Analytic Geometry

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Preface

To the Student

The study of analytic geometry is equally important for the science major, the engineering student, and the mathematics major. Its importance derives from the graphical insight that it gives to algebraic operations, to trigonometry, and to more advanced operations yet to be studied. Together with algebra and trigonometry it is the foundation on which calculus is built and on which numerous courses in science and engineering depend. The liberal arts student, also, finds analytic geometry interesting and profitable, for as he solves problems with a graphical explanation or a practical flavor he encounters mathematical concepts and applications hitherto unknown to him. To the student in any curriculum, the varied topics and multiple procedures of analytic geometry present the necessity of choice and of correlation, thus contributing to the development of his ability to reason.

The two basic objectives of this course are to learn to draw graphs of given equations and to learn to find the equations of certain curves. At the same time that the student is mastering these procedures he should be grasping how to reason in terms of graphs. For example, a thorough understanding of the locus-derivation method and of the use of basic equations of curves gives the student confidence that he can find the equation of a given curve. The study of general methods of sketching curves gives him an understanding of graphical processes that is invaluable in this and in later courses. Frequently in this book there will be more than one method available for sketching a given curve or for finding a required equation. As the student learns to choose between the available methods he will be developing mathematical maturity. His selection of the quickest method is not so important as his understanding of each step of the method chosen.

Many freshmen begin the course in analytic geometry without having already learned *how to study*. Numerous aids are given in this book to promote understanding and correlation of course content. The terse review in the appendix of some of algebra and trigonometry is more than a mere listing of formulas. The study hints, reminders, and "Exercises for the Student" throughout the book are all devices to encourage careful and *active* study. The student is expected not merely to read the derivations and illustrative problems in each assignment but to work them out with paper and pencil. Not until he has actively studied in this way is he ready to proceed to the solution of assigned problems. Some students, unfortunately, begin their study by trying to work assigned problems. When they have difficulty, they go back to read as little of the text and the illustrative material as is necessary to enable them to *get the answer* for a particular problem. The college student should never say he has "studied" when he has been guilty of this activity, which may in high school have erroneously been called "studying." He may have solved a few problems, but he has failed in his assignment, which was to learn some ideas.

The summaries at the ends of the chapters are intended to aid the student in learning the art of comprehensive and cumulative review. They will be extremely helpful for use before each quiz and before the final examination.

To the Teacher

There are many unusual features in this book in addition to those already mentioned. For example, the problems are unusually numerous and are variously adaptable. Some will serve for classroom drill, whether oral, at the blackboard, or at the seat. Others suggest future uses of mathematics and should stimulate the student's interest and motivate his study habits. Problems that correlate algebra, trigonometry, and analytic geometry aid in integrating the mathematics already learned. A few problems in each list, marked with an "S," are intended for assignment to the superior students, who should never be neglected. Because so many of the problems are novel and challenging, students have been found reading more than those assigned.

Many of the answers are unorthodox. The answers to the early problems in each list are of the traditional type. But answers to the later problems are frequently "partial answers." For example, if a student is asked to find the equation of a circle that goes through three given points, the "answer" may be a reminder that the circle should go through the three given points, or it may be a statement that the circle goes through a fourth point whose coordinates are given as the partial answer. Such answers, since they require the student to check his work, should help him to gain confidence and thoroughness. Moreover, they continually emphasize the fundamental principle of analytic geometry, that a curve will go through a point if the coordinates of the point satisfy the equation of the curve.

Many students confuse basic procedures of analytic geometry with applications within the subject, especially when one is taught in terms of the other. Wherever possible the author has tried to avoid such confusions. By separating the idea of translation of axes from its later applications to the conics, we can use this concept often enough in its introduction, its applications to the conics, its applications to transcendental curves, etc., so that understanding and retention of it are assured. The same is true of the idea of addition of ordinates and of multiplication of ordinates.

The content of the typical course in analytic geometry has increased in recent years because of the changing nature of courses in science, engineering, and mathematics. The author is convinced that we can achieve best results in analytic geometry by reducing the amount of memory work required and adhering to fundamentals. In some places in this book traditional formulas have been dropped if the associated problems could be solved by graphical reasoning. It is the author's firm belief, based on more than twenty years of teaching in mathematics (and on occasion in other departments), that it is much more important for the student to develop analytical and graphical powers of reasoning than to fill his mind with a maze of formulas that he will forget within a few months. We can help the student along these lines by assigning problems that ask him to interpret his graph by estimating a particular ordinate, by reading the simultaneous solutions of the equation of the curve and of a straight line, by using the graph as a table of values, etc. We can, by careful selection, choose a mode of solution that emphasizes fundamental concepts already learned and, at the same time, embraces the new idea. For example, the chapter on curve fitting can be covered in three or four assignments, depending on whether the article on interpolation is included. And, since the basic tool of this chapter is the method of selected points, a portion of this study will be valuable review. Moreover, this method is adequate for any problems that the undergraduate will encounter. When at a later time he needs an understanding of the methods of averages or of least squares, he will have the ability to learn those concepts. The single article on interpolation, like the article on the solution of an equation in one variable by the method of intersecting curves, is included because it, too, combines the review of old material with an introduction of the new and valuable idea.

The content of the two chapters on solid analytics is sufficient for the requirements of calculus and of later undergraduate courses in science and engineering. Here again the emphasis is on basic ideas of graphs and of equations.

There is enough material in this book for a typical 5-hour, 1-semester course. On the other hand, a few of the articles may be omitted and other topics covered less thoroughly so that the book may serve for a shorter course (such as the one given at North Carolina State College). And there are sufficient problems to allow for several assignments without overlapping.

This book is the result of 14 years of intermittent writing and rewriting to get a text that is teachable and at the same time productive of good study habits and mathematical reasoning. The author has had the benefit of generous suggestions and candid criticism from his colleagues at North Carolina State College. Professors R. C. Bullock, J. M. Clarkson, and C. G. Mumford read the manuscript before it was used in lithoprint form at this college. The chairman of the department of mathematics, Professor H. A. Fisher, encouraged the author as the book developed. Professor F. H. Miller of Cooper Union, who read the book in lithoprint form, made numerous valuable criticisms. To these and to his other colleagues the author expresses his sincere appreciation.

Failures to achieve the desired objectives are the sole responsibility of the author. He would welcome any suggestions or corrections from the teachers who may use this book.

Finally, the author desires to acknowledge the encouragement and assistance given by his wife both in the writing of the book and in reading the proof.

Raleigh, North Carolina
April, 1951

JOHN W. CELL

Contents

CHAPTER 1

Elementary Concepts of Plane Analytic Geometry

This first chapter is introductory, and contains fundamental concepts and formulas essential to the mastery of analytic geometry and to the study of succeeding courses in mathematics and the physical sciences.

1.1 Directed Line Segments

In this section we shall set up a coordinate system for locating points on a line. The student is already familiar with the thermometer and its readings above and below zero. We could, if we liked, say $+10°$ when we mean $10°$ above zero and $-20°$ when we mean $20°$ below zero. Also, a fall of $30°$ from any given temperature could be designated by $-30°$ and a rise of $40°$ by $+40°$. Thus, the sign prefixed to the temperature change would indicate the direction of that change in temperature. If no sign is prefixed, the positive sign is to be understood.

On the line segment shown in Fig. 1.1, the right-hand direction is to be positive and is so indicated by the arrow at the right-hand end

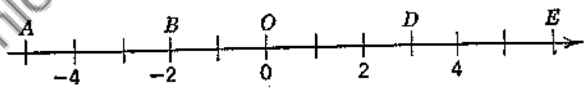


FIG. 1.1

of the line. The capital letters indicate the positions of several points, and the distances of these points from the point O can be determined by aid of the scale of numbers given below the line. Thus, B is 2 units to the *left* of O, and D is 3 units to the *right* of O.

On this diagram we can interpret $\overline{AB}$ as the magnitude of the distance from A to B, together with a sign prefixed to the magnitude.

If the direction from A to B is the same as that of the arrow on the line segment, the sign is positive; if the direction is opposite to this, the sign is negative. Thus, $\overline{AB} = +3$, $\overline{BD} = +5$, $\overline{EO} = -6$, and $\overline{BA} = -\overline{AB} = -3$.

Figure 1.2 is similar to Fig. 1.1 except that directed distances from the zero point, O, are indicated by small letters instead of by numbers.

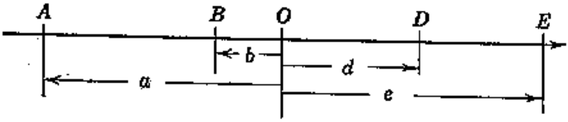


FIG. 1.2

On this diagram $\overline{OD} = d$, $\overline{OB} = b$, etc. (Notice carefully that a and b designate negative numbers in this figure.) Then

$$\overline{DE} = \overline{OE} - \overline{OD} = e - d, \quad \overline{AD} = \overline{AO} + \overline{OD} = \overline{OD} - \overline{OA} = d - a.$$

EXERCISE FOR THE STUDENT.* Verify each of the following in terms of Fig. 1.2: $\overline{BD} = d - b$, $\overline{EA} = a - e$, $\overline{AB} = b - a$.

The result, $\overline{AD} = d - a$, gives the directed distance (magnitude and direction) *from A to D*. The answer, $d - a$, is the directed distance from the origin, O, to the point D (*to which* the measurement is to be made) *minus* the directed distance from the origin to the point A (the point *from which* the measurement is to be made). Since the numerical value of a quantity may be denoted by $|a|$ (for example, $|-3| = |+3| = 3$), we may write $|\overline{AB}| = |\overline{BA}|$.

1.2 Cartesian Coordinates

We now proceed to develop a coordinate system to locate points in a plane. In Fig. 1.3 there are two directed lines drawn at right angles to each other and intersecting at a point O, called the "origin." The chosen positive directions are indicated by arrows at the right end of the horizontal line and at the upper end of the vertical line. In this book we shall refer to these two lines as the x-axis and the y-axis.

Uniform or ordinary scales have been laid off on both axes. To locate a point anywhere in the plane of these two lines, we need only tell how far to go from each of these two axes and in what direction.

* The careful student will perform these exercises as they occur and as an integral part of the preparation of his assignment.

Thus, instead of saying, "Start at O and go east 4 units and then go north 3 units," we can say, "Locate the point whose coordinates are (4, 3)." When we mean "Start at O and go west 3 units and then north 2 units," we say, "Locate the point whose coordinates are (−3, 2)."

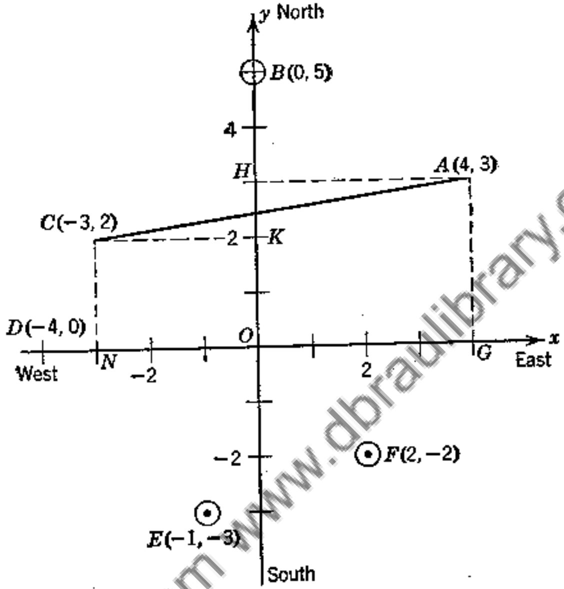


Fig. 1.3

The x-coordinate of a point is the directed perpendicular distance from the y-axis to the point and is called the *abscissa*. The y-coordinate of a point is the directed perpendicular distance from the x-axis to the point and is called the *ordinate*.*

The student has already used the terms abscissa and ordinate in his study of trigonometry and has learned to think of four quadrants,

* René Descartes (1596–1650) was a French mathematician who is given credit for introducing coordinate axes and combining geometry and algebra into what is now known as analytic geometry. The Greeks as far back as Apollonius, who died about 210 B.C., had studied the geometry of curves. Other mathematicians in the intervening years had developed much of what is now taught in high-school algebra and in college algebra. But it remained for Descartes to combine the two into a single study, and this he did only in substance. Succeeding mathematicians have developed the material for what is now given in the course in analytic geometry. The phrase "Cartesian coordinates" honors René Descartes for his contribution.

as numbered in Roman numerals in Fig. 1.4. He should observe that the ordinate of a point is positive if the point is above the x-axis, negative if below; that the abscissa is positive if the point is to the right of the y-axis, negative if to the left.

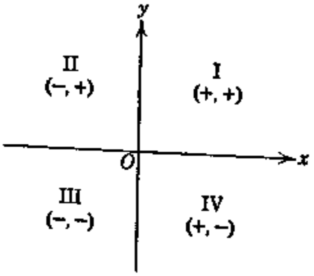


FIG. 1.4

1.3 Projections

The orthogonal projection (here, for brevity, called the projection) of a point on a line is the foot of the perpendicular drawn from the point to the line. The projection of a line segment $\overline{AB}$ upon a line L (see Fig. 1.5) is the directed line segment $\overline{A'B'}$ on L, where A' is the projection of the point A on the line L and B' is the projection of the point B. We shall denote the projection of $\overline{AB}$ on L by $(AB)_L$.

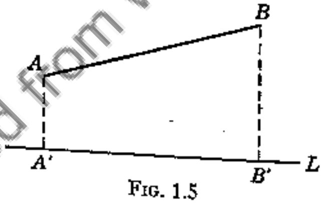


FIG. 1.5

We use Fig. 1.3 and see that $(CA)_x = \overline{NG} = +7$, $(CA)_y = \overline{KH} = +1$. Thus, to determine $(CA)_x$, we take the x-coordinate of the point A (*to which* the measurement is made) *minus* the x-coordinate of the point C (*from which* the measurement is made); or,

$$(CA)_x = (4) - (-3) = 7.$$

In a similar manner we can determine $(CA)_y$ by taking the y-coordinate of the point A minus the y-coordinate of the point C; or,

$$(CA)_y = (3) - (2) = 1.$$

EXERCISE FOR THE STUDENT. Use Fig. 1.3 and verify the following statements: $(EA)_x = 5$; $(EB)_y = 8$; $(FD)_x = -6$; $(BD)_y = -5$.

In Fig. 1.6, using letters for the ordinates and abscissas, we use the same process, as follows:

$$(P_2P_1)_x = \overline{EF} = \overline{EO} + \overline{OF}$$
$$= -\overline{OE} + \overline{OF} = -x_2 + x_1$$
$$= x_1 - x_2.$$

Thus, the directed length of the projection of $\overline{P_2P_1}$ *on the x-axis is the x-coordinate of the point* P_1 (to which *the measurement is made*) minus *the x-coordinate of the point* P_2 (from which *the measurement is made*).

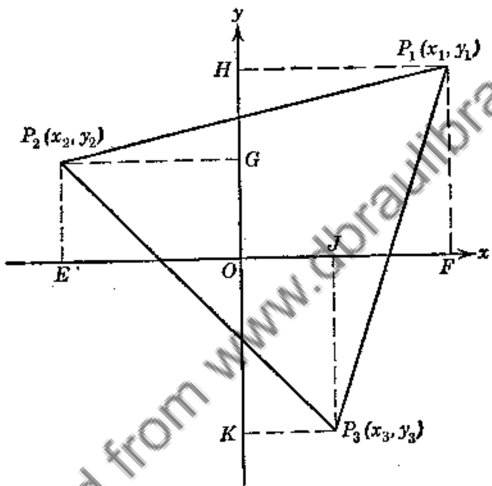


Fig. 1.6

Exercise for the Student. Use Fig. 1.6 and verify the following results:

$$(P_2P_1)_y = y_1 - y_2, \quad (P_3P_2)_x = x_2 - x_3, \quad (P_3P_2)_y = y_2 - y_3.$$

1.4 The General Distance Formula

The method of the preceding article may be used to find the directed length of a line segment that is parallel to either of the coordinate axes. To determine the numerical, or positive, value for the length of any inclined or oblique * line we may proceed as follows: Let the

* An oblique line is defined in this book to be any line in the xy-plane which is neither vertical nor horizontal, i.e., which is not parallel to either the x-axis or the y-axis.

ends of the line segment be respectively $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as shown in both Fig. 1.7 and Fig. 1.8.* Construct a right triangle

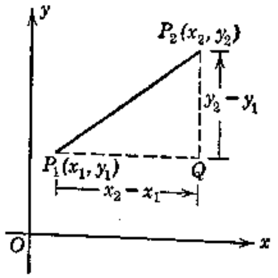


FIG. 1.7

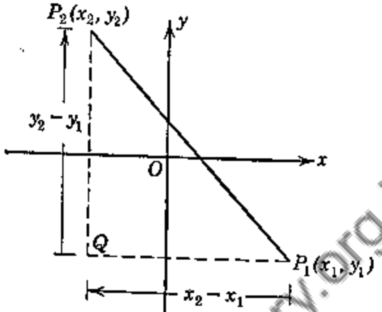


FIG. 1.8

P_1QP_2 that has this line segment for its hypotenuse. Then (for both figures)

$$\overline{P_1Q} = (P_1P_2)_x = x_2 - x_1, \quad \overline{QP_2} = (P_1P_2)_y = y_2 - y_1.$$

Use the Pythagorean theorem and obtain

$$(P_1P_2)^2 = (P_1Q)^2 + (QP_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Hence, L = length of $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

This result may be stated in words as follows: *the numerical or positive length of a line segment is equal to the positive square root of the sum of the square of the difference of the x-coordinates and the square of the difference of the y-coordinates.*

REMINDER. In college algebra it was agreed that, when there is no sign in front of a square-root symbol, the positive sign is to be understood.

If the student uses this general distance formula to find a *directed* distance (an unnecessarily long method) he *must* prefix the proper sign to his result. The student should learn, as he proceeds in his study of analytic geometry, to treat horizontal and vertical lines in one way and oblique lines in another.

* $\overline{P_1Q}$, for example, is dimensioned in both figures with a single arrow to emphasize that this is a directed quantity and to indicate the direction of measurement. This practice will be followed throughout this book.

It is of fundamental importance that the preceding formula for the length of an oblique line segment, and other formulas to be derived later, is valid irrespective of the quadrant in which a given point lies. The *principle of continuity*, which is the basis for the generalization of these relationships in analytic geometry, states that it is unnecessary to use a different length formula for each possible position (in the various quadrants) for the two points P_1 and P_2.

EXAMPLE

Given the three points $A(-4, -3)$, $B(2, -3)$, and $C(-4, 4)$. Determine the directed lengths $\overline{BA}$ and $\overline{AC}$ and the numerical length of $\overline{BC}$.

Solution. The student should draw a figure and label the three given points. Since $\overline{BA}$ is parallel to the x-axis:

$$\overline{BA} = (x\text{-coordinate of } A) - (x\text{-coordinate of } B) = -6.$$

Similarly, since $\overline{AC}$ is parallel to the y-axis, $\overline{AC} = +7$. To find the numerical or positive value for $\overline{BC}$, we may either use the formula of this article or take the positive square root of the sum of the squares of $\overline{BA}$ and $\overline{AC}$ (which amounts to the same thing). Thus,

$$L = \text{length of } BC = \sqrt{(-6)^2 + (7)^2} = \sqrt{85} \approx 9.22.*$$

The student should check the reasonableness of this result by use of his figure.

PROBLEMS

1. Given the triangle with vertices $A(-3, 2)$, $B(4, 2)$, $C(-3, -1)$. Draw the figure. Determine the directed lengths $\overline{AB}$ and $\overline{AC}$, and find the numerical length of $\overline{CB}$.

2. Determine the perimeters of each of the triangles with vertices as given:

(*a*) (3, 2), (7, 2), (7, 5).
(*b*) (1, 4), (0, 6), (−1, −2).
(*c*) (0, 0), (0, 5), (−12, 0).
(*d*) (2.57, 3.66), (1.44, 7.68), (−2.35, 4.78).
(*e*) (4, −4), (−2, −2), (2, 10).
(*f*) (1.75, 6.84), (5.92, 2.12), (7.63, −0.82).
(*g*) (1, 4), (4, 1), (−3, −3).
(*h*) (−3, 1), (−1, 3), (2, −2).

3. The abscissa of a point A is 4, of a point B is 7. The corresponding ordinates are each equal to 2 more than the square of the respective abscissas. Translate this last sentence into a mathematical equation; determine the ordinates of the points A and B and the straight-line distance $\overline{AB}$.

* The symbol ≈ means *is approximately equal*. Unless otherwise specified, all approximate results will be given in this book correct to the nearest three significant figures, that is, to slide-rule accuracy.

4. A triangle has vertices at $A(-4, 1)$, $B(x, 4)$, and $C(1, -2)$. Determine the abscissa for the vertex B so that the triangle will be isosceles with $\overline{AB} = \overline{CB}$.

5. Prove that each of the following triangles is a right triangle, and find the area:

(a) $(-2, 4), (6, 2), (5, -2)$.
(b) $(2, 3), (-2, -1), (1, -4)$.
(c) $(0, 0), (-6, 0), (0, -3)$.
(d) $(4, 1), (-2, 3), (2, -5)$.
(e) $(4a, 0), (0, -4a), (3a, -7a)$.
(f) $(0, 2b), (-3b, -b), (0, -4b)$.

6. The points $A(4, -5)$ and $B(-2, 3)$ are the ends of a diameter of a circle. Find the diameter and radius of the circle.

7. For each of the following quadrilaterals find the directed lengths of $\overline{AB}$ and $\overline{BC}$ and the positive lengths of $\overline{CD}$ and $\overline{DA}$:

(a) $A(-4, 4), B(-4, -2), C(1, -2), D(4, 2)$.
(b) $A(-3, 1), B(-3, -2), C(2, -2), D(4, 3)$.
(c) $A(5, 0), B(-6, 0), C(-6, -7), D(5, -2)$.
(d) $A(4, 2), B(4, -1), C(-3, -1), D(-5, 7)$.

8. Find the lengths of the diagonals for each of the quadrilaterals defined in Problem 7.

9. Express by an algebraic equation the statement that a point $P(x, y)$ shall always be 5 units distant from the point $A(3, -2)$.

10. Figure 1.9 shows a rectangle, a straight line, and a curve. Obtain a formula for the area of the rectangle in terms of y_1, y_2, and w.

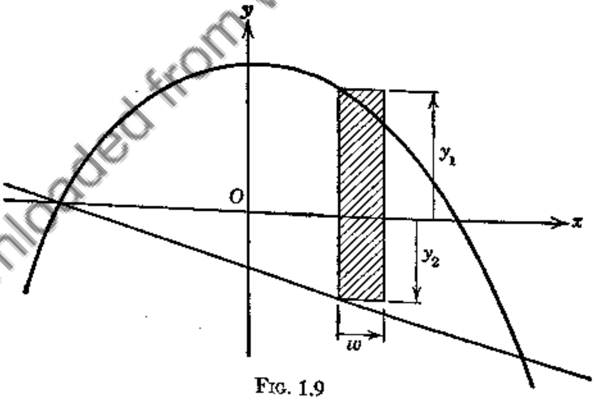


FIG. 1.9

11. Prove that the four points $A(7, -1)$, $B(-3, 2)$, $C(-6, -8)$, and $D(4, -11)$ are the vertices of a square, and find the radius of the inscribed circle.

12. If the product of the ordinate and abscissa is always 20, and if point A has an ordinate of 4 and point B has an ordinate of 10, find the distance between A and B.

1.5 Mid-Point Formulas

Let the ends of a line segment be at $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as shown in Fig. 1.10, and let the mid-point of this line segment be at

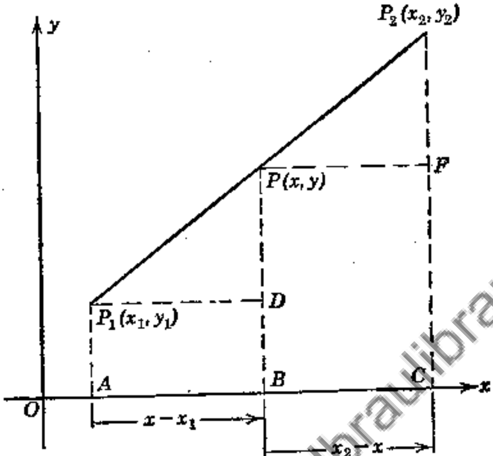


FIG. 1.10

$P(x, y)$. Since triangles P_1DP and PFP_2 are clearly congruent, we obtain $\overline{P_1D} = \overline{PF}$, or $x - x_1 = x_2 - x$.

Hence

$$x = \tfrac{1}{2}(x_1 + x_2).$$

Similarly

$$y = \tfrac{1}{2}(y_1 + y_2).$$

The student should, as an exercise, derive the formula for y. The mid-point coordinates are respectively the average of the x-coordinates and the average of the y-coordinates of the end points of the line segment P_1P_2.*

The same procedure † may be used to determine the coordinates of any point that divides a line segment into a given ratio.

* This is one of the few times that we add coordinates in a formula in analytic geometry. We usually *subtract.*

† Since one of the fundamental objectives of this course in analytic geometry is to learn to reason by aid of graphs, we omit the formulas frequently given for the coordinates of a point of division of a line segment. In this problem and in some subsequent topics we shall prefer the analysis based on the figure, even though that method of solution may be longer.

EXAMPLE

Given the two points $A(-1, 1)$ and $B(5, 3)$. Find the coordinates of the two points on the segment $\overline{AB}$ which trisect it.

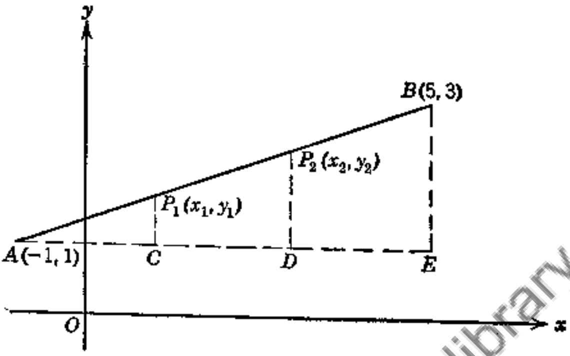


FIG. 1.11

Solution. We construct a figure such as Fig. 1.11 with the two required points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ located in their approximate positions. Since triangles ACP_1 and AEB are similar:

$$\frac{\overline{AC}}{\overline{AE}} = \frac{\overline{AP_1}}{\overline{AB}} \quad \text{or} \quad \frac{x_1 - (-1)}{6} = \frac{1}{3},$$

whence $x_1 = 1$;

$$\frac{\overline{CP_1}}{\overline{EB}} = \frac{\overline{AP_1}}{\overline{AB}}, \quad \text{or} \quad \frac{y_1 - 1}{2} = \frac{1}{3},$$

whence $y_1 = \frac{5}{3}$.

The required coordinates of P_1 are $(1, 5/3)$. The results could be checked exactly by aid of the distance formula. However, a glance at the figure shows that the coordinates are reasonable.

The student should use the same general procedure to show that the coordinates of P_2 are $(3, 7/3)$.

PROBLEMS

1. Given the triangle with vertices $A(-2, 1)$, $B(5, 1)$, $C(-2, -2)$. Determine the coordinates of the mid-point, D, of $\overline{BC}$, and then find the numerical values of the distances from this point to each of the three vertices.

2. Determine the coordinates of the mid-points of the sides of the following triangles:

(*a*) (3, 2), (7, 2), (7, 5).
(*b*) (1, 4), (0, 6), (−1, −2).
(*c*) (2.56, 3.66), (1.44, 7.68), (−2.36, 4.78).
(*d*) (1.75, 6.84), (5.91, 2.12), (7.63, −0.82).

3. The two points $A(8, 2)$ and $B(-4, -4)$ are the ends of a diameter of a circle. Find the coordinates of the center of the circle and the length of the radius.

4. Given the triangle $A(-2, 1)$, $B(4, 3)$, and $C(6, -1)$:

(*a*) Show that the mid-point on the side $\overline{AB}$ is $D(1, 2)$, on $\overline{BC}$ is $E(5, 1)$, and on $\overline{CA}$ is $F(2, 0)$.

(*b*) Find the coordinates of the point G on $\overline{AE}$ if G is twice as far from A as from E.

(*c*) Find the coordinates of the point H on $\overline{BF}$ which is twice as far from B as from F.

(*d*) Find the coordinates of the point I on $\overline{CD}$ which is twice as far from C as from D.

5. The ordinate of the point A is 2 and of the point B is 7. The abscissa and ordinate of each point form a solution of the equation $xy = 28$.

(*a*) Determine the abscissas of the two points.

(*b*) Determine the coordinates of the mid-point of the straight-line segment joining A and B.

(*c*) What is the ordinate of the point on the curve (whose equation is $xy = 28$) whose abscissa is the abscissa obtained in (*b*)?

(*d*) What is the abscissa of the point on the curve whose ordinate is the ordinate obtained in (*b*)?

6. Find the coordinates of the three points which divide the line segment from $A(-2, 8)$ to $B(10, 4)$ into four equal parts.

7. Given the two points $A(1, 1)$ and $B(7, 10)$. Find the coordinates of the point $P(x, y)$, on the line segment joining A and B, which is such that $\overline{AP} = 4\overline{PB}$.

8. Given that one end of a line segment is $A(4, -2)$ and that the mid-point is $M(7, 3)$. Find the coordinates of the other end of the line segment.

9. The mid-points of the three sides of a triangle are given in each of the following four cases. Find the coordinates of the vertices of the triangle. *Hint:* Assign literal coordinates to the three vertices and write down a sufficient number of simultaneous equations. Finally, check your results by finding the coordinates of the three mid-points from the coordinates of the three vertices which you have found.

(*a*) $(6, -1)$, $(-1, -2)$, $(1, -4)$.
(*b*) $(3, 5)$, $(-1, 6)$, $(0, 8)$.
(*c*) $(2, 3)$, $(4, 3)$, $(2, 6)$.
(*d*) $(1, 0)$, $(3, 4)$, $(-1, -2)$.

10. Given the two points $A(-1, 4)$ and $B(2, 1)$. Find the coordinates of a point P on $\overline{AB}$ extended through B such that P is:

(*a*) Twice as far from A as from B.

(*b*) Three times as far from A as from B.

11. Repeat Problem 10 for $A(2, 1)$ and $B(-1, 4)$.

12. Repeat Problem 10 for $A(-2, 1)$ and $B(4, 4)$.

13. Given $A(2, 5)$ and $B(4, 2.5)$.

(*a*) Use ordinary interpolation to find y when $x = 3$.

(*b*) Use the mid-point formula to find y when $x = 3$.

14. Prove that the three points $A(-2, 0)$, $B(2, -2)$, and $C(6, 6)$ form a right triangle, and find its area. Also determine the coordinates of the mid-point of the hypotenuse and verify that it is equidistant from the three vertices of the triangle.

15. A table of logarithms gives log 2.004 = 0.30190 and log 2.005 = 0.30211. Use the mid-point formulas to find the approximate value of log 2.0045, and check by ordinary interpolation. Note that the coordinates of the two points corresponding to the given data are (2.004, 0.30190) and (2.005, 0.30211).

16. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two given points. Let $P(x, y)$ be any point on the line which goes through P_1 and P_2, and let P be such that $\overline{P_1P}/\overline{PP_2} = r$, where $\overline{P_1P}$ and $\overline{PP_2}$ are directed line segments and r is a positive or negative number. Draw and label a schematic figure, and derive the following formulas:

$$x = \frac{x_1 + rx_2}{1 + r}, \quad y = \frac{y_1 + ry_2}{1 + r}.$$

17. Use the formulas derived in Problem 16 to find the coordinates of the point P in each of the following (a convenient check would be to find $\overline{P_1P}$ and $\overline{PP_2}$ by aid of your results):

(*a*) The line joining $P_1(1, -2)$ and $P_2(3, 2)$ is extended beyond P_2 to a point P, such that P is three times as far from P_1 as P is from P_2. Why does r, in this case, have the value -3?

(*b*) The point P is on the line segment between $P_1(0, 4)$ and $P_2(8, 0)$ and is three times as far from P_2 as it is from P_1. Why does $r = 1/3$ in this case?

(*c*) The line segment joining $P_1(-1, 3)$ and $P_2(5, 0)$ is to be trisected. Find the two points of trisection.

(*d*) The point P is on the line through $P_1(2, 4)$ and $P_2(5, 2)$ extended beyond P_1 so that $\overline{PP_1}$ is four times $\overline{P_1P_2}$.

1.6 Angle of Inclination. Slope

DEFINITION. *The angle of inclination (or the inclination, or the slope angle) of a line is the angle between $-90°$ and $+90°$ measured from the positive x-axis to the line and taken as positive if measured in the counterclockwise direction.* If the line is parallel to the y-axis the inclination will be taken as $+90°$.*

The *slope* of a line is defined to be the tangent of the angle θ of inclination (assuming that the angle is not 90°). Thus,

$$\theta = \text{arc tan (slope)},$$

and the principal value is to be used.

* Some books define the inclination to be between 0° and 180°. The definition of this book ensures agreement with the usual definition for the principal value for the inverse tangent function. It also ensures agreement with standard usages in certain topics in physics and engineering.

Figure 1.12 shows several lines with the angles of inclination and the slopes labeled. We observe that, if the axes are taken in the conventional position, lines sloping upward to the right will have positive

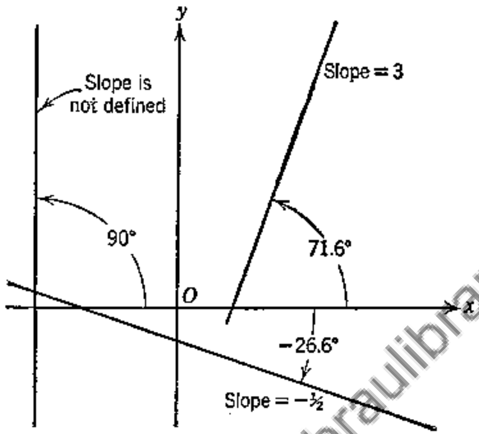


FIG. 1.12

slopes, and inclinations between 0° and 90°; lines sloping downward to the right will have negative slopes, and inclinations between 0° and −90°.

Since $\tan 0° = 0$ and $\tan 90°$ does not exist,* it follows that the slope of a horizontal line is zero and that the slope of a vertical line is not defined.

In order to determine the slope of an oblique line, we use a general oblique line as shown in Figs. 1.13 and 1.14. Let two points on this line be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Then in either of these figures θ is the angle of inclination and

$$\tan\theta = \frac{\overline{AP_2}}{\overline{P_1A}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2},$$

where the last fraction may be obtained from the one immediately preceding by multiplying the numerator and denominator by −1.

* There is a very real difference between the two statements: (1) the tangent of 90° is not defined; and (2) as the angle θ approaches 90° through values less than 90°, the value of $\tan\theta$ increases without limit. The first statement is concerned with the *nonexistence of any* value *at* 90°, and the other with what happens as the angle *approaches* 90°.

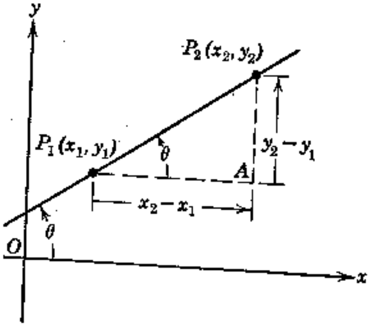


FIG. 1.13

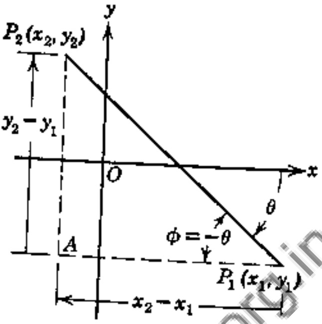


FIG. 1.14 *

Thus we see that the *slope of a line segment is the difference of the y-coordinates of the ends of the line segment divided by the difference of the x-coordinates, the differences being taken in the same order.*

1.7 Parallel and Perpendicular Lines. Angle between Two Lines

Figure 1.15 shows two oblique lines L_1 and L_2 which are at right angles to each other and whose slopes are respectively m_1 and m_2. Then

$$m_2 = \tan\theta$$

$$= \tan(90^\circ + \phi), \quad (\text{since } \theta - \phi = 90^\circ)$$

$$= -\cot\phi \quad (\text{why?})$$

$$= -\frac{1}{\tan\phi} \quad (\text{why?})$$

$$= -\frac{1}{m_1}.$$

Hence if two oblique lines are perpendicular their slopes are negative reciprocals of each other. Conversely, if two lines have slopes which are negative reciprocals of each other, then the two lines are perpendicular. To prove this statement we denote the two slopes by m_1 and $m_2 = -1/m_1$, and we assume that m_2 is positive. Then the inclinations of the two lines are given by $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.

* Notice that $\tan\phi = \overline{AP_2}/\overline{AP_1}$ and that $\tan\phi = -\tan\theta$.

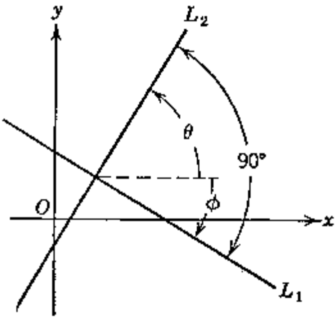


FIG. 1.15

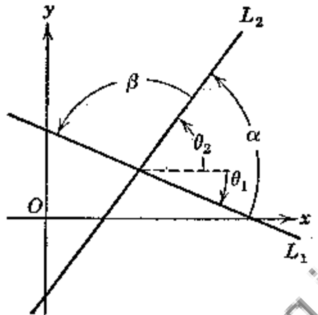


FIG. 1.16

It follows that

$$\tan\theta_2 = -\frac{1}{\tan\theta_1} = -\cot\theta_1 = +\tan(90° + \theta_1).$$

Hence $\theta_2 = 90° + \theta_1$, and the converse statement is demonstrated.

EXAMPLE 1

If the slope of one line is $2/3$, the slope of a perpendicular line is $-3/2$.

If two oblique lines are parallel, they have the same angle of inclination and hence they have the same slope.

We shall discuss two methods that may be used independently to find the angle measured from a line L_1 to a line L_2. One of the methods is to find the angles of inclination for the two lines and to combine these appropriately to determine the required angle. Thus, if the inclination angles, shown in Fig. 1.16, are respectively θ_1 and θ_2, and if the angle α is measured counterclockwise from L_1 to L_2, then $\alpha = \theta_2 - \theta_1$. If β is the required angle, as shown in Fig. 1.16, then

$$\beta = 180° - \alpha = 180° - \theta_2 + \theta_1.$$

The second method is first to determine some trigonometric function of α; the simplest function for the present purpose is the tangent function, since the slopes of the two lines are expressed in terms of this function. Then, since $\alpha = \theta_2 - \theta_1$,

$$\tan\alpha = \tan(\theta_2 - \theta_1)$$

$$= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1} \quad \text{(why?)}$$

$$= \frac{m_2 - m_1}{1 + m_1m_2},$$

where m_1 and m_2 are the respective slopes of lines L_1 and L_2.

Similarly, since $\beta = 180° - \alpha$,

hence, $$\tan \beta = \tan (180° - \alpha) = -\tan \alpha,$$

$$\tan \beta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

We observe that $\tan \alpha$ and $\tan \beta$ are equal numerically but are opposite in sign.

In either case the tangent of the angle between the two lines is equal to the slope of the line *to which* the angle is measured *minus* the slope of the line *from which* the angle is measured, all divided by 1 plus the product of the slopes (the angle in both cases being measured as positive in the counterclockwise direction).

EXAMPLE 2

A triangle is determined by its three vertices $A(5, -3)$, $B(2, 2)$, and $C(1, -3)$. Find each of the three vertex angles directly from the data, and then check by finding their sum.

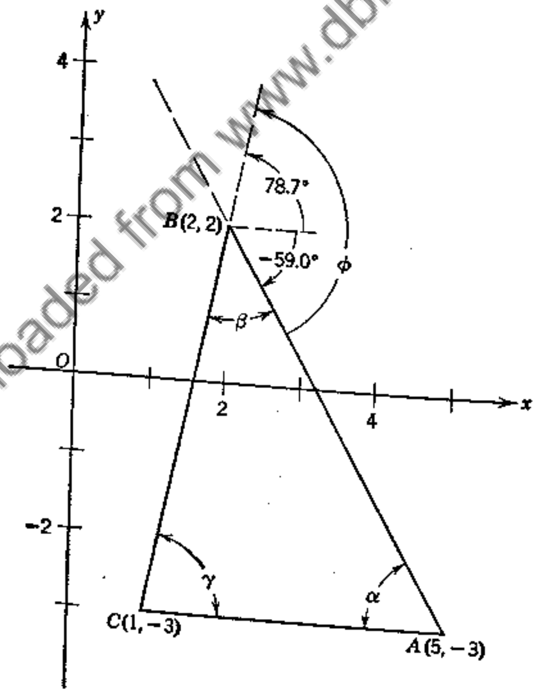


FIG. 1.17

Solution. The student should draw a figure such as Fig. 1.17, and verify that the slopes of the three sides are $m_{CB} = 5$, $m_{AB} = -\frac{5}{3}$, and $m_{CA} = 0$. The vertex angle at C is equal to the inclination of $\overline{CB}$, whence $\gamma = 78.7°$. The vertex angle at A is the negative of the inclination of $\overline{AB}$, whence $\alpha = 59.0°$. To determine the vertex angle at B, we use the first method and obtain

$$\phi = (78.7°) - (-59.0°) = 137.7°,$$

whence $\beta = 180° - \phi = 42.3°$. We check by finding that

$$\alpha + \beta + \gamma = 59.0° + 42.3° + 78.7° = 180.0°.$$

EXAMPLE 3

Given a line through $A(1, -1)$ and $B(3, 0)$ and a second line through A and $C(2, 3)$. Find the *exact* value of the tangent of the *acute* angle between the two lines.

Solution. Let the two inclination angles be α and β as indicated in Fig. 1.18. Then $\phi = \beta - \alpha$ and

$$\tan \phi = \tan (\beta - \alpha) = \frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha}.$$

But $\tan \beta = 4$ and $\tan \alpha = \frac{1}{2}$. Hence $\tan \phi = \frac{7}{6}$. Having determined the exact value of $\tan \phi$, we could obtain the exact value of any one of the other five trigonometric functions of ϕ either from a figure or by aid of the fundamental identities in trigonometry.

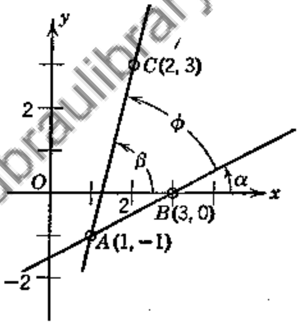


FIG. 1.18

Note that the same result would be obtained by using directly the formula for $\tan \alpha$ in this article.

PROBLEMS

1. Find the slopes of the lines determined by each of the following pairs of points:

(*a*) $(1, 4), (9, -6)$. (*b*) $(6, 9), (-4, -4)$.
(*c*) $(3, 7), (-4, 7)$. (*d*) $(8, 2), (8, -2)$.
(*e*) $(3.17, 2.79), (5.44, 8.44)$. (*f*) $(0, 4), (6, 0)$.
(*g*) $(2.17, -3.45), (-2.55, 3.17)$. (*h*) $(0, 0), (-4, 0)$.

2. Find the angle of inclination for each line in Problem 1.

3. Show by use of slopes that the three points $A(2, 2)$, $B(5, 1)$, and $C(3, 5)$ are the vertices of a right triangle, and then find its area.

4. The base of a triangle is the line segment joining $(\frac{1}{3}, 2)$ to $(-5, \frac{1}{2})$. What is the slope of the altitude drawn to this base?

5. Show that the line segment joining $(5, 1)$ to $(3, -3)$ is perpendicular to the line segment joining $(-6, 0)$ to $(-10, 2)$.

6. Show that the points $(-5, -3)$, $(2, \frac{2}{3})$, $(7, -6)$, $(1, -11)$ are the vertices of a trapezoid.

7. Three consecutive vertices of a parallelogram are $(-3, -1)$, $(3, 1)$, and $(9, 7)$. Find the coordinates of the fourth vertex.

8. Show that the perpendicular bisector of the line segment from $A(-4, 0)$ to $B(12, 2)$ passes through $C(5, -7)$:

(a) By showing that the lengths $\overline{AC}$ and $\overline{BC}$ are equal.
(b) By finding the slope of $\overline{AB}$, and of $\overline{DC}$ where D is the mid-point of $\overline{AB}$.

9. Draw straight lines passing through the origin and having the following slopes: (a) $+2$; (b) $-\frac{2}{3}$; (c) $+\frac{3}{4}$; (d) 0; (e) -4.17; and (f) 1.22.

10. The vertices of a triangle are $(6, -1)$, $(3, 8)$, and $(-3, -4)$.

(a) Plot to a large scale on graph paper, and measure the interior angles with a protractor.
(b) Find the vertex (or interior) angles by aid of the inclinations of the sides.
(c) Find the vertex angles by finding first the tangent of each of these interior angles.
(d) Prove that the triangle is a right triangle, and then find the two acute angles by aid of the right-triangle definitions of the trigonometric functions in terms of the lengths of the three sides.

11. Same as Problem 10 if the vertices are $(6, -1)$, $(1, -5)$, and $(-7, 5)$.

12. Given that the graph of $3x + 2y = 10$ is a straight line. By assigning an arbitrary value to one variable and determining the corresponding value of the other variable, find the coordinates of two different points on this straight line, and then find the slope of the line. Repeat with another pair of points.

13. Repeat Problem 12 for $2x - 5y = 20$.

14. Find the interior angles for each of the following triangles, each correct to the nearest tenth of a degree:

(a) $(3, 2)$, $(2, -1)$, $(-2, 1)$.
(b) $(-1, 1)$, $(-1, -2)$, $(1, -2)$.
(c) $(-2, 1)$, $(-1, -2)$, $(3, -1)$.
(d) $(-1, -1)$, $(5, -1)$, $(1, 1)$.
(e) $(-2, 4)$, $(6, 2)$, $(5, -2)$.
(f) $(2, 3)$, $(-2, -1)$, $(1, -4)$.

15. A line through $(1, -2)$ and $(5, 1)$ is perpendicular to a line through $(2, y)$ and $(1, -2)$. Find the value of y.

16. Given the curve $y = x^2 + 3x$. Determine the coordinates of the points on the curve which have abscissas respectively $x = 2$ and $x = 2.01$ (do not use a slide rule). Then determine the slope of the straight line through these two points.

17. A quadrilateral has its vertices in order at the four points $A(4, 3)$, $B(-3, 1)$, $C(-3, -2)$, and $D(2, -2)$.

(a) Find the slopes of the four sides.
(b) Find the inclinations of the four sides.
(c) Find the four interior angles.
(d) Find the slopes and inclinations of the lines joining consecutive mid-points of the sides.

1.8 Area of a Triangle

Let the three vertices of a triangle be $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ as shown in Fig. 1.19. We wish to derive a formula for the area of this triangle in terms of these given coordinates.

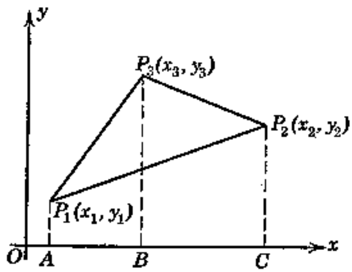


FIG. 1.19

The student might wish to compute the lengths of the three sides and then to make use of the "s-formula," namely,

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where

$$s = \tfrac{1}{2}(a+b+c).$$

Or the student might wish to determine the lengths of two sides and the vertex angle between these two sides and then to make use of the formula

$$\text{Area} = \tfrac{1}{2}ab \sin C.$$

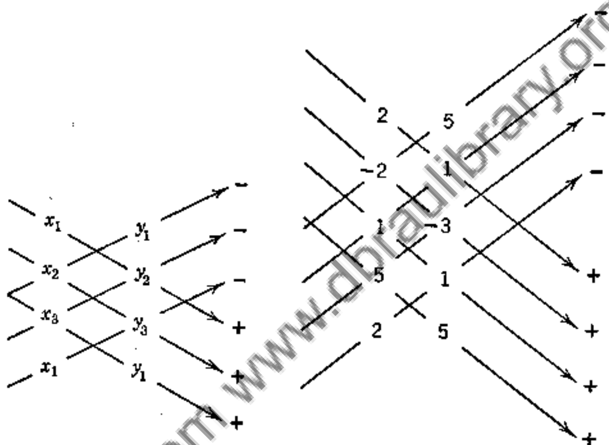
Although both methods are applicable, neither is as convenient as the geometric method about to be described.

From Fig. 1.19, we see that

$$\begin{aligned}\text{Area triangle } P_1P_2P_3 &= \text{Area trapezoid } ABP_3P_1 + \text{Area trapezoid } BCP_2P_3 - \text{Area trapezoid } ACP_2P_1 \\ &= \tfrac{1}{2}(y_3+y_1)(x_3-x_1) + \tfrac{1}{2}(y_2+y_3)(x_2-x_3) - \tfrac{1}{2}(y_1+y_2)(x_2-x_1) \\ &= \tfrac{1}{2}(x_3y_1 - x_2y_1 + x_1y_2 - x_3y_2 + x_2y_3 - x_1y_3).\end{aligned}$$

This is the required formula, but this form is difficult to memorize. To facilitate memorization we use the following rule.

RULE. Write the coordinates of the vertices in column form, giving the coordinates in counterclockwise order around the triangle and terminating with the coordinates of the first point. Obtain the products of the first number in each row and the second number in the row below, and find the sum of these products. Obtain the products of the first number in each row by the second number in the row above, and *subtract* the sum of these products from the sum previously obtained. Then take one-half of the result to obtain the required area.



EXERCISE FOR THE STUDENT. Write down the result obtained by this rule, and show that it yields the formula already derived for the area of the triangle.

That this same column scheme can be applied in exactly the same way to any polygon is not difficult to prove and is well known to civil engineers, who use this rule, for example, to determine the amount of earth to be excavated for a new roadway.

EXAMPLE

Determine the area enclosed by the quadrilateral with vertices (2, 5), (−2, 1), (1, −3), and (5, 1).

