difficult to formulate, though in particular cases the significance of this concept is usually clear enough. The implicit assumption in the text is that we are primarily concerned with *general* conditions only, i.e. we suppose that the effective parameters which specify a condition-figure Γ have generic values. *Specialization* of Γ is equivalent to imposing *internal* relations on these parameters.

we may write

$$nc = c+c'+c''+...+c^{(n-1)}.$$

Again, if c and c' are independent conditions such that $c = c'$, the product cc' may be called the square of c or c', and written as c^2 or c'^2. In a similar manner we may define the power c^n, where n is any positive integer.

nc and c^n denote classes of equivalent conditions.

1.4. With these definitions it is clear that, in so far as the conditions concerned are independent, the condition-symbols obey many of the fundamental laws of algebra. Thus, the sum and product are both commutative, and also the distributive law

$$c(c'+c'') = cc'+cc''$$

is satisfied. We can also interpret the relation $c-c' = c''$ as the equivalent of $c = c'+c''$; it being assumed that c, c', and c'' are of equal dimensions.

§ 2. The incidence calculus of Schubert

2. Before proceeding to the main examples we have chosen to illustrate the methods of enumerative geometry, we wish to give a brief indication of the way in which symbolic methods have been applied to the solution of *problems of pure incidence*, in which our sole object is to find the number of spaces $[k]$ of S_n which satisfy such a set of incidence conditions (with fixed sub-spaces of S_n) as exactly reduces their freedom to zero. The essentials of the method are

(i) the development—due to Schubert—of a formal representation of incidence conditions by symbols, and

(ii) the use of specializations of the condition-figure in order to effect systematic reduction of symbolic products.

We begin by giving in ordinary notation some simple examples of how the method works; and we shall then illustrate briefly the systematic symbolism which Schubert invented to cope with every possible incidence problem that can arise.

2.1. ***Lines of S_3.*** For the lines of ordinary space we shall use the following complete set of fundamental condition-symbols:

$$l, \quad l_P, \quad l_\pi, \quad l_s, \quad [l]. \tag{1}$$

Here l represents the *simple* condition that a variable line λ should meet a fixed line; l_P and l_π are the twofold conditions that λ

should pass through a fixed point and lie in a fixed plane respectively; l_s is the threefold condition that λ should belong to a fixed pencil; and $|l|$ is the fourfold condition that λ should be a specified line.

There are many obvious relations between these symbols: for example, the relations

$$l_P^2 = |l|, \qquad l_\pi^2 = |l|$$

express the facts that there is a unique line joining two given points or lying in two given planes. Plainly we have also $l_P l_\pi = 0$, since there is in general no line which passes through an assigned point and lies in an assigned plane.

There is, however, one essential reduction formula

$$l^2 = l_P + l_\pi \tag{2}$$

which depends on the specialization principle; the condition l^2 requires a line λ to meet two fixed lines a, b; and the formula (2) is obtained by allowing a and b to intersect, in which case λ either passes through their point of intersection or lies in their joining plane.

Using this we may write

$$l^4 = (l_P + l_\pi)^2 = l_P^2 + 2l_P l_\pi + l_\pi^2 = |l| + 2.0 + |l| = 2|l|. \tag{3}$$

This expresses symbolically the fact that *there are two lines which meet four assigned lines, in general position, in* S_3.

Examples

1. Show that $ll_P = ll_\pi = l_s$, $l^3 = 2l_s$.
2. Interpret the formula $l^2 = l_P + l_\pi$ in the most general manner.

2.2. *Lines of* S_4. There are nine fundamental types of incidence condition which can be imposed on a line λ of S_4. The corresponding symbols may be grouped as follows:

$$l;\ l_l,\ l_\Pi;\ l_P,\ l_{\Pi,l};\ l_\pi,\ l_{\Pi,P};\ l_{\pi,P};\ |l|. \tag{4}$$

Here l is the simple condition that λ meets a given plane,

l_l, l_Π are the twofold conditions that λ meets a line, and that λ lies in a solid respectively,

l_P, $l_{\Pi,l}$ are the threefold conditions that λ passes through a point, and that λ lies in a solid and meets a given line in this solid,

l_π, $l_{\Pi,P}$ are the fourfold conditions that λ lies in a plane, and that λ lies in a solid and passes through a given point of this solid,

$l_{\pi,P}$ is the fivefold condition that λ belongs to a plane pencil,
$|l|$ is the sixfold condition that λ is a specified line.

There are numerous relations between these symbols, for example, $l_l^3 = |l|$, expressing that three lines have a unique transversal, $l_\Pi^3 = |l|$, expressing that three solids have a common line. In addition to such obvious results we have important reduction formulae, of which one is

$$l^2 = l_l + l_\Pi. \tag{5}$$

To prove this we observe that the condition l^2 requires λ to meet two fixed planes; if these are specialized so that they meet in a line, then λ either meets their line of intersection or lies in the solid containing them.

It follows that we may write

$$\begin{aligned} l^6 &= (l_l + l_\Pi)^3 \\ &= l_l^3 + 3l_l^2 l_\Pi + 3l_l l_\Pi^2 + l_\Pi^3 \\ &= |l| + 3|l| + 3.0 + |l| \\ &= 5|l|. \end{aligned} \tag{6}$$

This means that *in S_4 there are five lines meeting six given planes in general position.* (Cf. Ch. X, §§ 3, 4.)

EXAMPLES

1. Establish the reduction formulae

$$ll_l = l_P + l_{\Pi,l}, \qquad ll_{\Pi,l} = l_l^2 = l_\pi + l_{\Pi,P},$$

and hence prove that

$$l^3 = l_P + 2l_{\Pi,l}, \qquad l^4 = 2l_\pi + 3l_{\Pi,P}, \qquad l^5 = 5l_{\pi,P}.$$

2. Show that, in S_5, 4 planes have 3 common transversal lines, and that 8 solids have 14 transversal lines.

2.3. *The general problem.* Immediately we pass from the above simple examples to the general problem of incidence for the $[k]$'s of S_n, the need for a systematic approach and symbolism becomes apparent. The general solution of the problem falls into three stages:

(*a*) The invention of a symbolic form which can represent every possible type of fundamental incidence condition on the $[k]$'s of S_n; so that, by this means, the general incidence *problem* can be represented in the form

$$c_1^{\lambda_1} c_2^{\lambda_2} \ldots c_m^{\lambda_m}, \tag{7}$$

where $c_1, c_2, \ldots, c_m$ are symbols representing all possible types of fundamental incidence conditions, the λ_i are non-negative integers, and the total dimension of the condition is

$$d = \sum_1^m \lambda_i d_i = (k+1)(n-k).$$

(*b*) The resolution, by systematic specialization of condition-figures, of every *product* $c_i c_j$ of two fundamental symbols into a *sum* of fundamental symbols, i.e. the construction of general reduction formulae of the type

$$c_i c_j = \sum_{\alpha=1}^m a_{ij,\alpha} c_\alpha \quad (i,j = 1,2,...,m), \tag{8}$$

where the $a_{ij,\alpha}$ are integers.

(*c*) The evaluation, by repeated use of the above formulae, of every symbol $c_1^{\lambda_1} c_2^{\lambda_2} ... c_m^{\lambda_m}$, of dimension $(k+1)(n-k)$, to the form Nc_m, where c_m is the unique fundamental symbol of dimension $(k+1)(n-k)$, expressing the fact that a $[k]$ is specified completely; so that N is then the *value* of the symbol, i.e. the solution of the enumerative problem.

The first stage was achieved with complete finality by Schubert, who recognized that every possible type of fundamental condition could be represented by a unique increasing sequence of $k+1$ positive integers of the form

$$(a_0, a_1,..., a_k), \quad \text{where} \quad 0 \leqslant a_0 < a_1 ... < a_k \leqslant n;$$

and that every such symbol represents a unique fundamental condition. The composite condition represented by the above incidence-symbol is that which requires a $[k]$ to satisfy such of the following conditions as are not nugatory:

to lie in an assigned $[a_k]$,
to meet in a $[k-1]$ an $[a_{k-1}]$ which lies in the $[a_k]$,
to meet in a $[k-2]$ an $[a_{k-2}]$ which lies in the $[a_{k-1}]$,
. . . .
to meet in a point an $[a_0]$ which lies in the $[a_1]$.

The first of these requirements is nugatory if $a_k = n$, the second if $a_{k-1} = a_k - 1$, and, generally, the $(s+1)$th if $a_{k-s} = a_{k-s+1} - 1$. In fact any part of the sequence which is a straight run of *consecutive* integers, say from a_{t+1} to a_s, is entirely nugatory except for the extreme term on the right (if this is not equal to n), which requires the $[k]$ to meet a certain $[a_s]$ in an $[s]$; and the requirement imposed by the *next term* a_t *on the left* reduces simply to that of meeting in a $[t]$ a certain $[a_t]$ which lies in $[a_s]$. Thus, in particular:

$(n-k,\dots,n)$ is the *null condition* (which is conventionally counted as a fundamental condition),
$(s,n-k+1,\dots,n)$ is the condition that $[k]$ meets an assigned $[s]$ in a point,
$(s-t,\dots,s,n-k+t+1,\dots,n)$ is the condition that $[k]$ meets an assigned $[s]$ in a $[t]$,
$(s-k,\dots,s)$ is the condition that $[k]$ lies in an $[s]$,
$(0,\dots,k)$ is the condition that $[k]$ is specified completely.

As regards the second and third stages referred to above, we shall go no farther than to illustrate, in the case $k = 1$, how the product of any two symbols may be replaced by a sum.

2.31. *Reduction of a product of two line incidence-symbols.* The general incidence-symbol for lines of S_n is of the form

$$(a,b) \quad (0 \leqslant a < b \leqslant n);$$

it represents the condition that a line λ should lie in a given $[b]$ and meet a given $[a]$ in that $[b]$. We have now to express any product such as $(a,b)(a',b')$ as a sum of incidence-symbols.

We show, in the first place, that there is no effective loss of generality by confining our attention to products of the form

$$(a,n)(a',n).$$

For the general product may be reduced to this case by observing that a line λ which lies in $[b]$ and in $[b']$ lies in the $[b+b'-n]$ common to these spaces; and if λ also meets the sub-spaces $[a]$ and $[a']$ of $[b]$ and $[b']$ respectively, then it must meet the spaces $[a+b'-n]$ and $[a'+b-n]$ in which $[a]$ and $[a']$ respectively meet $[b+b'-n]$. Hence

$$(a,b)(a',b') = (a+b'-n,\overline{b+b'-n})(a'+b-n,\overline{b+b'-n}), \tag{9}$$

where the bars over the expressions $b+b'-n$ indicate that *the* SAME *space* $[b+b'-n]$ *is to be taken for each of the two factors on the right.* Thus the product on the right is reduced to the required form, with $[b+b'-n]$ as the ambient space, instead of $[n]$.

Consider, then, a product of the form $(a,n)(a',n)$. If

$$a+a'+1 < n,$$

then every line λ meeting $[a]$ and $[a']$ lies in the space $[a+a'+1]$ joining these two, and we may write

$$(a, n)(a', n) = (a, \overline{a+a'+1})(a', \overline{a+a'+1}), \tag{10}$$

which merely makes $[a+a'+1]$ the ambient space instead of $[n]$. Thus we may suppose that $a+a'+1 \geqslant n$.

In this case $[a]$ and $[a']$ would normally meet in an $[a+a'-n]$ (or not at all if $a+a'-n = -1$); but we choose them *specially* so that they meet in an $[a+a'-n+1]$ and lie accordingly in a prime Π. The lines which meet $[a]$ and $[a']$ then either meet the $[a+a'-n+1]$ or lie in Π (and still meet $[a]$ and $[a']$). This gives the reduction formula

$$(a, n)(a', n) = (a+a'-n+1, n)+(a, \overline{n-1})(a', \overline{n-1}). \tag{11}$$

By applying this repeatedly until one or other of the null conditions $(a, a+1)$, $(a', a'+1)$ appears, the symbolic product is reduced finally to a sum.

To illustrate the procedure, we carry out the reduction of the product $(8, 16)(10, 19)$ for lines in $[20]$.

We have, by (9),

$$(8, 16)(10, 19) = (7, \overline{15})(6, \overline{15});$$

also, by (10),

$$(7, 15)(6, 15) = (7, \overline{14})(6, \overline{14}).$$

Then, by (11),

$$\begin{aligned}(7, 14)(6, 14) &= (0, 14)+(7, \overline{13})(6, \overline{13})\\ &= (0, 14)+(1, 13)+(7, \overline{12})(6, \overline{12})\\ &= \; . \quad . \quad . \quad . \quad . \quad .\\ &= \sum_0^5 (r, 14-r)+(7, \overline{8})(6, \overline{8}),\end{aligned}$$

whence, finally,

$$(7, 14)(6, 14) = \sum_0^6 (r, 14-r).$$

For further explicit formulae and details we refer the reader to the account given in Baker, vi, ch. ii, and to the works there quoted. We merely add that a complete solution of the general incidence problem has been given by Giambelli,† whose results

† *Mem. Torino*, (2) 52 (1902), 171.

have been rendered easier of application by Welchman.† In particular it has been shown that

(i) there are $\frac{1}{n-1}\binom{2n-2}{n}$ lines meeting $2n-2$ $[n-2]$'s in $[n]$,

and that

(ii) the number of lines of $[n]$ which meet $n-1$ $[n-3]$'s is

$$\binom{2n-4}{n-3}-\binom{n-1}{1}\binom{2n-7}{n-3}+\binom{n-1}{2}\binom{2n-10}{n-3}-\dots,$$

the series continuing until the terms become meaningless.

Examples

1. Transcribe all the symbols of (1) and (4) into the formal Schubert symbolism, and prove directly, by use of (11), that $(2,4)^6 = 5(0,1) = 5$.

2. If $a \leqslant a'$, and if μ denotes the lesser of the two numbers a and $n-a'-1$, prove that

$$(a,n)(a',n) = \sum_{i=0}^{\mu}(a-i, a'+1+i).$$

Deduce that, if $a+b' \leqslant a'+b$, then

$$(a,b)(a',b') = \sum_{i=0}^{\mu}(a+b'-n-i, a'+b-n+1+i),$$

where μ is the lesser of the numbers $b'-a'-1$ and $a+b'-n$.

3. The condition represented by the incidence-symbol $(a_0, a_1,\dots, a_k)$ is for some purposes more conveniently represented by the *condition-symbol*

$$\{\beta_0, \beta_1,\dots, \beta_k\},$$

where $\beta_i = n-k+i-a_i$, and where

$$n-k \geqslant \beta_0 \geqslant \beta_1 \geqslant \dots \geqslant \beta_k \geqslant 0.$$

Show that the dimension of the condition is $\sum_1^k \beta_i$. Show also how the special types of condition referred to in § 2.3 are represented in these condition symbols.

4. Express the results of Ex. 2 in terms of condition-symbols.

5. Express the order μ_0 and rank μ_1 of a curve of $[n]$ with the aid of Schubert symbols; and express similarly the elementary projective characters of a surface in $[n]$.

§ 3. The quadrics of S_3 and their representation on S_9

3. The three primary conditions. We shall now apply the basic principles of § 1 to solve various problems of enumerative geometry for the quadrics of [3]; and incidentally we shall obtain various results concerning the enumerative geometry of conics and cones in [3].

† *Proc. Camb. Phil. Soc.* **28** (1932), 18.

The quadric surfaces (F) of [3] form a system Σ of dimension 9, so that a finite number of quadrics can be expected to satisfy any set Γ of 9 given independent conditions. The types of condition we shall first consider are three in number, each being of dimension unity:

(1) The condition μ, that F should pass through a given point.
(2) The condition ν, that F should touch a given line.
(3) The condition ρ, that F should touch a given plane.

Evidently the condition ρ is dual to μ. We shall call these the *primary conditions*; and any condition-figure Γ representing a set of such conditions, of dimension $d \leqslant 9$, will be called a *primary condition-figure*.

In accordance with the notions of the symbolic calculus, as already described, we shall use the symbol μ^2 for the condition that a quadric F should pass through two given points; and so on. Thus our problem consists in evaluating all conditions of the form

$$\mu^r \nu^s \rho^{9-r-s},$$

where r, s and $9-r-s$ are all integers, positive or zero. Since the symbol ρ is dual to μ, a certain number of the results may be inferred from the remainder. It will be found that, taking this fact into account, there are 30 essentially distinct symbols to evaluate. From these calculations the numbers of quadrics satisfying other sets of conditions may be deduced, as we shall see later.

In order to avoid unnecessary detail, we shall give in each section the calculations for one or more typical formulae, leaving the rest as exercises for the reader.

3.1. ***The basic representation: the primal*** Δ. A considerable aid, in the sequel, is provided by a representation of the quadrics (F) which we shall now describe; besides that, the representation is interesting for its own sake.

Any quadric F of [3] has an equation of the form

$$\sum a_{ij} x_i x_j = 0 \quad (i,j = 0,1,2,3). \tag{1}$$

Thus F is uniquely specified by the ratios of the 10 coefficients $a_{ij} = a_{ji}$ in (1). Taking these as coordinate ratios, *we may represent the system* (F) *of quadrics by the points of a space* S_9. We shall call this the *basic representation*. (The reader may compare this with the representation, discussed in Ch. VII, § 3.13, of the conics of a plane.)

If (1) represents a point-cone, we have

$$\Delta \equiv \begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{vmatrix} = 0. \tag{2}$$

Hence, *to the cones of* (F) *there correspond the points of a quartic primal* Δ, i.e. a manifold V_8^4 in S_9. To the four points in which it meets a general line of S_9 correspond the four cones of a general pencil of quadrics.

When condition (2) is satisfied, the vertex (x_i) of the cone (1) is determined by any three of the four homogeneous equations

$$\sum_{i=0}^{3} a_{ij} x_i = 0 \quad (j = 0, 1, 2, 3). \tag{3}$$

3.11. *Generating spaces and tangent primes of* Δ. We now prove that Δ *is generated by a web of spaces* [5]; *and any tangent prime to* Δ *touches it at all points of a generating* [5].

Let $P(x_i)$ be any fixed point of S_3; then the equations (3), in which a_{ij} are variable, represent a space [5] lying entirely on Δ; this space corresponds to the linear ∞^5 system of cones having their vertices at P. The aggregate of generating [5]'s forms a web, since it is in (1, 1) correspondence with the points P.

Next, suppose that (a_{ij}) is a given (ordinary) point of Δ; then, if b_{ij} denote current coordinates, the equation of the tangent prime to Δ at (a_{ij}) is

$$\sum A_{ij} b_{ij} = 0, \tag{4}$$

where $A_{ij} = \dfrac{\partial \Delta}{\partial a_{ij}}$ is the cofactor of a_{ij} in Δ.

Assuming, as before, that the cone (a_{ij}) has vertex P, we obtain on solving the equations (3) the result

$$A_{ij} = \lambda x_i x_j,$$

where λ is a factor of proportionality.† Inserting the values of A_{ij} in (4), the latter becomes

$$\sum b_{ij} x_i x_j = 0.$$

It follows that the linear system of ∞^8 quadrics corresponding to the points of the tangent prime is the system of quadrics having

† This solution corresponds to the fact that the tangential equation

$$\sum A_{ij} u_i u_j = 0$$

of the cone must represent the repeated point P.

P for base point. Thus the tangent prime (4) touches Δ at each of the points corresponding to a cone with vertex at P, i.e. at all points of the generator representing the aggregate of all such cones. And, further, Δ *possesses only* ∞^3 *tangent primes, each of which touches it along a generating* [5].

3.12. *Multiple varieties on* Δ. Among the cones of (F) there are ∞^6 plane-pairs; their images in S_9 lie on Δ and form a double V_6 thereon; for in any pencil of quadrics which contains a plane-pair, the latter counts doubly as a cone.

There is also on Δ a manifold V_3, representing the ∞^3 planes of S_3, counted twice; clearly this manifold is triple for Δ.

The manifold V_6, being the image of the *unordered* plane-pairs (or point-pairs) of S_3, is known to be rational; and V_3 is plainly rational, being evidently the octavic Del Pezzo threefold considered in Ch. VIII, § 2.3, i.e. the projective model of all quadrics of S_3.

The order of V_6, i.e. the number of points in which it is met by a general [3], is equal to the number of plane-pairs contained in a general linear system of ∞^3 quadrics. If instead of a general system we consider the one defined by all quadrics passing through six given points, we see at once that the number of plane-pairs is ten; the number of plane-pairs in the general case will be equal to ten also, provided that none of the solutions in the special case is multiple; and this will be proved in § 3.5. Hence we may assert that V_6 is of order ten. In conclusion, then,

The primal Δ *representing the cones of* S_3 *has for double manifold a rational* V_6^{10}, *corresponding to the* ∞^6 *plane-pairs of* S_3, *and for triple manifold a Del Pezzo threefold* V_3^8, *corresponding to the double planes of* S_3.

3.13. *Relations of* Δ *and* V_6^{10} *with* V_3^8. Since V_3^8 is the projective model of all quadrics in S_3 and therefore by far the simplest of the three manifolds just described, it is convenient to be able to describe the other two in terms of V_3^8 alone. This we may do very simply by remarking first that V_3^8 contains ∞^4 conics and ∞^3 Veronese surfaces corresponding to the lines and planes of S_3 respectively; and by observing then that

(i) *the manifold* V_6^{10} *is the locus of the* ∞^4 *planes which meet* V_3^8 *in conics*,

(ii) *the primal* Δ *is the locus of the* ∞^3 [5]'*s which meet* V_3^8 *in Veronese surfaces.*

Furthermore, it is easy to see that if any point P of V_3^8 corresponds in the basic representation to a repeated plane u of [3], then the points of the tangent solid to V_3^8 at P all correspond to the pairs of planes of [3] of which u is one member; whence

(iii) *V_6^{10} is the locus of tangent solids to V_3^8, two such solids passing through every point of the manifold and being met in lines by the generating plane through that point.*

Finally, if uv and $(u+\epsilon u')(v+\epsilon v')$ are two consecutive plane-pairs of which the first has axis g, then the limiting pencil defined by these as ϵ tends to zero is $uv+\theta(u'v+uv')$, and this has g as base line. This means that if P is the point of V_6^{10} which represents uv, then the tangent [6] to V_6^{10} at P represents the ∞^6 system of quadrics which pass through g; and this [6] will therefore be the same for all points of the generating plane of V_6^{10} which represents all plane-pairs with axis g. Hence

(iv) *V_6^{10} has only ∞^4 tangent [6]'s in all; but each of these touches it at every point of a generating plane representing an axis g of plane-pairs of [3], and it represents the system of quadrics of [3] which have g as generator.*

3.2. *The condition ρ.* We proceed to obtain the representation in S_9 of the three primary conditions defined in § 3. As will soon appear, it is both natural and convenient to begin with the condition ρ.

First, denoting by ϕ the plane-envelope of the quadric F, we have for its equation, in plane coordinates (u_i),

$$\sum A_{ij}\,u_i\,u_j = 0. \tag{5}$$

Then the condition that the quadric F', with coordinates (b_{ij}), should be apolar to ϕ, is

$$\sum A_{ij}\,b_{ij} = 0. \tag{6}$$

Hence,

The locus, in S_9, of points corresponding to quadrics (ϕ) of S_3 which are apolar to a fixed quadric F' is a cubic primal, the first polar of the image (b_{ij}) of F' with respect to the primal Δ.

It follows from the properties of polar primals established in Ch. I, § 3.1, that this cubic primal passes simply through the double V_6^{10} and doubly through the triple V_3^8 of Δ. The manifold V_7^{12} in which it intersects Δ is the locus of points of contact of tangents to Δ from the point (b_{ij}).

Suppose now in particular that F' is a double plane whose equation is $(u_0 x_0 + u_1 x_1 + u_2 x_2 + u_3 x_3)^2 = 0$, so that $b_{ij} = u_i u_j$: then (6) becomes

$$\sum A_{ij} u_i u_j = 0. \tag{7}$$

This is the condition that the plane $\pi \equiv (u_i)$ should be touched by F. Since the double plane F' corresponds in S_9 to a point O of V_3^8, it follows that

The locus $U(\pi)$ in S_9, of points corresponding to quadrics of S_3 which touch a fixed plane π, is the first polar, in regard to Δ, of the point of V_3^8 representing that plane.

Thus $U(\pi)$ passes simply through V_6^{10} and doubly through V_3^8. It is clear, moreover, that $U(\pi)$ is a *conical* cubic primal; this follows either from the fact that Δ is a quartic monoid, and so the first polar of any triple point must be a cubic cone; or from the circumstance that, in the pencil formed by a general quadric and a double plane, no other quadric can be found to touch the plane than the double plane itself. This means, geometrically, that any line through O meets $U(\pi)$ in three points at O.

3.3. *The condition* ν. If l is a line of S_3, defined as the intersection of the planes (u_i) and (v_i), the condition that a quadric F should touch l is

$$\begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} & u_0 & v_0 \\ . & . & . & . & . & . \\ a_{30} & a_{31} & a_{32} & a_{33} & u_3 & v_3 \\ u_0 & u_1 & . & . & 0 & 0 \\ v_0 & v_1 & . & . & 0 & 0 \end{vmatrix} = 0. \tag{8}$$

This condition can be written in the form

$$\sum \frac{\partial^2 \Delta}{\partial a_{ij}\, \partial a_{kl}} b_{ij} b_{kl} = 0,$$

where $b_{ij} = u_i v_j$. Hence the corresponding locus in S_9 is the second polar, in regard to Δ, of the node of Δ which represents the plane-pair (u, v). Again, since the left-hand side of (8) may be expanded as a homogeneous quadratic in the variables $u_i v_j - u_j v_i$, it follows that the locus is the same for every plane-pair that can be drawn through l. Thus, in virtue of § 3.13, we see that

The locus, in S_9, of points corresponding to quadrics of S_3 which touch a given line l is a quadric primal $U(l)$, the second polar, in regard to Δ, of every point of the generating plane of V_6^{10} which corresponds to the system of plane-pairs with l as axis.

Thus $U(l)$ passes simply through V_3^8, as is otherwise clear, since every double plane formally satisfies the condition of tangency to l.

3.4. *The condition* μ. We have seen in § 3.11 that any tangent prime to Δ is represented by the system $U(P)$ of quadrics passing through a given point P of S_3. Conversely, any such system is represented in S_9 by the points of a prime $U(P)$ which, as we have shown, touches Δ at all points of the generating space [5] corresponding to the system of cones with their vertices at P.

3.5. *Contact properties.* We shall now prove three results which are of fundamental importance in the sequel.

THEOREM I. *The cubic primal $U(\pi)$, representing the quadrics which touch a fixed plane π, touches Δ along a V_7^6.*

Let O be any ordinary point of Δ lying on $U(\pi)$; then, corresponding to O, there will be a cone K whose vertex P lies on π. The tangent prime to $U(\pi)$ at O corresponds to the system of quadrics F such that, in the pencil formed by K and F, two of the quadrics touching π coincide with K: that is to say, F must pass through P. But, as we have seen, the system of quadrics through P represents the tangent prime to Δ at O. Thus $U(\pi)$ touches Δ at all ordinary common points.

THEOREM II. *The quadric primal $U(l)$, representing the quadrics which touch a fixed line l, touches the double V_6^{10} of Δ along a V_5^{10}.*

Let O be any ordinary point of V_6^{10} lying on $U(l)$; then, corresponding to O, there will be a plane-pair whose axis (or double line) g intersects l. The tangent prime to $U(l)$ at O corresponds to the system of quadrics F such that, in the pencil formed by F and the plane-pair, the two quadrics which touch l coincide with the plane-pair: that is, F must pass through the point where g meets l. Since all quadrics through g satisfy this condition, it follows, by § 3.13 (iv), that $U(l)$ touches V_6^{10} at all common points not lying on V_3^8.

THEOREM III. *The tangent prime $U(P)$ to Δ, representing the quadrics which pass through a fixed point P, touches the triple V_3^8 of Δ along a Veronese surface V_2^4.*

The reasoning is precisely as before; the surface of contact of $U(P)$ with V_3^8 corresponds (cf. § 3.13) to the ∞^2 double planes which pass through P: dually, we have the familiar plane representation of the Veronese surface.

Evidently $U(P)$ meets V_6^{10} in the points corresponding to the ∞^5 plane-pairs which pass through P; hence it does *not* touch the double manifold of Δ at all common points. This remark suffices to justify the method of § 3.12 by which the order of this manifold was obtained; for it is now clear that the space common to six given tangent primes to Δ will meet the double manifold in distinct points.

On the other hand, it follows from Theorem III that the space [6] common to any three given tangent primes to Δ will meet V_3^8 in one or more points, each of which, by the conditions of contact, must be an eightfold intersection. And since, in [3], a single plane can be drawn through three given points, we have an explanation of the fact that the triple manifold of Δ is of order eight.

EXAMPLES

1. Show that any three generating spaces [5] of Δ have in common a point of V_3^8.

2. Show that the cones whose vertices lie on a given line correspond to the points of a V_6^4, locus of ∞^1 generating spaces of Δ, which passes through V_3^8.

3. Prove that each quadric primal $U(l)$ touches Δ along a V_6^4.

4. From the fact that three quadrics of S_3 have in general 8 common points, deduce that 8 tangent primes can be drawn to Δ through an arbitrary plane.

5. Show that each of the ∞^2 tangent primes to Δ from a given point corresponds to a system of quadrics through a fixed point of a given quadric of S_3.

6. Show that the ∞^8 quadrics which touch a given quadric of S_3 correspond to the points of the tangent cone to Δ from a given point.

7. Find the character of the proper tangent cone to Δ from any point of V_6^{10}.

8. Show that every quadric $U(l)$ is a [6]-cone, whose vertex corresponds to the system of quadrics with l as a common generator.

9. Show that nine primals $U(\pi)$ meet residually to V_6^{10} in one point.

3.6. ***Geometrical interpretation of the condition-figures.*** The problem of evaluating a primary condition-symbol

$$\Gamma \equiv \mu^r \nu^s \rho^{9-r-s}$$

is that of finding the number of *irreducible* quadrics which satisfy the given set Γ of nine independent assigned conditions. It may happen in certain cases that a number of reducible quadrics also satisfy these conditions: for example, among the quadrics which pass through three given points and touch six given lines there is the plane of the three points, counted twice. Any such solution as this is, however, rejected as an improper solution of the problem.

We may interpret the given problem in terms of the basic representation as follows. Each point (μ) of Γ will correspond to a tangent prime $U(P)$ to Δ, each line (ν) to a polar quadric primal $U(l)$, and each plane (ρ) to a polar cubic primal $U(\pi)$ (§3.1). Hence we may say that

The number of irreducible quadrics which satisfy a set of primary conditions $\mu^r\nu^s\rho^{9-r-s}$ is equal to the number of free intersections of the r tangent primes, s polar quadric primals, and $9-r-s$ polar cubic primals associated with these conditions in the basic representation, external to the base points which they may possess on the multiple varieties V_6^{10} and V_3^8 of Δ.

Such, and only such, quadrics will be considered as solutions of the given problem.

Thus, geometrically speaking, the problem is equivalent to one in the theory of intersections of primals. If, as is always to be assumed in the sequel, the points, lines, and planes of Γ are in general position, none of the irreducible quadrics which satisfy Γ will be cones; for there are only ∞^8 cones in ordinary space and, as will appear in § 4, each condition of type μ, ν, or ρ effectively imposes one condition on all cones which do not degenerate into plane-pairs or double planes.

The following examples illustrate the kinds of base manifolds which the set of nine given primals may possess.

Examples

1. Consider the condition ρ^9: here we require to find the number of free intersections of nine polar cubic primals, residual to the simple base manifold V_6^{10}, and the double base manifold V_3^8.

2. For the condition $\nu^2\rho^7$, the base manifold is V_3^8, which is simple for the two quadric primals and double for the seven cubic primals.

3. Consider next the condition $\mu^2\nu\rho^6$: here we have two tangent primes, one quadric and six cubic primals. The base manifold is the curve in which the secundum common to the two primes meets V_3^8, and the curve is simple for the quadric and double for the cubics.

4. For the condition $\mu^3\nu^6$ the base points are the intersections with V_3^8 of the [6] common to the three tangent primes.

5. Consider finally the condition $\mu^5\nu\rho^3$: the three cubic primals all pass through V_6^{10}, and the quadric primal will meet this in a V_5, which again is met by the [4] common to the five tangent primes in a finite number of points; and these are base points for the system.

It is clear that, for any primary condition-figure of dimension $k < 9$, a similar geometrical interpretation may be given. For instance, the quadrics satisfying the eightfold condition $\mu^r\nu^s\rho^{8-r-s}$ correspond, in the basic representation, to a curve: this is the intersection of the r associated tangent primes, s polar quadric primals, and $8-r-s$ polar cubic primals, residual to any base curve they may possess on the multiple varieties of Δ.

It will appear later that many of the primary problems for quadrics can be solved merely by using the above result; for the more difficult problems, to which the following sections are preliminary, the basic representation is, moreover, of essential importance.

§ 4. The primary problems for cones and conics

4. The degenerate forms. In the calculations which follow, an indispensable role is played by those quadrics which, in a sense now to be defined, may be regarded as degenerate forms.

A general quadric surface F may be considered as an irreducible aggregate of points, planes, and generating lines in the following ways:

(a) as a locus of ∞^2 points;
(b) as an envelope of ∞^2 planes; this is dual to (a);
(c) as a regulus of generators (with, of course, the complementary regulus).

In each case there is an analytical representation in terms of the appropriate coordinates.

If now we assume that (a) acquires a double point,
or that (b) acquires a double plane,
or that (c) acquires a double line,

two of the three aggregates associated with the quadric are no longer of general character. In fact they may cease to be determinate and can be defined only by a limiting process. We call the three types of quadrics which are particularized in these ways the three *degenerate forms*; and we denote them by the symbols χ, ϕ, and ψ. Each of these forms is obtained by imposing a single condition on one or other of the three analytical representations of F, so that the systems (χ), (ϕ), and (ψ) must all be regarded as of dimension 8. We now consider them in turn.

4.01. *The form* χ. Suppose that a general quadric F, regarded as a locus of points, tends to a point-cone K as a limiting form: for the passage to the limit it suffices to consider the pencil $K+\lambda F$ of quadrics, and to let λ tend to zero. The limiting form has a double point at the vertex A of the cone K; and it is clear that its system of tangent lines is the aggregate of proper tangents to K. In order to find the envelope of tangent planes, we proceed as in Ex. 3, § 1.2, forming the tangential equation of the quadric $K+\lambda F$. We thus deduce that the envelope in question consists of the star of planes through A, each plane being counted twice.

In conclusion, then:

The points of χ are those of a quadric cone K, with vertex A.

The tangent planes of χ are the planes through A, each counted twice.

The tangent lines to χ are the proper tangents to K.

4.02. *The form* ϕ. This is dual to the preceding, and so the results may be inferred from the above. We thus see that:

The points of ϕ are those of a plane α, counted twice.

The tangent planes of ϕ are the planes which touch a fixed conic k lying in α.

The tangent lines to ϕ are the secants to k.

4.03. *The form* ψ. A ruled quadric, consisting of two complementary reguli, may be obtained as follows. Consider two homographic ranges of points on two skew lines l, l'; then the lines joining corresponding points in the homography are the generators of one regulus, and the lines incident to all these form the complementary regulus. The tangent planes to the quadric consist of all the planes through the generators.

In the general case the homography in question will be given by a relation of the form $a\lambda\lambda'+b\lambda+c\lambda'+d = 0$, where a, b, c, d are arbitrary. We now suppose that $ad = bc$, so that the relation takes the particular form $a(\lambda-\lambda_1)(\lambda'-\lambda_2) = 0$. In this case to a single point A of l there corresponds every point of l' and to a single point A' of l' there corresponds every point of l. It follows that the generators of one regulus of the quadric form two pencils, with centres A and A', lying in planes α and α' respectively; the complementary regulus then consists of the pencils (A, α') and

(A', α). The tangent planes to the surface form two stars, with centres A, A' respectively. Finally, taking a plane section of the figure, we see that the tangent lines consist of all the secants to the line $AA' = g$, say, each counted twice.

This result may also be obtained from the representation of the regulus by a conic on the quadric Ω of S_5 (Ch. X, § 2). If we suppose the regulus to acquire a double generator, the conic image has a double point, and so breaks up into a pair of intersecting lines. Thus the regulus consists of two pencils.

The limiting form defined in this way is called the form ψ. It has the following features:

The points of ψ are those of a plane-pair (α, α').

The tangent planes to ψ are the planes through one or other of two fixed points A, A' on the double line g of (α, α').

The tangent lines to ψ are the secants to g, each counted twice.

We call A, A' the *principal points* and g the *axis* of ψ.

4.1. *Summary of results.* The following table shows the various systems of points, lines, and planes which constitute the forms χ, ϕ, ψ.

	Points	*Tangent lines*	*Tangent planes*
χ	Quadric cone K, vertex A	Tangents to K	Planes through A, counted twice
ϕ	Plane α, counted twice	Secants to a conic k lying in α	Tangent planes to k
ψ	Plane-pair α, α', meeting in line g	Secants to g, counted twice	Planes through one of two fixed points A, A' on g

We note that, in the basic representation (which is based entirely on the point-locus concept of a quadric), each point of V_6^{10} corresponds† to ∞^2 forms ψ, and each point of V_3^8 corresponds to ∞^5 forms ϕ.

4.2. *Calculation of the primary conditions for χ, ψ, and ϕ.* In the following sections we shall determine the numbers of forms χ, ψ, and ϕ which satisfy the various sets of assigned primary conditions. Since each of these forms depends on eight parameters, the sets of conditions will be of the form $\mu^r \nu^s \rho^{8-r-s}$.

† It can be shown that the ∞^2 limiting forms ψ are associated with the ∞^2 primes through the tangent [6] to V_6^{10} at a point P, all approaches to P in directions which lie in any one of these primes giving the same form ψ. Similarly ∞^5 forms ϕ are associated with the ∞^5 primes which touch V_3^8 at a fixed point.

We shall also denote by χ, ψ, or ϕ the *condition* that a quadric should be of type χ, ψ, or ϕ; in this notation we have thus to evaluate all condition-symbols of the forms $\chi\mu^r\nu^s\rho^{8-r-s}$ and $\psi\mu^r\nu^s\rho^{8-r-s}$, since by the principle of duality we infer that $\phi\mu^r\nu^s\rho^{8-r-s} = \chi\rho^r\nu^s\mu^{8-r-s}$.

In the detailed calculations it is often convenient to denote the points, lines, and planes of the associated condition-figure Γ by the symbols $\bar{\mu}$, $\bar{\nu}$, $\bar{\rho}$, with distinguishing suffixes, if necessary. For example, the condition-figure for the symbol $\chi\mu^4\nu^2\rho^2$ will be written as $(\bar{\mu}_1,\ldots,\bar{\mu}_4;\ \bar{\nu}_1,\bar{\nu}_2;\ \bar{\rho}_1,\bar{\rho}_2)$, or briefly, as $(\bar{\mu}_i^4, \bar{\nu}_i^2, \bar{\rho}_i^2)$.

4.3. *Multiplicity of the solutions.* As in § 3.6, where we interpreted any primary problem for general quadrics as one in the intersection of manifolds, we may state the primary problems for the degenerate forms in geometrical terms.

Regarded simply as point-loci, the forms χ correspond to the ordinary points of the primal Δ; the forms ψ correspond to those of the double V_6^{10}, and the forms ϕ to those of the triple V_3^8. It follows that

The number of solutions of the problem $\chi\mu^r\nu^s\rho^{8-r-s}$ is equal to the number of intersections, suitably counted, of Δ with the corresponding r tangent primes, s polar quadrics, and $8-r-s$ polar cubic primals in the basic representation, residual to their common intersections with V_6^{10} and V_3^8.

The number of solutions of the problem $\psi\mu^r\nu^s\rho^{8-r-s}$ is equal to the number of intersections, suitably counted, of V_6^{10} with the corresponding r tangent primes, s polar quadrics, and $8-r-s$ polar cubic primals in the basic representation, residual to their common intersections with V_3^8.

The number of solutions of the problem $\phi\mu^r\nu^s\rho^{8-r-s}$ is equal to the number of intersections, suitably counted, of V_3^8 with the curve in which the corresponding r tangent primes, s polar quadrics, and $8-r-s$ polar cubic primals of the basic representation meet residually to V_3^8.

From these theorems many of the desired results can be obtained at once, without further resort to degeneration methods; examples of their use will follow shortly.

When the basic representation cannot be used directly, the first question to be decided is how to estimate the multiplicity of the intersections in any given case.

Consider, for example, the problem $\chi\mu^r\nu^{7-r}\rho$: here we have to determine the number of intersections of Δ with r tangent primes $U(\bar{\mu})$, $7-r$ polar quadrics $U(\bar{\nu})$, and one polar cubic $U(\bar{\rho})$. Now, by Theorem I of § 3.5, the polar cubic $U(\bar{\rho})$ touches Δ along a V_7^6; hence the value of $\chi\mu^r\nu^{7-r}\rho$ is *twice* the number of forms χ which satisfy the condition $\mu^r\nu^{7-r}\rho$. More generally, by a repetition of the argument, we see that

I. *The value of the condition-symbol $\chi\rho^r\nu^s\mu^{8-r-s}$ is equal to 2^r times the number of forms χ which satisfy the condition $\rho^r\nu^s\mu^{8-r-s}$.*

Dually, it follows that

II. *The value of the condition-symbol $\phi\mu^r\nu^s\rho^{8-r-s}$ is equal to 2^r times the number of forms ϕ which satisfy the condition $\mu^r\nu^s\rho^{8-r-s}$.*

Consider next the problem $\psi\mu^r\nu^s\rho^{8-r-s}$: here we must determine the number of intersections of V_6^{10} with r tangent primes $U(\bar{\mu})$, s polar quadrics $U(\bar{\nu})$, and $8-r-s$ polar cubics $U(\bar{\rho})$, residual to their base points (if any) on V_3^8. By Theorem II of § 3.5, each quadric $U(\bar{\nu})$ touches V_6^{10} along a V_5^{10}. It follows that

III. *The value of the condition-symbol $\psi\mu^r\nu^s\rho^{8-r-s}$ is equal to 2^s times the number of forms ψ which satisfy the condition $\mu^r\nu^s\rho^{8-r-s}$.*

4.4. *Evaluation of the condition-symbols for χ.* We come now to the systematic calculation of the condition-symbols for χ (and therefore ϕ). In this section we confine ourselves to those results which can be obtained without further use of degeneration methods: the more difficult problems for χ are considered in § 4.6.

(1) The first group of solutions is derived immediately from the table in § 4.1; there we see that if a form χ is to satisfy a condition ρ, the vertex (or double point) of χ must lie on $\bar{\rho}$. Hence

For all conditions of the type $\chi\rho^r\nu^s\mu^{8-r-s}$, where $r > 3$, the number of solutions is zero.