Solution. The student should draw the figure and locate the four points. The coordinates of the four points are tabulated in a column, beginning and ending with the coordinates of the chosen first point. The sum of the products of the first number in each row by the second number in the row below is given by

$$(2) + (6) + (1) + (25) = 34.$$

Then find the sum of the products of the first number in each row by the second number in the row above, and subtract the result from the preceding result:

$$(2) + (-15) + (1) + (-10) = -22, \quad (34) - (-22) = 56.$$

One-half of this result is the required area:

$$\text{Area} = \tfrac{1}{2}(56) = 28 \text{ square units.}$$

The student could, if he wished, solve this same problem by separating the quadrilateral into two triangles and applying the rule to each triangle. If he should put down the coordinates in clockwise order, instead of the specified counterclockwise order, the result would be a negative answer equal in numerical value to the required area.

1.9 Proofs of Theorems in Plane Geometry

We may use the methods of analytic geometry to establish the validity of theorems in plane geometry. But, as a more important objective in this section, we may use such applications to illustrate appropriate choices of axes to simplify the ensuing work.

EXAMPLE

Prove that lines joining consecutive mid-points of the sides of any quadrilateral form a parallelogram.

Solution. Draw a *general* quadrilateral (Fig. 1.20), i.e., a quadrilateral with no *special* property such as a right angle at one vertex or two sides parallel, etc.

Since the position of the axes will in no way affect the validity of the theorem, we may choose the axes in any position we like as suggested by Figs. 1.21, 1.22, 1.23, and 1.24. The choice of the origin at one of the vertices, as in Fig. 1.21, simplifies the coordinates of that particular vertex. If, in addition, we take one axis

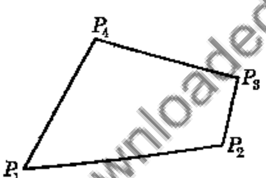


FIG. 1.20

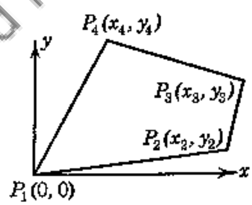


FIG. 1.21

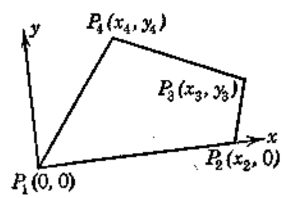


FIG. 1.22

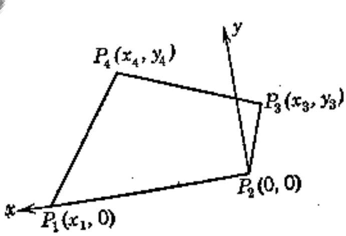


FIG. 1.23

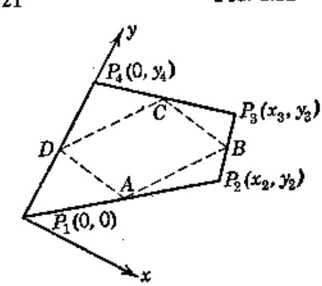


FIG. 1.24

along one of the sides of the quadrilateral, one coordinate of a second vertex will be simplified as shown in the other three figures. The student should notice the positive directions of the two axes as shown in Fig. 1.23, a choice which allows all but x_3 to denote positive numbers in that figure.

The student should verify that the mid-points in Fig. 1.24 have the following coordinates:

$$A\left(\frac{x_2}{2}, \frac{y_2}{2}\right), \quad B\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right), \quad C\left(\frac{x_3}{2}, \frac{y_3 + y_4}{2}\right), \quad D\left(0, \frac{y_4}{2}\right).$$

The slopes of the lines AB, BC, etc., are as follows:

$$m_{AB} = \frac{y_3}{x_3}, \quad m_{BC} = \frac{y_2 - y_4}{x_2}, \quad m_{CD} = \frac{y_3}{x_3}, \quad m_{AD} = \frac{y_2 - y_4}{x_2}.$$

Since opposite sides of $ABCD$ have the same slope, the figure is clearly a parallelogram.

PROBLEMS

1. Find the areas of the triangles whose vertices are as follows (wherever possible, check by a second method):

(a) (3, 4), (−1, −2), (−4, −2).
(b) (3, −2), (−6, 1), (−4, −8).
(c) (1.43, 2.57), (8.92, 4.76), (5.66, 11.78).
(d) (−4.11, −2.31), (4.77, 2.35), (−2.15, 7.74).
(e) (−2, 4), (−2, −2), (3, −2).
(f) (4, 4), (−1, 1), (1, −1).

2. Find the area of the quadrilateral with vertices at (−2, −2), (2, 2), (−5/2, 1/2), and (−1/2, 5/2).

3. Determine the coordinates of a point P *on the x-axis* such that the area of the triangle with vertices P, $A(3, 2)$, and $B(-6, 7)$ is 9 square units. (There are two answers to this problem.)

4. Determine the area of the triangle bounded by the three straight lines: $5y - 2x = 26$, $5y + 6x = 2$, and $14x + 5y = 58$. *Suggestion:* First solve each pair of equations simultaneously to determine the coordinates of the three vertices of the triangle.

5. Prove that the value of the adjoining determinant gives the area of a triangle with vertices at (x_1,y_1), (x_2,y_2), and (x_3,y_3).

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

6. If three points lie on a straight line, the triangle formed by these three points has an area zero. Hence find which of the following sets of three points are collinear (lie on a straight line):

(a) (−1, −1), (1, 1), (201, 201).
(b) (6, 6), (3, 7), (2, 8).
(c) (0, 5), (−1, 2), (1, 7).
(d) (−2, 4), (1, 0), (4, −4).

7. Prove the theorem of the example in Art. 1.9 by use of Fig. 1.22 and again by use of Fig. 1.23.

8. Prove that the mid-point of the hypotenuse of a right triangle is equidistant from each of the three vertices.

(*a*) Choose axes so that the right angle of the right triangle is at the origin and the two legs lie along the axes.

(*b*) Choose the origin at one vertex of the right triangle and the right-angle vertex on the positive part of the x-axis.

(*c*) Choose the origin at one vertex and the hypotenuse along the x-axis. The vertices will then be $A(0,0)$, $B(a,0)$, and $C(b,c)$. Since the angle at C is a right angle, write down the requirement in terms of slopes for $\overline{AC}$ to be perpendicular to $\overline{BC}$, and determine the relationship between a, b, and c.

9. Prove each of the following theorems from plane geometry:

(*a*) The line joining the mid-points of the nonparallel sides of a trapezoid is parallel to the two parallel sides and equal to one-half their sum.

(*b*) The diagonals of a parallelogram bisect each other. *Hint:* Can the vertices be taken at $(0, 0)$, $(a, 0)$, (b, c), and $(a + b, c)$?

(*c*) The diagonals of a rhombus are perpendicular.

(*d*) If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

(*e*) The two lines from a vertex of a parallelogram to the mid-points of the opposite sides trisect the opposite diagonal.

(*f*) Lines joining the mid-points of the sides of a triangle divide the triangle into four equal parts.

(*g*) Lines joining the mid-points of consecutive sides of a rectangle form a parallelogram whose area is one-half the area of the rectangle.

(*h*) The line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half the length of the third side.

(*i*) The sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of the diagonals.

(*j*) If the diagonals of a rectangle are perpendicular, it is a square.

1.10 Review

After he has completed a study of this first chapter, the student should be able to answer the following questions:

I *Lengths*

1. What is the rule for determining a directed length parallel to one of the coordinate axes?

2. What fundamental principle from plane geometry is used in deriving the general distance formula for the distance between two points? What is this distance formula? Why is it unwise to use this formula to find a directed length parallel to one of the coordinate axes?

II *Mid-Point*

1. What is the rule for finding the coordinates of the mid-point of a line segment by use of the given coordinates of the two ends of the segment? Derive it by aid of similar triangles.

III *Slope*

1. Define *inclination* and *slope.*

2. Could the slope of a line segment joining two points be determined from the difference of the "vertical" coordinates divided by the difference of the "horizontal" coordinates, the differences being taken in the same order?

3. What is the test for parallel lines? for perpendicular lines?

4. What formula from trigonometry is used in deriving the formula for the tangent of the angle between two lines?

5. Can you plot lines through a given point and with a given slope?

IV *Area*

1. What is the column scheme to find the area of a triangle? Illustrate with a problem of your own making.

REVIEW PROBLEMS

1. Plot to a large scale the quadrilateral determined by the four points $A(-2, 5)$, $B(0, -3)$, $C(4, -3)$, and $D(4, 3)$. Then solve the following problems:

(*a*) Find the length of $\overline{AD}$ and the *directed* length of $\overline{DC}$.

(*b*) Find the slopes of $\overline{AD}$ and $\overline{BC}$.

(*c*) Find the inclinations of $\overline{BA}$ and $\overline{CD}$.

(*d*) Find the interior angle of the quadrilateral at A.

(*e*) Write on your figure the coordinates of the mid-points of the four sides of the quadrilateral.

(*f*) Prove that the figure determined by the four mid-points found in (*e*), when taken in consecutive order, is a parallelogram.

(*g*) Find the area of the quadrilateral.

2. A rectangular bar of steel is placed in a testing machine and various stretching loads are applied. A graph is then drawn showing the load per square inch of cross-sectional area (s) as a function of the increase in length of each inch of original length (ϵ) of the test bar. One point on the straight part of the resulting graph, Fig. 1.25, has $s = 20{,}000$ lb./sq. in., and $\epsilon = 0.000{,}667$ in./in. Also, $s = 0$ when $\epsilon = 0$. Find the slope of the straight part of the curve. What is the inclination?

FIG. 1.25

3. Given that 0° C. = 32° F. and 100° C. = 212° F. If Fahrenheit temperatures are plotted vertically and centigrade horizontally, find the slope of the straight line joining the two points.

4. A hill is to be excavated for a roadway. If the reference point for one end of the cut is at the origin, the depths in feet to the horizontal roadway at various distances are given by the following sets of coordinates: (10, 10), (20, 15), (30, 18), (40, 13), and (50, 0). Find approximately the cross-sectional area of the cut in the direction of the measurements.

5. A bridge truss is shown in Fig. 1.26. The x-axis is to be chosen along the base and the y-axis along the mid-girder, DJ.

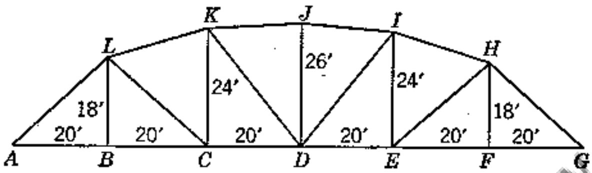


FIG. 1.26

(a) Find the coordinates of each pin (joint) of the truss.
(b) Find the length of each member of the truss.
(c) Find the area of the truss.
(d) Find the inclinations of members $\overline{LK}$ and $\overline{JI}$.

6. Given the triangle $A(12, -2)$, $B(-1, 4)$, $C(-2, -5)$. Let C represent the interior angle at C. Prove that

$$\tan \tfrac{1}{2}C = \tfrac{1}{3}(\sqrt{10} - 1)$$

by each of the following methods:

(a) Use the slopes of $\overline{AC}$ and $\overline{BC}$ to determine $\tan C$.
(b) Use the law of cosines to find $\cos C$.
(c) Prove that the triangle is isosceles, and then find $\tan C$ by aid of the right-triangle definitions for the trigonometric functions.
(d) Find the lengths of the three sides each correct to three significant figures, and then find the approximate value of $\tan (C/2)$ by aid of the relations from trigonometry:

$$\tan \frac{C}{2} = \frac{r}{s - c}, \quad s = \tfrac{1}{2}(a + b + c), \quad \text{and} \quad \text{area} = rs.$$

7S.* Two straight lines have slopes ¾ and 2 respectively. Find the *exact* value of the sine of the acute angle between the two given lines.

8S. Prove that the following projection formula gives the correct result for the area of a triangle with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$:

$$\text{Area} = \tfrac{1}{2}[(P_1P_2)_x(P_1P_3)_y - (P_1P_2)_y(P_1P_3)_x].$$

* Problems marked with an "S" are intended for assignment to the superior student.

CHAPTER 2

The Locus Derivation and the Straight Line

In this chapter we shall begin the study of the two fundamental problems of analytic geometry: to draw the graph of a given equation, and to find the equation of a curve defined by some property. After studying a general method for the finding of such equations, we shall apply that method to the derivation of two forms for the equation of a straight line and we shall study the properties of these equations for the straight line.

2.1 Graph or Locus of an Equation

The equation $y = 1 + x^2$ has the following solutions: (0, 1), (1, 2), (2, 5), (1.41, 3), (−1, 2). These were obtained by assigning values to x and determining the corresponding value of y in each case. We recall from our study of algebra that a *solution of an equation in two variables x and y is a pair of numbers (x, y) that together satisfy the equation*, i.e., that reduce the equation to a numerical identity.

If we determine a number of solutions for the preceding equation and plot the pairs of numbers thus obtained, the points so plotted will lie on a curve which is called the graph of the given equation (see Fig. 2.1). So far as this book is concerned, we shall make the following definition:

DEFINITION. *The graph of an equation consists of all those points, and only those points, with coordinates that are real numbers and that satisfy the given equation.**

* In certain senior and graduate physics and engineering courses (airplane wing design, fluid mechanics, electric and magnetic fields, etc.) it is convenient to use complex number solutions (instead of just real number solutions) and to plot a type of graph that is different from the graphs to be studied in this course. Modern electrical engineering especially would be almost impossible were it not for these complex numbers.

Sometimes the word *locus* is used instead of the word *graph*. Thus the graph of an equation is the locus of all those points, and only those points, with real coordinates that satisfy the given equation.

Paralleling the idea of the graph of an equation is the basic condition that a curve shall go through a point. This condition is the *fundamental principle of analytic geometry* and requires that *the coordinates*

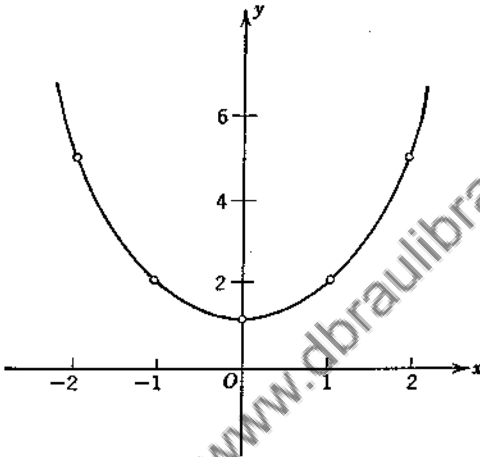


FIG. 2.1

of the point shall satisfy the given equation. In order to see how this principle will be utilized throughout the study of analytic geometry, let us study the following examples:

EXAMPLE 1

Find the value of m if the graph of $y = mx + 4$ goes through the point with coordinates (2, 3).

Solution. Since the graph of the equation is to go through the given point, the coordinates of the point must satisfy the given equation and hence $3 = 2m + 4$, whence $m = -\frac{1}{2}$. The resulting equation is $y = 4 - 0.5x$.

EXAMPLE 2

Find the values of a^2 and b^2 if the graph of the curve $x^2/a^2 + y^2/b^2 = 1$ goes through (4, 1) and (2, −2).

Solution. Substitute the two given sets of coordinates and obtain

$$\frac{16}{a^2} + \frac{1}{b^2} = 1, \quad \frac{4}{a^2} + \frac{4}{b^2} = 1.$$

Solve these two equations simultaneously by methods of algebra (it might be easier for the student to *let* $p = 1/a^2$ and $q = 1/b^2$ and to solve $16p + q = 1$ and $4p + 4q = 1$ simultaneously). Show that the resulting equation is $x^2 + 4y^2 = 20$.

EXAMPLE 3

Determine one set of values for a, b, and c so that the equation $ax + by + c = 0$ will be satisfied by (2, 5) and (6, −1).

Solution. Substitute the two pairs of coordinates, and obtain the two equations

$$2a + 5b + c = 0 \quad \text{and} \quad 6a - b + c = 0.$$

Both are linear homogeneous equations with three unknowns. Solve the two equations for some two of the variables in terms of the third (for example, for a and b in terms of c), and obtain $a = -3c/16$, $b = -c/8$. Now assign to c any arbitrary value other than zero, and compute the corresponding values for a and b. Thus, if $c = -16$, $a = 3$, and $b = 2$, then the equation becomes

$$3x + 2y - 16 = 0.$$

Alternatively, one could substitute the expressions in terms of c for a and b in the given equation and obtain

$$\left(\frac{-3c}{16}\right)x + \left(\frac{-c}{8}\right)y + c = 0.$$

This will simplify to the same final result.

EXAMPLE 4

Find the values of a, b, c if the graph of $x^2 + y^2 + ax + by + c = 0$ goes through (1, 3), (4, 2), and (−3, −5).

Solution. We substitute the three pairs of coordinates and obtain

$$a + 3b + c = -10,$$
$$4a + 2b + c = -20,$$
$$-3a - 5b + c = -34.$$

We subtract the third equation from each of the first two equations and obtain $4a + 8b = 24$ and $7a + 7b = 14$, which simplify to $a + 2b = 6$ and $a + b = 2$. We again subtract and find that $b = 4$, whence $a = -2$ and $c = -20$.

The student should verify that the resulting equation,

$$x^2 + y^2 - 2x + 4y - 20 = 0,$$

is satisfied by all three of the given pairs of coordinates. Indeed, the student should develop the habit of checking his final equation to be certain that the coordinates of each given point on a curve satisfy his equation for the locus.

PROBLEMS

1. Make a table of values for each of the following curves by assigning values to one variable and computing the corresponding values of the other variable, and then plot the graph of the equation. In each case, determine the coordinates

of *enough* points so that you can draw a smooth freehand curve through the plotted points. A slide rule will facilitate some of the computations and will be sufficiently accurate.

(*a*) $y = x^2 + 3x$ for x from -3 to $+2$ and plot for $x = -3, -2.5, -2, \cdots, 2$.
(*b*) $y = x/(x^2 + 1)$ for $0 \leqq x \leqq 5$.
(*c*) $y = 1/(x^2 + 1)$ for $-4 \leqq x \leqq 4$.
(*d*) $y^2 = 5x$.
(*e*) $x^2 - 3y^2 = 12$ for $-2 \leqq y \leqq 2$.
(*f*) $x^2 + 2y^2 = 9$.
(*g*) $x^2 + y^2 = 5$.
(*h*) $y = 2^x$ for $-3 \leqq x \leqq 2$.
(*i*) $y = \log_{10} x$ for $0.1 \leqq x \leqq 10$.
(*j*) $y = 2/x$ for $-5 \leqq x < 0, 0 < x \leqq 5$. (Note that no value for y corresponds to $x = 0$.)

2. Make a table of values for x and y for the curve

$$y = x - 1 + 2 \sin \pi x$$

by assigning the values $x = 0, 1, 2, 3, 4, 5$, and 6. Plot these points, and draw a smooth curve through them. Next compute y when $x = 0.5, 1.2, 2.5, 3.5, 4.5$, and 5.5, and draw a smooth curve through all 13 points. You will learn later how to draw the correct graph for this equation more easily.

3. Determine which of the following points are on the graph of $x + 3y = 9$: $(6, -1), (3, 2), (0, 3)$. What value must be assigned to a if the graph goes through $(8, a)$?

4. Determine r^2 if the graph of $x^2 + y^2 = r^2$ goes through $(3, -2)$.

5. Find a and b if the graph of $ax + by = 1$ goes through $(2, 5)$ and $(6, -1)$.

6. Determine a, b, and c if the graph of $x^2 + y^2 + ax + by + c = 0$ goes through $(1, 2)$, $(6, 3)$, and $(3, 0)$.

7. Determine k if the graph of $y^2 = kx$ goes through $(2, 5)$.

8. Determine a^2 and b^2 if the graph of $x^2/a^2 - y^2/b^2 = 1$ goes through $(4, 1)$, and $(6, 3)$.

9. Determine p if the graph of $y = p \log_{10} x$ goes through $(4, 2)$.

10. Determine a if the graph of $y = a \sin x$ goes through $(5\pi/6, 2)$.

11. Find the values of a and ϕ if the graph of $y = a \sin (x + \phi)$ goes through $(\pi/3, 3)$ and $(5\pi/6, 0)$. Choose the smallest positive value for ϕ.

12. Determine whether the four points $(2, 6)$, $(-3, 1)$, $(5, 5)$, and $(-1, -3)$ all lie on the graph of the curve

$$x^2 + y^2 + ax + by + c = 0.$$

13. Find the values of a, b, and c if the graph of $y = ax^2 + bx + c$ goes through $(1, 4)$, $(3, 7)$, and $(5, 4)$.

14. Find one set of values for a, b, and c if the graph of $ax + by + c = 0$ goes through $(2, 4)$ and $(6, 1)$.

15. Find one set of values for a, b, c, and d, if the graph of $ax + by + cz + d = 0$ goes through $(x = 4, y = 1, z = 0)$, $(4, -2, 2)$, and $(0, 6, -2)$.

2.2 The Locus-Derivation Method

In later chapters the student will learn how to sketch the graph of a given equation (the locus of points whose coordinates satisfy the given equation). In the present article, however, we are to have given certain geometrical properties from which we are to determine the equation of the curve. There is a very powerful method of procedure for all such problems that the student should learn. The steps in this procedure are as follows:

I. Sketch a figure and label it with the given data. Choose axes appropriately, to simplify the ensuing work.

II. Select on the graph a *general* point $P(x, y)$, which seems to satisfy the given conditions. (A general point is any point for which there are no peculiar or special properties. Thus the point should not be taken on an axis, etc.)

III. Make a geometrical statement which this point P is to satisfy according to the statement of the problem.

IV. Translate the geometrical statement into an algebraic statement, thus involving x and y.

V. Simplify the algebraic equation.

VI. Check your work by selecting some special point known to lie on the locus and determining whether its coordinates satisfy the final equation obtained in Step V. Or obtain a solution of that equation and show that the corresponding point satisfies the statement of the problem.*

In Step V, if the simplification entails squaring both sides of the equation or multiplying both sides by a quantity containing the variable, then the resulting equation may not be *equivalent* to the given equation. (From algebra, two equations are said to be equiva-

*The methods outlined in this section are extremely important in any engineering or physical science curriculum. In fact, this locus-derivation method is the first step in the student's mastery of the engineering or scientific method for solving problems. This scientific method consists of the following four steps:

Step I. Sketch figures, such as free-body diagrams; label all relevant points, lines, etc.

Step II. Determine what fundamental principle from science or from engineering applies (Newton's laws of motion, fundamental laws of electricity, similar triangles from plane geometry, etc.); apply this, and obtain a mathematical problem.

Step III. Perform the indicated mathematical operations and solve the mathematical problem.

Step IV. Discuss the results of the mathematical problem as they apply to the original problem and discuss the scientific or engineering consequences.

lent if every solution of one equation is a solution of the other equation, and conversely.) Thus, for example, the equations $(1/x) + (1/y) = 4$ and $y + x = 4xy$ are not equivalent. However, the locus defined by the second equation includes the locus defined by the first equation and, in addition, the point (0, 0). Similarly, the locus of the equation $x^2 + 2xy + y^2 = 4$, obtained by squaring $x + y = 2$, includes the locus defined by the equation $x + y = 2$ and, in addition, the locus defined by the equation $x + y = -2$. These two examples illustrate the fact that the equation which results from squaring both sides of an equation, or from multiplying by a quantity containing the variable, will always define the locus of the given equation and may, in addition, define extraneous loci. The student of analytic geometry should, as he increases in mathematical understanding, take note of facts such as these. The student should be extremely careful about dividing by a quantity containing the variable, for this process usually will yield a new equation that defines a locus that does *not* include the entire locus defined by the original equation.

EXAMPLE 1

Find the equation of the locus of a point that moves so that it is always equidistant from the two points (2, 3) and (−3, 1).

Solution. Step I. Sketch the figure, and label the two given points.

Step II. Select $P(x, y)$ on the perpendicular bisector of the line segment $\overline{AB}$ as shown in Fig. 2.2.

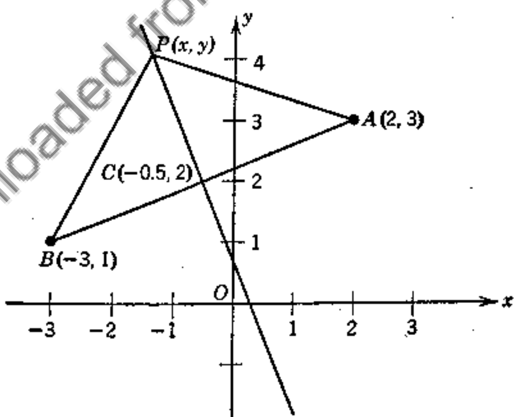


FIG. 2.2

Step III. $\overline{BP} = \overline{AP}$.

Step IV. $\sqrt{(x+3)^2 + (y-1)^2} = \sqrt{(x-2)^2 + (y-3)^2}$.

Step V. The student should perform the algebra to reduce the preceding result to $10x + 4y = 3$.

Step VI. Check by aid of the mid-point whose coordinates are $(-\frac{1}{2}, 2)$.

EXERCISE FOR THE STUDENT. Work the preceding example by aid of the geometrical statement: $\overline{CP}$ is perpendicular to $\overline{CA}$.

EXAMPLE 2

Determine the equation of the locus of a point that moves so that its distance from the point $(4, 0)$ is always numerically equal to $\frac{4}{5}$ of its (the moving point's) distance from the vertical line $x = \frac{25}{4}$.

Solution. Step I. Sketch axes, locate and name the given point $A(4, 0)$, and draw the vertical line $x = \frac{25}{4}$. Note that every point on this line has $\frac{25}{4}$ for its abscissa. (See Fig. 2.3.)

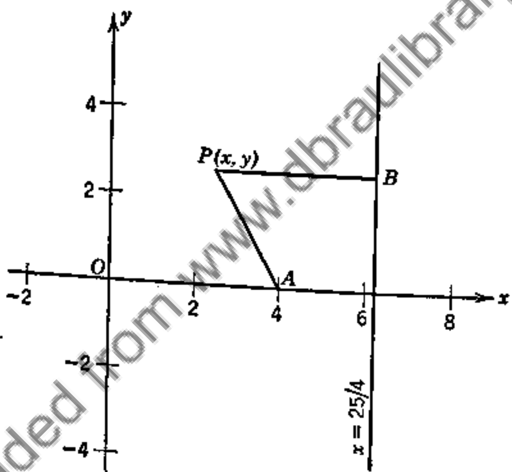


FIG. 2.3

Step II. Select $P(x, y)$ so that its distance from $(4, 0)$ seems to be about $\frac{4}{5}$ of its distance from the given line.

Step III. $\overline{AP} = \frac{4}{5}\overline{PB}$. (*Note:* $\overline{AP} = \frac{4}{5}\overline{BP}$ would be incorrect. Why?)

Step IV. $\sqrt{(x-4)^2 + y^2} = \frac{4}{5}(\frac{25}{4} - x)$.

Step V. Perform the necessary algebra to reduce this to $9x^2 + 25y^2 = 225$.

Step VI.* Check: Select a particular point whose coordinates satisfy this equation (for example, let $x = 4$, and find that $y = \frac{9}{5}$). Use this point as a *particular point P* and find the distances $\overline{AP}$ and $\overline{PB}$. Now determine whether the requirement of Step III is satisfied.

* There is an additional step in this locus-derivation method which is important if strict attention is not given to the idea of equivalent equations. The six steps in the locus derivation are concerned with the determination of the equation satisfied by the coordinates of every point on the locus. To complete the derivation,

It is important that the point $P(x, y)$ be chosen as a general point. In some cases there are two or more different positions for such a general point, and the locus is made up of different loci defined by quite different equations. We illustrate this idea in the next example.

EXAMPLE 3

Find the equation of the locus of a point that moves so that it is always 3 units farther from the point (0, 1) than from the horizontal line $y = 2$.

Solution. Step I. Sketch the figure as shown in Fig. 2.4, and label the given data.

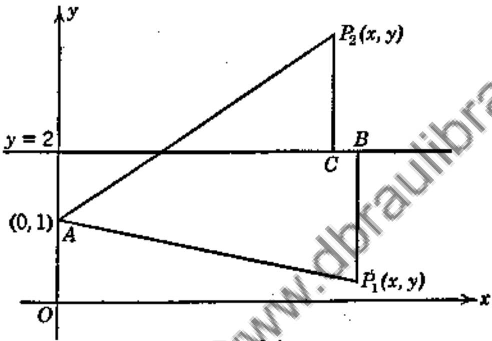


FIG. 2.4

Step II. Note the important fact that two different but general points $P_1(x, y)$ and $P_2(x, y)$ must be selected, one above the line $y = 2$ and the other below this line.

Step III. $y < 2$:

$$\overline{AP_1} = \overline{P_1B} + 3.$$

Step IV. $$\sqrt{x^2 + (y-1)^2} = (2 - y) + 3$$
$$= 5 - y.$$

$y > 2$:

$$\overline{AP_2} = \overline{CP_2} + 3.$$

$$\sqrt{x^2 + (y-1)^2} = (y - 2) + 3$$
$$= y + 1.$$

---

we should show that every point whose coordinates satisfy the final equation does lie on the given locus. (Or we could show that every point not on the locus has coordinates that do not satisfy the final equation.)

To show that every point $P_1(x_1, y_1)$ whose coordinates satisfy the final equation does lie on the given locus, we can reverse the order of the simplification in Step V and show, because of the equivalence of the successive equations, that P_1 is on the given locus. Otherwise, by utilizing the equation that the coordinates (x_1, y_1) satisfy, we can show that P_1 satisfies the statement of the locus.

Since this second part of the proof could be carried out by merely reversing the steps of the algebraic simplification, we shall usually omit this part of the derivation (assuming that due attention has been given to the idea of equivalent equations).

Step V. The student should simplify both of these and show that the one on the left-hand side becomes $x^2 = 24 - 8y$, which is valid if $y < 2$, and that the one on the right-hand side becomes $x^2 = 4y$, which is valid if $y > 2$.

Step VI. If $y < 2$, the figure suggests (4, 1) as a check value, and these coordinates satisfy the equation $x^2 = 24 - 8y$.

If $y > 2$, it is difficult to see a check point on the figure, so we assign $y = 4$ in the equation $x^2 = 4y$ and determine that (4, 4) is a point whose coordinates satisfy this equation. The distance from (0, 1) to (4, 4) is 5, and the distance from the horizontal line $y = 2$ to the point (4, 4) is 2. Hence, since $5 = 2 + 3$, the check is completed.

EXERCISE FOR THE STUDENT. Assign $y = 1$ in the equation $x^2 = 4y$, and show that neither point that results can satisfy the stated conditions of the problem. Then assign $y = 2.5$ in the equation $x^2 = 24 - 8y$, and show again that neither resulting point can satisfy the stated conditions of the problem.

PROBLEMS

1. Derive the equation of the locus of a point which moves so that:

(a) It is equidistant from (1, 2) and (5, 2).
(b) It is equidistant from (1, 4) and (5, 2).
(c) Its distance from (3, 2) is always 4.
(d) Its distance from the vertical line $x = 3$ is always equal to its distance from the point (5, 0).
(e) The sum of its distances from the points (3, 0) and (−3, 0) is always 10.
(f) Its distance from the point (3, 4) is always numerically equal to its distance from the point (1, 2).
(g) Its distance from the origin is numerically twice its distance from the horizontal straight line $y = 6$.
(h) Its ordinate is always three times its abscissa.
(i) Its distance from (0, 6) is always twice its distance from (3, 0).
(j) The difference of its distances from (5, 0) and (−5, 0) is always numerically equal to 8.
(k) The difference of its distances from (1, 1) and (−1, −1) is always numerically equal to 2.

2. Determine the equation of the locus of the vertex of an isosceles triangle whose base is the line segment joining the two points (−0.5, 2) and (3, 1.5).

3. Determine the equation of the locus of a point that moves so that the line joining the moving point to the fixed point $A(4, 1)$ always has an inclination of −45°.

4. Determine the equation of the locus of a point that moves so that its distance from the point (−1, 2) is always numerically equal to the slope of the line joining it to the point (−1, 2). *Suggestion:* In a locus derivation, such as this problem, it may be difficult to select a general point $P(x, y)$ which seems geometrically to satisfy the statement of the problem. In such a case, select an arbitrary but general point and proceed with the derivation.

5. Determine the equation of the locus of a point that moves so that the slope of the line joining it to the point (2, 3) is always 3 more than the slope of the line joining the moving point to the point (0, 3).

6. Use locus-derivation methods to determine the equations of the following straight lines:

(*a*) Through (0, 3) and slope 2. (*b*) Through (1, 2) and slope 4.
(*c*) Through (1, −3) and inclination −45°.
(*d*) Through (3, 2) and (5, −1). (*e*) Through (3, 2) and (3, 5).
(*f*) Through (4, 0) and (0, 6). (*g*) Through (x_1, y_1) and slope m.
(*h*) Through (0, b) and slope m. (*i*) Through (4, 7) and (−2, 7).

7. Derive the equation of the locus of a point that moves so that its distance from (−5, 0) is always 8 units more than its (the moving point's) distance from (5, 0). Show that the final result, after simplification, is the equation of the required locus *and* also the equation of the locus of a point that moves so that its distance from (−5, 0) is always 8 units *less* than its distance from (5, 0). Then show that the equation for the given locus may be written as $(x^2/16) - (y^2/9) = 1$ with the provision that x is positive, or that it may be written as $x = \frac{4}{3}\sqrt{9 + y^2}$.

8. A point moves so that its positive distance from (0, 1) is always 3 more than its *directed* distance measured *from* the horizontal line $y = 3$. Show that the equation of the locus is given by $x^2 = 2y - 1$.

9. A point moves so that its positive distance from (0, 1) is always 2 more than its *directed* distance measured *from* the horizontal line $y = 3$. Show that the required locus is that portion of the y-axis above the point (0, 1).

10. Determine the equation of the locus of the mid-points of the line segments joining the origin to points on the horizontal straight line $y = 2$.

11S. Find the equation of the locus described by the third vertex C of the triangle determined by $A(0, 2)$, $B(0, -2)$, and C if the interior angle at C is 45°. Note that the locus is given by one equation if $x > 0$ and by a different equation if $x < 0$.

12S. Determine the equation of the locus of a point that moves so that it is always 3 units farther from the point (2, 0) than from the vertical line $x = 1$. Show that the required locus is given by one equation for $x > 1$, and by a different equation when $x < 1$.

13S. Figure 2.5 shows a crank arm, $\overline{OA}$, of an engine (not shown); this crank arm revolves around the origin with a constant speed of 300 r.p.m. The con-

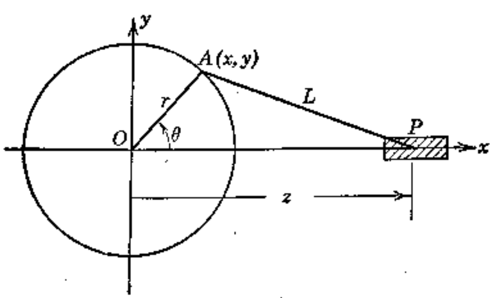


FIG. 2.5

necting rod, $\overline{AP}$, is joined to the crank arm at A; the other e
to a piston that moves along the x-axis. The lengths of the
crank arm are respectively L and r.

(a) Determine relations for x and y each in terms of r and
(b) Obtain a relation for z that may also involve r, L, and
(c) Obtain a relation for z that also involves r, L, and x.

2.3 Straight Lines Parallel to One of the Co

In this section we seek the general equation of a s
is parallel to one of the two coordinate axes. Throu
whenever we seek the equation of a new type of curve
ally resort to the use of the locus-derivation method.
prove the following theorem:

THEOREM. *The equation of a straight line parallel
the form $x = k$, where k is a constant and represents the
of the line from the y-axis.*

Proof: Step I. Sketch axes and an arbitrary strai
to the y-axis as shown in Fig. 2.6.

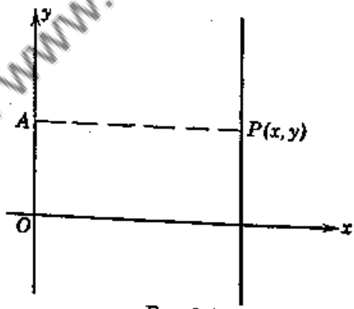


FIG. 2.6

Step II. Select a general point $P(x, y)$ on this strai
Step III. Draw the line $\overline{AP}$ perpendicular to th
$\overline{AP}$ = constant = k, where k is the directed distance
line from the y-axis.
Step IV. $\overline{AP} = x = k$.

The other steps are superfluous. The student sh
similar theorem which follows.

THEOREM. *The equation of a straight line parallel t
the form $y = h$, where h is a constant and represents the
from the x-axis to the line.*

2.4 The Slope Form of the Straight-Line Equation

In this article we seek the general equation of any oblique line (defining an oblique line to be a line that is not parallel to either axis). We again resort to the locus-derivation method. The problem, then, is to find the equation of a general oblique line that goes through a given point $P_1(x_1, y_1)$ and has a given slope m.

Step I. Sketch figure, locate the arbitrary given point, and draw the arbitrary oblique line as shown in Fig. 2.7.

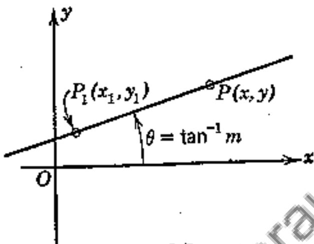


FIG. 2.7

Step II. Let $P(x, y)$ be any general point on this oblique line.

Step III. Make a geometric statement:

Slope of P_1P is m.

Step IV. Change to algebra:

$$\frac{y - y_1}{x - x_1} = m.$$

The other steps are superfluous. We have thus established the important theorem.

THEOREM. *The equation of the straight line which passes through $P_1(x_1, y_1)$ and has the slope m is $(y - y_1)/(x - x_1) = m$.*

We notice that this equation may be written in the alternative forms

$$y - y_1 = m(x - x_1) \quad \text{and} \quad y = mx + (y_1 - mx_1).$$

One can use the equation in the theorem to write the equation of a horizontal line which has a zero slope, but such a solution is too long. The equation of a vertical line *cannot* be written by use of this equation, since the slope of a vertical line is not defined. Thus, as in the first chapter, we find it convenient, as well as simpler, to use one method to write the equation of a line parallel to one of the axes and another method to write the equation of an oblique line.

Example

Find the equation of the straight line that goes through the points (2, 5) and (7, 3).

Solution. We choose as one point on the required line (2, 5), and find that the slope is $m = -2/5$. Then $(y - 5)/(x - 2) = -2/5$ or $2x + 5y = 29$. The student should, to develop a habit, check this final result by substituting the coordinates of both points, one pair at a time.

2.5 The Slope-Intercept Form of the Straight-Line Equation

Definition. *The x-intercept of a straight line is the abscissa (or directed distance) of the point where the line crosses the x-axis. Similarly, the y-intercept is the ordinate of the point where the line crosses the y-axis.*

We proceed to prove the following theorem:

Theorem. *The equation of a straight line with slope m and y-intercept b is* $y = mx + b$.

Proof. We do not need to resort to the locus-derivation method, which would be a correct procedure, since we have already established one type-form for the straight-line equation. The problem is to write the equation of a straight line that goes through $(0, b)$ and that has a slope m. We use the result of the preceding article and obtain:

$$\frac{y - b}{x - 0} = m \quad \text{or} \quad y = mx + b,$$

as stated in the theorem.

Example

Solve the example of the preceding article by aid of this type-form for the straight-line equation.

Solution. Since the straight line $y = mx + b$ is to go through (2, 5) and (7, 3), we obtain by applying the fundamental principle of analytic geometry: $5 = 2m + b$ and $3 = 7m + b$. We solve these two equations simultaneously and find that $m = -2/5$ and $b = 29/5$. Hence the required equation is

$$y = -\tfrac{2}{5}x + \tfrac{29}{5}.$$

2.6 The General Linear Equation in Two Variables

We proceed to prove the following theorem:

Theorem. *The graph of the linear equation*

$$Ax + By + C = 0$$

(*where A and B are not both zero*) *is a straight line.*

Proof. If $A = 0$, the equation reduces to $By + C = 0$; since $B \neq 0$, $y = -C/B$. This is the equation of a straight line that is parallel to the x-axis. Similarly, if $B = 0$ the equation reduces to $Ax + C = 0$ or to $x = -C/A$, since if $B = 0$ then $A \neq 0$. This is the equation of a straight line parallel to the y-axis.

If neither A nor B is zero, we may divide by B and write the equation in the form $y = -(A/B)x - C/B$. This, by the theorem of the preceding article, is the equation of a straight line with slope $m = -A/B$ and y-intercept $-C/B$. Hence the graph of every linear equation in two variables is a straight line.

An immediate consequence of this theorem is that the slope of a straight line, $Ax + By + C = 0$, may be found (if $B \neq 0$) by *solving for y in terms of x*. The coefficient of x in this solution is the required slope. If $B = 0$, the graph is a vertical line whose inclination is 90° and whose slope is not defined.

We next prove the theorem converse to the preceding one.

Theorem. *Every straight line has a first-degree equation.*

Proof. If the line is either horizontal or vertical the theorem is evident, since a horizontal line has an equation of the form y = constant, and a vertical line has an equation of the form x = constant. If the line is oblique, let its slope be m and its y-intercept b; then $y = mx + b$, and the theorem is evident.

The student learned in plane geometry that two points determine a line. Hence, to draw the graph of a straight line, the student may determine the coordinates of two points on the line, plot the two points, and then draw the line. Any two points will do, but usually the simplest two points are the points where the straight line crosses the two axes (which are given by the intercepts).

Example

Determine the slope, the inclination, and the two intercepts, and draw the graph of $2x + 3y = 5$.

Solution. When $x = 0$, $y = 5/3$, and when $y = 0$, $x = 5/2$. The two intercepts are $x = 5/2$ and $y = 5/3$. The graph can be drawn by aid of these two intercepts; no more information is needed, though a careful student will check by locating another point in addition to $(0, 5/3)$ and $(5/2, 0)$.

To determine the slope of the line, one may use the formula for the slope of a straight line through two points, or one can *solve for y in terms of x:*

$$y = -\tfrac{2}{3}x + \tfrac{5}{3};$$

whence the slope is $m = -2/3$ and the inclination is $\theta \approx -33.7°$.

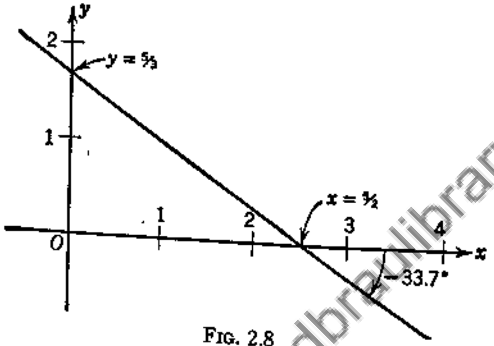


FIG. 2.8

This example shows how to draw rapidly the graph of a straight line by aid of its intercepts. If the points corresponding to the two intercepts are too close together, or if the line goes through the origin, the student may determine the coordinates of an additional point by assigning any value he wishes to x and computing the corresponding value for y. Or he might locate another point by use of the slope.

PROBLEMS

1. Determine the equation of each of the following straight lines:

(a) Through $(2, -3)$ and with slope 4.
(b) With y-intercept 3 and with slope 0.5.
(c) With x-intercept 2 and y-intercept 4.
(d) Through $(3, -1)$ and with slope -0.333.
(e) Through $(4, -2)$ and with inclination 68.4°.
(f) Through $(3, 2)$ and parallel to the x-axis.
(g) Through $(3, 2)$ and parallel to the y-axis.
(h) Through $(0, -2)$ and with inclination $-76.4°$.
(i) Through $(4, 3)$ and $(-4, 1)$.
(j) Through $(5, 2)$ and $(5, -4)$.
(k) Parallel to $7y + 4x = 11$ and with y-intercept 6.
(l) Perpendicular to $8y + 3x = 21$ and through $(7, 0)$.
(m) Perpendicular to $2x = 20 + 5y$ and through the point corresponding to the x-intercept for the given line.

2. Draw each of the following lines by use of the two intercepts and check by computing the coordinates of some point not on either axis. Also determine the slope and inclination, and label the angle of inclination on your figure.

(a) $y = 2x + 1$.
(b) $5x - y + 8 = 0$.
(c) $y + 3x = 2$.
(d) $10x = 2y + 3$.
(e) $x + 4y = 0$.
(f) $2x + 3 = 0$.
(g) $100x + 60y = 8{,}257$.
(h) $1.57x + 2.44y = 7.88$.
(i) $2.39x - 5.14y = 2.17$.
(j) $3.78y = 5.97$.

3. Obtain the equation of each of the following straight lines:

(a) Through $(-4, 0)$, and making an angle of $-60°$ with the x-axis.
(b) Through $(-5, 0)$ and $(0, -3)$.
(c) Perpendicular to $y = 3x + 6$ and through $(-4, 0)$.
(d) Perpendicular to $1.77x - 2.34y = 7.89$ and through $(0, 7.64)$.
(e) Through $(0, -2)$, the inclination angle being such that $\sec\theta = 4$ (note that there are two solutions).
(f) Through $(0, 2)$ and with inclination $90°$.
(g) Through $(-3, -2)$ and $(5, -2)$.
(h) Through $(1.89, 2.57)$ and $(7.68, 4.69)$.
(i) Through $(0.00267, 9.64)$ and $(0.00844, 42.56)$.

4. Determine the equation of the straight line through the point of intersection of $x + 4y = 7$ and $2x + y = 7$ and through $(1, 3)$.

5. Determine A so that the graph of $Ay = 2x + 4$ will be perpendicular to $2x + 5y = 13$.

6. Determine the equation of the straight line whose inclination is $26.3°$ more than the inclination of $4x - 11y = 18$, and that goes through the point $(-0.0784, 0.125)$.

7. The areas of a group of related plane figures (A sq. in.) are related to the lengths (L in.) by a linear formula. If $A = 8.00$ sq. in. when $L = 2.00$ in., and if $A = 46.0$ sq. in. when $L = 12.0$ in., obtain a formula for A in terms of L. Also find A when $L = 6.00$ in., and check by ordinary interpolation.

8. The monthly charge for gas in a certain city is \$0.60 plus \$1.50 for each 1000 cu. ft. actually consumed. Express the total cost (C) in terms of the number (N) of thousand cubic feet consumed. Draw the graph, and give the slope and intercepts. What physical meaning does the slope have?

9. Find what relations must hold between A, B, and C in the line

$$Ax + By + C = 0:$$

(a) If the x-intercept is 3.
(b) If the line is perpendicular to $2x - 3y = 8$.
(c) If the line is parallel to the x-axis.
(d) If the line is parallel to the y-axis.
(e) If the line is perpendicular to the y-axis.
(f) If the line goes through $(3, 5)$.

10. Determine the acute angle:

(a) Between $2y = x + 1$ and $3x + 4y = 40$.
(b) Between $2x - 5y = 1$ and the y-axis.
(c) Between $6x + y = 19$ and $5x - 2y = 3$.
(d) Between $2x + 5y = 4$ and the x-axis.
(e) Between $2x - 5y = 7$ and $5x + 2y = 4$.
(f) Between $3x + 6y = 8$ and $x - 2y = 5$.

11. Use the slope formula for the straight line to obtain the relation for Fahrenheit temperature in terms of centigrade temperature. (Recall that $F = 32°$ corresponds to $C = 0°$, and 212° F. to 100° C.) What temperature is the same on both scales?

12. Find the ordinate to a straight line corresponding to the abscissa 10 if the line goes through (2, 5) and has 4 as the x-intercept.

13. Use locus-derivation methods to derive the equations of the following general oblique lines:

(a) The line goes through (x_1, y_1) and (x_2, y_2).
(b) The line has intercepts $x = a$ and $y = b$.

14. Find the coordinates of the point that is equidistant from

(a) $(-1, 4)$, $(3, 2)$, and $(1, -3)$.
(b) $(0, 0)$, $(0, 6)$, and $(8, 0)$.
(c) $(4, 2)$, $(-3, 1)$, and $(-2, -6)$.
(d) $(0, 0)$, $(4, 0)$, and $(2, 2\sqrt{3})$.

15. Prove by methods of analytic geometry that the three altitudes of a general triangle intersect in a point.

16. Prove by methods of analytic geometry that the three medians of a general triangle intersect in a point that is a point of trisection of each median.

2.7 Perpendicular Distance from an Oblique Line to a Point

The purpose of this article is to explain a short method that can be used to find the perpendicular distance from a given oblique line $(L\colon ax + by + c = 0)$ to a point $P(X, Y)$.

We already know how to determine the directed perpendicular distance from the given line to a point if the line is parallel to either axis (for example, the abscissa of the point *to which* the measurement is made *minus* the abscissa of the point *from which* it is made). Hence we need to study only the case of an oblique line; that is, neither a nor b can be zero. We may therefore suppose that a represents a positive number, for otherwise we could first multiply through the equation by -1.

We shall prove that this required distance (d in Fig. 2.9) is given by the very simple formula:

$$d = \frac{aX + bY + c}{\sqrt{a^2 + b^2}}, \quad a > 0.$$

To prove this, we observe in Fig. 2.9 that $d = \overline{RP} \sin \theta$. But $\tan \theta$ is the slope of the given line and is $-a/b$. The figure is drawn with θ

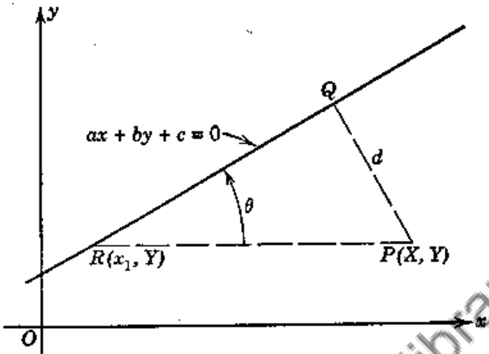


FIG. 2.9

a positive acute angle, and hence, for this figure, b represents a negative number. Then $\sin \theta$ is positive, and from Fig. 2.10, $\sin \theta = a/\sqrt{a^2 + b^2}$. Also $\overline{RP} = X - x_1$. Hence

$$d = (X - x_1)(\sin \theta) = \frac{aX - ax_1}{\sqrt{a^2 + b^2}}.$$

Since R is on the straight line, $ax_1 + bY + c = 0$, whence $ax_1 = -bY - c$, and the required relation is obtained.