(2) We consider next the group† $\chi\rho^3\nu^s\mu^{5-s}$; for all these, the vertex of χ is fixed at the point $\bar{\rho}_1\cdot\bar{\rho}_2\cdot\bar{\rho}_3$; and, by § 4.3, in each case the number of solutions is eight times the number of cones

† For conditions of this type the basic representation does not serve; we illustrate here a correspondence method which will be useful again later, and which has been employed by Ursell (*Proc. Lond. Math. Soc.* (2) **30** (1930), 322) to solve many of the primary problems for quadrics.

with fixed vertex which pass through $5-s$ given points and touch s given lines. We thus have $\chi\rho^3\mu^5 = 8$; to find $\chi\rho^3\mu^4\nu$ we now consider the $(1,1)$ correspondence set up on the line $\bar{\nu}$ by the cones of the system $(\bar{\rho}_i^3, \bar{\mu}_i^4)$. There are two coincidences in the correspondence, whence $\chi\rho^3\mu^4\nu = 16$. Considering similarly the $(2,2)$ correspondence set up on the line $\bar{\nu}_2$ by the cones of the system $(\bar{\rho}_i^3, \bar{\mu}_i^3, \bar{\nu}_1)$, we deduce that $\chi\rho^3\mu^3\nu^2 = 32$. We cannot, however, calculate $\chi\rho^3\mu^2\nu^3$ in this manner, since the plane $(\bar{\rho}_1 . \bar{\rho}_2 . \bar{\rho}_3)\bar{\mu}_1\bar{\mu}_2$ is an improper solution. The remaining results can be inferred by duality from the theory of conics in the plane, using the formulae just established.

(3) Next we have the conditions of type $\chi\rho^2\mu^r\nu^{6-r}$. Here the vertex of χ must lie on the fixed line $\bar{\rho}_1 . \bar{\rho}_2$.

From the basic representation we see that

$$\chi\rho^2\mu^6 = 4.3.3-2.10 = 16;$$

for the [3] common to the primes $U(\bar{\mu})$ meets V_6^{10} in points which are simple on each primal $U(\bar{\rho})$ and double on Δ. Whence, by a correspondence method or, equally well, from the representation, we have $\chi\rho^2\mu^5\nu = 32$, $\chi\rho^2\mu^4\nu^2 = 64$, and so on.

These results, as well as those which are obtainable in a similar manner, are shown in the following table, which may be verified by the reader.

TABLE I

$$\chi\mu^r\nu^s\rho^{8-r-s} = \phi\rho^r\nu^s\mu^{8-r-s}$$

$\chi\rho^r\nu^s\mu^{8-r-s}$	$= 0$ $(r > 3)$	$\chi\rho\mu^7$	$= 12$
$\chi\rho^3\mu^5$	$= 8$	$\chi\rho\mu^6\nu$	$= 24$
$\chi\rho^3\mu^4\nu$	$= 16$	$\chi\rho\mu^5\nu^2$	$= 48$
$\chi\rho^3\mu^3\nu^2$	$= 32$	$\chi\rho\mu^4\nu^3$	$= 96$
$\chi\rho^2\mu^6$	$= 16$	$\chi\mu^8$	$= 4$
$\chi\rho^2\mu^5\nu$	$= 32$	$\chi\mu^7\nu$	$= 8$
$\chi\rho^2\mu^4\nu^2$	$= 64$	$\chi\mu^6\nu^2$	$= 16$
$\chi\rho^2\mu^3\nu^3$	$= 96$	$\chi\mu^5\nu^3$	$= 32$
		$\chi\mu^4\nu^4$	$= 64$

4.5. *The primary conditions for* ψ. We come now to the conditions of the type $\psi\mu^r\nu^s\rho^{8-r-s}$. We recall (§ 4.1) that the form ψ, regarded as a point-locus, is a plane-pair (α, α'), having a double line or axis g; that its tangent lines are the secants to g, counted twice; and that its tangent planes form two stars having their centres at points A, A' (called principal points) of g.

As regards the multiplicities of the solutions, we have proved in § 4.3 that the value of $\psi\mu^r\nu^s\rho^{8-r-s}$ is 2^s times the number of forms ψ which satisfy the conditions $\mu^r\nu^s\rho^{8-r-s}$. How the latter may be determined is illustrated below.

(1) In the first place, since the form ψ is self-dual, we have in all cases $\psi\mu^r\nu^s\rho^{8-r-s} = \psi\rho^r\nu^s\mu^{8-r-s}$. Thus one set of results may be inferred from the other.

(2) Next, we observe that, when $r > 6$, the number of solutions is zero; for a pair of planes cannot be made to pass through more than six points in general position.

(3) Again, when $s > 4$, the number of solutions is zero; for the axis of a plane-pair which satisfies the conditions of the problem would be required to meet more than four lines in general position.

(4) In addition to the above, the number of solutions in many other cases is readily seen to be zero. In fact the procedure is much simpler here than in § 4.4, all the problems being essentially incidence problems for points, lines, and planes. Occasionally we may find it more convenient to use the basic representation, as suggested by the results of § 4.3. The following illustrations are typical.

TABLE II

$$\psi\mu^r\nu^s\rho^{8-r-s} = \psi\rho^r\nu^s\mu^{8-r-s}$$

$\psi\mu^8 = 0$	$\psi\mu^4\nu^4 = 0$	$\psi\mu^2\nu^6 = 0$
$\psi\mu^7\rho = 0$	$\psi\mu^4\nu^3\rho = 0$	$\psi\mu^2\nu^5\rho = 0$
$\psi\mu^7\nu = 0$	$\psi\mu^4\nu^2\rho^2 = 40$	$\psi\mu^2\nu^4\rho^2 = 32$
$\psi\mu^6\nu^2 = 0$	$\psi\mu^4\nu\rho^3 = 60$	$\psi\mu\nu^7 = 0$
$\psi\mu^6\nu\rho = 0$	$\psi\mu^4\rho^4 = 42$	$\psi\mu\nu^6\rho = 0$
$\psi\mu^6\rho^2 = 10$	$\psi\mu^3\nu^5 = 0$	$\psi\nu^8 = 0$
$\psi\mu^5\nu^3 = 0$	$\psi\mu^3\nu^4\rho = 0$	
$\psi\mu^5\nu^2\rho = 0$	$\psi\mu^3\nu^3\rho^2 = 48$	
$\psi\mu^5\nu\rho^2 = 20$	$\psi\mu^3\nu^2\rho^3 = 72$	
$\psi\mu^5\rho^3 = 30$	$\psi\mu^3\nu^5 = 0$	

EXAMPLES

1. To evaluate $\psi\mu^5\nu^2\rho$. We have to draw a pair of planes through five given points so that their line of intersection meets two given lines; and this is in general impossible. Thus $\psi\mu^5\nu^2\rho = 0$.

2. To evaluate $\psi\mu^4\nu^3\rho$. We have to draw a pair of planes through four given points so that their line of intersection meets three given lines. Hence $\psi\mu^4\nu^3\rho = 0$.

3. To evaluate $\psi\mu^5\rho^2\nu$. Using the basic representation, we have to determine the number of free intersections of the double V_8^{10} of Δ with the tangent primes $U(\bar{\mu})$, the cubic primals $U(\bar{\rho})$, and the quadric primal $U(\bar{\nu})$ of the condition. Now the primes have in common a [4] which meets V_8^{10}

in a decimic curve lying on $U(\bar{\rho})$ and meeting $U(\bar{\nu})$ in 20 points not on F_3^8. Hence $\psi\mu^5\rho^2\nu = 20$.

4. To evaluate $\psi\mu^3\nu^3\rho^2$. The axis of any of the required plane-pairs must meet the lines $\bar{\nu}_i$ and also one of the three lines joining the points $\bar{\mu}_i$ in pairs: the principal points of ψ are then uniquely determined. Hence there are six distinct solutions, each of them counting 8 times; so that

$$\psi\mu^2\nu^3\rho^2 = 48.$$

4.6. *The primary conditions for* χ (*continued*). In § 4.4 we considered a number of problems for the form χ which can be solved quite readily by using the basic representation. All these, however, as well as the more difficult cases which follow, may be treated by the method of degenerate forms, which we now apply to the form χ itself.

In § 4.01 we examined the system of tangent planes to χ, considered as the limiting form of a general quadric F. In view of the use ultimately to be made of the results now to be obtained, we define in connexion with a system Σ of ∞^1 cones (K) the following primary conditions:

μ, the condition that a cone of Σ should pass through a given point;

ν, the condition that a cone of Σ should touch a given line;

ρ, the condition† that a cone of Σ should have its vertex on a given plane.

As usual, the numbers of cones satisfying these conditions will be denoted by μ, ν, ρ respectively.

We now propose to calculate for the system of ∞^8 cones (K) those condition-symbols which we need to complete our previous results; the others may be obtained in a similar fashion.

4.61. *The forms* ψ' *and* ϕ'. For this purpose we require to consider the limiting forms assumed by a cone K when, considered as a point-locus, it tends (i) to a plane-pair, (ii) to a double plane.

We denote the corresponding degenerate forms by ψ' and ϕ' respectively.

As in § 4 we may prove

(i) The points of ψ' are those of a plane-pair (α, α').

The tangent lines to ψ' are the secants to the axis g of the plane-pair, counted twice.

† The symbol ρ is here used in a new sense; but, as will appear, it is convenient not to change our notation.

The vertex of K tends to a determinate point A on g.

(ii) The points of ϕ' are those of a double plane α.

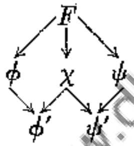
The tangent lines to ϕ' are the secants to one or other of two fixed lines g, g' in α.

The vertex of K tends to the point A of intersection of g and g'.

In each case we call A the *vertex* of the form; and we call g, g' the *axes* of the form ϕ'.

It will be noted that there are ∞^7 forms ψ', corresponding to the plane-pairs with an assigned point on each axis, and that there are ∞^7 forms ϕ', corresponding to the intersecting line-pairs in space. Thus each point of V_6^{10} corresponds to ∞^1 forms ψ', and each point of V_3^8 to ∞^4 forms ϕ'.

The relations between the general quadric F and the various degenerate forms are shown here:



We have now to calculate the conditions of the types $\psi'\mu^r\nu^s\rho^{7-r-s}$ and $\phi'\mu^r\nu^s\rho^{7-r-s}$ which are required to complete Table I (§ 4.4). The multiplicity of the solutions is settled by the theorems of § 4.3. We conclude in fact that

The value of the symbol $\psi'\mu^r\nu^s\rho^{7-r-s}$ is equal to 2^s times the number of forms ψ' which satisfy the condition $\mu^r\nu^s\rho^{7-r-s}$.

The value of the symbol $\phi'\mu^r\nu^s\rho^{7-r-s}$ is equal to 2^r times the number of forms ϕ' which satisfy the condition $\mu^r\nu^s\rho^{7-r-s}$.

The method of calculation is illustrated below.

Examples

1. To evaluate $\psi'\mu^4\nu^2\rho$. We observe that either one plane of ψ' contains three of the points $\bar{\mu}_i$, in which case the other plane contains the fourth; or each plane contains two of the given points. Taking one plane as $\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3$, say, we see that the axis of ψ' must be the transversal from $\bar{\mu}_4$ to $\bar{\nu}_1$ and $\bar{\nu}_2$. Thus there are four solutions of this type. Instead, if we take one plane through $\bar{\mu}_1\bar{\mu}_2$ and the other through $\bar{\mu}_3\bar{\mu}_4$, the double line must meet $\bar{\nu}_1$ and $\bar{\nu}_2$ and also the line common to the plane-pair. There are six solutions of this type. In each case, when the plane-pair is fixed, the vertex of ψ' is uniquely determined by the condition ρ. Since the multiplicity of the solution is 4, we have $\psi'\mu^4\nu^2\rho = 4(6+4) = 40$.

2. To evaluate $\phi'\nu^6\rho$. We have to determine all possible pairs of lines which meet on the plane $\bar{\rho}$, and which intersect one or other of the six lines $\bar{\nu}_1,\ldots,\bar{\nu}_6$. We obtain one type of solution by drawing the two transversals to $\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3, \bar{\nu}_4$ to meet $\bar{\rho}$ in P, Q, say, and drawing the transversals to $\bar{\nu}_5, \bar{\nu}_6$ from P, Q; there are $2\,{}^6C_4 = 30$ solutions of this kind. Again, the lines which meet $\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3$ and also $\bar{\nu}_4, \bar{\nu}_5, \bar{\nu}_6$ intersect on the common curve of the quadrics through $\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3$ and $\bar{\nu}_4, \bar{\nu}_5, \bar{\nu}_6$ respectively. This curve meets $\bar{\rho}$ in four points, each yielding one solution to the problem. There are $4\times 10 = 40$ solutions of this kind, so that altogether we have $\phi'\nu^6\rho = 70$.

3. To evaluate $\psi'\mu^5\rho^2$. The two planes $\bar{\rho}_1, \bar{\rho}_2$ of the condition must pass through the vertex of ψ'. Hence we draw the plane through $\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3$, say, find its intersection with $\bar{\rho}_1$ and $\bar{\rho}_2$, and through this point draw the plane containing $\bar{\mu}_4$ and $\bar{\mu}_5$. Thus there are ten solutions in all.

The complete set of results is given in the following table. It is obvious that all the condition-symbols containing a factor ρ^r, where $r > 3$, are zero. These are omitted from the table.

TABLE III

$\psi'\mu^r\nu^s\rho^{7-r-s}$ *and* $\phi'\mu^r\nu^s\rho^{7-r-s}$

$\psi'\mu^7 = 0$,	$\phi'\mu^7 = 0$	$\psi'\mu^3\nu^4 = 0$,	$\phi'\mu^3\nu^4 = 24$
$\psi'\mu^6\nu = 0$,	$\phi'\mu^6\nu = 0$	$\psi'\mu^3\nu^3\rho = 48$,	$\phi'\mu^3\nu^3\rho = 24$
$\psi'\mu^6\rho = 10$,	$\phi'\mu^6\rho = 0$	$\psi'\mu^3\nu^2\rho^2 = 24$,	$\phi'\mu^3\nu^2\rho^2 = 8$
$\psi'\mu^5\nu^2 = 0$,	$\phi'\mu^5\nu^2 = 0$	$\psi'\mu^3\nu\rho^3 = 6$,	$\phi'\mu^3\nu\rho^3 = 0$
$\psi'\mu^5\nu\rho = 20$,	$\phi'\mu^5\nu\rho = 0$	$\psi'\mu^2\nu^5 = 0$,	$\phi'\mu^2\nu^5 = 80$
$\psi'\mu^5\rho^2 = 10$,	$\phi'\mu^5\rho^2 = 0$	$\psi'\mu^2\nu^4\rho = 32$,	$\phi'\mu^2\nu^4\rho = 68$
$\psi'\mu^4\nu^3 = 0$,	$\phi'\mu^4\nu^3 = 0$	$\psi'\mu^2\nu^3\rho^2 = 16$,	$\phi'\mu^2\nu^3\rho^2 = 24$
$\psi'\mu^4\nu^2\rho = 40$,	$\phi'\mu^4\nu^2\rho = 0$	$\psi'\mu^2\nu^2\rho^3 = 4$,	$\phi'\mu^2\nu^2\rho^3 = 4$
$\psi'\mu^4\nu\rho^2 = 20$,	$\phi'\mu^4\nu\rho^2 = 0$	$\psi'\mu\nu^6 = 0$,	$\phi'\mu\nu^6 = 140$
$\psi'\mu^4\rho^3 = 3$,	$\phi'\mu^4\rho^3 = 0$	$\psi'\mu\nu^5\rho = 0$,	$\phi'\mu\nu^5\rho = 100$
$\psi'\nu^7 = 0$,	$\phi'\nu^7 = 140$	$\psi'\mu\nu^4\rho^2 = 0$,	$\phi'\mu\nu^4\rho^2 = 34$
$\psi'\nu^6\rho = 0$,	$\phi'\nu^6\rho = 70$	$\psi'\mu\nu^3\rho^3 = 0$,	$\phi'\mu\nu^3\rho^3 = 6$
$\psi'\nu^5\rho^2 = 0$,	$\phi'\nu^5\rho^2 = 20$		
$\psi'\nu^4\rho^3 = 0$,	$\phi'\nu^4\rho^3 = 3$		

4.7. *The primary conditions for cones.* Since the condition that a cone K should be of the form ψ' or ϕ' is simple, it follows that a system Σ of ∞^1 cones will in general contain finite numbers of forms ψ' and ϕ': we denote these by ψ' and ϕ' respectively.

In the basic representation, Σ will correspond to a curve C lying on Δ; evidently C meets V_6^{10} in ψ' points, and V_3^8 in ϕ' points; we shall assume that these are simple for C. Then the character μ of Σ represents the number of points in which C meets an arbitrary tangent prime to Δ; that is, μ is the *order* of C. Also ρ and ν are the respective numbers of free intersections of C with the cubic primal $U(\bar{\rho})$ and the quadric primal $U(\bar{\nu})$.

Consider the intersections of C with the quadric $U(\bar{\nu})$; since this passes simply through V_3^8, we have

$$2\mu = \nu + \phi'. \tag{1}$$

Again, consider the intersections of C with the cubic primal $U(\bar{\rho})$; this passes simply through V_6^{10} and doubly through V_3^8; moreover, by Theorem I (§ 3.5), it touches Δ along a V_7^8. Hence we have

$$3\mu = \psi' + 2\phi' + 2\rho. \tag{2}$$

Now the symbols appearing in (1) and (2) may equally well be regarded as conditions imposed on Σ; they may then be combined in accordance with the calculus of conditions as described in § 1.3: for example, they may be regarded as simultaneous equations for μ and ν. In that case, if we solve for ν we obtain

$$3\nu = 2\psi' + \phi' + 4\rho. \tag{3}$$

We may use these relations to determine the number of point-cones which satisfy any set Γ of conditions $\mu^r\nu^s\rho^{8-r-s}$. Geometrically speaking, we have to determine the number of free intersections with Δ of a set of r tangent primes, s polar quadrics, and $8-r-s$ polar cubic primals: which is the same as saying that we have to find the number of cones of the system Σ which satisfy the condition $\Gamma \equiv \mu^{r-1}\nu^s\rho^{8-r-s}$, and which also satisfy the condition μ. The curve C representing Σ is the intersection with Δ of the $r-1$ tangent primes, s polar quadrics, and $8-r-s$ polar cubic primals, residual to any base points they may possess on V_6^{10} and V_3^8. Evidently C will have only simple intersections with V_6^{10} and V_3^8, so that the previous formulae will apply. We thus have

$$\begin{aligned}\mu^r\nu^s\rho^{8-r-s} &= \mu\,.\,\mu^{r-1}\nu^s\rho^{8-r-s}\\ &= \tfrac{1}{3}(\psi'+2\phi'+2\rho)\mu^{r-1}\nu^s\rho^{8-r-s}, \quad \text{by (2)},\\ &= \tfrac{1}{3}\psi'\mu^{r-1}\nu^s\rho^{8-r-s}+\tfrac{2}{3}\phi'\mu^{r-1}\nu^s\rho^{8-r-s}+\tfrac{2}{3}\mu^{r-1}\nu^s\rho^{9-r-s}.\end{aligned}$$

The values of the first two symbols are known from Table III. As regards the third symbol, we carry out the calculations systematically so that the symbols containing the highest powers of ρ are evaluated first, then the next highest, and so on.

If the symbol μ does not occur in the given condition we may use (3) instead. Thus

$$\begin{aligned}\nu^s\rho^{8-s} &= \nu\,.\,\nu^{s-1}\rho^{8-s} = \tfrac{1}{3}(2\psi'+\phi'+4\rho)\nu^{s-1}\rho^{8-s}\\ &= \tfrac{2}{3}\psi'\nu^{s-1}\rho^{8-s}+\tfrac{1}{3}\phi'\nu^{s-1}\rho^{8-s}+\tfrac{4}{3}\nu^{s-1}\rho^{9-s}.\end{aligned}$$

Here also the calculations are performed systematically in the order described above.

EXAMPLES

1. To evaluate $\mu^4\nu^2\rho^2$.

We have
$$\mu^4\nu^2\rho^2 = \tfrac{1}{3}(\psi'+2\phi'+2\rho)\mu^3\nu^2\rho^2$$
$$= \tfrac{1}{3}(\psi'\mu^3\nu^2\rho^2+2\phi'\mu^3\nu^2\rho^2+2\mu^3\nu^2\rho^3)$$
$$= \tfrac{1}{3}(24+16+8),$$
since the value of $\mu^3\nu^2\rho^3$ is the number of conics of a plane which touch three given lines and pass through two given points. We thus obtain the formula $\mu^4\nu^2\rho^2 = 16$.

2. Suppose we require the value of ν^8; we begin by calculating $\nu^6\rho^2$, using the formula $\nu^6\rho^2 = \tfrac{2}{3}\psi'\nu^5\rho^2+\tfrac{1}{3}\phi'\nu^5\rho^2+\tfrac{4}{3}\nu^5\rho^3 = 8$.

Next we have $\nu^7\rho = \tfrac{2}{3}\psi'\nu^6\rho+\tfrac{1}{3}\phi'\nu^6\rho+\tfrac{4}{3}\nu^6\rho^2 = 34$.

Finally, we obtain
$$\nu^8 = \tfrac{2}{3}\psi'\nu^7+\tfrac{1}{3}\phi'\nu^7+\tfrac{4}{3}\nu^7\rho = 92.$$

Dually, the formula shows that there are 92 conics in space which meet eight given lines in general position.

The accompanying table gives the complete set of results for cones; from these we can infer by duality the numbers of conics in space which touch r given planes, intersect s given lines, and whose planes pass through $8-r-s$ given points.

TABLE IV

Conditions $\mu^r\nu^s\rho^{8-r-s}$ for Cones

$\mu^5\rho^3 = 1$	$\mu^6\rho^2 = 4$	$\mu^7\rho = 6$	$\mu^8 = 4$
$\mu^4\nu\rho^3 = 2$	$\mu^5\nu\rho^2 = 8$	$\mu^6\nu\rho = 12$	$\mu^7\nu = 8$
$\mu^3\nu^2\rho^3 = 4$	$\mu^4\nu^2\rho^2 = 16$	$\mu^5\nu^2\rho = 24$	$\mu^6\nu^2 = 16$
$\mu^2\nu^3\rho^3 = 4$	$\mu^3\nu^3\rho^2 = 24$	$\mu^4\nu^3\rho = 48$	$\mu^5\nu^3 = 32$
$\mu\nu^4\rho^3 = 2$	$\mu^2\nu^4\rho^2 = 24$	$\mu^3\nu^4\rho = 72$	$\mu^4\nu^4 = 64$
$\nu^5\rho^3 = 1$	$\mu\nu^5\rho^2 = 14$	$\mu^2\nu^5\rho = 76$	$\mu^3\nu^5 = 104$
	$\nu^6\rho^2 = 8$	$\mu\nu^6\rho = 52$	$\mu^2\nu^6 = 128$
		$\nu^7\rho = 34$	$\mu\nu^7 = 116$
			$\nu^8 = 92$

4.8 *Deduction of the condition-symbols for χ.* If now we wish to evaluate the symbol $\chi\mu^r\nu^s\rho^{8-r-s}$ by means of the preceding results, we have merely to recall that, by § 4.3, the value of any such symbol is equal to 2^r times the number of cones which satisfy the condition $\mu^r\nu^s\rho^{8-r-s}$. We may thus complete Table I by using the results of Table IV: the twelve formulae which remained to be determined are shown below.

TABLE V

$$\chi\mu^r\nu^s\rho^{8-r-s} = \phi\rho^r\nu^s\mu^{8-r-s}$$

$\chi\rho^2\mu^3\nu^3 = 96$	$\chi\rho\mu^3\nu^4 = 144$	$\chi\mu^3\nu^5 = 104$
$\chi\rho^2\mu^2\nu^4 = 96$	$\chi\rho\mu^2\nu^5 = 152$	$\chi\mu^2\nu^6 = 128$
$\chi\rho^2\mu\nu^5 = 56$	$\chi\rho\mu\nu^6 = 104$	$\chi\mu\nu^7 = 116$
$\chi\rho^2\nu^6 = 32$	$\chi\rho\nu^7 = 68$	$\chi\nu^8 = 92$

§ 5. ENUMERATIVE PROBLEMS FOR QUADRICS

5. The fundamental equations. We come now to the primary problems for general quadrics, i.e. the calculation of all ninefold conditions of the type $\mu^r\nu^s\rho^{9-r-s}$. As already stated, the preparatory results, shown in the previous tables, are essential to the work; we shall now see how they are to be utilized.

Consider a system Σ of ∞^1 quadrics: with this we associate the three characters μ, ν, ρ, as defined in § 3. Also Σ will in general contain finite numbers of degenerate forms of types χ, ψ, and ϕ; these numbers we denote by χ, ψ, ϕ respectively.

In the basic representation, Σ will correspond to a curve C; this meets the primal Δ in χ points which are simple on Δ, in ψ points of the double V_6^{10}, and in ϕ points of the triple V_3^8. We shall assume, as is generally the case, that all these intersections are simple for C. As in § 4.7, we see that the character μ represents the order of C, and that ν, ρ represent the respective numbers of its free intersections with the primals $U(\bar{\nu})$ and $U(\bar{\rho})$.

We now establish three equations which incorporate these facts.

(i) Considering the intersections of C with Δ, we have

$$4\mu = \chi+2\psi+3\phi. \tag{1}$$

(ii) Considering the intersections of C with a primal $U(\bar{\rho})$, we have

$$3\mu = \rho+\psi+2\phi. \tag{2}$$

(iii) From the intersections of C with a primal $U(\bar{\nu})$, we obtain

$$2\mu = \nu+\phi. \tag{3}$$

These equations provide the link between the results already established and those we now wish to obtain.

5.1. ***Evaluation of the primary condition-symbols.*** The above relations between the six characters of Σ (or C) may be combined in the manner described in § 4. In particular, they may be regarded as simultaneous equations for the symbols μ, ν, and ρ. When solved for the two latter they give

$$4\nu = 2\chi+4\psi+2\phi, \tag{4}$$

$$4\rho = 3\chi+2\psi+\phi. \tag{5}$$

We now show how these results are applied to evaluate the primary condition-symbols for quadrics.

Suppose that the system Σ is defined as that of the quadrics satisfying the eightfold condition $\mu^r\nu^s\rho^{8-r-s}$. The corresponding curve C is then the intersection of r tangent primes $U(\bar{\mu})$, s quadrics $U(\bar{\nu})$, and $8-r-s$ cubic primals $U(\bar{\rho})$, residual to their base varieties on Δ; it will therefore have only simple intersections with Δ, V_6^{10}, and V_3^8, so that the formulae of § 5 will apply.

In order to evaluate the symbol $\mu^r\nu^s\rho^{9-r-s}$ we have in effect to determine how many quadrics of Σ satisfy the condition ρ. Now

$$\begin{aligned}\mu^r\nu^s\rho^{9-r-s} &= \rho\,.\,\mu^r\nu^s\rho^{8-r-s}\\ &= \tfrac{1}{4}(3\chi+2\psi+\phi)\mu^r\nu^s\rho^{8-r-s}, \quad \text{by (5)},\\ &= \tfrac{1}{4}\{3\chi\mu^r\nu^s\rho^{8-r-s}+2\psi\mu^r\nu^s\rho^{8-r-s}+\phi\mu^r\nu^s\rho^{8-r-s}\}.\end{aligned}$$

Since the values of all three symbols within the brackets are known from the preceding tables, the value of $\mu^r\nu^s\rho^{9-r-s}$ follows.

Generally there is a choice of methods in any given case, according to which one of the relations (1), (4), and (5) is selected for use. Thus, if all three symbols μ, ν, ρ are present in the given condition $\mu^r\nu^s\rho^{9-r-s}$, we may equally well write

$$\mu^r\nu^s\rho^{9-r-s} = \mu\,.\,\mu^{r-1}\nu^s\rho^{9-r-s} = \nu\,.\,\mu^r\nu^{s-1}\rho^{9-r-s}.$$

When two or more methods are available for the same problem, they can be used to check one another. Finally, we note that, by the principle of duality, $\mu^r\nu^s\rho^{9-r-s} = \rho^r\nu^s\mu^{9-r-s}$. Hence, a certain number of the results can be inferred from the rest.

Examples

1. To evaluate $\mu^4\nu^3\rho^2$.

We have
$$\begin{aligned}\mu^4\nu^3\rho^2 &= \tfrac{1}{4}(3\chi+2\psi+\phi)\mu^4\nu^3\rho\\ &= \tfrac{1}{4}(3\chi\mu^4\nu^3\rho+2\psi\mu^4\nu^3\rho+\phi\mu^4\nu^3\rho)\\ &= \tfrac{1}{4}(3.96+0+0), \quad \text{by the tables,}\\ &= 72.\end{aligned}$$

2. To evaluate $\mu^2\nu^4\rho^3$.

We have
$$\begin{aligned}\mu^2\nu^4\rho^3 &= \tfrac{1}{4}(3\chi\mu^2\nu^4\rho^2+2\psi\mu^2\nu^4\rho^2+\phi\mu^2\nu^4\rho^2)\\ &= \tfrac{1}{4}(3.96+2.32+96), \quad \text{by the tables,}\\ &= 112.\end{aligned}$$

3. To evaluate ν^9.

Using (4),
$$\nu^9 = \tfrac{1}{4}(2\chi+4\psi+2\phi)\nu^8$$
$$= \tfrac{1}{4}(2\chi\nu^8+4\psi\nu^8+2\phi\nu^8)$$
$$= \tfrac{1}{4}(0+4.92+0) = 92.$$

TABLE VI

$$\mu^r\nu^s\rho^{9-r-s} = \rho^r\nu^s\mu^{9-r-s}$$

$\mu^9 = 1$	$\mu^5\nu^4 = 16$	$\mu^3\nu^6 = 56$
$\mu^8\nu = 2$	$\mu^5\nu^3\rho = 24$	$\mu^3\nu^5\rho = 80$
$\mu^8\rho = 3$	$\mu^5\nu^2\rho^2 = 36$	$\mu^3\nu^4\rho^2 = 112$
$\mu^7\nu^2 = 4$	$\mu^5\nu\rho^3 = 34$	$\mu^3\nu^3\rho^3 = 104$
$\mu^7\nu\rho = 6$	$\mu^5\rho^4 = 21$	$\mu^2\nu^7 = 80$
$\mu^7\rho^2 = 9$	$\mu^4\nu^5 = 32$	$\mu^2\nu^6\rho = 104$
$\mu^6\nu^3 = 8$	$\mu^4\nu^4\rho = 48$	$\mu^2\nu^5\rho^2 = 128$
$\mu^6\nu^2\rho = 12$	$\mu^4\nu^3\rho^2 = 72$	$\mu\nu^8 = 92$
$\mu^6\nu\rho^2 = 18$	$\mu^4\nu^2\rho^3 = 68$	$\mu\nu^7\rho = 104$
$\mu^6\rho^3 = 17$	$\mu^4\nu\rho^4 = 42$	$\nu^9 = 92$

5.2. *Direct calculation.* Many of the results shown in Table VI can be deduced immediately from the basic representation, as in the following examples.

EXAMPLES

1. To evaluate $\mu^3\nu^6$. We have to find the intersections of six quadrics $U(\bar{\nu})$, each passing simply through the triple V_3^8 of Δ, with three primes $U(\bar{\mu})$. The latter meet in a [6] which intersects V_3^8 in eight points. Thus $\mu^3\nu^6 = 2^6-8 = 56$.

2. To evaluate $\mu^5\nu\rho^3$. We observe that each cubic primal $U(\bar{\rho})$ passes simply through the double V_6^{10} of Δ, and that the quadric $U(\bar{\nu})$ touches this along a V_5^{10} (Theorem II, § 3.5). The five primes $U(\bar{\mu})$ give a [4]-section of the whole figure, whence
$$\mu^5\nu\rho^3 = 54-2.10 = 34.$$

3. To evaluate $\mu^3\nu^4\rho^2$. Here the [6] common to the three primes $U(\bar{\mu})$ meets V_3^8 in points which are simple on each primal $U(\bar{\nu})$ and double on each primal $U(\bar{\rho})$. Thus
$$\mu^3\nu^4\rho^2 = 9.16-4.8 = 112.$$

4. Establish the formulae $\mu^8\nu = 2$, $\mu^7\nu^2 = 4$,..., continuing the sequence as far as possible.

5. Establish the formulae $\mu^8\rho = 3$, $\mu^7\rho^2 = 9$, $\mu^7\nu\rho = 6$, $\mu^6\nu^2\rho = 12$, $\mu^5\nu^2\rho^2 = 36$.

6. Show that $\mu^4\nu^2\rho^3 = 68$, and that $\mu^3\nu^5\rho = 80$.

7. By means of the result $\mu^5\rho^4 = 21$, calculate the rank of the curve sections of V_6^{10}.

(The [4] common to the five primes $U(\bar{\mu})$ gives a section of the figure in which we have four cubic threefolds passing simply through a decimic curve and meeting residually in 21 points. Applying the equivalence formula of Ch. IX, § 1, we find that V_6^{10} is of rank 28.)

5.3. *Other conditions for quadrics.* So far we have considered only primary problems for quadrics; in this section we shall show how the previous results may be used to determine the numbers of quadrics which satisfy conditions of other types. We begin with those of dimension unity, then pass on to products of such conditions, and finally to conditions of higher dimension.

5.31. *Condition of tangency with a given surface.* (1) Consider in the first place the case where the given surface is a quadric; we wish to find how many quadrics of a simply infinite system Σ, with assigned characters μ, ν, ρ, will touch this surface.

Let F be the given quadric, and F' any quadric touching it; then, in the pencil formed by F and F', two of the cones are coincident; that is, in the basic representation, the line corresponding to this pencil is tangent to Δ. Hence, the system of quadrics touching F correspond to the points of the tangent cone K to Δ from the image point of F. And since the cone intersects Δ in points lying on a first polar, it follows that it passes doubly through V_6^{10} and sextuply through V_3^8.

Suppose, as in § 5, that the quadrics of Σ correspond to the points of a curve C, with the characters χ, ψ, ϕ; then, if N denotes the number of intersections of C with K, external to V_6^{10} and V_3^8, we have

$$12\mu = N+2\psi+6\phi.$$

Now, by § 5, equations (2) and (3),

$$\phi = 2\mu-\nu, \qquad \psi = 2\nu-\mu-\rho.$$

Hence the number N of quadrics of Σ which touch the quadric F is given by the formula

$$N = 2(\mu+\nu+\rho).$$

(2) We may obtain the same result by using specialization methods. Thus, suppose that F tends to the limiting form ψ, consisting of a plane-pair (α, α') meeting in an axis g, on which lie two principal points A, A' (§ 4.03). Then the quadrics of Σ which touch ψ belong to one of the following sets:

(i) those which touch either α or α';

(ii) those which pass through A or A';

(iii) those which touch g, each of these being counted twice.

Hence, the total number of quadrics touching ψ is

$$N = 2(\mu+\nu+\rho), \quad \text{as before.}$$

(3) We may use a similar method to find the number of quadrics of Σ which touch a general surface F^n of order n. For this purpose we let F^n tend to a limiting form consisting of n planes of a pencil, with axis g. It can then be shown that the tangents to this degenerate form consist of the secants to g, counted $n(n-1)$ times, and that the tangent planes are the planes which pass through one or other of $n(n-1)^2$ principal points A, A', A'',... situated on g. It follows that the number of quadrics touching F^n is equal to

$$n\mu+n(n-1)\nu+n(n-1)^2\rho.$$

(4) We can now apply the calculus of conditions in order to find the numbers of quadrics which satisfy a set of primary conditions, together with conditions of tangency to one or more surfaces. For example, the number of quadrics passing through seven given points and touching two given quadrics is $\mu^7(2\mu+2\nu+2\rho)^2$; expanding the symbol in brackets and using the results shown in Table VI, we may deduce the number in question. Or again, the number of quadrics touching seven given planes and two general surfaces F^m and F^n is given by the symbol

$$\rho^7\{m\mu+m(m-1)\nu+m(m-1)^2\rho\}\{n\mu+n(n-1)\nu+n(n-1)^2\rho\},$$

which may be evaluated by means of Table VI.

5.32. *Twofold conditions.* Consider now a system Σ of ∞^2 quadrics; of these, a finite number can be found to satisfy any two given conditions, and a finite number of degenerate forms contained in Σ will satisfy any one given condition. We shall now obtain formulae for two important conditions which are dual to one another:

The condition γ, that a quadric should touch a given plane on a given line in the plane.

The condition γ', that a quadric should touch a given line at a given point of the line.

To this end we observe that a finite number of quadrics of Σ will touch two given planes; this number is equal to the number of free intersections of the surface image M_2 of Σ with the corresponding cubic primals $U(\bar{\rho}_1)$ and $U(\bar{\rho}_2)$. Now let one of these primals tend to the other; the corresponding planes will then tend to coincidence, with a limiting line of intersection. Moreover, M_2 intersects Δ in a curve, not lying on V_6^{10}, and this will meet $U(\bar{\rho}_1)$ in $\chi\rho$ points. Hence

$$2\rho^2 = 2\gamma + \chi\rho.$$

Since, by § 5, $\chi = 2\rho - \nu$, we obtain

$$\gamma = \tfrac{1}{2}\nu\rho.$$

Dually we have

$$\gamma' = \tfrac{1}{2}\nu\mu.$$

These results may be combined with the preceding so as to give the numbers of quadrics satisfying a set of conditions of dimension nine, of any type so far considered.

5.33. *Threefold conditions.* We shall consider one example of these, namely, the condition ξ that a quadric should contain a given line. As shown in § 3.13, the quadrics which contain a given line form a linear system of freedom six which corresponds, in the basic representation, to a tangent [6] to V_6^{10}. Such a [6] will intersect V_6^{10} in a manifold V_4^4 and V_3^8 in a conic. The latter will of course lie on each quadric $U(l)$ but not in a general prime $U(P)$. Hence we obtain the results

$$\xi\mu^6 = 1, \quad \xi\mu^5\nu = 2, \quad \xi\mu^4\nu^2 = 4, \quad \xi\mu^3\nu^3 = 8, \quad \xi\mu^2\nu^4 = 16.$$

Since the condition ξ is self-dual, we can infer from the above formulae that $\xi\rho^6 = 1$, etc.

By the use of correspondence methods we may obtain an expression for ξ in terms of the symbols μ, ν, and ρ. The actual formula is

$$\xi = \tfrac{1}{4}\{2\nu^3 - 3(\mu+\rho)\nu^2 + 3(\mu^2+\rho^2)\nu + 2\mu\rho\nu - 2(\mu^3+\rho^3)\}.$$

For the proof of this result we refer the reader to Schubert's book.

EXAMPLES

1. With the notation of Table IV, show that the number of conics of a plane which touch five given conics is $\mu^3(2\nu+2\rho)^5$, and that this number is 3,264.

2. Show that the number of quadrics which touch nine given quadrics is $2^9(\mu+\nu+\rho)^9$, and that, by Table VI, this number is 666,841,088.

3. Prove that the number of quadrics of an ∞^2 system which contain a line of a given pencil is $\mu\rho$.

4. Show that any tangent [6] to V_6^{10} touches Δ along a V_5^2.

5. Prove that the condition ω that a quadric should touch a given plane at a given point is connected with ξ by the relation

$$2\xi + 4\omega = \mu\nu\rho.$$

Deduce a formula for ω from the previous section.

6. With the notation of Table IV, prove that the condition that a conic in space should pass through a given point is $\mu\nu-2\mu^2$.

7. By considering the intersection of the prime $U(P)$ with the quadric $U(l)$ in the case where P lies on l, show that $2\gamma' = \mu\nu$.

§ 6. The Specialization Principle

6. Specialization methods. In various sections of this book we have solved divorse problems in enumerative geometry by an appeal to a specialization argument. Thus:

(1) In Ch. I, § 3.2, we proved Bézout's Theorem for the intersection of primals in general position by breaking up each primal into a set of primes; and in the Examples on Ch. IX we used a similar method to find the number of intersections of r primals in $[r]$ residual to a common multiple curve.

(2) In Ch. IX, § 2, we found by specialization the order of the curve of contact of tangents to a surface in $[r]$ which meet a given $[r-2]$.

(3) In the Examples on Ch. IX we obtained the number of chords of a space curve which meet two given lines, by supposing the lines to be coplanar.

(4) In § 5.3 of the present chapter we obtained the number of quadrics of a simply-infinite system which touch a given quadric, by making the latter degenerate into a plane-pair.

Since some at least of the results so established are of essential importance to the theory, it is desirable to examine critically the assumptions on which our methods are based. As we shall see, this leads inevitably to a review of other questions concerning enumerative methods.

6.1. *The underlying assumption.* If the method used in each of the above problems is scrutinized, it would at first sight appear that, in every case, what we have asserted amounts to this:

Given a system of algebraic equations of which the number of solutions is in general finite, the number of those solutions will not change for any variation of the parameters occurring in the equations.

But this statement is certainly not true without qualification, as examples readily show.

Ex. 1. Let C be an irreducible curve of order n, situated in $[r]$. Any prime in general position meets C in n points, corresponding to the roots of an equation of the nth degree. But suppose that the prime is not in general position; for instance, let it have contact of a specified order with C or, again, let it pass through a multiple point of C. In either case the 'principle' asserted can remain valid only if we adopt some convention as to how the solutions of the resulting equation are to be counted.

Ex. 2. An instructive example occurs in § 3 of the present chapter. If Σ is a general linear system of ∞^6 quadric surfaces, we can show that it contains eight double planes. If, however, the parameters defining Σ are varied so that the system has three base points, it is clear that the number of double planes is reduced to one. A suitable convention will enable us in this case to count the plane as an eightfold solution.

6.2. *Proper and improper solutions.* A second factor in the problem of counting the solutions, assuming they are finite in number, appears from our previous work.

Ex. 1. Suppose we wish to find the number of quadrics satisfying the condition $\mu^3\nu^6$. Evidently the plane of the three given points, counted twice, will satisfy the conditions of the problem. However, we can state the problem in such a form as to exclude this from the result: thus we may say that we require the number of *irreducible* quadrics which satisfy the given conditions. In fact the double plane may be termed an improper solution to the problem, for any condition of type ν is automatically satisfied by such a degenerate quadric.

Ex. 2. Consider the problem of finding the triangles which are inscribed in one conic and circumscribed to another. Let S, S' be conics in general position: from any point P of S, draw the tangents to S', meeting S again in Q, Q_1. From Q, say, draw the remaining tangent to S', meeting S again in R; and from R draw the remaining tangent to S', meeting S again in P_1. We thus set up a $(2, 2)$ correspondence between P, P_1 on S; and the solutions to the problem in question must arise from the 4 coincidences of P with P_1. But these occur at the intersections of S with the residual tangents to S' from the points of contact with S of the common tangents to S and S', so that the required triangles are all degenerate.