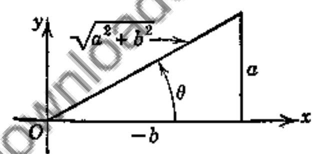


FIG. 2.10

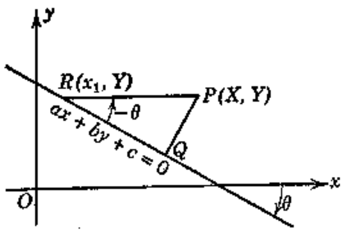


FIG. 2.11

If, in the equation $ax + by + c = 0$, a and b are both positive, then the straight line will slope downward to the right as shown in Fig. 2.11. Then $\overline{QP} = \overline{RP} \sin(-\theta)$. The remainder of the proof is similar to that already given for the case with $b < 0$.

Since d has the same sign as $\overline{RP}$, it follows that d is positive if the given point is to the right of the given line, and that d is negative if the point is to the left of the line.

EXAMPLE 1

Find the directed distance from the straight line $3x + 8y = 21$ to the point $(4, 9)$.

Solution.

$$d = \frac{(3)(4) + (8)(9) - 21}{\sqrt{9 + 64}} \approx 7.37.$$

EXAMPLE 2

Derive the equation of the locus of a point that moves so that its distance from the point $(2, 3)$ is always equal to its distance from the line $3x + 5y = 4$.

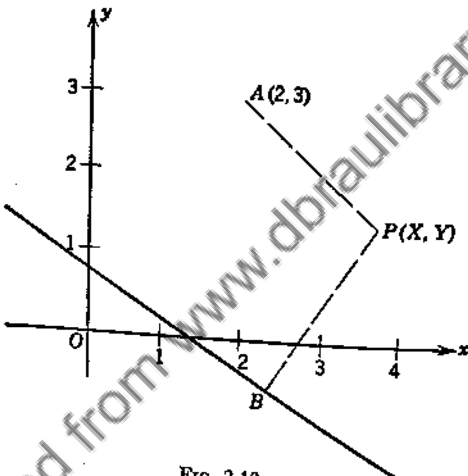


FIG. 2.12

Solution. Step I. Sketch figure, and label the given data.
Step II. Select $P(X, Y)$ equidistant from the point and line.
Step III. $\overline{AP} = \overline{BP}$. (Note that $\overline{AP} = \overline{PB}$ would be incorrect since $\overline{AP}$ is necessarily positive, being an oblique distance between two points, and $\overline{BP}$ is positive since P is to the right of the given line.)
Step IV. Obtain

$$\overline{AP} = \sqrt{(X - 2)^2 + (Y - 3)^2} \quad \text{and} \quad \overline{BP} = (3X + 5Y - 4)/\sqrt{34}.$$

Step V. The student should equate these two expressions, square both sides, and simplify to

$$25X^2 - 30XY + 9Y^2 - 112X - 164Y + 426 = 0.$$

Step VI. Check by finding a Y-intercept ($Y \approx 3.14$ or 15.08), and determine the distance from one of the corresponding points to the given point and to the given line.

PROBLEMS

1. Determine the perpendicular distance from the line $3x + 4y = 10$ to each of the following points, and in each case tell on which side of the line the given point lies:

(a) (5, 7). (b) (−1, 4). (c) (9, 1). (d) (0, 0).
(e) (1, −2). (f) (−2, 0). (g) (0, −1). (h) (0, −5).

2. Figure 2.13 shows a bridge truss. Insert axes with origin at J, and determine the perpendicular distances:

(a) From the member DI to the pin E.
(b) From the member DI to the pin J.
(c) From the member IE to the pin F.
(d) From the member HE to the pin F.
(e) From the member GF to the pin J.
(f) From the member GF to the pin I.

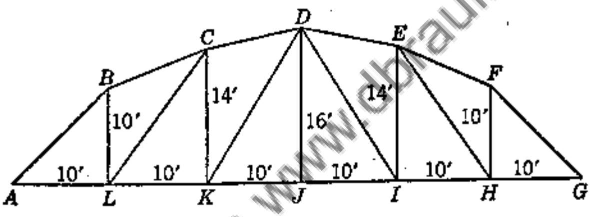


FIG. 2.13

3. Find the coordinates of two points on the y-axis, each of which is 4 units numerically distant from the line $5x + 12y = 20$.

4. Prove that the following pairs of lines are parallel, and then find their distances apart by finding the distance from one line to a particular point on the other line:

(a) $2x - y = 10$ and $4x - 2y + 3 = 0$.
(b) $4x + 3y = 7$ and $8x + 6y + 11 = 0$.
(c) $3x + y = 4$ and $6x + 2y + 5 = 0$.
(d) $1.78x - 4.56y = 8.96$ and $y = 0.390x - 4.37$ (to slide-rule accuracy).

5. Derive the equation of the locus of a point which moves so that its distance from $(-1, 1)$ is always numerically equal to its distance from $x + y = 4$.

6. Determine the equations of the bisectors of the angles between the two lines $4x + 3y = 9$ and $5x + 12y = 12$:

(a) By finding (1) the inclinations of the two lines; (2) one angle between the two lines; (3) half of this angle; (4) the inclinations of the two bisectors; (5) the coordinates of the point of intersection of the two lines; (6) the required equations of the two bisectors.

(*b*) By a locus-derivation method. Note that, using Fig. 2.14, we have $\overline{DP_1} = \overline{CP_1}$ and $\overline{AP_2} = -\overline{BP_2}$. We must prefix the minus sign to $\overline{BP_2}$, since P_2 is to the left of the line through B and the method of this article yields a negative number for the distance.

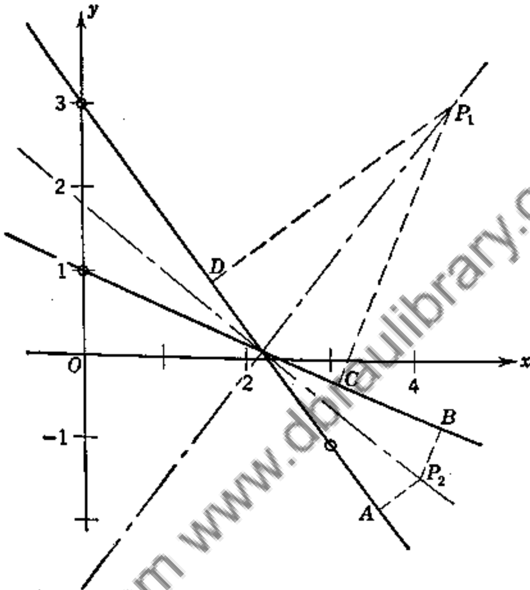


FIG. 2.14

7. Find the equations of the bisectors of the angles between

(*a*) $4x + 3y = 12$ and $3x + 4y = 8$.
(*b*) $2x + y = 5$ and $x - 2y = 4$.
(*c*) $3x + 4y = 12$ and $5x + 12y = 36$.
(*d*) $2x + 4y = 45$ and $2x + y = 12$.

8. A triangle has its three vertices at $A(-8, 1)$, $B(13, 1)$, and $C(-2, 9)$.

(*a*) Find the equation of the bisector of the interior angle at each vertex.

(*b*) Show that these bisectors meet in a common point; find its coordinates and the distance from each side to this common point. (This point will be the center of the inscribed circle.)

(*c*) Find the lengths of the three sides of the triangle and the radius of the inscribed circle by aid of the formula from trigonometry:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

9. Find the equations of the bisectors of the interior angles for each of the following triangles:

(*a*) $(3, 1), (-1, 1), (-1, -2)$.
(*b*) $(4, 2), (0, -1), (1, -2)$.

10. Given each of the following triangles as defined by the equations of its sides. Find (1) the coordinates of the vertices; (2) the lengths of the altitudes; (3) the area, by use of each of the altitudes in turn, and check by the column scheme:

(a) $2x - 3y = 3$, $x + 5y + 5 = 0$, $3x + 2y = 11$.
(b) $x - 5y = 8$, $2x - y + 2 = 0$, $x + y = 2$.

11. Find the value of k in $3x + ky = 5$ if the numerical distance from this line to (1, 3) is 2 units. Draw both of the resulting straight lines.

12S. Find, by a locus derivation, the equation of a straight line if the perpendicular distance from the origin to the line is 5 units long and if the perpendicular from the origin to this line has an inclination of 60°. Then find the equation if the perpendicular distance is p and the inclination of the perpendicular is ϕ.

13S. Use Fig. 2.15 and derive the formula of this article for the distance from the oblique line $ax + by + c = 0$ to the point $P(X, Y)$. Use $L = \overline{BP}$ and $L =$ (projection of $\overline{OC}$ on $\overline{OA}$) + (projection of $\overline{CP}$ on $\overline{OA}$) $- \overline{OA}$. Hence show that $L = X \cos\phi + Y \sin\phi - \overline{OD}\cos\phi$; determine $\tan\phi$ in terms of $\tan\theta$; then substitute for $\cos\phi$, $\sin\phi$, and $\overline{OD}$; and simplify.

2.8 Families of Lines

In this article we shall discuss the graphs of linear equations (in two variables) with the additional provision that these equations shall involve one other variable or *parameter*. For each value assigned to the parameter the graph of the equation will be a straight line. We are interested in any property that is common to all of the straight lines obtained by assigning values to this parameter. For example, the graph of

$$y = 2x + b$$

for each value assigned to the parameter b is a straight line whose slope is 2. The totality of graphs obtained from this equation by assigning different values to the parameter is a family of parallel straight lines.

As a second example, the graph of

$$y - 2 = m(x - 1),$$

for each value assigned to the parameter m, is a straight line that goes through (1, 2). The equation, then, describes a family of lines all of which go through a common point.

As a third example, let us consider the two linear equations:

$$x - 2y - 1 = 0 \quad \text{and} \quad x + y - 4 = 0.$$

These intersect in the point (3, 1); i.e., $x = 3$ and $y = 1$ is the simultaneous solution of the two given equations. If we multiply the second equation by some arbitrary number and add to the first equation, we will obtain a new linear equation whose graph will be a straight line that will go through (3, 1). Thus,

$$x - 2y - 1 + k(x + y - 4) = 0,$$

for any particular value assigned to the parameter k, is a straight line that will have as one solution that solution common to the two given equations. This is easy to see since for such a common solution the

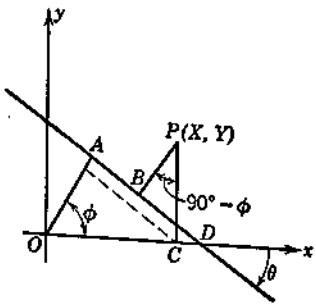


Fig. 2.15

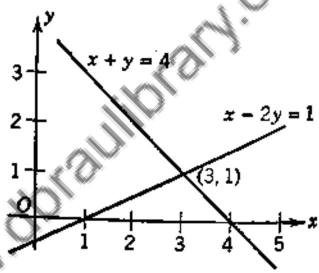


Fig. 2.16

values of the quantities $x - 2y - 1$ and $x + y - 4$ will both be zero. Hence we have in this single equation the description of the family of lines all of which go through the common solution of the two given equations. Moreover, we can write this equation without any need for actually finding that common solution (3, 1).

Example

Find the equation of the straight line that goes through the point of intersection of $x - 2y - 1 = 0$ and $x + y - 4 = 0$ and that goes through (0, 6).

Solution. We could find the coordinates of the point of intersection and then write the equation of the required straight line by aid of the two points. Instead, we first use the equation of the family of lines through the point of intersection of the two given lines:

$$x - 2y - 1 + k(x + y - 4) = 0.$$

The member of this family that also goes through (0, 6) can be found by substituting $x = 0$, $y = 6$; and we find $-13 + 2k = 0$, whence $k = {}^{13}\!/_{2}$ and the required equation for the straight line reduces to $5x + 3y = 18$.

PROBLEMS

1. Write the equation for each of the following families of lines:

(a) All the lines go through (2, 5).
(b) All the lines have a slope of 7.
(c) All the lines have an x-intercept of 9.
(d) All the lines are parallel to $2x + 5y = 7$.
(e) All the lines are perpendicular to $2x - 3y = 4$.
(f) All the lines go through the point of intersection of $2x + 3y = 4$ and $x - y = 6$.

2. Describe the common property for each of the following families of lines, and sketch several members:

(a) $3x - 4y = q$.
(b) $y = 4(x - 1) + c$.
(c) $y = 2x + 2a$.
(d) $y - g = 2(x - g)$.
(e) $s = 2t + k$.
(f) $2s - 3t = h$.
(g) $x - 3y - 7 + h(2x - y - 14) = 0$.
(h) $k(x - 4) + y - 3x = 2$.
(i) $(2 + k)x + (3 - k)y = 4 + 2k$.
(j) $hy - hx = 4$.

3. Show that the equation $4x + 5y = k$ is the equation of the family of lines all of which are parallel to $4x + 5y = 10$. Use this fact to find the equation of the straight line that is parallel to $4x + 5y = 10$ and that goes through $(1, -2)$.

4. Find the equation of the line that goes through (1, 1) and that has as a solution the solution that is common to the two equations $3x + y = 5$ and $x - 4y = 6$.

5. Find the equation of the straight line that goes through the point of intersection of $2x + y = 7$ and $x - y = 5$ (use the coordinates of the point of intersection only as a check) and that

(a) Goes through (1, 1).
(b) Has a y-intercept of 7.
(c) Has a slope of 2.
(d) Is parallel to $2x + 3y = 4$.
(e) Is perpendicular to $2x + 3y = 4$.
(f) Is perpendicular to the y-axis.
(g) Is 4 units numerically distant from the origin.
(h) Goes through the origin.

6. Obtain the equation of the family of lines that are perpendicular to the line through $(1, -3)$ and (5, 1). Then select the member that bisects the line segment joining the two given points.

7. Draw several members of $F = ma$ if m is the parameter, F is the dependent variable, and a is the independent variable.

8. Find the coordinates of a point on $y = 2x + 1$ that is 4 units upward along the line from (1, 3): (a) by use of the equation of the family of lines perpendicular to the given line; (b) by trigonometry.

9. Find the equation of the family of lines through the point of intersection of $2x - 3y = 6$ and $x + y = 4$. Prove that every member of this family passes

through (3.6, 0.40). What has this point to do with the original two lines? Then select the member of this family:

(a) Whose x-intercept is $x = -5$.
(b) Whose slope is 2.
(c) Whose perpendicular distance from the origin is numerically equal to 2.

10S. Every quadratic equation in one variable may be written in the form: $x^2 + bx + c = 0$. But we may interpret this last equation as a linear equation in the variables b and c with x as the parameter. Draw the members of this family for the values $x = -5, -4, \cdots, 4, 5$ and use 1 in. = 2 units on both the b- and c-axes. Label each member with the associated value of x. The resulting graph may be used to solve quadratic equations in one variable, provided that we first reduce the coefficient of the squared term to $+1$.

11S. Repeat Problem 10 for the similar equation $x^3 + bx + c = 0$.

2.9 Cumulative Review

1. *Lengths.* A directed distance parallel to the x-axis is the abscissa of the point to which the measurement is made minus the abscissa of the point from which it is made. A directed distance parallel to the y-axis is equal to the ordinate of the point to which the measurement is made minus the ordinate of the point from which it is made.

The length of an oblique line segment (a positive number) is equal to the square root of the sum of the squares of two quantities: the difference of the abscissas and the difference of the ordinates.

The directed distance from the straight line $ax + by + c = 0$ to the point $P(X, Y)$ is given by

$$d = \frac{aX + bY + c}{\sqrt{a^2 + b^2}}, \quad a > 0.$$

2. *Mid-Point.* The mid-point of a line segment has for its abscissa the average of the abscissas of the ends of the segment; for its ordinate, the average of the ordinates of the ends of the segment.

3. *Slope.* The slope of a horizontal line is zero. The slope of a vertical line does not exist. The slope of an oblique line segment through two points is equal to the difference of the ordinates of the two points divided by the difference of the abscissas, the differences being taken in the same order.

4. *Inclination.* The inclination is the angle whose tangent is the slope, i.e., $\theta = \arctan m$, and the principal value for the inverse tangent is to be chosen.

5. *Area.* The area of a polygon is obtained by a convenient column scheme (do not forget to take one-half of the result).

6. *Fundamental Principle of Analytic Geometry.* A curve goes through a particular point if the coordinates of that point satisfy the equation of the curve.

7. *Locus Derivation* involves six steps:

I. Sketch the figure and label it with the given data.
II. Select a general point $P(x, y)$.
III. Make a geometric statement that P must satisfy.
IV. Translate Step III to an algebraic statement.
V. Simplify.
VI. Make a numerical check.

8. *Equations of Straight Lines.* If the line is parallel to the x-axis, write the equation as $y =$ some constant; if it is parallel to the y-axis, write $x =$ some constant; if it is an oblique line use either the slope method or the slope-intercept form.

9. *Graph of a Straight Line.* Determine the points corresponding to the intercepts (and, perhaps, one other point), and draw the line.

10. *Slope of an Oblique Line.* Solve for y in terms of x; the coefficient of x in this equation is the required slope.

REVIEW PROBLEMS

1. The natural length of a spring (L in.) is 8 in., and a force (F lb.) of 40 lb. is required for each inch of compressing or lengthening the spring. Obtain a formula for F in terms of L, and draw the graph. Also find the area between this straight line, the L-axis, $L = 9$ in., and $L = 12$ in.; and give the proper units for that "area."

2. A train leaving a railroad station has an acceleration of $a = 0.4 + 0.03t$ ft./sec.^{2} Draw a graph for a in terms of t, and determine the area between the straight line, the t-axis, $t = 0$, and $t = 100$ sec. Also find the area between the line, the t-axis, $t = 0$, and $t = t$ sec.

3. The boiling point (T) of water decreases as the altitude (h) above sea level increases. At $h = 500$ ft., $T = 211°$ F.; and at $h = 2500$ ft., $T = 207°$. Assuming a linear relationship, as a first approximation, express the altitude h in terms of the boiling-point temperature T. What is the boiling-point temperature when $h = 12{,}000$ ft.? What are the intercepts, and what is the physical significance of the T-intercept?

4. Given the three points $A(2, 5)$, $B(2, -1)$, $C(-2, 2)$.

(a) Show that this triangle is an isosceles triangle by finding the lengths of the three sides.

(b) Find the slopes and inclinations of all three sides.

(c) Find the coordinates of the mid-points of all three sides.

(*d*) Find the interior angle at C, correct to slide-rule accuracy, by aid of the inclinations of the adjacent sides.

(*e*) Find the interior angle at C, correct to the nearest minute, by first finding $\tan C$ by aid of the slopes of the adjacent sides.

(*f*) Find the interior angle at C by aid of the law of cosines.

(*g*) Find the interior angle at C by use of right-triangle relationships.

(*h*) Find the equations of all three sides, and check each result.

(*i*) Find the equations of all three altitudes.

(*j*) Find the equations of the perpendicular bisectors of the sides.

(*k*) Find the equations of the medians.

(*l*) Find the equation of the bisector of angle B.

(*m*) Find the area by at least two different methods.

5. Determine the equation of the locus of a point that moves so that its perpendicular distance from the line $y = 1$ is always equal to its (the moving point's) distance from the point $(0, 3)$.

6. Find the perpendicular distance (*a*) from the line $2x - 3y + 4 = 0$ to the point $(5, 1)$; (*b*) from the line $3y + 2 = 0$ to the point $(5, 7)$.

7. The speed (V) in feet per second at which sound travels in air is approximated by the equation $V = 1090 + 1.14\,(F - 32)$, where F is temperature in degrees Fahrenheit. Find the temperature if $V = 1150$ ft./sec.; if $V = 1050$ ft./sec. Draw the graph.

8. A copper wire of length 100 ft. has a resistance of $R = 4.00$ ohms at $T = 20°$ C. For temperatures from about $-50°$ C. to $+100°$ C., R is a linear function of T. If this line segment were continued to its point of intersection with the T-axis, this intercept would be $T = -234°$ C. Find the equation for R in terms of T, and give a physical interpretation for the slope of this line.

9. A cylinder 12 ft. long and 4 in. in diameter is lying on its curved side. The curved part is thoroughly heat-insulated with asbestos material. The temperature at one end of the cylinder is kept at 100° C. and at the other end at 10° C. If the temperature at any interior point is a linear function of the distance of the point from one end of the cylinder, find the temperature (T) at a distance of x ft. from the colder end.

10. A graduated income-tax scale is to be devised. A net income of \$1500 or less is not to be taxed. A net income of \$4000 is to be taxed 6 cents on the dollar. All other incomes above \$1500 are to be taxed according to a linear formula, and this formula is to give the correct tax for the two stated incomes. Express tax (T) in dollars as a function of net income (I) for I from 0 to \$6000; draw the graph.

11. The vertices of a triangle are $A(-3, -4)$, $B(3, -4)$, $C(3, 4)$.

(*a*) Find the equations of all three sides.

(*b*) Find the equations of all three altitudes, and show that they intersect in a point D.

(*c*) Find the equations of the medians, and show that they intersect in a point E.

(*d*) Find the equations of the perpendicular bisectors of the sides, and show that they intersect in a point F.

(*e*) Find the equations of the bisectors of the interior angles, and show that they intersect in a point G.

(*f*) Find the radius of the circumscribed circle (the center is at F).

(*g*) Find the radius of the inscribed circle (the center is at G).

(*h*) Show that three of the four points D, E, F, and G are collinear.

(*i*) Find the three vertex angles, each correct to the nearest minute.

(*j*) Check your results in all the preceding problems by aid of a figure drawn to a large scale.

12. Find the x-intercept of the straight line that goes through $(5, -2)$ and is perpendicular to $3x + 4y = 11$.

13. A point on the straight line that goes through $(3, 15)$ and $(11, -3)$ has an abscissa of 5; find its ordinate.

14S. The equations of the four sides of a parallelogram are $x - 3y + 5 = 0$, $3x + 2y + 4 = 0$, $x - 3y = 6$, and $3x + 2y = 18$. Without finding the coordinates of the vertices of the parallelogram, find the equations of its two diagonals. Then check by use of the coordinates of the vertices.

CHAPTER 3

Introduction to Curve Sketching

3.1 Introduction

The student has already learned in his high-school mathematics and again in college algebra how to plot curves by plotting a number of points. This is a satisfactory method for very simple curves but has four faults when applied to more complicated curves:

1. The method takes too much time on more complicated curves.
2. It gives no information about the important properties of the curve.
3. It fails to indicate proper choices of units on the two axes.
4. It does not give the certainty that one has plotted enough points to draw the correct graph.

For subsequent purposes in calculus and in numerous courses in engineering and science, it is unnecessary to plot a careful graph. All that is needed is a quick sketch, a sketch that shows the general shape and characteristics of the curve.

The methods to be studied in this chapter do not have the faults listed above and can be used to obtain either a quick sketch or a more accurate plot. These methods can be used to sketch any curve. *After* the student has thoroughly learned the various methods to be discussed in this chapter, he will use only such properties and plot only such points as may enable him to obtain the required graph in a short time.

In this discussion we shall consistently use x for the independent (horizontal) variable and y for the dependent (vertical) variable. Since other variables are in more common use in science and engineering, the student should be able to apply the methods of this chapter so that he can sketch a curve defined by other variables than x and y.

3.2 Intercepts

The points where a curve crosses either axis are frequently easy to determine. The x-intercepts for a given curve are the directed distances along the x-axis, or the abscissas, of the points where the curve meets that axis, and are obtained by placing $y = 0$ in the given equation of the curve and solving for x. The y-intercepts are defined in a similar manner and are obtained by setting $x = 0$ and solving for y.

EXAMPLE

Determine the intercepts for the two curves:

$$(a)\ x^2 + 4y^2 - x + 6y + 2 = 0, \quad (b)\ xy^2 + 4y = 8.$$

Solution. (a) Set $x = 0$, and obtain $4y^2 + 6y + 2 = 0$, whence the y-intercepts are $y = -1$ and $y = -\frac{1}{2}$. Set $y = 0$, and obtain $x^2 - x + 2 = 0$, both of whose roots are imaginary. Hence the graph of the given equation does not meet the x-axis.

(b) Set $x = 0$, and obtain $4y = 8$, whence the y-intercept is $y = 2$. Set $y = 0$, and obtain $0 = 8$. But such an equation states an impossibility, and we arrived at this absurdity by setting $y = 0$. Hence we conclude that y cannot be zero, and hence that the curve does not meet the x-axis.

3.3 Symmetry

DEFINITIONS. *Two points are said to be symmetrical with respect to a line if that line is the perpendicular bisector of the line segment joining the two points. Two points are said to be symmetrical with respect to a third point if this third point is the mid-point of the line segment joining the first two points. A curve is said to be symmetrical with respect to a line if for each point A on the curve there exists a second point B that also is on the given curve, and that is such that the given line is the perpendicular bisector of the line segment joining A and B. Symmetry of a curve with respect to a point is defined in a similar manner using point symmetry instead of line symmetry for each pair of points A and B.*

A circle is clearly symmetrical with respect to each diameter and with respect to its center. Figure 3.1 shows a curve that is symmetrical with respect to the origin, with respect to the line $y = x$ or the 45° line, and with respect to the line $y = -x$ or the $-45°$ line.

Now suppose that a given curve has an equation that may be written in the form $f(x, y) = 0$,* and suppose that it goes through a general

* The notation $f(x, y) = 0$ is used in algebra to designate an equation in two variables as, for example,

$$x^2 + 3xy + 2y^2 - 4x - 6 = 0.$$

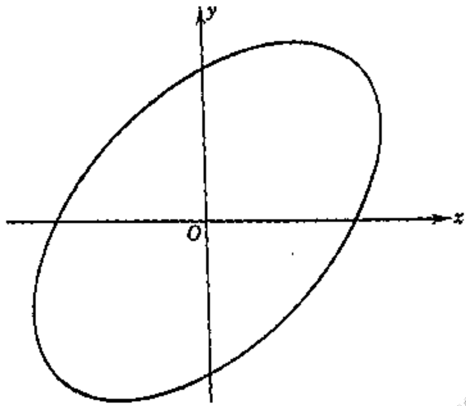


FIG. 3.1

point $P(x, y)$ (see Fig. 3.2). Then $f(x, y) = 0$. Suppose that it also goes through a second point $Q(x, -y)$ and hence that $f(x, -y) = 0$. Since the x-axis is the perpendicular bisector of the line segment QP, and since we assumed that P was a general point, it follows that the given curve is symmetrical with respect to the x-axis. Hence the test for symmetry with respect to the x-axis is to replace y by $-y$; if the resulting equation is equivalent to the given equation, the curve is symmetrical with respect to the x-axis.

Figure 3.2 shows the point $P(x, y)$ and three other points $Q(x, -y)$, $R(-x, y)$, and $T(-x, -y)$. These three points indicate three basic tests for symmetry. In each case, if the equation (that results from the stated substitution) is equivalent to the given equation, the graph has the indicated symmetry.

I. *The test for symmetry with respect to the x-axis is to replace y by $-y$.*
II. *The test for symmetry with respect to the y-axis is to replace x by $-x$.*
III. *The test for symmetry with respect to the origin is simultaneously to replace x by $-x$ and y by $-y$.*

If the equation is algebraic, the foregoing rules may be restated as follows.*

I. If the equation contains only *even powers* of y, the graph is symmetrical with respect to the x-axis.

II. If the equation contains only *even powers* of x, the graph is symmetrical with respect to the y-axis.

* There are certain trivial exceptions or additions that need not be discussed here.

III. If the total degrees in x and y together of the terms of the equation are *all* even or are *all* odd, then the graph is symmetrical with respect to the origin.

We observe that, if a curve has any two of the three types of symmetry, it necessarily has also the third type of symmetry.

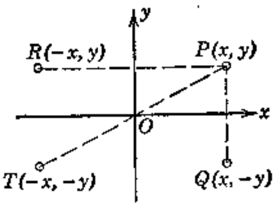


FIG. 3.2

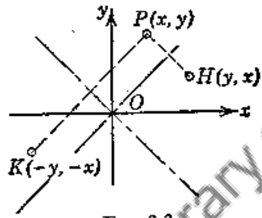


FIG. 3.3

Less frequently, we have occasion to use two other types of symmetry. These are indicated in Fig. 3.3, and the statements of the tests are as follows:

IV. The test for symmetry with respect to the 45° line through the origin (i.e., the line $y = x$) is to replace simultaneously y by x and x by y.

V. The test for symmetry with respect to the −45° line through the origin (i.e., the line $y = -x$) is to replace simultaneously y by $-x$ and x by $-y$.

The proofs of these two tests for symmetry will be valuable exercises for the student.

EXAMPLE

Test for symmetry:

(*a*) $x^2 + 5y = 0$. (*b*) $x^2 - 3xy + y^2 = 5$.

(*c*) $x^2 + 4y^2 = 9$. (*d*) $y = 3 \sin 2x$.

Solution.

(*a*) Symmetry with respect to the y-axis (since only even powers of x occur).

(*b*) Symmetry with respect to the origin, the 45° line, and the −45° line.

(*c*) Symmetry with respect to the x-axis, the y-axis, and the origin.

(*d*) Symmetry with respect to the origin, since the equation obtained by replacing x by $-x$ and y by $-y$, namely $(-y) = 3 \sin(-2x)$, can be reduced to the given equation by aid of rules from trigonometry and algebra.

3.4 Horizontal and Vertical Asymptotes

In college algebra, the student learned that in mathematics *we never divide by zero.* Let us see where that principle leads us in curve sketching. We start by plotting the curve $y = 1/(x - 2)$ by plotting points and make up a table of values such as that given in the adjoining table. When we try to substitute $x = 2$ we are led to a forbidden operation. Let us compute values of y for x near $x = 2$ and see what we can conclude. If we take values of x larger than 2 and let the values get closer to 2, we see from the table that the value of y is

x	y	x	y
−3	−0.20	2.001	1000
−2	−0.25	2.01	100
−1	−0.33	2.1	10
0	−0.50	2.5	2
1	−1	3	1
1.5	−2	4	0.5
1.9	−10	5	0.33
1.99	−100	6	0.25

increasing without limit in a positive direction. Similarly, if we let x take on values smaller than 2 and let these values approach 2, the values of y get *numerically* larger, again without limit.

An inspection of Fig. 3.4 shows that the curve approaches closer and closer to the vertical line, whose equation is $x = 2$, as the values of x get closer and closer to 2, and at the same time the numerical values of y increase without limit. This line, $x = 2$, is called an asymptote for the given curve.

DEFINITION. *An asymptote (or asymptotic line) is a line that the curve approaches as the numerical value of one of the variables increases without limit; that is, the perpendicular distance from a point on the curve to the line must approach zero.*

If we solve the given equation for x in terms of y, we obtain

$$x = \frac{2y + 1}{y}.$$

This time the value that would cause division by zero is $y = 0$.

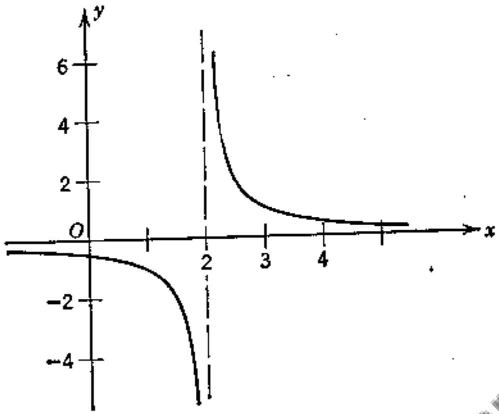


FIG. 3.4

Figure 3.4 shows that, as the value of y gets nearer and nearer the value zero, the numerical value of x increases without limit.

This example illustrates the following rule by which we may determine the equations of the vertical and horizontal asymptotes:

RULE. *To determine the equations of the vertical asymptotes, solve for y in terms of x, equate the denominator to zero, and solve. To determine the equations of the horizontal asymptotes, solve for x in terms of y, equate the denominator to zero, and solve the resulting equation.* If no real solution to such an equation exists, there is no asymptote of this type.

The equation of the horizontal asymptote in the preceding example, $y = 0$, could also be obtained in another manner. We could ask what happens to the value of the fraction $y = 1/(x - 2)$ as x increases numerically. Thus, if x takes on successively the values 10, 100, 1000, 10,000, etc., the value of y clearly gets closer and closer to zero. Hence we conclude that one can determine the equations of the horizontal asymptotes from the form of the equation that gives y in terms of x by examining the behavior of the x-expression as x increases numerically without limit.

In the preceding example, it is important that the student realize that there is no value that can be assigned to y when $x = 2$. We have seen that, as x approaches the value 2, the value of y increases numerically without limit. The student is warned to distinguish between the statements that "zero divided by any other number is zero" and "one cannot divide by zero." Also he should understand the distinction between stating that something is *not defined* and stating that it has the value *zero*.

EXAMPLE

Determine the equations of the horizontal and vertical asymptotes for $y^2(x^2 - 2x - 3) = 4x^2$.

Solution. We solve for y^2 in terms of x: *

$$y^2 = \frac{4x^2}{(x-3)(x+1)};$$

and for x in terms of y from $x^2(y^2 - 4) - 2xy^2 - 3y^2 = 0$:

$$x = \frac{2y^2 \pm \sqrt{4y^4 - 4(y^2-4)(-3y^2)}}{2(y^2-4)}.$$

We equate the denominators to zero, and obtain as the equations of the horizontal and vertical asymptotes: $x = 3$, $x = -1$, $y = 2$, and $y = -2$. The alternative method of determining the equations of the horizontal asymptotes may be illustrated by this example. We use the solution for y^2 in terms of x, and divide the numerator and denominator by the highest power of x that occurs in either the numerator or denominator (in this case it is x^2) and obtain

$$y^2 = \frac{4}{1 - (2/x) - (3/x^2)}.$$

As x increases without limit (let $x = 10$, then $x = 100$, then $x = 1000$, etc.), the value of y^2 approaches the value 4. Hence the two horizontal asymptotes are given by $y^2 = 4$, or by $y = 2$ and $y = -2$.

3.5 Excluded Regions

A very helpful step in sketching curves is to exclude regions where *there are no points of the locus.* We shall explain the procedure by means of an example.

EXAMPLE

Determine the excluded regions for the curve whose equation is $x^2y = 4y + x^2$.

Solution. We first solve for y in terms of x (this is a linear equation in y), and obtain $y = x^2/(x^2 - 4)$. We determine the *critical numbers, those numbers which make either the numerator or the denominator zero*, to be $x = -2$, $x = 0$, and $x = +2$.

When $x < -2$, both $x^2 - 4$ and x^2 are positive, and the fraction is therefore positive. Hence y is positive for all values of x less than -2, and so *cannot be negative.* Figure 3.5 shows crosshatching for y negative and $x < -2$, a region where there can be no graph.

* Solving for y^2 in terms of x and solving for y in terms of x gives the same *denominator* factors.

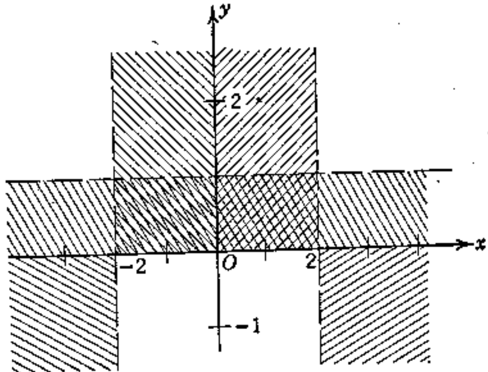


FIG. 3.5

When $-2 < x < 0$ (x has any value between -2 and 0), the numerator is positive and the denominator is negative, and y is therefore negative in this region and hence *not positive*. When $0 < x < 2$, y is negative and hence not positive. When $x > 2$, y is positive and hence not negative. All these results are shown by the crosshatching on the figure to indicate regions where there is no graph.

We next solve for x in terms of y (the given equation is a quadratic equation in x or a linear equation in the variable x^2), and obtain

$$x = \pm\sqrt{\frac{4y}{y-1}}.$$

The critical numbers that reduce to zero the numerator or denominator (of the fraction under the radical) are $y = 0$ and $y = 1$. When $y < 0$, both the numerator and denominator are negative, the fraction is positive, and therefore x is real and there is *no* excluded region for $y < 0$. When $0 < y < 1$, the numerator is positive, the denominator is negative, and the fraction is negative; hence x is imaginary in this region and this region is crosshatched in the figure. When $y > 1$, x is again real.

As a check, we observe that the graph of the original equation has symmetry with respect to the y-axis, and that the crosshatching satisfies this condition of symmetry. Stated differently, we could have disregarded all negative values of x in the study of excluded regions.

This illustrative example shows two types of excluded regions: positive-negative, and imaginary.

RULE. *To determine the excluded regions for a given curve, first solve for y in terms of x and for x in terms of y. If either solution involves an* **even root**, *examine the quantity under that radical and determine the range of values that will make the other variable imaginary. If no even root is present, find the range of values that will make the other variable positive and the range of values that will make it negative.*

3.6 Choice of Scales on the Two Axes

In science and in engineering, it is a rare exception to see a graph with the same scale on both axes. Also, it is unusual to discover a graph which has 1 square on the sheet of graph paper equal to 1 unit. The ideas discussed in the preceding articles can be used to give suggestions for the choice of units on the two axes.

The problem itself may indicate the range of values for one or both variables and hence imply the choice of units.

EXAMPLE

Determine the proper scales on the axes for the variables p and v for a graph of the gas law $pv = 20$. The range of pressure is from $p = 10$ lb./sq. in. to $p = 80$ lb./sq. in. The corresponding range for the volume is from $v = 0.25$ cu. ft. to $v = 2$ cu. ft. (and the equation is valid for p and v in the stated units). The graph is to be made on a sheet of 8.5 by 11 in. paper, and p is to be the dependent variable.

Solution. Since the range of both variables is to be positive, we can place the origin at the lower left-hand corner of the sheet. Since in this example it is more convenient to take the shorter axis for the v-axis, the paper should be placed in normal position (and not turned sideways). A convenient scale for p (the available length on the vertical scale is about 10 in.) would be 1 in. = 10 lb./sq. in. If v is to range from 0 to 2 (the available scale length is about 7 in.), use 2.5 in. = 1 cu. ft. This makes the scale length 5 in. and allows easy graphical interpolation.

If standard cross-section paper is used for a graph, the scale chosen should allow easy decimal interpolation. This implies that 1 square on the graph may be used to represent 1, 2, 5, 10, 100, 0.5, 0.2, 0.1, 0.01, etc., units for the variable. One should avoid, for example, 1 square = 3 units or 3 squares = 1 unit.

In the absence of information about the range of values for either variable, the resulting graph should display the important characteristics of the given curve. The discussion method yields certain critical values for each variable (intercepts, equations of horizontal and vertical asymptotes, critical numbers for excluded regions). We may use these numbers to guide the choice of scales on the two axes. In the graphs that follow we shall frequently use different scales on the two axes.

3.7 Summary of Curve Sketching by the Discussion Method

We summarize the material of the preceding articles as follows:

I. *Intercepts.* Let $x = 0$ and solve for y; let $y = 0$ and solve for x.

II. *Symmetry* (five types). Determine what substitutions can be made in the original equation that will yield the same or an equivalent equation, and then determine what type of symmetry this indicates.

The tests for symmetry are as follows:

Test (substitution)	Symmetry
$-x$ for x	y- axis
$-y$ for y	x-axis
$-x$ for x and $-y$ for y	origin
x for y and y for x	45° line
$-x$ for y and $-y$ for x	$-45°$ line

III. *Asymptotes.* Solve for y in terms of x and for x in terms of y. Equate each denominator to zero, and solve for all the real solutions; or, alternatively, determine whether one variable approaches a definite value as the other variable increases without limit.

IV. *Excluded regions.* Use the solutions for y in terms of x and for x in terms of y. Examine for the range of values for one variable that yield imaginary values for the other variable (arising from even roots), and for the range of values for one variable that make the other variable positive or negative, and crosshatch the regions where there is no curve.

V. Choose scales for the two axes on graph paper, and sketch the curve.

VI. The first five steps are often sufficient for the purpose of a quick sketch. If a more accurate plot is desired, the quick sketch will suggest the positions of necessary points; compute the coordinates of these points, plot them, and then draw the curve.

It is unnecessary to write out the discussion after the student understands each of these steps, since it is desirable to obtain the graph as quickly as possible. The final sketch should show the positive directions of the two axes, the variable plotted on each axis, and the scale used on each axis.

A very convenient check is given by the following theorem:

Theorem. *The graph of a linear equation can intersect the graph of an nth-degree algebraic equation in at most n points.*

To prove this theorem we may solve the nth-degree equation simultaneously with the equation of a general straight line

$$Ax + By + C = 0.$$

When one variable is eliminated by the method of substitution, the resulting equation in the other variable will be of degree n or less. Such an equation of degree n, by the extension of the fundamental theorem of college algebra, will have precisely n roots; however, some of these may be imaginary roots and others may be repeated roots. Corresponding to each real solution there will be a point on the curve. Thus, there cannot be more than n of these points of intersection.

EXAMPLE

Discuss for intercepts, symmetry, asymptotes, and excluded regions, and sketch the locus of $x^2y + x^2 = 4y$.

Solution. Intercepts. When $x = 0$, $y = 0$; when $y = 0$, $x = 0$.

Symmetry. The curve will be symmetrical with respect to the y-axis, since the replacement of x by $-x$ yields an equivalent equation.

Asymptotes. From $y = x^2/(4 - x^2)$, we see that the equations of the vertical asymptotes are $x = 2$ and $x = -2$, and that the equation of the horizontal asymptote is $y = -1$. From $x = \pm 2\sqrt{y/(y + 1)}$, we may read the equations of the same asymptotes.

Excluded regions. The critical numbers for $y = x^2/(4 - x^2)$ are $x = 0$, $x = -2$, and $x = +2$. The critical numbers for $x = \pm 2\sqrt{y/(y + 1)}$ are $y = 0$ and $y = -1$. The statements for the excluded regions are expressed as follows in the form of inequalities:

When $x < -2$, $y < 0$; $-2 < x < 0$, $y > 0$; $0 < x < 2$, $y > 0$; $2 < x$, $y < 0$.

When $y < -1$, x has two values; $-1 < y < 0$, x is imaginary; $0 < y$, x has two values.

The locus is shown, together with the crosshatching, in Fig. 3.6. As a check, we observe that the original equation is of the third degree and that no straight line can be drawn on the xy-plane which will intersect the curve in more than three points.

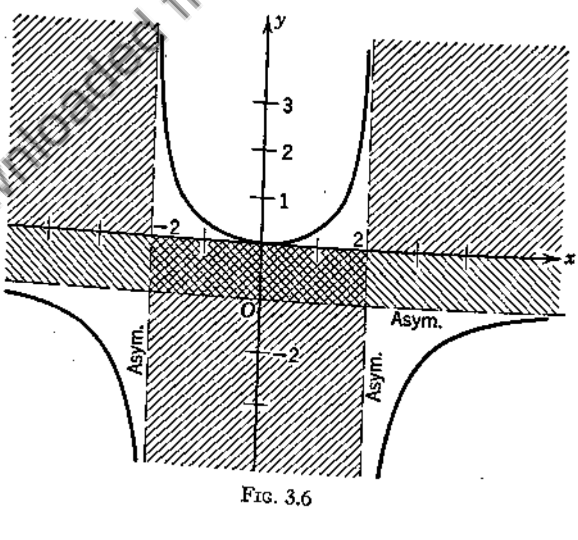


FIG. 3.6

PROBLEMS

1. Determine the intercepts for each of the following curves:

(a) $x^2 + 2xy = 4$.

(b) $x^2 - 3xy + y^2 - 4x - 5y = 6$.

(c) $x^2 - 4y^2 = 16$.

(d) $x^2 + 4y^2 = 16$.

2. State what symmetry each of the following curves possesses:

(a) $x^2 + 4y = 6$.

(b) $xy^2 + x^3y^4 = 6$.

(c) $x^2 + 2xy = y^2 + 6$.

(d) $xy^2 + x^2y = 8$.

(e) $y = \cos x$.

(f) $y = 2^x + 2^{-x}$.

3. Crosshatch the xy-plane to show the region in which x is imaginary:

(a) $x = 4y \pm \sqrt{y - 1}$.

(b) $x = 2y + 3 \pm \sqrt{\dfrac{y - 1}{y + 2}}$.

(c) $x = 3 \pm \sqrt{2y - 1 \pm \sqrt{y + 2}}$. (*Note:* Ignore all but the innermost square root.)

4. Determine the regions in which y is positive and the regions in which y is negative; crosshatch the xy-plane accordingly:

(a) $y = 2(x - 1)(x + 2)$.

(b) $y = \dfrac{x - 1}{3x}$.

(c) $y = \dfrac{x(x - 4)(x + 6)}{(x - 2)(x + 2)}$.

(d) $y = \sqrt[3]{8 - x}$.

5. Determine the equations for the horizontal and vertical asymptotes of the following curves:

(a) $y = \dfrac{4x - 3}{x - 2}$.

(b) $y = x^2 - 4x$.

(c) $y = \dfrac{2}{x - 3}$.

(d) $y = \dfrac{x^2 - 3}{x + 4}$.

(e) $s = \dfrac{t^2 - 4t - 5}{t^2 - 3t + 2}$.

(f) $s^2 = \dfrac{9t - 7}{4t - 5}$.

6. Show that the following curves have no horizontal or vertical asymptotes:

(a) $y = a + bx, b \neq 0$.

(b) $y = a + bx + cx^2, c \neq 0$.

(c) $y = a + bx + cx^2 + dx^3, d \neq 0$.

Note. These three examples imply the theorem: The graph of y as a polynomial in x has no horizontal or vertical asymptotes.

7. Use the discussion method in order to verify the graphs in Figs. 3.7, 3.8, and 3.9.

(a) $y = x^4 - 4x^2$ or $x = \pm\sqrt{2 \pm \sqrt{4 + y}}$.

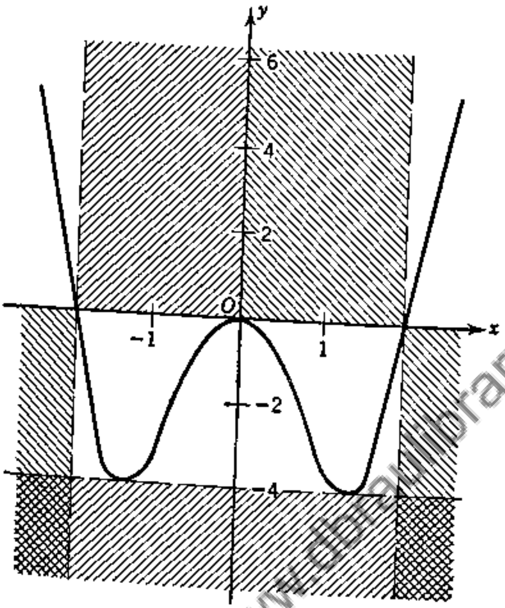


FIG. 3.7

(b) $xy^2 + y^2 = x$.

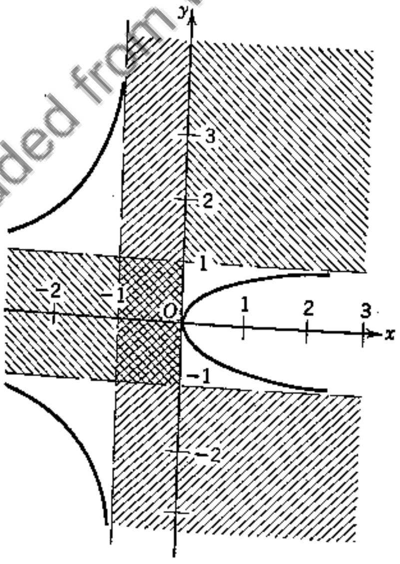


FIG. 3.8

(c) $x^2y + y = 2x$,

or:

$$x = \frac{1 \pm \sqrt{1 - y^2}}{y}, \; y = \frac{2x}{x^2 + 1}.$$

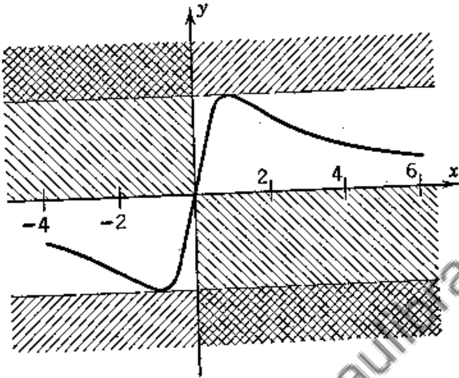


FIG. 3.9

8. The equation of a certain locus has the following properties: y is a single-valued function of x while x is a double-valued function of y, and the equation is of the third degree in x and y together; the intercepts are $x = 2$, $x = -2$, and $y = 4$; the locus is symmetrical with respect to the y-axis; the asymptotes are $x = 1$, $x = -1$, and $y = 1$; for $x < -2$, $y > 0$; for $-2 < x < -1$, $y < 0$; for $-1 < x < 1$, $y > 0$; for $1 < x < 2$, $y < 0$; for $2 < x$, $y > 0$; for $1 < y < 4$, x is imaginary. Draw a locus that satisfies these conditions.

9. Discuss completely and sketch each of the following curves. Do not plot any points other than those obtained in the discussion.

(a) $y = 8x - x^2$.
(b) $y = 4 - x^2$.
(c) $x^2y + y = 4$.
(d) $x^2y - 4y = 3x^2$.
(e) $x^2 + y^2 = 1$.
(f) $s^2 + t^2 = 6$.
(g) $x^2 - y^2 = 4$.
(h) $u^4 + v^4 = 1$.
(i) $xy^2 + 4y^2 = 4x$.
(j) $pv = 5$.
(k) $y^2 = x(x - 4)^2$.
(l) $y(x^2 + 2x - 8) = x + 2$.
(m) $x^2y + y = x - 1$.
(n) $y^2 = x^3 - 3x$.
(o) $(x + 1)^2y = (x - 1)^2$.
(p) $y(x^2 - 4) = x^3 - x$.
(q) $x^2 - xy + y^2 = 3$.
(r) $y^2 + 16x^2 = x^4$.
(s) $xy - 4y = x - 1$.
(t) $x^2 + y^3 = 6x$.
(u) $st + 2s = 2t + 2$.
(v) $ts^2 + 4t = 4s$.
(w) $p^2 + pq + q^2 = 4$.
(x) $x^2 + 4xy + 8y^2 = 4$.