It should be noted that, unless we make a further stipulation, these triangles, unlike the double plane of Ex. 1, are to be regarded as proper solutions. If we introduce into the problem the qualification that the required triangles must be non-degenerate, we reduce the number of solutions to zero. The question as to whether we may legitimately make this stipulation is discussed in the next paragraph.

6.3. *Permissible and impermissible conditions.* In order to settle the question just raised, it may be helpful at this point to emphasize an assumption that underlies all our work but which has until now remained merely implicit. It is of course supposed throughout that we are dealing with a system of *algebraic* equations: that is to say, as stated in § 1, that any condition imposed on our system Σ of forms must be algebraic. This essential restriction not only excludes problems which obviously imply conditions of a non-algebraic type, but it necessitates a closer examination of certain conditions which at first sight appear to be permissible.

Ex. 1. Consider, for example, the problem of determining how many lines meet a space curve in four points that are *distinct* from one another. In terms of the parameters the restriction to distinct points of intersection means that certain solutions must not be coincident; and this condition cannot be expressed by means of an algebraic equation.

Ex. 2. On the other hand, it may be permissible to stipulate that certain solutions are to be excluded as improper, provided that, in the 'basic representation' of the problem, analogous to that used in the present chapter, the points corresponding to these solutions lie on an assigned algebraic manifold. This is precisely what we do in finding, for example, the number of irreducible cones which satisfy the condition $\mu^3\nu^5$; here we reject the solution corresponding to any common point in which the primes $U(\bar{\mu}_1)$, $U(\bar{\mu}_2)$, $U(\bar{\mu}_3)$ meet the triple V_3^8 of Δ. This is equivalent to assigning V_3^8 as base manifold for all the primals concerned in the basic representation of the problem. And so in general.†

Ex. 3. Suppose that we wish to find the number of bitangents to a nodal plane quartic; in the basic representation of the lines of the plane we assign the ∞^1 lines passing through the node as a base curve, and so obtain only the proper solutions to the problem.

6.4. *Tentative formulation of principle.* Suppose, then, that we have decided on a convention for counting the solutions to a given problem, and that we have also decided, in the manner described, which solutions are to be classed as improper. A little reflection will show that even now the statement made in § 6.1 is open to objection.

Consider, for example, a reducible curve C in $[r]$; by adopting a suitable convention we can say that any prime which contains no part of C meets the curve in the same number of points. But it is obvious that no convention will avail to preserve the number of intersections if the prime happens to contain one of the components of C.

Again, in § 1.1, we determined the number of lines which meet

† We may add that, for a manifold representing a *specialized* condition-figure, certain precautions may be necessary to ensure the proper counting of solutions. On this point see Severi, *Atti Ist. Veneto*, 75 (1916), 1121.

four given lines of [3] by supposing that two of the lines were coplanar. If, however, we further specialized the figure by making three of the lines coplanar, we should obtain an infinity of solutions.

These examples suffice to indicate one correction that has obviously to be made to the proposed principle. We shall therefore amend it to read as follows:

The number of solutions of a system of algebraic equations, if it is finite for one set of the parameters occurring in the equations, will remain constant for variations of those parameters, provided

(i) *the solutions are properly counted, and*

(ii) *the variations are never such as to render the number of solutions infinite.*

Numerous illustrations of this principle will occur to the reader. Thus, we know that it is impossible to draw a chord of a Veronese surface from a general point of the ambient space; the points from which chords can be drawn lie on a cubic primal of [5], and through each such point there passes an infinity of chords.

Before we subject the principle to further examination it is necessary to enlarge the scope of our inquiry by some additional illustrations.

6.5. ***The collineations of a line.*** Fresh light is thrown on the nature of enumerative problems in general by the study of the group $x' = (ax+b)/(cx+d)$ of collineations of the points of a line. We obtain a basic representation of the group by taking (a, b, c, d) to be homogeneous coordinates in S_3. Each collineation then has for image a single point of S_3; but not every point of S_3 represents a collineation—in fact, the totality of points lying on the quadric $ad-bc = 0$ have no corresponding collineations. We shall call this the *fundamental quadric* of the representation; the solutions of any determinate problem concerning collineations will correspond in S_3 to a number of points not lying on the fundamental quadric (cf. § 6.3).

The collineations which transform a given point x into another given point x' correspond to the points, external to the fundamental quadric, of the plane whose equation is

$$x' = (ax+b)/(cx+d).$$

There is a unique collineation which changes three distinct points x_1, x_2, x_3 into three distinct points x_1', x_2', x_3', namely,

$$\frac{(x'-x_1')(x_2'-x_3')}{(x'-x_3')(x_2'-x_1')} = \frac{(x-x_1)(x_2-x_3)}{(x-x_3)(x_2-x_1)}. \tag{1}$$

In particular, it follows that there is a unique collineation which leaves a point x_1 fixed and interchanges a pair of points x_2, x_3.

To the group of involutory transformations there correspond the points of the plane $a+d = 0$, external to the fundamental quadric. It follows from (1) that there is a unique involutory transformation which interchanges a pair of points x_1, x_2, and also a pair x_3, x_4.

Ex. 1. We consider first the following problem: how many collineations of a line leave invariant a given triad of points?

It is clear that there are six solutions to the problem, namely, (i) the identical transformation, (ii) the involutory transformations which leave one point of the triad fixed and interchange the other two, and (iii) the transformations which permute all three points.

The geometrical representation of the problem is interesting. Let us suppose, for simplicity, that the triad is given by the equation $x^3+1 = 0$. If $x' = (ax+b)/(cx+d)$ is a collineation of the desired type, the equation $(ax+b)^3+(cx+d)^3 = 0$ must be equivalent to $x^3+1 = 0$. Thus $a^3+c^3 = b^3+d^3$, $a^2b+c^2d = 0$, $ab^2+cd^2 = 0$. These equations represent three surfaces, whose intersections give the solutions to the problem. It will be noticed, however, that the surfaces intersect in a *curve*, consisting of the three lines common to $a^3+c^3 = b^3+d^3 = 0$ and the fundamental quadric $ad-bc = 0$. Residually to these the surfaces meet in six points, giving the solutions

$$x' = x, \quad x' = \omega x, \quad x' = \omega^2 x, \quad x' = 1/x, \quad x' = \omega/x, \quad x' = \omega^2/x,$$

where $\omega = e^{2i\pi/3}$.

Ex. 2. Consider next the question of determining the number of collineations which leave invariant a given tetrad of points.

We know that any such collineation must change the tetrad into another having the same cross-ratio. Hence, for a general tetrad, we deduce that there are four solutions: (i) the identical transformation, and (ii) the involutory transformations which interchange one pair of the tetrad and then interchange the remaining pair.

Thus, for example, in the case of the tetrad given by the equation $x^4+x^2+1 = 0$, the required transformations are

$$x' = x, \quad x' = 1/x, \quad x' = -x, \quad x' = -1/x.$$

Here again the geometrical aspect is illuminating. If the collineation $x' = (ax+b)/(cx+d)$ leaves the given tetrad invariant, the equation $(ax+b)^4+(ax+b)^2(cx+d)^2+(cx+d)^4 = 0$ must be equivalent to $x^4+x^2+1 = 0$. This leads to *four* relations between a, b, c, d; hence, in the basic representation, we have to find the points common to four surfaces. Thus the problem does not come within the category of typical problems as defined in § 1; for we have no reason to expect that four surfaces will have any common points whatever. Nevertheless, in the present instance it is easily verified that they meet in a curve lying on the fundamental quadric, and also in four points external to it, representing the proper solutions to the problem.

Ex. 3. A still more interesting situation arises when the given tetrad is harmonic or equianharmonic. We know that in the first case there are 8 permutations of the set, including identity, which give the same cross-ratio; these fall into two classes, namely, the tetrads of the type described in Ex. 2, and those such as $(x_1 x_4 x_3 x_2)$, obtained by interchanging a single pair of non-adjacent points and leaving the other pair unaltered.

In the equianharmonic case (when the cross-ratios are either $-\omega$ or $-\omega^2$) there are 12 permutations which give the same cross-ratio; for in addition to those already stated, we now have such permutations as $(x_1 x_3 x_2 x_4)$, in which a pair of adjacent points is interchanged.

Thus, to conclude, we may summarize these results as follows:

The number of collineations which leave invariant the points of a tetrad defined by the equation

$$a_0 x^4+a_1 x^3+a_2 x^2+a_3 x+a_4 = 0$$

is 4 *in general but, for continuous variation of the parameters* (a_i), *may become* 8 *or* 12.

As a matter of historical interest it may be noted that the above example of the failure of the specialization principle, as usually applied, called attention to the need for a closer examination of its range of validity.

6.6. *Conditions of validity of the specialization principle.* The principle, as formulated in § 6.4, was used explicitly by Schubert and others to obtain the bulk of the known results in enumerative geometry; clearly it might have been invoked, with some degree of plausibility at least, to justify our methods in the problems we have considered in earlier sections of this book. Nevertheless, the conclusion of § 6.5 indicates that it is not valid without some further restriction.

The question is thus raised: can we establish a criterion which will enable us to determine *a priori* when the specialization method may legitimately be used? This question was answered by Severi,† and we shall indicate below the substance of his exposition. First, however, we consider a simple kind of enumerative problem in which the conditions imposed are resoluble into a product of two conditions of complementary dimensions. As will appear, the proof of our statements depends upon certain general propositions concerning algebraic systems of manifolds traced on a given variety; for these the reader is referred elsewhere.‡

6.61. *The representative manifold of a system. Condition-manifolds.* Consider an algebraic system Σ of geometrical forms (F) in a space S_r; in infinitely many ways we can represent the forms (F) by the points of a manifold V, which we shall call the *representative manifold* of the system. If Σ is of dimension d, so is V; and if, as we shall always suppose, Σ is irreducible, so likewise is V. In order to apply the theory we must assume that V is free from multiple points. Further, the representation may require that certain varieties on V be assigned as base manifolds, whose points do not correspond to members of Σ.

Ex. 1. In the line geometry of ordinary space, V may be taken to be a general quadric primal of [5].

Ex. 2. In the present chapter, for the representation of the quadrics of S_3, V has been taken to be a space S_9; for the representation of quadrics which are not cones, V is a space S_9 with a primal Δ assigned as base manifold; and for the representation of all irreducible quadrics, V is a space S_9 with a base manifold V_6^{10}.

Ex. 3. For the problems concerning the collineations of a line, V is a space S_3 with an assigned base quadric.

† *Rend. Palermo*, **33** (1912), 313; a more recent elaboration of the theory is given by Severi in *Annali di Mat.* (4) **19** (1940), 153.

‡ Severi, *Abh. Math. Sem. Hamburg*, **9** (1933), 335 (§§ 11–14) and *Annali di Mat.* (4) **26** (1947), 221.

With any condition-figure Γ of prescribed character, there is associated a condition c (which we shall assume to be of permissible type). Suppose, moreover, that the dimension k of c is less than d. There corresponds to c on V an algebraic manifold $U_{d-k} = U$, the *condition-manifold* relative to c; and we suppose that this is in general irreducible. As Γ varies continuously, in the relative positions of its components and in its position in S_r, U will vary in an algebraic system on V. Among the possible positions for Γ, some will be of quite general character and others will be particularized in various ways; in the application of specialization methods we are concerned only with the latter kind. There are now four possibilities to consider:

(1) It may happen that, for all specializations of Γ, the condition-manifold U is irreducible and of dimension $d-k$.

(2) Instead, for certain specializations of Γ, U may break up into a number of irreducible components, some of them possibly multiple, but all of dimension $d-k$: that is to say, the condition-manifold is *pure* (cf. Ch. I, § 4).

(3) Or, for various specializations, U may break up into a number of components of different dimensions: we then say that it is *impure*.

(4) Finally, for certain specializations, it may happen that U, though remaining irreducible, is of dimension exceeding $d-k$.

·**6.62.** *The criterion of validity.* Suppose that c and c' are conditions, of complementary dimensions k and $d-k$, relative to condition-figures Γ and Γ' which are generally situated with respect to each other. Suppose also that Γ and Γ' are of general character, so that the corresponding condition-manifolds U and U' are irreducible varieties of dimensions $d-k$ and k respectively. These varieties will meet, residually to the base manifolds of V, in a certain number of points which, counted with their proper multiplicities, represent the number of forms of Σ which satisfy the d-fold condition cc'. We now wish to determine whether this number will remain unchanged as U assumes various positions corresponding to specializations of Γ.

According to the general theory of manifolds, to which we have alluded above, this will be the case so long as U remains irreducible, or reducible and pure, of dimension $d-k$. It will certainly not be true in case (4), and it will not necessarily be true in case

(3), even if none of the components of U has dimension greater than $d-k$. We may thus state the following condition as sufficient† to ensure the validity of the specialization method:

For any typical problem, in which c and c' are conditions of complementary dimensions, and c' is relative to a general condition-figure, the number of forms satisfying the condition $c.c'$ will remain unaltered for any specialization of the condition-figure relative to c for which the corresponding condition-manifold is (i) *irreducible or* (ii) *reducible and pure, of the same dimension as for the generic c.*

From what has been said, it is clear that, with similar restrictions for the condition-figure relative to c', we can apply simultaneous specializations to c and c' in the given problem. Again, though we do not need this fact in our work, it can be shown that the number cc' will be unchanged for specializations of Γ, even if Γ and Γ' are not generally situated with respect to each other, provided that U is irreducible, or reducible and pure, and that it does not meet U' in an infinity of points.

6.63. *Applications and examples.* We now examine, in the light of the above criterion, various specialization processes which we have hitherto used in our work.

(1) *The intersection of primals. Bézout's Theorem.* Suppose that the representative manifold V is a space $[r]$; then the primals of any order in that space form a linear system of pure manifolds on V. If we are given r primals on V, in general position, of orders $n_1, n_2,..., n_r$ respectively, and we seek to determine the number of their intersections by breaking up each primal into a set of primes, we are in effect appealing to the specialization principle in its primitive form, i.e. as part of the theory of algebraic manifolds mentioned in § 6.6. The same remark applies to the method of finding the number of intersections of r primals residual to a common multiple curve, which was suggested in the Examples on Ch. IX.

(2) *Line geometry in* [3]. Taking the representative manifold V to be a general quadric primal of [5], we see that

(i) The manifold corresponding to the condition of intersecting a given line is the section of V by a tangent prime; it is thus always irreducible.

† It may be proved that the condition is also necessary; see the works of Severi quoted above.

(ii) The condition of meeting two given lines corresponds to a [3]-section of V; this is a quadric surface, which will break up into a pair of planes if the given lines intersect. Thus the condition-manifold in this case is either irreducible or reducible and pure. Hence we may apply such specialization methods as in the Examples on Ch. IX, where we found the number of chords of a space curve which meet two given lines.

(iii) The condition of meeting three given lines is in general represented by a conic or line-pair on V. But if the three lines are coplanar, it corresponds to a plane. Hence this type of specialization must be excluded.

(3) *The projective characters of surfaces.* In Ch. IX we obtained two fundamental results in the enumerative geometry of surfaces:

(i) The order μ_1 of the curve of contact of tangent planes to a surface in $[r]$ which meet a given $[r-3]$.

(ii) The number $\mu_2+\nu_2$ of tangent planes to the surface which meet two given spaces $[r-3]$.

These were both deduced from specialization of the respective condition-figures. In order now to justify this procedure we need only project the given surface on to ordinary space, from a general $[r-4]$, and apply the results concerning line geometry of [3].

6.7. *Severi's analysis.* We now leave the reader to discuss the remaining methods used, and we pass to Severi's analysis of the problem. As before, let F be a generic form of the system Σ, and let F' be the specific figure with which it is to have the relation c. Besides the manifold V of § 6.61 we now consider the manifold V' whose points represent all possible figures (F') in S_r. Then the condition c gives rise to an algebraic correspondence ω between the points of V and V', in which homologous points are those for which the relative entities F, F' satisfy c; ω will be mapped by a certain manifold M on the variety which represents the pairs of points of V and V'. It may happen that V' is reducible, and that c itself is the sum of a number of conditions. We suppose that any one of these (c_1, say) is relative to an irreducible part V'_1 of V', and gives rise to a correspondence ω_1 between V and V'_1. We then assume that the manifold M_1 which maps ω_1 is irreducible or reducible and pure; in this case we say that ω_1 itself is pure.

To a generic F' there corresponds (by ω_1) a variety U on V; this will be pure, of dimension e, say, the dimension of c_1 thus being $d-e$. Suppose now that c_1 varies continuously and that F' tends to a specialization $\bar{F}'$ which is not fundamental (i.e. one with more than ∞^s homologues F in ω_1). Then Severi has shown that the specialization $\bar{F}'$ is valid. Similar considerations apply to the other components $M_2, M_3,\ldots$ of M; and Severi's conclusion is that *the specialization process is legitimate if and only if the correspondence* ω *is pure.*

Ex. Consider the condition imposed on an involution F which is to leave invariant a given tetrad F' on a line (§ 6.5); this sets up a correspondence between the representative (irreducible) varieties V_2 and V'_4 which is reducible and impure. For, if we fix a particular involution F, we obtain one manifold W_1 consisting of the ∞^1 tetrads F' formed by two double points and a pair of mates in F, and another (distinct) manifold W_2 of the ∞^2 tetrads formed by two pairs of mates in F. As F moves in V_2, W_2 invades the whole of V'_4 but W_1 covers only the part V'_3 which maps the harmonic sets in V'_4. Thus the correspondence between V_2 and V'_4 is the sum of two irreducible correspondences of different dimensions; this explains why, in § 6.5, the number of solutions in the harmonic case differs from the number obtained in general.

From the indications we have given it will be clear that any analysis of the methods of enumerative geometry must involve deep questions of algebraic geometry. For a systematic discussion of such questions we refer the reader generally to the work of van der Waerden, and in particular to his long series of papers 'Zur algebraischen Geometrie', I–XV, *Math. Annalen* (1933–8), which are largely concerned with fundamental enumerative problems.

BOOKS RECOMMENDED FOR FURTHER READING AND REFERENCE

Baker, *Principles of geometry*, vi, ch. ii.
Room, *The geometry of determinantal manifolds*, ch. ix.
Schubert, *Kalkül der abzählenden Geometrie.*
Zeuthen, *Lehrbuch der abzählenden Methoden.*

CHAPTER XII

GEOMETRY ON A CURVE

§ 1. Sets of points on a curve

1. In earlier chapters of this book we have been concerned for the most part with what we may call projective algebraic geometry, or the geometry of algebraic manifolds relative to the projective spaces in which they are immersed. In the sequel we shall be dealing more particularly with invariantive algebraic geometry, i.e. with those properties of algebraic manifolds which are invariant under birational transformation. From this point of view we attempt to consider as a single unit all manifolds which are birationally transformable into one another, and to regard them as different *projective* representations or models of the same abstract entity.

Invariantive geometry has two main developments:

(*a*) the formal development of relations of equivalence between sub-manifolds of a given manifold V, and the study of fundamental invariant and covariant systems of equivalent sub-manifolds;

(*b*) existence theorems and numerical relations between invariant numbers which largely characterize geometry on V.

In this and the following chapter we give only an introduction to some of the principal properties of curves and surfaces, referring the reader to the classical treatises and memoirs for further details.

For brevity we shall assume, without further qualification, that all the manifolds under discussion are algebraic.

1.1. *Representation of point-sets.* Invariantive geometry on an algebraic curve is essentially the study and classification of zero-dimensional manifolds on an algebraic one-dimensional manifold, i.e. the study of series of sets, each consisting of a finite number of points, on an algebraic curve C. The curve C will be supposed, here and throughout the whole of this chapter, to be irreducible.

The primary characteristic of a set of points (distinct or not) on C is the number of points in the set; this we call the *grade* or *order* of the set. This gives a preliminary classification of all sets on C, according to their order, in totalities H_1, H_2, H_3,... of all single points, pairs of points, triads of points,... of C. The first of these totalities has C itself as algebraic image. We now indicate a general method of constructing an image-manifold Ω_n whose points shall represent univocally all the sets of H_n for any n.

We suppose C to lie in a space S_k, so that H_n is subordinate, in the first instance, to the totality Π_n of all sets of n points of S_k. Now Π_n is itself subordinate to the totality of envelopes W_n of class n of S_k; for any W_n is defined by the vanishing of a polynomial of degree n in the coördinates of a prime of S_k, and it degenerates into a set of n points when the polynomial breaks up into n linear factors. By regarding the coefficients of the general polynomial defining a W_n as homogeneous coordinates of a point of S_ν, where $\nu = \binom{n+k}{k} - 1$, we obtain an unexceptional representation of all the W_n of S_k on points of S_ν; and in this representation, since the conditions for a polynomial of degree n to break up into n linear factors are algebraic, Π_n will be represented by an algebraic manifold in S_ν. Finally, the conditions that any set of points of S_k should lie on C are also algebraic; and hence H_n will be represented birationally on an image manifold Ω_n contained in that of Π_n. Hence

THEOREM I. *The totality of all sets of any given order n on a curve C is representable birationally on the totality of points of an algebraic image manifold Ω_n.*

Since C is irreducible, Ω_n will also be irreducible and of dimension n.

1.2. *Algebraic series.* The most general kind of series of sets with which we shall be concerned is defined as follows:

DEFINITION. An *algebraic series* of grade n on C is any system of sets of n points of C whose members are in (1, 1) correspondence with the points of an algebraic manifold U. The series has dimension ρ equal to that of U.

An algebraic series of grade n and dimension ρ is called a γ_n^ρ. Thus, the totality of sets of n points of C is the γ_n^n whose image manifold is Ω_n; and every other algebraic series of grade n is represented on Ω_n by an algebraic sub-manifold of Ω_n. The individual points of C constitute a γ_1^1, while the same points, each counted s-fold, form a γ_s^1.

An algebraic series γ_n^ρ has a further important character, namely, its *index* λ; in the case where the generic set of the series consists of n distinct points,† this is the number of distinct sets of the series which contain ρ generic points of C.

A simply-infinite algebraic series of index unity is called a *pencil* of sets on C, or an *involution* on C. Precisely one set of a pencil contains any given point of C, and contains it once only. The simplest example of a pencil is the γ_1^1 of the individual points of C; but the generators of any ruled surface evidently cut a pencil of sets on any irreducible curve of the surface.

A *base point* of a series is a point common to all the sets of the series.

1.3. *Linear series.* Among algebraic series on a curve, we now single out a special type of far-reaching importance, to which we apply the adjective *linear* in recognition of its origin and characteristic structure.

DEFINITION. A *linear series* of sets of points on a curve C, immersed in a space S_k, is a series of sets of points cut on C by the primals of a linear system of S_k, given by an equation of the form

$$\Phi \equiv \lambda_0 \phi_0 + \lambda_1 \phi_1 + \ldots + \lambda_r \phi_r = 0, \tag{1}$$

where λ_i are variable parameters, and ϕ_i are polynomials of the same degree in the coordinates of a point of S_k.

It should be noted that these sets will in general be residual to a fixed set Γ of points on C.

C is often called the *carrying curve* or *carrier* of the series.

A linear series of order n and dimension ρ is denoted by g_n^ρ. For example, if C is of order n, and if it does not lie in any prime of S_k, the primes of S_k cut on it the sets of a g_n^k; while those primes which pass through h generic points of C cut the curve residually in the sets of a g_{n-h}^{k-h}.

The dimension of the linear series cut on C by Φ will not necessarily be the same as the dimension of Φ; but this discrepancy may always be avoided in virtue of

THEOREM II. *The linear system of primals which is to cut any given linear series on C may always be chosen so that each set of the series is cut on C by one and only one primal of the system.*

† To define λ in all cases we must employ the general notion of intersection multiplicity. Thus if U is the image manifold in Ω_n of any γ_n^ρ, and if U' is the image manifold in Ω_n of the series of all sets of order n which contain ρ generic points of C, then the index λ of γ_n^ρ is the number of intersections, suitably counted, of U with U'.

To prove this we suppose that the linear series is given originally, as in the above definition, by the intersection of C with the system Φ of primals defined by (1). We suppose also

(a) that the polynomials ϕ_i are linearly independent,

(b) that h linearly independent primals of Φ contain C entirely.

By (a), the dimension of Φ is r; and by (b), supposing, as we may, that the h linearly independent primals in question are given by $\phi_{r-h+i} = 0$ $(i = 1,..., h)$, we are enabled to write the equation of Φ in the form

$$\Phi \equiv \Phi_1 + \Phi' = 0, \tag{2}$$

where Φ' vanishes on C, and the system Φ_1, given by

$$\Phi_1 \equiv \lambda_0 \phi_0 + \lambda_1 \phi_1 + ... + \lambda_{r-h} \phi_{r-h} = 0, \tag{3}$$

is such that no primal of the system contains C. In the first place, then, any set cut on C by a primal Φ is cut by a primal Φ_1 for which $\lambda_0,..., \lambda_{r-h}$ have the same values as in Φ. And in the second place, no two primals of Φ_1 can meet C in the same set; for if such were the case, the primals of the linear pencil containing the two in question would all meet C in the same set, and one of them would therefore contain C entirely, contrary to hypothesis. Thus Φ_1 is the required sub-system of Φ whose individual primals cut separately all the individual sets of the given series on C.

The following deductions from this theorem are important:

COROLLARY 1. *The dimension of the linear series cut on C by any linear system Φ of freedom r is $r-h$, where h is the number of linearly independent primals of Φ which contain C.*

COROLLARY 2. *If a linear series on a curve has dimension r, then exactly one set of the series contains r generic points of C.*

In other words, the index of any linear series is unity. This new and simple interpretation of the dimension of a linear series leads us to replace the word *dimension*, in the case of a linear series, by *freedom*.

COROLLARY 3. *The sets of any g_n^r on a curve can be represented linearly on the individual points of a space S_r, in the sense that any g_n^i contained in the g_n^r is represented in S_r by an i-dimensional space S_i.*

This follows at once from the theorem. For if each set of the g_n^r is cut on C by a unique primal of a linear system Φ, of freedom r, whose equation is $\Phi \equiv \sum_0^r \lambda_i \phi_i = 0$, then there is a birational correspondence between the sets of the g_n^r and the points $(\lambda_0,..., \lambda_r)$ of an r-dimensional space S_r; and any g_n^i contained in the g_n^r, being cut on C by a linear sub-system of Φ, is represented in S_r by a linear space S_i of S_r.

In virtue of the structural identity thus established between a g_n^r and the space S_r, we sometimes speak of the *intersection*, inside a g_n^r, of subordinate series g_n^ρ and g_n^σ, meaning by this the linear series, usually of dimension $\rho+\sigma-r$, whose image in S_r is the intersection of the $[\rho]$ and the $[\sigma]$ representing the series in question. In a similar way we also speak of the (minimum) *join* of a g_n^ρ and a g_n^σ in the g_n^r, this being usually a $g_n^{\rho+\sigma+1}$ when the dimension indicated does not exceed r.

1.31. *Rational functions.* From a more strictly analytical aspect, the basis of the theory of linear series on a curve is the concept of a *rational function* of a point on the curve.

If C is any curve in S_k, and if L, M are any two polynomials of the same degree μ, say, in the coordinates $x_0, x_1,..., x_k$, which do not vanish identically on C, the ratio $L:M$ has a unique finite value at the generic point of C; this ratio is said to be a rational function $\mathscr{R}$ of a point of C.

Since we are concerned only with the values of $L:M$ at points of C, it is clear that any other polynomial ratio $L':M'$ which is identically equal to $L:M$ on C (in virtue of the equations of C) represents the same rational function $\mathscr{R}$ on C. Thus $\mathscr{R}$ is representable in many different ways as the ratio of two polynomials on C.

The function $\mathscr{R} \equiv L:M$ vanishes at some points of C and becomes infinite at others; these points are called its *zeros* and *poles* respectively. If C is of order n, each of L, M has $n\mu$ zeros (or their equivalent) on C; and if some of these, constituting a set Γ, are common to L and M, so that they cancel each other out, the residual sets, G_1 and G_2 say, will be the true sets of zeros and poles of $\mathscr{R}$ on C.

We observe, however, that G_1, G_2 are the sets cut on C, residual to the set Γ, by the primals $L = 0$ and $M = 0$ respectively. Hence

THEOREM III. *Any two sets (without common points) belonging to one and the same linear series on a curve C are the set of zeros and the set of poles respectively of a rational function on C.*

For we have only to take as L and M the polynomials whose vanishing defines the pair of primals of the linear system which cut the two sets in question on C.

The rational function $\mathscr{R}$ has the same number of zeros as it has of poles, both reckoned with the proper multiplicities. If λ is any constant, the set of zeros of the rational function $\mathscr{R}-\lambda$ is called a *set of constant level* of $\mathscr{R}$; evidently the set is cut on C, residual to Γ, by the primal $L-\lambda M = 0$, belonging to the linear pencil of primals defined by L and M. Hence,

The sets of constant level of any rational function, which is not a constant, on C, form a linear pencil (or g_n^1). Conversely, the sets of any g_n^1 on C are the sets of constant level of a rational function on C.

More generally, since any g_n^r can be cut on C by a linear system of primals Φ of the form

$$\Phi \equiv \lambda_0 \phi_0 + \lambda_1 \phi_1 + \ldots + \lambda_r \phi_r = 0,$$

it can be regarded as the aggregate of all zero sets of rational functions of the form

$$\mathscr{R} \equiv \lambda_0 + \lambda_1 \mathscr{R}_1 + \lambda_2 \mathscr{R}_2 + \ldots + \lambda_r \mathscr{R}_r,$$

where $\mathscr{R}_i = \phi_i/\phi_0$; or it can be regarded as the aggregate of all sets of constant level of rational functions of the form

$$\lambda_1 \mathscr{R}_1 + \lambda_2 \mathscr{R}_2 + \ldots + \lambda_r \mathscr{R}_r.$$

In applying this definition we must agree to regard all rational functions of the system $\mathscr{R}$ as having the same assigned set of poles: so that if, for example, a particular rational function of the system loses some of these poles by cancellation against superimposed zeros, this cancellation must be regarded as not having taken place. The same convention enables us to include the case of a linear series with *fixed points* common to every set of the series.

1.32. *Criterion for linearity of an algebraic series.* We now prove

THEOREM IV. *The necessary and sufficient conditions that a series γ_n^1 should be a g_n^1 are that it should be rational and of index unity.*

The conditions in question are clearly necessary. Suppose then that we are given a γ_n^1 whose sets are in $(1, 1)$ correspondence with the values of a parameter λ, and which is such that one and only one of its sets contains (simply) a generic point of the carrying curve C. To each point of C there corresponds a unique value of λ; and λ is therefore a rational algebraic function of a point of C. The points of C at which λ has any assigned value are those of the set of γ_n^1 associated with this value. Thus the sets of γ_n^1 are the sets of constant level of a rational function on C; and the γ_n^1 is therefore a g_n^1, by the result of the previous section.

The above theorem can be extended to show that, if $r \geqslant 2$, every γ_n^r of index unity is a g_n^r.

1.33. *Invariance of linear series under birational transformation.*

THEOREM V. *In any birational transformation of a curve, linear series transform into linear series.*

This fact, of fundamental importance, is an immediate consequence of the simple remark that any rational function $\mathscr{R}$ on a curve C transforms to a rational function $\mathscr{R}'$ on any birational transform C' of C; for if this be so, any linear series g_n^r of C, since it is the system of sets of constant level of a system of rational functions of the form

$$\lambda_1 \mathscr{R}_1 + \lambda_2 \mathscr{R}_2 + \ldots + \lambda_r \mathscr{R}_r,$$

transforms to the system of sets of constant level of the corresponding system of rational functions

$$\lambda_1 \mathscr{R}'_1 + \lambda_2 \mathscr{R}'_2 + \ldots + \lambda_r \mathscr{R}'_r$$

on C', and its transform is therefore a g_n^r on C'. Hence, to complete the proof of the theorem, we verify the above remark, as follows.

The equations of any birational transformation of C into a curve C' are of the form

$$\rho y_i = \phi_i(x_0, \ldots, x_k) \quad (i = 1, \ldots, j),$$

where $(x_0, x_1, \ldots, x_k)$, $(y_0, y_1, \ldots, y_j)$ are the coordinates of general points of the ambient spaces of C and C', and the ϕ_i are polynomials of the same degree. These equations (in so far as they concern only the curves C, C') must be reversible in the form

$$\rho x_i = \psi_i(y_0, \ldots, y_j) \quad (i = 1, \ldots, k),$$

where the ψ_i are polynomials of like degree. By means of these last equations, however, it is evident that any rational function $\mathscr{R}$ on C, given as the ratio of two polynomials of like degree in x_i, transforms to a rational function on C' given as the ratio of the transformed polynomials in the y_i. Thus the theorem is proved.

This result justifies the introduction of linear series into invariantive geometry on a curve, and prepares the way for the fundamental role they will play in the subsequent theory. In addition, it justifies us, when investigating properties of linear series, in confining our attention to *plane* curves.

1.4. *Complete linear series.* A linear series on a curve is said to be *complete* if it is not contained in any more ample linear series of the same order.

This definition at once raises the question as to whether complete linear series may or may not overlap; the answer is contained in

THEOREM VI. *Any two linear series on a curve C which have a set in common are contained in one and the same linear series of the same order on C.*

To prove this we assume that C is a plane curve (§ 1.33) whose equation, in homogeneous coordinates x, y, z, is $f = 0$. We suppose that the generic set G_1 of the first series is cut on C, residual to a fixed set A, by a curve of the system

$$\Phi \equiv \lambda_0 \phi_0 + \lambda_1 \phi_1 + \ldots + \lambda_r \phi_r = 0,$$

while the generic set G_2 of the second series is cut on C, residual to a fixed set B, by a curve of the system

$$\Psi \equiv \mu_0 \psi_0 + \mu_1 \psi_1 + \ldots + \mu_s \psi_s = 0.$$

Further, we suppose that the set, G^* say, common to the two series, is that cut by $\phi_0 = 0$ in the first case and by $\psi_0 = 0$ in the second.

Consider now the sets of points in which C is met by curves of the linear system

$$\Gamma \equiv \psi_0(\lambda_0 \phi_0 + \lambda_1 \phi_1 + \ldots + \lambda_r \phi_r) + \phi_0(\mu_0 \psi_0 + \mu_1 \psi_1 + \ldots + \mu_s \psi_s) = 0,$$

where all the λ_i and the μ_i are regarded as arbitrary parameters. Since every term in Γ contains both a ϕ_i and ψ_j as factor, the generic curve of the system passes through the sets A and B; and, since one of the factors in question is either ϕ_0 or ψ_0, the curve in question also contains G^*. Thus curves of Γ meet C in the fixed set $A + B + G^*$ and residually in a linear series of sets which reduces to the first of the two given series by taking the μ_i to be all zero, and to the second by taking the λ_i to be all zero. The two series are therefore both contained in this new series, and so the theorem is proved.

It follows at once that complete linear series on a curve cannot overlap. Every linear series containing a given set of points G on a curve C belongs to a unique complete linear series containing G.

It may, of course, well happen that G is *isolated* on C, i.e. it belongs to no linear series of freedom at least unity. In this case we preserve the generality of our previous statement by regarding any single set of points on C as defining a *linear series of freedom zero.*

1.5. *Linear equivalence.* In order to formalize the theory of complete linear series, we introduce a relation of linear equivalence as follows:

DEFINITION. Any two sets of points on a curve C are said to be *linearly equivalent* if there exists a linear series on C containing them both.

If A, B are the sets in question, we denote the relation by $A \equiv B$. The definition amounts to asserting that $A \equiv B$ on C if A and B, after abstracting any points common to both sets, are the sets of poles and zeros respectively of a rational function on C.

The properties of linear equivalence are summarized in

THEOREM VII. *The relation of linear equivalence is* (*a*) *symmetrical,* (*b*) *reflexive,* (*c*) *transitive, and* (*d*) *additive.*

Of the four properties mentioned (*a*) is self-evident. So also is (*b*), expressing that $G \equiv G$, where G is an arbitrary set; for G itself forms a linear series of freedom zero. As regards (*c*), we observe that if $A \equiv B$ and $B \equiv C$, then the sets A, B, C belong to the complete linear series defined by B; and hence $A \equiv C$. As regards (*d*), we note first that by the sum of two sets A, B on a curve we mean the set of all the points of A together with those of B, each point in the sum-set being counted to the sum of its multiplicities in the original sets. Supposing then that A', B' are sets such that

$$A \equiv A', \qquad B \equiv B',$$

there will exist sets X, Y such that $A+X$, $A'+X$ are the complete sets of intersection of C with curves $L = 0$, $L' = 0$ of the same order, while $B+Y$, $B'+Y$ are the complete intersections of C with curves $M = 0$, $M' = 0$, of the same order. Since $A+B+X+Y$ and $A'+B'+X+Y$ are then the complete sets of intersection of C with the composite curves $LM = 0$, $L'M' = 0$, it follows that

$$A+B \equiv A'+B'.$$

Thus the equivalence relation is additive; and the theorem is completely established. The additive property of equivalence can also be regarded as an expression of the fact that the product of two rational functions on C is itself a rational function on C.

In general we use the symbol $|A|$ to denote the *complete* linear series to which any given set A belongs. For example, $|A+B|$ denotes the complete series typified by the set $A+B$, which is the same as that typified by any sum of a set of $|A|$ and a set of $|B|$. Similarly $|A+A|$ or $|2A|$ denotes the complete series defined by any sum of a pair of sets (or one repeated set) of $|A|$; and a similar interpretation holds for the symbol $|nA|$, where n is any positive integer.

1.51. *Subtraction of sets. Virtual sets.* In contrast to the operation of addition just described, the *subtraction* of a set B from a set A on a curve is evidently possible only when B happens to be entirely contained in A. The subtraction of B from itself—which is always possible—gives the unique *null-set* on the curve, a set which must be regarded as effective, and which we usually denote by the symbol 0.

Thus if we denote the whole field of effective sets (including the null set) on a curve by $\mathscr{H}$, the operation of addition is universally possible in $\mathscr{H}$, but that of subtraction is not; the situation in this respect is analogous to that obtaining in the arithmetic of cardinals. A wider field $\mathscr{K}$ of virtual sets, in which both addition and subtraction are always possible, can be defined as follows:

DEFINITION. Any formal difference $A-B$ of two effective sets A, B on a curve is regarded as defining a *virtual set* on the curve, on the understanding that two formal differences $A-B$ and $A'-B'$ define the same virtual set if $A+B' = A'+B$.

The equality sign used here signifies, of course, absolute identity. The *order* of the virtual set $A-B$ is the difference between the orders of A and B.

In the field $\mathscr{K}$ of virtual sets so defined there is (*a*) a null-set $A-A$ (independent of A), (*b*) a unique set $A-0$ (or simply A) corresponding to every effective set A, and (*c*) a purely negative set $0-A$ (or $-A$) associated likewise with every effective set A.

Addition and subtraction in $\mathscr{K}$ are defined by the obvious rules

$$(A-B)+(C-D) = \overline{A+C}-\overline{B+D},$$

$$(A-B)-(C-D) = \overline{A+D}-\overline{B+C},$$

and obey the ordinary laws.

Finally, a relation of linear equivalence is defined for virtual sets by the assertion that

$$(A-B) \equiv (C-D)$$

if and only if the sets $A+D$ and $B+C$ are effectively equivalent.

The reader will have no difficulty in verifying that the equivalence relationship so defined for virtual sets possesses the four properties referred to in Theorem VII. Thus, if $\alpha = A-A'$, $\beta = B-B'$, $\gamma = C-C'$ are three virtual sets such that

$$\alpha \equiv \beta, \qquad \beta \equiv \gamma,$$

then, by hypothesis,

$$A+B' \equiv A'+B, \qquad B+C' \equiv B'+C,$$

whence, by addition and suppression of $B+B'$ on each side of the relation, we obtain

$$A+C' \equiv A'+C,$$

so that

$$\alpha \equiv \gamma.$$

1.52. *Series of equivalence.* From the transitive property just established there arises the concept of a series of equivalence which is the analogue, in the field of virtual sets, of a complete linear series in the field of effective sets.

DEFINITION. A series of equivalence is a totality of mutually equivalent virtual sets which may or may not contain a nucleus of effective sets forming a complete series.

If A is any virtual set we usually denote the series of equivalence defined by it by the symbol A itself.

It should be noted that, although every virtual set $A-B$ has an *order*, equal to the difference (positive or negative) between the orders of A and B, it is not in the nature of such a set to have members; and although in practice we tend to ignore the distinction between an effective set A and the corresponding 'effective virtual set' $A-0$, there is in fact a sharp logical distinction between the two. The real motive for introducing virtual sets and series of equivalence is to facilitate the investigation of equivalence relationships in the effective field by utilizing the simpler symbolic calculus of the wider virtual field.

1.6. ***Simple and complex linear series.*** A linear series g_n^r is said to be complex (as opposed to simple) if those of its sets which contain an arbitrary point P of the carrying curve C always contain further points P' of C. The definition is significant for $r \geqslant 2$.

In such a case, suppose that the number of points P' is $\mu-1$; then each of the μ points P, P' evidently imposes the same linear condition on curves (or primals) of the linear system which cuts the g_n^r on C; and the μ points form a set of an algebraic series γ_μ^1 of index unity.

Supposing, then, that the g_n^r is without base points, each of its sets must consist of a fixed number, n_1, say, of sets of γ_μ^1, where $n = n_1 \mu$; and if Γ is any curve whose points map the individual sets of γ_μ^1, the image on Γ of sets of g_n^r is a $g_{n_1}^r$. Thus,

Any complex series g_n^r, without fixed points, and of freedom $r \geqslant 2$, is composed of a linear series of sets of a rational or irrational pencil.

Particularly important is the case of a linear series composed of groups of sets of a rational (and therefore linear) pencil g_μ^1. Examples of either kind are the series cut by the generators, or by linear series of generators, of a rational or irrational cone on the section of the cone by any surface of order μ.

EXAMPLES

1. The points of any rational curve are in (1, 1) correspondence with the values of a parameter x, say; and all sets of any order n on the curve are given by the equation

$$\lambda_0 x^n + \lambda_1 x^{n-1} + \ldots + \lambda_n = 0,$$

which is linear in the parameters λ_i. Hence,

The totality of sets of order n on a rational curve is a g_n^n.

Conversely, if a curve contains a g_n^n $(n \geqslant 1)$ it is rational. For it then contains a g_1^1, obtained by fixing $n-1$ points for sets of the g_n^n; and the rationality of the g_1^1 implies the rationality of the curve.