10. (a) Sketch a graph from which you can read the value of $\tan 2\theta$ when given the value of $\tan\theta$. Thus, since $\tan 2\theta = (2\tan\theta)/(1 - \tan^2\theta)$, let $y = \tan 2\theta$ and $x = \tan\theta$, and sketch $y = 2x/(1 - x^2)$.

(*b*) Sketch similar graphs using

(1) $\cos 2\theta = 2\cos^2\theta - 1.$

(2) $\cos^2(\theta/2) = (1 - \cos\theta)/2.$

(3) $\tan(\theta + 45°) = (\tan\theta + 1)/(1 - \tan\theta).$

11. First sketch the curve $y = 4 - x^2$. Then inscribe a rectangle with two of its vertices on this curve and with its base on the x-axis. The width of the base is to be $2X$. Express the area of this rectangle in terms of X; sketch A as a function of X, assuming that $0 \leqq X \leqq 2$.

12S. A rectangular piece of cardboard measures 10 by 18 in. Squares of width x inches are cut out of each corner, and the remaining cardboard is bent into a rectangular box with no top. Express the volume of the box in terms of x, and sketch. Then estimate the value of x that yields the largest volume for the box.

13S. The outer radius of an annular or ring-shaped area is R in. and the thickness of the ring is t in. (hence the radius of the inner circle is $R - t$). Show that the area of the ring between the two circles can be written as

$$\frac{A}{R^2} = 2\pi\frac{t}{R} - \pi\frac{t^2}{R^2},$$

and sketch a graph of A/R^2 in terms of t/R. Why should the final graph show only that portion of the curve for $0 \leqq t/R \leqq 1$?

14S. Many algebraic curves have names that have been given to them for one reason or another. The following equations and the similar set in Problem 17S at the end of this chapter are listed with the names of the curves. The student is to sketch each curve. In many cases it will be convenient first to reduce the equation to another form involving ratios or dimensionless variables, and then to introduce new variables, as illustrated in the first few equations.

(*a*) $y^2 = x^3/(2a - x)$, the cissoid of Diocles, who lived about 100 B.C. We rewrite the equation in the form $(y/a)^2 = (x/a)^3/[2 - (x/a)]$, introduce $X = x/a$ and $Y = y/a$, sketch the graph of $Y^2 = X^3/(2 - X)$, and finally rescale the two axes so that $x = a$ when $X = 1$, $x = 2a$ when $X = 2$, etc.

(*b*) $y^2 = x^2(a - x)/(a + x)$ or $Y^2 = X^2(1 - X)/(1 + X)$, the strophoid.

(*c*) $x^2y + b^2y = a^2x$ or $Y = X/(X^2 + 1)$, where $X = x/b$ and $Y = by/a^2$, the serpentine.

(*d*) $y^2 = x^2(3a + x)/(a - x)$, the trisectrix of Maclaurin.

(*e*) $x^2y = a^2(a - y)$, the witch of Agnesi.

(*f*) $(a^2/x^2) - (b^2/y^2) = 1$, bullet-nosed curve.

(*g*) $(a^2/x^2) + (b^2/y^2) = 1$, cross curve.

(*h*) $y^2(x^2 + y^2) = a^2x^2$, the kappa curve.

15S. A point moves so that the product of its distances from $A(a, 0)$ and $B(-a, 0)$ is always equal to a constant b^2. Find the equation of the locus in terms of x and y. Then let $X = x/a$ and $Y = y/a$, and show that the resulting equation may be written in the form $(X^2 + Y^2)^2 + 2(Y^2 - X^2) = (b/a)^4 - 1$. Finally, sketch the locus and treat the three possible cases: $b < a$, $b = a$, and $b > a$.

16S. Show that the graph of the equation $x + y = xy$ and the graph of the related equation $(1/x) + (1/y) = 1$ are not precisely equivalent. What algebraic operation must be performed on the second equation to obtain the first equation?

3.8 Graphs of the Power Law: $y = ax^n$

In this article, we shall apply the discussion method to sketch graphs of equations of the form $y = ax^n$. We shall suppose that $a > 0$ in this discussion, but it should be clear that the graph with a negative would be the mirror image in the x-axis of the corresponding curve plotted with a positive.

I. *Intercepts.* If n is positive, all the curves go through the origin; if n is negative, the curves do not cross either axis.

II. *Symmetry.* If n is an even integer, whether positive or negative, the curve will be symmetrical with respect to the y-axis; if n is an odd positive or negative integer, the curve is symmetrical with respect to the origin. If n is a decimal number, this number will probably be an approximation for the correct value, in which case the graph should be drawn only in the first quadrant. If n is a fraction such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, etc., y is always positive for x positive since (from algebra) the principal value of the square root, or fourth root, etc., is to be used.

III. *Asymptotes.* If n is a positive number, there are no asymptotes; if n is a negative number, both axes are asymptotes.

IV. *Excluded regions.* This step is unnecessary, since the complete graph may be obtained by symmetry from the graph drawn in the first quadrant.

The first-quadrant graphs of some of these curves are shown in Fig. 3.10. These graphs may be separated into three groups as shown

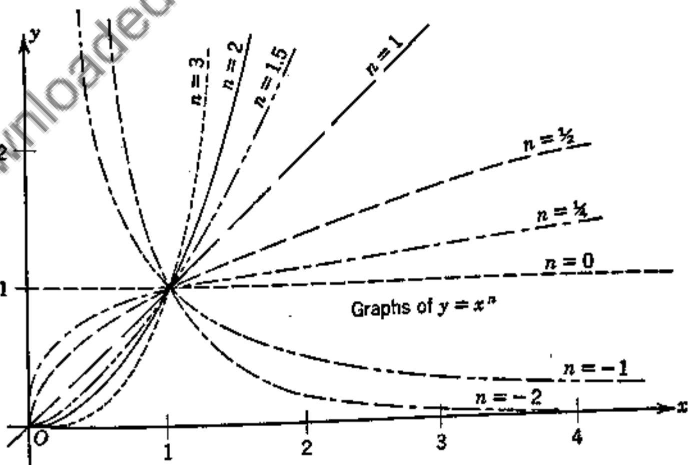


FIG. 3.10

in Fig. 3.11. If n is larger than 1, the curves go through the origin and at that point are all tangent to the x-axis. If $0 < n < 1$, the curves go through the origin and at the origin are tangent to the y-axis. If n is negative, the curves have both axes as asymptotes.

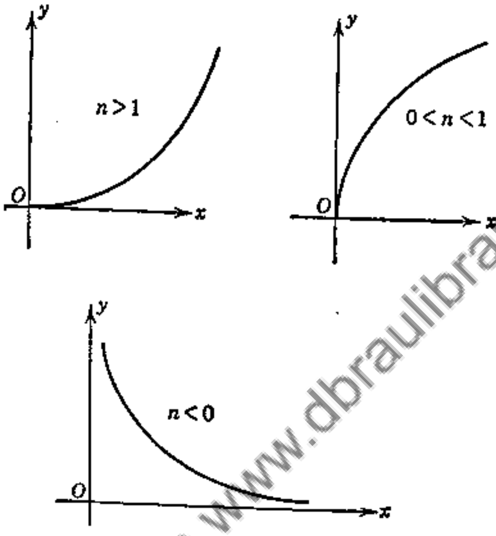


FIG. 3.11

We observe that the special case in which $n = 1$ becomes $y = ax$, which is a straight line. If $n = 0$, the resulting equation is $y = a$, which is a horizontal straight line. (We shall see later that the curves for which $n = 2$, ½, and -1 have other names in addition to being power-law curves.)

PROBLEMS

1. Plot the following curves, each set on one graph. In each case show the complete graph and plot enough points to enable a smooth curve to be drawn.

(a) $y = 2x^n$ for $n = 1$, 1.5, 2, and 3.
(b) $y = 2x^n$ for $n = 0$, 0.333, 0.5, and 1.
(c) $y = 2x^n$ for $n = -1$, -0.5, and -2.

2. Repeat Problem 1 for the equation $y = 0.5x^n$.

3. Sketch the following curves in the first quadrant without locating any points at all:

(a) $y = 2x^{1.78}$.
(b) $y = 3x^{0.78}$.
(c) $y = 0.8x^{-1.4}$.
(d) $y = x^5$.
(e) $y = 1.92x^{2.11}$.
(f) $y = 4x^{-0.111}$.

4. Sketch the graph of $pv^m = k$, where m and k are both positive constants and p is the dependent variable.

5. Sketch $pv^m = k$ for the three cases $m = +1$, $m = +2$, and $m = +1.41$, k being a positive constant. Use $y = p/k$ as the dependent variable, and show only that portion of the graph in the first quadrant. Use three points on each of the first two curves, and sketch the third curve between the first two.

6. (a) Sketch on the same axes in the first quadrant: $y_1 = 4.56x^{-2.13}$ and $y_2 = 4.56x^{-2}$.

(b) Which ordinate is larger for $x > 1$?

(c) Determine one value of x such that for all larger values of x the ordinate y_1 will be smaller than 0.001 (*Hint:* use your result in (b)).

7. Sketch $y = ax^n$ for $n = 2$ and $a = 1, 2, -1$, and -2, and show all four curves on the same graph.

8. If x is any number between zero and one, which is the larger: the cube root, or the square root? Which is larger if $x > 1$? Sketch graphs to prove your conclusions.

9. Sketch on the same graph:

(a) $y = x$ and $y = 1/x$.
(b) $y = x^2$ and $y = x^{1/2}$.
(c) $y = x^3$ and $y = x^{1/3}$.

10. (a) If y varies directly as the square of x, and if $y = 4$ when $x = 5$, sketch the graph of y as a function of x.

(b) What is the graph if y varies inversely as the square of x, and if $y = 4$ when $x = 5$?

11. Estimate from a graph one simultaneous solution of $y = 2x^{1/3}$ and $y = 4x^{-2}$.

12. Sketch appropriate graphs, and estimate to two significant figures the values of the following:

(a) $2x^{1/2}$ when $x = 0.4$.
(b) $2x^{-3/2}$ when $x = 1.3$.
(c) $2x^{1.6}$ when $x = 0.8$.
(d) $2x^{-4}$ when $x = 0.9$.
(e) $2(1.5)^{0.25}$.
(f) $2(2.5)^{-1}$.

3.9 Addition of Ordinates

In this article, we shall study a second powerful method by which we may sketch curves. The basic idea for this second method is that of sketching complicated curves by use of the graphs of simple curves, such as the power-law curves. In subsequent articles we shall use this fundamental method for curve sketching. In this article we shall study the method and make use of the power-law curves and the discussion method.

Suppose that we wish to sketch the graph of $2xy = x^2 - x + 1$. One procedure would be to use the discussion method. Suppose, instead, that we solve the given equation for y in terms of x:

$$y = \frac{x - 1}{2} + \frac{1}{2x}.$$

We could then plot the graphs of $y_1 = (x - 1)/2$ and $y_2 = 1/(2x)$, as shown in Fig. 3.12, as dotted curves. In order to plot points on the required curve we could proceed as follows: If, for example, $x = 2$, the required value of y would be the sum of the arithmetic values of $(x - 1)/2$ and $1/2x$ or $0.5 + 0.25 = 0.75$. We could then plot the point (2, 0.75), which would be one point on the required curve.

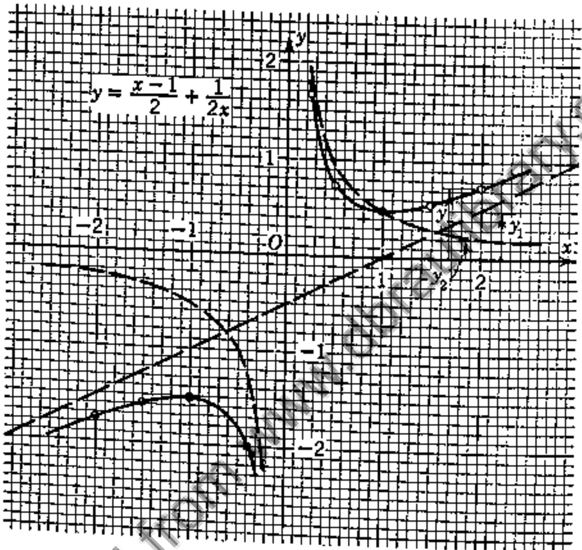


FIG. 3.12

However, all we need to know is the position of this point—we do not care what its y value is. We could determine this position *graphically* by *adding the ordinates to the two simpler curves*. Moreover, this addition could be effected either with a pair of dividers or by the addition of squares, if the two curves are plotted on graph paper. Thus, in Fig. 3.12, the required y value at $x = 2$ is shown as the sum of y_1 and y_2. The student should use a pair of dividers and a straightedge in order to satisfy himself that the circled points have been correctly located by this process. He should observe that, since the original curve is symmetrical with respect to the origin, he need plot only half of the curve by this new method.

Since the two *component curves* (the simpler curves in this example) were easy to plot or to sketch, the required curve could be obtained more rapidly by this new method of addition of ordinates than by the former discussion method. The method of addition of ordinates can be used whenever the original equation, when solved for y in terms of x, can be split into two or more parts, each of which can be sketched easily. Thus, if the original equation can be written in the form $y = f(x) + g(x)$, and if the two component curves $y_1 = f(x)$ and $y_2 = g(x)$ are easy to sketch, then the required curve may be obtained by adding ordinates.

PROBLEMS

In each of the following problems, plot the component curves and then add ordinates to obtain the graph of the given equation. As a part of your solution, give the equations of the component curves.

1. $y = (x - 1)/2 + (x - 2)/3$. Check your result in this problem by drawing the graph of the original equation in simplified form.

2. $y = x + 1$.

3. $xy + 3x = 4$.

4. $y = x^2 + (x - 2)/3$.

5. $x^2 - xy = 2$.

6. $y = x + (1/x^2)$.

7. $y = (x/2) + (-2/x)$.

8. $y = 1 + x + x^2 + (1/x)$. Use three component curves.

9. $5x^2 - 4xy + 4y^2 + 4y - 2x = 63$. First show that

$$y = \frac{x - 1}{2} \pm \sqrt{16 - x^2}.$$

Then plot by addition and subtraction of ordinates using $y = y_1 + y_2$ and $y = y_1 - y_2$, where $y_1 = (x - 1)/2$ and $y_2 = \sqrt{16 - x^2}$. The second equation has a graph that is the top half of a circle with center at the origin and radius 4.

10. $y = x - 1 \pm \sqrt{x}$. Note that $y_2 = \sqrt{x}$ is a power-law curve and that the principal value of the square root is to be used whenever no sign is shown. Finish the graph by both addition and subtraction of ordinates.

11. $xy + y^2 = 4$. Solve for x in terms of y and add *abscissas*. Why is this a better method in this particular problem than to solve for y in terms of x and then add ordinates?

12. Sketch $y(x^2 - 4) = x^3 - x$ by writing the equation in the form $y = x + 3x/(x^2 - 4)$. Use the discussion method to sketch one of the component curves.

13. The volume of a tin can (right circular cylinder) is 80 cu. in. Express the total amount of tin (two ends and curved part) as a function of the radius, and then sketch the graph of the amount of tin in terms of the radius.

14. Plot $y = x + \sqrt{4 - x^2} + \sqrt{6 - x^2}$, where $y_2 = \sqrt{4 - x^2}$ is the top half of a circle with center at the origin and radius 2, and $y_3 = \sqrt{6 - x^2}$ is the top half of another circle with center at the origin and radius $\sqrt{6} \approx 2.45$. Use 1 in. = 1 unit on both axes. Finally estimate the x-intercepts of the final curve. What algebraic equation are you solving by aid of your graph? How otherwise could you solve this equation?

15. Repeat Problem 14 for $y = x + \sqrt{4 - x^2} - \sqrt{6 - x^2}$.

16. Solve the equation $(4/x^2) - 3\sqrt{x} = 0$ to graphical accuracy by drawing a graph of the equation $y = (4/x^2) + (-3\sqrt{x})$.

3.10 Translation of Axes

As we are learning in this chapter, there are three fundamental and distinct methods that can be used to sketch a given curve. One of these three methods should be used whenever the name of the curve is not immediately evident so that special methods may be employed.

As an introduction to this article, the student is asked to study the following pairs of equations and to determine, before he reads further, which of the two he would prefer to sketch if he were given a choice.

1. $(x - 2)^2 + (y + 1)^2 = 16$, or $X^2 + Y^2 = 16$.
2. $(y + 3)^2 = (x - 1)/[4 + (x - 1)^2]$, or $Y^2 = X/(4 + X^2)$.
3. $y - 1 = 1.78(x - 2)^{0.577}$, or $Y = 1.78X^{0.577}$.

The answer, in each case, is the second equation, because the second equation possesses some symmetry or has some simple appearance that the first equation does not have. The purpose of this article is to explain how to simplify the problem of sketching such graphs as those of the first equations in these three cases. We shall make recurrent use of this idea in subsequent chapters.

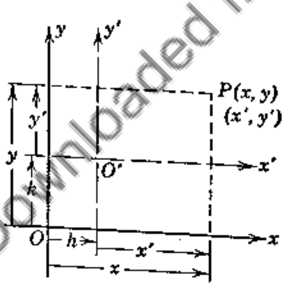


FIG. 3.13

In Fig. 3.13 there are shown two sets of axes labeled x, y and x', y'. An arbitrary point P has coordinates (x, y) with respect to the one set of axes and (x', y') with respect to the other set. We shall suppose that the coordinates of the origin in the primed set of axes, with respect to the other set, are (h, k). From the dimensioning on the figure we see that $\mathbf{x' = x - h}$ and $\mathbf{y' = y - k}$. Moreover, it is important that the coefficients of x and y are each the number *one*. Also the same scale must be used on the x'-axis that is used on the x-axis, and the same scale must be used on the y'-axis that is used on the y-axis.

EXAMPLE 1

Given $2x + y = 6$. Translate axes to a new origin at $(2, -1)$, determine the new equation, and make a sketch showing both sets of axes and the graph of the given equation.

Solution. The equations of translation are $y' = y - (-1) = y + 1$ and $x' = x - 2$. We substitute these in the given equation and obtain

$$2(x' + 2) + (y' - 1) = 6,$$

or $2x' + y' = 3$. The graph is shown in Fig. 3.14. One could check this particular problem by noting the intercepts on the primed axes and by computing them from the new equation.

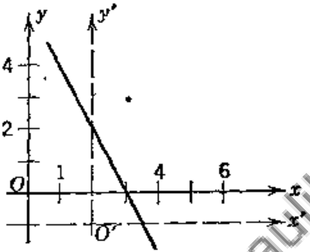


FIG. 3.14

EXAMPLE 2

Sketch $2y = 3 + 0.5(x - 1)^2$.

Solution. We rewrite the equation in the form $2(y - 1.5) = 0.5(x - 1)^2$. The form of this equation suggests the equations of translation: $x' = x - 1$ and $y' = y - 1.5$, which means that we are to translate axes so that the new origin is at $(1, 1.5)$. The new equation is $2y' = 0.5x'^2$, or $y' = 0.25x'^2$. This is a power-law curve and is symmetrical with respect to the y'-axis. The graph is shown in Fig. 3.15.

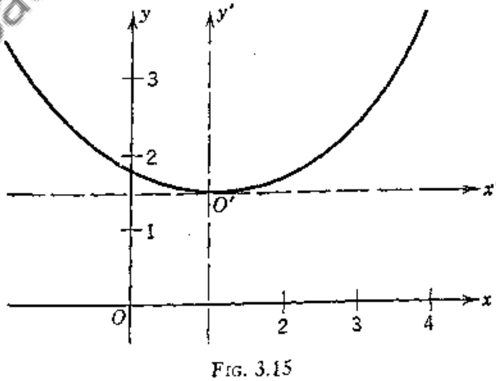


FIG. 3.15

EXAMPLE 3

Sketch $xy + 2x - 3y = 4$.

Solution. The easy solution is to rewrite the equation in the form

$$(x - 3)(y + 2) = 4 - 6 = -2,$$

translate axes so that the new origin is at $(3, -2)$, and sketch on the new axes the graph of $x'y' = -2$ (which is a power-law curve with negative exponent).

A second method of solution is to substitute from $x' = x - h$ and $y' = y - k$ into the given equation and obtain

$$(x' + h)(y' + k) + 2(x' + h) - 3(y' + k) = 4,$$

or

$$x'y' + x'(k + 2) + y'(h - 3) + (hk + 2h - 3k - 4) = 0.$$

This last equation would be simpler if the first-degree terms were missing. Hence we *choose* $h = 3$ and $k = -2$, and the equation becomes $x'y' + 2 = 0$ as in the first solution.

This second solution is longer than the first, but both methods are correct. To complete this example, the student should sketch a figure and show both sets of axes and the curve.

The second and third of these illustrative examples show that, if we wish to sketch a curve whose equation involves x everywhere as $x - h$ and y everywhere as $y - k$, then we can introduce the equations of translation $x' = x - h$ and $y' = y - k$, which implies translation to a new origin at (h, k); and finally we can sketch the graph of the new equation on the primed or auxiliary axes. Notice that, when the graph has been drawn by aid of these equations of translation, the primed axes have served their purpose and could even be erased.

PROBLEMS

1. Transform $2x + 3y = 6$ by translating axes so that the new origin is at $(-3, 4)$; sketch the straight line and both sets of axes.

2. Sketch $2(x - 1) + 5(y + 2) = 8$ by using the method of translation of axes.

3. Sketch the following curves by use of the method of translation of axes:

(*a*) $y = 1 + 2/x$.
(*b*) $(y - 1)^2 = 1/(x - 2)$.
(*c*) $y = 2 + (x - 1)^{0.75}$.
(*d*) $y = 2(x - 1)^3 + 4$.
(*e*) $(y - 1)(x + 2) = 7$.
(*f*) $R - 17.2 = 0.00463(T + 234)$.
(*g*) $s = 2 + 3(t - 1)^2$.
(*h*) $v = \sqrt{64h - 1280}$.
(*i*) $(y + 1)^2 = 4x^3$.
(*j*) $y - 2 = 2(x - 1)$.

4. By the proper translation of axes, remove the first-degree terms from the equation $xy - 2x - y + 8 = 0$. Then sketch both pairs of axes and the curve. Now solve the original equation for y in terms of x, and determine the equations

of the horizontal and vertical asymptotes. What is the relationship between the equations of the asymptotes and the coordinates of the new origin in the method of translation of axes?

5. What do the equations $2x + y = 1$ and $5x + 3y = 5$ become if axes are translated so that the new origin is at their point of intersection?

6. What do the following equations become if axes are translated so that the new origin is at the indicated point?

(a) $y = 4 \sin 2x$, $(\pi/4, 3)$.
(b) $y = 1 + 3 \cos 4(x - \frac{2}{3})$, $(\frac{2}{3}, 1)$.
(c) $y = 2 + 5 \log (x - 7)$, $(7, 2)$.
(d) $y = 3 \tan (2x - 8)$, $(4, 0)$.

7. Sketch the following by use of a combination of the methods of translation of axes and the discussion method:

(a) $y = 2 + (3x - 3)/(x^2 - 2x + 5)$.
(b) $y = 4 - 2/(x^2 + 2x + 2)$.
(c) $y = (2x^2 - 2x + 4)/(x^2 - 2x + 3)$. (*Hint:* first divide.)
(d) $r = (2t - 2)/(t^2 - 2t - 3)$.

8. Derive the equation of the locus of a point that moves so that its directed distance from the line $x = 1$ is always equal to the oblique distance from (3, 1) to the moving point. Then translate axes so that the new origin is at (2, 1), show that the resulting curve is a power-law curve, and sketch the curve and both sets of axes.

9S. Derive the equation of the locus of a point that moves so that its distance from (−1, 2) is always numerically equal to the slope of the line joining the point (−1, 2) to the moving point. Then sketch the curve.

3.11 Graphs of Polynomials in One Variable

We consider in this article the graph of y as a polynomial in x:

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where $a_0, a_1, \cdots, a_n$ are constants, and $a_n \neq 0$, $n > 1$.

Since the denominator of the right-hand member is 1 (and hence cannot be zero), there are no vertical asymptotes. As the numerical value of x increases, the numerical value of a_nx^n will become larger than all the other terms of the right-hand side combined, and hence the shape of the curve for large numerical values of x will be governed by the values of that term of highest degree. Since n is a positive integer larger than 1, we know from our knowledge of the graphs of the power-law curves that the shape of the curve for these polynomials and for large numerical values of x will be the shape of a power-law curve with exponent larger than 1. We conclude that there can be no horizontal asymptotes for these curves.

When the absolute value of x is small, the value of y will be governed by the value of $a_0 + a_1x$ if $a_1 \neq 0$, or of $a_0 + a_2x^2$ if $a_1 = 0$ and $a_2 \neq 0$, etc. Hence we can approximate the curve for small values of x by a straight line or by a translated power-law curve.

We can obtain further information about the shape of a curve for y a polynomial in x if we rewrite the polynomial in factored form. We shall illustrate with an example.

EXAMPLE

Sketch a graph of $y = 3x(x + 2)^3(x - 1)^2$.

Solution. The x-intercepts are $x = 0$, $x = -2$, and $x = 1$. The y-intercept is $y = 0$. If we were to expand the right-hand side, the term of highest degree would be $3x^6$, and we conclude that, when the numerical value of x is large, the required graph will have the shape of the graph of $y = 3x^6$. If we apply the method of excluded regions we see that $y > 0$ when $x < -2$; that $y < 0$ when $-2 < x < 0$; that $y > 0$ when $0 < x < 1$; and that $y > 0$ when $1 < x$.

If the value of x is near zero, the value of y will be near the value of $3(x)(2)^3(-1)^2$, or $24x$. Hence the required graph can be approximated for small values of x by the graph of the straight line $y = 24x$. If, on the other hand, the value of x is near the value $x = -2$, then the value of y will be approximated by the value of $y = 3(-2)(x + 2)^3(-2 - 1)^2 = -54(x + 2)^3$. Hence, when x is near $x = -2$, the required graph can be approximated by the graph of the translated power-law curve $y = -54(x + 2)^3$.

By the same sort of reasoning, we see that when x is near $x = 1$ the curve can be approximated by $y = 3(1)(3)^3(x - 1)^2$, or by $y = 81(x - 1)^2$, which is another power-law curve translated to $(1, 0)$ as the new origin. With this information we are in a position to sketch the shape of the curve. We need to plot additional points only if we wish a more precise graph for the curve. We show the sketch in Fig. 3.16.

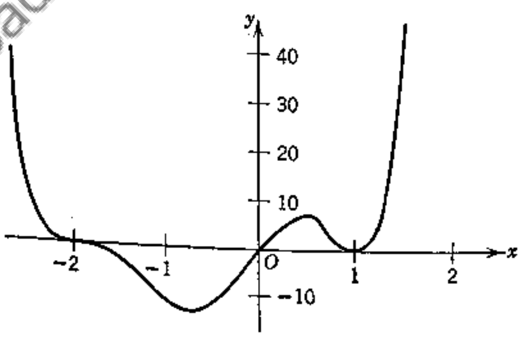


FIG. 3.16

The reasoning used in this example is more detailed than is necessary when we understand the use of power-law curves at the x-intercepts.

Thus, there are two steps required to sketch a polynomial-type curve:

I. Determine the shape of the curve as x increases through positive values and as x decreases through negative values (toward the left) by inspecting the sign of the coefficient and the exponent of the term of largest degree in x.

II. Determine the behavior at each x-intercept by inspecting the exponent of the factor which involves each such x-intercept. Thus, for example, if the factor is $(x - 4)^3$, we would know that the curve crosses the x-axis at $x = 4$ and is also tangent to the x-axis at that crossing point. Whether the curve crosses going upward or going downward would depend on the sign of the coefficient of this factor evaluated at $x = 4$, but this question could be answered otherwise by utilizing the information obtained in I in combination with the knowledge of the types of crossing at all the x-intercepts.

PROBLEMS

1. Sketch the graph of each of the following:

(a) $y = 2x(x - 2)^3$.
(b) $y = 0.5x(x - 1)(x + 1)$.
(c) $y = 2x^2(x + 1)^2(x - 1)^2$.
(d) $y = 2x^3(x - 1)^3(x + 1)^3$.
(e) $s = t^4(t - 2)^3$.
(f) $s = 0.25(t^3 - 1)^2$.

2. Plot a graph of $s = t^3 - 6t^2 + 9t - 4$ for $-2 < t < 6$. Obtain some information about the graph by rewriting the right-hand side in factored form. Also, make use of synthetic division to find coordinates of the additional points required to make an accurate and neat graph.

3. Plot each of the following by the method outlined in Problem 2:

(a) $y = x^3 + 2x^2 + x$.
(b) $y = x^5 + 3x^4 + 3x^3 + x^2$.
(c) $s = t^4 - 6t^3 + 11t^2 - 6t$.
(d) $s = t^6 - 200t^4 + 10{,}000t^2$.

4. Plot on the same sheet of graph paper the graphs of the following:

(a) $s = t^4 - 6t^2 - 7$. (b) $v = 4t^3 - 12t$. (c) $a = 12t^2 - 12$.

Show the graphs for $-3 < t < 3$; use 1 in. = 1 unit on the t-axis, 1 in. = 10 units on the s-axis, 1 in. = 20 units on the v-axis, and 1 in. = 40 units on the a-axis; let the s-, v-, and a-axes coincide.

5. Sketch $y = x(x - 1)^2$ by multiplication of ordinates, i.e., by sketching the two component curves $y_1 = x$ and $y_2 = (x - 1)^2$ and then multiplying ordinates.

6S. Examine the function $y = \dfrac{x^4(x - 1)^2(x + 3)^3}{(x - 2)^3(x + 2)^2}$ for its behavior near each x-intercept and near each vertical asymptote, and then sketch the complete graph.

3.12 Locus by Factoring

Rarely can a given algebraic equation whose graph is desired be factored into the product of two or more factors on one side and the very important number zero on the other side. When this factoring can be done, the graph can be obtained by aid of the following theorem:

THEOREM. *If a given equation $f(x, y) = 0$ can be written in the form $g(x, y) \cdot h(x, y) = 0$, then the graph of the given equation $f(x, y) = 0$ is the same as the combined graphs of $g(x, y) = 0$ and $h(x, y) = 0$ drawn on the same axes.*

The proof of this theorem is elementary, since the required graph consists of all the pairs of numbers (x, y) that satisfy the given equation and no other pairs of numbers. Any such pair of numbers that satisfies the given equation will necessarily satisfy one of the two derived equations; conversely, any pair of numbers which satisfies one of the derived equations will satisfy the given equation.

EXAMPLE

Sketch the graph of $x^2 - 4xy + 4y^2 = 9$.

Solution. This equation may be written in the form

$$(x - 2y)^2 - 9 = 0,$$

or

$$(x - 2y - 3)(x - 2y + 3) = 0.$$

Hence the graph of the given equation is the pair of straight lines whose equations are $x - 2y - 3 = 0$ and $x - 2y + 3 = 0$.

3.13 Review of the General Methods of Curve Sketching

The student is urged to use the following outline as a review of the methods of curve sketching. (There are some spaces in this outline that remain to be filled after study of material in subsequent chapters.) The student must know the general methods. In addition, he must later memorize certain types of curves and know how to sketch rapidly any given curve by aid of these types, or by a combination of the general methods, or by both.

GENERAL METHODS

I. DISCUSSION METHOD.

1. Intercepts.
2. Symmetry.
3. Asymptotes.
4. Excluded regions.

II. COMBINATION OF ORDINATES.

1. Addition of ordinates.
2. ....
3. ....

III. TRANSFORMATION OF VARIABLES.

1. Translation of axes.
2. ....
3. ....

IMPORTANT SPECIAL CURVES

I. STRAIGHT LINES.

II. POWER-LAW CURVES: $y = ax^n$.

1. $n > 1$.
2. $0 < n < 1$.
3. $n < 0$.

III. SECOND-DEGREE CURVES.

1. ....
2. ....
3. ....
4. ....

IV. TRANSCENDENTAL CURVES.

....

REVIEW QUESTIONS

1. How does one determine the x-intercepts for a given curve? Sketch a figure from which one may read the tests for symmetry with respect to the x-axis, the y-axis, and the origin. What are the two methods by which one may find the equations of the asymptotes for a given curve that are parallel to the x-axis? What are the critical numbers to be used in discussing the excluded regions for the curve whose equation is

$$y = \pm\sqrt{\frac{x-2}{x(x^2-9)}}?$$

2. Why is point plotting when used by itself an *unsatisfactory* method for curve sketching? How can point plotting be employed appropriately after the discussion method has been used?

3. What algebraic process should be used to solve $xy^2 + x = 2y$ for x in terms of y? What process should be used to solve for y in terms of x? Does this equation have as many as six different solutions, and how is the method of point plotting related to the idea of a solution?

4. Illustrate the three possible shapes in the first quadrant for power-law curves $y = ax^n$ if $a > 0$. What do these shapes become if $a < 0$, and in which quadrant would they lie? From what graph may a value of an expression such as $2(1.46)^{1/3}$ be estimated?

5. What are the equations of the two component curves if $2xy = 2x^2 + x + 4$ is to be drawn by the method of addition of ordinates? When should the method of addition of ordinates be used for sketching the locus of a given equation?

6. Illustrate the method of curve sketching by translation of axes with the locus of $y = 2 + 3(x-1)^2$. What are the coordinates of the point to which the origin should be translated for the equation $xy - 2x + 3y = 8$ in order that the

new equation will contain no first-degree terms, and what are the equations of translation?

7. What are four solutions of $xy = 2$? What are the two simultaneous solutions of $xy = 2$ and $x + y = 4$?

REVIEW PROBLEMS

1. (*a*) Determine the intercepts of $xy + 3y = 12$.

(*b*) Tell what symmetry exists:

$$(1)\quad x^2 - xy + 2y^2 = 4.$$

$$(2)\quad x^2y - xy^2 = 3.$$

$$(3)\quad x^2 + 2y = 4.$$

(*c*) Given $y = (x - 1)/(x - 4)$. For what range of values of x is y positive? negative? Answer by crosshatching the xy-plane.

(*d*) If $y = x \pm \sqrt{4 - x^2}$, for what range of values of x is y imaginary? Answer by crosshatching the xy-plane.

(*e*) Determine the equations of all horizontal and vertical asymptotes of the curve $y = 1/(x^2 - 4)$.

2. Sketch $y = 2x/(x^2 + 4) + 1/(x^2 + 1)$. Sketch each of the two component curves by the discussion method, and then add ordinates.

3. Sketch $y - 2 = \dfrac{8(x + 1)}{(x + 1)^2 + 4}$.

4. The coordinates of four points are given by $A(1, 5)$, $B(4, -3)$, $C(-7, 6)$, and $D(-3, -8)$. What are the new coordinates of these four points if axes are translated to a new origin at $(2, -1)$? Check from a figure.

5. Sketch $y = 3 + x^2 + 4/(x - 1)$.

6. Sketch $(x - 2)(y - 3) - (x - 2)^2 = 4$.

7. Sketch $(y + 2)^2 = (x + 3)/(x + 1)$ by translating axes to either $(-3, -2)$ or $(-1, -2)$.

8. Sketch $8x^2 - 8xy + 4y^2 + 4x - 12y + 9 = 0$. First show that

$$y = x + 1.5 \pm \sqrt{1 - (x - 1)^2}.$$

The graph of the second component curve, $y_2 = \sqrt{1 - (x - 1)^2}$, is the top half of a circle with center at $(1, 0)$ and radius 1.

9. Show, by solving for y in terms of x, that the equation

$$x^2 - 2xy + y^2 - 5x + 5y = 6$$

can be reduced to two linear equations; then sketch the graph.

10. Sketch $y + x^2y = 4$ and $y = 0.2x^2$ on the same graph. Show that these two curves intersect at $(2, 0.8)$ and $(-2, 0.8)$. Then determine the area of the

rectangle whose sides are parallel to the coordinate axes and that circumscribes the area between the two given curves.

11. Sketch $100y + x^2y = 100x$ and $y = 0.5x$ on the same graph. In how many points does the straight line intersect the curve?

12. Sketch $y = x^2/(x^2 + 4)$ by use of the discussion method and the idea that, when x is small, y behaves like $x^2/4$.

13. Sketch $25y + x^2y = 200x$. For what positive value of x is it true that for all larger positive values the value of y is less than 1% of the value of y at $x = 5$?

14. Derive the equation of the locus of a point that moves so that its distance from the point $(-1, 2)$ is always numerically equal to the slope of the line joining the moving point to this fixed point. Then sketch the locus.

15. The cross section of a loud-speaker horn is given by the equation $y^2 = (2.18x + 1)^4$, between $x = 0$ and $x = 1$. Plot the outline of the horn.

16S. Discuss and sketch the graph of $x/x_{st} = 1/(1 - \omega^2/\omega_n^2)$ using ω/ω_n as the independent variable and x/x_{st} as the dependent variable. This equation arises in a problem in vibrations; it is important to be able to find the positive values of ω for which "resonance" occurs, these being the values of ω that yield vertical asymptotes; what are they?

17S. Sketch each of the following curves, which have historical interest. It may be convenient to introduce new variables X and Y as ratios relating the given variables and the constants, as, for example, $X = x/a$ and $Y = y/b$. (These new variables are sometimes called dimensionless variables.)

(a) $(x^2 + y^2)^2 - a^2(x^2 - y^2) = 0$, the lemniscate.
(b) $(x^2 + y^2 + ax)^2 - a^2(x^2 + y^2) = 0$, the cardioid.
(c) $(x^2 + y^2 + 4ay - a^2)(x^2 - a^2) + 4a^2y^2 = 0$, the cocked hat.
(d) $x^2y^2 = (b^2 - y^2)(a + y)^2$, the conchoid.

18S. Let a variable line through $A(0, -a)$ intersect the x-axis at B; locate the two points P_1 and P_2 on the line and on opposite sides of the x-axis such that $\overline{P_2B} = \overline{PB}_1 = k =$ constant. Construct three different curves by use of a ruler for the three cases $k = a$, $k = a/2$, and $k = 2a$. Then find the equation of all such curves.

CHAPTER 4

Conics

In the preceding chapter, three different methods of sketching curves were presented—general methods that apply to all curves. In the present chapter we shall give the names and graphs of four important kinds of curves that will occur so frequently in scientific courses that they merit a special study. These four kinds of curves all have equations of the second degree. In this chapter, then, we shall study special cases of the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C cannot simultaneously be zero.

4.1 The Circle

The circle is the simplest of the four kinds of curves that are to be studied, and a mathematical definition for the circle is given first.

DEFINITION. *A circle is the locus of a point that moves in a plane so that its distance from a given fixed point (called the center) is a constant (called the radius).*

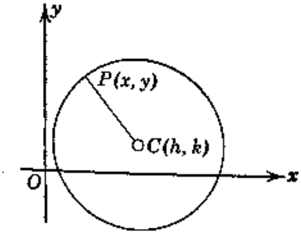


FIG. 4.1

Using the locus-derivation method, we derive the equation of a circle with center at (h, k) and radius r.

Step I. Sketch axes and label $C(h, k)$.
Step II. Select a general point $P(x, y)$ on the circle of radius r.
Step III. $\overline{CP} = r$, or, better, $\overline{CP}^2 = r^2$.
Step IV. $(x - h)^2 + (y - k)^2 = r^2$.

The other steps are superfluous.

We conclude that the general equation of a circle with center at (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

This general equation is easy to remember, since it asserts only that the square of the distance from the center of the circle to any point on the circumference is equal to the square of the radius. This form of the equation of a circle should be used whenever we wish to obtain it from data that yield almost immediately the coordinates of the center and the radius.

If the center of the circle is at the origin, this equation becomes

$$x^2 + y^2 = r^2.$$

This first general form for the equation of a circle may also be expressed in the following equally important and useful form:

$$x^2 + y^2 + ax + by + c = 0.$$

This second form should be used to determine the equation of a circle from data that do not yield the center and radius quickly.

That these two general forms are equivalent is a matter of elementary algebra, for, if we expand the first form, we obtain an equation of the second form. Conversely, if we complete the squares on x and y in the second form, we obtain the first form. Thus,

$$x^2 + ax + \frac{a^2}{4} + y^2 + by + \frac{b^2}{4} = -c + \frac{a^2}{4} + \frac{b^2}{4},$$

$$\left(x + \frac{1}{2}a\right)^2 + \left(y + \frac{1}{2}b\right)^2 = \frac{a^2 + b^2 - 4c}{4}.$$

This is a circle if $a^2 + b^2 - 4c > 0$; its radius will then be

$$\frac{1}{2}\sqrt{a^2 + b^2 - 4c}.$$

If $a^2 + b^2 - 4c = 0$, the locus consists of the single point

$(-a/2, -b/2)$, and is sometimes referred to as a point circle. If

$$a^2 + b^2 - 4c < 0,$$

the equation has no locus; that is, there are no pairs of numbers (x, y) that both are real numbers and satisfy the given equation.

When an equation is given in this second form and the graph is desired, we complete the squares on x and y as illustrated in the second example that follows.

EXAMPLE 1

Write the equation of the circle that has the ends of a diameter at (3, 1) and (−5, 7).

Solution. The center is at the mid-point of the diameter and has coordinates (−1, 4). The diameter, obtained by aid of the formula for the distance between two points, is $d = 10$, and hence the radius is $r = 5$. The required equation is then

$$(x + 1)^2 + (y - 4)^2 = 25,$$

or

$$x^2 + y^2 + 2x - 8y = 8.$$

The student should, as a matter of habit, check to be certain that the circle does go through the two given points.

EXAMPLE 2

Identify and sketch $2x^2 + 2y^2 - 3x + 4y = 1$.

Solution. The graph will be a circle or will have no locus, since, dividing by 2, we obtain

$$x^2 + y^2 - 1.5x + 2y = 0.5,$$

which is in the second form for the equation of the circle. To determine the center and radius, we complete the squares on x and y as follows:

$$(x^2 - 1.5x + 0.75^2) + (y^2 + 2y + 1) = 0.5 + 0.5625 + 1,$$

or

$$(x - 0.75)^2 + (y + 1)^2 = 2.0625.$$

Hence the center is at (0.75, −1) and the radius is $r = \sqrt{2.0625} \approx 1.44$. The student should complete the problem by drawing the circle.

EXAMPLE 3

Find the equation of the circle that goes through (1, 3), (4, 2), and (−3, −5).

Solution. The coordinates of the center are not readily obtainable; hence we start with the second form of the equation of the circle, require the circle to go through the three given points, and thus obtain from $x^2 + y^2 + ax + by + c = 0$:

$$a + 3b + c = -10,$$
$$4a + 2b + c = -20,$$
$$-3a - 5b + c = -34.$$

The student should solve these equations simultaneously by the method of addition-subtraction (see the illustrative examples in Art. 2.1) and show that the final result is $x^2 + y^2 - 2x + 4y = 20$. *As a habit*, the student should show that this equation is satisfied successively by the three given pairs of numbers, that is, that this resulting circle really does go through the three given points.

PROBLEMS

1. Determine the center and the radius and draw the circle:

(*a*) $x^2 + y^2 + 6x - 4y = 3$.
(*b*) $x^2 + y^2 - 6x + 4y = 12$.
(*c*) $4x^2 + 4y^2 - 4x + 2y + 1 = 0$.
(*d*) $x^2 + y^2 = 4x - 2y + 5$.
(*e*) $x^2 + y^2 - 2.54x - 3.28y = 1.72$ (use slide rule).
(*f*) $s^2 + t^2 - 74.8s - 56.4t = 7450$.
(*g*) $p^2 + v^2 - 20p - 50v = 864$.
(*h*) $s^2 + t^2 = 40(s + t) + 45$.
(*i*) $(s + t)^2 + (s - t)^2 + 5(s + t) + 3(s - t) = 10$.

2. Determine the equation of the circle:

(*a*) With center at $(2, -3)$ and radius 5.
(*b*) With ends of a diameter at $(5, 6)$ and $(-5, 6)$.
(*c*) With center at $(3, 4)$, tangent to $y = 7$.
(*d*) With center at $(3, 4)$, tangent to $x + 2y = 20$.
(*e*) Passing through the four points $(3, 0)$, $(1, 0)$, $(1, 4)$, and $(3, 4)$.
(*f*) With center at $(0, 3)$, tangent to the x-axis.
(*g*) With center at $(-1, -3)$, the circle going through $(1, 2)$.
(*h*) With center at $(2, -3)$, tangent to $3x + 4y = 20$.
(*i*) Circumscribing the equilateral triangle with two vertices at $(0, 0)$ and $(6, 0)$ and the third vertex in the first quadrant.

3. Find the equation of the circle that goes through the three points:

(*a*) $(5, 4)$, $(3, 2)$, and $(-3, 0)$.
(*b*) $(4, 0)$, $(0, 3)$, and $(0, 0)$.
(*c*) $(0, 2)$, $(0, 7)$, and $(3, 1)$.
(*d*) $(6, -6)$, $(-1, -5)$, and $(7, -5)$.
(*e*) $(1.50, 3.16)$, $(6.50, 3.95)$, and $(2.51, -1.02)$.
(*f*) $(-2, 1)$, $(0, 1)$, and $(-2, -4)$.

4. Use the second general form for a circle and try to find the equation of the circle that goes through $(1, 5)$, $(2, 7)$, and $(-1, 1)$. Why do you get into trouble in solving the three simultaneous equations?

5. Derive the equation of the locus of a point that moves so that it is always twice as far from $(4, 5)$ as from $(-2, 3)$. Then identify and draw the locus.

6. Determine by locus-derivation methods the equation of the circle that has the ends of a diameter at $(4, 1)$ and $(-5, -5)$. Use the fact that a triangle is a right triangle if it is inscribed in a circle and has a diameter for one side.

7. Find the equation of the circle that circumscribes the triangle with vertices at the given points:

(*a*) $(0, 0)$, $(0, -4)$, $(3, 0)$.
(*b*) $(1, -2)$, $(5, -4)$, $(4, -3)$.
(*c*) $(2, 4)$, $(0, 2)$, $(4, -2)$.
(*d*) $(1, -2)$, $(2, 1)$, $(4, -1)$.

8. Find the equation of the circle that goes through the two given points and whose center is on the given line:

(a) $(4, 2), (-3, 1), y + 2x = 0$.
(b) $(2, 1), (0, 5), x + 2y = 3$.
(c) $(1, 4), (3, 0), 2x + y = 6$. (What is peculiar about Part (c)?)

9. The center of a circle is on $x + 2y = 5$, the abscissa of the center is 7, and the radius of the circle is 3; what is the equation of the circle?

10. (a) Determine the area between $y = \sqrt{16 - x^2}$ and the x-axis. Why is the graph of the locus entirely above the x-axis?

(b) Determine the area enclosed by $x^2 + y^2 = 6y$.

(c) Determine that area that is outside $x^2 + y^2 + 21 = 6x + 8y$ and inside $x^2 + y^2 - 10x - 12y + 25 = 0$.

11. A point moves so that its abscissa is always 3 less than the square of its distance from the point $(-3, 2)$. Identify and draw the locus.

12. A thin strip of metal is bent without being stretched so that its ends are at $(0, 0)$ and $(6, 0)$, and so that its lowest point is at $(3, -3)$. If the curve is an arc of a circle, what was the original length of the strip? What is the equation of the semicircle?

13. (a) Show that the graph of $x^2 + y^2 - 2x + 6y + 10 = 0$ is a point-circle, i.e., a circle with zero radius.

(b) Show that there is no graph of $x^2 + y^2 - 4x + 2y + 14 = 0$.

14. Identify and draw the graph of $I_1^2 + I_2^2 = (E/R_1)I_1 + (E/R_2)I_2$, if I_1 is the independent variable, and $E = 100$ volts, $R_1 = 2$ ohms, and $R_2 = 10$ ohms.

15. A wheel rests on the level ground, touches a vertical wall, and just touches the upper corner of a 2-in. wide and 4-in. high projection at the foot of the wall. What is the radius of the wheel?

16S. Draw the circle that has its center at the point $C(2, 0)$ and a radius $r = 2$ (so that the circle goes through the origin O). Then draw the line $x = 1$. Next use the markings on a ruler, to locate a large number of points, and draw the locus of the following rule: On each chord (or chord extended) $\overline{OB}$ of the circle, locate

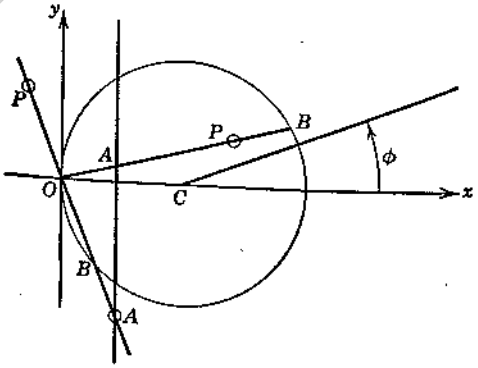


FIG. 4.2

the point of intersection A with the line $x = 1$, and then locate a point P so that $\overline{OP} = \overline{AB}$ (where both $\overline{OP}$ and $\overline{AB}$ are directed lengths).

(a) Find the equation of the locus of P.