2. Let C be a general plane cubic; since C is not rational, its individual points are mutually disequivalent. The pairs of points of C fall into ∞^1 distinct series g_2^1, each generated by the pencil of lines through a fixed point of C; and the totality of pairs, which is a γ_2^2, is generated simply by these g_2^1.

The ∞^3 triads of points of C, forming a γ_3^3, separate themselves into ∞^1 series g_3^2, any one of which can be obtained (in infinitely many ways) as the series of sets in which C is cut residually by the net of conics through three fixed points of the curve. More precisely, if K is any set of the g_6^5 cut on C by all conics of the plane, and if X, Y are complementary triads such that $X+Y = K$, then the complete series $|X|$ and $|Y|$ are such that each set of either is complementary to every set of the other. The particular g_3^2 cut on C by the lines of the plane is projectively, not invariantively, distinguished from the others; it loses its prominence if C is transformed into another plane cubic by a quadratic transformation whose three base points lie on C.

3. If C is a plane quartic with a node O, any two individual points are disequivalent; in general two point-pairs also are disequivalent, the

exception occurring for point-pairs belonging to the g_2^1 cut on C by lines through O.

The ∞^3 triads of points of C belong to ∞^2 series g_3^1, any one of which is generated by a pencil of conics through O and three other fixed points of C. These include the ∞^1 series g_3^1 generated by pencils of lines through one fixed point of C, which are not invariantively different from the rest.

4. If C is a non-singular plane quartic, every individual point, every point-pair, and also the generic point triad, is isolated as regards equivalence. C contains, however, ∞^1 special series g_3^1, each of which is generated by the pencil of lines through a fixed point of C.

The generic tetrad of points of C belongs to a g_4^1; but C contains a special g_4^2, namely, that cut on C by the lines of the plane.

5. *Elliptic and hyperelliptic curves.* Any curve which is transformable birationally into a non-singular plane cubic is called elliptic (Ch. II, §4); such curves and—as will appear later—only such curves possess a simply-infinite family of g_2^1. Any curve, such as the uni-nodal plane quartic, which is neither rational nor elliptic, and which possesses a g_2^1, is said to be *hyperelliptic*. A more general example of such curves is a plane curve of order n $(n > 4)$ whose only singularity is an $(n-2)$-fold point. For the properties of these curves see § 4.

6. *Projective model of a* g_n^r $(r \geqslant 2)$. If the linear system of curves (or primals) which cuts a g_n^r on C has equation

$$\Phi \equiv \lambda_0 \phi_0 + \lambda_1 \phi_1 + \ldots + \lambda_r \phi_r = 0,$$

then the equations

$$X_0/\phi_0 = X_1/\phi_1 = \ldots = X_r/\phi_r$$

transform any point P of C into the point P', in space S_r, whose homogeneous coordinates are $(X_0, X_1, \ldots, X_r)$. As P describes C, P' describes a curve C' in S_r, in such a way that to the set of the g_n^r cut on C by the generic Φ there corresponds the set of points cut on C' by the prime whose equation is

$$\lambda_0 X_0 + \lambda_1 X_1 + \ldots + \lambda_r X_r = 0.$$

Thus g_n^r becomes the linear series cut on C' by primes: C' is therefore called the *projective model* of g_n^r.

All projective models of the same g_n^r on C are projectively equivalent, i.e. transformable into each other by (non-singular) collineations of S_r. For two different systems Φ and Ψ which cut the same g_n^r on C define the same system of rational functions on C, and the two projective models derived from the same system Φ (by different choice of base curves $\phi_0, \ldots, \phi_r$) are plainly related by a linear homogeneous transformation of the X_i.

If the given series is simple (§ 1.6) the correspondence between C and C' is birational, and C' is of order n; but if it is complex, and compounded of a γ_μ^1, then each set of γ_μ^1 transforms to a single point of C', and C' is of order $n_1 = n/\mu$.

Any linear series contained in the g_n^r is represented on C' by the series cut out by a linear system of primes of S_r, such primes forming a star with vertex S_k $(0 \leqslant k \leqslant r-2)$.

If the given g_n^r is complete, the curve C' is normal in S_r.

7. Using the projective model for a simple series g_n^r, show how to define the *tangent linear pencil* g_n^1 at the set G of an algebraic series γ_n^1 immersed in a g_n^r.

Prove that, if G, G_1 are consecutive sets of γ_n^1 having consecutive s-fold points P, P_1 respectively, then the tangent g_n^1 to γ_n^1 at G has P as $(s-1)$-fold base point. Deduce Bertini's Theorem, that the generic set of g_n^r can have no multiple point which is not a base point of the series.

§ 2. Jacobian and canonical series

2. Jacobian set of a g_n^1. Consider, on a curve C, a series g_n^1 of which G is a generic set. We define a *double point* of the g_n^1 as a point of C which counts twice in the set of the series to which it belongs; the set of all double points of the g_n^1 is then termed the *Jacobian set* of the series in question, and is denoted by $J(G)$.

We assume for simplicity that C is a plane curve, having equation

$$f(x,y,z) = 0,$$

and endowed with only ordinary multiple points O_i of multiplicities λ_i; for any plane curve can be transformed birationally into a curve of this kind (cf. Ch. III, § 3.2), and so there is no loss of generality from the invariantive point of view. If the g_n^1 is cut on C, residually to a fixed set K, by a pencil Φ of curves with equation

$$\Phi \equiv \phi + \lambda\phi' = 0,$$

we observe that a double point of g_n^1 is a point of contact (not in general belonging to K) of C with a curve of Φ. Hence $J(G)$ is the set of points cut on C, residually to K and O_i, by the curve $J(f,\Phi)$ whose equation is

$$J(f,\Phi) \equiv \frac{\partial(f,\phi,\phi')}{\partial(x,y,z)} = 0.$$

Note (i). If, in particular, the g_n^1 is cut on C (supposed now of order n) by the pencil of lines $L+\lambda L' = 0$ through a generic point $O(X,Y,Z)$ of the plane, then $J(f,\Phi)$ is the first polar of O with regard to C, having equation

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0.$$

This is a curve of order $n-1$, *adjoint* to C (Ch. II, § 2), i.e. having a (λ_i-1)-fold point at every λ_i-fold point O_i of C; as we know, it meets C elsewhere in the points of contact of tangents to the curve from O.

Note (ii). *An s-fold point of g_n^1, which is not a base point of the* *series, counts* $(s-1)$*-fold in the Jacobian set of the* g_n^1. This means that if a particular g_n^1 happens to possess a set in which one point U counts s-fold as a member ($s > 2$), then U counts $(s-1)$-fold as a member of the Jacobian set.

In the case where C is rational, the proof of this statement is elementary, and can be left as an exercise to the reader.

In the general case, the result follows from an application of Zeuthen's rule for calculating the multiplicity of a coincidence in a simply-infinite system of point-pairs (Ch. IV, § 4). For if (P, P') is any pair of points belonging to the same set of the g_n^1, the Jacobian set $J(G)$ is the set of coincidence points in the system of all pairs such as (P, P'). On a generic line passing close to U there is one point P; and of the $n-1$ points P' corresponding to P, $s-1$ will lie, with P, near U. By the rule in question, then, the multiplicity of U as a coincidence is equal to the sum of the infinitesimal orders, relative to PU, of the $s-1$ corresponding infinitesimals $P'U$; and this is $s-1$, since the infinitesimal distances involved are all comparable in length.

2.1. *Jacobian series of a* g_n^r. Consider next a g_n^2 whose sets are cut on C, residually to a fixed set K, by the net Φ whose equation is

$$\Phi \equiv \lambda\phi_1+\mu\phi_2+\nu\phi_3 = 0.$$

Any g_n^1 contained in the g_n^2 is cut on C by a linear pencil Φ^* of curves of Φ, having an equation of the form

$$\Phi^* \equiv \Phi_1+\theta\Phi_2 = 0,$$

where $\Phi_1 \equiv a_1\phi_1+b_1\phi_2+c_1\phi_3$ and $\Phi_2 \equiv a_2\phi_1+b_2\phi_2+c_2\phi_3$; and its Jacobian set is that cut on C, residually to K and the multiple points O_i, by the curve $J(f,\Phi^*)$ whose equation is

$$J(f,\Phi^*) \equiv \partial(f,\Phi_1,\Phi_2)/\partial(x,y,z) = 0.$$

Now this latter equation is easily seen to reduce to the form

$$J(f,\Phi^*) \equiv A\frac{\partial(f,\phi_2,\phi_3)}{\partial(x,y,z)}+B\frac{\partial(f,\phi_3,\phi_1)}{\partial(x,y,z)}+C\frac{\partial(f,\phi_1,\phi_2)}{\partial(x,y,z)} = 0,$$

where A, B, C are the numbers $b_1c_2-b_2c_1$, etc.; so that, as the pencil Φ^* varies in Φ, the curve $J(f,\Phi^*)$ describes a net. Hence the Jacobian sets of all g_n^1 contained in the given g_n^2 are cut on C, residually to K and the points O_i, by the curves of a linear net, and they are therefore equivalent sets on C.

The extension of this result to a linear series g_n^r, of any freedom

r, is immediate. For we can pass from any g_n^1 in the g_n^r to any other g_n^1 in the g_n^r via an intermediary g_n^1 containing a set of each of the given series g_n^1; whence, by the result just established, and by the transitive property of equivalence, it follows that the Jacobian sets of the two series g_n^1 are equivalent. Hence

THEOREM VIII. *The Jacobian sets of all g_n^1 contained in a g_n^r are mutually equivalent.*

It should be noted that, except in the case $r = 2$, the Jacobian sets are only *contained* in a linear series, inside which they actually constitute an algebraic series of dimension $2r-2$. In defining the Jacobian series of a g_n^r, we go still farther and envisage the *complete* linear series to which all the sets in question belong:

DEFINITION. The Jacobian series of a g_n^r is the complete linear series $|J(G)|$ containing all Jacobian sets of g_n^1 contained in the g_n^r.

Ex. If C is of order n, the Jacobian sets of g_n^1 contained in the g_n^2 cut on C by the lines of the plane form the linear series cut on C (residually to the multiple points) by the net of first polars: its order is the class of C. The corresponding Jacobian series is that cut on C by all curves of order $n-1$ adjoint to C, and, as will appear later, this series is in fact complete.

2.2. *Jacobian set of a g_n^1 with base points.* If O is any simple base point of a g_n^1, belonging to every set of the series, then one set of the g_{n-1}^1 obtained by discarding O contains O; hence one set of the g_n^1 contains O doubly. Thus O is a member of the Jacobian set of the g_n^1; we now ask whether it is a simple or a multiple member of that set. Reverting to the notation of § 2, we have to find the number of intersections of C and $J(f, \Phi)$ which fall at a common zero of f, ϕ, ϕ'.

If f is of degree m, and if ϕ, ϕ' are of degree p, the equation of $J(f,\Phi)$, in virtue of the Euler identities

$$mf \equiv xf_x+yf_y+zf_z, \quad \text{etc.},$$

can be written in the form

$$zJ(f,\Phi) \equiv p \begin{vmatrix} \frac{m}{p}f & \phi & \phi' \\ f_x & \phi_x & \phi'_x \\ f_y & \phi_y & \phi'_y \end{vmatrix} = 0.$$

Taking z to be unity, and supposing O to lie at $x = 0$, $y = 0$, we observe that the intersections of $f = 0$ with $J(f,\Phi)$ will be identical with those of $f = 0$ and the modified curve J^* whose equation is

$$J^* = \begin{vmatrix} f & \phi & \phi' \\ f_x & \phi_x & \phi'_x \\ f_y & \phi_y & \phi'_y \end{vmatrix} = 0.$$

On writing the equations of f, ϕ, ϕ', J^* in the forms

$$f \equiv u^{(1)}+u^{(2)}+\dots, \qquad J^* \equiv U^{(1)}+U^{(2)}+\dots,$$
$$\phi \equiv v^{(1)}+v^{(2)}+\dots, \qquad \phi' \equiv w^{(1)}+w^{(2)}+\dots,$$

where each of $u^{(i)}$, $v^{(i)}$, $w^{(i)}$, $U^{(i)}$ is homogeneous of degree i in x and y, we find that

$$U^{(1)} \equiv |u^{(1)}\ u^{(1)}_x\ u^{(1)}_y|,$$

where the second and third rows of the determinant are obtained by replacing u by v and w respectively; and we see at once that $U^{(1)}$ vanishes identically. Furthermore, we find, by using the Euler identities, that

$$U^{(2)} \equiv |u^{(2)}\ u^{(1)}_x\ u^{(1)}_y|+|u^{(1)}\ u^{(2)}_x\ u^{(1)}_y|+|u^{(1)}\ u^{(1)}_x\ u^{(2)}_y|$$
$$\equiv -|u^{(2)}\ u^{(1)}_x\ u^{(1)}_y|.$$

Thus J^* has a double point at O; and its nodal tangents, given by $U^{(2)} = 0$, are in general distinct† from the tangent to C at O. The number of intersections of $J(f, \Phi)$ with C at O is therefore two. Hence

THEOREM IX. *A simple base point of a g^1_n counts twice as a member of the Jacobian set of the series.*

2.3. *Addition formulae for Jacobian series.* From Theorem IX we deduce at once a remarkable additive property of Jacobian series. If $|G_1|$ and $|G_2|$ are series (of positive freedom), then the Jacobian series of their complete sum $|G_1+G_2|$ is completely determined by the Jacobian set of *any* subordinate linear pencil and, in particular, by that of a pencil whose generic set consists of a fixed set G_2 together with a set varying in a linear pencil of $|G_1|$. The Jacobian points of a pencil of this kind will, however,

† The only conditions under which $U^{(2)}$ has $u^{(1)}$ as factor are (*a*) that O is a double base point of the g^1_n, every curve of ϕ touching C at O, or (*b*) that O is a triple point of a set of the g^1_n, the curve ϕ^* of the pencil ϕ which touches C at O having actually three-point contact with C at O. These alternatives are easily verified by taking ϕ to be ϕ^*, and by taking the common tangent of ϕ^* and C to be $y = 0$.

be those of the linear pencil of $|G_1|$, together with the points of the fixed set G_2, counted twice (Theorem IX). Hence

$$J(G_1+G_2) \equiv J(G_1)+2G_2;$$

whence, by symmetry,

$$J(G_1+G_2) \equiv J(G_1)+2G_2 \equiv J(G_2)+2G_1.$$

2.4. *The canonical series.* From the formulae just obtained we deduce that, if $|G_1|$ and $|G_2|$ are any pair of linear series, of freedom at least unity, then

$$J(G_1)-2G_1 \equiv J(G_2)-2G_2;$$

so that the series of equivalence $J(G_1)-2G_1$ is actually independent of G_1. Hence

THEOREM X. *If $|G|$ is any linear series of positive freedom on a curve C, and if $J(G)$ is any set of the Jacobian series of $|G|$, then the series of equivalence X, defined by*

$$X \equiv J(G)-2G,$$

is independent of G.

The series X is called the *canonical series of equivalence* of C, while the linear series $|X|$, if it exists, is simply the *canonical series* of C; both are by nature invariant for birational transformations of C.

To calculate the grade of X, which will be a numerical invariant for geometry on C, we take C to be a plane curve of order n with only ordinary multiple points O_i, of multiplicities λ_i; and we suppose $|G|$ to be the g_n^2 cut on C by the lines of the plane. Then the Jacobian set $J(G)$ of any g_n^1 contained in $|G|$ is, as previously remarked, the set of points of contact of tangents to C from a point P of the plane. Hence its order is the class m of C, given by

$$m = n(n-1)-\sum \lambda_i(\lambda_i-1) = 2n+2p-2,$$

where p is the genus of C, previously defined (Ch. II, § 2) by the formula

$$p = \tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\sum \lambda_i(\lambda_i-1).$$

Hence the grade of X is $m-2n = 2p-2$. This result provides, incidentally, a totally new proof of the invariance of p under birational transformation, already established in Ch. IV, § 5.

The set $J(G)$ in question is cut on C by the first polar of P and therefore belongs to the linear series cut on C, residually to the points O_i, by adjoint curves of order $n-1$. Hence, if there exist curves of order $n-3$ adjoint to C, each of these, since it forms with any two lines an adjoint of order $n-1$, meets C, residually to O_i, in a set of the series $|J(G)-2G|$, or, in other words, in an effective canonical set. This proves

THEOREM XI. *The canonical series of a curve C of genus p has grade $2p-2$; and if C is a plane curve of order n with ordinary singular points O_i, curves of order $n-3$ (if such exist), adjoint to C, cut C, residually to the points O_i, in effective canonical sets.*

In Ch. II, § 2.1, we showed that the freedom of curves of order $n-3$ adjoint to C is at least $p-1$, from which it follows that the canonical series $|X|$ is of freedom $p-1$ at least. In a later section (§ 4) we shall show that the freedom of $|X|$ is, in fact, exactly $p-1$, the series cut on C by all adjoints of order $n-3$ being always the complete canonical series.

Examples

1. On any rational curve, canonical sets are purely virtual (of order -2), and they are equivalent to the negative of an arbitrary point-pair of the curve.

2. The canonical series of a plane non-singular cubic (or any elliptic curve) is the g_0^0 whose unique set is the null-set of the curve.

3. The canonical series of the uni-nodal plane quartic $C^4(O^2)$ is the g_2^1 cut on the curve by lines through O. The canonical series of the hyperelliptic curve $C^n(O^{n-2})$ is cut on the curve by sets of $n-3$ lines through O, and is therefore compounded of the unique g_2^1 cut on the curve by lines through O; its freedom is $n-3$ and its grade $2n-6$.

4. The canonical series of the general plane quartic is the g_4^2 cut on the curve by the lines of the plane.

§ 3. Correspondences with valency

3. One of the most important applications of the invariantive geometry of curves is to the theory of correspondences, either between the points of two curves or between the points of a single curve. In an earlier chapter on correspondences the various problems treated had to be reduced, often by cumbersome methods, to rational correspondences: in this section we develop an elegant theory of correspondences on irrational curves, leading at once to the solution of many geometrical problems which could only be attacked with great difficulty by more elementary methods.

3.1. ***The general correspondence.*** We consider, in the first place, algebraic correspondences between variable points P, P' of two curves C, C' respectively, it being understood that, unless the contrary is stated, everything we say applies equally to the case when C and C' are the same curve. It may be assumed in general that C and C' are irreducible plane curves, since no loss of generality is thereby involved.

DEFINITION. An (α, α') correspondence between points (P, P') of (C, C'), in which to a generic point P of C there corresponds a group G' of α' points of C', while to a generic point P' of C' there corresponds a group G of α points of C, is said to be *forward-regular* if the sets G' are all equivalent on C', and *backward-regular* if the sets G are all equivalent on C.

It should be noted that forward-regularity, for example, implies only that the sets G' belong to the same $g^r_{\alpha'}$; they will form a $g^1_{\alpha'}$ only if C is rational and $\alpha = 1$. In general the sets G' form an algebraic series $\gamma^1_{\alpha'}$, whose index is α. In this connexion we note the following result:

THEOREM XII. *In any $(1, \alpha')$ correspondence between C and C', any g^1_n on C transforms to a $g^1_{n\alpha'}$ on C'.*

Let the coordinates of corresponding points (P, P') be (x_0, x_1, x_2) and (y_0, y_1, y_2) respectively; and let the g^1_n be cut on C by the pencil

$$\phi + \lambda\phi' = 0. \tag{1}$$

The corresponding series on C' is obtained by effecting a rational substitution on x_0, x_1, x_2 in (1), in terms of y_0, y_1, y_2; hence it is clear that the series on C' is likewise cut out by a linear pencil of curves. Moreover, its order is obviously $n\alpha'$.

THEOREM XIII. *If a correspondence is forward-regular, it is also backward-regular, and vice versa.*

For, by hypothesis, the sets G' will be cut on C', residually to a fixed set, by the curves of a simply-infinite family having equation $\phi(y_0, y_1, y_2) = 0$. Each point (x_0, x_1, x_2) determines one group G' and therefore one curve ϕ, so that the coefficients in the equation $\phi(y_0, y_1, y_2) = 0$ must be rational functions of x_0, x_1, x_2. Hence the equation must be of the form

$$\Phi(x_0, x_1, x_2; y_0, y_1, y_2) = 0,$$

where Φ is a polynomial homogeneous in y_i and also in x_i $(i = 0, 1, 2)$. But the equation as so written defines a correspondence between points P, P' of the two curves; and, on fixing the coordinates of P', we obtain a curve which must cut C, residually to a fixed set of this curve, in a generic set G. Thus the sets G are all members of the series cut on C by the linear system of curves obtained by replacing the coefficients in Φ (regarded as a polynomial in x_i) by constants. These sets are therefore equivalent to each other, and so the theorem is established.

In virtue of this result we may speak of *regular* correspondences between C and C'.

3.11. *Image curve of a correspondence.* If the equations of C and C', in non-homogeneous coordinates, are

$$F(x,y) = 0, \qquad F'(x',y') = 0,$$

we may represent the generic point-pair (P, P') by the single point P_1 of space S_4 whose (non-homogeneous) coordinates are (x,y,x',y'). As P and P' describe C and C' respectively, P_1 will describe a surface Ω of S_4, the image of the totality of ∞^2 point-pairs of C, C'.

Consider now any (α, α') correspondence $\mathbf{T}$ between P and P'; the aggregate of ∞^1 pairs of corresponding points (P, P') will be represented on Ω by a curve Γ which we may call the *image curve* of $\mathbf{T}$. We then say that a correspondence $\mathbf{T}$ is *irreducible* if its image curve Γ in S_4 is irreducible.

We note further that

(i) the curve C is in $(1, \alpha')$ correspondence with Γ, while Γ is in $(\alpha, 1)$ correspondence with C';

(ii) if $\mathbf{T}$ is regular (forward or backward), its equation is of the form $\Phi(x, y; x', y') = 0$, where Φ is a polynomial. Hence Γ is the complete intersection of Ω with the primal $\Phi = 0$.

Observation (i) is extremely useful as it enables us to *factorize* any (α, α') correspondence between C and C' into a $(1, \alpha')$ correspondence between C and Γ and an $(\alpha, 1)$ correspondence between Γ and C'.

3.2. We now prove the following

LEMMA. *Any rational algebraic series γ_n^1, of index $\lambda \geqslant 1$, of sets of points on a curve is contained in a linear series g_n^r $(r \geqslant 1)$.*

We may suppose the sets G of a rational γ_n^1 on C to be put in $(1, 1)$ correspondence with the points P of a line L. This establishes an (n, λ) correspondence between C and L which is necessarily forward-regular, since sets of λ points of L are mutually equivalent. Hence, by Theorem XIII, the correspondence is also backward-regular; and the sets G, being mutually equivalent, therefore belong to a g_n^r $(r \geqslant 1)$.

From this result follows (cf. Theorem XII)

THEOREM XIV. *In any* $(\alpha, 1)$ *correspondence between two curves, equivalent sets transform to equivalent sets.*

We next prove

THEOREM XV. *In any* (α, α') *correspondence between curves* C, C', *equivalent sets on either curve correspond to equivalent sets on the other.*

Consider the $(1, \alpha')$ correspondence between C and the image curve Γ defined in § 3.11. If Γ is irreducible, it follows from Theorem XII that equivalent sets on C transform to equivalent sets on Γ; and, by means of the $(\alpha, 1)$ correspondence between Γ and C', these transform to equivalent sets on C' (Theorem XIV).

If Γ is reducible, the result still follows, since it holds for each irreducible component of Γ, and since equivalence is an additive property. Thus the theorem is proved in all cases.

3.3. *Relation between canonical sets in a correspondence. Zeuthen's Theorem.* Consider, in the first place, a $(1, \alpha')$ correspondence between C and C'. We shall denote the forward operation, of passing from a point P of C to the corresponding set of α' points P' on C', by the symbol $\mathbf{T}$, and the reverse operation, giving a single point P corresponding to a point P' of C', by $\mathbf{T}^{-1}$. Then, on C we shall have a set Ω_C of *branch-points* of $\mathbf{T}$, any one of which is such that the corresponding set on C' contains a double point. The set of *coincidence-points* is denoted by $K_{C'}$.

Let g_n^1 be any series, without fixed points, on C, and G its generic set. By Theorem XII, this series transforms to a $g_{n\alpha'}^1$ of C' whose generic set $G' = \mathbf{T}G$. Any double point of g_n^1 gives rise to α' double points of $g_{n\alpha'}^1$; but, in addition, $g_{n\alpha'}^1$ has double points arising from the branch-points on C, each point of $K_{C'}$ being double in the set of $g_{n\alpha'}^1$ to which it belongs. Thus the Jacobian set of the latter series is given by

$$J(G') \equiv \mathbf{T}J(G)+K_{C'}.$$

If $X(C)$ and $X(C')$ denote canonical sets on C and C', we have

$$J(G) \equiv X(C)+2G, \qquad J(G') \equiv X(C')+2G'.$$

Hence $\mathbf{T}X(C)+2G'+K_{C'} \equiv X(C')+2G'$, and

$$X(C') \equiv \mathbf{T}X(C)+K_{C'}. \tag{1}$$

For the reverse transformation we may, by Theorem XV, multiply symbolically by $\mathbf{T}^{-1}$. Then, observing that

$$\mathbf{T}^{-1}\mathbf{T}X(C) = \alpha' X(C), \qquad \text{and} \qquad \mathbf{T}^{-1}K_{C'} = \Omega_C,$$

we obtain

$$\mathbf{T}^{-1}X(C') \equiv \alpha' X(C)+\Omega_C. \tag{2}$$

Hence, if $\mathbf{S}$ denotes an $(\alpha, 1)$ correspondence between C and C', and we write $\mathbf{S} = \mathbf{T}^{-1}$ in (2), interchanging C and C', α and α', we have

$$\mathbf{S}X(C) \equiv \alpha X(C')+\Omega_{C'}. \tag{3}$$

The case of a general (irreducible) correspondence $\mathbf{T}$ between C and C' can now be dealt with by considering it as the product of a $(1, \alpha')$ correspondence $\mathbf{T}_1$ between C and the image curve Γ, and an $(\alpha, 1)$ correspondence $\mathbf{T}_2$ between Γ and C'. By (1) and (3),

$$X(\Gamma) \equiv \mathbf{T}_1 X(C)+K_\Gamma,$$

$$\mathbf{T}_2 X(\Gamma) \equiv \alpha X(C')+\Omega_{C'}.$$

Thus

$$\mathbf{T}_2\mathbf{T}_1 X(C)+\mathbf{T}_2 K_\Gamma \equiv \alpha X(C')+\Omega_{C'}.$$

And since $\mathbf{T}_2 K_\Gamma = K_{C'}$, $\mathbf{T}_2\mathbf{T}_1 = \mathbf{T}$, we finally obtain

$$\mathbf{T}X(C)+K_{C'} \equiv \alpha X(C')+\Omega_{C'}. \tag{4}$$

Hence

THEOREM XVI. *In any irreducible (α, α') correspondence between curves C, C', the transform of a canonical set of C with the set of coincidence points on C' constitute a set which is equivalent to any set composed of α canonical sets of C' together with the set of branch-points on C'.*

The numerical content of equation (4) is expressed by

COROLLARY I. *If C, C' have genera p, p', and if the numbers of branch-points on C, C' are z, z' respectively, then*

$$\alpha'(2p-2)+z = \alpha(2p'-2)+z'. \tag{5}$$

This, it will be observed, is Zeuthen's Theorem (Ch. IV, § 6), which has now been deduced by a completely different method.

For a $(1, \alpha')$ correspondence between two curves of equal genera p, equation (5) becomes

$$(\alpha'-1)(2p-2)+z = 0.$$

Since z is non-negative, $(\alpha'-1)(p-1) \leqslant 0$. Hence

COROLLARY 2. *Any uni-rational correspondence between two curves of the same genus $p > 1$ must be birational.*

3.4. *Correspondences on a curve.* We proceed now to discuss correspondences on a single curve C of arbitrary genus. We shall assume that the correspondences in question are not *degenerate*, i.e. such that to a point of C there corresponds every point of C; and that the set of points corresponding to a generic point P does not include P itself. A correspondence on C will therefore have in general a finite number of united points, besides a set of branch-points of the forward correspondence $\mathbf{T}$ and a similar set for the backward correspondence $\mathbf{T}^{-1}$.

3.41. *Calculus of operators.* If $\mathbf{S}$ is an (α, α') correspondence between points P, Q on C, we write, symbolically,

$$\mathbf{S}P = \sum_1^{\alpha'} Q_i, \qquad \mathbf{S}^{-1}Q = \sum_1^{\alpha} P_i.$$

For many purposes it is convenient to regard C as a pair of overlapping curves, loci of P and Q respectively; the symbols $\mathbf{S}$, $\mathbf{S}^{-1}$, when used as operators, then represent transformations in one direction.

(1) *Sum* $\mathbf{S}+\mathbf{T}$. If $\mathbf{T}$ denotes a (β, β') correspondence between points P, R, so that $\mathbf{T}P = \sum_1^{\beta'} R_i$, we define the operator sum $\mathbf{S}+\mathbf{T}$ by the equation

$$(\mathbf{S}+\mathbf{T})P = \sum_1^{\alpha'} Q_i + \sum_1^{\beta'} R_i. \tag{1}$$

Thus addition is evidently commutative and associative.

Since, in (1), P is related to any one of the $\alpha'+\beta'$ points $Q+R$ by the condition that it corresponds to such points either by $\mathbf{S}^{-1}$ or $\mathbf{T}^{-1}$, it follows that

$$(\mathbf{S}+\mathbf{T})^{-1} = \mathbf{S}^{-1}+\mathbf{T}^{-1}. \tag{2}$$

(2) *Product* $\mathbf{ST}$. In the same way, if

$$\mathbf{T}P = \sum_1^{\beta'} Q_i, \quad \text{and} \quad \mathbf{S}Q_i = \sum_{j=1}^{\alpha'} R_{ij},$$

then $\mathbf{ST}$ is defined by the relation

$$\mathbf{ST}P = \sum_{j=1}^{\alpha'} \sum_{i=1}^{\beta'} R_{ij}. \tag{3}$$

Evidently multiplication is associative, and obeys the first distributive law,

$$(\mathbf{T}_1+\mathbf{T}_2)\mathbf{T}_3 = \mathbf{T}_1\mathbf{T}_3+\mathbf{T}_2\mathbf{T}_3.$$

However, it is not commutative. Also $\mathbf{T}^{-1}\mathbf{T}P$ is not the point P, but a set of $\beta\beta'$ points in which P counts β'-fold.

In (3) P is related to any one of the points R_{ij} by the condition that it is one of the points corresponding by $\mathbf{T}^{-1}$ to one of the points corresponding by $\mathbf{S}^{-1}$ to R_{ij}. Hence

$$(\mathbf{ST})^{-1} = \mathbf{T}^{-1}\mathbf{S}^{-1}.$$

3.5. *Correspondences with valency.* Correspondences on irrational curves are divided into two classes: those which possess a *valency* (in a sense now to be defined) and those—called *singular*—which do not. The former class, which is much the more important, will claim our attention here.

DEFINITION. On a curve C a one-way correspondence which makes correspond to any point P a group of points $\sum_1^{\alpha'} Q_i$ is said to possess a *valency* if there exists an integer γ (positive, negative, or zero) such that, as P varies, the set $\gamma P+\sum_1^{\alpha'} Q_i$ varies in a series of equivalence on C. The number γ is called the valency of the correspondence.

Ex. If to any point P of a plane curve of order n we make correspond the set of $n-2$ points Q in which the tangent at P meets the curve again, then all the sets $2P+\sum Q_i$ belong to the g_n^2 cut on the curve by lines of the plane; thus $\gamma = 2$.

For correspondences on a rational curve, the concept of valency is nugatory, since all sets of the same order are equivalent, and every integer possesses the property required of γ. As against this, however, the following result is important:

THEOREM XVII. *If a correspondence on an irrational curve possesses a valency, this valency is unique.*

For suppose that a correspondence $\mathbf{T}$ on C has valencies γ_1 and γ_2. Then, as P varies on C, each of the sets

$$\gamma_1 P+\mathbf{T}P, \qquad \gamma_2 P+\mathbf{T}P$$

varies in a series of equivalence; hence $(\gamma_1-\gamma_2)P$ likewise varies in a series of equivalence. If we suppose that $\gamma_1-\gamma_2 = k > 0$, this means that each point of C, counted k times, is a set of a g_k^r on C. If $k = 1$, then $r = 1$, and C is certainly rational. If $k > 1$, then

$r > 1$, since even on a rational curve multiple points do not form a linear series, and the g_k^r is certainly not compounded of any involution of order $\nu > 1$. Thus, if we represent the g_k^r on the prime sections of a birational transform C^* of C, then C^* will be a curve of order k in $[r]$ possessing the property that an infinity of primes have k-point contact. For this to be possible we must have $r \geqslant k$; and since r cannot exceed k, C^* must be a curve of order k in $[k]$, and therefore rational. Hence, if C is not rational, γ is unique.

3.51. *Properties of valency.* In the calculations which follow we shall frequently have to express the fact that a certain set U, say, derived from a point P of a curve C, varies in a series of equivalence on C. We express this formally by writing $U \equiv$ const.

THEOREM XVIII. *The valency of the sum of two correspondences is equal to the sum of their valencies, and the valency of their product is minus the product of their valencies.*

If **S** and **T** are the two correspondences, of valencies γ_1 and γ_2 respectively, then

$$\gamma_1 P + \mathbf{S}P \equiv \text{const.}, \tag{1}$$

$$\gamma_2 P + \mathbf{T}P \equiv \text{const.}, \tag{2}$$

whence
$$(\gamma_1+\gamma_2)P+(\mathbf{S}+\mathbf{T})P \equiv \text{const.} \tag{3}$$

This proves the first part of the theorem.

Again, if Q is any point of $\mathbf{T}P$,

$$\gamma_1 Q + \mathbf{S}Q \equiv \text{const.},$$

whence, summing for every Q, we obtain

$$\gamma_1 \mathbf{T}P + \mathbf{ST}P \equiv \text{const.} \tag{4}$$

Multiplying (2) by γ_1 and subtracting from (4), we find

$$-\gamma_1\gamma_2 P + \mathbf{ST}P \equiv \text{const.} \tag{5}$$

which proves the second part of the theorem.

EXAMPLES

1. The identical correspondence has, of course, valency -1. If O is a fixed point of a plane cubic, and $\mathbf{T}P$ is the remaining intersection of OP with the curve, $\mathbf{T}$ has valency 1, and $\mathbf{T}^2P$ has valency -1.

2. If $\mathbf{T}_1$, $\mathbf{T}_2$ are two different correspondences of the above type, verify again that $\mathbf{T}_1\mathbf{T}_2$ has valency -1.

THEOREM XIX. *On any curve there exist composite correspondences of every valency.*

On a curve C any g_n^1 (without fixed points) determines a correspondence $\mathbf{E}$ by which, to each point P, there correspond the $n-1$ further points of C which form with P a set of the g_n^1. Such a correspondence is called *elementary*; we observe that $\mathbf{E}^{-1}$ is identical with $\mathbf{E}$, and that $\mathbf{E}$ has valency 1.

If, then, $\mathbf{E}_1$, $\mathbf{E}_2$,..., $\mathbf{E}_\gamma$ are γ elementary correspondences, it follows that $\sum \mathbf{E}_i$ is a (composite) correspondence of valency γ; also the product of $\sum \mathbf{E}_i$ by a further elementary correspondence $\mathbf{E}$ has valency $-\gamma$; while the correspondence $\sum \mathbf{E}_i + \mathbf{E} \sum \mathbf{E}_i$ has valency zero. Since γ is an arbitrary positive integer, the theorem is proved.

THEOREM XX. *A correspondence and its reverse have the same valency.*

This theorem is certainly true for a correspondence $\mathbf{T}$ of valency zero; for such a correspondence is regular (§ 3.1) so that, by Theorem XIII, $\mathbf{T}^{-1}$ also has valency zero.

The theorem is also true for any composite correspondence which is constructed, as above, from elementary correspondences, since each of the latter is self-inverse.

From these facts the theorem may be proved for any correspondence $\mathbf{T}$ of valency γ. For if $\mathbf{S}$ is a composite correspondence of valency $-\gamma$, then $\mathbf{S}+\mathbf{T}$ has valency zero; and therefore $(\mathbf{S}+\mathbf{T})^{-1} = \mathbf{S}^{-1}+\mathbf{T}^{-1}$ also has valency zero. Since $\mathbf{S}^{-1}$ has valency $-\gamma$, it follows that $\mathbf{T}^{-1}$ has valency γ.

3.52. *United points of a correspondence. The Cayley–Brill formula.* We come now to the central theorem of the subject, which characterizes the set of united points of a correspondence on a curve:

THEOREM XXI. *If U is the set of united points of the correspondence* $\mathbf{T}$, *of valency γ, on a curve C, then*

$$U \equiv \mathbf{T}P+\mathbf{T}^{-1}P+\gamma X+2\gamma P,$$

where P is any point of C, and X is a canonical set.

We prove the result by stages, as follows:

(a) *The theorem is true if $\gamma = 0$.* For the method of § 3.1, when applied to a correspondence on a single plane curve C, shows that the relation between corresponding points $(x) \equiv (x_0, x_1, x_2)$ and $(x') \equiv (x'_0, x'_1, x'_2)$ of C is given by an equation of the form

$$F(x,x') \equiv \sum_0^r \phi_i(x_0,x_1,x_2)\psi_i(x'_0,x'_1,x'_2) = 0,$$

in which ϕ_i is a polynomial of degree N, say, and ψ_i a polynomial of degree N'. The sets $\mathbf{T}(x)$ all belong to the series cut on C, residually to a base set A', by curves of the linear system

$$L' \equiv \sum_0^r \mu_i \psi_i = 0,$$

while the sets $\mathbf{T}^{-1}(x')$ all belong to the series cut on C, residually to a base set A, by curves of the linear system

$$L \equiv \sum_0^r \lambda_i \phi_i = 0.$$

All sets on C of the form $\mathbf{T}(x)+\mathbf{T}^{-1}(x')$ are contained in the linear series cut on C, residually to $A+A'$, by curves of the system

$$L^* \equiv \sum_0^r \sum_0^r \nu_{ij} \phi_i \psi_j = 0,$$

the polynomials ϕ_i, ψ_j here being functions of the same current coordinates (x). This system evidently contains the curve

$$F(x,x) = \sum_0^r \phi_i \psi_i = 0$$

which cuts C, residually to $A+A'$, in the set U of united points. Hence, when $\gamma = 0$, $U \equiv \mathbf{T}(x)+\mathbf{T}^{-1}(x)$.

(b) *The theorem is true for an elementary correspondence.* In this case $\mathbf{T}$ is generated by a g_n^1 and $\gamma = 1$; and U is the Jacobian set of the series. Thus, if G is a generic set,

$$U \equiv 2G+X.$$

And since
$$G \equiv P+\mathbf{T}P \equiv P+\mathbf{T}^{-1}P,$$

we may write
$$U \equiv \mathbf{T}P+\mathbf{T}^{-1}P+X+2P.$$

(c) *The theorem is true for a composite correspondence of positive valency* γ. For if $\mathbf{T} = \sum_1^\gamma \mathbf{T}_i$, where $\mathbf{T}_i$ is elementary, then $U = \sum_1^\gamma U_i$; and hence

$$U \equiv \sum \mathbf{T}_i P+\sum \mathbf{T}_i^{-1}P+\gamma X+2\gamma P,$$

or
$$U \equiv \mathbf{T}P+\mathbf{T}^{-1}P+\gamma X+2\gamma P.$$

(d) *The theorem is true for any correspondence of negative valency* $-\gamma$. For if $\mathbf{T}$ has valency $-\gamma$, and if $\mathbf{T}'$ is a composite correspondence of valency γ, then $\mathbf{T}+\mathbf{T}'$ has valency zero; hence, by (a), its set $U+U'$ of united points is such that

$$U+U' \equiv (\mathbf{T}+\mathbf{T}')P+(\mathbf{T}^{-1}+\mathbf{T}'^{-1})P.$$

But, by (*c*), $$U' \equiv \mathbf{T}'P+\mathbf{T}'^{-1}P+\gamma X+2\gamma P.$$

Hence $$U \equiv \mathbf{T}P+\mathbf{T}^{-1}P-\gamma X-2\gamma P.$$

(*e*) *The theorem is true for any correspondence of positive valency* γ. For if $\mathbf{T}$ has positive valency γ, we may construct a composite correspondence $\mathbf{T}'$ of valency $-\gamma$. Then, as in (*d*),

$$U+U' \equiv (\mathbf{T}+\mathbf{T}')P+(\mathbf{T}^{-1}+\mathbf{T}'^{-1})P,$$

whence, by (*d*), we have

$$U \equiv \mathbf{T}P+\mathbf{T}^{-1}P+\gamma X+2\gamma P.$$

Thus the theorem is true in all cases.

The numerical content of this proposition is expressed in the following

COROLLARY. *If* $\mathbf{T}$ *is an* (α, α') *correspondence of valency* γ, *on a curve of genus* p, *the number* u *of coincidences is given by*

$$u = \alpha+\alpha'+2\gamma p.$$

This is known as the Cayley–Brill formula.

3.53. *Common corresponding pairs of two correspondences.* Consider two correspondences (P, Q) and (P, R) on C, having indices (α, α') and (β, β') and valencies γ_1, γ_2 respectively. Denoting by $\mathbf{S}$ and $\mathbf{T}$ the respective forward correspondences, we wish to determine the set W of coincidences of a point of the set $\mathbf{S}P$ with a point of the corresponding set $\mathbf{T}P$.

The correspondence (Q, R) is represented by $\mathbf{T}\mathbf{S}^{-1}$ and its reverse (R, Q) by $\mathbf{S}\mathbf{T}^{-1}$. Its indices are thus $(\alpha\beta', \beta\alpha')$ and its valency is $-\gamma_1\gamma_2$. Hence, by Theorem XXI,

$$W \equiv \mathbf{T}\mathbf{S}^{-1}Q+\mathbf{S}\mathbf{T}^{-1}Q-\gamma_1\gamma_2 X-2\gamma_1\gamma_2 Q.$$

From this relation follows

THEOREM XXII. *The number of common corresponding pairs of points of an* (α, α') *correspondence of valency* γ_1 *and a* (β, β') *correspondence of valency* γ_2 *on the same curve is* $\alpha\beta'+\alpha'\beta-2\gamma_1\gamma_2 p$.

3.54. *Coincidence set of a correspondence.* A coincidence point (as distinct from a united point) of a correspondence $\mathbf{T}$ is one which counts twice in a set $\mathbf{T}P$. We now investigate the set K of such points when $\mathbf{T}$ is an (α, α') correspondence of valency γ.