(b) Lay off an arbitrary angle ϕ with vertex at C as shown in Fig. 4.2, and let one side of this angle intersect the locus of P at P'. Show that the inclination of OP' is one-third the inclination of CP' and hence that one can trisect an arbitrary angle by aid of this curve—which is called the trisectrix of Maclaurin.*

17S. The linkage shown in Fig. 4.3 consists of six rigid rods: $\overline{BC} = \overline{BE} = a$, and $\overline{FE} = \overline{ED} = \overline{DC} = \overline{CF} = b$. Point B is fixed on the circle whose center is

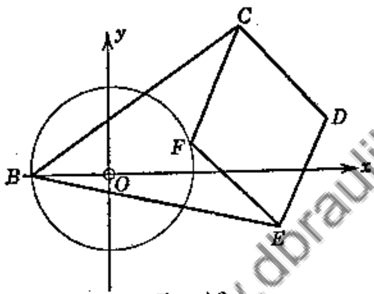


FIG. 4.3

at O. The linkage is pinned at B, C, D, E, and F. As F moves along an arc of the circle, find the locus of the point D. This is Peaucellier's linkage, invented in 1864. *Hint:* First show that $\overline{BF} \cdot \overline{BD} = \text{Constant} = a^2 - b^2$.

18S. Draw a circle with center at the origin and radius r. Locate the points $A(0, -r)$ and $B(6r, -r)$. Next locate the point E on the y-axis, whose ordinate is the same as the ordinate of the point F in the first quadrant, which is the intersection of the given circle and a second circle with center at $C(r, 0)$ and radius r. What is the error in using the length of BE in place of the circumference of the given circle? Note that this is a convenient graphical method by which we can approximate the circumference of a given circle

4.2 Families of Circles

In this article we shall study the geometrical interpretation of the equation of a circle that also involves one arbitrary parameter. We observe, by way of an elementary example, that the equation

* Angles may be trisected by aid of several different curves, such as the conchoid $x^2y^2 = (a + y)^2(b^2 - y^2)$, or the limaçon $(x^2 + y^2 - 2x)^2 = x^2 + y^2$ (which will be treated later, in the chapter on polar coordinates). All these curves can be drawn by aid of a marked ruler and compasses. The trisection problem, which was started by the Greeks, is to trisect an angle by aid of an *unmarked* straightedge and compasses, and this problem can be shown by methods of advanced mathematics to be impossible of solution.

$x^2 + y^2 = r^2$ could be interpreted as the equation of the family of circles with centers at the origin. Again, $x^2 + y^2 - ax - ay = 0$ is the equation of the family of circles that have centers on the line $y = x$ and that all go through the origin.

The equation $x^2 + y^2 - 2ay = 1$ (see Fig. 4.4) can be given the following physical interpretation in addition to being described as a

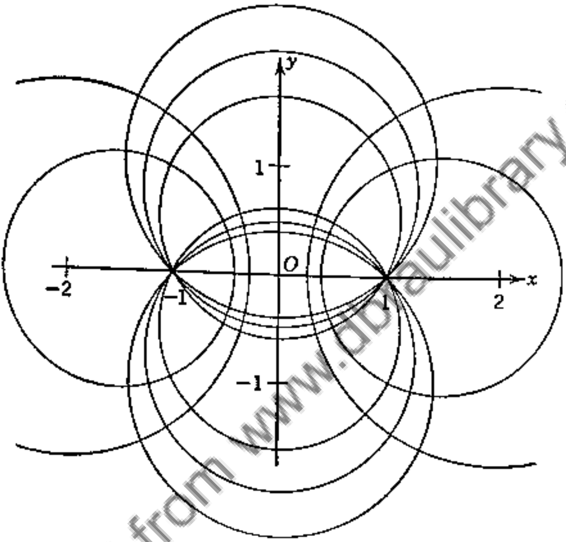


FIG. 4.4

family of circles whose centers are on the y-axis and that all go through the two points (1, 0) and (−1, 0). Imagine a large horizontal sheet of metal with an upturned faucet placed at (1, 0) and a sink set at (−1, 0). When the water flows from (1, 0), the individual particles of water will flow along arcs of this family of circles until these particles flow down the sink at (−1, 0). The related family of circles $x^2+y^2+1-2bx=0$ has centers at $(b, 0)$ and radii given by $r = \sqrt{b^2 - 1}$. This second family of circles intersects each member of the first family at right angles and has the physical interpretation that all the particles of water moving along the arcs of the first family of circles have the same speed as they cross one member of the second family.*

* There are other physical interpretations for these two families of circles taken together, as, for example, the electric and magnetic fields with a north pole at (1, 0) and a south pole at (−1, 0).

We consider next what geometric interpretation we can give to an equation obtained by adding to the equation of one circle the result of multiplying the equation of a second circle by a parameter. Suppose that the two given equations are

$$x^2 + y^2 + 4x - 2y = 5$$

and

$$x^2 + y^2 - 12x - 10y = -11.$$

If we combine these two equations as indicated, we obtain

$$x^2 + y^2 + 4x - 2y - 5 + k(x^2 + y^2 - 12x - 10y + 11) = 0.$$

We may expand and collect to obtain

$$(1 + k)x^2 + (1 + k)y^2 + (4 - 12k)x - (2 + 10k)y + (11k - 5) = 0.$$

For all values of k except $k = -1$, this is a circle, and when $k = -1$ the graph is clearly a straight line. All the circles and the straight line

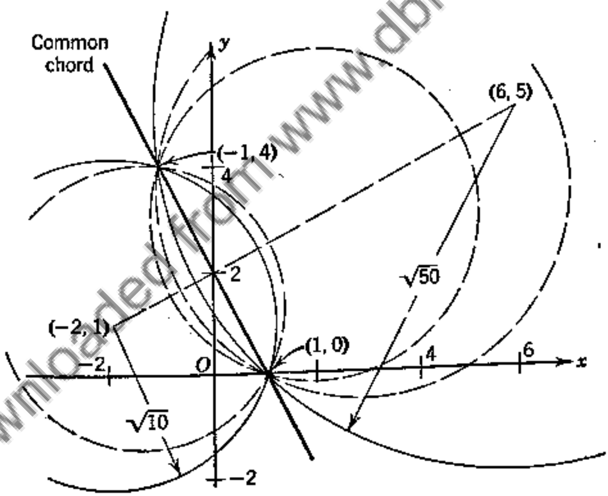


FIG. 4.5

necessarily go through the two points of intersection of the two circles because of the manner in which we formed this last equation. In Fig. 4.5 we show the two given circles, the straight line, and a few members of the family of circles.

We observe that the straight line is the *common chord* for the two circles and that the equation of that common chord is obtained by choosing the particular value of k that eliminates the terms in x^2 and y^2 from the equation of the family of curves.

DEFINITIONS. *If the equations of two circles*

$$x^2 + y^2 + a_1x + b_1y + c = 0$$

and

$$x^2 + y^2 + a_2x + b_2y + c_2 = 0$$

are combined to yield

$$x^2 + y^2 + a_1x + b_1y + c_1 + k(x^2 + y^2 + a_2x + b_2y + c_2) = 0,$$

the graph of the particular member of this family for $k = -1$ *is a straight line, called the common chord if the two circles intersect and the radical axis otherwise.* Clearly the straight line will be perpendicular to the line joining the centers of the two given circles.

EXAMPLE 1

Find the equation of the radical axis for the two circles $2x^2 + 2y^2 + 2y - 5 = 0$ and $x^2 + y^2 - 10x + 20 = 0$.

Solution. The student should show that these two circles do not intersect, by finding the coordinates of their centers, the distance between the centers, and the sum of the two radii. We form the equation

$$2x^2 + 2y^2 + 2y - 5 + k(x^2 + y^2 - 10x + 20) = 0.$$

The graphs for various values of k are circles except for $k = -2$; in this case we obtain $20x + 2y - 45 = 0$, and this is the equation of the radical axis.

EXAMPLE 2

Find the equation of the circle that goes through $(-1, 1)$ and the two points of intersection of the circles

$$x^2 + y^2 - 8x + 2y + 8 = 0 \quad \text{and} \quad x^2 + y^2 - 2x + 4y + 1 = 0.$$

Solution. The long solution would be to solve simultaneously the two equations for the circles, and then to find the equation of the circle that goes through those two points of intersection and the given point. The easy solution is first to form the equation of the family of circles that go through the two points of intersection:

$$x^2 + y^2 - 8x + 2y + 8 + k(x^2 + y^2 - 2x + 4y + 1) = 0,$$

and then to find the particular member of this family that goes through $(-1, 1)$. The student should go through the requisite substitution and algebra to show that $k = -{}^{20}\!/_{9}$ and that the required equation is

$$11x^2 + 11y^2 + 32x + 62y - 52 = 0.$$

PROBLEMS

1. Determine the equation of the family of circles:

(*a*) That have their centers at $(2, -3)$.
(*b*) That have their centers on the line $y = 2x$ and are tangent to the y-axis.
(*c*) That have their centers on the x-axis and all go through $(0, 2)$.
(*d*) That have their centers on the y-axis and all go through the origin.
(*e*) That have their centers on the line $y = 2x$ and are tangent to the line $3x + 4y = 10$.

2. Find the equation of the circle that goes through the points of intersection of $x^2 + y^2 + 6x + 2y = 27$ and $x^2 + y^2 - 10x - 2y + 13 = 0$, and that satisfies the following conditions:

(*a*) The circle goes through the origin.
(*b*) The circle goes through $(5, -1)$.
(*c*) The circle has the x-coordinate of its center at $x = 13$.
(*d*) The circle has its center at the mid-point of the line segment joining the centers of the two given circles.
(*e*) The circle goes through $(0, 3)$.

3. (*a*) Show that the two circles $x^2 + y^2 + 6x - 6y + 2 = 0$ and

$$x^2 + y^2 - 4x + 4y + 2 = 0$$

do not intersect in points with real coordinates. Find the simultaneous solutions.
(*b*) Find the equation of the radical axis.
(*c*) Find the equation of the circle that goes through $(1, 0)$ and through the two points of intersection (with imaginary coordinates) of the two given circles.

4. Find the equation of the radical axis or common chord for each of the following, and state which name is correct for your result:

(*a*) $x^2 + y^2 + 2x + 4y = 8$ and $x^2 + y^2 - 6x - 6y + 10 = 0$.
(*b*) $x^2 + y^2 - 4x - 6y + 5 = 0$ and $x^2 + y^2 + 8x + 10y = 11$.
(*c*) $x^2 + y^2 + 2x + 4y = 1$ and $x^2 + y^2 - 6x - 8y + 20 = 0$.
(*d*) $x^2 + y^2 + 2x - 2y = 0$ and $2x^2 + 2y^2 - 4x - 8y + 9 = 0$.
(*e*) $2x^2 + 2y^2 + 3x + 4y - 5 = 0$ and $3x^2 + 3y^2 - 6x - 11y + 3 = 0$.
(*f*) $4x^2 + 4y^2 - 4x + 4y = 3$ and $2x^2 + 2y^2 + 0.5x - 4y = 0.75$.

5. Find the shortest distance from the circle $x^2 + y^2 + 8x = 6y$ to the circle $x^2 + y^2 - 16x + 4y + 43 = 0$.

6. Find the length of the common chord of the two circles

$$x^2 + y^2 - 2x - 4y = 20 \quad \text{and} \quad x^2 + y^2 + 2x = 52.$$

7. Find the equation of the common chord or radical axis for the two circles $x^2 + y^2 + 5 = 4x + 6y$ and $x^2 + y^2 + 8y = 2x + 1$. Then find the radii for the two circles and the distances from the centers to the common chord or radical axis. What conclusion can you draw about this straight line?

8S. Determine the radius of the circle shown in Fig. 4.6.

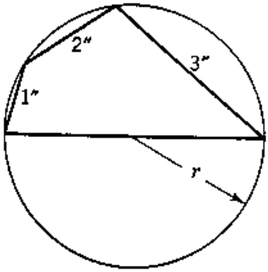


FIG. 4.6

9S. A circle has its center at (h, k) and radius r. From a point $P(x_1, y_1)$, which is exterior to the circle, two tangent lines are drawn to the circle. Prove that the lengths of these tangent lines are given by

$$L^2 = (x_1 - h)^2 + (y_1 - k)^2 - r^2.$$

Then prove the corollary, that, if the equation of the circle is

$$x^2 + y^2 + ax + by + c = 0,$$

then the length of the tangents from the exterior point is given by

$$L^2 = x_1^2 + y_1^2 + ax_1 + by_1 + c.$$

4.3 The Parabola

We shall be concerned in this article with a study of curves whose equations may be reduced to the form:

$$(\text{One variable})^2 = (\text{Some constant}) \text{ times } (\text{Other variable}).$$

DEFINITION. *A parabola is the locus of a point that moves in a plane so that its distance from a fixed line is always equal to its distance from a fixed point not on the line.*

In order to derive the equation of this locus we shall use the locus-derivation method. To introduce as much symmetry as possible in the final equation we shall take the fixed line (called the *directrix*) to be $x = -p/2$ and the fixed point (called the *focus*) to be at $(p/2, 0)$, so that the distance from the directrix to the focus is p.

Step I. Sketch axes and label the given data.
Step II. Select a general point $P(x, y)$ as indicated in Fig. 4.7.

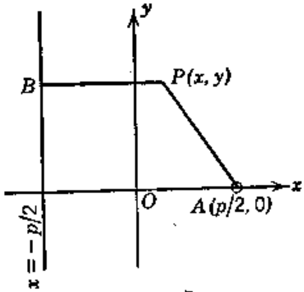


FIG. 4.7

Step III. $\overline{BP} = \overline{AP}$ (where $\overline{AP}$ is positive by definition and $\overline{BP}$ is chosen instead of $\overline{PB}$, so that we are equating two positive numbers).

Step IV. $x + p/2 = \sqrt{(x - p/2)^2 + y^2}$.

Step V. Square both sides, collect, and obtain $y^2 = 2px$.

Step VI. The student should check by aid of any of the check points $(0, 0)$, $(p/2, p)$, and $(p/2, -p)$.

If we had taken the fixed line to be $y = -p/2$ and the fixed point to be at $(0, p/2)$, the resulting equation would have been $x^2 = 2py$. The equations $y^2 = -2px$ and $x^2 = -2py$ are also equations of parabolas. Therefore the general form of the equation of a parabola that is symmetrical with respect to one of the two axes and that goes through the origin is

$$(\text{One variable})^2 = (\text{Some constant}) \text{ times } (\text{Other variable}).$$

We observe that (in the derivation) p is the directed distance from the fixed line (directrix) to the fixed point (focus), and enters into the general equation in the form: $2p$ times the variable of first degree.

We next apply the discussion method to sketch the graph of the equation $y^2 = 2px$, assuming p to be a positive constant.

I. Intercepts. $x = 0$ and $y = 0$.

II. Symmetry. x-axis.

III. Asymptotes. Since $y = \pm\sqrt{2px}$ and $x = y^2/(2p)$, and, since there is no variable term in either denominator, there are no horizontal or vertical asymptotes.

IV. Excluded regions. Both the equations in Step III show that for all values of y the value of x is never negative.

The graph is shown in Fig. 4.8. The student should notice that the curve is an open curve. Moreover, $y = \pm\sqrt{2p}\, x^{1/2}$ are power-law curves. The related curve $x^2 = 2py$, or $y = x^2/(2p)$ is likewise a power-law curve.

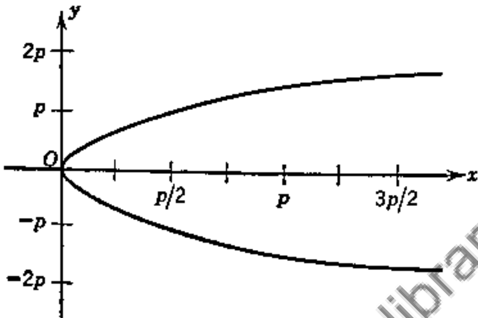


FIG. 4.8

DEFINITIONS. *The line of symmetry for the parabola is called the axis of the parabola; the intersection of this axis with the parabola is called the vertex;* the directrix and focus have already been defined.

For most problems in science and engineering it is unnecessary to associate any geometrical significance with the coefficient $2p$, so that the equation of the parabola may be remembered in the form

$$y^2 = kx (\text{or } x^2 = ky).$$

EXAMPLE 1

Identify and sketch the curve $x^2 - 2x - 3y = 5$.

First Solution. The presence of terms in x of degrees one and two indicates that we may complete the square. We do this and obtain $(x-1)^2 = 3(y+2)$. The form of this equation suggests that we employ translation of axes with $x' = x - 1$ and $y' = y + 2$, so that we are translating axes to a new origin at $(1, -2)$. The new equation is $x'^2 = 3y'$, and we identify this as a parabola. We observe that this last equation indicates symmetry with respect to the y'-axis, and that, when y' is negative, x' is imaginary (or that y' is positive for all values of x'). Hence the parabola opens *upward* around the y'-axis. We compute x' when y' is some convenient number, say $y' = 3$, plot the two points thus obtained, and finally sketch the curve.

Second Solution. We first identify the curve as a parabola, since we could reduce the equation to the standard form for a parabola by completing the square and translating axes. We solve for the first-degree variable in terms of the other variable, i.e., in this example for y in terms of x, and obtain $3y = x^2 - 2x - 5$. If we were to complete the square and translate axes, we would find the curve to be symmetrical with respect to the y'-axis. We determine the x-intercepts to be $x = 1 \pm \sqrt{6}$ or 3.45 and -1.45. The axis of the parabola is therefore parallel to the y-axis and bisects the line segment joining the points corresponding to these

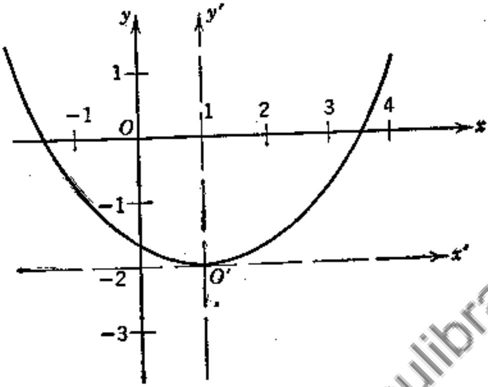


FIG. 4.9

two intercepts; hence the axis is $x = 1$. When $x = 1$, $y = -2$; these are the coordinates of the vertex. The graph may then be sketched by aid of the vertex and the two intercepts.

EXAMPLE 2

Choose axes and determine the equation for the parabolic gate (in a dam) shown in Fig. 4.10.

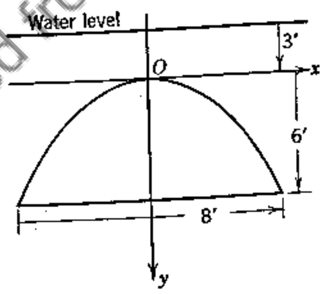


FIG. 4.10

Solution. We choose the vertex to be at the origin, the y-axis to be the axis of the parabola, and the positive direction to be downward, so that the numbers entering into the solution will be positive.

Since the curve is symmetrical with respect to the y-axis, the equation will involve the square of x, and so we use $x^2 = ky$. The curve goes through (4, 6), hence $16 = 6k$ or $k = 8/3$; the required equation is $3x^2 = 8y$.

EXAMPLE 3

A headlight is frequently designed so that a cross section through its axis is a parabola. The filament of the bulb is placed at the focus of the parabola in order that all rays of light coming from the filament shall be reflected by the parabolic reflector as rays parallel to the axis of the parabola. A particular headlight is to be 8 in. in diameter and 3 in. in depth; the bulb has a spherical globe of diameter 0.800 in. Determine the length of the fitting to house the shank of the bulb, the required length to be measured from the reflector to the bulb.

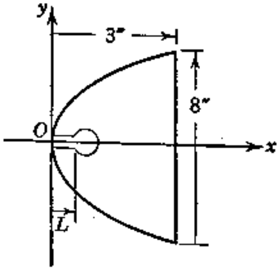


FIG. 4.11

Solution. Axes are chosen as shown in Fig. 4.11, and the equation of the parabola is found to be $y^2 = 5.333x$. Hence the focal distance is one-fourth of the coefficient 5.333, or 1.333 in.; and the required distance is $L = 1.333 - 0.400 = 0.933$ in.

PROBLEMS

1. Sketch rapidly:

(*a*) $y^2 = -5x$.
(*b*) $2x^2 + 3y = 0$.
(*c*) $s^2 + 4t = 0$.
(*d*) $y = 2x^2$.
(*e*) $y^2 = x/4$.
(*f*) $s = 0.46t^2$.

(*g*) $s = gt^2/2$ where g is approximately 32.2 ft./sec.^{2}

(*h*) $V = \sqrt{2gh} = \sqrt{64.4h}$. Show only positive values for V. Why?

2. Sketch by translating axes:

(*a*) $(y - 2) = 4(x + 1)^2$.
(*b*) $y^2 + 16y + x = 4$.
(*c*) $2x^2 + 3x + 4y = 5$.
(*d*) $y = 4x - x^2$.
(*e*) $y + x^2 = 6x - 5$.
(*f*) $x^2 + 3.78y = 7.88x + 9.68$.
(*g*) $y = 1 + \sqrt{x - 1}$.
(*h*) $y = 2 - \sqrt{x - 3}$.
(*i*) $y^2 + 2 = 2x + 4y$.
(*j*) $x^2 + 3y + 4 = 2x$.
(*k*) $2s^2 = 3t + 4s$.
(*l*) $s = 2t^2 - 2t + 2$.

3. Sketch the curves of Problem 2 without translating axes; i.e., find the intercepts and vertex.

4. Determine the equations of the parabolas with vertex at the origin, with axis the x-axis, and

(*a*) Through (4, 5).
(*b*) Through (−4, 5).
(*c*) Through (−0.5, −0.75).
(*d*) Through (14.8, 27.6).
(*e*) With focus at (3, 0).
(*f*) With focus at (−2, 0).
(*g*) With directrix $x = 3$.
(*h*) With directrix $x = -2$.

5. Water is spouting from the end of a horizontal pipe placed 30 ft. above the ground. Ten feet below the line of the pipe, the stream of water has curved outward 12 ft. beyond a vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground? The assumption that the curve (that the water describes) is a parabola is proved in fluid mechanics.

6. Determine the equation of the locus of a point that moves so that its distance from the line $y = -3$ is always numerically equal to its distance from the point (6, 3). Identify and sketch the locus.

7. s varies directly as the square of time t, and, when $t = 1.5$ sec., $s = 36$ ft. Determine the equation for s in terms of t, identify, and sketch.

8. Find the equation of the parabola whose axis is parallel to the y-axis (why is the equation then necessarily of the form $y = ax^2 + bx + c$?) and that goes through

(*a*) $(-1, 0)$, $(1, 4)$, and $(2, 3)$.
(*b*) $(1, 3)$, $(2, 7)$, and $(3, 13)$.
(*c*) $(0, 4)$, $(2, -2)$, and $(6, -38)$.
(*d*) $(1, -1)$, $(2, 1)$, and $(3, 5)$.
(*e*) $(0, 3)$, $(1, 5)$, and $(2, 7)$. Why does this last set of three points lead to difficulty?

9. Crosshatch the area between $y = 4 - x^2$ and $x + y + 2 = 0$, and give a rough estimate for that area. In order to make such an estimate, circumscribe a rectangle about the required area such that the sides of the rectangle are parallel to the coordinate axes, and compute the area of this rectangle.

10. Sketch, give the coordinates of the vertex and focus, and give the equations of the axis and directrix in each of the following:

(*a*) $y^2 - 6x + 4y + 22 = 0$.
(*b*) $4x^2 + 4x + 16y = 47$.
(*c*) $y^2 + 8x + 9 = 2y$.
(*d*) $x^2 + 6 = 2x + 5y$.

11. Given the parabola $y^2 = 2px$. Determine the coordinates of the points on this curve that are directly above and below the focus. What is the length of the straight-line segment joining these two points (the focus bisects this segment)? This particular line segment is called the latus rectum of the parabola.

12. (*a*) Find the length of the chord (straight-line segment) of the parabola $y^2 = 6x$ that joins the points of intersection of the parabola and the line $x = 2$.

(*b*) Sketch $y^2 = 4 + x$ and $x + y = 2$ on the same pair of axes; find the length of the chord or straight-line segment that joins the two points of intersection.

13. A roadway 40 ft. wide is 1 ft. lower at the sides than in the middle. Determine the distance from a horizontal line that just touches the top of the roadway at the middle, to the curve of the roadway at a distance of 10 ft. from either side of the road, if the crown of the roadway is

(*a*) An arc of a parabola. (*b*) An arc of a circle. (*c*) An isosceles triangle.

14. Find the equation of the parabola whose axis is parallel to the x-axis (why is the equation necessarily of the form $x = Ay^2 + By + C$?) and that goes through

(*a*) $(0, 1)$, $(2, 2)$, and $(8, -1)$. (*b*) $(3, 1)$, $(7, 2)$, and $(13, 3)$. (*c*) $(0, 1)$, $(3, 2)$, and $(7, 4)$.

15. When a gun is fired so that the shell has an initial speed of v ft./sec., and makes an angle of θ with the horizontal, a first approximation for the trajectory or curve described by the shell is given by

$$y = -\frac{g \sec^2 \theta}{2v^2} x^2 + x \tan \theta,$$

where $g = 32.2$ ft./sec.^{2} approximately.

(*a*) Sketch the trajectory if $\theta = 45°$ and $v = 1000$ ft./sec.

(*b*) Determine the range of the shell, that is, the value of x other than $x = 0$ when $y = 0$.

(*c*) Determine the maximum height reached by the shell; i.e., find the coordinates of the vertex. The easy solution for this problem is to make use of your results in (*b*).

(*d*) Determine the equation of the directrix and observe that this equation is independent of the angle θ. Hence the ordinate to the directrix is the distance the shell would rise if fired vertically upward.

16. A culvert is to be built as shown in Fig. 4.12. A form must be made, steel reinforcements inserted, and concrete poured. The curve of the inside is an arc of a parabola.

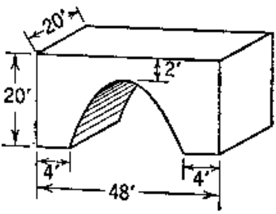


FIG. 4.12

(*a*) Determine the heights to the curved part at 5-ft. intervals across the culvert opening.

(*b*) Given that the cross-sectional area of the parabolic area (such as in this problem) is equal to two-thirds the area of the circumscribing rectangle. Determine the number of cubic yards of concrete that should be ordered for the culvert (neglect shrinkage of the concrete).

17. Two telephone poles are 200 ft. apart, and a wire is fastened to each pole 50 ft. above the level ground. Halfway between the poles the wire is 40 ft. above the ground. Find the height of the wire 50 ft. from either pole. A correct first approximation for the shape or curve of the wire is a parabola, and this assumption will give an accuracy of three significant figures in this problem.

18. Prove that the abscissa of the vertex of the parabola $y = ax^2 + bx + c$ is $x = -b/(2a)$.

19. Use the results of the theorem in Problem 18 to determine the coordinates of the vertex for each of the following parabolas:

(*a*) $y = x^2 - 2x - 3$.
(*b*) $y = 2x^2 - 6x - 5$.
(*c*) $2y = x^2 + 3x + 1$.
(*d*) $x = y^2 - 4y - 1$.

20S. Sketch a graph of $y^2 = 3x$ according to the following directions:

(*a*) Draw $Y = 3x$ for $0 < x < 4$.

(*b*) Let $Y = y^2$ or $y = \sqrt{Y}$. Put new scale numbers along the Y-axis according to the following scheme: Since $y = 2$ when $Y = 4$, place the number $y = 2$ at the point on the Y-axis corresponding to $Y = 4$. Repeat this for $Y = 0.25$, $Y = 1$, $Y = 2.25$, $Y = 4$, and so on. The resulting graph can be used as a graph of the given equation. It is a distorted graph since one scale is non-uniform.

21S. A telephone company agrees to install telephones in a rural community according to the following agreement: If there are 500 or fewer subscribers, the installation charge will be \$4.00 for each telephone; if there are more than 500 subscribers, the company will deduct ½ cent for each subscriber in excess of 500 from the charge on every telephone. Plot a graph of the total revenue in terms of the number of subscribers. What number of subscribers will give the largest revenue? Should the company limit the total number of subscribers? Why?

22S. The two curves $y = x^2 + 1$ and $y = 4 - 2x^2$ intersect at the point (1, 2). By methods of calculus, we can show that the slope of the line tangent to the first curve at a point with abscissa x, is $m_1 = 2x$, and to the second curve is $m_2 = -4x$. Sketch the two curves, draw the tangent lines at the given point, and find the acute angle between these two tangent lines.

23S. Figure 4.13 shows the graphs of $y^2 = 4x + 4$ and $y = 2x - 2$. The rectangle shown in the figure has t for the positive value of its height and x_L and x_C as the abscissas of its two ends. Determine expressions in terms of y and t for

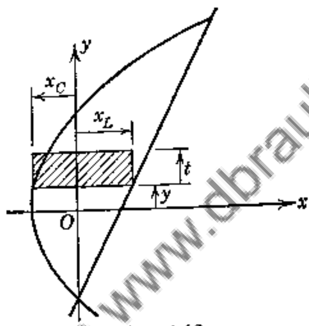


Fig. 4.13

(a) The area of the rectangle.
(b) The product of the area and the abscissa to the center of the rectangle.

4.4 The Ellipse

We shall be concerned in this article with a study of equations that can be reduced to the form:

$$Ax^2 + By^2 = C,$$

where A, B, and C are all *positive* numbers.

Definition. *An ellipse is the locus of a point that moves in a plane so that the sum of its distances from two fixed points is a constant.*

We proceed to derive an equation for this locus, and we shall use the locus-derivation method. The final equation will be simple and symmetrical if we choose axes so that one axis goes through the two fixed points and the other axis is the perpendicular bisector of the line segment joining those two points. Figure 4.14 shows one such choice of axes.

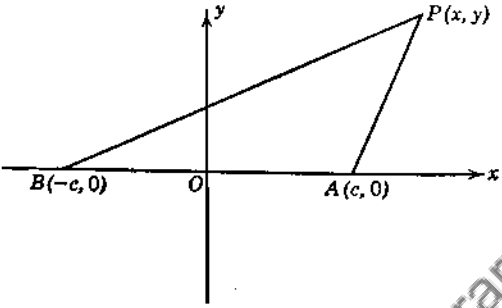


FIG. 4.14

I. We sketch the figure and label the fixed points $A(c, 0)$ and $B(-c, 0)$.

II. We locate a general point $P(x, y)$ that satisfies the statement of the problem.

III. $\overline{AP} + \overline{BP} = 2a$, where the sum of the distances, a constant by the definition, has the value $2a$. We use the multiplier 2 in order to simplify the subsequent algebra.

IV. $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$.

V. We transpose one radical, say the first, square both sides, and obtain

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2.$$

We collect, transpose the single radical to one side, simplify, and obtain

$$a\sqrt{(x-c)^2 + y^2} = a^2 - cx.$$

(Notice how the 4's canceled, because of the choice of $2a$ for the constant.) Now we square both sides again, collect, and obtain

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2).$$

Notice the expression $a^2 - c^2$. It should remind us of the Pythagorean theorem from plane geometry. With this suggestion we may simplify the preceding equation by introducing a new letter $b^2 = a^2 - c^2$ and obtain

$$b^2x^2 + a^2y^2 = a^2b^2 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

VI. Check. It is left to the student to determine the two intersections of this curve with the x-axis and to determine from the original figure that each such point satisfies the requirements in the definition of the ellipse.

Exercise for the Student. Apply the discussion method of curve sketching to the curve

$$b^2x^2 + a^2y^2 = a^2b^2$$

and verify the following conclusions:

I. Intercepts. $x = \pm a$, $y = \pm b$.

II. Symmetry with respect to both axes and the origin.

III. Asymptotes. No horizontal nor vertical asymptotes.

IV. Excluded regions. The values of y are imaginary if $x^2 > a^2$. Also, the values of x are imaginary if $y^2 > b^2$.

V. Additional points.

x	0	$0.25a$	$0.50a$	$0.75a$	a
y	b	$0.968b$	$0.866b$	$0.662b$	0

The graph is shown in Fig. 4.15 along with a right triangle that emphasizes the relation between a, b, and c.

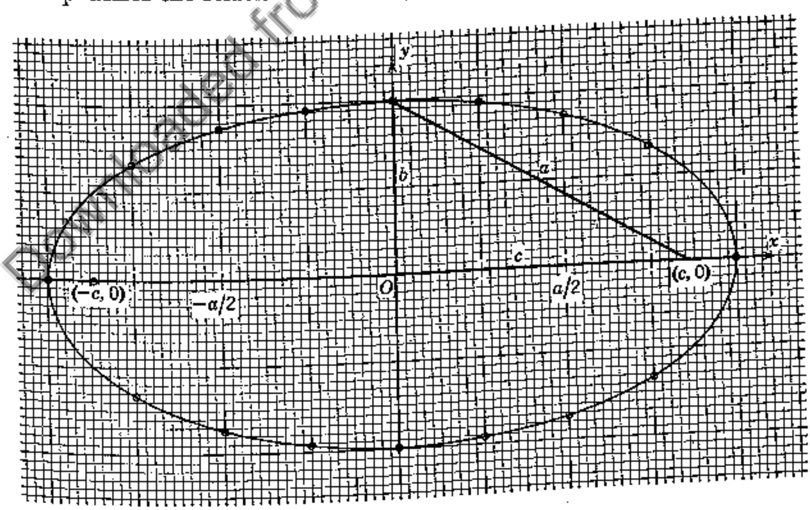


Fig. 4.15

DEFINITIONS. *The two fixed points in the definition of the ellipse are called the foci of the ellipse; the length 2a is called the major axis; the length 2b is called the minor axis; and the vertices are the ends of the major axis.* Notice in the figure that the major axis is the longest "diameter" while the minor axis is the shortest "diameter." Notice also that the length of the semimajor axis is a. Notice further that each y-crossing of the curve is equidistant from the two foci and hence is at a distance of a from each focus.

If we had taken the two fixed points on the y-axis, and the x-axis as the perpendicular bisector, the resulting equation for the ellipse would clearly be the same as our preceding equation with x replaced by y and y by x, i.e., $y^2/a^2 + x^2/b^2 = 1$. We may apply the method of translation of axes to see that the general form of the equation of an ellipse with center at (h, k) and major axis parallel to the x-axis is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

A similar equation could be written if the major axis were parallel to the y-axis.

The student should observe that a is the positive square root of the larger denominator if the right-hand member of the equation is $+1$. Also, the foci are, because of the derivation, on the major axis.

The numerical value of c/a can be used to determine the shape of an ellipse. Thus, as c approaches zero and hence as b approaches a, the ellipse approaches the form of a circle. On the other extreme, as c approaches a and hence as b approaches zero, the ellipse becomes long and narrow. This ratio, c/a, is called the *eccentricity* e of the ellipse. We observe that in the right triangle shown in Fig. 4.15 the eccentricity e is the cosine of the acute angle at the focus.

EXAMPLE 1

Identify and sketch $25x^2 + y^2 - 50x + 20y = 500$.

Solution. First complete the squares on x and y as follows:

$$25(x^2 - 2x + 1) + (y^2 + 20y + 100) = 500 + 25 + 100,$$

$$25(x-1)^2 + (y+10)^2 = 625,$$

or

$$25x'^2 + y'^2 = 625.$$

The curve is an ellipse with center at $(1, -10)$. A quick sketch may be obtained merely by the use of the intersections of the curve and the translated axes. If a more accurate plot is desired, the student may compute additional points, judiciously choosing the value of one variable. The graph is shown in Fig. 4.16, and the student will notice that in this problem a choice of different scales on the two axes is almost mandatory. However, care must be taken in the interpretation of such graphs with different scales.

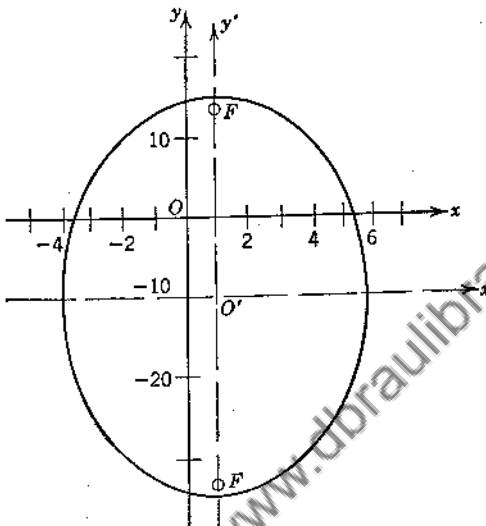


Fig. 4.16

In many problems in science where the ellipse is encountered, it is unnecessary to locate the foci. These are located in the figure and were found by computing c from $a = 25$, $b = 5$, and

$$c = \sqrt{a^2 - b^2} = \sqrt{600} \approx 24.5.$$

The student should notice that no graphical method can be used to locate the foci in this figure, since the scales on the two axes are different.

Example 2

Determine the equation of the ellipse with center at the origin, symmetrical with respect to both axes, and passing through (2, 2), and (4, −1).

Solution. The student may start with $x^2/a^2 + y^2/b^2 = 1$. However, since nothing has been said or asked about the foci, it would be simpler to write the equation as $px^2 + qy^2 = 1$ (where the choice of letters for coefficients is immaterial). Since the curve is to go through the two given points, p and q must satisfy $4p + 4q = 1$ and $16p + q = 1$. When he has found p and q, the student can show that the required equation is $x^2 + 4y^2 = 20$.

Exercise for the Student. Solve the preceding example by use of the equation $x^2/a^2 + y^2/b^2 = 1$.

PROBLEMS

1. Sketch the following curves (be careful to label the units on both axes):

(a) $4x^2 + 9y^2 = 25$.
(b) $9x^2 + y^2 = 45$.
(c) $3u^2 + 5v^2 = 30$.
(d) $5u^2 + 5v^2 = 23$.
(e) $2.78s^2 + 56.9t^2 = 864$.
(f) $4.56x^2 + 3.72y^2 = 11.6$.
(g) $9(x - 3)^2 + 16(y + 1)^2 = 0$.
(h) $(x - 1)^2 + 4(y + 2)^2 = 16$.
(i) $5(s + 1)^2 + 2(t - 3)^2 = 20$.
(j) $4x^2 + 9y^2 - 40x + 54y + 145 = 0$.
(k) $2x^2 + y^2 = 4x + 4y + 10$.
(l) $10x^2 + 200y^2 + 400y = 41 + 20x$.
(m) $s^2 + 10t^2 - 10s - 200t + 925 = 0$.

2. Determine the equation of the ellipse:

(a) That is symmetrical with respect to both axes and that goes through (0, 40) and (5, 0).

(b) That has its center at (3, 2), the ends of the major axis at (3, 9) and (3, −5), and the ends of the minor axis at (1, 2) and (5, 2).

(c) That has its center at the origin, that is symmetrical with respect to both axes, and that goes through (7, 1) and (2, 5).

(d) Same as (c) but goes through (2, 6) and (3, 2).

(e) That is the locus of a point moving so that the sum of its distances from (6, 0) and (−6, 0) is always 16.

(f) That is the locus of a point moving so that the sum of its distances from (0, 4) and (0, −4) is always 12.

(g) That has its center at (4, −1), one focus at (4, 1), and one vertex at (4, 3).

(h) That has its foci at (6, 0) and (−6, 0), and its vertices at (8, 0) and (−8, 0).

(i) That has its foci at (5, 1) and (5, −3), and that goes through (1, −1).

(j) That has its foci at (0, 4) and (0, −4), and an eccentricity of $e = \frac{2}{3}$.

(k) That has the ends of its minor axis at (5, −3) and (5, 1), and that has an eccentricity of $e = \frac{1}{4}$.

3. Plot carefully $4x^2 + y^2 = 16$ by the following graphical method: Use 1 in. = 1 unit on both axes. Locate the points of intersection with the axes, and then locate the foci on the y-axis by use of compasses set with center at one point of intersection with the x-axis and with the radius $a = 4$. Now fasten thumb tacks at each focus and tie one end of a string at one thumb tack and the other end of the string at the second thumb tack in such a way that the length of string between the tacks is $2a = 8$ in. Place a pencil against the string, drawing it taut, and describe a curve with the point of the pencil by moving it against the taut string. When the curve is completed it will necessarily be an ellipse because the pencil point describes a locus of points whose sum of distances from the two tacks is a constant.

4. Use the graph drawn in Problem 3 for $4x^2 + y^2 = 16$, or a graph drawn by plotting sufficient points to ensure accuracy. Select several distinct points on the ellipse, and use a straightedge to draw the tangent lines to the ellipse at these points. Next draw straight-line segments from one such point to the two foci. What does the figure suggest to be true about the angles between these two focal lines and the tangent line? Try it for the other points.

If the curve of the ellipse were a cross section of a curved mirror and a long neon light were placed at one focus perpendicular to the plane of the cross section, what would you suspect to be true about the amount of light received along a line through the other focus (and perpendicular to the plane of the curve) as compared with the amount received along any other line?

If the ellipse were a horizontal cross section of a room and the wall were made of a material which would reflect sound, where would be a good place to listen to a conversation carried on at one focus?

5. Determine the equation of the locus of a point that moves so that the sum of its distances from (0, 0) and (6, 0) is always 10.

6. Determine the equation of the locus of a point that moves so that the square of its distance from $y = 2$ is always equal to 3 times the square of its distance from (0, 0). Identify and sketch the locus.

7. Determine the equations of the ellipses shown in Figs. 4.17, 4.18, and 4.19 (in each case be careful to notice the indicated positive directions of the axes).

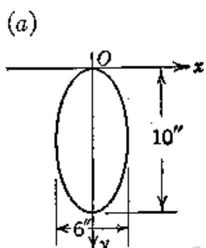


FIG. 4.17

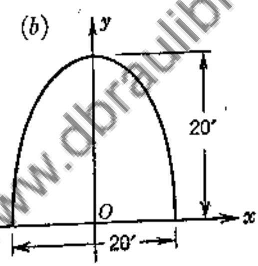


FIG. 4.18

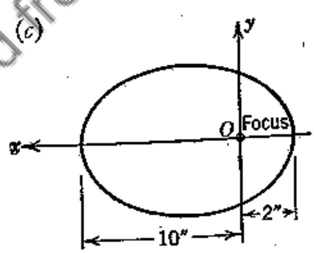


FIG. 4.19

8. Draw $x^2/a^2 + y^2/b^2 = 1$, using 2 in. = a units on the x-axis and 2 in. = b units on the y-axis.

9. Find the length of the chord of $x^2 + 3y^2 = 9$ that is parallel to the x-axis and that goes through (1, 1).

10. Crosshatch the area to the right of $3x + y = 3$ and inside $4x^2 + y^2 = 12$.

11. Determine the area to the right of $x + 2y = 4$ and inside $x^2 + 4y^2 = 16$. Make use of the fact that the total area enclosed by an ellipse is area $= \pi ab$.

12. Sketch each of the following ellipses, and determine the coordinates of the foci, the length of the major and minor axes, and the coordinates of the vertices:

(a) $9x^2 + 25y^2 = 225$.
(b) $3x^2 + y^2 = 9$.
(c) $16x^2 + 25y^2 + 100y + 16 = 32x$.
(d) $5.88x^2 + 2.47y^2 = 11.24$.
(e) $x^2 + 4y^2 - 4x + 8y = 8$.
(f) $9x^2 + 4y^2 + 36x + 24y + 36 = 0$.

13. An ellipse is symmetrical with respect to both axes and the origin and goes through (0, 5) and (−2, 0). Find the abscissa to the curve in the third quadrant that corresponds to an ordinate of −3.

14. An ellipse is symmetrical with respect to the x-axis and goes through (0, 0), (8, 0), and (4, 2). Find the "spread" of the ellipse along the line $x = 2$ and also along the line $y = 1$.

15. An airplane strut is 6.00 ft. long. Every section is an ellipse, the width at the center being 1.000 in. and the thickness 0.750 in. At each end, the width is 0.750 in. and the thickness 0.500 in. The strut is tapered uniformly from the middle towards both ends (i.e., the thickness and width are each linear functions of the distance z in. from the center of the strut).

(a) Determine the semimajor and semiminor axes at a distance of z in. from the center of the strut.

(b) Determine the cross-sectional area at z in. from the center and in particular at 1 ft. from the center. The area of an ellipse is πab.

(c) Sketch a graph of the cross-sectional area as a function of z for z from 0 to 36 in. Identify the curve.

16. Sketch $x^2 + 4y^2 = 20$ and $x - 2y + 2 = 0$ on the same pair of axes, and find the area of the triangle that has one vertex at the origin and the other two of its vertices at the points corresponding to the simultaneous solutions of the two equations.

17S. Show that the equation of the locus of the mid-points of a family of parallel chords in an ellipse is a line segment that goes through the center of the ellipse.

18S. Plot $x^2 + 2y^2 = 4$ according to the following directions: First plot $X + 2Y = 4$ on a sheet of graph paper (use a convenient decimal scale of about 1 in. = 1 unit or 2.5 in. = 1 unit on both axes). Next, since $X = x^2$ and $Y = y^2$, change the scales on the two axes so that with the distorted scales the graph in the first quadrant will be a distorted graph of the given equation. For example, when $x = 0.5$, $X = 0.25$, so that one should mark $x = 0.5$ at the position on the X-axis that corresponds to $X = 0.25$.

19S. Use the graph of Problem 18S, and draw on the same distorted scales the graph of $4x^2 + 4y^2 = 9$. Then solve the two equations simultaneously by aid of this peculiar graph.

4.5 The Hyperbola

In this article we shall be concerned with a study of equations of the second degree which can be reduced to the form

$$px^2 + qy^2 = m,$$

where p and q are of *opposite* sign and none of p, q, and m is zero.

Definition. *A hyperbola is the locus of a point that moves in a plane so that the difference of its distances from two fixed points is a constant.*

We proceed to find the equation of this locus by the locus-derivation method. For simplicity, just as in the case of the ellipse, we take one axis through the two fixed points and the other axis as the perpendicular bisector of the line segment joining the two fixed points. Let the two fixed points be at $A(-c, 0)$ and $B(c, 0)$, and let the difference of the distances be numerically equal to $2a$.

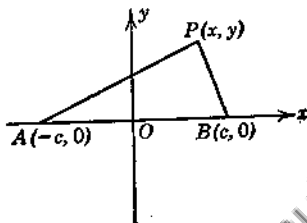


Fig. 4.20

I and II. See Fig. 4.20.

III. $\overline{AP} - \overline{BP} = \pm 2a$. We shall show the work using the positive sign. The student should show that he arrives at the same final result if he starts with the minus sign in the right-hand member.

IV. $\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$.

V. The student should perform the requisite algebra and show that this can be simplified to $(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$. We may further simplify this result if we choose c to be the hypotenuse of a right triangle with a as one leg and b as the other leg. Then $b^2 = c^2 - a^2$, and we may write the equation in either of the forms:

$$b^2x^2 - a^2y^2 = a^2b^2 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

VI. The numerical check is left to the student as an exercise.

Exercise for the Student. Apply the discussion method of curve sketching to verify the following facts about the curve whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

I. Intercepts. $x = \pm a$, no y-intercepts (or the y-intercepts are imaginary).

II. Symmetrical with respect to both axes and the origin.

III. Asymptotes. No horizontal or vertical asymptotes.

IV. Excluded regions. y is imaginary for all values of x between $x = -a$ and $x = a$. Every line parallel to the x-axis intersects the curve in two distinct points.

V. Additional points. The following points are on this curve:

x	a	$1.5a$	$2a$	$2.5a$
y	0	$1.12b$	$1.73b$	$2.29b$

We plot these points and such other points as may be obtained because of symmetry, and draw the curve as shown in Fig. 4.21.

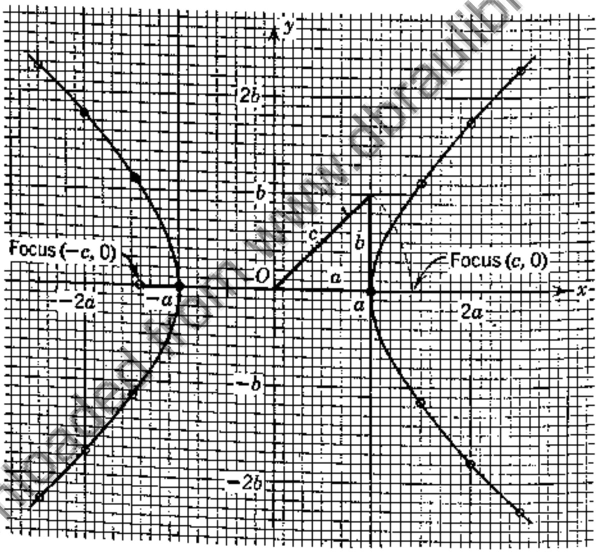


FIG. 4.21

The student will notice that the resulting curve consists of two separate pieces, or branches, as they are called. Both branches are open curves, and both branches satisfy Step III of the derivation, one with the positive sign in the right-hand member and the other with the negative sign.

DEFINITIONS. *The foci of a hyperbola are the two fixed points in the definition—they are located at* $(c, 0)$ *and* $(-c, 0)$ *in Fig.* 4.21. *The* *transverse axis is the line segment which, in this figure, joins the two vertices at* $(a, 0)$ *and* $(-a, 0)$. *The conjugate axis is of length* $2b$ *and joins the points* $(0, -b)$ *and* $(0, b)$. *The center of a hyperbola is the mid-point of the transverse axis—the origin in the present example. The two hyperbolas*

$$\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1 \quad \text{and} \quad \frac{y^2}{q^2} - \frac{x^2}{p^2} = 1$$

are called conjugate hyperbolas.

If the *same scale is used on both axes,* the foci may be located easily by aid of compasses, as shown in Fig. 4.21.

If the foci are on the y-axis, and if the curve is symmetrical with respect to both axes and the origin, the equation of the hyperbola will be

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} - \frac{y^2}{a^2} = -1.$$

(How may one arrive at these equations without repeating the derivation?) Hence the basic equation of a hyperbola with center at the origin and symmetrical with respect to both axes is $px^2 + qy^2 = 1$, where p and q are *opposite in sign.* The method of translating axes may then be used to obtain typical equations for a hyperbola with center at (h, k) and symmetrical with respect to lines parallel to the coordinate axes.

Exercise for the Student. Write the general equation of a hyperbola with center at (h, k) and with transverse axis parallel to the x-axis.

We compare the ellipse and hyperbola and notice that for the ellipse a is always larger than b. For the hyperbola, a is the real intercept value and may be smaller than b, equal to b, or larger than b. In the case of the ellipse, c is one leg of a right triangle whereas it is the hypotenuse in the case of a hyperbola.

One essential difference between a parabola and one branch of a hyperbola is that the hyperbola has straight-line asymptotes that are oblique to the axes. To prove this statement, we prove the following theorem.

Theorem. *The two straight lines* $y = bx/a$ *and* $y = -bx/a$ *are asymptotes for the hyperbola* $x^2/a^2 - y^2/b^2 = 1$.

Proof. Because the graph of the hyperbola is symmetrical with respect to both axes and the origin, it will only be necessary to prove that in the first quadrant the line $y = bx/a$ is an asymptote (see

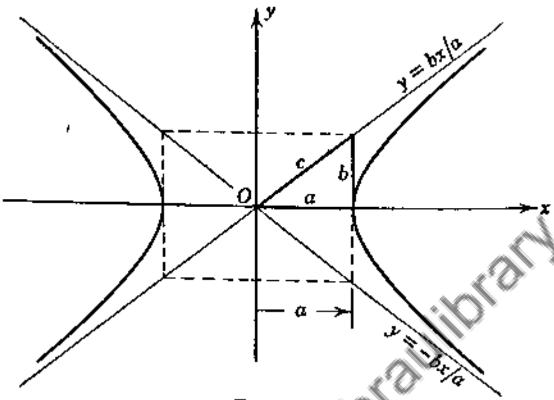


FIG. 4.22

Fig. 4.22). We solve the equation of the hyperbola for y in terms of x and obtain

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2},$$

and we will use the positive sign in the ensuing discussion. We then divide inside the radical by x^2 and multiply outside by x, and obtain

$$y = \frac{bx}{a}\sqrt{1 - \left(\frac{a}{x}\right)^2}.$$

From this last equation we see that, as x increases without limit, the value of $(a/x)^2$ approaches zero. Hence we infer that the value of y approaches the value given by (bx/a) and therefore that $y = bx/a$ is an asymptote. Let us complete the proof since the preceding reasoning is incomplete.* We shall show that in the first quadrant the perpendicular distance between a point on the hyperbola and the line $y = bx/a$ approaches zero as the point moves away from the origin (so that the distance from the origin to the point increases without

* The same reasoning applied to $y = \pm x\sqrt{1 - (2/x) - (1/x^2)}$ would imply that $y = \pm x$ are asymptotes, but this is *not* true. We shall study hyperbolas of this type in a later article.

limit). If $P_1(x_1, y_1)$ is the point on the hyperbola, the perpendicular distance from this point to the line $bx - ay = 0$ is given by

$$q = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}}.$$

But $x_1^2/a^2 - y_1^2/b^2 = 1$. Hence $ay_1 = +b\sqrt{x_1^2 - a^2}$ and

$$q = \frac{b}{\sqrt{a^2 + b^2}} \frac{x_1 - \sqrt{x_1^2 - a^2}}{1}.$$

We rationalize the numerator of the second fraction, simplify, and obtain

$$q = \frac{b}{\sqrt{a^2 + b^2}} \frac{a^2}{x_1 + \sqrt{x_1^2 - a^2}}.$$

From this last result we see that, as x_1 increases without limit, the denominator also increases without limit and the value of q then approaches zero. This completes the proof of the theorem.

This theorem suggests a rapid method for sketching hyperbolas. If we observe that the intercepts for $x^2/a^2 - y^2/b^2 = 1$ are $x = \pm a$ and $y = \pm ib$, and that the two asymptotes go through the points (a, b) and $(a, -b)$ respectively, as well as the origin, then sketching a hyperbola resolves itself into locating the real intercepts, locating the points (a, b) and $(a, -b)$ and drawing the asymptotes, and then sketching the two branches of the hyperbola (see Fig. 4.22). The same general procedure applies for the other form for the hyperbola $y^2/a^2 - x^2/b^2 = 1$. Moreover, it is unnecessary, for purposes of sketching, to put the equation into either of these standard forms. What is necessary (after completing the squares and translating axes) is to determine the real and the imaginary intercepts.

EXAMPLE 1

Sketch $x^2 - 25y^2 - 6x - 50y + 9 = 0$.

Solution. The locus is clearly a hyperbola since the equation is of the second degree and the coefficients of x^2 and y^2 are opposite in sign.* We complete the squares on both x and y as follows:

$$(x^2 - 6x + 9) - 25(y^2 + 2y + 1) = -9 + 9 - 25,$$

or

$$(x - 3)^2 - 25(y + 1)^2 = -25.$$

* The locus would be a degenerate hyperbola, i.e., a pair of intersecting straight lines, if the right-hand member of the third equation (in this example) were 0 instead of -25.

The form of this equation suggests the equations of translation $x' = x - 3$ and $y' = y + 1$, so we translate axes to $(3, -1)$ and obtain $x'^2 - 25y'^2 = -25$ for the new equation. The intercepts are $x' = \pm 5i$ and $y' = \pm 1$. The rectangle, the diagonals or asymptotes, and the curve are shown in Fig. 4.23. We note that $a = 1$, $b = 5$, and $c = \sqrt{26}$. Hence the foci are at (3, 4.10) and (3, −6.10) and cannot be located by aid of compasses, since the scales on the two axes are unequal. The vertices are at (3, 0) and (3, −2).

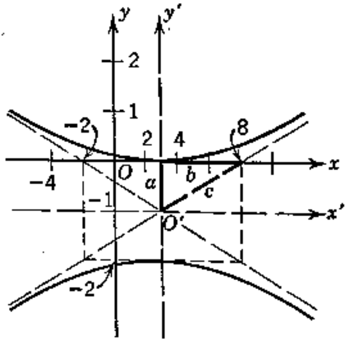


FIG. 4.23

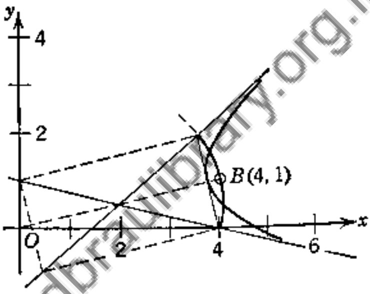


FIG. 4.24

EXAMPLE 2

Two listening posts are located at $O(0, 0)$ and $B(4, 1)$, the unit being in miles. Microphones located at these two points show that a gun is 3.60 miles closer to B than to O. Sketch, by graphical means, a curve which goes through the gun's location.

Solution. The microphones are located at the foci of a hyperbola, and $2a = 3.60$ whence $a = 1.80$ (why?). We plot the figure and locate the two microphones, using the same scale on both axes. We locate the mid-point of the line segment $\overline{OB}$ or the center of the hyperbola.

We next measure $a = 1.80$ units along this line segment in both directions from the center and locate the two vertices of the hyperbola. Finally, we use compasses and construct the associated rectangle, the diagonals or asymptotes, and the *one branch* of the hyperbola that is nearer B. The graph is shown in Fig. 4.24. In actual practice several microphones could be used and only the asymptotes would be drawn, since the gun would probably be some distance away from each microphone and hence almost on one of the two asymptotes for each pair of microphones.

4.6 Equilateral Hyperbolas

The particular case $a = b$ in the general equation $x^2/a^2 - y^2/b^2 = 1$ yields

$$x^2 - y^2 = a^2,$$

and the graph is called an *equilateral hyperbola* or a *rectangular hyperbola*. The associated rectangle is a square, and the associated triangle is an isosceles right triangle.

EXERCISE FOR THE STUDENT. Sketch $x^2 - y^2 = a^2$ using 1 in. = a units on both axes. Label the coordinates of the vertices and foci, showing the associated right triangle, and determine the equations of the asymptotes.

If the foci are at (m, m) and $(-m, -m)$, and if $a = b = m$, the equation that results by the locus derivation will be $xy = m^2/2$ and the asymptotes will be the x- and y-axes. This equation, xy = constant = k, is a special power-law curve $y = ax^n$ with $n = -1$.

The ratio $e = c/a$, which is called the *eccentricity*, may be used as a measure for the shape of a hyperbola. If $c/a = \sqrt{2}$, which implies that $a = b$, the hyperbola is an equilateral hyperbola. As the value of c approaches that of a so that b approaches zero, the hyperbola $x^2/a^2 - y^2/b^2 = 1$ becomes very slender around the x-axis. As the value of c/a increases (while the value of a stays fixed), and hence as the value of b increases, the hyperbola becomes fatter around the x-axis or slenderer around the y-axis. The eccentricity is between zero and one for the ellipse, and is greater than one for the hyperbola. (We note that the eccentricity is the secant of one of the acute angles in the associated right triangle.)

PROBLEMS

1. Sketch the following curves:

(a) $x^2 - y^2 = 20$.
(b) $x^2 - 16y^2 + 16 = 0$.
(c) $y^2 - 4x^2 = 16$.
(d) $s^2 - 400t^2 = 400$.
(e) $3x^2 - y^2 - 6x - 4y + 11 = 0$.
(f) $xy = 4$.
(g) $25x^2 - 144y^2 = 2400$.
(h) $xy + 8 = 0$.
(i) $(y - 2)^2/4 - (x + 1)^2/12 = 1$.
(j) $pv = 560$.
(k) $y^2 + 4y + 6x + 13 = x^2$.
(l) $x^2 + 10y + 2 = 5y^2 + 4x$.
(m) $2x^2 - 5y^2 + 8x - 20y - 32 = 0$.
(n) $(p - 2)(v - 3) = 4$.

2. Sketch the curve $xy - 3x + 4y = 6$ by first rewriting the equation in the form $(x + 4)(y - 3) = ?$ and then translating axes.

3. Given the equation $16y^2 - 9x^2 - 64y + 18x + 19 = 0$.

(a) Determine the values of a, b, and c, and sketch the curve.
(b) Determine the coordinates in terms of the original variables for the center, the foci, and the vertices or ends of the transverse axis; label these on the graph.
(c) Determine the equations of the asymptotes.

4. Repeat Problem 3 for each of the following equations:

(a) $x^2 + 2x + 2y = 4 + y^2$.
(b) $16x^2 + 160 = 9y^2 + 32x$.
(c) $xy = 4$.
(d) $4x^2 - y^2 - 8x + 2y + 7 = 0$.

5. Determine the equations of the following hyperbolas:

(*a*) With center at the origin, transverse axis along the y-axis, and $a = 5$ and $b = 7$.

(*b*) With center at (1, 2), ends of transverse axis at (1, 7) and (1, −3), ends of the conjugate axis at (−1, 2) and (3, 2).

(*c*) With ends of the transverse axis at (−3, 4) and (5, 4), and with $c = 6$.

(*d*) With center at the origin, symmetrical with respect to both axes, and passing through (0, 6) and (2, 8).

(*e*) With center at the origin, symmetrical with respect to both axes, and passing through (4, 1) and (5, 3).

(*f*) Passing through (0, 4), (0, −4), (1, 5), (−1, 5), (1, −5), and (−1, −5).

(*g*) With center at the origin, one vertex at (4, 0), one asymptote going through (6, 9), and the hyperbola symmetrical with respect to both axes.

(*h*) Having both axes as asymptotes and going through (2, 5).

(*i*) Having both axes as asymptotes and going through (−2, 7).

(*j*) Having vertices at (4, 3) and (4, −1), and having an eccentricity of $e = 2$.

(*k*) Having $y = 2x$ and $y = -2x$ as asymptotes, an eccentricity of $e = \sqrt{5}$, and one vertex at (4, 0).

(*l*) Having the ends of its conjugate axis at (2, 6) and (2, 0), and an eccentricity of $e = 2$.

6. Determine the equation of the locus of a point that moves so that the difference of its distances from the two points (0, 4) and (0, −4) is always 6. Identify and sketch the locus.

7. Determine the equation of the locus of a point that moves so that its distance from the point (3, 0) is always twice its *directed* distance from $4x - 3 = 0$. Identify and sketch the locus.

8. Three listening posts are located at $A(0, 0)$, $B(4, 1)$, and $C(1, 3)$, the unit being 1 mile. Microphones located at these points show that a gun is 2.20 miles closer to C than to A, 3.60 miles closer to B than to A, and 1.50 miles closer to B than to C. Locate the gun by use of the asymptotes of the three hyperbolas thus defined.

9. Sketch $4y^2 = x^2 + 16$ and $3x + 2y = 4$ on the same pair of axes, and find the length of the line segment that joins their two points of intersection.

10. Show that the equation $xy + ax + by + c = 0$ can be rewritten as $(x + b)(y + a) = ab - c$ and is an equilateral hyperbola if $ab - c \neq 0$. What are the equations of the asymptotes? What is the graph if $ab - c = 0$?

11. Two loran stations are located at $A(0, 0)$ and $B(0, 500)$, the unit being 1 mile. An airplane pilot knows from the signals that at a certain instant he is 300 miles nearer B than A. What is the equation of the curve that these data define?

12. A hyperbola is symmetrical with respect to both axes and the origin and goes through (0, 3) and (8, 6). Find the ordinate to this curve in the second quadrant corresponding to an abscissa of −4.

13. Prove the statement in the second paragraph of Art. 4.6; that is, derive the equation of the locus of a point that moves so that the difference of its distances from (m, m) and $(-m, -m)$ is always $2m$.

14. State Boyle's law in chemistry, and sketch the associated graph.

15. Let $P(x, y)$ be any point that lies on the curve $xy = 1$, and let $P_1(x, 0)$ be the projection of P on the x-axis. Prove that the area of the triangle OPP_1, where O is the origin, is independent of the position of P on the curve. Hence show that this particular equilateral hyperbola is the locus of a point that moves so that the area of the triangle OPP_1 is a constant.

16S. Prove that the product of the perpendicular distances from every point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ to the two asymptotes is a constant.

17S. Plot a graph of the family of curves $x^2 + y^2/(1 - c^2) = 1$, and show the members that correspond to $c = 0, 0.25, 0.5, 0.75, 1.25, 1.50$, and 2. What property do all these curves have in common?

18S. Plot a graph of the family of curves $x^2/a^2 + y^2/(a^2 - 1) = 1$, and show the members that correspond to $a = 0.25, 0.5, 0.75, 1.25, 1.50$, and 2. What property do all these curves have in common?

19S. Plot the graphs of $x^2 - 4y^2 = 4$ and $x^2 + 4y^2 = 16$ using distorted scales on both axes constructed according to the following scheme: Let $Y = y^2$ and $X = x^2$, and lay off "square" scales on both axes—for example, when $x = 2$, measure to $X = 4$ but put down the original number $x = 2$; again, when $x = 3$, measure to $X = 9$ and put down $x = 3$. The X and Y scales are, of course, ordinary scales.

4.7 A General Locus Definition for Conics

The student should write out a complete outline of curve sketching including methods for sketching rapidly each of the curves previously discussed in this chapter. A part of this latter discussion is given in the following outline:

The *circle*. An equation of the form $x^2 + y^2 + ax + by + c = 0$. Complete the squares on x and y, locate the center, find the radius, and draw the circle.

The *parabola*. An equation of the form $y = ax^2 + bx + c$ or of the form $x = Ay^2 + By + C$. Complete the square and translate axes *or* determine the intercepts and locate the axis of symmetry and the vertex.

The *ellipse*. An equation of the form $Ax^2+By^2+Cx+Dy+E=0$ in which A and B are both positive and unequal in value. Complete the squares and translate axes.

The *hyperbola*. An equation of the form $Ax^2+By^2+Cx+Dy+E=0$ in which A and B are of opposite sign. Also an equation of the form $xy + ax + by + c = 0$. In exceptional circumstances, either of these equations may have as its graph a pair of intersecting straight lines.

The name *conic* is derived from the fact that the circle, ellipse, parabola, and hyperbola are all possible curves of intersection of a double cone and a plane, as shown in Fig. 4.25.

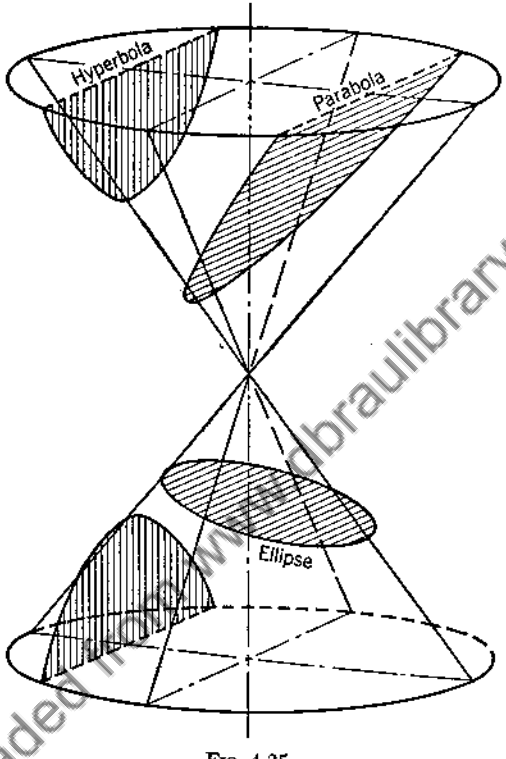


FIG. 4.25

The equations for these conics may all be derived by a single locus derivation from the following definition:

DEFINITION. *The locus of a point that moves in a plane so that its distance from a fixed point (a focus) divided by its distance from a fixed line (a directrix) is a non-negative constant (the eccentricity), is a conic section.* It is assumed that the fixed point is not on the fixed line.

I and II. We take the fixed line to be the y-axis and the fixed point to be the point $F(q, 0)$, $q > 0$, as shown in Fig. 4.26. Then, with $P(x, y)$ as a general point:

III. $e = \overline{FP}/\overline{AP}$ or $e \cdot \overline{AP} = \overline{FP}$.

IV. $ex = \sqrt{(x - q)^2 + y^2}$.

V. The student should show that this may be simplified to

$$x^2(1 - e^2) - 2xq + q^2 + y^2 = 0.$$

VI. The student should check with the evident check point (q, eq).

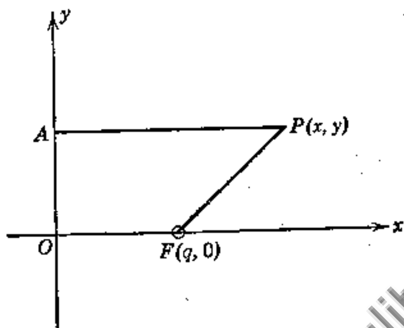


FIG. 4.26

There are three general cases to be considered for the equation obtained in Step V.

Case 1. $e = 1$. In this case the equation reduces to $y^2 = 2q(x - q/2)$, which is a parabola.

Case 2. $e < 1$. The equation may be reduced, by completing the square, to the form:

$$(1 - e^2)\left(x - \frac{q}{1 - e^2}\right)^2 + y^2 = \frac{q^2e^2}{1 - e^2}.$$

This is an ellipse with $a = qe/(1 - e^2)$ and $b = qe/\sqrt{1 - e^2}$. It is easy to compute c and find that $c = qe^2/(1 - e^2)$. Hence $c/a = e$, as was stated in Art. 4.4.

As a particular case with $e < 1$, we note that, if $e = 0$, the locus is the single point $(q, 0)$, or a point circle.

Case 3. $e > 1$. The equation may be reduced to the form:

$$(e^2 - 1)\left(x + \frac{q}{e^2 - 1}\right)^2 - y^2 = \frac{q^2e^2}{e^2 - 1}.$$

The locus is a hyperbola. We compute the following values:

$$a = \frac{qe}{e^2 - 1}, \quad b = \frac{qe}{\sqrt{e^2 - 1}}, \quad c = \frac{qe^2}{e^2 - 1}.$$

We observe that $c/a = e$, as was stated in Art. 4.6.

We note that the fixed point in the definition of the locus is a *focus*. The fixed line is the *directrix* in the case of the parabola, and we give it the same name for the ellipse and hyperbola. Because of symmetry, the ellipse and the hyperbola each have two directrices. Moreover, in both cases we may use the preceding equations to see that the distance from the center to a directrix is given by $q/|1 - e^2|$, which may be simplified to

$$d = \frac{a}{e}.$$

We have thus established the following theorem:

Theorem. *The ellipse and the hyperbola each have two directrices. These directrices are perpendicular to the axis through the foci and are each $d = a/e$ distant from the center.*

PROBLEMS

1. Derive, by locus-derivation methods, the equation of the locus of a point that moves so that its distance from the point (4, 0) is always equal to e times its distance from the y-axis. Simplify your result for the following three cases, and draw the locus:

(a) $e = \frac{1}{2}$. (b) $e = 1$. (c) $e = 2$.

2. Draw the graphs of the following conics; label with coordinates the important details such as the center, the foci, and the vertices; give the equations of the directrices and of the asymptotes; state the value of the eccentricity in each case:

(a) $x^2 = 8y$.
(b) $x^2 + 4y^2 = 16$.
(c) $x^2 - 4y^2 = 16$.
(d) $x^2 - y^2 = 4$.
(e) $xy = 2$.
(f) $y^2 + 6x + 2y = 5$.
(g) $25x^2 + 9y^2 - 50x + 18y = 191$.
(h) $4x^2 + y^2 - 16x + 4y + 16 = 0$.
(i) $9x^2 - 16y^2 - 18x - 32y + 137 = 0$.
(j) $4x^2 - y^2 - 16x - 4y + 8 = 0$.

4.8 The General Conic by Addition of Ordinates

We may use the method of addition of ordinates to sketch any curve whose equation has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C are not simultaneously zero. If B is zero, the curve may be sketched rapidly by methods already explained in this chapter. If $B \neq 0$, we may use the method of addition of ordinates to sketch the curve. That method of addition of ordinates requires that we solve for y in terms of x, sketch the component curves, and finally add ordinates to obtain the graph of the given equation.

EXAMPLE

Sketch $x^2 - 2xy + y^2 - x + 2y = 4$.

Solution. We rewrite the equation in standard quadratic form with y as the variable, and obtain

$$y^2 + y(2 - 2x) + (x^2 - x - 4) = 0.$$

We use the quadratic formula, simplify, and obtain

$$y = x - 1 \pm \sqrt{5 - x}.$$

The two component curves are $y = x - 1$ (which is a straight line) and $y = \sqrt{5 - x}$ (which is the top half of the parabola $y^2 = 5 - x$). The graph is shown in Fig. 4.27.

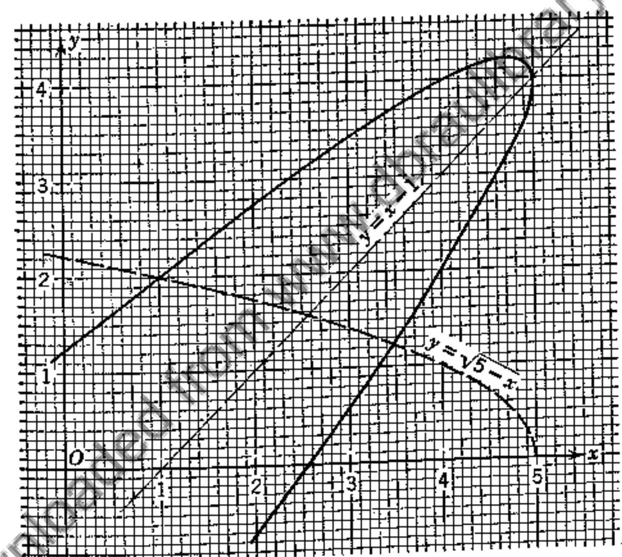


FIG. 4.27

Instead of adding and subtracting the ordinates of the top half of the parabola to the corresponding ordinates of the straight line, an alternative solution would be to draw the entire parabola and to add ordinates.

If $C \neq 0$ in the general equation, the solution for y in terms of x may be written in the form

$$2Cy = -(Bx + E) \pm \sqrt{(B^2 - 4AC)x^2 + 2x(BE - 2CD) + (E^2 - 4CF)}.$$

One of the component curves is a straight line. The other component curve is of *elliptic type* if the *indicator* $B^2 - 4AC$ is negative, of *para-*

bolic type if $B^2 - 4AC = 0$, and of *hyperbolic type* if $B^2 - 4AC$ is positive. We shall see later that the graph of every second-degree equation in two variables is either a conic (circle, ellipse, parabola, or hyperbola), an imaginary locus, or a pair of straight lines. Assuming this for the present, the sign of *the indicator* gives us a quick method by which we may determine the type of conic with which we are dealing. If the graph is a pair of straight lines, this will become apparent when we solve for y in terms of x; in this case the quantity under the radical will be a perfect square and the whole solution for y will reduce to two linear expressions in x.

EXAMPLES

Identify the following by aid of the indicator:

(1) $x^2 + 2xy + 3y^2 - 4x = 5$.
(2) $x^2 + 4xy + 5y = 8$.

Solution. (1) $B^2 - 4AC = 4 - 12 = -8$. Hence the locus is either an ellipse or a point ellipse, or there is no locus.

(2) $B^2 - 4AC = 16$. The locus is either a hyperbola or two intersecting lines.

If the student should happen to forget how the sign of the indicator distinguishes the various conics, let him apply the test to simple conics whose names he knows.

EXERCISE FOR THE STUDENT. Use the sign of the indicator to identify each of the following conics:

$$7x^2 + 2y^2 = 5, \quad x^2 - 4y^2 = 11, \quad xy = 6.$$

PROBLEMS

1. Identify and sketch the following curves:

(a) $y = x - 4 \pm \sqrt{1 - x}$.
(b) $y = 1 - 0.5x \pm \sqrt{4x^2 - 8x}$.
(c) $y = \frac{1}{3}x + 5 \pm \sqrt{9 - x^2}$.
(d) $4x^2 - 4xy = 1$.
(e) $x^2 + xy + y^2 = 8$.
(f) $x^2 - 2xy + y^2 + 2y - x = 1$.
(g) $4x^2 - 3xy - y^2 = y - x$.
(h) $3x^2 - 4xy + 8x = 1$.
(i) $x(x - y) = 2$.
(j) $y^2 - 2xy + x^2 + 16 = 4x$.
(k) $y^2 - 4xy - 8y + 16x + 32 = 0$ by addition of abscissas.

2. Identify the following curves by inspection, by aid of the indicator test, or by solving for y in terms of x and identifying the component curves. Then sketch the curve.

(a) $x^2 - 2xy + 4y^2 = 6x + 2$.
(b) $x^2 - y^2 = 3x + 7y + 11$.
(c) $s = \frac{1}{2}gt^2 + v_0t + s_0$ if $g = 32.2$ ft./sec.^{2}, $v_0 = 400$ ft./sec., and $s_0 = 0$.

(d) $x^2 + 4y = 5x$.
(e) $y = 2x + 7$.
(f) $x^2 - 4xy + 4y^2 = 5x - 10y$.
(g) $pv = 243$.

(h) $u^2 + v^2 = 5$, if the variables are u^2 and v^2 (*not* u and v).
(i) $u^4 - v^4 = 16$, if the variables are u^2 and v^2.
(j) $\sin^2\theta + \cos^2\theta = 1$, if the variables are $\sin\theta$ and $\cos\theta$.
(k) $\sec^2\theta = 1 + \tan^2\theta$, if the variables are $\sec\theta$ and $\tan\theta$.
(l) $\sin^2\theta = (1 - \cos 2\theta)/2$, if the variables are $\sin\theta$ and $\cos 2\theta$.
(m) $uv + 40u - 70v = 200$.

3. Show that the graph of $4y^2 - 4xy - 3x^2 + 4y - 10x + 13 = 0$ is a hyperbola. Plot this curve by the method of addition of ordinates, and determine the equations of the asymptotes.

4S. Explain how to draw a graph of

$$x^2 - 4xy + 4y^2 - 6x + 4y + 1 - k = 0$$

for $k = 0, 1, 2, 3, 4, -1, -2, -3$, and -4, by first constructing a template or pattern. Draw the curves.

4.9 Rotation of Axes

If only the graph of a general second-degree equation is desired, the method of solution is to use addition of ordinates. Sometimes, however, certain properties in addition to the graph itself are needed, and sometimes these cannot be found from the graph obtained by the method of addition of ordinates. We proceed in this article to develop a method that will yield both the graph and these properties for every second-degree equation. The method involves a second type of change of variable (translation of axes was the first) and has some applications in science and engineering.

Suppose that the x- and y-axes are rotated through an angle θ. The new axes, the x'- and y'-axes, are shown in Fig. 4.28. Let a

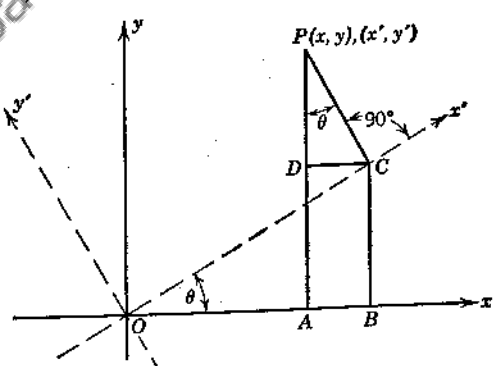


FIG. 4.28

general point P have coordinates (x, y) referred to the original axes, and coordinates (x', y') referred to the new or rotated axes. Then

$$x = \overline{OA} = \overline{OB} - \overline{AB} = \overline{OB} - \overline{DC} = x' \cos\theta - y' \sin\theta,$$

$$y = \overline{AP} = \overline{AD} + \overline{DP} = \overline{BC} + \overline{DP} = x' \sin\theta + y' \cos\theta.$$

We have thus shown that the equations for rotating axes through an angle θ are

$$\mathbf{x = x' \cos\theta - y' \sin\theta,}$$

$$\mathbf{y = x' \sin\theta + y' \cos\theta.}$$

EXAMPLE

What does the equation $xy = 4$ become if the axes are rotated through an angle of 45°?

Solution. Since $\sin 45° = \cos 45° = 1/\sqrt{2}$, the equations of rotation become $x = (x' - y')/\sqrt{2}$ and $y = (x' + y')/\sqrt{2}$. We substitute these in the given equation and obtain $x'^2 - y'^2 = 8$, which is again a hyperbola, as it *must* be, since a rotation of axes does not change the curve but does change the orientation of the curve with respect to the axes.

PROBLEMS

1. Solve the two equations of rotation for x' and y' in terms of x and y.

2. What does the equation $x^2 - 2xy + y^2 = 4$ become if the axes are rotated through an angle of 45°? Sketch both sets of axes and the graph of the given equation.

3. Transform $3x + 4y = 6$ by rotating axes through an acute angle $\theta = \arctan \frac{4}{3}$. Sketch the given straight line and both sets of axes.

4. Holes are punched in a steel plate at $A(0, 0)$ and $B(4, 1)$ as a first operation. A second operation requires the adjustment of a pattern and then the drilling of additional holes at $C(3, 5)$ and $D(-1, 5)$. One operator believes that these four holes would be more accurately located with respect to each other if the x-axis were located on the line AB (with the origin still at A). What are the new coordinates of B, C, and D, each correct to the nearest second decimal, if the axes are rotated so that B is on the x'-axis?

5. Given the equation $22x^2 + 12xy + 13y^2 = 50$. Rotate axes through an acute angle $\theta = \arctan \frac{1}{2}$, and simplify.

6. Transform the equation of the illustrative example in Art. 4.8,

$$x^2 - 2xy + y^2 - x + 2y = 4,$$

by rotating axes through 45°, and obtain $4y'^2 + x'\sqrt{2} + 3y'\sqrt{2} = 8$. Then, by the proper choice for translation of axes, change the equation to $4y''^2 = -x''\sqrt{2}$. Finally, draw all three sets of axes and the graph of the given equation.

7. What do the coordinates of the point $(-2, 5)$ become if the axes are rotated through an angle of 36.4°? Solve by aid of the equations of rotation, and check by a graphical solution.

8. Figure 4.29 is intended to suggest an airplane climbing upward. Let L be the lift force, D the drag force, N the force perpendicular to the direction of climb, and T the force in the direction of climb. Show that $N = L\cos\phi + D\sin\phi$ and $T = -L\sin\phi + D\cos\phi$, and also determine a similar set of formulas for L and D in terms of N, T, and the angle of climb ϕ.

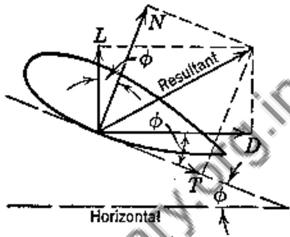


FIG. 4.29

9S. Given the equation $x^2 + y^2 + ax + by + c = 0$. If axes are rotated through an angle θ, let the new equation be $x'^2 + y'^2 + Ax' + By' + C = 0$. Show that $A^2 + B^2 - 4C$ is equivalent to $a^2 + b^2 - 4c$, independent of the angle θ through which the axes are rotated. Then give a geometric reason why this should be so.

10S. In many algebra textbooks, in the chapter on simultaneous equations, a method is given for solving simultaneously two equations that are both unchanged when x and y are interchanged. The method is to introduce two new variables that are defined by $x = u - v$ and $y = u + v$.

(a) Show that the stated test is equivalent to the requirement of one type of symmetry, and identify the type.

(b) Show that the substitutions in terms of u and v amount to a rotation of axes (together with a change of scale), and tell through what angle the axes are rotated.

4.10 The General Conic by Rotation of Axes

We start with the general equation of the second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C cannot simultaneously be zero. We rotate axes through an angle θ by use of

$$x = x'\cos\theta - y'\sin\theta$$

and

$$y = x'\sin\theta + y'\cos\theta$$

and obtain after some simplification:

$$\begin{aligned} x'^2(A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta)& \\ + x'y'(-2A\sin\theta\cos\theta + B\cos^2\theta - B\sin^2\theta + 2C\sin\theta\cos\theta)& \\ + y'^2(A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta)& \\ + x'(D\cos\theta + E\sin\theta) + y'(-D\sin\theta + E\cos\theta) + F = 0.& \end{aligned}$$

If we wish to simplify the resulting equation so that no term in $x'y'$ is present, that is, so that it will be a simple conic such as we studied earlier in this chapter, we can try to choose the angle θ so that the coefficient of $x'y'$ will be zero. Then

$$B(\cos^2\theta - \sin^2\theta) + (C - A)(2\sin\theta\cos\theta) = 0$$

or

$$B\cos 2\theta + (C - A)\sin 2\theta = 0.$$

If $A - C \neq 0$, we can write this equation in the form

$$\tan 2\theta = \frac{B}{A - C}.$$

If $A - C = 0$ and if $B = 0$, the graph is a circle, and rotation of axes is unnecessary. If $A - C = 0$ and $B \neq 0$, the choice of θ such that $\cos 2\theta = 0$, whence $\theta = 45°$ is one solution, will make the coefficient of $x'y'$ zero. The preceding equation shows that a proper choice of θ can always be made to eliminate the term in $x'y'$. The basic equation, stated in words, is

$$\tan 2\theta = \frac{\text{Coefficient of } xy}{\text{Coefficient of } x^2 \text{ minus Coefficient of } y^2}.$$

If it is desired to carry through an *exact* solution, then it will be convenient to use any *one* of the three half-angle identities that the student learned in trigonometry ($\sin\theta = \pm\sqrt{(1 - \cos 2\theta)/2}$, $\cos\theta = \pm\sqrt{(1 + \cos 2\theta)/2}$, or $\tan\theta = (1 - \cos 2\theta)/\sin 2\theta$). For example, the values of $\sin 2\theta$ and $\cos 2\theta$ can be determined from the value of $\tan 2\theta$ and an appropriate reference triangle; from

$$\sin\theta = \pm\sqrt{\frac{1 - \cos 2\theta}{2}}$$

the exact values of $\sin\theta$ and $\cos\theta$ may be found with the use of a second reference triangle; and, finally, the equations of rotation may be written and substitutions made in the original equation. The student should always perform the simplifications to be certain that the coefficient of $x'y'$ is actually zero.

It is only an exceptional problem in which an exact solution can be performed without a large amount of arithmetic. Usually it will be more feasible to determine the angle θ to the requisite accuracy from the value of $\tan 2\theta$, thence to obtain the values of $\sin\theta$ and $\cos\theta$ from tables (or slide rule if only rough accuracy is desired), and finally to perform the algebraic substitution and simplification by aid of tables or a computing machine.*

The equation that results from this rotation of axes will have the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0.$$

Both A' and C' cannot simultaneously be zero, since we started with an equation of the second degree. If either A' or C' is zero, the locus is a parabola (or it may degenerate to a pair of parallel straight lines). If A' and C' are of the same sign, the locus will be an ellipse (or the locus may be a point ellipse or there may be no locus). If A' and C' are of opposite sign, the locus will be a hyperbola (or the locus may be a pair of intersecting straight lines). The exceptional cases are readily apparent, whether the method of addition of ordinates or the method of rotation of axes is used to draw the locus. With these remarks, we complete the proof for the use of the indicator $B^2 - 4AC$ that was described in Art. 4.8.

EXAMPLE

Rotate axes to eliminate the xy-term in the equation

$$4x^2 + 24xy + 11y^2 - 40x - 70y - 5 = 0.$$

Then draw the graph and determine the coordinates of the vertices in terms of the original axes.

Solution. In this case, $\tan 2\theta = -24/7$ and we may *choose* 2θ to terminate either in the second quadrant or in the fourth quadrant. Figure 4.30 shows one reference triangle for 2θ and the corresponding reference triangle for θ, since

$$\sin\theta = -\sqrt{\frac{1 - \cos 2\theta}{2}} = -3/5.$$

Thence $x = (4x' + 3y')/5$ and $y = (-3x' + 4y')/5$. The student should substitute these in the given equation and simplify it to $4y'^2 - x'^2 - 16y' + 2x' = 1$ or

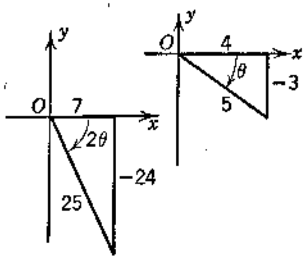


FIG. 4.30

* An aid to machine or table computation is given in the following reduction, made by rotating axes and partially simplifying the result:

Given $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and $\tan 2\theta = B/(A - C)$, then the new equation is

$$x'^2(A + C + B \csc 2\theta) + y'^2(A + C - B \csc 2\theta)$$
$$+ 2x'(D\cos\theta + E\sin\theta) + 2y'(-D\sin\theta + E\cos\theta) + 2F = 0.$$

$4(y' - 2)^2 - (x' - 1)^2 = 16$. The form of this last equation suggests that we employ the method of translation of axes and that the equations of translation are $x'' = x' - 1$ and $y'' = y' - 2$. We translate axes, therefore, to (1, 2) in the primed-axis notation, and obtain $4y''^2 - x''^2 = 16$. The graph of this hyperbola, together with all three sets of axes, is shown in Fig. 4.31.

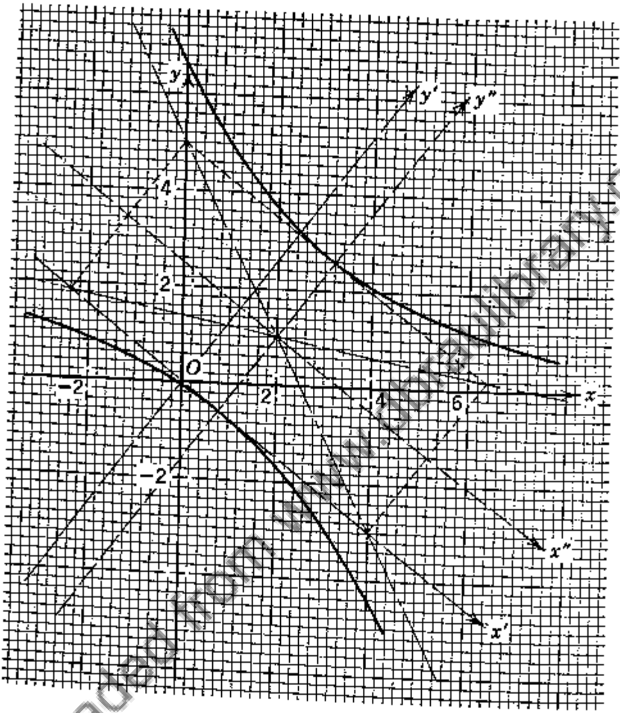


FIG. 4.31

In order to find the coordinates of the vertices in terms of the original axes we observe that the vertices are $x'' = 0$, $y'' = 2$, and $x'' = 0$, $y'' = -2$. The equation of translation may be rewritten in the form $x' = x'' + 1$ and $y' = y'' + 2$, and from these we may see that the coordinates of the vertices in terms of x' and y' are (1, 4) and (1, 0). We employ the equations of rotation to find the corresponding coordinates in terms of the original variables to be $(\frac{16}{5}, \frac{13}{5})$ and $(\frac{4}{5}, -\frac{3}{5})$. These could be checked on the graph.

The student may find it easier to draw his axes so that the primed axes are in horizontal and vertical positions on his sheet of graph paper and the original axes are inclined.

Second Solution. This particular problem involves an equation in which the coefficients are such that an exact solution is relatively easy. We use the numerical method as the second solution. Tables show that $2\theta = -73° 44.4'$,

$\theta = -36° 52.2'$, $\sin \theta = -0.60000$, and $\cos \theta = +0.80000$; the resulting equation is the same as before.

An excellent check on the entire process is to compute the coordinates of one or two points on the original locus and to determine whether these points are on the curve in the final graph.

PROBLEMS

1. (*a*) Work the illustrative example of this article with the angle 2θ such that $90° < 2\theta < 180°$.

(*b*) Draw the graph for the equation of this illustrative example by the method of addition of ordinates.

2. Rotate axes to eliminate the xy-term. Then translate axes and sketch. Show all three sets of axes on your figure. The numbers in the following equations have been chosen carefully so that either the exact method or the numerical method may be used. (Your answer may be the same as the answer given at the end of the text or it may have x'' instead of y'' and y'' instead of x'' with, perhaps, a change in the signs. These differences depend upon the particular choice made for the quadrant in which the angle 2θ terminates.)

(*a*) $x^2 - xy + y^2 - 4x - 4y = 20$.
(*b*) $24xy - 7y^2 + 36 = 0$.
(*c*) $4xy + 3x^2 = 4$.
(*d*) $9x^2 - 24xy + 16y^2 - 50x + 9 = 0$.
(*e*) $3x^2 - 4xy + 8x = 4$.
(*f*) $11x^2 - 24xy + 4y^2 + 30x + 40y = 45$.
(*g*) $3x^2 + 2xy + 3y^2 - 16x + 16y + 52 = 0$.
(*h*) $19x^2 + 6xy + 11y^2 - 26x + 38y + 31 = 0$.

3. Rotate axes to eliminate the xy-term and use a slide rule or tables to aid in the computation. Then translate axes and sketch the curve. Show all three sets of axes on your graph.

(*a*) $x^2 + 2xy + 3y^2 + 4x + 5y - 6 = 0$.
(*b*) $2x^2 + 3xy + 4y^2 + 5x - 6y = 7$.
(*c*) $2xy - 3y^2 - 4x - 7y = 11$.
(*d*) $x^2 - 4xy + 4y^2 + 3x - 6y = 4$.

4. From the equation $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$, determine by the exact method and by an approximate method the values of the sine and cosine of an angle of rotation to eliminate the xy-term.

5. Determine an angle of rotation to eliminate the xy-term in the equation $\sqrt{3}\, x^2 - 3xy + 3x + 4 = 0$.

6. Use the results of the illustrative example of this article to solve the following problems:

(*a*) Obtain the equations that express x and y in terms of x'' and y''.

(*b*) Use the results from (*a*), and find the coordinates of the center and vertices with respect to the original variables.

(*c*) Find the equations of the asymptotes in terms of x and y.

(*d*) Sketch the curve by the method of addition of ordinates, and determine from this graph the equations of the asymptotes.

7. Find the coordinates of the focus of

$$9x^2 + 24xy + 16y^2 - 140x - 20y + 400 = 0.$$

8. Find the length of the major axis of $17x^2 - 12xy + 8y^2 = 80$.

9. Find (*a*) by rotation of axes and (*b*) by addition of ordinates the equations of the asymptotes of $4x^2 - xy = 2$.

10. Find, correct to three significant figures, the acute angle between the positive x-axis and one asymptote of

$$5y^2 = 4xy + x^2 + 12.$$

11S. Derive the equation of rotation of axes that is given in the footnote in this article.

12S. Start with the equation $I_y x^2 + 2P_{xy}xy + I_x y^2 = k$ and solve the following problems:

(*a*) If axes are rotated through an angle θ such that the coefficient of $x'y'$ is zero and such that the new equation is $I_y'x'^2 + I_x'y'^2 = k$, obtain expressions for I_y' and I_x' in terms of the original coefficients and the angle θ, and show that $I_x + I_y = I_x' + I_y'$.

(*b*) If axes have been rotated as directed in (*a*), find the values of I_x' and I_y' and the angle θ if $I_x = 13.48$ in.^{4}, $I_y = 4.90$ in.^{4}, and $P_{xy} = -4.76$ in.^{4}

This problem arises in physics and in engineering mechanics.

4.11 Review

The student should review the outline in Art. 3.13, and add rotation of axes and the names of the various conics in their proper places in that outline.

REVIEW QUESTIONS

1. What are the two basic forms for the general equation of a circle? Derive one of these by a locus-derivation method. Which form would we use to find the equation of a circle that has its center at $(2, -3)$ and that is tangent to $x + 2y = 4$? Which form would we use to find the equation of a circle that goes through two given points and that is tangent to the y-axis?

2. What is the locus definition of a parabola? If the equation is in the form $x^2 = ky$, what is the distance between the vertex and the focus? What are two methods by which one may obtain the equation of the parabola whose axis is the y-axis, if the parabola goes through $(0, 4)$, $(2, 0)$, and $(-2, 0)$? What algebraic steps are required to change $y^2 + 6x = 4y + 2$ to $y'^2 = -6x'$? Could we check this algebraic manipulation by finding the coordinates of a point on the given locus (in particular, one of the intercepts) and by determining whether the curve, as drawn, does go through that point? Could we check the work in this example by finding the coordinates of the vertex and showing that these coordinates satisfy the original equation?

3. What is the locus definition of the ellipse? What is the fundamental relation between a, b, and c? Sketch an ellipse, and show and label the associated right triangle. What algebraic steps are necessary to change

$$2x^2 + 4y^2 - 4x + 8y - 19 = 0$$

to $2x'^2 + 4y'^2 = 25$? How could we check this algebraic manipulation after we have drawn the graph of an ellipse such as the preceding example? What is a procedure for finding the equation of the ellipse that goes through (6, 2) and (1, 4), and that is symmetrical with respect to both axes? What are two different methods for finding the equation of the ellipse that has its vertices at (4, 2) and (4, −8) and its foci at (4, 0) and (4, −6)?

4. What is the locus definition of the hyperbola? What equation relates a, b, and c? Sketch a hyperbola, and show and label the basic right triangle. What algebraic steps are necessary to change $7x^2 - 4y^2 - 28x + 8y + 54 = 0$ to $4y'^2 - 7x'^2 = 30$? What is a method to be used to find the equation of a hyperbola with a focus at (4, 1), a vertex at (4, −1), and its center at (4, −2)? How may we obtain the equation of a hyperbola that is symmetrical with respect to both axes, that has $2y = x$ as one asymptote, and that goes through (1, 5)? Can we obtain the equations of the asymptotes of a hyperbola, assuming that we have drawn its graph, if the coordinates of the center are known and if a and b are known?

5. What methods may be used to sketch the locus of the equation

$$4x^2 - 4xy + y^2 - x - 2y + 1 = 0?$$

How may we show that it is either a parabola or a degenerate locus? What methods may be used to obtain the coordinates of the vertex and the equation of the axis of this parabola (assuming that it is not a degenerate locus)?

6. If axes are to be rotated through a positive acute angle whose tangent is ½, what are the equations of rotation? Through what positive acute angle (correct to the nearest tenth of a degree) should the axes be rotated in order to eliminate the xy-term in the equation $x^2 + 2xy + 3y^2 + 4x + 5y + 6 = 0$? What is the value of the indicator $B^2 - 4AC$ for this equation, and what would this suggest as the locus of the preceding equation? What else might it possibly be as a locus?

REVIEW PROBLEMS

1. Identify and sketch the following curves:

(a) $2x^2 + 2y^2 + 4x = 5y + 3$.
(b) $xy + 2x + 3y = 4$.
(c) $3x^2 + 5y^2 + 6x = 4 + 20y$.
(d) $y = 4x - x^2$.
(e) $2x^2 - 11y^2 = 25x + 7$.
(f) $2x^2 + 3xy + y^2 = 7 - 4x$.
(g) $y = 3(x - 2)^2$.
(h) $(x - 2y)(x + 2y) = 5$.
(i) $5x^2 - 4xy + 4y^2 - 6x - 4y = 11$.
(j) $x^3 + y^3 = 8$.

(k) $x^4 + y^4 = 16$ and $x^2 + y^2 = 4$ on the same graph (compute y when $x = 1$ on both curves).

(l) $\tan(\theta/2) = (1 - \cos\theta)/(\sin\theta)$ if the dependent variable is $\tan(\theta/2)$ and the independent variable is $\sin\theta$.

(m) $y = 2(x - 1)^2 + 1.45(x + 1)^3$.

(n) $y = 2x(x + 1)(x - 1)^2$.

(o) $E = Ir$, first, if r is a positive constant and second, if E is a positive constant.

2. Find the equation of the locus of a point that moves so that its numerical distance from the line $x = 2$ is always 3 less than its distance from the point (4, 0). Show that the locus is made up of arcs of two parabolas, and draw the locus.

3. A ladder is 20 ft. long and has 19 rungs spaced 1 ft. apart and each end rung is 1 ft. from its end of the ladder. The ladder is placed against a vertical wall (more than 20 ft. high), and the bottom end of the ladder is pulled outward along the horizontal ground in such a way that the upper end of the ladder is always in contact with the wall. Show that the locus for each of the 19 rungs is an ellipse.

4. Draw the locus of $17x^2 + 12xy + 8y^2 = 50$ by both addition of ordinates and rotation of axes. Use the same scales, so that your two graphs will be identical.

CHAPTER 5

Transcendental Curves

The graph of any polynomial that can be written in the form:

$$(\text{Polynomial in } x \text{ and } y) = 0, \quad \text{or} \quad P(x, y) = 0,$$

is called an *algebraic curve*. The straight line and the conics are all examples of such curves. Equations in x and y that are not representable in this form lead to transcendental curves, which are to be studied in this chapter. Examples of such equations are $y = \sin x$, $y = \tan x$, $y = \log_a x$.