If Q, Q' are any pair of points of the same set $\mathbf{T}P$, we denote the symmetrical correspondence between Q, Q' by $\mathbf{S}$. Evidently

$$\mathbf{S}Q = \mathbf{T}\mathbf{T}^{-1}Q-\alpha Q,$$

so that the indices of $\mathbf{S}$ are $\alpha(\alpha'-1)$. Also, since $\mathbf{TT}^{-1}$ has valency $-\gamma^2$, the set $\mathbf{TT}^{-1}Q-\gamma^2 Q$ varies in a series of equivalence, as, therefore, does $(\alpha-\gamma^2)Q+\mathbf{S}Q$. Hence $\mathbf{S}$ has valency $\alpha-\gamma^2$, so that, by Theorem XXI,

$$K \equiv 2\mathbf{TT}^{-1}Q+(\alpha-\gamma^2)X-2\gamma^2 Q.$$

Hence

THEOREM XXIII. *In any (α,α') correspondence of valency γ between points P, Q of a curve C, the number of coincidences occurring in the sets (Q) corresponding to points P is in general*

$$2\alpha(\alpha'-1)+2p(\alpha-\gamma^2).$$

3.55. *Note on multiple coincidences.* Zeuthen's rule (Ch. IV, §4) for calculating the multiplicities of a coincidence for a correspondence on a line (or rational curve) still holds good for correspondences on any curve, provided—as can always be arranged—that the coincidences occur at simple points of the curve. This follows essentially from the differential character of the criterion, and from the fact that the correspondence is subordinated to a rational correspondence by joining corresponding points to a fixed point.

APPLICATIONS

1. *Characters of a plane curve with ordinary singularities.* In Ch. IV, §4.2, we obtained Plücker's equations for a plane curve endowed with nodes and cusps; we consider now the analogous problem for a plane curve C having ordinary multiple points. We denote the order, genus, class, number of inflexions, and number of double tangents by n, p, m, ι, τ respectively. In the following applications of the correspondence principle it should be noted that a λ-fold point of general type represents λ distinct (accidentally overlapping) points of C, so that a coincidence on the curve can fall at such a point only if a pair of corresponding points tend to coincidence along the same branch.

(i) Let two points on C correspond if their join passes through a generic fixed point O, not on C. For this correspondence, evidently, $\alpha = \alpha' = n-1$ and $\gamma = 1$. Also true coincidences arise only from tangents to C which pass through O. Hence

$$m = 2n-2+2p.$$

(ii) Consider next the correspondence $\mathbf{T}$ which makes correspond to any point P of C the $n-2$ points Q in which the tangent to C at P meets the curve again. Evidently $\alpha' = n-2$, and $\gamma = 2$. Also $\mathbf{T}^{-1}P$ is the set of points of contact (distinct from P) of tangents from P to C, whence

$$\alpha = m-2 = 2n+2p-4.$$

The coincidences in this case are inflexions of C, so that

$$\iota = (n-2)+(2n+2p-4)+4p = 3n+6p-6.$$

The set of coincidences occurring in the sets $\mathbf{T}P$ is that of points of contact of double tangents to C. Hence (Theorem XXIII),

$$2\tau = 2(2n+2p-4)(n-3)+2p(2n+2p-4-4),$$

or

$$\tau = 2(n-2)(n-3)+2p(2n+p-7).$$

Ex. 1. Show, by Zeuthen's rule, that a cusp (replacing a double point) reduces the above value of ι by 2.

Ex. 2. Prove that, for an (α, α') correspondence of valency γ, the joins of corresponding points envelop a curve which is in general of class $(n-1)(\alpha+\alpha')-2\gamma p$. Explain how this result is modified for a symmetrical correspondence, and illustrate by reference to the plane cubic (cf. Ch. IV, § 1.2).

Ex. 3. Show that, on a curve of genus p, a g^1_μ and a g^1_ν have in general $(\mu-1)(\nu-1)-p$ common pairs.

2. *Multiple secants of space curves.* In the applications which follow we consider a non-singular space curve C, of order n and genus p, with $h = \frac{1}{2}(n-1)(n-2)-p$ apparent nodes; we assume further that it possesses ∞^1 trisecants and a finite number of quadrisecants. The following preliminary results will be required.

(*a*) *If G is the set of intersections of C with a plane, and X a canonical set, then any* CHORDAL *set H (of $2h$ points), cut on C by the chords through a point O not on C, belongs to the fixed series of equivalence*

$$(n-3)G-X.$$

For, on projecting C from O into a plane n-ic curve C^* with h nodes, we find that the chordal set projects into the set of nodes; and, by § 2.4, any curve of order $n-3$ through these nodes cuts, residually, a canonical set on C^*.

It follows, incidentally, that any surface of order $n-3$ through a chordal set cuts residually a canonical set on C.

(*b*) *Any* TRISECANT *set T, cut residually on C by the trisecants which pass through a given point P of C, belongs to the series of equivalence*

$$(n-4)(G-P)-X.$$

For, if C projects from P into a curve C^* of order $n-1$, sections of C^* by lines belong to the series $|G-P|$, and the set T projects into the nodes of C^*. The required result then follows as in (*a*). We note also that a generic cone of order $n-4$, with vertex P and passing through T, cuts C residually in a canonical set.

3. *Common chords of two space curves.* Let C, C' be two non-singular space curves having characters n, n', etc.; and suppose, first, that C and C' do not intersect. From any point P of C we draw the chords of C' to meet the curve in $2h'$ points P'; and from each of the points P' we draw chords of C, meeting C in $4hh'$ points Q. Consider the correspondence (P, Q) on C, with indices (α, α'), say. We have $\alpha' = 4hh'$. To find α, we

note that QP' is any one of the $n'(n-1)$ common generators of the cones projecting C, C' from Q. Hence $\alpha = nn'(n-1)(n'-1)$. Further, the correspondence has valency zero; for, starting from P, the chordal set (P') varies in a series of equivalence (by 2(a)) and, similarly, the total set (Q),

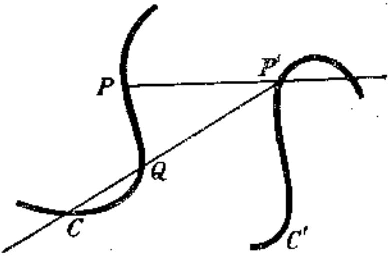


FIG. 15.

being the sum of a fixed number of chordal sets of C, varies in a series of equivalence on C. The coincidences of P and Q arise from common chords of C, C', any such chord absorbing 4 coincidences. Thus the required number τ is given by

$$\tau = \tfrac{1}{4}\{4hh'+nn'(n-1)(n'-1)\}.$$

By a slight modification of the argument it is easily seen that i simple intersections of C, C' reduce the above value of τ by

$$i(n-1)(n'-1)-\tfrac{1}{2}i(i-1).$$

4. *Scroll of joins of corresponding point-pairs on C.* If (P, P') is an (α, α') correspondence of valency γ on C, we require the order ν of the scroll generated by lines PP'. Let l be a generic line; we establish a new correspondence (P, Q) on C by the condition that the plane (l, Q) should meet C again at one of the points P'. The new correspondence is the product of the original correspondence (P, P') and the correspondence (P', Q) of point-pairs whose joins meet l; and since the latter has indices $(n-1, n-1)$ and valency 1, the correspondence (P, Q) has indices $\{\alpha(n-1), \alpha'(n-1)\}$ and valency $-\gamma$. Each join PP' which meets l gives a simple coincidence of P with Q. Hence

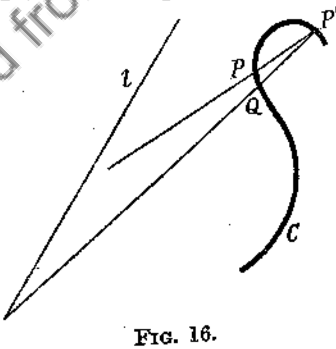


FIG. 16.

$$\nu = (n-1)(\alpha+\alpha')-2\gamma p.$$

5. *The trisecant scroll of C.* To find the order t of the trisecant scroll we apply Ex. 4, suitably interpreted, to what we may call the trisecant correspondence $\mathbf{T}_1$ on C, in which any two points (P, P') correspond if they lie on the same trisecant. By projecting C from P it is clear that through P there pass λ trisecants, where

$$\lambda = \tfrac{1}{2}(n-2)(n-3)-p.$$

Thus $\mathbf{T}_1$ (being symmetrical) has indices $(2\lambda, 2\lambda)$. Also, by 2 (*b*), the trisecant set $\mathbf{T}_1 P$, augmented by $(n-4)P$, varies in a fixed series of equivalence on C, so that $\mathbf{T}_1$ has valency $n-4$. Finally, any trisecant counts 6 times as join of a pair of corresponding points in $\mathbf{T}_1$—twice for each pair of the three intersections with C since $\mathbf{T}_1$ is symmetrical. Hence, by Ex. 4,

$$6t = (2\lambda+2\lambda)(n-1)-2(n-4)p,$$

or

$$t = (n-2)\{\tfrac{1}{3}(n-1)(n-3)-p\}.$$

A finite number σ of tangents to C will meet the curve again. These are evidently associated with united points of the correspondence $\mathbf{T}_1$, and hence

$$\sigma = 2\lambda+2\lambda+2(n-4)p = 2(n-2)(n-3)+2(n-6)p.$$

6. *Stationary trisecants.* A finite number ξ of trisecants of C will be such that at two of their three intersections with C the tangents to C intersect. Such a trisecant will evidently count twice in the set of λ trisecants from its third intersection with C, and may thus be called stationary. To calculate ξ we consider the bitangential correspondence $\mathbf{T}_2$ on C, which makes correspond every pair of points at which the tangents intersect. Starting from a point P, the corresponding points P' are the points of contact of planes drawn through the tangent at P to touch the curve elsewhere; thus $\mathbf{T}_2 P$ is the Jacobian set of the g^2_{n-2} cut on C by planes through the tangent at P. Hence

$$\mathbf{T}_2 P \equiv 2(G-2P)+X,$$

where G is a plane section of C. It follows that $\mathbf{T}_2$ has both indices equal to $2n+2p-6$, and valency 4.

Consider now the possible coincidences of a point of $\mathbf{T}_2 P$ with a point of the corresponding trisecant set $\mathbf{T}_1 P$. Such a coincidence is either (i) a point forming with P the bitangential point-pair of a stationary trisecant, or (ii) the point of contact of a tangential trisecant from P, the latter point counting twice as a coincidence since it occurs doubly in the set $\mathbf{T}_1 P$. And since each stationary trisecant provides two coincidences, the total number is $2\xi+2\sigma$; hence, by Theorem XXII,

$$2\xi+2\sigma = 2.2\lambda(2n+2p-6)-2.4.(n-4)p.$$

Substituting for σ from Ex. 5, we obtain after reduction

$$\xi = 2(n-2)(n-3)(n-4)-2p(n^2-10n+26-2p).$$

Ex. By considering the united points of $\mathbf{T}_2$, show that the number of stationary osculating planes of C is $4(n+3p-3)$.

7. *Quadrisecants.* The number δ of quadrisecants of C may be evaluated

by considering the coincidence set W (as in § 3.53) of the trisecant correspondence $\mathbf{T}_1$. Any coincidence Q of two members of the set $\mathbf{T}_1 P$ is either

(i) the point of contact of a tangential trisecant from P, or

(ii) a tangential point of a stationary trisecant, counting twice in the set of trisecants from P, or

(iii) a point on a quadrisecant from P.

The coincidences (i) and (ii) are simple, their respective numbers being σ and 2ξ. As regards (iii), we observe that each intersection of a quadrisecant $P_1 P_2 P_3 P_4$ with C counts 6-fold in W; for the point P_2, for example, counts twice in the trisecant set from P_1, as belonging to the trisecants $P_1 P_2 P_3$ and $P_1 P_2 P_4$ through P_1; and the formal set of points belonging with P_1 to *some* trisecant set $\mathbf{T}_1 P_1$, since it consists of the trisecant sets, *diminished* by P_1, of all points of the trisecant set of P_1, clearly contains P_1 6-fold. Thus 24δ coincidences arise from the quadrisecants, and hence, by Theorem XXIII, we have

$$\sigma+2\xi+24\delta = 2.2\lambda(2\lambda-1)+2p\{2\lambda-(n-4)^2\}.$$

Substituting for λ, σ, ξ, we find on reduction

$$\delta = \tfrac{1}{12}(n-2)(n-3)(n-4)-\tfrac{1}{2}p(n^2-7n+13-p).$$

The curve of least order with a quadrisecant is the rational quintic, for which $\delta = 1$. The existence of this quadrisecant can be verified by representing on a plane a cubic surface which contains the curve.

A non-singular curve of order n and genus p, situated in [4], has in general a finite number of trisecants. By methods similar to the above, it may be shown that the number in question is

$$\tfrac{1}{6}n(n-4)(n-5)-(n-4)(p-1).$$

8. *Tritangent planes.* Another notable (symmetric) correspondence on C is that between points P, P' through which a plane can be drawn to touch C twice elsewhere. Through P can be drawn a number τ of planes to touch C twice elsewhere; this is the number of bitangents to the projected curve, so that $\tau = 2(n-3)(n-4)+2p(2n+p-9)$. The correspondence $\mathbf{T}_3$ in question has indices $(n-5)\tau$. Its valency may be calculated from the facts that (i) $\mathbf{T}_2$ has valency 4, (ii) the correspondence between a point of contact of a bitangential plane and one of the $n-4$ ordinary intersections of this plane with C has valency $2(2n+2p-10)$; so that the reverse correspondence has also this valency. It follows that the valency of $\mathbf{T}_3$ is $\tau-4(2n+2p-10)$. The united points of $\mathbf{T}_3$ give the points of contact of the ω tritangent planes to C, whence

$$3\omega = 2\tau(n-5)+2p\{\tau-4(2n+2p-10)\}.$$

This gives

$$\omega = 8\binom{n-3}{3}+4\binom{p}{3}+4p(n-5)(n-5+p).$$

For the curve ${}^4C^6$ which is the complete intersection of a quadric and a cubic surface, we have $\omega = 120$; thus there are 120 triads of points which, counted twice, form sets of the (canonical) g_6^4 cut by planes on the curve.

9. *Correspondence without valency.* It will be shown in § 5.1 that a curve of genus p ($p > 0$) possesses a finite number of *moduli*, i.e. birationally invariant continuous parameters such that two curves of equal genus are birationally equivalent if, and only if, these parameters have the same values for both curves. For an elliptic curve there is a single modulus which, for a plane cubic, is the (rationalized) cross-ratio of the four tangents that can be drawn to the curve from a point of itself.

We may then speak of a curve of genus p with general moduli, in contrast to curves of genus p which, in virtue of special values of the moduli, or certain relations between them, have geometrical properties not possessed by the general curve. It can be shown† that, *on a curve with general moduli, every correspondence has a valency.* On the other hand, singular correspondences may exist on curves with special moduli, as the following examples show.

Ex. 1. Show that, if the equation of a plane cubic is reduced to the form $y^2 = x(x-1)(x-\lambda)$, one of the six cross-ratios of the pencil of tangents referred to above is equal to λ.

Ex. 2. Show that, if $\lambda = -1$ (the case of the harmonic cubic), the curve carries a singular correspondence of period 4, given by $x' = -x$, $y' = iy$.

Ex. 3. Prove that the equianharmonic cubic $y^2 = x^3-1$ carries a singular correspondence of period 3, given by $x' = \omega x$, $y' = y$, where $\omega^3 = 1$.

Ex. 4. Prove that, on a curve of genus $p > 1$, every birational transformation of the curve into itself is either generated by a g_2^1, or a correspondence without valency.

§ 4. The Riemann–Roch Theorem

4. Characteristic property of the adjoint curves. In what follows the fundamental problem is that of constructing, on a given curve, the complete series g_n^r characterized by a set G of n given points. Suppose now that f is an irreducible plane curve of order m, having only ordinary multiple points, and that none of the points G lies at a multiple point of f. We then proceed as follows: through G we draw an adjoint curve ϕ, of sufficiently high order l, meeting f elsewhere at the multiple points and in a residual set H; and through H we draw all the adjoints of order l. We shall prove that these cut the complete series g_n^r containing G.

For suppose that G is cut on f by a curve ψ of a linear system (ψ) and that a generic set G' of g_n^r is cut out by a curve ψ' of the same system. The curves ψ and ψ' will in general have a number of common points on f, but we assume that none of these is included in G.

Let P be a typical k-fold point of f which is h-fold for ψ and

† Hurwitz, *Math. Ann.* **28** (1886), 561.

hence also for ψ'. Since P is $(k-1)$-fold (at least) for the adjoint ϕ, the composite curve $\phi\psi'$ has at least an $(h+k-1)$-fold point at P; so that, by Noether's Theorem (Ch. V, § 1),

$$\phi\psi' = \phi'\psi+\theta f, \tag{1}$$

where ϕ' is a curve of order l, having a $(k-1)$-fold point at each k-fold point of f which lies on ψ. Now let Q be a k-fold point of f which is not on ψ; this certainly is $(k-1)$-fold for the adjoint ϕ, so that, by (1), it must be $(k-1)$-fold for ϕ'. Thus ϕ' also is an adjoint curve. Moreover, ϕ' cuts out the set G'; for, by (1), all points common to ψ' and f must lie on ϕ' or ψ, and, by hypothesis, no point of G' lies on ψ.

Suppose, then, that the series g_n^r is cut by the linear system

$$\lambda_0\psi_0+\lambda_1\psi_1+...+\lambda_r\psi_r = 0. \tag{2}$$

Each of the curves ψ_i $(i = 0, 1,..., r)$ will determine a curve ϕ_i, as described above, giving the relations

$$\phi\psi_i = \phi_i\psi+\theta_i f \quad (i = 0, 1,..., r). \tag{3}$$

From this we obtain, by addition,

$$\phi\{\lambda_0\psi_0+...+\lambda_r\psi_r\} = \psi\{\lambda_0\phi_0+...+\lambda_r\phi_r\}+\Theta f, \tag{4}$$

where the λ_i are arbitrary parameters, and $\Theta \equiv \sum \lambda_i\theta_i$.

Hence, in conclusion, the series g_n^r defined by (2) is equally well defined by the system

$$\lambda_0\phi_0+\lambda_1\phi_1+...+\lambda_r\phi_r = 0, \tag{5}$$

which establishes the result. It is easily seen that this holds even when the g_n^r has fixed simple points; for, by (1), if ϕ, ψ, and ψ' have such a common point, it must lie on ϕ' also. Hence

THEOREM XXIV. *The system of all adjoint curves of any given order l cuts on f, residually to the multiple points and (possibly) a group of fixed simple points, a complete series. Conversely, any complete series can be cut out by adjoint curves of some order.*

COROLLARY 1. *The system of all curves of order l, having a k-fold point at each k-fold point of f, cuts a complete series on f.*

For this system is obtained from that of the adjoints of order l by imposing $\sum k$ linear conditions on the curves, where the summation extends to all the multiple points.

Applying this result to the case $l = m$, we have the important

COROLLARY 2. *The* CHARACTERISTIC SERIES *of any complete linear system of plane curves is complete.*

4.1. *Freedom of a complete series. The Riemann–Roch Theorem.* Let g_n^r be a complete series cut on f by adjoint curves ϕ of a given order l, and suppose, in the first place, that these curves have no fixed points at simple points of f. The grade n of the series is thus given by

$$n = ml - \sum k(k-1). \tag{1}$$

The genus p of f is given by

$$p = \tfrac{1}{2}(m-1)(m-2) - \tfrac{1}{2}\sum k(k-1). \tag{2}$$

In evaluating the freedom r there are two cases to consider:

(i) If $l \geqslant m$, we must allow for the fact that ϕ may break up into f and a residual curve of order $l-m$. Thus

$$r \geqslant \tfrac{1}{2}l(l+3) - \tfrac{1}{2}\sum k(k-1) - \tfrac{1}{2}(l-m)(l-m+3) - 1. \tag{3}$$

It will be noted that (3) holds also in the cases $l = m-1$, $l = m-2$, so that the inequality is valid for $l \geqslant m-2$.

Now, by (1) and (2), the second member of (3) is identically $n-p$, whence we have

$$r \geqslant n-p.$$

(ii) If $l < m-2$, we have instead

$$r \geqslant \tfrac{1}{2}l(l+3) - \tfrac{1}{2}\sum k(k-1).$$

By (1) and (2), we therefore have

$$\begin{aligned} r &\geqslant n-p+\tfrac{1}{2}(m-1-l)(m-2-l) \\ &\geqslant n-p. \end{aligned}$$

Hence, in both cases, $r \geqslant n-p$.

Suppose, secondly, that g_n^r is cut out by curves ϕ having a certain number ν of fixed points on f; then, if ρ is the freedom of the complete series $g_{n+\nu}$ cut by the curves ϕ, we have, by the previous result,

$$\rho \geqslant n+\nu-p.$$

Also, since the sets of g_n^r are all contained† in sets of $g_{n+\nu}^\rho$, it follows that

$$r \geqslant \rho-\nu,$$

whence

$$r \geqslant n-p.$$

Hence

THEOREM XXV. *If g_n^r is a complete series, then $r \geqslant n-p$.*

4.2. This result can be made more precise by distinguishing between those series which can be cut out by adjoints of order

† We shall say, briefly, that g_n^r is *contained* in $g_{n+\nu}^\rho$.

$l \leqslant m-3$, and those which cannot be so obtained. Since the adjoints of order $m-3$ cut out the canonical series ($p > 1$), we may state the distinction as follows.

DEFINITION. A series is *special* if its sets are contained in those of the canonical series; in the contrary case it is called *non-special*.

It should be noted that the order of a special series will in general be less than that of the canonical series; in this case there will be a residual series of some order.

In what follows we shall refer to the adjoints of order $m-3$ as the Φ-curves, or the *canonical adjoints*.

With each complete special series there is associated an *index of speciality*, i.e. the number of linearly independent Φ-curves which contain a generic set of the series. A non-special series has index of speciality zero. Since, for the canonical series, $n = 2p-2$, a special series has order not exceeding $2p-2$.

Before proceeding to the Riemann–Roch Theorem, which replaces the inequality of Theorem XXV by an equation, we require the following preliminary result:

THEOREM XXVI (*Noether's Reduction Theorem*). *If G is any set of a complete special series g_n^r, and if P is a point of f not lying on all the Φ-curves through G, then the complete series $|G+P|$ is also of freedom r, i.e. it has P as fixed point.*

Suppose that the series g_n^r is cut out by Φ-curves passing through a set H of fixed points; and let $|a|$ denote the pencil of lines through P which meet f in residual sets A of $m-1$ points. If G is a generic set of g_n^r, there is, by hypothesis, at least one Φ-curve through G and H which does not contain P; adding to this the line a, we obtain an adjoint curve of order $m-2$ whose complete intersection with f consists of $G+H+P+A$. Hence the series $|G+P|$ is cut by adjoints of order $m-2$ passing through the fixed points $H+A$. Hence P must be a fixed point for this series; for the adjoints all contain the $m-1$ points A, and therefore break up into the line a and a curve Φ.

4.3. From this follows the Riemann–Roch Theorem:

THEOREM XXVII. *If g_n^r is a complete series having index of speciality i, then $r = n-p+i$.*

We have already shown, in § 4.1, that $r \geqslant n-p$; hence, to prove the theorem for $i = 0$, we have to show that, if $r > n-p$, the series is special. This is obvious when $r = 0$, for then $n \leqslant p-1$, so that we can always draw a Φ-curve through the set of n points since, by § 4.1, the system $|\Phi|$ has freedom $p-1$ at least.

The general result is now obtained by induction: we assume that it holds for any g_{n-1}^{r-1} and prove it for a g_n^r. We suppose that $r > n-p$, so that $r-1 > (n-1)-p$. Let P be a point of f not common to all the sets of g_n^r, and consider the series $|g_n^r - P| = g_{n-1}^{r-1}$. By hypothesis, this is special; and, by Theorem XXVI, if P is not common to all the Φ-curves through a set G of g_{n-1}^{r-1} the series $|G+P|$ is a complete g_n^{r-1}. But this series is contained in g_n^r, so that P must be common to all the Φ-curves through G, and g_n^r is therefore special.

Thus the theorem is completely established for the case $i = 0$.

Next, suppose that $i > 0$; then, by definition, a unique Φ-curve can be found to pass through a set G of g_n^r and $i-1$ arbitrarily assigned points of f. Hence, if $i = 1$, no Φ-curve can be made to contain the set $G+P$, so that the series $|G+P|$ is non-special. Also, by Theorem XXVI, its freedom is r; and since its order is $n+1$, it follows from what has just been proved that

$$r = (n+1)-p = n-p+1.$$

Thus the theorem is true when $i = 1$. Now assume that it holds for series of index $i-1$. Let g_n^r be a series of index i; then, if G is a generic set of g_n^r and P any point of f, the series $|G+P|$ has order $n+1$ and index $i-1$, so that its freedom r' is given by

$$r' = (n+1)-p+(i-1) = n-p+i.$$

And since P does not lie on all the Φ-curves containing G, it follows from Theorem XXVI that $r' = r$. Thus the Riemann-Roch Theorem is established for all values of i.

COROLLARY 1. *A complete series g_n^r is non-special if $r = n-p$, and special if $r > n-p$.*

COROLLARY 2. *The canonical series has freedom $p-1$.*

For $i = 1$ and $n = 2p-2$.

COROLLARY 3. *A complete series g_n^r is non-special if $n > 2p-2$, or if $r > p-1$, and it is special if $n < p$.*

The first case has already been noted; in the second it is obvious that the system $|\Phi|$, of freedom $p-1$, cannot cut out the series. And in the third, since $r \geqslant 0$, we must have $i > 0$.

Note. An alternative method of developing the theory of series on a curve, which is independent of Noether's Theorem and the consideration of adjoint curves, has been given by Castelnuovo.† A third method, based on an essentially different definition of the genus, has been evolved by Severi.‡

EXAMPLES

1. Show that the adjoints of order $l \geqslant m-3$ form a regular system.

2. Show that the number $p+1$ is the minimum value of n for which a generic set of n points on f belongs to a g_n^1. (This result is taken as the definition of p by Weierstrass.)

3. On a rational curve, every complete series of order $n > 0$ is of freedom n and, on an elliptic curve, it is of freedom $n-1$. Conversely, prove that a curve containing a g_n^n is rational, and that a curve containing a complete g_n^{n-1} $(n > 2)$ is elliptic.

4. Prove that, on a curve of genus 2, every complete series of order $n > 2$ has freedom $n-2$.

5. Prove that a curve which contains two distinct series g_2^1 is either elliptic or rational. (Transform the curve into a plane curve on which the series are cut by pencils of lines.)

6. Show that, if the series g_n^r has index of speciality i, then $i < p$.

4.4. ***Some properties of special series.*** We begin by considering the canonical series itself.

THEOREM XXVIII. *The canonical series is the sole g_{2p-2}^{p-1} existing on f; and it is without fixed points.*

In the first place, a series of order $n = 2p-2$ and freedom $r = p-1$ is necessarily special, since $r > n-p$; and since the Φ-curves cut out ∞^{p-1} canonical sets, the g_n^r in question must coincide with the canonical series. In the second place, if the series had a fixed point P, the series $|g_{2p-2}^{p-1}-P| \equiv g_{2p-3}^{p-1}$ would, with the addition of another point Q, give a g_{2p-2}^{p-1} distinct from the canonical series, and this we have seen to be impossible.

Next, we inquire under what conditions the canonical series can be compounded of an involution. It has been shown in § 1.6 that, if a series g_n^r is compounded of an involution γ_μ^1, n must be a multiple of μ. On the curve Γ which maps the involution, the series g_n^r is represented by a series of order n/μ and freedom r, from which it follows that $n/\mu \geqslant r$. Moreover, if $n/\mu = r$, the

† *Atti Accad. Torino*, **24** (1889), 346; *Memorie scelte*, 19.
‡ *Trattato di geometria algebrica*, ch. V.

curve Γ must be rational, and in that case the algebraic series γ_μ^1 is a linear series g_μ^1.

Applying this result to the canonical series, for which $n = 2p-2$, $r = p-1$, we have $n/\mu \geq n/2$. Thus, if the series is complex, μ must have the value 2 and, since the sign of equality holds, the involution is a g_2^1.

Conversely, if a curve of genus $p > 1$ contains a g_2^1, the latter must be special. A Φ-curve passing through a point of one set must pass through the residual point; hence the canonical series itself is composed of the involution g_2^1. It is clear, besides, that there can be one and only one g_2^1 on the curve.

We know that a rational and an elliptic curve contain respectively a double and a single infinity of series g_2^1. In the case $p = 2$, the canonical series is a g_2^1, but for $p > 2$ there is in general no such series on the curve. As has already been stated (Ex. 5, § 1.6) a curve of genus $p > 1$ which contains a g_2^1 is called *hyperelliptic*.

In conclusion, then, we have

THEOREM XXIX. *The canonical series of a curve is simple except when the curve is hyperelliptic; and in this case it is composed of a g_2^1.*

4.41. *Clifford's Theorem.* We pass now to special series of lower dimension. We first prove the following

LEMMA. *The sets of a special complete series g_n^r each impose $n-r$ conditions on the Φ-curves which are required to contain them.*

Suppose that the series has index of speciality i; then the number of conditions imposed on the Φ-curves in question is clearly

$$(p-1)-(i-1) = p-i = n-r,$$

by the Riemann–Roch Theorem. From this we now deduce

THEOREM XXX (*Clifford's Theorem*). *For any special series g_n^r, $n \geq 2r$.*

For r generic points of the carrying curve, imposing r conditions on the Φ-curves, fix a set of the g_n^r; and this set cannot therefore impose less than r conditions on the Φ-curves required to contain it. Hence, by the lemma, $n-r \geq r$, if the given series g_n^r is complete; so that $n \geq 2r$. If the series is incomplete, with freedom $\rho < r$, then $n \geq 2\rho$, *a fortiori*.

It will appear from the next paragraph that, if the carrying curve is not hyperelliptic, then $n > 2r$, except of course in the case where $r = p-1$.

4.5. *Canonical curves.* We have seen that, for any curve f which is not hyperelliptic, the canonical series is simple; its projective model (§ 1, Ex. 6) is therefore a projectively unique C^{2p-2}, normal in $[p-1]$, which is in birational correspondence with f. This curve is called a *canonical curve* of genus p. Evidently its projective properties correspond to invariant properties of all curves birationally equivalent to f. Conversely, any curve of order $2p-2$ and genus p, situated in $[p-1]$, is necessarily a canonical curve; for the primes of the ambient space cut on it a series g_{2p-2}^{p-1}, which must be the canonical series.

Suppose now, if possible, that f, and therefore C^{2p-2}, contains a special g_{2r}^{r} with $r < p-1$. Any set of such a series would impose r conditions on canonical sets required to contain it (§ 4.41); this would mean that all the primes which pass through r generic points of C^{2p-2} would contain the whole set of the g_{2r}^{r} defined by those points; in other words, the $[r-1]$ joining r generic points of C^{2p-2} would always meet the curve in r further points. This is manifestly impossible for any $r < p-1$, as may be seen, for example, by projecting the curve from $r-2$ of its points. Hence

The sign of equality in Clifford's Theorem is only attained, for non-hyperelliptic curves, in the case of the canonical series.

This result leads at once to an important property of canonical curves:

THEOREM XXXI. *A canonical curve has no multiple points.*

Suppose that P is a k-fold point of C^{2p-2}; then the ∞^{p-2} primes through P cut a special series g_{2p-2-k}^{p-2} on the curve, whence, by the result just established,

$$2p-2-k > 2(p-2),$$

so that

$$k < 2.$$

This means, in other terms, that the canonical series contains no *neutral pairs*, such that sets containing one point of a pair will automatically contain the other.

The proof of the above theorem shows how questions concerning birational transformation can be translated, by means of the canonical model, into the language of projective geometry. Another important result which may be established by similar means is the following:

A non-hyperelliptic curve of genus $p \geqslant 2$ cannot possess an infinity of birational self-transformations.

This is deduced from the proposition† that the only algebraic curves possessing an infinity of self-collineations are rational. This in turn is a consequence (for non-singular curves at least) of the formula of § 5.2 below, which shows that the number of hyper-osculating primes of a non-singular ${}^pC^n$ of $[r]$ is

$$n(r+1)+r(r+1)(p-1).$$

If $p > 0$, this number is not less than $n(r+1)$, but in any case is finite.

If the points of contact of these primes are all distinct, or form a set of $N > r+1$ linearly independent points, there is no collineation, other than identity, for which they are united. Moreover, any self-collineation of the curve must leave invariant the set of N points taken as a whole; hence the number of these collineations cannot exceed $N!$.

For further details, and for the discussion of the case in which the curve possesses singular points, the reader is referred to the treatise by Enriques–Chisini.

EXAMPLES

1. If g_n^r is a complete special series, and $g_{n'}^{r'}$ is its residual with respect to the canonical series, prove that $n+n' = 2p-2$, $n-n' = 2(r-r')$.

2. Prove that a hyperelliptic curve of genus p can be represented on a double curve C^{p-1}, normal in $[p-1]$, and hence upon a double line; and that there are $2(p+1)$ branch-points in the representation.

3. Show that a general hyperelliptic curve is birationally equivalent to a curve with an equation of the form

$$y^2 = (x-a_1)(x-a_2)...(x-a_{2p+2}).$$

4. Show that, for $p = 3$, the general non-hyperelliptic canonical curve is a plane quartic; that, for $p = 4$, it is the complete intersection of a quadric and a cubic surface; and that, for $p = 5$, it is the complete intersection of three quadrics of [4].

5. Show that the general canonical ${}^4C^6$ can be projected into a plane bi-nodal quintic, or a plane sextic with two triple points. Show also that the general canonical ${}^5C^8$ projects into a plane sextic with 5 nodes but that, if the curve contains a g_3^1, it can be transformed to a plane nodal quintic.

§ 5. CURVES IN S_r

5. The results obtained by considering linear series on a plane curve have many important extensions to curves in general; to some of these the present section is devoted.

† See Enriques–Chisini, iii. 239.

Let C be a curve of order n and genus p, situated in $[r]$. The primes of the ambient space cut on C a series g_n^r; if the series is complete, C is normal, while, if it has deficiency δ, C is the projection of a curve of the same order, normal in $[r+\delta]$.

The curve C is said to be *special* or *non-special*, according as the series g_n^r is special or non-special. Then the Riemann–Roch Theorem, applied to g_n^r, gives

THEOREM XXXII. *A non-special curve ${}^pC^n$ is normal in $[n-p]$; a special curve of index i is normal in $[n-p+i]$.*

COROLLARY 1. *A rational curve of order n, situated in $[r]$, where $r < n$, is the projection of a C^n of $[n]$; and an elliptic curve of order n, situated in $[r]$, where $r < n-1$, is the projection of a C^n of $[n-1]$.*

The first of these results was obtained by a different method in Chapter II.

COROLLARY 2. *Any curve of genus p can be birationally transformed to a non-singular curve of order $n > 2p$, normal in $[n-p]$.*

For, taking any complete, non-special, simple series g_n^r on the curve, for which $n > 2p$, we can represent it birationally on a C^n of $[r]$, where $r = n-p$. If this curve had a k-fold point $(k > 1)$, the primes through this point would cut on C a g_{n-k}^{r-1} which would be special, since $r = n-p$, and hence $r-1 > n-k-p$. But since $n > 2p$, we have $r > p$, whence $r-1 > p-1$, so that g_{n-k}^{r-1} cannot be special. This proves the result.

5.1. *Moduli of a curve.* We have already alluded to the moduli of a curve of genus $p \geqslant 1$, and remarked that there is a single modulus in the case $p = 1$. For $p > 1$ the number of moduli is determined by

THEOREM XXXIII. *A general curve of genus $p > 1$ depends on $3p-3$ moduli.*

First, suppose that $p > 2$; and let the given curve of genus p be represented by a non-singular curve C^n situated in its normal space S_r: if, as we assume, n is sufficiently great, we shall have $r = n-p$. If C^n is projected on to a plane from $r-2$ generic points, we obtain a curve Γ^n having d ordinary double points, where

$$d = \tfrac{1}{2}(n-1)(n-2)-p.$$

It can be proved† that, if n is sufficiently large, the generic plane

† See Severi, *Vorlesungen*, Anhang G.

curve of order n with d nodes will be irreducible and will depend on precisely ρ parameters, where

$$\rho = \tfrac{1}{2}n(n+3)-d = 3n+p-1.$$

We shall assume this result in what follows. Now the projection Γ^n was obtained by selecting a vertex $[r-3]$ in S_r; and since, by Ch. I, § 1.2, the freedom of spaces $[r-3]$ in S_r is $3(r-2)$, each curve C^n gives rise to $\infty^{3(n-p-2)}$ projections. Again, C^n was obtained from the given curve of genus p by selecting a particular g_n^r from the totality of such series on the curve; and such a selection can be made in ∞^{n-r} ways. The number of possible representations by curves Γ^n is therefore ∞^ν, where

$$\nu = 3(r-2)+(n-r) = 3n-2p-6.$$

If, then, C^n does not possess an infinity of birational self-transformations (which, by § 4.5, is certainly the case if $p > 2$), two series g_n^r will give rise in general to projectively distinct curves. On the other hand, since there are ∞^8 collineations in the plane, each g_n^r gives rise to ∞^8 projectively equivalent curves. It follows that the number $N(p)$ of moduli is

$$N(p) = \rho-\nu-8 = 3p-3.$$

Suppose, finally, that $p = 2$. As remarked in Ex. 2, § 4.5, the curve may then be represented on a double line with 6 branch-points. Since there are ∞^3 collineations of the line into itself, we have $N(2) = 3$.

Ex. 1. Use the above method to show that $N(1) = 1$, adopting a plane cubic for model.

Ex. 2. Show that $N(2) = 3$, by using a plane nodal quartic.

5.2. *The $(r+1)$-fold points of a g_n^r.* Let the sets of a simple series g_n^r $(r > 1)$ be represented on a curve ${}^pC^n$ of $[r]$; then the $(r+1)$-fold points of the series correspond to the points of contact with ${}^pC^n$ of the hyperosculating primes of $[r]$. Now the ∞^1 spaces $[i]$ $(1 \leqslant i \leqslant r-1)$ having $(i+1)$-point contact with ${}^pC^n$ generate a manifold V_{i+1} of a certain order n_{i+1}, say. Moreover, the primes which pass through a given $[r-i-1]$ in generic position cut on ${}^pC^n$ a series g_n^i whose $(i+1)$-fold points are the n_{i+1} points of contact of the osculating $[i]$'s which meet $[r-i-1]$.

Consider now the section of $V_{r-1}^{n_{r-1}}$ by a generic plane ϖ; this is a curve whose order μ_0 and rank μ_1 are obviously n_{r-1} and n_r respectively. The curve has a certain number κ of cusps arising from the n_{r-2} osculating $[r-3]$'s which meet ϖ; and its inflexional tangents are the section by ϖ of the n_{r+1} hyperosculating primes.

Applying Plücker's equations (Ch. IV, § 4.2) to the curve, we have

$$n_{r+1} = \kappa - 3(\mu_1 - \mu_0),$$

whence

$$n_{r+1} = n_{r-2} + 3(n_r - n_{r-1}). \tag{1}$$

From this relation n_{r+1} can be found by induction, starting from the known results $n_1 = n$, $n_2 = 2(n+p-1)$. Let us assume, then, that

$$n_i = in + i(i-1)(p-1), \quad \text{for all } i \leqslant r.$$

From (1),

$$\begin{aligned} n_{r+1} &= \{(r-2)+3r-3(r-1)\}n+ \\ &\qquad + \{(r-2)(r-3)+3r(r-1)-3(r-1)(r-2)\}(p-1) \\ &= (r+1)n + r(r+1)(p-1). \end{aligned}$$

Thus the result holds also for $i = r+1$. Hence

THEOREM XXXIV. *The number of $(r+1)$-fold points of a simple g_n^r on a curve of genus p is in general $(r+1)\{n+r(p-1)\}$.*

COROLLARY. *The osculating spaces $[i]$ of a non-singular ${}^pC^n$ generate a manifold of order $(i+1)\{n+i(p-1)\}$.*

This result is due to Veronese,† who first gave the relation (1), which clearly holds for the curve sections of all manifolds V_{i+1} $(i = r-3, r-4, \ldots, 1)$.

Ex. 1. Establish the above formula for the $(r+1)$-fold points by means of a valency correspondence on the curve.

Ex. 2. Prove that the rank of V_{i+1} is $2(n_{i+1}+p-1)$.

5.3. *The postulation of a curve.* Another problem whose solution depends on the Riemann–Roch Theorem is that of determining the *postulation* of a given curve for primals of a given order.

Let C be an irreducible non-singular curve of order n and genus p, situated in S_r. The primals of any given order m cut on C a series g_{mn}^{ρ}; this may or may not be complete, but in any case those primals which contain $\rho+1$ arbitrary points of C thereby contain it entirely, so that the postulation $P(m)$ of C for primals V_{r-1}^{n} is $\rho+1$. It should be noted that all such primals may be reducible.

If the series g_{mn}^{ρ} is special, we cannot in general obtain an upper limit to $P(m)$ unless we know the index of speciality of the series. But if it is non-special, as is certainly the case when $mn > 2p-2$,

† *Math. Ann.* 19 (1881), 161.

an upper limit to $P(m)$ is given by $mn-p+1$, by the Riemann-Roch Theorem. Thus we may write

$$P(m) = mn-p+1-\delta \quad (\delta \geqslant 0).$$

We shall now consider the case where C is known to be the intersection, complete or partial, of $r-1$ primals of S_r; here it is possible to state a precise value for $P(m)$ for all $m > N$, say. Beginning with space curves, in the first place we prove

THEOREM XXXV. *If C is the complete intersection of two surfaces F_1 and F_2, of orders N_1 and N_2 respectively, situated in* [3], *the complete canonical series is cut on C by surfaces of order $N = N_1+N_2-4$; and if C is residual to a curve C', the canonical series is cut on C by surfaces of order N containing C'.*

We may confine our attention to the second part of the theorem, which includes the first as a particular case. We shall assume that C' is an irreducible curve, of order n' and genus p', and we suppose that it meets C in i points (I), where $i > 0$. We denote by $|A|$ the series of plane sections of C; then a Jacobian set $J(A)$ of $|A|$ consists of the points of contact of tangents to C which meet a given line l.

We now have recourse to the method by which, in Ch. IX, § 1.2, the formula for the rank of C was obtained: we consider, that is to say, the Jacobian of F_1, F_2 and any two planes through l. This is a surface of order N_1+N_2-2, which meets C in the points (I) and in a set $J(A)$. Thus surfaces of order N_1+N_2-2 cut sets of the series $|J(A)+I|$ on C, and therefore surfaces of order $N = N_1+N_2-4$ cut sets of the series $|J(A)-2A+I|$ on C. Such surfaces therefore cut on C a series of order $2p-2+i$, which is non-special and of freedom

$$\rho = p-2+i-\delta \quad (\delta \geqslant 0).$$

Similarly, surfaces (F) of order N cut on C' a non-special series of order $2p'-2+i$ and freedom $\rho' = p'-2+i-\delta'$ $(\delta' \geqslant 0)$. Hence the surfaces (F) which contain $\rho'+1$ generic points of C' will contain it entirely. Such surfaces will cut canonical sets on C, forming a series of freedom $p-1-\delta''$ $(\delta'' \geqslant 0)$; and if they are made to contain $p-\delta''$ generic points of C, they will contain the composite curve $C+C'$. Such surfaces form a system $|F|$ of freedom

$$\binom{N+3}{3}-1-(p+p'+i-1)+\delta'+\delta''. \tag{1}$$

Now by Noether's Theorem for surfaces, which is established in an analogous manner to that for plane curves, all surfaces (F) containing C and C' have an equation of the form

$$F \equiv A_1 F_1 + A_2 F_2 = 0. \tag{2}$$

Computing the number of arbitrary coefficients in the form (2), we find that the freedom of F is

$$\binom{N+3}{3} - 1 - \{\tfrac{1}{2}N_1 N_2(N_1+N_2-4)+1\}. \tag{3}$$

Now the last term in (3) represents the virtual (or numerical) genus of the composite curve $C+C'$ (Ch. IV, § 8); and this, again, is equal to $p+p'+i-1$. Hence, comparing (1) and (3), we deduce that $\delta' = \delta'' = 0$.

This proves the theorem (which is due to Noether) on the assumption that the curve C' residual to C is irreducible; but in fact such an hypothesis is unnecessary, for we may extend the concept of canonical series to the case of a composite curve, and then reason precisely as before (cf. Ch. XIII, § 8.4).

COROLLARY. *Surfaces of order N which pass through the points (I) cut canonical sets on C and C'.*

With the same notation and methods we can prove

THEOREM XXXVI. *If C is the complete intersection of F_1 and F_2, the postulation formula for C, namely, $P(l) = ln-p+1$, holds for all surfaces of order $l > N$; and if C is a partial intersection, the formula holds for all $l \geqslant N$.*

For we can show that, in each case, surfaces of order l cut a complete non-special series on C.

Finally, by the same methods, and with the aid of Noether's Theorem for primals in S_r, we can prove the following results:

If C is the complete intersection of $r-1$ primals F_i of orders N_i $(i = 1, 2,..., r-1)$ in S_r, the complete canonical series is cut on C by primals of order $N = \sum N_i - r - 1$; and if C is residual to a curve C', the complete canonical series is cut on C by primals of order N through C'.

In the first case the postulation formula holds for all primals of order $l > N$; and in the second, for all primals of order $l \geqslant N$.

It has further been shown by Severi† that, if C is a non-singular curve, and if C' is any curve which forms with C a complete

† *Rend. Palermo*, **17** (1903), 73.

intersection, then primals of any order through C' cut a complete series on C. Such primals may therefore be termed *adjoint* to C.

Ex. Examine, by means of Theorem XXXVI, the validity of the formula $P(l)$ for all non-singular curves C^n ($n \leqslant 6$) of S_3.

NOTES AND EXAMPLES ON CHAPTER XII

1. *Specification of space curves.* In space S_3, three surfaces of orders n_i ($i = 1, 2, 3$) which pass through a non-singular curve ${}^pC^n$ will in general meet elsewhere in a finite number N of points, given by the formula (Ch. IX, § 1.4)

$$N = n_1 n_2 n_3 - n(n_1+n_2+n_3-4)+2p-2.$$

We may show that, for given n and p, it is not in general possible to choose n_i so that $N = 0$. Consider, for example, the curve ${}^1C^5$; since it does not lie on a quadric, we may write $n_i = m_i+3$ ($m_i \geqslant 0$). We thus obtain

$$N = m_1 m_2 m_3 + 3(m_2 m_3 + m_3 m_1 + m_1 m_2) + 4(m_1+m_2+m_3)+2.$$

Since this expression is always positive, it follows that a space curve requires in general a system of four equations for its representation.

2. *Monoidal representation of space curves.* Prove that any space curve, situated generically with regard to the fundamental tetrahedron, lies on a surface having an equation of the form $t\phi-\psi = 0$, where ϕ, ψ are homogeneous polynomials in the coordinates x, y, z. Show that this surface is rational, and obtain its plane representation.

Prove also that, if the given curve is of order n, then ϕ and ψ may be assumed to be of orders $n-3$ and $n-2$ respectively.

3. *Parameters of a curve in* $[r]$. By means of Theorem XXXIII it can be shown that *the non-special curves* ${}^pC^n$ *of* $[r]$ ($n \geqslant p+r$) *depend on* $(r+1)n-(r-3)(p-1)$ *distinct parameters.* The proof rests upon the following considerations:

(i) All birationally equivalent curves ${}^pC^n$ of $[r]$ are obtained from one and the same curve by selecting on it a g_n^r; and since, by hypothesis, $n \geqslant p+r$, each set of n points on the curve defines a unique complete g_n^{n-p}. The number of parameters on which the latter depends is $n-(n-p) = p$.

(ii) The freedom of series g_n^r contained in a given g_n^{n-p} is that of spaces $[r]$ in $[n-p]$, that is, $(r+1)(n-p-r)$. (Ch. I, § 1.2.)

(iii) Each g_n gives rise to $\infty^{(r+1)^2-1}$ projectively equivalent curves.

(iv) Finally, each birationally distinct curve depends on $3p-3$ moduli ($p > 1$). Thus, if $p > 1$, the number N of distinct parameters is

$$N = p+(r+1)(n-p-r)+(r+1)^2-1+3(p-1)$$
$$= (r+1)n-(r-3)(p-1).$$

Ex. Prove that this formula holds also for $p = 0$, $p = 1$, and, in the case $p = 0$, deduce the result from the generation of a rational normal curve by the intersection of projectively related pencils of primes.

4. *The postulation of a curve.* Let C be a non-singular curve of order n and genus p, situated in $[r]$. If the primals of order m cut a non-special series on C, its postulation for these primals will be at most $mn-p+1$;

and it will be precisely this if the series cut out is complete. In this connexion we note the following theorems, due to Castelnuovo†:

The primals of order $m \geqslant n-2$ *cut a complete non-special series on* C.

The primals of order $m \geqslant \frac{n-r}{r-1}$ *cut a non-special series on* C.

It follows from the first theorem that the postulation is exactly $mn-p+1$ for all $m \geqslant n-2$, and, from the second, that it is *at most* $mn-p+1$ for $m \geqslant \frac{n-r}{r-1}$.

Ex. 1. Show that the postulation of a rational normal C^n for quadrics is $2n+1$.

Ex. 2. Show that the postulation of a general ${}^4C^6$ for the quadrics of [3] is precisely 9.

5. *Maximum genus of a curve.* Using similar methods to those employed in the proof of the last two theorems, Castelnuovo‡ has obtained a formula for the maximum genus p of a curve containing a g_n^r, or, what is the same thing, of a curve of given order n in [r]; his result is that

$$p \leqslant \chi\{n-r-\tfrac{1}{2}(r-1)(\chi-1)\},$$

where χ is the least integer not less than $(n-r)/(r-1)$. In the case $r = 3$, this formula gives $p = (n-2)^2/4$ or $(n-1)(n-3)/4$, according as n is even or odd. In the first case, the curve is the complete intersection of a quadric with a surface of order $n/2$ and, in the second, it is the intersection of the quadric with a surface of order $(n+1)/2$, residual to a generator.

For $r > 3$ and $n \geqslant 2r$, the curve lies on $\binom{r-1}{2}$ linearly independent quadric primals; when $n = 2r$, we have a canonical curve of genus p.

Ex. Verify that the curves of the stated orders, lying on a quadric surface, have maximum genera. Also determine the maximum genus of a curve of given order on a quadric.

6. *The projective characters of a curve. Classification of curves.* The systematic classification of space curves, which had hitherto been carried only as far as those of order 6, was first attempted by Noether§ and Halphen.‖ Their work reveals the following facts:

(1) For a given value of the order n there does not in general exist a *non-singular* curve for every value of the genus p between zero and the maximum.†† Thus, consider the case $n = 9$, $p = 11$: a curve with these characters could not lie on a quadric and, since cubic surfaces would cut on it a non-special series g_{27} of freedom not exceeding 16, the curve would be the complete intersection of at least ∞^2 cubic surfaces; which is impossible.

(2) There exist distinct families of curves, of equal generality, having the same characters n and p. Consider, for example, a curve for which

† *Rend. Palermo*, **7** (1893), 89; *Memorie scelte*, 95.

‡ *Atti Accad. Torino*, **24** (1889), 346; *Memorie scelte*, 19.

§ *Berlin Akad. Abh.* (1882), 1. ‖ *Journ. École Polytechnique*, **52** (1882), 1.

†† Although Comessatti (*Atti Ist. Veneto*, (8) $\mathbf{17}_2$ (1915), 1685) has shown that there exist *nodal* curves of every genus between these values.

$n = 9$, $p = 10$; through it there pass at least ∞^1 cubic surfaces so that, if the curve does not lie on a quadric, it is the complete intersection of two cubic surfaces. If, however, it lies on a quadric, we see that it is the intersection of the quadric with a sextic surface, residual to three skew lines. In the first case the series cut on the curve is the canonical series, while in the second it is non-special. Both types of curve depend on 36 parameters: the first, because there are ∞^{36} pencils of cubic surfaces, and the second, because every quadric contains ∞^{27} such curves, as we find from the plane representation of the surface.

It is, however, possible to distinguish between these curves by introducing an additional projective character; thus, if ν denotes the minimum order of a cone which contains the $\frac{1}{2}(n-1)(n-2)-p$ chords of the curve issuing from a generic point, Halphen finds that $\nu = 4$ for the first type and that $\nu = 5$ for the second.

(3) Halphen has shown that even the additional character does not suffice to distinguish in general between the various types of curves; for example, in the case $n = 15$, $p = 28$, $\nu = 9$, there exist two distinct families of curves which require a fourth character for their specification. It thus seems likely that no finite number of projective characters will prove sufficient for the classification of space curves.

7. *Limiting forms of curves in* $[r]$. Further progress in the classification of curves has been made by Severi,† by considering the degenerate forms of a curve, in particular those forms which consist of n lines having a suitable number of simple intersections (connected polygons). Severi shows that every family of non-special irreducible curves in $[r]$ of order n and genus p contains as limiting forms all possible types of connected n-gon with $n+p-1$ vertices. This result has not, however, been extended to special curves, except for certain restricted ranges of n. There is, moreover, a further difficulty: two non-isomorphic n-gons may each be a limiting form of one and the same irreducible curve, so that the classification into families by means of these polygons is not readily achieved.

8. *The postulation of a multiple curve.* Let ${}^pC^n$ be a non-singular curve in [3], and suppose that it is required to find the postulation of ${}^pC^n$ as an i-fold curve for surfaces of a given order m. By using an inductive method Campedelli‡ has shown that the postulation, for all sufficiently large values of m, is

$$\frac{i(i+1)}{6}\{(3m-4i+4)n-(2i+1)(p-1)\}.$$

By assuming that ${}^pC^n$ can degenerate into an n-gon with $n+p-1$ vertices, Todd§ has extended this result to primals of order m in any space $[r]$ which are to contain as i-fold curve a given non-singular ${}^pC^n$ of the space. The virtual postulation is found to be

$$\frac{1}{r}\binom{i+r-2}{r-1}\{[mr-(i-1)(r+1)]n-(2i+r-2)(p-1)\}.$$

† Severi–Löffler, *Vorlesungen*, Anhang G. ‡ *Rend. Palermo*, **55** (1931), 198.
§ *Proc. Camb. Phil. Soc.* **36** (1940), 27.

B. Segre† has subsequently obtained the above formula without recourse to a degeneration argument.

† Ibid. **38** (1942), 368.

BOOKS RECOMMENDED FOR FURTHER READING

BAKER, *Principles of geometry*, v.
COOLIDGE, *Algebraic plane curves*, bk. iii.
ENRIQUES–CHISINI, *Teoria geometrica delle equazioni*, iii.
SEVERI, *Trattato di geometria algebrica.*
SEVERI–LÖFFLER, *Vorlesungen über algebraische Geometrie*, chs. i–vi.
VAN DER WAERDEN, *Algebraische Geometrie.*

CHAPTER XIII

GEOMETRY ON AN ALGEBRAIC SURFACE

THE subject of the present chapter is a very large one indeed, and the literature behind it, extending over most of a century, is both vast and varied in its many different methods of approach. In the limited space at our disposal, we shall attempt no more than an introduction to basic ideas of which some are foreshadowed in earlier chapters of this book, and we hope that this introduction will make it easier for the reader to approach the many classical memoirs which give the main results already achieved. In the first part of the chapter we show how the concepts of linear series and linear equivalence on a curve may readily be extended to surfaces, or indeed to algebraic manifolds of arbitrary dimension, and we exhibit as clearly as we can the fundamental novelty of exceptional curves arising in birational transformations of one surface into another; and in the rest of the chapter we give some account of the canonical system of a surface and of some simple types of surface singularities.

§ 1. CURVE SYSTEMS

1. Preliminary observations. In investigating the invariant geometry of an algebraic surface F, we are concerned, for the most part, with properties of systems of curves on F. Many such properties are immediate extensions of corresponding properties of series of sets of points on a curve C; and where this is the case we shall often content ourselves with a formal statement of results, leaving the reader to make the necessary modifications of the proofs given in Ch. XII.

In certain fundamental respects, however, the theory for surfaces is radically different from that for curves. Thus, for example, we were able to prove quite simply that a plane curve with arbitrary singularities could be transformed, by a succession of quadratic transformations, into one with only ordinary singularities; and hence (Ch. XII, § 5) into a non-singular space curve. For surfaces, on the other hand, the problem of transforming a surface with arbitrary singularities into a non-singular surface is one of extraordinary complexity and quite outside our present scope.†

† For the history and final solution of this problem, see Zariski, *Algebraic surfaces*, ch. i, § 6 and *Ann. of Math.* (2) 43 (1942), 583.

We are compelled, therefore, to limit our discussions, for the most part, to non-singular surfaces or to surfaces which possess only *normal* singularities, i.e., in ordinary space, to surfaces which possess at most a double curve with a finite number of triple points, triple also for the surface.

We shall, however, be able to include—at a later stage—surfaces which possess certain simple types of *resoluble singularities*, i.e. singularities which we know how to remove by a finite sequence of simple resolutions. The question as to whether any surface singularity is resoluble in this sense is still open, even though the general problem of removing the singularities of a surface, by other methods, has now been solved.

1.1. ***Algebraic curve systems on a surface.*** If an algebraic curve C, lying on a surface F, is defined by a system of equations E in which some of the coefficients are allowed to vary, subject to a set of algebraic relations E', then the equations E and E' together define an algebraic system (C) of curves on F. From the relations E' there can be constructed an algebraic image-manifold U of the system, such that the curves of (C) are in birational correspondence with the points of U. The manifold U may, of course, be reducible and may consist even of irreducible components of different dimensions; we introduce therefore the following definitions:

(i) (C) is a *continuous* system, if U is connected, so that every pair of points of U can be joined by a continuous path on U;

(ii) (C) is *irreducible*, if U is irreducible;

(iii) (C) has *dimension* ρ, equal to that of U.

Thus, for example, if F is an ordinary quadric surface, the totality of straight lines (curves of order unity) on F is the sum of two disconnected irreducible systems—the λ-generators and the μ-generators—while the totality of curves of order two on F consists of three irreducible systems, namely, those containing all plane sections, all pairs of λ-generators, and all pairs of μ-generators, these systems being of dimension 3, 2, 2 respectively.

Besides the dimension ρ, defined above, the general irreducible algebraic system (C) on F has in general three further fundamental invariant characters which are

(*a*) its *grade* γ, which is the number of intersections (supposed finite) of a generic pair of curves of (C);

(*b*) its *genus* π, which is the genus of a generic irreducible curve of (C);

(*c*) its *index* λ, which is the number of curves of (C), properly reckoned, which pass through ρ generic points of F.

A *pencil* of curves of F is any irreducible algebraic system for which $\rho = \lambda = 1$. If F is ruled, its generators evidently form a pencil.

An *isolated curve* of F is one which belongs to no continuous system of positive dimension. Thus, for example, each of the 27 lines on a general cubic surface is isolated.

1.2. *Linear systems of curves.* The preceding definitions apply in particular to linear systems of curves on F, which we define as follows:

DEFINITION. A *linear system* $|C|$ on a surface F, immersed in space S_k, of dimension $k \geqslant 3$, is a system of curves C cut on F, residual usually to a fixed curve Γ, by the primals of a linear system Φ of S_k.

In a manner exactly analogous to that used in Ch. XII, § 1.3, it may be shown that the linear system cutting $|C|$ on F can always be so chosen that each curve of $|C|$ is cut on F by a unique primal of Φ; that, when Φ, so chosen, has freedom r, $|C|$ is also of freedom (dimension) r and is representable linearly on the points of a space S_r; and that $|C|$ is such that one and only one of its curves passes through r generic points of F.

If Φ has base points or base curves on F, other than the assigned curve Γ, then such points or curves will give rise to *base points* (simple or multiple) or *base curves* (simple or multiple fixed components) of $|C|$.

1.3. *Linear equivalence on a surface.* In the development of the theory of linear curve-systems on a surface, we can take over, with little more than the appropriate verbal modifications,

(*a*) the whole theory of rational functions on a curve and the deductions made from it (Ch. XII, § 1.31);

(*b*) the theory of complete linear series and the combinatory properties of linear series under the relation of equivalence (Ch. XII, § 1.31).

Thus, in regard to (*a*), we note that a rational function $\mathscr{R}$ (ratio of polynomials) of a variable point P of a surface F has a *curve of zeros* and a *curve of poles*, as also, more generally, a set of *curves* *of constant level* on F; that these latter form a *linear pencil* of curves on F, every linear pencil being capable of such a characterization; and, more generally, that any linear system of curves on F can be regarded (with suitable conventions) as the set of level curves of a linear system $\sum_1^r \lambda_i \mathscr{R}_i$ of rational functions, where the λ_i are arbitrary parameters. The theorems for surfaces, analogous to Theorems IV and V for curves, are as follows:

THEOREM I. *In order that a simply-infinite algebraic system of curves may be a linear pencil it is necessary and sufficient that it should be* (i) *rational,* (ii) *of index unity.*

THEOREM II. *In any birational transformation of a surface, linear curve-systems transform always into linear curve-systems.*

The general truth embodied in this last theorem requires, and will receive in the sequel, careful interpretation in relation to fundamental points of the transformation.

Under the heading (*b*) above referred to we have

THEOREM III. *Every curve C on a surface F belongs to a unique complete linear system (possibly of conventional freedom zero) on F.*

This theorem, as in Ch. XII, § 1.5, is the basis of a relation of (linear) *equivalence* of curves on F, by which we write $C \equiv D$ whenever C and D belong to the same complete linear system of curves on F. Also two linear systems $|C|$ and $|D|$ have a unique complete *sum-system* $|C+D|$, and the relation of equivalence is additive with respect to this operation.

We note, finally, that from the class of all effective curves of F we may derive, exactly as in Ch. XII, § 1.5, a wider class of *virtual curves* of F, defined as formal differences $C-D$ of effective curves; this contains, in particular, the unique *null-curve* $C-C$ of the surface. In this wider class, in which subtraction as well as addition is a universal operation, the relation of equivalence is still a well-defined relation, and each totality of mutually equivalent virtual curves is called a *system of equivalence* of (virtual) curves of F. Any such system may or may not have a core of effective curves, forming a complete linear system.

1.4. ***Simple, complex, and compound linear systems.*** By a method exactly analogous to that used (Ch. VI, §§ 2, 4) for plane systems of curves, we may distinguish different types of linear systems $|C|$ on a surface F by reference to the projective models of such systems. Thus if $|C|$ is of freedom $r \geqslant 2$ and is cut on F by primals of the system Φ whose equation is

$$\Phi \equiv \lambda_0 \phi_0 + \lambda_1 \phi_1 + \ldots + \lambda_r \phi_r = 0,$$

then the equations $$\frac{X_0}{\phi_0} = \frac{X_1}{\phi_1} = \ldots = \frac{X_r}{\phi_r}$$

transform F into a locus F^* which is the projective model of $|C|$ on F; and the character of the representation of F on F^* specifies the type of $|C|$. In particular

(a) $|C|$ is *simple* if the correspondence between F and F^* is birational,

(b) $|C|$ is *complex* if F is in $(\mu, 1)$ correspondence with F^*, where $\mu > 1$, and

(c) $|C|$ is *compound* if F^* is a curve instead of a surface, each point of F^* corresponding to a curve of F.

If C is simple and of grade γ, then F^* is a surface of order γ, and this surface is normal if $|C|$ is complete. If $r = 2$ then $|C|$ must be homaloidal ($\gamma = 1$), and any surface containing such a net is necessarily rational.

If $|C|$ is complex, then F^* is of order γ/μ, and the points of F^* are in birational correspondence with the sets of an involution I_μ on F (cf. Ch. VI, § 4), and $|C|$ is said to be *compounded* of I_μ. Any surface which is not rational and which yet contains a rational I_2 is said to be *hyperelliptic*; it is representable on a double-plane with a branch-curve whose points correspond to coincident point-pairs of I_2.

If $|C|$ is compound (and free from fixed components), then all curves of $|C|$ which pass through a generic point of F contain as component a whole curve, L say, through this point, and all the curves L which so arise form a pencil (L), rational or irrational, on F. In this case each curve of $|C|$ consists of a set of curves of (L), in number ν say; the projective model F^* of $|C|$ is a curve of order ν, whose points are in birational correspondence with the curves of (L); and the system $|C|$ is said to be *compounded of the pencil* (L) on F. The simplest example of such a linear system of curves is that provided by any linear 'series' of generators of a ruled surface (cf. Exx. 2, 3, 4 below).

1.5. *Adjoint surfaces.* If F is a surface with only normal singularities in ordinary space, then any surface through the double curve D of F is said to be *adjoint* to F; this is a particular case of more general relations of subadjointness and adjointness to which we refer later (§ 9.2). By a straightforward generalization to surfaces of Noether's $Af+B\phi$ Theorem and by the method of Ch. XII, § 4, it may be shown that adjoint surfaces have the following property:

Adjoint surfaces of any given order intersect F, residually to D, in a COMPLETE *linear system of curves.*

It follows, of course, that the complete system $|C|$ containing any given curve C on F may be constructed by means of adjoints of any sufficiently high order which contain a fixed curve A residual to C.

Remarks and Examples

1. *Linear conditions on curves of a linear system.* If $|C|$ is a linear system cut on F by a linear system of primals Φ, we envisage a linear condition, or set of linear conditions, applicable to curves of $|C|$ as derived, in the first instance, from a linear condition or conditions applied to primals of Φ. But the results, as they affect $|C|$, are so similar to those obtaining in regard to linear systems of curves in a plane that we can regard the conditions as being imposed directly on $|C|$ instead of on Φ.

To justify this we may consider the linear system, $|L_N|$ say, cut on F by all primals Φ, of order N. This has, as its projective model, a surface Ω_N in space S_ν, of dimension ν given by

$$\nu = \binom{N+k}{k} - 1 - \sigma_N,$$

where $\sigma_N - 1$ is the freedom of N-ic primals through F; and every linear system $|C|$ which can be cut on F by N-ic primals has a projective model F^* which is a projection of Ω_N from a suitable linear space Π.

In particular, to any point O of S_ν, regarded as a condition on primes, there corresponds a simple linear condition on L_N, this condition assuming the character of a simple assigned base-point if O lies on Ω_N; and if $|C|$ is the linear sub-system of $|L_N|$ obtained by imposing the condition in question, its projective model F^* is the projection of Ω_N from O.

Similarly if $|C|$ is obtained by imposing an s-fold base point O_1 on L_N, Π is the s-tangential space of Ω_N at the corresponding point O of this surface (cf. Ex. 5 below); and F^* is the projection of Ω_N from Π.

Again, if $|C|$ is obtained by imposing a base curve Γ_1 on L_N, Π is the least (linear) space containing the corresponding curve Γ of Ω_N.

In the general case the vertex Π is the least space containing all the separate vertices corresponding to the various individual conditions imposed.

2. *Generation of compound systems.* If (L) is any pencil of irreducible

curves (irreducible pencil) of F whose members are mapped by points of a curve Γ, then *the sets of ν curves of (L) which correspond to sets of a linear series on Γ form a linear system of (composite) curves of F.*

To prove this it is sufficient to observe that any rational function of a point varying on Γ defines a corresponding rational function on F whose value is constant on each curve of (L). The composite curves C are clearly the level curves of a linear system of such rational functions on F. As a particular case we note the result:

Any linear series of sets of generators on a ruled surface is a linear system of (composite) curves of the surface.

3. *Reducible pencils.* If (H) is a pencil of sets on the curve Γ of the previous example, then (H) defines a *reducible pencil* of curves of F. A linear series on Γ which is compounded of (H) defines a linear system on F which is compounded directly of the reducible pencil and only indirectly of the basic irreducible pencil (L).

4. *System compounded of a linear pencil.* If $\phi+\lambda\phi' = 0$ is the equation of a linear pencil $|L|$, without fixed components, then the most general linear system compounded of curves of $|L|$ has an equation of the form

$$(\lambda_0,\ldots,\lambda_r)(\phi,\phi')^r = 0.$$

5. *Tangential space of a surface at a point.* If F is a surface in S_k, the homogeneous coordinates x_i $(i = 0,\ldots,k)$ of a variable simple point P of F may be taken to be analytic functions $x_i(u,v)$ of two parameters u, v in a limited region R of F; and we may write

$$P \equiv \{x_i(u,v)\}.$$

In R a relation of the form $\qquad \psi(u,v) = 0,$

where ψ is analytic, in general defines a curve on the surface; and this curve has an s-fold point at P if ψ and its derivatives of all orders up to those of the $(s-1)$th vanish at P.

Applying this to the section of F by the prime π whose equation is $\sum\limits_0^k a_i x_i = 0$, we take

$$\psi(u,v) = \sum_0^k a_i x_i(u,v),$$

and we find that the section has an s-fold point at P if π contains all the points

$$P,\quad \frac{\partial P}{\partial u},\quad \frac{\partial P}{\partial v},\quad \frac{\partial^2 P}{\partial u^2},\quad \frac{\partial^2 P}{\partial u\partial v},\quad \ldots,\quad \frac{\partial^{s-1} P}{\partial v^{s-1}}.$$

If s is small enough to allow these $\frac{1}{2}s(s+1)$ points to be in a proper subspace of S_k, they define a space $\Pi^{(s)}$ (of dimension $\frac{1}{2}(s+2)(s-1)$ in general) which we call the *s-tangential space of F at P.* Hence:

A necessary and sufficient condition that a prime should meet F in a curve having an s-fold point at a simple point P of the surface is that the prime should contain the s-tangential space of F at P.

Evidently $\Pi^{(1)}$ is P itself; $\Pi^{(2)}$ is the tangent plane at P; and generally $\Pi^{(s)}$ contains the complete neighbourhoods of P on F of every order up to the $(s-1)$th. Also $\Pi^{(s)}$ contains the osculating $[s-1]$ at P of every curve on F through P and can be defined by this property.

From the fact that the osculating $[s-1]$'s to a curve at consecutive points P, P' meet in the osculating $[s-2]$ to the curve at P we deduce the result:

The s-tangential spaces at consecutive points P, P' of a surface F meet in the $(s-1)$-tangential space of F at P or in some space containing it.

6. *Bertini's Theorem.* By applying the preceding theorem to the projective model of a linear system $|C|$, of freedom $r \geqslant 2$ and not compounded of a pencil—restrictions which are easily removed afterwards—we may show that:

If any algebraic simply-infinite curve-system (C), immersed in a linear system $|C|$, is such that consecutive curves C, C_1 of (C) have consecutive s-fold points P, P_1 respectively, then the 'tangent' linear pencil to (C) at C has P as $(s-1)$-fold base point.

From this we may deduce

Bertini's Theorem. The generic curve of a linear system has no multiple points which are not base points of the system (cf. Ch. VI, § 1.2).

As regards the restriction $r \geqslant 2$ imposed above, we need only remark that any pencil, when augmented by a suitable fixed component, can be regarded as immersed in a linear system of greater freedom; and the previous proof then applies.

7. An important corollary of Bertini's Theorem states that

Any linear system $|C|$, without fixed components, whose generic curve is composite is compounded of a pencil.

§ 2. Exceptional Curves

2. One of the main difficulties in the theory of birational transformations of surfaces, as compared with the corresponding theory for curves, is that in any birational transformation of a surface F into a surface F' there may be (and usually are) certain points of F which transform exceptionally into curves of F' and certain curves of F which transform into points of F'. The existence of these fundamental points and curves, at which the general (1, 1) character of the transformation breaks down, makes it difficult to regard F and F' as, in any ordinary sense, abstractly identical; but if we wish to pass freely from geometry on F to geometry on F', it is essential to have some systematic interpretation of the phenomena which will enable us to use a common formal symbolism for both surfaces.

The general mechanism of birational transformations of one surface F into another surface F' has already been explained and amply illustrated for the case when the surfaces are rational (cf. Chs. VI and VII); and since this work, so far as it goes, is typical of birational surface transformations in general, we can take it as a starting-point for the analysis to follow.

2.1. ***Birational transformations.*** We suppose in the first place that we are dealing with the case in which F and F' are both non-singular surfaces. The surface F' is then the projective model of some simple linear system $|C|$ on F; and we suppose—for the sake of simplicity—that $|C|$ possesses only a finite number of distinct base points O_i of multiplicities k_i ($i = 1,..., s$). In the same way F is the projective model of a simple linear system $|C'|$ on F', whose base points O'_j we suppose to be distinct and of multiplicities k'_j ($j = 1,..., t$). Then

(i) the neighbourhood of O_i on F ($i = 1,..., s$) transforms into a rational fundamental curve E'_i, of order k_i, of $|C'|$, i.e. E'_i has no variable intersections with the curves of $|C'|$ and imposes only one condition on curves of this system which are required to have it as a component,

(ii) the neighbourhood of O'_j on F' ($j = 1,..., t$) corresponds likewise to a rational fundamental curve E_j, of order k'_j, of $|C|$,

(iii) the two sets of fundamental curves, E_j and E'_i, are all of index 1, i.e. each E_j has one free intersection with its residual curves in $|C|$, and each E'_i has one free intersection with its residual curves in $|C'|$.

We say, then, that any curve of F (such as E_j) which can be transformed into the neighbourhood of a simple point of a birationally equivalent surface F' is an *exceptional curve* of F.

To sum up then we may say that any birational correspondence between two non-singular surfaces F and F' may be expected to possess a certain number of fundamental points O_i on F which correspond to exceptional curves E'_i on F', and a further set of fundamental points O'_j on F' which correspond to exceptional curves E_j on F. This leads to a classification of all such transformations into

(*a*) *unexceptional transformations*, in which there are no fundamental points O_i or O'_j;

(*b*) *transformations of the first kind*, in which none of the curves E_j passes through any of the points O_i, and none of the curves E'_i passes through any of the points O'_j;

(*c*) *transformations of the second kind*, in which the condition of the previous case is not satisfied.

The significance of this classification rests on a remarkable theorem, which we do not prove here, to the effect that the only surfaces which admit birational transformations of the second kind are those which are either rational or birationally transformable into ruled surfaces.

2.2. *Resolution of a point of F.* In order to study exceptional curves, it is convenient to define a standard operation by which any one given point of F may be transformed into an exceptional line of a surface F', while the correspondence between F and F' is otherwise unexceptional. *Such a standard resolution of a point O of F is defined by taking F' to be the projective model of the totality of curves in which F is met by quadrics*† *through O.*

By this transformation, namely, the neighbourhood of O is transformed into a line E of F', and we say that O is *resolved* into E.

If we consider now the more general birational transformation of the preceding section, with sets of fundamental points O_i on F and O'_j on F', the surfaces, F_1 and F'_1 say, which are obtained by resolving successively all the points O_i and all the points O'_j respectively, will plainly be in *unexceptional* birational correspondence. Thus any birational transformation of the straightforward type defined in § 2.1 can be reduced to an unexceptional birational transformation by resolving a finite number of points of each of the surfaces concerned. This will be found to be a useful result.

2.3. *The invariant transform of a curve.* Let us consider now in more detail the consequences of resolving a point O of F into an exceptional line E of F'.

If a curve C of F passes simply through O, then its *ordinary (proper) transform* is a curve $\bar{C}$ which meets E in the point corresponding to the direction of C at O. But if C is a member of any continuous system (C) whose generic member does not pass through O, then its *invariant transform*, as a member of the corresponding system on F', is the composite curve $C' = \bar{C}+E$. The order of $\bar{C}$, in fact, is less by one than that of the transform of a generic curve of (C). In the same way, if C has a λ-fold point at O, its invariant transform is $C' = \bar{C}+\lambda E$.

† The dimension of the space containing F' will naturally be very much higher than that containing F; but this is irrelevant, and in any case F' can be projected generally into [5] without any danger of complications. Also the use of quadrics, instead of primals of any other order $n > 2$, has no special significance.

To reconcile the two aspects, we make a distinction between *projective curves* of F, which are curves in the ordinary sense, and *invariant curves* of F, which are neighbourhoods of projective curves, i.e. curvilinear origins of curve-branches on F. In the above example, then, since every branch on F which has its origin on C transforms into a branch on F' which has its origin either on $\bar{C}$ or on E, the composite invariant curve $\bar{C}+E$ is seen to be the actual transform of the invariant curve C. The neighbourhood of O, then, which we denote by $\tilde{O}$, is to be regarded, in the invariantive sense, as a *latent curve* of F which is transformed to the actual (invariant) curve E by resolution of O.

The curve $C-\tilde{O}$, which transforms into the (proper) invariant curve $\bar{C}$ of F', is an example of what we call an *improper curve* of F.

2.4. *Fictitious points of F.* The points of E, as we have seen, correspond to directions at O, in the sense that all curve-branches on F' which originate in a point O_1 of E correspond to curve-branches of F which originate in O and have a fixed tangent at O. It is convenient then to regard every point such as O_1 as corresponding to a *fictitious point*, usually denoted by the same symbol O_1, in the first neighbourhood of O on F. (It is a little unfortunate that the word neighbourhood has to be used here in two quite distinct senses, (a) as a totality of curve-branches, and (b) as a class of fictitious points of a certain rank.) The neighbourhood $\tilde{O}_1$ of O_1 on F', which is latent in the *first degree* on F', corresponds to the totality of curve-branches through O and O_1 on F; we call this totality the neighbourhood of O_1 on F, and it is to be regarded as an invariant curve which is latent *in the second degree* on F. By resolving the point O_1 of F' into a line E_1 of a surface F'', we obtain a realization of the doubly latent curve $\tilde{O}_1$ of F.

Plainly, now, the above process may be extended indefinitely so as to define a strict hierarchy of formal neighbourhoods of O on F, the first consisting of ∞^1 fictitious points O_1, the second of ∞^2 fictitious points O_2 of which each belongs to the first neighbourhood of a definite O_1, the third consisting of the ∞^3 fictitious points O_3 in the first neighbourhoods of the points O_2, and so on; each point O_r of the rth neighbourhood has a unique train of *ancestors* $O_{r-1}, O_{r-2},..., O_1, O$, and it is representable by an actual point of the surface $F^{(r)}$ obtained by successively resolving $O, O_1, ..., O_{r-1}$. The neighbourhood (in the other sense) of O_r is an invariant curve $\tilde{O}_r$ which is latent in the $(r+1)$th degree on F and is first realized as an exceptional line E_r on a surface $F^{(r+1)}$. If O_r is a *generic* point of the rth neighbourhood, then $\tilde{O}_r$ represents a totality of curve-branches of F which have $(r+1)$-point contact at O; but every neighbourhood of O after the first contains *special* points (of ever-increasing complexity) whose neighbourhoods represent totalities of singular (non-linear) branches at O (cf. Notes and Examples at the end of § 2.6).

2.5. *Structure of an invariant curve.* It is important to realize that an invariant curve C on F is not merely the 'sum' of the neighbourhoods of all its points. If O is any simple point of C, then there is a definite consecutive sequence of points $O, O_1, O_2,...$ on C, such that the generic curve-branch through $O, O_1,..., O_r$ has $(r+1)$-point contact with C at O. We observe then that *the latent curve* $\tilde{O}_r$ $(r = 1, 2,...)$ *is contained r-fold in C.* For if $O, O_1, O_2,...$ are successively resolved, the successive invariant transforms of C are composite curves of the forms

C_1+E, where E and C_1 meet in O_1,

$(C_2+E_1)+(E'+E_1)$, where E_1 and C_2 meet in O_2,

$(C_3+E_2)+2(E_1'+E_2)+E''$, where E_2 and C_3 meet in O_3,

and so on, and the result follows at once.

The above result implies that the subtractions indicated in a symbol of the form $C-\tilde{O}-\tilde{O}_1-\tilde{O}_2-...-\tilde{O}_r$ can be regarded as effective; the symbol, in fact, represents an actual (though improper) curve of F which is transformed into the proper (irreducible) homologue of C on the surface $F^{(r+1)}$ obtained by resolving successively the points $O, O_1,..., O_r$.

Similarly, if O is a λ-fold point of C, then $\tilde{O}$ is contained λ-fold in C; and the improper curve $C-\lambda\tilde{O}$ of F transforms into the proper homologue of C on F'.

To sum up we may say that all possible proper curves on all birational transforms of F are actual transforms, in the invariantive sense, of proper or improper or latent curves of F. In this limited sense, then, we may regard all birational transforms of F as abstractly identical with F.

2.6. *Linear systems and their bases.* We begin by defining a *free* linear system on a surface as one which has no fixed components and no base points.

If $|C|$ is any linear system of positive freedom on a surface F, which has base points O_i of multiplicities k_i (some of which may be fictitious) but no (proper) fixed components, then $|C|$ may be regarded as having a set of *latent fixed components* $\sum k_i \bar{O}_i$. If we discard these, we are left with a linear system

$$|C_1| = |C - \sum k_i \bar{O}_i|$$

—improper on F—which is a proper free linear system on the surface F' obtained by resolving successively all the points O_i; and it is often convenient to replace $|C|$ by this reduced system $|C_1|$.

To represent symbolically systems with an *assigned base*, we proceed as follows. We regard any complete linear system $|C|$ as being *virtually free*, and its base elements—if it happens to possess any in spite of being complete—as virtually inexistent. Those curves of $|C|$ which contain an assigned base Θ—assigned points and fixed components to assigned multiplicities—are said to form a linear system, $|\bar{C}|$ say, which is *complete relative to the base* Θ. The virtually free reduced form of $|\bar{C}|$, shorn of its assigned proper and latent base curves, is $|C-\Theta|$, and we may express this symbolically by writing

$$|\bar{C}| = |C-\Theta|+\Theta.$$

The system $|C-\Theta|$ is often used in place of $|\bar{C}|$. It should be remembered, of course, that $|\bar{C}|$, by being made to contain Θ, may have acquired additional base elements, either in the form of extra multiplicity (beyond the assigned) at base elements of Θ or entirely new base elements; but these again are to be regarded as virtually inexistent.

Notes and Examples

1. If a point O $(= O_0)$ of F is resolved into E on F', and O_1 on E (representing a fictitious point O_1 of F) is then resolved into E_1 on F'', the proper (irreducible) transform of E on F'' is a curve e, and this meets E_1 in a point which we may denote by $O_2^{(0)}$. This represents a special *satellite point* $O_2^{(0)}$ in the first neighbourhood of O_1 on F, distinguished from all the ordinary points O_2 of this neighbourhood by the fact that any linear curve-branch of F'' with $O_2^{(0)}$ as origin corresponds to an ordinary *cuspidal branch* on F, with OO_1 as cuspidal tangent (cf. Ch. III, § 2.2).

The composite curve $e+E_1$ represents the *total neighbourhood* $\bar{O}$, while e represents the *diminished neighbourhood* $\bar{O}-\bar{O}_1$. If $|C''|$ is the linear system on F'' representing prime sections of F, then the systems $|C''-e-E_1|$ and $|C''-e-2E_1|$ represent respectively sections of F by primes through O and by primes through O and O_1.

2. Show that in the first neighbourhood of the point $O_2^{(0)}$ of the previous example there are two satellite points $O_3^{(0)}$ and $O_3^{(1)}$ which lie respectively in the (irreducible) diminished neighbourhoods (on F''') of O and O_1; and show generally that any point O_r (except O itself) has either one or two satellites in its first neighbourhood according as it is a free point (non-satellite) or a satellite.

3. A simple sequence $O\ (= O_0)$, O_1, O_2,... on F is a sequence of points of which each, after the first, is in the first neighbourhood of its immediate ancestor (predecessor in the sequence). We may suppose the points to be successively resolved on surfaces F', F'',..., and we denote by $e^{(i)}, e_1^{(i)},..., e_{i-1}^{(i)}$ the irreducible curves which represent the diminished neighbourhoods of O, O_1,..., O_{i-1} on $F^{(i)}$, the last of these curves being the (undiminished) exceptional line E_{i-1}. We say then that O_i is *proximate* to O_j $(j < i)$ if and only if the corresponding point O_i of E_{i-1} lies on $e_j^{(i)}$.

Verify that any point O_i $(i > 0)$ is always proximate to its immediate ancestor O_{i-1}, and that it may be proximate (if it is a satellite point) to one remote ancestor O_α $(0 \leqslant \alpha < i-1)$; also that the points of the sequence which are proximate to any given point O_α are a set $O_{\alpha+1}$, $O_{\alpha+2}$,..., $O_{\alpha+s}$ $(s \geqslant 1)$ following without a break after O_α.

4. *Puiseux expansions.* In the neighbourhood of any singular point (x_0, y_0) of a plane curve C, the points of C lie on a finite number of *branches*, of which any one has a parametric representation of the form

$$x-x_0 = t^\mu, \qquad y-y_0 = a_0 t^\nu + a_1 t^{\nu+\nu_1} + \dots \quad (|t| < k),$$

where $\mu, \nu, \nu_1 \dots$ are positive integers with no common factor (cf. Ch. III, § 3.4). This representation is called a *Puiseux expansion* of the branch, and the *order* of the branch is the lesser of the numbers μ, ν (cf. Severi, *Vorlesungen*, Kap. ii, § 3, and Enriques–Chisini, *Teoria geometrica* . . ., ii, Libro quarto, cap. 1). The same result holds at any simple point O of a surface F, when O is a singular point of a curve C lying on F, and $x-x_0$, $y-y_0$ are suitable intrinsic parameters of a point of F near O.