5.1 Change of Scale

Before beginning a study of some further types of curves it will be convenient in the first two articles of this chapter to extend two of the basic methods of curve sketching.

The method of translating axes to simplify a given equation requires that we replace $x - h$ by x' and $y - k$ by y'; the coefficients of x and y must each be 1. When rotating axes we replace x by $x' \cos\theta - y' \sin\theta$, and y by $x' \sin\theta + y' \cos\theta$. In both cases we may be able to simplify the given equation by replacing each of x and y by expressions involving two new variables. We consider the graphical meaning of another type of change of variable when we substitute x' for αx and y' for βy. We shall see in the following two examples that the graphical meaning of these replacements is to stretch or to compress the graph but not to change its essential shape.

EXAMPLE 1

Sketch the graph of $x^2 + 4y^2 = 25$ by use of the graph of $X^2 + Y^2 = 1$.

Solution. We may write the original equation as $(x^2/25) + (4y^2/25) = 1$. The form of this equation suggests the replacements $X = x/5$ and $Y = 2y/5$. We make the indicated substitutions and obtain the second of the two given equations, whose graph is a circle with center at the origin and radius 1. From $X = x/5$, we note that when $X = 1$, $x = 5$; when $X = 2$, $x = 10$; when $X = 0.1$, $x = 0.5$; etc. This means that we are to stretch the circle in the X-direction in the ratio 5 to 1. Likewise, from $Y = 2y/5$ we see that for $Y = 1$, $y = 5/2$; for $Y = 2$, $y = 5$;

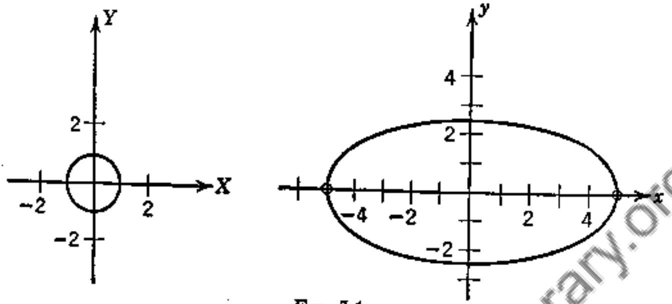


FIG. 5.1

etc. Thus we are to stretch the circle in the Y-direction in the ratio 5 to 2. The graphs of the circle and the resultant curve, the ellipse, are both shown in Fig. 5.1.

EXAMPLE 2

Sketch the graph of $y = cx/(x^2 + n^2)$.

Solution. We change the equation by the rules of algebra to the form

$$\frac{ny}{c} = \frac{x/n}{(x/n)^2 + 1}.$$

We substitute y' for ny/c and x' for x/a, and obtain the new and simpler equation $y' = x'/(x'^2 + 1)$. We sketch its graph by the discussion method and show the result in Fig. 5.2. In order to obtain the scales for the graph of the given equation, we note that when $y' = 1$, $y = c/n$; when $x' = 1$, $x = n$. We show the x- and y-scales alongside the x'- and y'-scales in the figure.

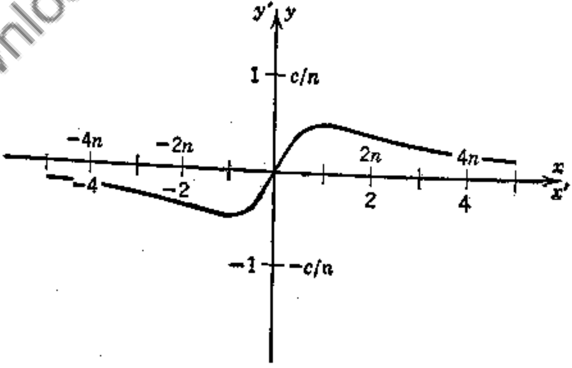


FIG. 5.2

These two examples illustrate the following theorem.

THEOREM. *The effect of the substitutions x' for αx and y' for βy is to distort the graph in both directions.* If $\alpha > 1$, the graph in the $x'y'$ plane must be compressed in the x' direction in the ratio of α to 1, since for $x' = 1$, $x = 1/\alpha$; for $x' = 2$, $x = 2/\alpha$; etc. If $\alpha < 1$, the graph in the $x'y'$ plane must be stretched in the x' direction in the ratio of α to 1.

It is unnecessary to memorize whether the effect is to stretch or to compress, or in what ratio, since this will be easy to determine in any problem. The essential use of this theorem will be in visualizing the shape of the graph of a given equation from that of a simpler equation.

We note that the effect of a replacement such as $x' = ax + h$ would be both to translate axes and to distort the scale. Thus, we could first replace ax by X (a distortion), and then $X + h$ by x' (a translation). Or, since $x' = a(x + h/a)$, we could first replace $x + h/a$ by X (a translation), and then aX by x' (a distortion).

5.2 Composition of Ordinates

In the third chapter we learned to sketch rapidly any curve whose equation could be written in the form $y = f(x) + g(x)$, providing the graphs of $y_1 = f(x)$ and $y_2 = g(x)$ were easy to sketch. In this article we shall extend the idea of addition of ordinates by the following theorems whose proofs are self-evident:

THEOREM. *The graph of $y = f(x) \cdot g(x)$ may be obtained from the graphs of $y_1 = f(x)$ and $y_2 = g(x)$ by multiplying the ordinates of these two component curves.*

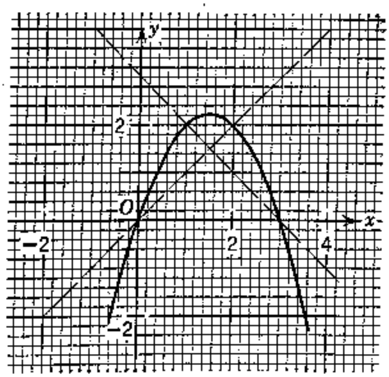


FIG. 5.3

EXAMPLE 1

Sketch the graph of $y = 3x - x^2$ by multiplying ordinates.

Solution. We rewrite the equation in the form $y = x(3 - x)$ and introduce $y_1 = x$ and $y_2 = 3 - x$. We show the graphs of these two straight lines in Fig. 5.3. The ordinates of these two straight lines for a given abscissa may be read from the graph and multiplied mentally or by aid of a slide rule. For the given abscissa, the resulting product is used as the ordinate of a point on the required curve. We repeat this operation until we have located enough points to draw the curve. The student should notice that the three numbers, -1, 0, and $+1$, are especially convenient multipliers.

THEOREM. *The graph of $y = f(x)/g(x)$ may be obtained from the graphs of $y_1 = f(x)$ and $y_2 = g(x)$ by division of ordinates.*

The basic ideas expressed in these two theorems and in the process of addition of ordinates may be extended to still other operations. For example, we could obtain the graph of $y = \sqrt[3]{4 - x^2}$ from the graph of $Y = 4 - x^2$ by finding the cube roots of the ordinates to the second curve. We could also obtain the required graph from the graph of $y' = \sqrt{4 - x^2}$ by raising each ordinate to the $\frac{2}{3}$ power. (A slide rule would be helpful in either case.)

Similarly, we could obtain the graph of $y = 1/(x^2 + 1)$ from the graph of $Y = 1/(X + 1)$ by determining the square roots of the abscissas X, since $X = x^2$ when $Y = y$, and $x = \pm\sqrt{X}$. Alternatively, we could obtain the required graph by finding the reciprocals of the ordinates to the curve $Y = X^2 + 1$.

EXAMPLE 2

Sketch the graph of $y = \frac{1}{4}(1 - x^2)^2$ by squaring ordinates.

Solution. In Fig. 5.4 we show the graph of the parabola $Y = \frac{1}{2}(1 - x^2)$. Since $y = Y^2$, we read several ordinates of this parabola, square them, plot the results for the corresponding abscissas, and obtain the second curve shown in that figure.

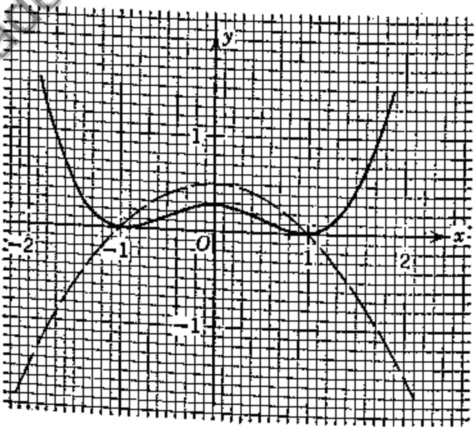


FIG. 5.4

PROBLEMS

1. Sketch the graph of $y = 3/x$ by use of the graph of $Y = 1/x$.
2. Sketch the graph of $y = 2x - 4x^2$ by aid of the graph of $y = X - X^2$.
3. Sketch the graph of $y = \sqrt{4 - 9x^2}$ by use of the graph of $y = \sqrt{4 - X^2}$.
4. Sketch the graphs of the following by use of multiplication of ordinates:

(a) $y = x\sqrt{4 - x^2}$.
(b) $y = x(4x - x^2)$.
(c) $y = x^2(4 - x)$.
(d) $y = (x - 1)(x^2 + 1)$.
(e) $y = (4 - x^2)^{1/2}(9 - x^2)^{1/2}$.
(f) $y = (1/x)(4 - x)$.

5. Sketch the graph of $Y = 6x - x^2$ and use this graph to obtain the graph of $y = \sqrt{6x - x^2}$.
6. Use the graph of $Y = 6x - x^2$ to obtain the graph of $y = (6x - x^2)^2$. What is true of the squares of numbers less than 1?
7. Sketch the graph of $y = 1/(x^2 + 1)$ by use of the graph of $y = 1/(X + 1)$.
8. Sketch $y^2 = x^4 - 16x^2$ by multiplication of ordinates.
9. Sketch the graph of $X^2 - Y^2 = 1$ using the same scale on both axes. Then, without changing anything but the scales on the two axes, give the distorted graph of each of the following:

(a) $4x^2 - 9y^2 = 1$.
(b) $x^2 - 2y^2 = 1$.
(c) $4x^2 - 8y^2 = 27$.
(d) $x^2 - y^2 = 4$.

10. Draw the graph of $X^2 + Y^2 = 1$. Then, without changing anything but the scales on the two axes, alter the same graph so that it will be a graph of:

(a) $x^2 + y^2 = 25$.
(b) $x^2 + y^2 = 0.0001$.
(c) $4x^2 + 9y^2 = 1$.
(d) $4x^2 + 9y^2 = 36$.

5.3 The Graphs of the Sine and Cosine Functions

The student has already learned in trigonometry what the graphs of $y = \sin x$ and $y = \cos x$ look like. These, however, are shown in Fig. 5.5. If the student has not previously plotted undistorted graphs of these two curves, he should do so as an aid to memorizing them. The graphs show the abscissa labeled in radian measure, and both curves are periodic or repeat after 2π radians.

DEFINITION. *A graph of $y = f(x)$ is said to be periodic of period p if p is the smallest positive number such that $f(x + p) \equiv f(x)$.*

The smallest value of p that will make true both $\sin(x + p) \equiv \sin x$ and $\cos(x + p) \equiv \cos x$ is $p = 2\pi$. We conclude that the graphs of $y = \sin x$ and $y = \cos x$ are both periodic, of period 2π.

By aid of the graphs of the two basic equations, $y' = \sin x'$ and $y' = \cos x'$, we can sketch rapidly the graphs of $y = a \sin bx$ and

$y = a \cos bx$. Assuming that $a > 0$ and $b > 0$, we may make the substitutions $y' = y/a$ and $x' = bx$ to reduce the new equations to the form of the basic equations. We see, therefore, that the graphs of these new equations will have the same general shapes as those of the basic equations. Thus, the graph of $y = a \sin bx$ will have a zero ordinate when $bx = 0$, a peak or maximum value when $bx = \pi/2$, another zero ordinate when $bx = \pi$, a minimum or trough value when $bx = 3\pi/2$, and another zero value when $bx = 2\pi$. If $a < 0$, one cycle of the sine wave, for example, will start at the origin and proceed

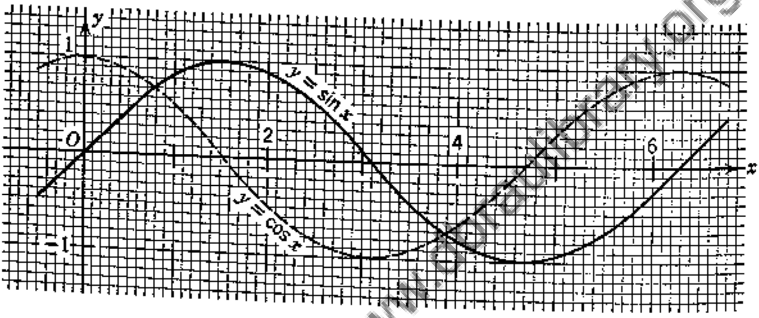


FIG. 5.5

in a *downward* direction. These two curves, $y = a \sin bx$ and $y = a \cos bx$ are *periodic*, of period $2\pi/b$. The largest numerical value of the ordinate, which is $|a|$ in both cases, is the *amplitude* or peak value; the amount of the curve traversed in 1 abscissa unit is the *frequency* $= f = 1/\text{period} = b/2\pi$.

From these properties the student can sketch the graphs of $y = a \sin bx$ and $y = a \cos bx$ with rapidity and with fair accuracy. A suggested sequence of steps is as follows (assuming $a > 0$):

1. Determine the amplitude a, which is the multiplier of the sine or cosine function.

2. Determine the period $p = 2\pi/b$ either by memorizing this relationship or by comparing the equation of the required curve with the equation of the basic curve. Thus, for example, since the graph of $y = \sin x$ has a period of 2π, the graph of $y = 2 \sin 5x$ will have a period that can be determined by solving $5x = 2\pi$.

3. Lay off an abscissa equal in length to the period p. Notice that the sine wave starts at the origin and the cosine wave starts at its peak value on the y-axis (assuming that the coefficient of the cosine function is positive).

4. Next locate the peak or maximum points and the trough or minimum points for this first period and the points on the x-axis where the curve crosses. Additional points may be plotted if greater accuracy is desired (special angles such as 30° or 60° yield quick results). Additional periods or cycles may be drawn if desired.

EXAMPLE 1

An electric generator delivers a voltage that is given approximately by $e = 169 \sin 120\pi t$ volts. Sketch the wave for one period.

Solution. The required graph, shown in Fig. 5.6, is the graph of a typical so-called 120-volt, 60-cycle household voltage wave. Actually the household voltage

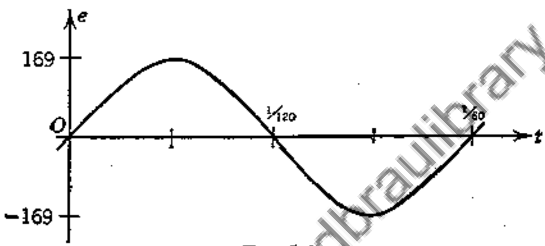


FIG. 5.6

wave is likely to have some higher harmonics present (see Problem 9 at the end of this article). The reason that the voltage is called 120 volts when the peak value is about 169 volts is explained in courses on electricity.

EXAMPLE 2

A weight vibrates on a spring in such a way that its distance measured from a reference position is always given by $y = 4 + 1.70 \cos 40t$, where y is in inches and t is in seconds. Sketch the graph for one period.

Solution. The graph is shown in Fig. 5.7. Notice that, whereas the graph could be obtained by using the method of translation of axes, it is just as easy to think of adding $y = 4$ to $y = 1.70 \cos 40t$ by addition of ordinates. In this particular problem the peak displacement is 5.7 in., whereas the amplitude of vibration is 1.7 in.

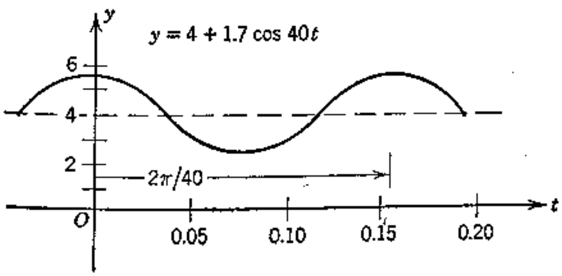


FIG. 5.7

The graph of $y = a \sin (bx + \theta)$, where a, b, and θ are constants, could be obtained by the method of translating axes. Thus, we could rewrite $y = 4 \sin (5x + 60°)$, for example, or the preferable form,* $y = 4 \sin (5x + \pi/3)$, in the form

$$y = 4 \sin 5(x + \pi/15) = 4 \sin 5(x + 0.209);$$

we could then translate axes to $(-0.209, 0)$, and sketch $y' = 4 \sin 5x'$. An easier method for many such curves is first to sketch

$$y = a \sin (X + \theta),$$

where the translation is evident, and then to change the abscissa scale by aid of the substitution $X = bx$ (which implies replacing the abscissa numbers X by X/b).

EXAMPLE 3

Sketch the graph of $i = 4.77 \sin (120\pi t + \pi/3)$, where t is in seconds and i is in amperes.

Solution. We first sketch the graph of $i = 4.77 \sin (X + \pi/3)$ and then change the scale on the axis of abscissas by dividing each scale number by 120π. Figure 5.8 shows the graph of the required curve and also that of $i = 4.77 \sin 120\pi t$ as a

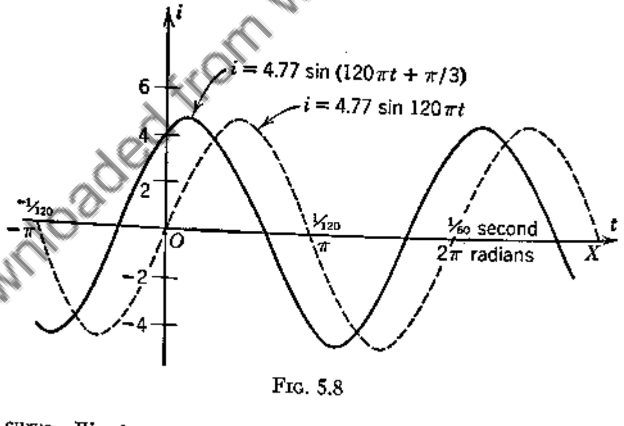


FIG. 5.8

dotted curve. We observe that the graph of $i = 4.77 \sin (120\pi t + \pi/3)$ *leads* the graph of $i = 4.77 \sin 120\pi t$ by $\pi/3$ radians on the X-scale or by $(\pi/3) \div 120\pi = 1/360$ sec. on the time scale.

* Since bx is to be measured in radians the angle θ should likewise be measured in radians. In the example, $120\pi t$ is in radians and t in seconds so that the coefficient 120π is in radians per second.

DEFINITION. *The graph of* $y = a \sin (bx + \theta)$ *leads the graph of* $y = a \sin bx$ *by the angle* θ *if* θ *is positive, and lags if* θ *is negative.* In electrical engineering this angle θ is called the phase angle.

Referring again to Fig. 5.8, we observe that the solid curve crosses the time axis $\frac{1}{360}$ sec. ahead of, or in front of, or at an earlier time than, the dotted curve. This terminology corresponds to the statement that one runner *leads* a second runner in a race if he reaches the finish mark at an earlier time than the second runner, and *lags* if he reaches the finish mark at a later time.

We may draw the graph of Fig. 5.8 by another line of reasoning that goes as follows: We first draw the graph of $i = 4.77 \sin X$ and use this graph as the basis from which we draw the required graph. Since the comparison implies that $X = 120\pi t + \pi/3$, we may solve for the crossing points and peak and trough points for one period by equating $120\pi t + \pi/3$ to the X-value of each such point and solving for t. Thus, in the present example, one full period of $i = 4.77 \sin X$ starts at $X = 0$ and ends at $X = 2\pi$. Then a full period of the required curve starts at $120\pi t + \pi/3 = 0$, or $t = -\frac{1}{360}$, and ends at $120\pi t + \pi/3 = 2\pi$, or $t = \frac{5}{360} = \frac{1}{72}$. From this information we could draw the required curve, since we know it will have the same general shape as the basic sine curve.

This method of reasoning illustrates the idea that we may sketch the graphs of some curves by comparing them with basic curves whose graphs we know or can draw easily. We could utilize the same sequence of steps to sketch, for example, $y = 2 \sin (x^2)$, or, alternatively, in this example, we could take the square roots of the abscissas to the curve $Y = 2 \sin X$ and lay off the new curve (or we could introduce a distorted scale which would have the same effect).

PROBLEMS

1. Sketch rapidly each of the following curves, and show at least two periods. List the period, frequency, and amplitude.

(a) $y = 4 \sin 3x$.
(b) $y = 5 \cos 10x$.
(c) $y = 2 + 3 \cos \pi x$.
(d) $e = 50 + 25 \sin 60\pi t$.
(e) $y = 3 + 2 \sin (2x + 4)$.
(f) $y = 5 - 2 \cos (3x + 6)$.
(g) $F = 50 - 20 \sin 120\pi t$.
(h) $F = 2.56 \sin (1000\pi t - \pi/4)$.
(i) $y = 4 - 5 \cos (3x + 7)$.
(j) $L = 465 \sin 500\pi t$.

2. Sketch on adjoining graphs all three of the following curves: $x = 2 \sin 10\pi t$, $v = 20\pi \cos 10\pi t$, and $a = -200\pi^2 \sin 10\pi t$.

3. A tuning fork gives off the tone for middle C and hence the frequency is 256 cycles per second with an amplitude a that depends on several things. Sketch the graph of $y = a \sin (256)(2\pi t)$ for two cycles. Then sketch on the same axes the graph for C above middle C if its frequency is 512 cycles per second.

4. A radio station has a carrier frequency of 800 kilocycles or 800,000 cycles per second. Sketch a graph of $e = E \sin 1{,}600{,}000\pi t$ for two cycles (assume that E is a positive constant).

5. Sketch the graph of $y = 10 \sin 2x + 5 \cos 2x$ by addition of ordinates. Then rewrite the equation as follows:

$$y = \sqrt{100 + 25}\left[\frac{10}{\sqrt{125}} \sin 2x + \frac{5}{\sqrt{125}} \cos 2x\right] = \sqrt{125} \sin (2x + \phi),$$

where $\cos \phi = 10/\sqrt{125}$, $\sin \phi = 5/\sqrt{125}$, whence $\tan \phi = \frac{1}{2}$. Then $\phi = 26.6° = 0.464$ radian and $y = 11.18 \sin (2x + 0.464)$. Now sketch this last curve. Your two graphs must, of course, be the same.

6. Sketch by either method suggested in Problem 5:

(a) $y = 40 \sin 3x + 30 \cos 3x$.
(b) $y = 100 \sin 20\pi t + 20 \cos 20\pi t$.
(c) $F = 60 \sin 100\pi t - 80 \cos 100\pi t$.

Notice that the angles in the two trigonometric functions must be the same if the second method is to be used.

7. Draw a careful graph of $y = \sin x$, and then draw the tangent line at the origin (use a straightedge and draw the tangent line by eye). If accurately done, the tangent line should go through the point (1, 1). What does this graph assert about an approximate relationship between $\sin x$ and x radians for small angles x?

8. Sketch rapidly:

(a) $y = 2 \cos (\pi x/4)$.
(b) $y = 1 - \sin x$.
(c) $y = 1.41 (\sin x + \cos x)$.
(d) $y = \sin (x + \pi/6)$.
(e) $y = \sin x + \sin (\pi/6)$.
(f) $y = \cos^2 x = (1 + \cos 2x)/2$.
(g) $\sin x = \frac{1}{2}$.
(h) $\cos x + 1 = 0$.

9. An electric voltage is composed of a direct-current component of 40 volts, a fundamental or first harmonic of 30 volts at 60 cycles per second, and a third harmonic of 10 volts that lags by $60° = \pi/3$ radian. Stated in equation form, this voltage is

$$e = 40 + 30 \sin 120\pi t + 10 \sin (360\pi t - \pi/3).$$

Use addition of ordinates to sketch this curve. *Suggestion:* Let $X = 120\pi t$ and sketch the curve $e = 40 + 30 \sin X + 10 \sin (3X - \pi/3)$. Then change the X-scale into a t-scale by use of the equation relating the two variables.

10. *Estimate* the area under one arch of each of the following curves:

(a) $y = 4 \sin 2x$.
(b) $y = 3 \cos \pi x$.
(c) $y = 2 \sin (10\pi x - \pi/4)$.

11. Sketch on the same graph:

(a) $y = 2 \sin x$ and $y = \sin 2x$.
(b) $y = \sin (x + \pi/4)$ and $y = \sin x + \sin (\pi/4)$.
(c) $y = \cos x + 2$ and $y = \cos (x + 2)$.
(d) $y = (\sin 2x)/2$ and $y = \sin x$.
(e) $y = \sin x$ and $y = \sin^2 x$.
(f) $y = \cos^2 x$ and $y = 0.5 + 0.5 \cos 2x$.

12. Estimate to one decimal from a graph, and check by tables, the values of the following (the angle is in radian measure):

(a) $\sin (1)$.
(b) $\sin (5)$.
(c) $\cos (8)$.
(d) $\sin (2\pi + 2)$.
(e) $\cos (2.5\pi + 1)$.

13. Sketch the following curves by comparing them with one of the basic graphs ($y = a \sin x$ or $y = a \cos x$):

(a) $y = 2 \sin (x^2)$.
(b) $y = \sin (1/x)$.
(c) $y = \cos (1/x)$.
(d) $y = \sin \sqrt{x}$.
(e) $y = \cos (x^3)$.
(f) $y = \sin (2 \sin x)$.

14. Prove algebraically that the graphs of $y = a \sin bx$ and $y = a \cos bx$ are symmetrical with respect to the origin and y-axis respectively.

15S. Sketch on the same graph for $-\pi < x < \pi$:

(a) $y = (\pi/4)(\pi/2 - x)$ for $0 < x < \pi$ and $y = (\pi/4)(\pi/2 + x)$ for $-\pi < x < 0$.
(b) $y = \cos x$.
(c) $y = \cos x + (1/9) \cos 3x$.
(d) $y = \cos x + (1/9) \cos 3x + (1/25) \cos 5x$.
(e) $y = \cos x + (1/9) \cos 3x + (1/25) \cos 5x + (1/49) \cos 7x$.

Can you guess what the graph of $y = \cos x + (1/9) \cos 3x + \cdots$ (with no last term) looks like between $x = -\pi$ and $x = +\pi$?

16S. Given $e = 141.4 \sin (157t + \pi/4)$ volts and $i = 7.07 \sin (157t - \pi/6)$ amp. Sketch graphs of e in terms of t, i in terms of t, and $p = ei$ in terms of t. Then show that the equation for p can be reduced to

$$p = 500 \cos 75° - 500 \cos (314t + \pi/12).$$

5.4 Graphs of the Other Trigonometric Functions and of the Inverse Trigonometric Functions

The graphs of $y = a \tan bx$, $y = a \cot bx$, $y = a \sec bx$, and $y = a \csc bx$ can all be obtained from the graphs of the sine and cosine functions by use of division of ordinates. For example, we can sketch a graph of $y = 4 \tan 2x$ by sketching $y_1 = 4 \sin 2x$ and $y_2 = 1 \cos 2x$. Then, since $\tan 2x = \sin 2x/\cos 2x$, it follows that $y = y_1/y_2$; hence we are to divide the ordinates to the sine curve by the corresponding ordinates to the cosine curve. Whenever the cosine function is zero, the graph of the tangent function will necessarily have a vertical asymptote. (Why?) The graph is shown in Fig. 5.9.

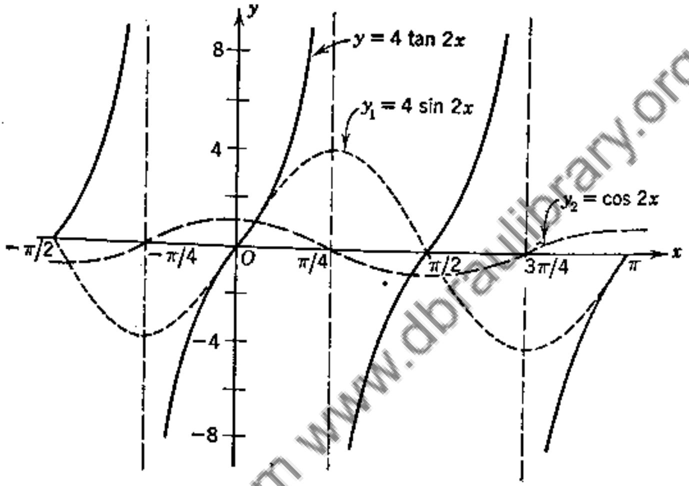


FIG. 5.9

EXERCISE FOR THE STUDENT. Show that the graph of $y = a \tan bx$ has a period of $p = \pi/b$ and that it has *no* amplitude, but that a is the value of y when $bx = \pi/4 = 45°$.

The graphs of the inverse trigonometric functions are easy to obtain from the graphs of the standard trigonometric functions. Thus, $y = \text{arc} \sin x$ or $y = \sin^{-1} x$ is equivalent to $x = \sin y$. Similarly, $y = 2 \text{ arc} \cos 3x$, or y is twice an angle whose cosine is $3x$. may be written successively in the following forms:

1. $y/2 = \text{arc} \cos 3x$, or y divided by 2 is an angle whose cosine is $3x$;
2. $\cos (y/2) = 3x$, or the cosine of the angle $(y/2)$ is $3x$;
3. $(\frac{1}{3}) \cos (y/2) = x$. This has an amplitude of $\frac{1}{3}$ and a period (which we obtain by equating $y/2$ to 2π) of $y = 4\pi$. The student should sketch the graph to complete the exercise.

Example

Sketch the graph of $y = 1 + 2\cos^{-1}(x+1)/3$.

Solution. We rewrite this equation successively in the following forms:

$$y - 1 = 2\cos^{-1}\frac{x+1}{3}; \quad \frac{y-1}{2} = \cos^{-1}\frac{x+1}{3};$$

$$\cos\frac{y-1}{2} = \frac{x+1}{3}; \quad \text{and} \quad x + 1 = 3\cos\tfrac{1}{2}(y-1).$$

We translate axes to $(-1, 1)$ and sketch $x' = 3\cos 0.5y'$ on the new axes. The amplitude is 3 and the period is 4π. The graph is shown in Fig. 5.10.

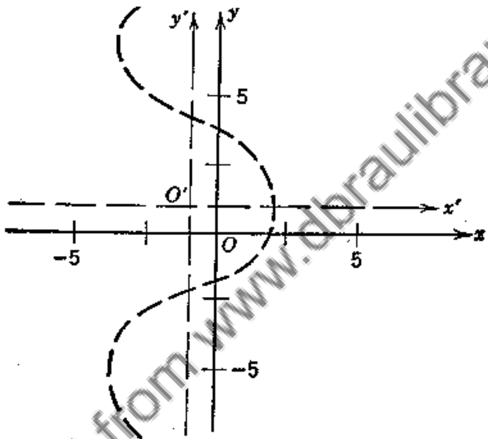


Fig. 5.10

PROBLEMS

1. Sketch rapidly the graphs of the following equations; show at least one complete period and give the value of the period:

(*a*) $y = 0.9\tan 2x$.
(*b*) $y = 1.3\cot(x-1)$.
(*c*) $y = \tan(x + \pi/4)$.
(*d*) $y = \tan x + \tan(\pi/4)$.
(*e*) $y = 2\tan 2\pi x$.
(*f*) $y = 1 + 3\cot(x/2)$.
(*g*) $y = 2 \text{ arc } \sin x$.
(*h*) $y = 1 + 0.5 \text{ arc } \cos 2x$.
(*i*) $y = \text{arc } \tan(0.5x + 1) - 1$.
(*j*) $y = \cot^{-1} 0.5x$.
(*k*) $s = 2\sec 2t$.
(*l*) $y = 3\csc \pi x$.
(*m*) $y = 4\csc 5\pi t$.
(*n*) $F = 4.25\,|\sec 10\pi t\,|$.
(*o*) $y = 2\,|\sin 100\pi t\,|$.
(*p*) $F = 3\,|\cos 200\pi t\,|$.
(*q*) $y = (\frac{1}{2}) \text{ arc } \sin 2x$.
(*r*) $y = (\frac{1}{3}) \text{ arc } \cos 3x$.
(*s*) $y = (1/\pi) \text{ arc } \sin(x/3)$.
(*t*) $y = (2/\pi)\cos^{-1}(x/10)$.
(*u*) $y = 2 + 2\sin^{-1}(x-1)$.
(*v*) $s = \pi + \frac{1}{2} \text{ arc } \cos(t+1)$.

2. Sketch the graph for the principal values of the following inverse trigonometric functions (the range for the principal values is listed with each function):

(*a*) $y = \text{arc sin } x, -\pi/2 \leqq y \leqq \pi/2$.
(*b*) $y = \text{arc cos } x, 0 \leqq y \leqq \pi$.
(*c*) $y = \text{arc tan } x, -\pi/2 < y < \pi/2$.
(*d*) $y = \text{arc cot } x, 0 < y < \pi$.
(*e*) $y = \text{arc sec } x, 0 \leqq y < \pi/2$ and $-\pi \leqq y < -\pi/2$.
(*f*) $y = \text{arc csc } x, 0 < y \leqq \pi/2$ and $-\pi < y \leqq -\pi/2$.

3. Estimate the area between the curve $y = 2 \text{ arc sin } 0.5x$, the y-axis, $y = 0$ to $y = 2\pi$.

4. Using only principal values, sketch by addition of ordinates the graph of $y = \text{arc sin } x + \text{arc sin } 2x$.

5. Sketch a graph of $y = \tan x + \cot x$ for $-\pi/2 < x < \pi/2$. Then show that this function of x may also be expressed as $y = 2 \csc 2x$.

6. Plot careful graphs of the following equations for $0 < x < 4$:

$$y = 2 \sin (\pi x/4), \quad y = x(4 - x)/2, \quad \text{and} \quad y = \sqrt{4x - x^2}.$$

7. Plot on the same graph for $-1 < x < 1$: $y = \tan (\pi x/2)$ and $y = 3x/(2 - 2x^2)$. Plot enough points to show that the two curves are distinct curves.

8. Plot on the same graph for $-\pi/2 < x < \pi/2$: $y = \tan x$ and $y = x + (x^3/3)$.

9S. Plot a careful graph of one arch of $y = \sin x$ (where x is in radians), using the same scale on both axes. Use a straightedge to draw by eye the lines tangent to this curve at the following points, and then draw a graph of the slope of the tangent line as a function of x; next tabulate the corresponding values of $\cos x$: $x = 0$, $x = \pi/6$, $x = \pi/4$, $x = \pi/3$, $x = \pi/2$, $x = 2\pi/3$, $x = 3\pi/4$, $x = 5\pi/6$, and $x = \pi$.

10S. Plot a careful graph of $y = \tan x$ (where x is in radians) from $x = -\pi/2$ to $x = \pi/2$, using the same scale on both axes. Then use a straightedge to draw by eye the lines tangent to this curve at $x = -\pi/4$, $x = 0$, and $x = \pi/4$. Compute the approximate values of the slopes of these tangent lines, and compare with the corresponding values of $\sec^2 x$.

5.5 Graphs of the Exponential Functions

The student may have learned in college algebra or in trigonometry about natural logarithms, or logarithms to the base e, as contrasted with common logarithms to the base 10. This number has an exact value that is usually denoted by e, though electrical engineers (preferring e for voltage) use the corresponding Greek letter ϵ (epsilon). When written in decimal form this number e never terminates and never repeats.* An approximate value for e, far more accurate than

* In fact, the number e, as well as the number π, is a transcendental number; that is, it cannot be the root of an algebraic equation with integral coefficients. It is from this fact (that the number π is transcendental), proved by a German named Lindemann in 1882, that mathematicians were able to show that one cannot

the student is ever likely to need, is given by

$$e \approx 2.71828 \quad 18284 \quad 59045 \quad 23536 \quad 02874.$$

This number appears in diverse places, as will be evidenced by some of the problems. The electrical engineer uses this number constantly; the chemical engineer uses it in the description of chemical reactions; the biologist uses it in his studies of growth and decay; and the statistician uses it frequently in his deductions.

We proceed first of all to discuss the graph of $y = e^x$.

I. Intercepts. When $x = 0$, $y = 1$; there is no x-intercept, because a positive number raised to a real-number power is always a positive number.*

II. Symmetry. There is none.

III. Asymptotes. The x-axis is an asymptote since y approaches zero as x decreases algebraically through negative values.

IV. Excluded regions. y is always positive, since a positive number raised to a real power is positive. In fact, when x is positive, $y > 1$ and y increases rapidly as x increases; when x is negative, y is positive but less than 1.

V. We plot a few points using either $e \approx 2.72$ or tables of powers of e (see table on page 310).

The graph is shown in Fig. 5.11. The associated graph of $y = e^{-x}$ is obtainable from the graph of $y = e^x$ since the two curves are symmetrical to each other with respect to the y-axis.

EXERCISE FOR THE STUDENT. Plot the graphs of $y = e^x$ and $y = e^{-x}$ on graph paper (10 or 20 squares per inch), using about 1 in. = 1 unit on both axes, and place the x-axis at the bottom of the graph sheet and the y-axis in the middle of the sheet. Compute and plot for $x = \frac{1}{2}$, and then combine the value for $x = 1$ and the value for $x = \frac{1}{2}$ by the laws of exponents to yield the values for $x = 1.5$, $x = 2$, $x = 2.5$, $x = -0.5$, $x = -1$, etc.; or use the tables on page 310.

It is now quite easy to sketch the graphs of equations of the form $y = ae^{bx}$. Thus, if a and b are positive numbers, we may introduce

square a circle, that is, that one cannot construct by use of compasses and an *unmarked* straightedge a square whose area is demonstrably equal to the area of a given circle. (See Cajori, *A History of Mathematics*, The Macmillan Co., 1926, p. 446.)

* The curious relationship $e^{i\pi} = -1$ illustrates the necessity of requiring the exponent to be a real number.

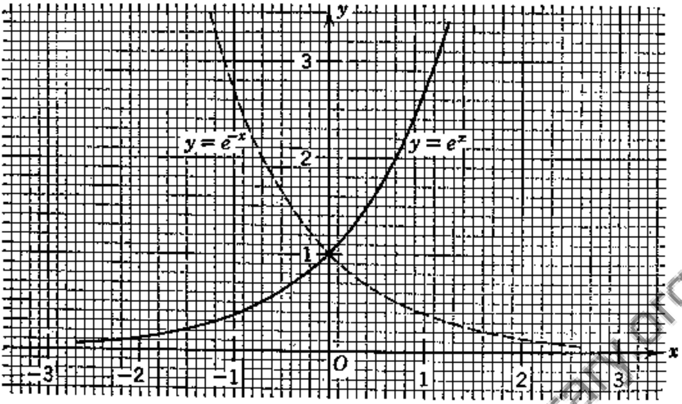


FIG. 5.11

the new variables $X = bx$ and $Y = y/a$, from which we obtain $Y = e^X$. If either a or b is negative, we may proceed in a similar manner. Hence the required graph is of the same general shape as that of $Y = e^X$, or a variation of it, such as $Y = e^{-X}$, or $Y = -e^X$, or $Y = -e^{-X}$. Therefore, to sketch such a curve we may compute y for a very few values of x (say $x = 0$ and one positive and one negative value), plot the three points, and then sketch the curve.

It is important that the student realize that every equation of the form $y = ae^{bx}$ can also be written in either of the two other forms: $ae^{bx} = a(10^{cx}) = a(d^x)$. The student should have no difficulty in changing from one form to the other by aid of logarithms.

EXAMPLE 1

Sketch rapidly $y = 0.7e^{-0.8x}$.

Solution. When $x = 0$, $y = 0.7$; when $x = 1.25$, $y = (0.7)(e^{-1}) \approx 0.26$; when $x = -1.25$, $y = (0.7)(e) \approx 1.9$. We plot these three points and sketch the curve as shown in Fig. 5.12.

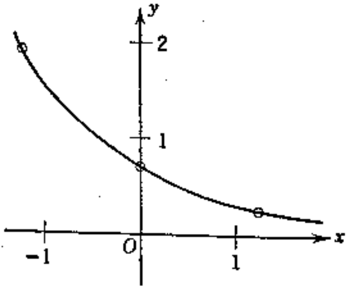


FIG. 5.12

EXAMPLE 2

Sketch $y = -2 - 0.5(3^{1-0.4x})$.

Solution. We rewrite the equation in the form $y + 2 = -0.5[3^{-0.4(x-2.5)}]$. We translate axes to (2.5, −2) and obtain $y' = -0.5(3^{-0.4x'})$. We compute y' for $x' = 0$, 2.5, and −2.5, plot the resulting points, and sketch the curve (Fig. 5.13).

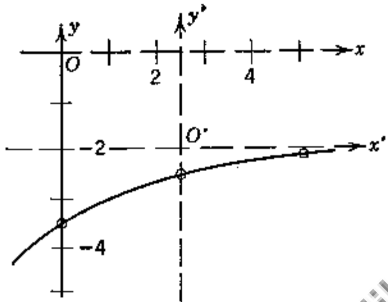


FIG. 5.13

5.6 Graphs of the Logarithmic Functions

In a previous course the student may have plotted a graph of $y = \log_{10} x$, and this graph is shown in Fig. 5.14 along with that of the equation $y = \log_e x$ (or $y = \ln x$, as we shall write it). In plotting the graph of $y = \log_{10} x$, or $y = \log x$ (as we shall denote it), the student should realize that

$$\log_{10} \tfrac{1}{2} = \log (0.5) \approx 9.699 - 10 = -0.301.$$

The graph of $y = \ln x$ may be obtained directly from the graph of $y = e^x$, since $y = \ln x$ is equivalent (from the definition of a logarithm) to $x = e^y$. Thus the graph of the natural logarithm function $y = \ln x$ is the same curve as $x = e^y$ and this may be obtained from the graph of $y = e^x$ by interchanging the x- and y-axes.

The student should notice that $y = \log_a x$, with $a > 1$, is equivalent to $x = a^y$; and this graph is an exponential-function graph of x in terms of y, and hence is of the same general shape as the graphs of the two logarithmic curves shown in Fig. 5.14.

EXERCISE FOR THE STUDENT. Use Fig. 5.14 and read off the ordinates to $y = \ln x$ and $y = \log x$ when $x = 3.5$, and determine the value of $(\ln x)/(\log x)$. Repeat for $x = 2.5$, $x = 1.5$, $x = 0.5$. Your result in each case should be about 2.30; this result may be established by algebraic methods (see Problem 16S at the end of this article).

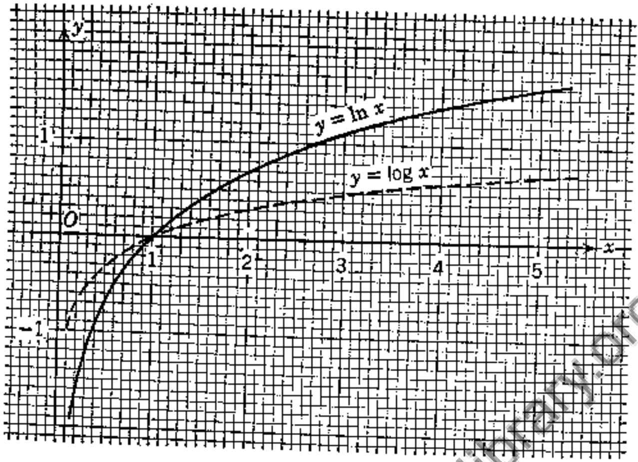


FIG. 5.14

We conclude that one may determine natural logarithms by looking up the logarithm of the required number to the base 10 and then multiplying the result by 2.30 (a more accurate value is 2.30259).

EXAMPLE

Sketch the graph of $y = \ln(x^2 - 1)$.

Solution. We observe that the required graph will be symmetrical with respect to the y-axis. To complete the analysis we shall compare the function $x^2 - 1$

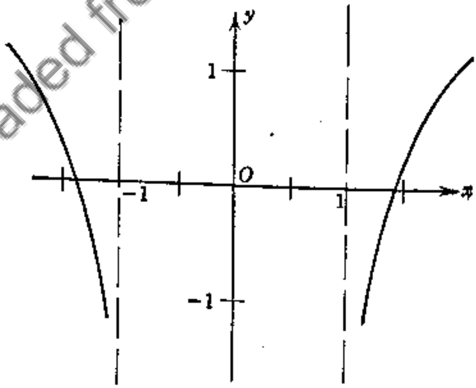


FIG. 5.15

with the function x for the basic graph of $y = \ln x$. The basic curve has an asymptote at $x = 0$; hence the required graph will have asymptotes at $x^2 - 1 = 0$, or at $x = 1$ and $x = -1$. The basic curve has an x-intercept at $x = 1$; hence the required curve will have x-intercepts: $x^2 - 1 = 1$, or $x = \pm\sqrt{2}$. The ordinate to $y = \ln x$ is negative when $0 < x < 1$; hence the ordinates to the required curve will be negative when $0 < x^2 - 1 < 1$, or when $1 < x^2 < 2$, which implies that either $-\sqrt{2} < x < -1$ or $1 < x < \sqrt{2}$. Similarly, we may see that the ordinates to the required graph are positive if $x^2 > 2$ and that there will be no graph for $x^2 - 1 < 0$, or for $x^2 < 1$.

Second Solution. If $x > 1$, so that $x - 1$ and $x + 1$ are both positive numbers, we may rewrite the given equation in the form

$$y = \ln (x - 1)(x + 1) = \ln (x - 1) + \ln (x + 1).$$

Hence we may sketch part of the required curve by the method of addition of ordinates and the remainder from symmetry. The student should do this to complete this second method of solution. The completed graph is shown in Fig. 5.15.

PROBLEMS

1. Sketch rapidly (plot three points in each case):

(*a*) $y = 2e^{-x/2}$.
(*b*) $y = 3e^{x/4}$.
(*c*) $y = 4e^{-0.25x}$.
(*d*) $y = 2e^{0.3x}$.
(*e*) $y = 1 + 0.8(2^{0.5x})$.
(*f*) $y = 2 - 1.2e^{1-0.5x}$.
(*g*) $q = 4(5^{-r})$.
(*h*) $y = \ln (x - 1)$.
(*i*) $y = \log (x + 2)$.
(*j*) $s = 2 + \ln (t + 2)$.
(*k*) $y = 2 + 3^{x+1}$.
(*l*) $y = \log_2 (x - 1)$.
(*m*) $Q = \log (2h - 1)$.
(*n*) $y = \log [(x - 3)/100]$.
(*o*) $y = \ln (x + 1) + \ln (e^2)$.
(*p*) $y = \ln [(x - 1)/(x + 2)]$ for $x > 1$.
(*q*) $y = 2 \ln (x + 3)$.
(*r*) $s = 4 \ln (2 - t)$.

2. Sketch each of the following sets of exponential curves on the same graph:

(*a*) $y = ae^x$ for $a = \frac{1}{2}$, 1, and 2.
(*b*) $y = e^{bx}$ for $b = \frac{1}{2}$, 1, and 2.
(*c*) $y = e^{bx}$ for $b = -\frac{1}{2}$, -1, and -2.
(*d*) $y = 2^x$, $y = 3^x$, and $y = e^x$.

3. Sketch each of the following exponential curves:

(*a*) Growth of bacteria: $N = 1000e^{0.06t}$, where N is number and t is in minutes.

(*b*) Radium disintegration: $Q = 1000e^{-0.000439T}$, where Q is in milligrams, T is in years.

(*c*) Intensity of light x ft. below the surface of a lake: $I = e^{-0.86x}$.

(*d*) Healing of a deep wound: $S = 5.2e^{-0.15T}$, where S is in centimeters, T is in days.

(*e*) Density of earth's atmosphere: $D = 0.59e^{-0.18x}$, where x is miles above sea level and D is tons per cubic mile.

(*f*) Population of Portland, Oregon: $P = 8300e^{0.08(T-1870)}$, where P is number of people at year T (valid for $1870 < T < 1915$).

(*g*) Application of Fourier's heat law: $x = 2.34e^{-0.00538\theta}$, where θ is temperature.

(*h*) Ceiling of an airplane in a climb: $H = 20{,}000 - 20{,}000e^{-0.0017t}$, where H is feet above sea level and t is time in seconds.

(*i*) Relative amount of dye in frog's capillaries: $A = e^{-0.24t}$, where t is the time in minutes after the injection.

(*j*) Belt friction: $T_2 = T_1e^{-0.4\theta}$, where θ is the angle of contact of the belt around a circular shaft and T_1 and T_2 are the two "pulls" on the ends of the belt.

(*k*) Speed of a chemical reaction in terms of temperature: $V = e^{0.0693T}$.

(*l*) Atmospheric pressure: $p = 14.7e^{-0.000,037,7h}$, where p is pressure in pounds per square inch and h is feet above sea level.

(*m*) Example of Newton's law of cooling: $\theta = 40e^{-0.00144t}$, where t is in seconds and θ is difference in temperature.

(*n*) Valuation of 100 trucks in dollars at T years: $V = 250{,}000e^{-0.40T}$.

(*o*) Series electrical circuit: $i = (E/R)(1 - e^{-Rt/L})$; sketch $y = Ri/E$ in terms of $x = Rt/L$.

(*p*) Relative speed of propagation of nerve impulse with changing temperature: $V = e^{0.06t}$.

(*q*) Brightness of stars: $I = I_1e^{-0.92M}$, where M is the magnitude of the star and I_1 is the intensity of light of a star of the first magnitude.

(*r*) Speed of rotating wheel after power is cut off: $V = 1000e^{-0.15t}$.

4. (*a*) Discuss $y = e^{-x^2}$ for intercepts, symmetry, and asymptotes; show that y is never negative; and sketch.

(*b*) Sketch $y = e^{-x^2}$ by comparing with the graph of $Y = e^{-X}$.

5. Sketch on the same graph $y = 2^x$ and $y = \log_2 x$, and show that the two curves are symmetrical together with respect to the 45° line.