By taking $x_0 = y_0 = 0$ and applying *resolutions*, of one or other of the forms $x' = x$, $y' = y/x$ and $x' = x/y$, $y' = y$ (cf. Ch. III, Exx. 5, 9), find the general Puiseux expansion of a branch which passes through the origin O, the point O_1 of the x-axis in the first neighbourhood of O, and which is primitive at one or other of the three satellite points $O_2^{(0)}, O_3^{(0)}, O_3^{(1)}$ referred to in Ex. 2. (A branch is *primitive* at a fictitious point O_i if its transform on $F^{(i)}$ is a linear branch in generic direction through O_i.)

[*Solution*: The three types of branch required are

(i) $x = t^2, \qquad y = a_3 t^3 + a_4 t^4 + \dots \quad (a_3 \neq 0),$

(ii) $x = t^3, \qquad y = a_4 t^4 + a_5 t^5 + \dots \quad (a_4 \neq 0),$

(iii) $x = t^3, \qquad y = a_5 t^5 + a_6 t^6 + \dots \quad (a_5 \neq 0).$]

5. Show that the general rhamphoid cuspidal branch

$$x = t^2, \qquad y = a_2 t^2 + a_4 t^4 + a_5 t^5 + \dots \quad (a_5 \neq 0)$$

is primitive at the satellite point in the first neighbourhood of an ordinary free point O_2 (following O, O_1).

§ 3. Characters of $|A+B|$

3. The three primary numerical characters of a linear system on a surface are its freedom, grade, and genus. If these have the respective values r_1, n_1, π_1 and r_2, n_2, π_2 for two complete systems $|A|$ and $|B|$, we may inquire how these numbers are related to the characters r, n, π of the complete sum-system $|A+B|$.

For the freedom r of $|A+B|$, we have only the inequality

$$r \geqslant r_1+r_2, \tag{1}$$

expressing that $|A+B|$ certainly includes every composite curve consisting of a curve of $|A|$ and a curve of $|B|$; and the equality sign is only attained if $|A+B|$ consists entirely of such curves.

The grade n of $|A+B|$ is given by

$$n = n_1+n_2+2i, \tag{2}$$

where i is the number of intersections, supposed finite, of a generic curve of $|A|$ with one of $|B|$. This follows at once by selecting two composite curves A_1+B_1 and A_2+B_2 and enumerating their intersections.

To find the genus π of $|A+B|$, we consider the more general problem of estimating the genus π of a non-singular irreducible curve C which tends continuously to a limit composed of two curves A, B, likewise irreducible and non-singular, which have genera π_1, π_2 and which intersect in i distinct points P_α $(\alpha = 1,..., i)$. We denote by F the surface generated by C; and we suppose, as we evidently may, that F lies in [3].

If h, h_1, h_2 are the numbers of chords that can be drawn to C, A, B from an arbitrary point O, and if N_1, N_2 are the orders of A and B, then

$$\pi_1 = \tfrac{1}{2}(N_1-1)(N_1-2)-h_1, \qquad \pi_2 = \tfrac{1}{2}(N_2-1)(N_2-2)-h_2,$$

$$\pi = \tfrac{1}{2}(N_1+N_2-1)(N_1+N_2-2)-h.$$

As C approaches $A+B$, the congruence of chords of C tends to a limiting congruence which clearly includes (i) chords of A, (ii) chords of B, and (iii) lines which meet A and B in distinct points, or which pass through a point P_i and lie in the tangent plane to F at this point. The exclusion of the totality of lines through P_i is based on the evident fact that any chord of C whose end points both tend to P_i can only tend to a tangent to F at P_i.

Now the number of chords from O to C remains the same as the limit is approached; and hence, plainly,

$$h = h_1+h_2+(N_1 N_2-i).$$

From the four equations written down, there follows at once the relation

$$\pi = \pi_1+\pi_2+i-1, \tag{3}$$

and this gives us, for the special case in which C varies in a linear system on F, the relation between the genera† of $|A|$, $|B|$, and $|A+B|$.

Clearly the relations (1), (2), (3) continue to hold for systems which are complete relative to a set of assigned base points; for they apply directly to the corresponding free systems got by resolving all the base points.

3.1. *Canonical number.* If C is any curve of the surface for which the complete system $|C|$ has a well-defined genus π and grade n, then the number $\vartheta = 2\pi-2-n$ is called the *canonical number* of C, and it has the same value for all curves equivalent to C. Its significance depends on the fact, easily deduced from (2) and (3) above, that if its value for $|A|$, $|B|$, and $|A+B|$ are ϑ_1, ϑ_2, and ϑ, then $\vartheta = \vartheta_1+\vartheta_2$. Hence:

The canonical number is an additive character of curves on a surface.

§ 4. Virtual characters of curves on F

4. A *virtual numerical character* of one or more curves on F (virtual or effective) is one whose value depends only on the systems of equivalence to which the curves belong and not on the curves themselves. The first such character we have to define is the virtual number of intersections of two curves A, B, a number which we shall denote by AB.

4.1. *Definition of AB.* If A and B are *effective* curves of F, even though either or both may be improper or latent, then we may suppose them transformed, by such resolutions as may be necessary, into proper curves of a birationally equivalent surface F'. On F' the complete systems $|A|$ and $|B|$ may be such that generic curves of $|A|$ and $|B|$ intersect in a finite number of points; and if so, we call this number the ***virtual intersection number AB***

† In a later section (§ 8.4) we shall give an independent proof of the formula for the genus of a composite curve.

of A and B. We have now to extend this definition to the case when A and B are not restricted in any way whatsoever.

We observe first that product symbols such as AB, in so far as they are already defined, obey the commutative and distributive laws of algebra. Thus in particular $A(B+C) = AB+AC$; and formula (2) of § 3 may be written in the form

$$(A+B)^2 = A^2+2AB+B^2. \tag{1}$$

For any pair of effective curves A, B to which the ordinary definition of AB may be inapplicable, we define AB as the value of

$$A(B+X)-AX,$$

where X is any curve such that both $A(B+X)$ and AX are already defined.† To justify this definition, we observe that if X' is any other curve such that $A(B+X')$ and AX' are defined, then $A(B+X+X')$ is also certainly defined (in the restricted sense); and

$$A(B+X+X') = A(B+X)+AX' = A(B+X')+AX,$$

whence $$A(B+X)-AX = A(B+X')-AX'.$$

Also if X is chosen such that X^2 is defined, then

$$(A+X)(B+X)-AX-BX-X^2$$
$$= A(B+X)-AX = B(A+X)-BX.$$

Having thus defined AB unambiguously for any pair of *effective* curves A, B, and in such a way that the product symbols concerned still obey the commutative and distributive laws, we extend our definition to *virtual* curves by assuming forthwith the complete universality of these laws. Thus if $A = A_1-A_2$ and $B = B_1-B_2$, where A_1, A_2, B_1, B_2 are all effective, we write

$$AB = A_1B_1-A_1B_2-A_2B_1+A_2B_2,$$

observing that AB, as so defined, is invariant as A, B vary in systems of equivalence on F. Hence:

Any two curves A, B on a surface have a unique virtual intersection number AB; and the product symbolism involved obeys the commutative and distributive laws.

Thus, for example, even an isolated curve C on F has a *virtual*

† In this and similar instances we leave it to the reader to prove that an auxiliary X with the required properties can always be assumed to exist.

grade C^2, equal to $(C+X)C-XC$ for any X such that $(C+X)C$ and XC exist in the restricted sense. More generally:

The virtual grade of any curve of the form $\sum \lambda_i C_i$, where the C_i are effective and the λ_i are integers, positive or negative, is given by

$$n = \sum \lambda_i^2 n_i + 2 \sum_{i \neq j} \lambda_i \lambda_j d_{ij}, \tag{2}$$

where n_i is the virtual grade of C_i and d_{ij} is the virtual number of intersections of C_i with C_j.

4.2. *Virtual genus* $\pi(C)$. We begin by defining the virtual genus $\pi(C)$ of *any* curve C of a complete irreducible linear system $|C|$, free from base points, as the actual (effective) genus of the *generic* curve of $|C|$. The actual genus of any particular curve C may be less than $\pi(C)$ by $\frac{1}{2}\sum k_i(k_i-1)$, where k_i is the multiplicity of a typical one of its multiple points; and C may also, of course, be composite so that it has no actual genus. We have now, as in the preceding case, to extend the scope of this restricted definition.

To cover the case of any *effective* curve C on F, we apply the formula (3) of § 3. For a pair of proper curves A, B, such that the complete systems $|A|$, $|B|$, $|A+B|$ are all irreducible, the formula gives

$$\pi(A+B) = \pi(A)+\pi(B)+AB-1,$$

it being assumed that generic curves of $|A|$ and $|B|$ meet in distinct points. We define the virtual genus $\pi(C)$ of any effective curve C as the constant value of

$$\pi(C+X)-\pi(X)-CX+1, \tag{3}$$

where X is any curve such that each term in this expression is defined in the restricted sense. To verify that this value is independent of X, we observe that if X, X' both give a meaning to the expression in question, then the complete system $|C+X+X'|$ is certainly also irreducible and

$$\pi(C+X+X') = \pi(C+X)+\pi(X')+X'(C+X)-1$$
$$= \pi(C+X')+\pi(X)+X(C+X')-1;$$

so that

$$\pi(C+X)-\pi(X)-CX+1 = \pi(C+X')-\pi(X')-CX'+1.$$

Having thus defined $\pi(C)$ for any effective curve, we extend our definition to virtual curves by assuming the universal validity of (3). Thus if A and B are effective, we obtain $\pi(A-B)$ by writing $A = (A-B)+B$, so that, by (3),

$$\pi(A) = \pi(A-B)+\pi(B)+B(A-B)-1,$$

whence

$$\pi(A-B) = \pi(A)-\pi(B)-B(A-B)+1; \tag{4}$$

and it is easy to verify that $\pi(A-B)$, as so defined, is invariant if $C = A-B$ varies in a system of equivalence. Hence:

Every curve on F has a virtual genus obeying the addition law (3).

From (4) we deduce incidentally that

(i) the *null curve*, $A-A$, on F has virtual genus unity,

(ii) any *negative curve*, $-A$, on F has virtual genus $2+A^2-\pi(A)$.

Also, as in § 3.1, we may define for any curve C a *virtual canonical number* $\vartheta(C)$, given by

$$\vartheta(C) = 2\pi(C)-2-C^2, \tag{5}$$

whose additive property—a consequence of (1) and (4)—still holds good. If C is any curve of the form $\sum \lambda_i C_i$, this additive property gives

$$\vartheta(C) = \sum \lambda_i \vartheta(C_i) \tag{6}$$

and this gives the following general result:

The virtual genus $\pi(C)$ of any curve $C = \sum \lambda_i C_i$, where the λ_i are integers, positive or negative, is given by

$$2\pi(C)-2-C^2 = \sum \lambda_i\{2\pi(C_i)-2-C_i^2\}. \tag{7}$$

§ 5. Numerical properties of neighbourhoods

5. Let O be any simple point of F, and E the exceptional line into which it is resolved on a surface F'; and let $|C|$ denote the system of prime sections on F and also the transform of this system on F'. Then $|C-E|$ on F', representing the system of sections of F by primes through O, consists of curves which meet E in one point; and we have, plainly,

$$CE = 0, \qquad (C-E)E = 1,$$

whence

$$E^2 = -1.$$

The virtual genus of E, equal to its effective genus, is 0. Hence

THEOREM IV. *The virtual grade, genus, and canonical number of any total neighbourhood of a point (proper exceptional curve) are* -1, 0, -1 *respectively.*

From the identities

$$(C-\lambda\tilde{O})^2 = C^2-2\lambda C\tilde{O}+\lambda^2\tilde{O}^2 = C^2-\lambda^2,$$
$$\vartheta(C-\lambda\tilde{O}) = \vartheta(C)-\lambda\vartheta(\tilde{O}) = \vartheta(C)+\lambda,$$

we deduce

THEOREM V. *An assigned λ-fold base point reduces the virtual grade of a linear system by λ^2 and its virtual genus by $\frac{1}{2}\lambda(\lambda-1)$.*

The results in Theorem IV are applicable to the most general composite exceptional curve obtained by resolving a point O and any set of neighbouring points. A converse theorem—characterizing the most general (composite) curve of virtual grade -1 and virtual genus 0 which is exceptional—has been established by Barber and Zariski† and by Du Val (l.c. on p. 417).

It should be remembered also (cf. § 2.1) that on surfaces which are rational or transformable into scrolls, there can exist *improper* exceptional curves, e.g. a line of the plane diminished by the neighbourhoods of two of its points is such a curve (cf. Exx. 4, 5 below).

5.1. *Orthogonal property of neighbourhoods.* If C is any proper curve of F, and O is any actual or fictitious point of F which is not contained in C, then plainly $C\tilde{O} = 0$. Rather surprisingly, this result still holds if $\tilde{O}$ is contained in C. For if O is an actual λ-fold point of C, then, by resolution of O, C becomes a composite curve of the form $C_1+\lambda E$, where C_1 meets E in λ points; whence

$$C\tilde{O} = (C_1+\lambda E)E = C_1E+\lambda E^2 = \lambda-\lambda = 0;$$

and the case when O is fictitious can be reduced to the above by resolutions. This gives

THEOREM VI. *A proper curve of F has virtually no intersections with the* (*total*) *neighbourhood of any actual or fictitious point of the surface.*

The above theorem leads to an important general property of total neighbourhoods—the orthogonal property—which is as follows:

THEOREM VII. *If O, O' are any two distinct points, actual or latent, of a surface F, then $\tilde{O}\tilde{O}' = 0$.*

† *Amer. J. of Math.* **57** (1935), 119.

To prove this we note first that if O and O' are actual distinct points of F, or fictitious neighbours of two such points, then the result is obvious by resolutions; and the same is true if the above situation is reached after the resolution of any ancestors which O and O' may have in common. The only other possibility is that O', for example, may lie in some neighbourhood of O; and then a suitable series of resolutions transforms $\tilde{O}$ into an exceptional curve and O' into a point of this curve, whence, by Theorem VI, the result follows at once.

COROLLARY. *Assigned base points O_i of multiplicities k_i reduce the virtual grade and genus of a linear system by $\sum k_i^2$ and $\frac{1}{2}\sum k_i(k_i-1)$ respectively.*

For we have the relations

$$\tilde{O}_i^2 = -1, \qquad C\tilde{O}_i = 0, \qquad \tilde{O}_i\tilde{O}_j = 0 \quad (i \neq j),$$

whence $$(C-\sum k_i\tilde{O}_i)^2 = C^2-\sum k_i^2;$$

and $$\vartheta(C-\sum k_i\tilde{O}_i) = \vartheta(C)-\sum k_i\vartheta(\tilde{O}_i) = \vartheta(C)+\sum k_i,$$

whence $$\pi(C-\sum k_i\tilde{O}_i) = \pi(C)-\tfrac{1}{2}\sum k_i(k_i-1).$$

Notes and Examples

1. Find the virtual grade and genus of $\lambda\tilde{O}$, where λ is any integer, positive or negative.

2. If O_1 is a first neighbour of O, discuss the composition of the proper exceptional curve, transform of $\tilde{O}$, obtained by resolving O and O_1. Prove that the diminished neighbourhood $\tilde{O}-\tilde{O}_1$ becomes a proper irreducible curve of virtual grade -2.

3. Deduce from Ex. 2 that the projective model of quadric sections of F through O and O_1 has a double-point corresponding to $\tilde{O}$.

4. The *plane*, as a particular rational surface, contains no proper exceptional curves but an infinity of improper ones. Thus, for example, $L-\tilde{O}_1-\tilde{O}_2$ is an improper exceptional curve, where L is any line of the plane and O_1, O_2 are either actual points of L or an actual point of L and its consecutive point on L; for any such improper curve is transformable into the whole neighbourhood of a simple point of a quadric or quadric cone.

5. In a general quadratic Cremona transformation of the plane, the precise invariant transform of the system of lines is a system of diminished conics $|2L-\tilde{O}_1-\tilde{O}_2-\tilde{O}_3|$, L being a line, and O_1, O_2, O_3 the fundamental points.

Deduce from this and Ex. 4 that any conic of the plane, diminished by the neighbourhood of 5 points of itself, is an exceptional curve.

6. *Any generator of a ruled surface R has virtual grade zero.* For if C is a prime section and L a generator of R, then, clearly, $CL = 1$; and since

any prime through L meets R residually in another simple directrix curve, $(C-L)L = 1$; so that $L^2 = 0$. Each generator of R is an isolated curve (for linear equivalence) if R is irrational.

7. *Projection of a ruled surface R.* If R is projected from a point O of the generator L into a ruled surface R_1, it follows easily from Ex. 6 that L projects into a simple point O_1 of R_1; and the tangent plane at O projects into the generator L_1 of R_1 which passes through O_1. More precisely $\bar{O}$ transforms into the improper exceptional curve $L_1-\bar{O}_1$ on R_1, while $\bar{O}_1$ is the transform of the improper exceptional curve $L-\bar{O}$ of R. In general it can be seen that:

Any generator L of a ruled surface (not a plane), diminished by the neighbourhood of an actual point of itself, is an improper exceptional curve of the surface.

8. *The general λ-curve on a ruled surface.* Any curve which meets every generator of R in λ points is called a λ-curve of R. By constructing a suitable rational function on R, it can be shown that:

If K is any λ-curve of a ruled surface R, then $K \equiv \lambda C+G$, where C is a prime section of R and G is a set of generators effective or virtual.

9. Using (2) of § 4.1 and (7) of § 4.2, deduce the following corollaries from Ex. 8:

If R is of order n and genus p, then the virtual grade γ and the virtual genus π of a λ-curve of order k of R are given by

$$\gamma = \lambda(2k-n\lambda), \qquad \pi = \lambda p-\tfrac{1}{2}(\lambda-1)(2k-2-n\lambda),$$

and the number i of intersections of a λ_1-curve of order k_1 with a λ_2-curve of order k_2 is given by

$$i = \lambda_1 k_2+\lambda_2 k_1-n\lambda_1\lambda_2.$$

10. On a general cubic surface of S_3, the proper exceptional curves are the 27 lines of the surface; and on the rational normal cubic scroll, the only proper exceptional curve is the directrix line.

11. If a non-singular quartic surface in S_3 contains a line, verify that this line has virtual grade -2.

12. *Characteristic matrices of a consecutive sequence of points.*† Let O $(= O_0)$, $O_1,\ldots, O_r$ be any simple consecutive sequence of points on a surface F (cf. § 2, Ex. 3), and let $E_0,\ldots, E_r$ be the (composite) exceptional curves into which the *total* neighbourhoods of the points are resolved on a surface F^* $(= F^{(r+1)})$. Also let $e_0,\ldots, e_r$ be the *irreducible* curves of F^* which represent the *diminished* neighbourhoods of $O_0,\ldots, O_r$, i.e. the neighbourhoods of these points diminished by those of any subsequent points which are proximate to them.

If $\mathbf{E}$ and $\mathbf{e}$ are the single-column matrices whose components are E_i and e_i respectively, then there exists a triangular matrix $\mathbf{m}$—*the proximity matrix* of the sequence—such that

$$\mathbf{e} = \mathbf{m}\mathbf{E},$$

where $m_{ij} = 1$ if $j = i$, $m_{ij} = 0$ if $j < i$, and $m_{ij} = -1$ or 0, if $j > i$, according as O_j is or is not proximate to O_i.

† Cf. Du Val, *Amer. J. of Math.* 58 (1936), 285.

There exists then also an inverse triangular matrix $\mathbf{n} = \mathbf{m}^{-1}$—the *multiplicity matrix* of the sequence—such that

$$\mathbf{E} = \mathbf{ne},$$

where $n_{ij} = 1$ if $j = i$, $n_{ij} = 0$ if $j < i$, and n_{ij} is a positive integer if $j > i$. Plainly the numbers in the last column of $\mathbf{n}$ represent the multiplicities at the points O_i of a primitive branch through O_r: more generally, n_{ij} is the multiplicity at O_i of a primitive branch through O_j.

By the orthogonal properties of neighbourhoods we have

$$(E_i E_j) = \mathbf{E}\tilde{\mathbf{E}} = -\mathbf{I},$$

where $\tilde{\mathbf{E}}$ denotes the transposed of $\mathbf{E}$, and $\mathbf{I}$ is a unit matrix; and hence the *intersection matrix* $\mathbf{i}$ of the irreducible curves e_i is given by

$$\mathbf{i} = \mathbf{e}\tilde{\mathbf{e}} = \mathbf{mE}\tilde{\mathbf{E}}\tilde{\mathbf{m}} = -\mathbf{m}\tilde{\mathbf{m}}.$$

13. *On any curve-branch through a simple sequence of points O_0, O_1,..., the multiplicity of the branch at O_i is always equal to the sum of its multiplicities at points proximate to O_i.*

Prove this result, and show that, if the proximity relations of a simple sequence O_0,..., O_r are known, the above rule is sufficient to determine the multiplicities at all the points of a branch which is primitive at O_r.

14. Verify that for an ordinary cuspidal branch, primitive at the satellite point $O_2^{(0)}$, we have

$$\mathbf{m} = \begin{pmatrix} 1 & -1 & -1 \\ & 1 & -1 \\ & & 1 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 1 & 1 & 2 \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix};$$

and find the corresponding characteristic matrices of the sequences $(O, O_1, O_2^{(0)}, O_3^{(0)})$ on a cuspidal branch of order 3, and $(O, O_1, O_2^{(0)}, O_3^{(1)})$ on a cubo-quadratic cuspidal branch (cf. § 2, Exx. 1, 2, 4).

15. Show that the sequence $(O, O_1, O_2, O_3^{(1)})$ on a rhamphoid cuspidal branch (cf. § 2, Ex. 5) has characteristic matrices

$$\mathbf{m} = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & -1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 1 & 2 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

§ 6. Singularities of Surfaces

6. The general subject of surface singularities is extremely interesting, but vastly more complicated than the corresponding theory for curves; and many of its problems are still unsolved. In this section we can only touch on the general theory and exhibit some elementary or especially interesting simple cases.

We classify possible singularities of F, to begin with, into (*a*) *singular curves*, i.e. multiple curves which may have other multiple curves of F in their neighbourhoods; (*b*) *isolated point singularities*, i.e. multiple points, not lying on any multiple curve, which may have other multiple points or multiple curves of F in their neighbourhoods; and (*c*) *special point singularities* which are special singular points on one or more singular curves of F.

Resolution of a point O *of* F consists in transforming the whole ambient space S_k by means of all quadrics through O; the resulting V_k, projective model of the quadrics in question, contains a linear space Π_{k-1} on which the whole first neighbourhood of O is homographically represented; and F is transformed into a surface F' on which the whole first neighbourhood of O on F is represented by the curve in which F' meets Π_{k-1}. The order of this curve is equal to that of the nodal cone at O, and equal therefore to the multiplicity of F at O.

Resolution of a curve Γ of F consists in transforming the whole ambient space S_k by means of all primals, of some sufficiently high order l, which pass through Γ; the resulting V_k, projective model of the primals in question, contains a 'scrollar' manifold R_{k-1}, locus of ∞^1 spaces Π_{k-2}, which represents the whole neighbourhood of Γ, the individual points of any Π_{k-2} representing (homographically) sections of the neighbourhood of one point of Γ by planes through the tangent to Γ at this point; and F is transformed to a surface F' on which the whole first neighbourhood of Γ on F is represented by the curve in which R_{k-1} meets F'. This curve, if Γ is of multiplicity λ on F, meets every generator Π_{k-2} in λ points, distinct or coincident.

The two concepts just defined, resolution of a point and of a curve, have already been discussed, for the case $k = 3$, in Ch. VIII, § 3.

6.1. ***The general singular curve.*** We consider first the general singular curve Γ, of multiplicity λ, on F; and we suppose this to be irreducible and free from multiple points, since such could be removed by point-resolutions. We resolve Γ, as above, into a curve Γ_1, section of F' by R_{k-1}; and we suppose, for generality, that the irreducible components of Γ_1 are (*a*) simple curves $\Gamma_{1,\alpha}$ of F' which may count multiply in Γ_1 as curves of contact of R_{k-1} with F', and (*b*) multiple curves $\Gamma_{1,\beta}$ of F' which may likewise count more than their multiplicities because of contact conditions.

Any $\Gamma_{1,\alpha}$ represents one of the distinct *curvilinear branches* composing the curvilinear singularity of F along Γ, and this branch is completely resolved on F'; the multiplicity of the branch is equal to the number of points, $\lambda_{1,\alpha}$ say, in which $\Gamma_{1,\alpha}$ meets a space Π_{k-2}, or it may be a multiple of this number if $\Gamma_{1,\alpha}$ is a curve of contact, the branch being then cuspidal; and finally, if $|C|$ is the system of curves on F' which represent prime sections of F, the sets of $\lambda_{1,\alpha}$ points in which $\Gamma_{1,\alpha}$ meets the spaces Π_{k-2} (or multiples of these sets in the contact case) are *neutral* for $|C|$.

Any $\Gamma_{1,\beta}$, of multiplicity λ_β for F', represents a (fictitious) curve of F, of multiplicity λ_β, in the first (curvilinear) neighbourhood of Γ; and further resolutions are required to separate out the component curvilinear branches of F' (and F) through $\Gamma_{1,\beta}$.

The whole process runs parallel with the resolution of the singular point on a generic prime section of F at any one of the points where the prime meets Γ, and it terminates when all the curvilinear branches composing the original singularity of F are separated out and resolved into simple curves of a birational transform of F.

6.2. *The general isolated point singularity.* If O is any point of multiplicity λ of F, not lying on a multiple curve, then the resolution of O transforms the first neighbourhood of O on F (cone of 'nodal directions' at O) into a curve Γ of order λ of F'. We regard this curve Γ as corresponding to an *infinitesimal curve*, also denoted by Γ, in the neighbourhood of O on F.

The curve Γ of F' may have multiple points or multiple components which may or may not be multiple for F'; but since Γ is the complete section of F' by the space Π_{k-1} which represents the neighbourhood of O, it follows that

The multiplicity for F' of any point or component of Γ cannot exceed λ.

Any points or component curves of Γ which are multiple for F' are regarded as corresponding to *fictitious multiple points* or *infinitesimal multiple curves* of F in the first neighbourhood of O; and no one of these, by what we have just said, can have greater multiplicity than that of O itself.

If no point or component of Γ is multiple for F', we say that O is an *isolated λ-fold point* of F; and the singularity may be invariantively described as an (infinitesimal) fundamental curve Γ

of the system $|C|$ of prime sections of F. The multiple point is *ordinary* only in the very special case when Γ is irreducible and non-singular.

If F has multiple points or curves in the neighbourhood of O, then these too may be resolved; but it is not yet known whether any isolated point singularity can be completely removed by resolutions alone. Any singularity for which the process is known to terminate is said to be *resoluble*.

6.3. *Construction of point singularities.* So far we have only sketched very briefly a tentative method of resolving given surface singularities into simple curves of a birationally equivalent surface. We propose now to reverse this procedure, i.e. we start from a non-singular surface ψ, and we discuss the birational transformations of ψ which give surfaces F possessing various types of point-singularity.

Any birational transform F of ψ is the projective model of a simple irreducible linear system $|C|$ on ψ; and we may suppose that $|C|$ is free from base points, since these latter could be removed by resolutions. Let Ω be a total fundamental curve of $|C|$, composite possibly, with components of various multiplicities; and let O be the point of F which corresponds to Ω. Further, let $|C_1| \equiv |C-\Omega|$ be the system† residual to Ω in $|C|$; this is of freedom one less than that of $|C|$, it is without fixed components, and we suppose it to be free from base points.‡ Since $|C|$ is free from base points, we have $C\Omega = 0$; and since the curves C_1 represent sections of F by primes through O, the multiplicity λ of O on F is given by

$$\lambda = C^2-(C-\Omega)^2 = 2C\Omega-\Omega^2 = -\Omega^2.$$

Hence: *A total fundamental curve of grade $-\lambda$ of $|C|$ represents the neighbourhood of a λ-fold point of the projective model of $|C|$.*

We now consider the composition of the singular point of F at O; and for this purpose we construct the projective model F' of the system $|C'| \equiv |2C-\Omega|$ which represents all sections of F by quadrics§ through O. Any curve Ω_1 (possibly composite) which is wholly contained in Ω and which is a total fundamental curve,

† We need not suppose, naturally, that either of the systems $|C|$ or $|C_1|$ is complete.

‡ If $|C_1|$ has a base point A, the neighbourhood of A should be added to Ω, as a latent component, to preserve the totality of Ω as a fundamental curve.

§ Here also the system $|C'|$ need not be the complete system $|2C-\Omega|$.

of grade $-\lambda_1$ say, of $|C'|$, represents a λ_1-fold point of F', O_1 say; or it may be regarded as representing a fictitious λ_1-fold point O_1 of F infinitely near to O. Of necessity, as already remarked, $\lambda_1 \leqslant \lambda$. If O_1 is resolved, its neighbourhood will become a curve on a surface F'', projective model of a system $|C''| \equiv |2C'-\Omega_1|$; and any curve Ω_2 subordinate to Ω_1, which is totally fundamental for $|C''|$ will represent a second fictitious multiple point of F, consecutive to O_1; and so on.

We return, however, to another possibility, namely, that some part, Λ say, of Ω may represent a multiple curve of F', corresponding to an infinitesimal multiple curve Γ of F in the neighbourhood of O; this happens if Λ contains an infinity of sets of points which are neutral for $|C'|$. The resolution of Γ, by primals of sufficiently high order l which pass through it, will exhibit its first neighbourhood explicitly on a surface which is the projective model of a system $|C''| \equiv |lC'-\Lambda|$.

By proceeding in this way, we have at least a tentative procedure for analysing any singularity of F arising from a total fundamental curve Ω of ψ; but the possible complications are manifold.

6.4. ***Improper and foliate singularities.*** Two special categories of point singularities may be distinguished as follows:

(i) A singularity of F at O is *improper* if the section of F by a generic prime through O has the same genus as a generic prime section of F.

(ii) A singularity of F at O is *foliate* if the total neighbourhood of O on F can be transformed into a finite sum of latent (exceptional) curves on a non-singular transform of F.

If a non-singular surface in [5] is projected from a point of one of its chords or of one of its tangent lines, the *improper double point* or *improper cusp* so arising is an example of an improper singular point (which happens also to be foliate).

An example of a foliate singularity which is not improper is obtained when any non-singular non-ruled surface in higher space is projected from a generic line of one of its tangent planes; this gives, on the projected surface, a point of multiplicity 4 (with a repeated quadric cone as its nodal cone), and the genus of the section of the surface by a generic prime through the point is one less than that of the generic prime section.

6.5. *Simple types of singularity.* We now proceed to describe in detail some of the simpler well-known types of singularity of a surface. We suppose in all cases that the surface F possessing the singularity is the transform of a non-singular surface ψ, and that prime sections of F are represented on ψ by the curves of a linear system $|C|$ which is free from base points.

(i) *An irreducible double curve D of F* corresponds to a curve Δ of ψ containing an involution γ_2^1 of pairs of points which are neutral for $|C|$. The genera p, P of D and Δ are related by

$$2P-2 = 2(2p-2)+\nu,$$

where ν is the number of pinch-points on D (cf. Ch. XII, §3.3).

(ii) *An irreducible cuspidal curve K of F* corresponds to a curve Λ of ψ, such that the curves of $|C|$ through any point of Λ have a fixed tangent at that point.

(iii) *An isolated λ-fold point O of F* corresponds to a fundamental curve Ω, of grade $-\lambda$, of $|C|$. Since $C\Omega = 0$ and $\Omega^2 = -\lambda$, we deduce that $(C-\Omega)\Omega = \lambda$, and hence

$$\pi(C) = \pi(C-\Omega)+\pi(\Omega)+\lambda-1.$$

It follows that the deficiency of the genus of prime sections through O (as compared with that of general prime sections) is

$$\delta = \pi(\Omega)+\lambda-1,$$

and the multiple point is improper if $\pi(\Omega) = 1-\lambda$.

If F is in [3], δ has its maximum possible value $\frac{1}{2}\lambda(\lambda-1)$, and $\pi(\Omega) = \frac{1}{2}(\lambda-1)(\lambda-2)$, which is the *virtual* genus of the nodal cone.

(iv) *Two consecutive multiple points*, O, O_1, of multiplicities λ, λ_1 for F $(\lambda_1 \leqslant \lambda)$, correspond on ψ to a total fundamental curve Ω of $|C|$, of grade $-\lambda$, and to a total fundamental curve Ω_1 of $|2C-\Omega|$, which is part of Ω and of grade $-\lambda_1$.

We consider only the *simplest case* in which F is in [3], the line OO_1 is the only multiple generator—and of multiplicity λ_1—of the nodal cone at O, and the neighbourhood of O_1 is non-singular and therefore of genus $\frac{1}{2}(\lambda_1-1)(\lambda_1-2)$. In this case the curve $\omega = \Omega-\Omega_1$ represents the original nodal cone at O, and it will therefore meet Ω_1 in λ_1 points and have (virtual) genus

$$\pi(\omega) = \tfrac{1}{2}(\lambda-1)(\lambda-2)-\tfrac{1}{2}\lambda_1(\lambda_1-1).$$

By simple calculation we find that

$$\omega^2 = -\lambda-\lambda_1, \qquad \pi(\Omega) = \tfrac{1}{2}(\lambda-1)(\lambda-2),$$

and that the consecutive multiple points O, O_1 behave in many ways like a pair of distinct multiple points of the surface.

(v) *An ordinary binode O* of a surface F in [3] is an isolated double point at which the nodal cone is a pair of distinct planes α, β intersecting in a line. The singularity is therefore represented on ψ by a pair of rational curves, ω_1 and ω_2, which meet in a point and together constitute a total fundamental curve of $|C|$, of grade -2.

The section of F by a generic prime through O has a double point there, with one branch touching α and one touching β. Thus

$$1 = (C-\omega_1-\omega_2)\omega_1 = (C-\omega_1-\omega_2)\omega_2,$$

and, since $\omega_1\omega_2 = 1$, it follows that

$$\omega_1^2 = \omega_2^2 = -2.$$

(vi) *Binode B_s of order s.* A point singularity of F which consists of σ consecutive double points which lie on a linear curve-branch is either a *binode of even order* B_s with $s = 2\sigma$, or a *binode of odd order* B_s with $s = 2\sigma+1$, according as the last node, on being made explicit by resolution, is an ordinary conical node or an ordinary binode. An ordinary conical double point (though not a binode) is included in the series as a B_2; the ordinary binode discussed in (v) is a B_3.

We find it convenient in this case to denote the first (actual) node by O_1 and the others of the sequence by $O_2,..., O_\sigma$.

The nature of the total fundamental curve Ω_1 which represents this singularity on ψ is easily found from the process of resolution; for as soon as any node O_i, excepting O_σ, becomes explicit, its nodal cone—a pair of planes meeting in a line—is resolved into a pair of lines which intersect in O_{i+1}, and finally O_σ is resolved into a conic or into an intersecting line-pair according as it is an ordinary conical node or a binode. It follows then that Ω_1 is a simple open chain of $s-1$ rational curves, say

$$\Omega_1 = \sum_{i=1}^{s-1} \omega_i,$$

where $\omega_i\omega_{i+1} = 1$ $(i = 1,..., s-2)$, and $\omega_i\omega_{i+\alpha} = 0$ $(i = 1,..., s-3,$ $\alpha > 1)$. This whole curve is fundamental for $|C|$ and of grade -2.

Similarly the curve

$$\Omega_2 = \sum_{i=2}^{s-2} \omega_i$$

represents the B_{s-2} obtained by resolving O_1, and it is therefore fundamental for $|C'| \equiv |2C-\Omega_1|$ and likewise of grade -2; and generally

$$\Omega_\alpha = \sum_{i=\alpha}^{s-\alpha} \omega_i \quad (\alpha = 1,\dots,\sigma)$$

is of grade -2 and fundamental for $|C^{(\alpha-1)}| \equiv |2C^{(\alpha-2)}-\Omega_{\alpha-1}|$.

Observing that $\omega_\alpha^2 = \omega_{s-\alpha}^2$ by symmetry, and that

$$\omega_\alpha \Omega_{\alpha+1} = 1 = \omega_{s-\alpha}\Omega_{\alpha+1}, \qquad \omega_\alpha \omega_{s-\alpha} = 0 \quad (\alpha < \sigma),$$

it follows that

$$-2 = \Omega_\alpha^2 = (\Omega_{\alpha+1}+\omega_\alpha+\omega_{s-\alpha})^2 = 2+2\omega_\alpha^2,$$

so that $\omega_\alpha^2 = -2$, this result still holding for $\alpha = \sigma$. Hence:

A binode B_s of F can be represented on ψ by a simple open chain of $s-1$ rational curves, each of grade -2, which together form a total fundamental curve, of grade -2, of $|C|$.

Binodes of any order s occur naturally in various connexions. Thus, for example, if a variable surface in [3] is made to have contact of order $s-1$ with a given surface along a given curve, then it will have, in general, a finite number of variable binodes B_s on the curve.†

(vii) *The general proper unode.* A general *unode* of a surface F in ordinary space is a double point O of F whose only speciality is that the nodal cone at O is a repeated plane. If we take O to be $(0,0,0,1)$ and write the equation of F in the form

$$0 = z^2t^{n-2}+u_3t^{n-3}+\dots,$$

and if we resolve O by a Cremona quadratic transformation with O as isolated fundamental point (cf. Ch. VIII, § 4.3), we find that F has, in general, 3 conical double points, O_1, O_2, O_3, infinitely near to O, in different directions in the unodal plane; on this basis we *define the general proper unode, in space of any number of dimensions, as a double point of F which has three other ordinary (conical) nodes in its first neighbourhood, these three lying in a plane through the first (the unodal plane).*

By resolving O, O_1, O_2, O_3, we find that the total fundamental curve of $|C|$ which represents the singularity is of the form $\Omega = 2\omega+\Omega_1+\Omega_2+\Omega_3$, where Ω_1, Ω_2, Ω_3 are the three non-intersecting rational curves (of grade -2) which represent the neigh-

† Cf. Hudson, *Cremona transformations*, p. 243, and Semple, *Proc. Roy. Irish Acad.* **43** (1936), 49–71.

bourhoods of O_1, O_2, O_3, while ω is another rational curve, meeting each of Ω_1, Ω_2, Ω_3 in one point, which represents the pencil of directions in the unodal plane. This curve ω is easily found to be likewise of grade -2, and this completes the description of Ω.

(viii) *The tacnode.* A surface F is said to have a tacnode at O if it has a double point at O and an infinitesimal double line Γ in the first neighbourhood of O (with no further complications). The nodal cone at O is the repeated tacnodal plane $O\Gamma$.

If δ is the genus deficiency of a generic section of F by a prime through O, then δ is always 2 if F is in [3], but it can be either 2, 1, or 0 if F is in higher space. We have, therefore, three species of tacnode: (*a*) the *proper tacnode*, for which $\delta = 2$, (*b*) the *tacnode of deficiency* 1, and (*c*) the *improper tacnode*, for which $\delta = 0$.

To resolve the tacnodes completely we first resolve O, so that Γ becomes an explicit double line on a surface F', and we then resolve Γ, as in (i), into a simple curve of a surface F''. It appears then that the tacnode can be represented on a non-singular transform ψ of F by a fundamental curve Ω of $|C|$, of grade -2, which contains a rational involution g_2^1 of pairs of points which are neutral for a system $|C'| = |2C-\Omega|$. Also

$$\delta = \pi(C)-\pi(C-\Omega) = \pi(\Omega)-\Omega^2-1 = \pi(\Omega)+1,$$

so that Ω *is elliptic, rational, or of virtual genus* -1 *according as the tacnode is proper, of deficiency* 1, *or improper.*

For the improper tacnode it is easy to see that Ω consists of two non-intersecting exceptional curves, transformable into the neighbourhoods of two points, and that the neutral point-pairs are given by a homographic correspondence between the points of these two curves.

An example of a tacnode of deficiency 1 is obtained by projecting any conical node of a surface in higher space from a generic point of the solid containing the nodal cone.

An example of an improper tacnode is obtained by choosing a surface in higher space which possesses a bitangent solid (solid containing two tangent planes) and projecting it from a point of the chord of contact of this solid.

§ 7. THE CANONICAL SYSTEM

7. The Jacobian curve of a net. We propose now to extend to surfaces the theory of Jacobian series and the canonical series for a curve, and the essential foundation for this extension will be the definition of the Jacobian curve $J(C)$ of a linear net $|C|$ on a surface F. The definition and properties of $J(C)$ for the case when F is a plane have already been discussed (Ch. VI, § 5.1).

The preliminary definition of $J(C)$, of restricted application, is that $J(C)$ is the *node-locus* of $|C|$; or, which amounts to the same thing, that it is the *locus of coincidence points of characteristic sets* of $|C|$.

To obtain a stronger definition, we take a parametric representation of F, valid in a region R on the surface, by which the coordinates x_i $(i = 0,..., k)$ of a variable point of F are proportional to analytic functions $f_i(u, v)$ of two parameters u, v; and we suppose that the curves of the linear net $|C|$ are given by the equation

$$C(u,v) \equiv \lambda_1 \phi_1(u,v) + \lambda_2 \phi_2(u,v) + \lambda_3 \phi_3(u,v) = 0, \tag{1}$$

where ϕ_1, ϕ_2, ϕ_3 are analytic in R. Any double point of a curve of the net must satisfy the three conditions

$$C(u,v) = \frac{\partial C}{\partial u} = \frac{\partial C}{\partial v} = 0;$$

and we therefore define $J(C)$ in R as the curve whose equation is

$$J(C) \equiv \left| \phi_i \quad \frac{\partial \phi_i}{\partial u} \quad \frac{\partial \phi_i}{\partial v} \right| = 0, \tag{2}$$

the values 1, 2, 3 of i giving the rows of the determinant.

If $|C|$ has a simple base curve Γ whose equation is $\sigma(u,v) = 0$, we write $\phi_i = \sigma\psi_i$ $(i = 1, 2, 3)$, and then

$$J(C) \equiv \left| \sigma\psi_i \quad \frac{\partial \sigma}{\partial u}\psi_i + \sigma\frac{\partial \psi_i}{\partial u} \quad \frac{\partial \sigma}{\partial v}\psi_i + \sigma\frac{\partial \psi_i}{\partial v} \right| = \sigma^3 \left| \psi_i \quad \frac{\partial \psi_i}{\partial u} \quad \frac{\partial \psi_i}{\partial v} \right|.$$

Hence

THEOREM VIII. *If the linear net $|C|$ has a simple base curve Γ, then $J(C)$ consists of the Jacobian curve of the residual net $|C-\Gamma|$, together with Γ counted threefold.*

7.1. *Jacobian system.* Suppose now, as we may, that F is in [3], with equation $F(x, y, z, t) = 0$, and that it has (at most) normal singularities with D as its double curve. Also let $|C|$ be cut on F, residually to a fixed simple curve A, and possibly also to D, by the linear net of surfaces Φ whose equation is

$$\Phi \equiv \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3 = 0.$$

A node O of a curve of $|C|$ is either (a) a point of D, in which case it must in general be regarded as virtually inexistent, or (b) a point of contact of F with a surface of Φ; and hence, applying Theorem VIII, we may define $J(C)$ in general as the curve in which F is met by the surface

$$J(F, \Phi) \equiv \frac{\partial(F, \phi_1, \phi_2, \phi_3)}{\partial(x, y, z, t)} = 0 \tag{3}$$

residually to D and to the curve A counted threefold.

From this we deduce, as in Ch. XII, § 2.1,

THEOREM IX. *The Jacobian curves of all linear nets of curves of any linear system $|C|$ on F belong to one and the same complete linear system on F.*

The complete linear system just referred to is called the *Jacobian system* $|J(C)|$ of $|C|$; but if $|C|$ has an assigned base, this definition must be modified (cf. § 7.2) by assigning a corresponding base for the Jacobian system.

By a simple extension of the method of Ch. XII, § 2.3, we deduce from Theorems VII and VIII an addition law for Jacobian systems, namely,

$$J(A+B) \equiv J(A)+3B \equiv J(B)+3A. \tag{4}$$

We note finally that if Ω is any irreducible fundamental curve of $|C|$, then every Jacobian curve of a linear net of $|C|$ contains Ω; for the generic point of Ω formally satisfies the condition that curves of $|C|$ passing through it should have a fixed tangent there. This does not imply, however, that the *complete* Jacobian system $|J(C)|$ has Ω as fixed component (cf. § 7.5).

7.2. *Relative covariance of Jacobian system.* The Jacobian system $|J(C)|$ is only relatively covariant over birational transformations of F, i.e. it is absolutely covariant for unexceptional birational transformations but liable to be augmented or diminished by exceptional curves in transformations which are not unexceptional.

To show this, we suppose, for simplicity, that one point O of F is resolved into an exceptional line E of F', and that O is not a base point of $|C|$. The generic curve of $|C|$ through O transforms into a composite curve C_1+E of F', and this has a formal double point at the intersection (variable in general) of C_1 with E. Thus

E is a new component of the Jacobian curve, relative to F', of a generic net of curves of $|C|$; and if we denote corresponding curves (in the invariant sense) on F and F' by the same symbols, we may write

$$J'(C) = J(C)+E, \tag{5}$$

where J and J' refer to Jacobians relative to F and F' respectively. We shall return to this subject in § 7.4.

7.3. *The canonical system.* If $|A|$ and $|B|$ are any pair of linear systems of freedom not less than 2 on F, it follows from (4) that

$$J(A)-3A \equiv J(B)-3B,$$

and this means, evidently, that the system of equivalence X defined by

$$X \equiv J(A)-3A \tag{6}$$

is independent of $|A|$. We call X the *canonical system of equivalence* on F; and if X has any effective members forming a linear system $|X|$, we call this the *canonical system* of F. In addition

(i) the *geometric genus* p_g of F is defined as the number of linearly independent canonical curves on F,

(ii) the *canonical genus* ω of F is defined as $\pi(X)$, and

(iii) the *canonical grade* of F is defined as X^2.

Suppose now that F is a surface of order n in ordinary space, and that it has (at most) ordinary singularities with D as double curve. If $|C|$ denotes the system of plane sections of F, any net in $|C|$ will be generated by planes through a point A of space, and its Jacobian curve will be the locus of points of contact of proper tangent planes through A. This is the curve in which F is met, residually to D, by the first polar of A; and it follows then that the residual curves cut on F by surfaces of order $n-1$ through D all belong to the complete Jacobian system $|J(C)|$. From this we deduce, by (6), that surfaces of order $n-4$ through D (if such exist) meet F residually in canonical curves. Hence

THEOREM X. *If F is a surface of order n in ordinary space and possesses only normal singularities, then surfaces of order $n-4$ through the double curve (if such surfaces exist) meet F residually in canonical curves.*

It is, in fact, the case that the surfaces in question cut the *complete* canonical system on F. This follows from the theorem, to which reference has already been made in § 1.5, that the surfaces of any given order which pass through the double curve of a surface such as F cut a complete linear system of curves on the surface.

EXAMPLES

1. *The plane.* If F is a plane whose lines form a linear net $|L|$, then $J(L)$ is the null-curve and X is the system of equivalence defined by $-3L$. We have $p_g = 0$; and by applying (2) of § 4.1 and (7) of § 4.2, we find that $\pi(X) = 10$, $X^2 = 9$.

2. For a *general quadric* we verify that $X \equiv -2C$, where C is a plane section. Also $\pi(X) = 9$, $X^2 = 8$, $p_g = 0$.

3. The *non-singular F^4 in* [3] has one adjoint surface of order $n-4$, namely, the null-surface; and it has therefore one (effective) canonical curve, namely, the null-curve. Hence $p_g = 1$, and we find that $\pi(X) = 1$, $X^2 = 0$.

4. *The quintic surface with a double line* has a canonical system consisting of the pencil of plane cubics in which the surface is met by planes through the double line. Thus $p_g = 2$, $\pi(X) = 1$, $X^2 = 0$.

5. *A quintic surface with a double conic* has a unique effective canonical curve, namely, the residual line l in which the plane of the double conic meets the surface. This surface is the projection of a sextic intersection of a quadric and a cubic primal in [4] from a point of itself, the line l representing the neighbourhood of the point of projection. We have $p_g = 1$, $\pi(X) = 0$, $X^2 = -1$.

7.4. *Relative invariance of the canonical system.* To see how the canonical system behaves under birational transformations with fundamental points, we consider first the case in which a non-singular surface F is transformed into another non-singular surface F' in such a way that there are n distinct points $O_1, \ldots, O_n$ of F which transform into exceptional curves $E'_1, \ldots, E'_n$ of F', but no points of F' which correspond to curves on F. The argument of § 7.2 shows then that if J is the Jacobian curve of any linear net $|C|$ without base points on F, and if J' denotes the Jacobian curve of the corresponding system $|C|$ on F', then J' is the transform of J augmented by all the curves $E'_1, \ldots, E'_n$ as new disjoint components. If we follow our usual plan of representing curves which correspond (invariantively) on F and F' by the same symbol, we may write

$$J' = J + \sum E'_i, \qquad J E'_i = 0;$$

and, if X, X' are canonical curves of F, F' respectively, we have

$$X' + 3C \equiv X + 3C + \sum E'_i, \qquad (X+3C)E'_i = 0,$$

so that

$$X' \equiv X + \sum E'_i, \tag{7}$$

and, by using $CE'_i = 0$ and $E'^2_i = -1$,

$$X'E'_i = -1 \quad (i = 1,..., n). \tag{8}$$

Also, from (7) and (8) we deduce that

$$X'^2+n = X^2, \qquad \pi(X')+n = \pi(X). \tag{9}$$

Consider now the type of birational transformation in which there are e irreducible curves E_i of F which transform into points O'_i of F' $(i = 1, 2,..., e)$, and e' irreducible curves E'_j of F' which correspond to points O_j of F $(j = 1,..., e')$, this being the type of transformation considered in § 2.1. If F_1, F'_1 denote the surfaces, in unexceptional correspondence, into which F and F' are transformed by resolving the O_j and the O'_i respectively, then, by (7),

$$X - \sum E_j \equiv (X_1 - \sum E'_i) - \sum E_j,$$
$$X' - \sum E'_i \equiv (X'_1 - \sum E_j) - \sum E'_i,$$

where X_1, X'_1 denote canonical curves of F_1 and F'_1; and since $X_1 \equiv X'_1$ by the unexceptional correspondence between F_1 and F'_1, it follows that

$$X - \sum E_j \equiv X' - \sum E'_i. \tag{10}$$

In the same way, from (8) and (9) we deduce that

$$X^2+e = X'^2+e', \qquad \pi(X)+e = \pi(X')+e'. \tag{11}$$

These results give

THEOREM XI. *In any birational transformation between non-singular surfaces F and F' in which e irreducible exceptional curves E_i of F transform to points of F' and e' irreducible curves E'_j of F' transform to points of F, the relative invariance of the canonical system and of its virtual grade and genus are characterized by* (10) *and* (11).

It appears incidentally from the above proof that if E is one of the exceptional curves on F, then $XE = -1$ (cf. equation (8)) and hence $(X-E)E = 0$. For a surface which possesses an effective canonical system $|X|$, these relations imply that E is a fixed component of $|X|$ and that E has virtually no intersections with curves of the residual system $|X-E|$; also the number of such exceptional curves E must be finite. The system $|X-\sum E|$, where the summation extends to all (proper) exceptional curves of F, is called the *pure canonical system* $|\mathfrak{X}|$ of F. We have then

THEOREM XII. *If a surface F with an effective canonical system $|X|$ contains only irreducible exceptional curves E, then these are finite in number and they are all fixed components of $|X|$; also the pure canonical system $|\mathfrak{X}| = |X - \sum E|$ (when it exists) is absolutely invariant.*

If the canonical system $|X|$ of F happens to have an unassigned base point O (simple and isolated), then the neighbourhood of O is a fixed component of the pure canonical system of F; and this means, for example, that if O is resolved into a curve E of F', then E counts twice (as fixed component) in the canonical system $|X'|$ of F'.

7.5. *Jacobian system of a system with assigned base.* We consider next a relatively complete linear system $|\bar{C}|$, obtained by imposing assigned base points O_i of multiplicities k_i on a complete linear system $|C|$ which is free from base points; and we propose to define a suitable Jacobian system for $|\bar{C}|$. For this purpose we substitute for $|\bar{C}|$ the linear system $|C - \sum k_i \bar{O}_i| = |C_1|$ which is proper and virtually free from base points on a surface F' on which the O_i have been resolved.

We have then, by (4),

$$J'(C) \equiv J'(C_1 + \sum k_i \bar{O}_i) \equiv J'(C_1) + 3 \sum k_i \bar{O}_i;$$

and since

$$J'(C) \equiv X' + 3C \equiv X + \sum \bar{O}_i + 3C \equiv J(C) + \sum \bar{O}_i,$$

we have

$$J'(C_1) \equiv J(C) - \sum (3k_i - 1)\bar{O}_i. \tag{12}$$

Plainly $J'(C_1)$ will contain the proper transform on F of every Jacobian curve of a net of $|\bar{C}|$, and hence

We define the formal Jacobian system $|J(\bar{C})|$ as that derived from $|J(C)|$ by imposing an assigned $(3k-1)$-fold base point at each assigned k-fold base point of $|C|$.

This definition may be confirmed by observing that when the parametric definition of § 7 is used to find the multiplicity of the Jacobian curve of a net of $|\bar{C}|$, at any isolated k-fold base point of $|\bar{C}|$, this multiplicity is found in fact to be $3k-1$. As against this, however, it can happen (cf. the example below) that when $|\bar{C}|$ has consecutive base points, the observed multiplicities of a

Jacobian curve at these are different from those formally assigned in the above definition; and this is due to the fact that in the observed Jacobian curve there should have been included one or more latent components which would have augmented it to a curve of the formal Jacobian system.

EXAMPLE

If $|\overline{C}|$ has consecutive k-fold base points, O and O_1, then the visible Jacobian curve of a net of curves $|\overline{C}|$ has a $3k$-fold point at O and a $(3k-2)$-fold point at O_1; but this becomes a curve of the formal Jacobian system, with $(3k-1)$-fold base points at each of O, O_1, when it is augmented by a latent component $e = \tilde{O}-\tilde{O}_1$. To justify this we resolve O and O_1 into exceptional curves $E = e+E_1$ and E_1 of a surface F'; and we observe that, whereas e is merely a fundamental curve of the formal system

$$|J'(C_1)| = |J(C)-(3k-1)(E+E_1)|,$$

the same curve e is necessarily an actual component of the Jacobian curve of any net of $|C-k(E+E_1)|$, since it is in fact a fundamental curve of this system (cf. § 7.1). Hence the latent curve $\tilde{O}-\tilde{O}_1$ must be added, as a fixed component, to the apparent Jacobian of any net of $|\overline{C}|$, and this clears up the difficulty.

§ 8. ADJOINT CURVES

8. Intersection series. From the definition (by rational functions) of the relation of equivalence for sets of points on a curve and for curves on a surface, it follows at once that if two equivalent curves cut determinate sets on an irreducible curve C of F, then these sets are equivalent on C. We deduce from this the following result:

If L is any curve (effective or virtual) on a surface F, then the system of equivalence defined by L determines an intersection series of equivalence on any irreducible curve C of F.

We shall denote this series of equivalence on C by $L[C]$; and a relation of the form

$$L[C] \equiv M[C]$$

will therefore signify that the sets (effective or virtual) cut on C by any curves equivalent to L and M respectively are equivalent.

The symbol $C[C]$ will denote the *characteristic series of equivalence* on C.

8.1. *Definition of adjoint systems.* If C is any effective curve on a surface F, then all the curves (virtual or effective) which belong to the system of equivalence $A(C) \equiv C+X$, where X is a canonical curve of F, are called *free adjoints* of C, and $A(C)$ is the *adjoint system of equivalence* of C. If the complete systems $|C|$ and $|A(C)|$ are effective, then $|A(C)|$ is the *linear system adjoint to* $|C|$. Finally, if $|\bar{C}|$ is the system obtained by imposing assigned base points O_i to assigned multiplicities k_i on $|C|$, then the system $|A(\bar{C})|$ obtained by imposing the same base points to multiplicities k_i-1 on $|A(C)|$ is called the *linear system adjoint to* $|\bar{C}|$ *with assigned base.*

8.2. The principal theorem of the present section is

THEOREM XIII. *If* $|C|$ *is any (proper) irreducible linear system on* F, *whose base points (if such exist) are all regarded as assigned, then curves of the adjoint system* $|A(C)|$, *if this exists, meet the generic curve of* $|C|$, *apart from intersections at the base points, in canonical sets of this curve.*

The proof of this theorem is by three stages as follows.

(i) *Suppose first that* $|C|$ *is free from base points and of freedom* $r \geq 2$. We consider then a general linear net N of curves of $|C|$; and we denote by $J(C)$ the Jacobian curve of the net, and by C a generic curve of the net. If n is the grade of $|C|$, the curves of N will cut a g_n^1 on C; and this will have a Jacobian set, Γ say, whose points must lie on $J(C)$, since the latter is the locus of all points of coincidence in characteristic sets of N. Also, plainly, every intersection of $J(C)$ with C is a point of Γ, since N is free from base points. Hence, on the one hand,

$$\Gamma \equiv X_0(C)+2C[C],$$

where $X_0(C)$ denotes a canonical set on C; and, on the other hand,

$$\Gamma \equiv J(C)[C] \equiv (A(C)+2C)[C] \equiv A(C)[C]+2C[C].$$

By comparing these, we have the required result

$$A(C)[C] \equiv X_0(C).$$

(ii) *Suppose next that* $|\bar{C}|$ *is a system with base points, all regarded as assigned, but that* $|\bar{C}|$ *has still freedom* $r \geq 2$. Let the base points of $|\bar{C}|$ be O_i $(i = 1,..., s)$, of multiplicities k_i, and let the corresponding complete system (free from assigned base points) be $|C|$. On resolving the neighbourhoods $\tilde{O}_i$ into explicit curves of a surface F', we obtain in place of $|\bar{C}|$ the system $|C_1| \equiv |C-\sum k_i \tilde{O}_i|$ which is proper, irreducible, and free from base points on F'; and the generic C_1 is therefore met in canonical sets by curves of the adjoint system $|A'(C_1)|$ as already shown, such canonical sets corresponding to canonical sets on the associated $\bar{C}$ on F. Now

$$X' \equiv X + \sum \tilde{O}_i,$$

and hence

$$A'(C_1) \equiv X' + C_1 \equiv X + \sum \tilde{O}_i + C - \sum k_i \tilde{O}_i \equiv A(C) - \sum (k_i - 1)\tilde{O}_i,$$

so that $|A'(C_1)|$ corresponds to the adjoint system $|A(\bar{C})|$ on F, as already defined for a system with assigned base. It follows therefore that curves of $|A(\bar{C})|$ cut canonical sets on those of $|\bar{C}|$ apart from intersections at base points.

(iii) *Suppose, finally, that $|C|$ has freedom* 0 *or* 1. We may assume here that the generic C (or C itself if $r = 0$) is non-singular, the contrary case being reducible to this by resolutions.

We choose then an auxiliary system $|K|$ such that (*a*) the complete system $|C+K|$ is free from base points and of freedom $r \geqslant 2$, and (*b*) the generic K meets the generic C in a finite number i of points. Let $C+K$ be any composite curve consisting of a generic C and a generic K, and let $|L|$ be a general linear net, containing $C+K$, from the system $|C+K|$. If n is the virtual grade of $|C|$, the curves of $|L|$ cut a g^1_{n+i} on C; and the Jacobian curve $J(L)$ of $|L|$ will meet C in (*a*) the Jacobian set Γ of the g^1_{n+i}, and (*b*) the set $K[C]$ of intersections of K with C; for any point of either of these sets counts twice in the characteristic set of $|L|$ to which it belongs. Hence, as in (i), we have the two relations

$$\Gamma \equiv X_0(C) + 2L[C] \equiv X_0(C) + 2K[C] + 2C[C],$$

$$\Gamma + K[C] \equiv J(L)[C] \equiv (X+3L)[C] \equiv A(C)[C] + 2C[C] + 3K[C],$$

whence, by subtraction,

$$A(C)[C] \equiv X_0(C).$$

This completes the proof of the theorem. In applying it to obtain canonical sets on any particular (irreducible) curve C of F, every k-fold point of C must be accounted an assigned $(k-1)$-fold base point of the appropriate adjoint curves.

8.3. *Virtual canonical series.* We have just shown in the preceding section that if an irreducible curve C of F has any multiple points, then the curves of the unrestricted adjoint system

$A(C)$ do not cut canonical sets on C; but for many purposes it is convenient to consider the series which they do cut on C. This leads us to the definition:

The *virtual canonical series* (or *virtual canonical series of equivalence*) on any curve C of F is the complete series (or series of equivalence) defined by any one of the sets in which C is met by a free adjoint curve $A(C) \equiv C+X$.

From now on we shall use the symbol $X_0(C)$ to denote a virtual canonical set of C, in the sense just explained; and we shall distinguish this from an effective (i.e. true) canonical set of C by denoting this latter by $\bar{X}_0(C)$. The grades of the corresponding series are $2\pi(C)-2$ and $2\bar{\pi}(C)-2$, where $\pi(C)$ and $\bar{\pi}(C)$ are the virtual and effective genera of C, equal only when C is non-singular.

8.4. *Canonical series on a composite curve.* The preceding definition only applies, in the first instance, to irreducible curves of F; but in order to bring out the consequences, in certain limiting cases, of the arguments leading to Theorem XIII, it is convenient to give it an interpretation also for composite curves. This requires some convention as to the meaning of equivalent sets and linear series on such curves.

To define *equivalent sets on any composite curve* C of F, we proceed as follows:

(i) We regard the irreducible components C_i $(i = 1,..., h)$ of C as all completely separate, whether they merely intersect or whether some of them overlap to form a multiple component of C; so that any point common to C_i, C_j $(j \neq i)$ represents two distinct points of C.

(ii) We say that two sets $G = \sum G_i$ and $G' = \sum G'_i$, where G_i and G'_i are sets of C_i, are equivalent on C if and only if $G_i \equiv G'_i$ on C_i for $i = 1,..., h$.

On this basis we may define the virtual canonical series on C exactly as in § 8.3. If an irreducible curve C', varying in $|C|$, tends continuously to C, then the virtual canonical sets cut by $|A(C)|$ on C' tend to virtual canonical sets on C; and the grade of $|X_0(C)|$ is still given by $2\pi(C)-2$. This remains true, likewise, if C is isolated or if $|C|$ is reducible, for we can still apply the usual arguments employing an auxiliary system $|K|$ such that $|K|$ and $|C+K|$ are both irreducible.

If we consider now the case in which $C = C_1+C_2$, where C_1 and C_2 are irreducible and intersect in i points, we have

$$X_0(C) \equiv (C_1+C_2+X)[C] \equiv (C_1+C_2+X)[C_1]+(C_1+C_2+X)[C_2],$$

i.e. $$X_0(C) \equiv \{X_0(C_1)+C_2[C_1]\}+\{X_0(C_2)+C_1[C_2]\}.$$

Hence

A virtual canonical set of the composite curve C_1+C_2 is equivalent to the sum of virtual canonical sets of C_1 and C_2 together with the sets cut by each of C_1, C_2 on the other.

If we equate the orders of the sets concerned, we have

$$2\pi(C_1+C_2)-2 = \{2\pi(C_1)-2\}+\{2\pi(C_2)-2\}+2i,$$

whence $$\pi(C_1+C_2) = \pi(C_1)+\pi(C_2)+i-1;$$

and this constitutes an independent proof of the formula for the (virtual) genus of a composite curve on a surface (cf. § 4.2).

We note finally that the (virtual) number of points in which any curve C of F is met by a canonical curve is

$$XC = (X+C)C-C^2 = 2\pi(C)-2-C^2 = \vartheta(C),$$

i.e. *the canonical number $\vartheta(C)$ of any curve on F is the virtual number of intersections of C with a canonical curve of F.* This explains, evidently, the additive property of ϑ.

8.5. *Curves adjoint to plane sections.* Consider now, in [3] a surface F^n of order n with normal singularities. The adjoint F^{n-4} (passing through the double curve) meet F, as we have seen, in canonical curves X; and hence adjoint F^{n-3} meet F in curves of the system $C+X$, where C is a plane section of F, i.e. in curves adjoint to the plane sections. Hence:

If F^n has plane sections of genus p, the F^{n-3} adjoint to F^n cut this surface in curves of order $2p-2$ adjoint to the plane sections.

More generally, the adjoint F^{n-4+l} meet F^n in curves adjoint to the complete sections of F^n by surfaces of order l; and we note that all the adjoint systems so obtained are complete (cf. § 1.5).

To calculate the canonical number $\vartheta(K)$ of an arbitrary curve K of order ϵ_0 on F^n, we make use of the class of immersion ζ of K in F^n, this being the number of tangent planes to F at points of K which pass through an arbitrary point O of space. Plainly ζ is the number of points, not on the double curve of F^n, in which K is met by the first polar of O; in other words, ζ is the number of intersections of K with the Jacobian $J(C)$ of sections of F^n by planes through O. Thus

$$\zeta = \{J(C)\}K = (X+3C)K = XK+3CK = \vartheta(K)+3\epsilon_0;$$

and this formula, which is independent of any possible singularities of K, is true for a surface in space of any number of dimensions, since ζ is unaltered by general projection. Hence:

The (virtual) canonical number $\vartheta(K)$ of any curve of order ϵ_0 and class of immersion ζ on a surface F is given by $\vartheta(K) = \zeta-3\epsilon_0$.

8.6. *Relation between the grade and genus of X.* A further deduction from Theorem XIII is that the virtual genus $\omega = \pi(X)$ and virtual grade X^2 of the canonical system of a surface are not independent invariants. For the system adjoint to $|X|$ is $|2X|$; so that, if $|X|$ is effective,

$$X_0(X) \equiv 2X[X],$$

and by equating the virtual grades of both sides we have the numerical relation

$$2\omega-2 = 2X^2.$$

When the canonical system is wholly virtual, the above relation still holds, as can be proved in the usual way by use of an auxiliary system $|K|$ such that $|K+X|$ is effective; and this proves

THEOREM XIV. *If $\omega = \pi(X)$ is the virtual genus of the canonical system, then the virtual grade of this system is $\omega-1$.*

This result is confirmed by the examples given in § 7.3.

By use of the above relation and the formulae of §§ 4.1, 4.2, we obtain the further useful numerical results:

(i) *the grade $\gamma^{(n)}$ and genus $\omega^{(n)}$* OF THE PLURICANONICAL SYSTEM *$|nX|$ are given by $\gamma^{(n)} = n^2(\omega-1)$, $\omega^{(n)} = \frac{1}{2}n(n+1)(\omega-1)+1$;*

(ii) *the grade γ' and genus π' of the system $|A(C)|$ adjoint to a given system $|C|$ of grade γ and genus π are given by*

$$\gamma' = 2(2\pi-2)-\gamma+\omega-1, \qquad \pi' = 3(\pi-1)+\omega-\gamma.$$

§ 9. SUMMARY OF FURTHER DEVELOPMENTS

9. We shall now conclude this brief introduction to the invariantive geometry of surfaces by giving a short summary of a few of the further developments which cannot here be discussed in detail. It is hoped that this may stimulate the reader's interest and serve in some measure as a guide to his further reading. The summary can make no pretence of being complete; but it will indicate some of the many directions in which the invariantive theory of surfaces has been developed.

9.1. ***The definitions of adjoint and sub-adjoint surfaces.*** We have already referred to the fact that for a surface F with normal singularities in [3], surfaces of any given order through the double curve cut a complete system of curves on F, the basis of this result being a straightforward extension to surfaces of Noether's $Af+B\phi$ Theorem for curves.

To extend the scope of the theorem, by the same method, a surface is first defined as being *sub-adjoint* to a surface F with arbitrary singularities in [3] if it passes $(i-1)$-fold (at least) through every explicit i-fold curve of F; and this leads to the theorem:

The sub-adjoint surfaces of any given order of a surface F cut a complete linear system of curves on F.

The definition of *proper adjoint surfaces* of a surface with arbitrary singularities is a much more delicate problem: but a criterion sufficient to define them is that which runs as follows:†

The surfaces Φ^l, *of order* l, *adjoint to a given surface* F, *of order* n *in* [3], *are to be defined in all cases in such a way that they cut on* F *the system of curves adjoint to* $|(l-n+4)C|$, *where* $|C|$ *is the system of plane sections of* F.

On the basis of this criterion it is found that (i) adjoint surfaces must pass $(i-1)$-fold through every i-fold curve of F (including those which are infinitely near to explicit multiple curves); (ii) if O is any isolated s-fold point of F, the adjoint surfaces must have an $(s-2)$-fold point at O; (iii) successive multiple points, O, O_1 say, of multiplicities s, s_1 on F, impose the same conditions on adjoint surfaces as if they were distinct; (iv) adjoint surfaces pass simply through an ordinary tacnode of F; (v) more generally, if F has an s-fold point at O and an s_1-fold line γ infinitely near to O, then if $s_1 < s$, the adjoint surfaces pass $(s-2)$-fold through O and (s_1-1)-fold through γ, but if $s_1 = s$, they pass $(s-1)$-fold through O and $(s-2)$-fold through γ.

The adjoint surfaces of order $n-4$ of F, if such exist, cut the canonical system on F; and we shall call these the Φ-surfaces.

9.2. ***Plurigenera of F.*** The multiples $|2X|$, $|3X|$,..., of the canonical system are called the pluricanonical systems of F (cf.

† Cf. Enriques–Campedelli, *Teoria delle superficie algebriche*, p. 125.

§ 8.6), and their effective freedoms augmented by unity are called the *plurigenera* $P_2, P_3, \ldots$ of F.

In particular the bicanonical system $|2X|$, of freedom P_2-1, is cut on F by *biadjoint surfaces* of order $2n-8$. Biadjoint surfaces of any order are determined by the conditions, for example, that (i) they pass doubly through a double curve of F; (ii) more generally, if F has an ordinary i-fold curve, the biadjoint surfaces pass i-fold through this curve and touch $(i-2)$-fold each of the sheets of F along it; (iii) if O is an isolated s-fold point of F, they are subject to no conditions if $s = 2$, they have a double point at O if $s = 3$, a 4-fold point at O if $s = 4$, and if $s \geqslant 5$, they have an s-fold point and contact with F of order $s-5$ at O, and (iv) if F has a tacnode at O, they touch the tacnodal plane at O.

9.3. *The arithmetic genus* p_a. Let F be a surface with ordinary singularities in [3], its projective characters being $\mu_0, \mu_1, \mu_2, \nu_2$, and let its double curve Γ be of order ϵ_0 and rank ϵ_1, with t triple points which are triple also for F.

The postulation of Γ for all surfaces of sufficiently high order l (i.e. the number of conditions that must be imposed on such surfaces to make them contain Γ) is found to be

$$l\epsilon_0-(\tfrac{1}{2}\epsilon_1-\epsilon_0)-2t;$$

and for unrestricted l, this formula defines the *virtual postulation* of Γ, which may be either less than or greater than the actual.

For $l = \mu_0-4$, the above formula gives immediately the *virtual freedom* of special adjoint surfaces Φ; and if we denote this by p_a-1, we have

$$p_a = \binom{\mu_0-1}{3}-(\mu_0-4)\epsilon_0+\tfrac{1}{2}\epsilon_1-\epsilon_0+2t.$$

The number p_a so defined is called the *arithmetic genus* of F; it is equal to the geometric genus p_g if and only if the postulation formula for Γ is valid down to $l = \mu_0-4$.

By using the results of Ch. IX, § 3, we can express ϵ_0, ϵ_1, t in terms of $\mu_0, \mu_1, \mu_2, \nu_2$ and so obtain the formula

$$p_a = \mu_0-1-\tfrac{2}{3}\mu_1+\tfrac{1}{6}\mu_2+\tfrac{1}{12}\nu_2.$$

The fact that p_a, as so defined, turns out to be an absolute invariant of F is one of the most surprising discoveries in the theory of surfaces; but such is in fact the case, and p_a takes its place beside p_g as one of the two most important invariants of a surface F. A surface for which $p_g = p_a$ is said to be *regular*. For an irregular surface the difference $q = p_g - p_a$ is always positive and it is called the *irregularity* of the surface.

The invariance of p_a follows at once, for example, from the justly celebrated

THEOREM OF PICARD. *If $|C|$ is any linear system of irreducible curves on F, of freedom $r \geqslant 1$, then the* DEFICIENCY *of the series cut on the generic curve of $|C|$ by the adjoint system $|A(C)|$ (i.e. the deficiency of the freedom of this series as compared with that of the complete series to which it belongs) is always equal to the irregularity $p_g - p_a$ of F.*

This remarkable theorem was proved by Picard† in 1905 by transcendental methods; later, however, a proof of an algebro-geometrical character was given by Severi,‡ based on the concept of continuous non-linear systems of curves. On the generic curve of such a system the proximate curves cut a linear series, called the 'characteristic series' of the system; and in the course of Severi's demonstration it is assumed that the *characteristic series of any complete continuous system of curves is complete.* The validity of this assumption, however, and of the various attempts to justify it, is still open to question; and in the algebro-geometric development of the theory of surfaces, the invariance of p_a is usually established by proving the theorem—weaker than Picard's—that $p_g - p_a$ is the *least upper bound* of the deficiency of the series cut by $|A(C)|$ on the generic curve of an irreducible system $|C|$.

The theorem of Picard implies that on any regular surface, the system $|A(C)|$ adjoint to a given system $|C|$ always cuts the complete canonical series on a generic curve of $|C|$.

9.4. *Irregular surfaces.* It is easily verified that every rational surface is regular, with $p_g = p_a = 0$; and every non-singular surface in [3] is likewise regular. But for a ruled surface of genus p, with normal singularities, we have $p_g = 0$ (the existence of special adjoints Φ in [3] being easily seen to be impossible), while it is found that $p_a = -p$. Hence:

Every ruled surface of genus $p > 0$ is irregular, with irregularity p.

† Picard, *Journal für Math.* **129** (1905), 275; cf. also Picard and Simart, *Théorie des fonctions algébriques de deux variables indépendantes* (Paris, 1906), vol. ii, ch. xiii, § 16.

‡ *Rend. Lincei*, (5), **17** (1908)$_2$, 465.

For the rest, irregular surfaces which are not birationally equivalent to scrolls are rather hard to find; but it is known, for example, that any surface which contains an irrational pencil of curves of genus π has irregularity $q \geqslant \pi$. Thus it is found, for example, that the surface in which a general cubic line-cone in [4] is met by a general cubic primal has $p_g = 3$, $p_a = 2$. Again any surface of order $2n$ which has n-fold points at the eight intersections of three quadrics $P = 0$, $Q = 0$, $R = 0$ has an equation of the form†

$$f_n(P, Q, R) = 0,$$

and for this surface $p_g = \frac{1}{2}n(n-1)$, $p_a = n-1$. Finally it can be shown‡ that the image surface of pairs of points P, P' of two curves C, C', of genera π, π' has

$$p_g = \pi\pi', \quad p_a+1 = (\pi-1)(\pi'-1), \quad \omega = 2(2\pi-2)(2\pi'-2)+1.$$

9.5. *The Riemann–Roch Theorem.* The Riemann–Roch Theorem for curves admits of generalization for linear systems on a surface, but only in the weaker form of an inequality, and by arguments of much greater depth. We define the *index of speciality* i of a linear system $|C|$ on F as the number of linearly independent curves of the canonical system $|X|$ which contain a generic C (as component), and the theorem in question for surfaces may then be expressed as follows:

THEOREM. *If $|C|$ is any complete linear system of curves on F of virtual characters n and π, index of speciality i, and freedom r, then*

$$r \geqslant n-\pi+p_a+1-i.$$

The number $\sigma = r-(n-\pi+p_a+1-i)$ is called the *superabundance* of $|C|$. The system is said to be *regular* if $\sigma = 0$ (cf. Severi, *Fondamenti di geometria algebrica*, p. 106).

The superabundance of the canonical system itself is easily seen to be equal to the irregularity of the surface.

Plainly regular systems are particularly simple, and in this connexion we have a very useful result (which depends on Picard's Theorem):

Every system which is adjoint to some irreducible system is regular.

This means that irreducible linear systems which are more ample than the canonical system (i.e. contain it partially) are regular.

† Castelnuovo, *Rend. Ist. Lombardo*, (2) **24** (1891), 137.
‡ Cf., for example, Baker, *Principles of geometry*, vol. vi. 282–5.

Another important result which can be derived from the Riemann–Roch Theorem is the celebrated

THEOREM OF CASTELNUOVO.† *On every surface F, the deficiency of the characteristic series of any complete linear system does not exceed the irregularity of F; and there exist systems for which this maximum deficiency is attained.*

It follows from this, of course, that on a regular surface every complete linear system has a complete characteristic series.

9.6. *Exceptional curves.* We mention here, for completeness, a result, due to Castelnuovo and Enriques,‡ to which we have already referred:

The only surfaces from which it is impossible to remove all exceptional curves by birational transformation are those birationally equivalent to (rational or irrational) scrolls.

One incidental result of the method of proof of this theorem is that on any non-scrollar surface any rational irreducible curve must have negative grade.

9.7. *The problem of rationality.* The problem of finding necessary and sufficient conditions for a surface to be rational is plainly one of fundamental importance; but its solution has not proved to be easy. It was soon found, for example, that it was not even sufficient to have p_g and p_a both zero.

The first stage in the solution of the problem was Noether's Theorem,§ to the effect that *if F contains a linear pencil of rational curves, then it is rational*; and then Castelnuovo proved the stronger result that *if F contains a net of elliptic or hyperelliptic curves whose characteristic series is non-special, then F is rational*, the method which he used being that of representing such a surface on a double-plane.

Finally, on the basis of the above results, Castelnuovo solved the problem completely; and the result is expressed in

CASTELNUOVO'S THEOREM.‖ *The necessary and sufficient conditions for the rationality of F are that $p_a = P_2 = 0$.*

The fact that the conditions $p_a = p_g = 0$ do not involve $P_2 = 0$ is illustrated by the existence of a type of sextic surface which

† *Annali di Mat.* (2) **25** (1897), 235; *Memorie scelte*, 361.

‡ *Annali di Mat.* (3) **6** (1901), 165; cf. also Enriques, *Teoria delle superficie algebriche*, p. 262.

§ *Math. Ann.* **3** (1870), 161.

‖ *Mem. della società italiana per le scienze* (3) **10** (1896), 103, or *Memorie scelte*, 307.

passes doubly through the six edges of a tetrahedron; for the geometric and arithmetical genera of such a surface are both zero, but the faces of the tetrahedron form a unique biadjoint surface of order 4, so that $P_2 = 1$.

A further result of Castelnuovo's,† this time a generalization of Lüroth's Theorem (Ch. IV, § 1.3), states that *all plane involutions are rational*; this implies that if the coordinates of a point on a surface F are expressible as rational functions of two parameters, then F is rational.

9.8. *The postulation of a surface for primals.* If F is an irreducible surface in $[r]$ $(r > 3)$, without multiple points, proper or improper, and if the system $|C|$ of prime sections of F has grade n and genus p, then the *virtual postulation* $P(l)$ of F for primals of order l (equal to the actual postulation for all sufficiently large l) is given by

$$P(l) = n\binom{l+1}{2} - l(p-1) + p_a + 1.$$

The difficult problem here is to find a useful lower limit to the values of l for which $P(l)$ is also the effective postulation.

One useful criterion is that *if $|kC|$ is regular and cuts a non-special series on C, then $P(l)$ is the effective postulation for $l \geqslant k$.*

But Severi, with the help of Picard's Theorem, has obtained much more useful and remarkable results for the case in which F is defined as a partial intersection of primals, residual to another non-singular surface; thus:

SEVERI'S THEOREM.‡ *If F and F' are irreducible surfaces, without multiple points, which together constitute the complete intersection of $r-2$ primals of orders n_i $(i = 1,..., r-2)$, then* (i) *the complete canonical system $|X|$ of F (if this exists) is cut on F by primals of order $\sum n_i - r - 1$ which pass through F'*, (ii) *primals of the same order cut canonical sets on the curve of intersection of F and F'*, (iii) *primals of any order $l > \sum n_i - r - 1$ which pass through F' cut on F the complete system of adjoints to a multiple of the system of prime sections of F*, and (iv) *the formula for $P(l)$ gives the effective postulation of F (or F') for $l \geqslant \sum n_i - r$, and, also, if F and F' are both regular, for $l = \sum n_i - r - 1$.*

† *Math. Ann.* **44** (1894), 125.

‡ *Rend. Palermo*, **17** (1903), 73.

In the case when $r = 4$, F (and therefore F') has a number d of improper nodes; and it may be shown then that, for

$$l \geqslant n_1+n_2-4,$$

the effective postulation of F is

$$P(l) = n\binom{l+1}{2}-l(p-1)+p_a+1-d;$$

and here again, if F and F' are regular, the formula holds also for $l = n_1+n_2-5$.

NOTES AND EXAMPLES ON CHAPTER XIII

1. We have already referred, in Ch. IX, § 7.1, to the *Zeuthen–Segre invariant* I of a surface, given by

$$I = \mu_2-2\mu_1+3\mu_0-4 = 12p_a-\omega+9.$$

If two surfaces F, F' are in birational correspondence, prove that

$$I-e = I'-e',$$

where e, e' are the numbers of exceptional curves of F, F' respectively which transform into points of the other.

2. Obtain the following values of p_a for the sextic surfaces of [3] whose double curves are respectively

(i) a conic: $p_a = 5$;
(ii) two skew lines: $p_a = 4$;
(iii) three concurrent lines: $p_a = 3$;
(iv) the edges of a tetrahedron: $p_a = 0$;
(v) an elliptic quartic and a line not meeting it: $p_a = -1$.

3. If two surfaces F, F', of arithmetical genera p_a, p'_a respectively, have for partial intersection a curve C of genus π, prove that the arithmetical genus of the composite surface $F+F'$ on which C is included in the double curve is $p_a+p'_a+\pi$.

4. Prove that two irreducible systems of curves which are mutually residual with respect to the canonical system have the same virtual dimension and the same superabundance.

5. Show that the postulation formula for a surface is valid for all l in the cases where F is (i) a quadric, (ii) a Segre quartic surface, (iii) the complete intersection of a quadric and a cubic primal.

Show also that if F is the complete (non-singular) intersection of primals of orders n_1, n_2 in [4], then, except when $n_1 = n_2 = 2$ or $n_1 = 2$, $n_2 = 3$, the complete pluricanonical system $|iX|$ is cut on F by the primals of order $i(n_1+n_2-5)$, and deduce that

$$P_i = \tfrac{1}{2}i(i-1)(\omega-1)+p_a+1.$$

6. Show that the postulation formula is universally valid for the following general surfaces of [4]: (i) the cubic scroll, (ii) the rational surface ${}^2F^5$, (iii) the intersection ${}^3F^6$ of two cubic primals residual to a cubic scroll.

Show also that the modified formula, allowing for d improper nodes, is valid, for all values of l, for the following surfaces in [4]: (i) the general projected rational quartic scroll; (ii) the surface ${}^3F^6$ residual, for a quadric and a quartic primal, to two skew planes; (iii) the elliptic quintic scroll ${}^1R^5$; (iv) the projection on [4] of the normal elliptic sextic scroll ${}^1R^6$ in [5]; (v) the projected Del Pezzo surface ${}^1F^5$.

7. Show that the intersection of two cubic primals in [4], residual to a plane, is a surface F^8 having $p_g = p_a = 2$, $\omega = 1$, on which $|X|$ is a pencil of elliptic quartics.

8. In [4], two cubic primals containing the projection ${}^0F^4$ of a Veronese surface meet elsewhere in a scroll ${}^1R^5$; the curve common to ${}^0F^4$ and ${}^1R^5$ is a ${}^6C^{10}$. On this the primes cut a g^4_{10} which is special and therefore of deficiency 1, equal to the sum of the irregularities of ${}^0F^4$ and ${}^1R^5$.

9. Determine the character of the canonical system for the intersection of a cubic and a quartic primal of [4] residual to (i) a scroll ${}^0R^3$, (ii) two skew planes, (iii) a Segre ${}^1F^4$, (iv) a scroll ${}^0R^4$.

10. A *canonical surface* Φ is one whose prime sections form its complete canonical system. Verify that the following surfaces satisfy this definition:

for $p_g = 4$, $\omega = 6$, Φ is a general quintic surface;

for $p_g = 4$, $\omega = 7$, Φ is a sextic surface with a double plane cubic.

If primals of orders n_1, n_2 in [4] intersect in a surface F, show that F is a canonical surface if $n_1 + n_2 = 6$, and extend this result to higher space.

BOOKS RECOMMENDED FOR FURTHER READING

BAKER, *Principles of geometry*, vi.

CONFORTO, *Le superficie razionali.*

ENRIQUES–CAMPEDELLI, *Lezioni sulla teoria delle superficie algebriche.*

LEFSCHETZ, *L'analysis situs et la géométrie algébrique.*

PICARD and SIMART, *Théorie des fonctions algébriques de deux variables indépendantes*, i and ii.

SEVERI, *Fondamenti di geometria algebrica.*

ZARISKI, *Algebraic surfaces.* (This work contains a bibliography which is complete up to 1935.)

PRINTED IN
GREAT BRITAIN
AT THE
UNIVERSITY PRESS
OXFORD
BY
CHARLES BATEY
PRINTER
TO THE
UNIVERSITY