6. A hospital staff has on hand 0.012 gram of radium. They know that the amount (A gram) they will have on hand t years later (composed solely of the remnant of the starting amount and assuming no loss from handling) will be given by the equation

$$A = 0.012e^{-0.000,439t}.$$

Sketch a graph of A in terms of t. Will the loss during the first 10 years be serious?

7. Sketch separate graphs of $y = \log (x^2)$ and $y = 2 \log x$. Note that a fundamental assumption made in deriving the laws of logarithms is that only logarithms of positive numbers are to be considered.

8. Sketch a graph of $y = 25^x - 5^x$, and estimate the value of x that corresponds to $y = 10$.

9. When an iron rod is heated, its length (L inches) is given in terms of the temperature of the rod (T degrees Fahrenheit) by $L = 40e^{0.000,01T}$. Sketch and give the lengths of the rod when $T = 0°$ F., 300° F., and 0° absolute (which is nearly $-460°$ F.).

10. Use the graphs of $y = e^x$, $y = \ln x$, and $y = \log x$ (see Figs. 5.11 and 5.14), and estimate the values of the following (you may check your results by use of tables):

(*a*) e^x when $x = -1.5, -0.4, -0.2, 0.2$, and 0.5.
(*b*) $\ln x$ when $x = 0.1, 0.5, 0.9, 1.1$, and 1.5.
(*c*) $\log x$ when $x = 0.2, 0.5, 0.8, 1.2$, and 1.5.
(*d*) x if $e^{-100} = x$.
(*e*) x if $\ln x = -100$.

11. Determine approximate values for a and b if the curve $y = ae^{bx}$ goes through the following pairs of points:

(a) (0, 100) and (10, 7).
(b) (1, 3) and (5, 11).
(c) (2, 0.45) and (12, 0.016).
(d) (0, 40) and (10, 5).

12. Sketch on the same graph: $y = \log_2 x$, $y = \log_3 x$, and $y = \log_4 x$.

13. The equation $y = 1000e^{0.0005x} + 1000e^{-0.0005x} - 1970$ is the equation of a power-line wire between two towers 800 ft. apart. The x-axis is along the level ground, and the y-axis is the perpendicular bisector of the line segment joining the bases of the towers.

(a) Sketch the curve of the wire.

(b) Find the height of the wire at $x = 0$ and at $x = 400$ ft., and then find the sag of the wire that is the difference between these two heights.

14. The population of a town is 8000 and is expected to increase by 5.5% each year. The population at the end of t years will then be $P = 8000(1.055^t)$. Sketch, and also express the equation in the form $P = ae^{bt}$. What is P when $t = 10$ years?

15. Sketch by use of combination of ordinates the graphs of the six *hyperbolic* functions that are defined in terms of the exponential function as follows:

(a) $\sinh x = (e^x - e^{-x})/2$.
(b) $\cosh x = (e^x + e^{-x})/2$.
(c) $\tanh x = \sinh x/\cosh x$.
(d) $\coth x = \cosh x/\sinh x$.
(e) $\text{sech}\, x = 1/\cosh x$.
(f) $\text{csch}\, x = 1/\sinh x$.

16S. Derive the change of base relations:

$$\log_a b = (\log_c b)/(\log_c a) = (\log_c b)(\log_a c).$$

If $a = e$, show that $\ln b = (\ln 10)(\log x) \approx 2.30259 \log x$.

5.7 Damped Waves. Boundary Curves

Curves with equations of the form $y = ae^{-bx} \sin cx$, where a, b, and c are usually positive numbers, are met frequently in problems in vibrations, in electrical engineering, etc. As the graphs will show, these curves are similar to the graph of the simple sine function except that the "amplitude" decreases with each "period," and approaches zero as a limit as x increases.

These curves are most easily sketched by combination of ordinates, since the graphs of the component curves $y = ae^{-bx}$ and $y = \sin cx$ are basic curves whose graphs have been memorized. Multiplication of ordinates will yield the required curve, and, if we use the obvious multipliers of 0, 1, and -1 from the sine curve, we can secure a number of points quite rapidly. When the multiplier is -1 the position of the required point is on the curve $y = -ae^{-bx}$ directly below the position it would have if the multiplier were $+1$. Hence it is convenient to sketch in as a dotted curve the graph of this reflection in the x-axis of the exponential curve before proceeding to multiply ordinates.

DEFINITION. *The two curves* $y = ae^{-bx}$ *and* $y = -ae^{-bx}$ *are called boundary curves for the graph of* $y = ae^{-bx} \sin cx$. *More generally, the two curves* $y = f(x)$ *and* $y = -f(x)$ *are boundary curves for the graphs of* $y = f(x) \sin cx$ *and* $y = f(x) \cos cx$.

EXAMPLE

A weight of 10 lb. hangs from a spring; a shock absorber is attached to the weight, the effect of the shock absorber being eventually to stop the weight (in a physical sense), and then to keep it from vibrating. The weight is pulled downward 2 in. and released. Its distance below a convenient measuring position (its position if it were at rest) is given by the equation $y = 2e^{-0.4t} \cos \pi t$ (approximately); y is in inches and is measured positive downward, and t is in seconds. Sketch the graph, and then determine the time required for y to decrease to an amount that is thereafter always less than 0.005 in., when the weight may be considered to have actually stopped. Stated differently, this equation applies to this physical problem only over a time interval that we are asked to determine.

Solution. We sketch the two boundary curves $y = 2e^{-0.4t}$ and $y = -2e^{-0.4t}$ and the cosine curve $y = \cos \pi t$, and then multiply ordinates to obtain the final curve as shown in Fig. 5.16. To complete the problem we observe that y will

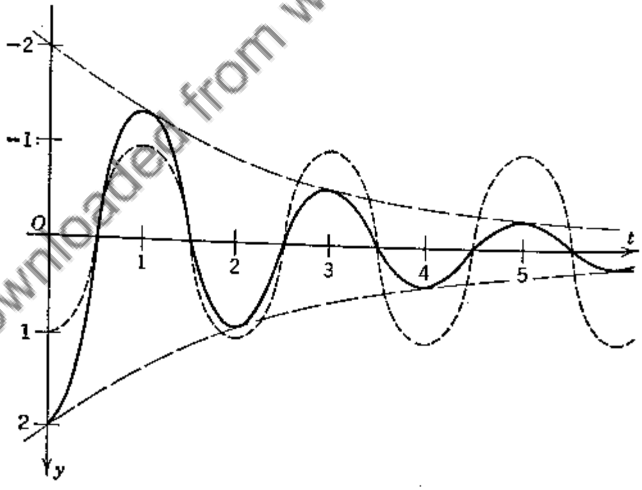


FIG. 5.16

always be less than 0.005 when $2e^{-0.4t}$ is less than 0.005, since this curve is a boundary curve. Hence we solve $2e^{-0.4t} = 0.005$ or $e^{+0.4t} = 400$ and $0.4t = 5.99+$; therefore $t = 15.0$ sec.

PROBLEMS

1. Sketch by multiplication of ordinates and show both boundary curves:

(a) $y = (x/2) \sin \pi x$ for $-2 < x < 4$.
(b) $y = 2^x \cos 2x$ for $-3 < x < 3$.
(c) $y = 2e^{-0.4x} \sin \pi x$ for x from 0 to 5.
(d) $y = 0.8e^{-0.5x} \cos (\pi x/2)$ for x from 0 to 8.
(e) $F = 40e^{-0.002t} (\cos 0.004t + 3 \sin 0.004t)$ for $0 < t < 3000$.
(f) $Q = 4e^{-0.25t} \sin (2\pi t + \pi/4)$ for t from -1 to 4.
(g) $L = 0.75e^{-40t} \cos (100\pi t - \pi/4)$ for $0 < t < 0.06$.
(h) $xy = \cos \pi x$ for $-4 \leq x < 0$ and $0 < x \leq 4$.

2. Sketch $y = (1/x) \sin 2\pi x$ by multiplication of ordinates. Prove algebraically that this curve is symmetrical with respect to the y-axis. Compute the value of y when $x = 0.1$ and when $x = 0.01$. What value does y *approach* as x approaches zero? (Notice that this equation does not define a value of y when x *equals* zero.)

3. Sketch on the same graph:

(a) $y = \sin 2\pi x + 2e^{-0.4x}$ and $y = 2e^{-0.4x} \sin 2\pi x$ for $-1 < x < 2$.
(b) $y = \sin 2x + 2 \cos 3x$ and $y = 2 \sin 2x \cos 3x$ for $0 < x < 2\pi$.
(c) $y = \sin 2x + (1/x)$ and $y = (1/x) \sin 2x$ for $-\pi < x < 2\pi$.

4. Sketch:

(a) $y = (\sin x) \ln (x - 1)$.
(b) $y = \ln (x - 1) + \sin x$.
(c) $ye^{0.4x} = \sin x$ for $-2 < x < 6$.
(d) $y = (\sin \pi x)/(2x)$ for $-2 < x < 4$.
(e) $x^2y = \sin (x - \pi/2)$ for $-2 < x < 4$.

5. Use the illustrative example of this article and work the following problems:

(a) Determine the *exact* values of y when $x = 0, 1, 2, 3,$ and 4, and let these values be $A, B, C, D,$ and E respectively.

(b) Determine the exact numerical values of the ratios A/B, B/C, C/D, and D/E. Hence show that $A, B, C, D,$ and E form a geometric progression.

(c) Determine the corresponding value for the ratios if the equation of the curve is $y = ae^{-bx} \cos cx$ (use $cx = 0, \pi, 2\pi$, etc.).

6. Sketch the graph of $e = 2(1 + 0.5 \cos 120\pi t) \cos 360\pi t$, where t is in seconds and e is in volts. Use multiplication of ordinates, and first sketch $y_1 = 2 + \cos 120\pi t$ and $y_2 = \cos 360\pi t$. Then rewrite the equation successively as

$$e = 2 \cos 360\pi t + \cos 360\pi t \cos 120\pi t$$

$$= 2 \cos 360\pi t + 0.5 \cos 480\pi t + 0.5 \cos 240\pi t.$$

Notice that the same graph could be sketched from this last equation by the method of addition of ordinates.

7. Sketch the graphs of the following curves:

(a) $y = 1.5(1 + 0.8 \cos 2\theta)(\cos 3\theta)$, where $\theta = 1000\pi t$.
(b) $y = 50(1 + 0.6 \cos 5000t - 0.3 \cos 10{,}000t)(\sin 50{,}000t)$.

8S. Sketch i amp. $= 0.75(1 + 0.4 \cos \theta)(\cos 20\theta)$, where $\theta = 10{,}000\pi t$. You may distort the final graph by drawing it with a straightedge, and the final graph will give some idea of an *amplitude-modulated* radio wave.

9S. Sketch the graphs of $y = A \sin (Bt + C)$ subject to the following conditions:

(a) $A = 100$, $B = 1{,}000{,}000\pi t$, $C = \pi/4$.
(b) $A = 100 \sin 100{,}000\pi t$, $B = 1{,}000{,}000\pi t$, $C = \pi/4$ (amplitude-modulated wave).

Note: The graph with $A = 100$, $B = 1{,}000{,}000\pi t$, $C = 0.2\pi \sin 100{,}000\pi t$, would yield a frequency-modulated wave.

5.8 Irrational Roots Obtained by the Method of Intersecting Graphs

The student learned in algebra how to solve certain equations for their irrational roots. One method was to write the equation in the form $f(x) = 0$, to plot a graph of $y = f(x)$, and then to read the x-intercepts of the resulting graph. A different graphical method is given by the following theorem.

THEOREM. *The real solutions of the equation $f(x) = g(x)$ are obtained as the x-coordinates of the points of intersection of the two curves $y = f(x)$ and $y = g(x)$.*

We may prove this theorem by aid of the concept of addition of ordinates. Thus, the graph of $y = f(x) - g(x)$ could be obtained from the component curves $y = f(x)$ and $y = g(x)$ by subtracting ordinates. The x-intercepts of the combination curve would then clearly be the x-coordinates of the points of intersection of the two component curves.

If a given equation can be manipulated algebraically into the form given in the theorem and if the two required or component curves are easily sketched, then a set of approximate solutions can be determined rapidly.

EXAMPLE 1

Estimate the roots to the nearest tenth, in the equation

$$2 \ln (x + 4) + x^2 - 4 = 0.$$

Solution. We let $y = 2 \ln (x + 4)$. Then the given equation becomes $y + x^2 - 4 = 0$, or $y = 4 - x^2$. We draw the first curve by translation of axes ($y' = 2 \ln x'$), and by plotting y' for $x' = 0$ and $x' = e$. We recognize the second curve to be a parabola, notice that it is symmetrical with respect to the y-axis, compute the intercepts, and draw the curve. The student should complete the solution by sketching the two curves (1 in. = 1 unit on the x-axis would be an appropriate scale). He should estimate the x-coordinates of the points of intersection and find that $x = 0.9$ and $x = -1.4$ approximately. He could check the entire solution by substituting these values in the *original* equation and by determining if that equation is approximately satisfied by these values.

The student should notice that it would have been equally correct to start the solution by rewriting the given equation in the form $\ln(x + 4) = \frac{1}{2}(4 - x^2)$, and then by forming the two equations $y = \ln(x + 4)$ and $y = \frac{1}{2}(4 - x^2)$.

Example 2

We are given the equation $i = (E/R)(1 - e^{-Rt/L})$, where i is current in amperes, t is time in seconds, E is voltage in volts, R is resistance in ohms, and L is inductance in henries. (Your understanding of the mathematics of this problem does not depend on an understanding of these terms, which are defined in a course in physics.) We are also given that $E = 10$ volts and that $L = 2$ henries. When $t = 3$ seconds an ammeter shows that $i = 4.90$ amp. We are to determine the resistance R correct to the nearest 0.1 ohm.

Solution. Substitute the given numbers and obtain

$$4.9 = \frac{10}{R}(1 - e^{-1.5R}).$$

Rewrite this equation in the form $0.49R = 1 - e^{-1.5R}$, and then in the form $e^{-1.5R} = 1 - 0.49R$. Plot the graphs of $y = e^{-1.5R}$ and $y = 1 - 0.49R$ as shown in Fig. 5.17, and read the R-coordinates of their points of intersection.

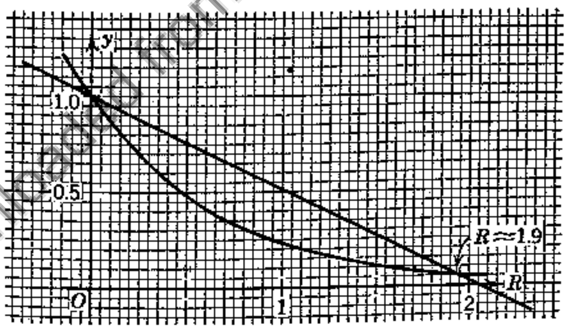


FIG. 5.17

The required answer is $R = 1.9$ ohms. Note that $R = 0$ is not a solution of the *original equation*.

PROBLEMS

1. Solve the following equations by the method of intersecting graphs:

(*a*) $\sin x - 0.6x = 0$.
(*b*) $e^{0.4x} + x - 4 = 0$.
(*c*) $\tan x + 1 - x = 0$ (estimate two solutions).
(*d*) $\cos x - \ln x = 0$.
(*e*) $2x^{1.5} - 4 - x = 0$.
(*f*) $(2/x) + 3 - x = 0$.
(*g*) $2e^{-0.4x} - \sin \pi x = 0$ (estimate four solutions).

2. The volume of a spherical segment is given by $V = \pi h^2 R - \pi h^3/3$, where R is the radius of the sphere and h is the height of the segment. Determine h to the nearest second decimal if $V = 10$ cu. ft. and $R = 5$ ft.

3. The formula $d^6 = Ad + B$ is used to determine the diameter in inches of pipe required to discharge each second a given quantity of water. If $A = 20$ and $B = 450$, use the method of intersecting graphs to obtain an estimate for d.

4. Solve for the real roots of the equation $2x^3 - 6x^2 - 2x - 1 = 0$ by the following sequence of steps:

(1) Decrease the roots by 1 (in the language of algebra). This is equivalent (why?) to translating the axes in

$$y = 2x^3 - 6x^2 - 2x - 1 \text{ to } (1, 0).*$$

(2) Solve this new equation, $x'^3 = 4x' + 3.5$, by the method of intersecting graphs.

(3) Obtain the required solutions of the original equation by adding 1 to each estimate made in (2).

* The short method given in algebra for decreasing the roots as applied to the given problem is as follows, and the new equation is $2x'^3 - 8x' - 7 = 0$.

2	−6	−2	−1	1
	2	−4	−6	
2	−4	−6	−7	
	2	−2		
2	−2	−8		
	2			
2	0			

An examination of this method should explain why the number 1 was chosen. Three subtractions of the product of the coefficient of x^3 by the chosen number (by which the roots are to be decreased) are to be made from the coefficient of x^2. For simplicity in the method of intersecting graphs, we choose that number which makes the coefficient of x'^2 zero. Thus, we let $x' = x + a_1/3a_0$ where the equation is $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$.

5. Solve correct to two significant figures by the method explained in Problem 4, and by aid of the same graph of $y = x'^3$ that was used in that problem:

(a) $2x^3 - 12x^2 + 2x = 21.$ (b) $x^3 - 5x = 11.$

(c) $x^3 + 3x^2 + 3x - 5 = 0.$ (d) $2x^3 + 4x = 9.$

(e) $x^3 - 9x^2 + 24x - 16 = 0.$ (f) $x^3 - 6x^2 + 12x = 24.$

6S. Solve $\sin 120\pi t = 0.289e^{-218t}$ (t in seconds) for the two smallest positive solutions.

7S. Estimate the real solutions of $x^4 + 4x^3 + 7x^2 + 2x - 14 = 0$ from the graphs of one circle and two parabolas. *Hint:* First increase the roots by 1: then replace x'^4 by y^2.

5.9 Review

At this stage the student should write out a comprehensive outline of this course to date. The process of answering the following review questions will aid him in making up this outline.

REVIEW QUESTIONS

1. Given the two points $A(-1, 7)$ and $B(3, 5)$. State in words how to determine the length of $\overline{AB}$, the slope and inclination of $\overline{AB}$, the coordinates of the midpoint of $\overline{AB}$, the equation of $\overline{AB}$, the equation of the line perpendicular to $\overline{AB}$ and passing through A, and the area of the triangle with vertices at A, B, and the origin.

2. A point moves so that its abscissa is always equal numerically to its *vertical* distance from the curve $y = \ln x$. Explain the steps in the determination of the equation of the locus of this point.

3. What operations comprise the discussion method of curve sketching? What is the first step to be used if a curve is to be sketched by either addition or multiplication of ordinates? If axes are to be translated, what must be the coefficients of x and y? If x is replaced by $2x'$ and y by $3y'$, what is the effect on the graph?

4. From the realization that the x-intercepts of $y = 6 - x - x^2$ are $x = 2$ and $x = -3$, explain how to determine the coordinates of the vertex of a parabola without completing the square. Then sketch the curve.

5. What are the methods to be used when only a rapid sketch of an ellipse is desired? of a hyperbola? (The first step is to complete the squares to facilitate translation of axes.)

6. What is the significance of $B^2 - 4AC$ for the general conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0?$$

Write down examples (with xy-terms present) of equations whose graphs are ellipses, hyperbolas, and parabolas. What are the two methods that may be used to draw the graph of such an equation?

7. Outline short methods for sketching curves of the following forms: $y = a \sin bx$; $y = a \cos bx$; $y = ae^{bx}$; and $y = \log_4 x$.

8. What is the locus of $y = 2\sin(3x + 4)$; what are the period, the amplitude, the value of y when $x = 0$, the value of x when $3x + 4 = 0$, and the value of x when $3x + 4 = 2\pi$? What are two different methods that may be used to draw the locus of $y = 2\cos^2(\pi x/2)$? How may we draw the loci of equations such as $y = 2\tan \pi x$ and $y = \frac{1}{3}\text{arc}\cos 2x$?

9. How may we sketch an accurate graph of $y = 2e^{-x/4}$ without plotting more than three points (for the range from $x = -6$ to $x = +6$)? What is the relation between the graphs of $y = e^x$ and $y = \ln x$; of $y = 10^x$ and $y = \log x$; of $y = 2^x$ and $y = \log_2 x$? Can we sketch the graph of $y = 2\ln(2x + 1)$ by reference to the graph of $Y = \ln X$, and can we sketch the same graph by translating *both* axes?

10. By aid of the basic graphs of $y = e^x$, $y = \ln x$, $y = \sin x$, and $y = \cos x$, can you estimate solutions of equations such as the following: $e^x = 0.04$, $e^{-1.5} = x$, $\ln 0.2 = x$, $e^{-400} = x$, $\sin 5 = x$, and $\cos(10\pi + 1) = x$?

11. Explain how each of the following equations may be used as the first step of a graphical process to solve the equation $x^3 + 2x - 4 = 0$:

(a) $x^3 = 4 - 2x$.
(b) $x^2 = (4/x) - 2$.
(c) $x^2 = +\sqrt{2 - 2(x - 1)^2}$.

12. If we desired to sketch $y = 5e^{-0.2x}\sin 0.4\pi x$ by aid of boundary curves, why would it be incorrect to use for boundary curves $y = e^{-0.2x}$ and $y = -e^{-0.2x}$, and for the other component curve $y = 5\sin 0.4\pi x$? What is the correct choice for the equations for the boundary curves, and why? Could we sketch the required curve by multiplying the ordinates to the two curves $y = e^{-0.2x}$ and $y = \sin 0.4\pi x$, and then changing the y-units each to read five times its former value?

13. Are the following loci the same or different? Explain any difference.

(a) $(x^2 + 4y^2)(x - y) = 0$ and $x - y = 0$.
(b) $y = \sqrt{x}$ and $y^2 = x$.
(c) $x^2 = y^2$ and $x = y$.
(d) $1/x + 1/y = 1$ and $x + y = xy$.

REVIEW PROBLEMS

1. Identify and sketch rapidly each of the following curves (s is the dependent variable in every case):

(a) $st = 4$.
(b) $2s^2 + 2t^2 = 7$.
(c) $2s^2 + 5t = 11$.
(d) $2s^2 - 5t^2 = 11$.
(e) $4s^2 = t^3$.
(f) $s^2 - t^2 = 17$.
(g) $s = 3t^{-2}$.
(h) $s^2 + 4st + t^2 = 4$.
(i) $s^2 + 4st + 4t^2 = 4$.
(j) $s^2 + st + t^2 = 4$.
(k) $s^2 + 2st + t^2 = 4t$.
(l) $s = 4t$.
(m) $(s - 1)^2 + (t + 2)^2 = 5$.
(n) $(s - 1)^2 + 3(t + 1)^2 = 7$.
(o) $s^2 = 3.24t$.
(p) $s = 4t^2 + 8t - 12$.
(q) $s = 0.5t$.
(r) $(s - 2)^2 - 3(t + 1)^2 = 6$.

(s) $(s-1) = 4(t-2)^{2.47}$.
(t) $(s-1) = 4(t-2)^2$.
(u) $(s-1) = 4(t-2)^{0.56}$.
(v) $(s-1) = 4(t-2)^{-1.73}$.
(w) $(s-1)^2 = 4(t-2)$.
(x) $(s-1)^2 = 4(t-2)^2$.

2. Identify and sketch the following curves (s is the dependent variable in each case):

(a) $s = \ln t$.
(b) $s = 2 + \log(t-1)$.
(c) $s = 4 \sin \pi t$.
(d) $s = 10 + 10 \cos 12\pi t$.
(e) $s = 0.005 \cos 10{,}000\pi t$.
(f) $t \cos 65° + s \sin 65° = 7$.
(g) $s = 4 \sin \pi(t + 0.5)$.
(h) $s = 4 \cos(200\pi t - \pi/6)$.
(i) $s = 2^t$.
(j) $s = 0.5^t$.
(k) $s = 3e^{0.25t}$.
(l) $s = 3 - 2e^{-0.5t}$.
(m) $s = |t|$.
(n) $s = 2 \text{ arc sin } (t/2)$.
(o) $s = (2 + t^2)/(2 - t)$.
(p) $s = 4 \cos^{-1} \pi t$.
(q) $s = 2 \tan 3\pi t$.
(r) $s = 2 \cot 2t$.
(s) $s = 0.5(e^{0.25t} + e^{-0.25t})$.
(t) $s = 2/(\sec 2\pi t)$.
(u) $s = 4/(\csc 0.25t)$.
(v) $s = \ln(1/t)$.
(w) $s = (2/t) \sin \pi t$.
(x) $s = 3e^{-t/4} \cos(\pi t + 0.25\pi)$.

3. Make a complete table of *type equations* for the various curves that have been studied thus far. Assume that the method of translation of axes is perfectly understood so that, for example, one type equation for a circle is $x^2 + y^2 = r^2$.

CHAPTER 6

Polar Coordinates

Thus far in this course points have been located by the description: "Go east 4 blocks and north 3 blocks," that is, by (4, 3). But directions that would enable us to arrive (at least theoretically) at the same destination are: "Go 5 blocks along a line making an angle of approximately 37° with the east direction." This is the fundamental idea of this chapter. We shall again be concerned with the two basic problems of analytic geometry, which are to draw the graph of a given equation and to find the equation of a given locus.

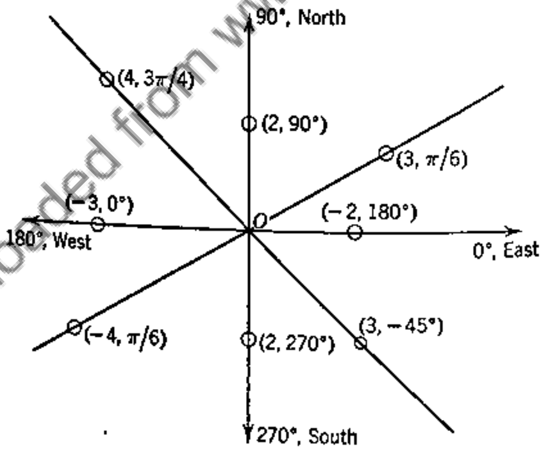


Fig. 6.1

6.1 Introduction

As in trigonometry, we shall use an angle θ to be positive when measured in the counterclockwise direction (negative when measured clockwise), and to have its initial side along the positive x-axis. Also, we shall use the radius vector distance from trigonometry, but we shall allow it to be positive or negative, taking it to be positive when measured along the terminal side of the angle θ. Thus, the polar coordinates of a point, (r, θ), give an angle θ to be laid off, and then the directed distance r to be measured from the origin. In Fig. 6.1 we show the location of a number of points, each labeled with its polar coordinates.

DEFINITIONS. *The axis (usually horizontal) that is the initial side of the angle θ is called the polar axis. The origin, as it has been called in the system with rectangular coordinates, is called the pole in this new topic. The vertical line through the pole and perpendicular to the polar axis is called the 90° line.*

PROBLEMS

1. Plot the following points: $(3, 150°)$; $(2, \pi)$; $(-3, 45°)$; $(2, 1)$, which implies an angle of $\theta = 1$ radian; $(4, 2)$; $(5, 3\pi/2)$.

2. Plot the following points and label each point with its coordinates: $(3, 3\pi/4)$; $(3, -5\pi/4)$; $(-3, -\pi/4)$; $(-3, 7\pi/4)$; $(3, 11\pi/4)$.

3. Plot the following points: $(2, \pi)$; $(-2, 0)$; $(-2, 2\pi)$; $(2, -\pi)$; $(2, 3\pi)$.

4. Plot the points $(2, \pi/6)$; $(2, 5\pi/6)$; $(-2, \pi/6)$; $(2, -\pi/6)$. What is the symmetrical relationship between the first point and each of the other three points?

5. Locate an arbitrary point in the first quadrant and label its coordinates as (r, θ). Then locate the points $(-r, \theta)$, $(r, -\theta)$, and $(r, \pi - \theta)$. What is the symmetrical relationship between the original point and each of these three points?

6. Locate an arbitrary point in the second quadrant and label its coordinates (r, θ). Then see that the following would also serve for its coordinates, and give three more such pairs of numbers: $(-r, \theta - \pi)$; $(-r, -3\pi + \theta)$; $(r, -2\pi + \theta)$.

7. Construct a dummy sheet of polar-coordinate paper according to the following directions. (You can then place a sheet of thin paper over your dummy sheet and sketch or plot on it any of the curves assigned in the remainder of this chapter.) Use a sheet of 8.5 by 11-in. heavy drawing paper and India ink. With center at the center of this sheet draw heavy circles with radii 1 in., 2 in., 3 in., and 4 in. Then draw light circles with radii $r = 0.2$ in., 0.4 in., etc., up to 3.8 in. Next draw as heavy lines the horizontal and vertical lines through the center and also the 45° and 135° lines. Then draw as light lines the lines at 15°, 30°, etc., but omit that part of each line that lies inside the circle with 1-in. radius.

6.2 Plotting Polar-Coordinate Curves by Plotting Points

The most elementary method for plotting curves determined by an equation in polar coordinates,* though it is often cumbersome and tedious, is by plotting points. In this article we shall first study how

* The locus of an equation in polar coordinates is the totality of points, each of which has at least one pair of polar coordinates that satisfies the given equation.

to plot curves with polar-coordinate variables by plotting points, and in a subsequent article we shall discuss a better method. The present method will be to solve the equation for one of the variables in terms of the other, to substitute values for one variable and compute the other, and finally to plot the points and the curve.

In polar coordinates it is generally easier to solve for r in terms of θ and then to assign values for θ. Also, the common-sense method for finding r would be to substitute the special angles of trigonometry as much as possible, or else to utilize a slide rule.

EXAMPLE 1

Plot $r = 4 \sin \theta$.

Solution. We make up a table of values as follows:

θ	0°	30°	45°	60°	90°	120°
r	0	2	2.83	3.46	4	3.46

The student should complete this table for the values $\theta = 135°, 150°, 180°, 210°, 225°, 240°$, etc., and should see that the curve repeats as θ takes on values above 180°.

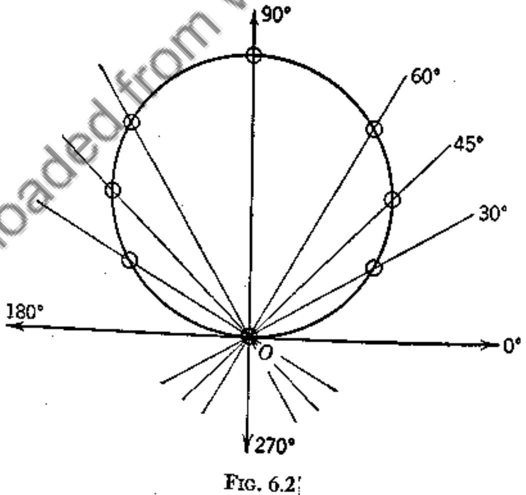


FIG. 6.2

The student will notice that the curve in Fig. 6.2 looks like a circle. Could it be that the equation of this example is exactly the polar-coordinate equation of a particular circle? The answer to this question will be given in a later article.

EXAMPLE 2

Plot the curve $r = 10 \sin 2\theta$.

Solution. We make use of special angles in this example by assigning values to θ at every 15°, for then 2θ will take on values at intervals of 30°. It is convenient to show the work in tabular form, being careful to do the computations for each column before proceeding to the next column.

θ	2θ	$\sin 2\theta$ *	r
0°	0°	0	0
15	30	0.5	5.00
30	60	0.866	8.66
45	90	1	10.00
60	120	0.866	8.66
75	150	0.5	5.00
90	180	0	0
105	210	−0.5	−5.00
120	240	−0.866	−8.66
135	270	−1	−10.00
etc.			

* Notice that the memorized graph of $y = \sin x$ may be used in writing down the values in this column.

The graph, as shown in Fig. 6.3, is a four-leaf rose or clover.

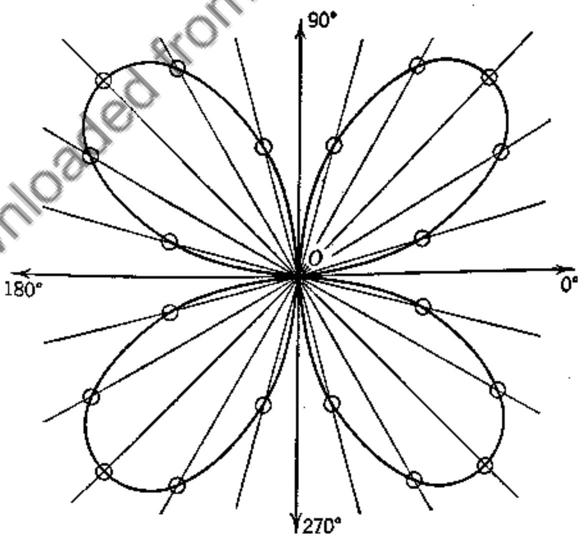


FIG. 6.3

PROBLEMS

Make a table of values (use trigonometric functions of special angles as much as possible) and plot the graphs of the following polar-coordinate equations:

1. $r = 4 \cos \theta$.
2. $r = 1 + \theta$ for $0 < \theta < 4\pi$ radians.
3. $r = 2/(2 + \cos \theta)$.
4. $r = 4 - \cos \theta$.
5. $r = 2 + 2 \cos \theta$.
6. $r = 1 + 2 \sin \theta$.
7. $r = 5 \cos 2\theta$.
8. $r = 10 \cos 3\theta$.
9. $r \cos \theta = 2$.
10. $r \sin 2\theta = 4$.
11. $r = e^{-0.25\theta}$ for $-2\pi < \theta < 4\pi$.
12. $r = \ln \theta$.
13. $r^2 = 4 + 4 \sin \theta$.
14. $r^2 = 4 \cos 2\theta$.
15. $r^2 = 4 \sin \theta$.
16. $r = 4 \cos^2 2\theta$.
17. $r = 2 \sin 2\theta$.
18. $r = 2 \cos 3\theta$.

6.3 The Discussion Method for Sketching Polar-Coordinate Curves

Plotting by points is satisfactory when the equation is simple. But the first illustrative example in the preceding section would be much simpler if we knew that the graph was going to be a circle passing through the pole, with its center on the 90° axis and with a diameter of 4 units. Then we would need only to reach for compasses. Even if we did not know that the graph was a circle, it would be almost as easy to obtain the graph if we knew that the curve was symmetrical with respect to the 90° line and that it repeated every 180°. We shall, therefore, proceed to discuss simple rules that will facilitate the sketching of polar-coordinate graphs.

I. Symmetry (see Fig. 6.4).

(*a*) If one can substitute $-\theta$ for θ and obtain the same equation, the graph is symmetrical with respect to the polar axis.
(*b*) If the equation is unchanged when θ is replaced by $\pi - \theta$, the graph is symmetrical with respect to the 90° line.
(*c*) If the equation is unchanged when r is replaced by $-r$, the graph is symmetrical with respect to the pole.

In all three cases, if the resulting equation is different, the graph *may still be symmetrical with respect to the part of the coordinate system considered.*

We have already seen that there are an infinite number of different sets of coordinates for a single point. As a consequence there are an infinite number of different tests for symmetry with respect to the polar axis, for example. Two of these are as follows:

(1) Substitute $2\pi - \theta$ for θ.
(2) Substitute $\pi - \theta$ for θ and $-r$ for r.

The student should learn the three principal tests for symmetry as stated in (a), (b), and (c), and at the same time should realize that there may still be symmetry though any test fails.

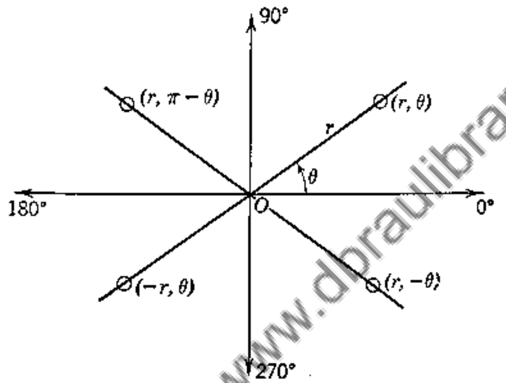


FIG. 6.4

II. Closed or Open Curve. If the values of r are bounded, the curve is closed;* if the values of r can increase without limit, the curve is open.

III. Variation. Determine the behavior of r as the *total angle* increases from 0° to 90°, from 90° to 180°, etc. The student has already memorized the graphs of $y = \sin x$ and $y = \cos x$ and can therefore make use of this knowledge in this new connection.

There are other rules that could be learned, but these are the important ones. One convenient fact to know is that, if $r = 0$ for some particular value of θ, then the curve goes through the pole and has there that θ-line for a tangent. Thus, for example, one part of the graph of $r = 1 + 2 \sin \theta$ passes through the pole and is there tangent to the line $\theta = -30°$, and a different part of the curve goes through the pole and is there tangent to the line $\theta = 210°$.

* With respect to closed curves we are assuming that each trigonometric function is a function of some integral or fractional part of θ. The graph of $r = 2 \sin \pi\theta$ would lie entirely within $r = 2$ but would clearly never repeat.

EXAMPLE 1

Discuss and sketch $r = 4\cos 2\theta$.

Solution. I. Symmetry. Since $\cos(-2\theta) = \cos 2\theta$, the curve is symmetrical with respect to the polar axis, and hence we need to discuss the curve only for θ from 0° to 180° and to obtain the remainder of the curve by symmetry. Since $\cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = +\cos 2\theta$, the curve is symmetrical with respect to the 90° axis and hence we need to discuss the graph (taking both symmetries into account) only for θ from 0° to 90°. If we substitute $-r$ for r, the equation becomes $-r = 4\cos 2\theta$, which is a different equation. Hence the *test* for symmetry with respect to the pole fails, but in this case we already know that the curve *is* symmetrical with respect to the pole (because the curve is symmetrical with respect to *both* the polar axis and the 90° line).

II. Closed or Open Curve. The largest numerical value that r can have is 4. Hence the curve is a closed curve.

III. Variation. The *total* angle is 2θ. As 2θ increases from 0° to 90° (θ from 0° to 45°), the cosine function decreases from 1 to 0, and hence r decreases from 4 to 0. This result is shown in a schematic manner in Fig. 6.5 by the several radius

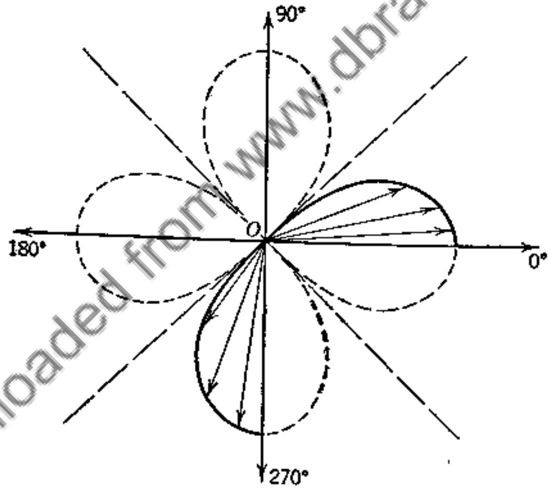


FIG. 6.5

vectors from $\theta = 0°$ to $\theta = 45°$. Again as 2θ increases from 90° to 180° (θ from 45° to 90°), the cosine of 2θ decreases from 0 to -1, and hence θ decreases from 0 to -4; this result is shown in Fig. 6.5 by the negative radius vectors from $\theta = 45°$ to 90°. We could continue this type of analysis and draw the entire curve. This, however, is unnecessary in this example because of the symmetries that are known.

This type of analysis may be summarized as follows:

2θ	0° to 90°	90° to 180°	etc.
θ	0° to 45°	45° to 90°	
$\cos 2\theta$	1 to 0	0 to -1	
r	4 to 0	0 to -4	

Since $r = 0$ when $\theta = 45°$, the curve is tangent to the 45° line as it goes through the pole. The portion of the curve obtained from the variation discussion is shown as a solid curve, and the remainder of the curve, obtained by symmetry, is dotted (this dotted portion would, of course, ordinarily be shown as a continuous curve).

Example 2

Discuss and sketch $r = 2/(1 + 2\cos\theta)$.

Solution. I. Symmetry. With respect to the polar axis, since $\cos(-\theta) = +\cos\theta$.

II. Closed or Open Curve. As θ approaches 120° (or 240°), $\cos\theta$ approaches $-\frac{1}{2}$, and hence the denominator approaches zero; thus the value of r increases without limit and the curve is open.

III. Variation. We need to discuss this only for θ from 0° to 180°:

θ	0° to 90°	90° to 120°	120° to 180°
$\cos\theta$	1 to 0	0 to −0.5	−0.5 to −1
$1 + 2\cos\theta$	3 to 1	1 to 0	0 to −1
r	$\frac{2}{3}$ to 2	2 without limit through positive values	Increases through negative values to −2

The student should study this discussion in terms of the curve as shown in Fig. 6.6. That the curve is a hyperbola will be shown in a later article. For con-

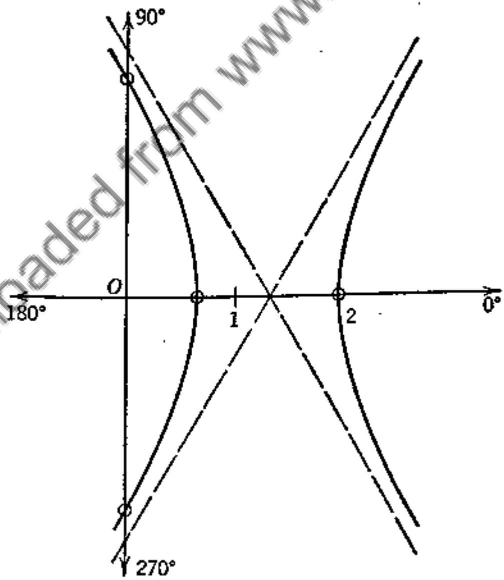


Fig. 6.6

venience, the asymptotes of the hyperbola are shown, and the student will notice that $\theta = 120°$ and $\theta = 240°$ (see Step II in the discussion) define lines through the pole parallel to these two asymptotes.

EXAMPLE 3

Discuss and sketch $r = 4 + \sin(3\theta/2)$. Then sketch on a separate graph $y = 4 + \sin(3x/2)$ in rectangular coordinates and compare the two graphs.

Solution. I. Symmetry. All three tests for symmetry fail.

II. Closed or Open Curve. The curve is closed since r can never be numerically more than 5.

III. Variation. The *total* angle is $3\theta/2$. As $3\theta/2$ increases from 0° to 90° (θ increases from 0° to 60°), the sine of the total angle increases from 0 to 1, and hence r increases from 4 to 5. The student should complete this discussion in tabular form until $\theta = 720°$, after which the curve will start repeating.

The completed curve is drawn in Fig. 6.7, and the rectangular coordinate graph of the associated equation is drawn in Fig. 6.8. The student will notice that the

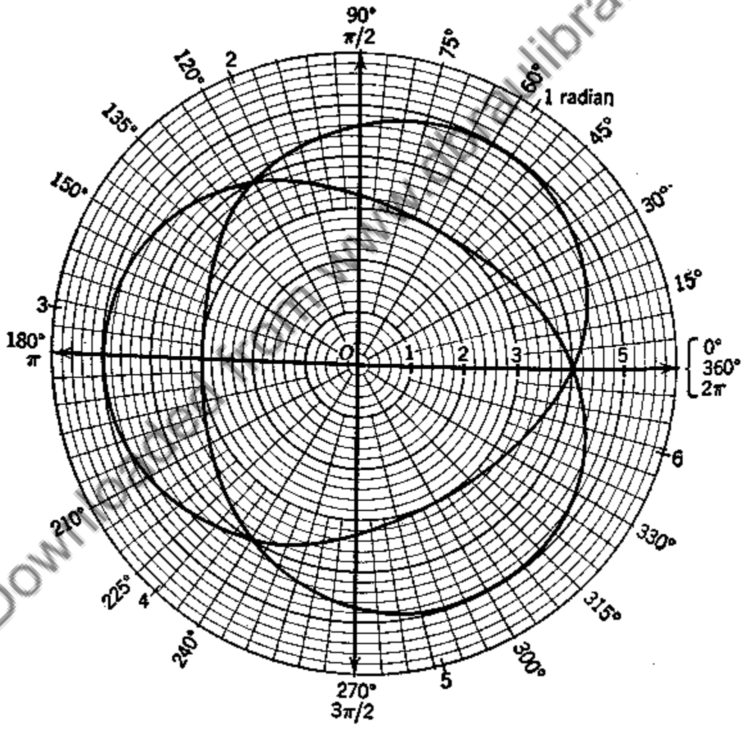


FIG. 6.7

rectangular-coordinate graph has a periodicity of $4\pi/3$, whereas the polar-coordinate graph requires $4\pi = 720°$ before that curve starts repeating. However, if the portion of the polar-coordinate curve between 0° and 240° could be rotated counterclockwise through 240° and again through 2(240°), the entire polar-coordinate graph would be obtained.

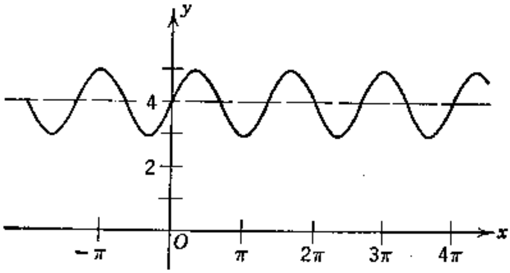


Fig. 6.8

Engineers find frequent use for polar coordinates in their study of machine design. The polar-coordinate curve in Fig. 6.7 could be the curve traced by the center of the "follower" in the motion of a cam mechanism. The rectangular-coordinate graph in Fig. 6.8 would then be the "layout" or "developed diagram" for the cam motion.

PROBLEMS

1. Verify and complete the following discussion for the equation $r = 1 + 2 \sin \theta$, and sketch the curve:

I. Symmetry. Ninety-degree line.
II. Closed or Open Curve. The curve is a closed curve.
III. Variation. Discuss for θ increasing from $-90°$ to $0°$ and from $0°$ to $+90°$.

Also note that the curve goes through the pole tangent to the line $\theta = -30°$ and also, by symmetry, through the pole tangent to the line $\theta = 210°$.

2. Discuss and sketch the following polar-coordinate curves:

(a) $r \cos \theta = 3$.
(b) $r \sin \theta + 2 = 0$.
(c) $r = 4 \cos \theta$.
(d) $r = 3 \sin \theta$.
(e) $r = 2 - \sin \theta$.
(f) $r = 2 + \cos \theta$.
(g) $r = 3 + 3 \cos \theta$.
(h) $r = 4 - 2 \sin \theta$.
(i) $r = a(1 - \cos \theta)$.
(j) $r = 3 \cos \theta + 4 \sin \theta$.
(k) $r = 1 + 2 \cos \theta$.
(l) $r = 2 + \sin 2\theta$.
(m) $r = 2 \cos (3\theta/2)$.
(n) $r = 2/(3 + \cos \theta)$.
(o) $r = 2/(1 + \sin \theta)$.
(p) $r = 2/(1 + 2 \sin \theta)$.
(q) $r = 2 \sin 2\theta$.
(r) $r = 2 \cos 3\theta$.

3. Sketch rapidly:

(a) $\theta = 120°$.
(b) $\theta = 2$ (radians understood).
(c) $r = 2$.
(d) $r = \theta$ (a spiral curve).

4. Discuss and sketch the following polar-coordinate curves:

(a) $r = 6 \tan\theta \sin\theta$ (the cissoid).
(b) $r = 2\,|\cos\theta|$.
(c) $r = 4\sin^2(\theta/2)$.
(d) $r^2 = 4\sin^2\theta$.
(e) $r^2\cos 2\theta = 4$.
(f) $r^2 = 4\cos\theta$.
(g) $r = 10 - 5\sin(3\theta/2)$.
(h) $r = 5 + 10\cos(3\theta/2)$.
(i) $2r = 3 - 2\cos 2\theta$.
(j) $r = 4 + 2\cos(\theta/2)$.
(k) $r = 40 + 20\cos\theta - 10\cos 2\theta$.
(l) $r = 3\cos(\theta/2)$.
(m) $r = (4\pi\theta - \theta^2)/\pi^2$.

5. (a) Show that the graph of $r = (2\pi - \theta)^2 + \theta^2$ *is* symmetrical with respect to the polar axis.

(b) Show that the graph of $r = (3\pi - \theta)^3 + \theta^3$ *is* symmetrical with respect to the 90° line.

6. Sketch $r = a \sin n\theta$ for $n = 1, 2, 3$, and 4 (assume a to be a positive number). Can you then guess what the curve will look like if n is an even integer? if n is an odd integer?

7. Estimate the area enclosed by one loop of $r^2 = 4\cos 2\theta$.

8. Estimate the area enclosed by one loop of $r = 4\sin 2\theta$.

9. Draw the lines tangent to the curve $r^2 = 4\sin 2\theta$ at $\theta = 45°$ and at $\theta = 30°$ and estimate their slopes.

10. Sketch on adjacent graphs (polar and rectangular coordinates):

(a) $r = 4\sin\theta$ and $y = 4\sin x$.
(b) $r = 4\sin 2\theta$ and $y = 4\sin 2x$.
(c) $r = 2\sec\theta$ and $y = 2\sec x$.

11. Several smooth slides are arranged with varying angles θ of inclination. Blocks placed at the tops of the slides are simultaneously released. It can be shown by principles of physics that one second later each block has traveled $16\sin\theta$ ft. down its slide. Draw the locus of the blocks at this instant.

12. (a) Draw a circle of radius 1.5 in. whose center is on the horizontal center line and 3 in. from the left edge of a sheet of 8.5 by 11 in. plain paper. Let the left end of the horizontal diameter of this circle be the point B. Draw a large number of circles centered on the circumference of the circle and passing through the point B; make their centers about ¼ in. apart.

(b) Plot on a new sheet of paper the graph of $r = 3(1 + \cos\theta)$ and use 1 in. = 1 unit for the radius vector.

(c) What is common between the two graphs?

13S. Discuss the set of curves $r = a + b\cos\theta$, assuming that a and b are both positive numbers. Discuss three cases and draw a sample curve for each case: (1) $a > b$, sketch for $a = 2b$; (2) $a = b$; (3) $a < b$, draw the curve for $b = 2a$.

14S.* Verify the entries in the following table for the graph of

$$r = \frac{\cos(\pi/2 \cos\theta)}{\sin\theta},$$

and complete the table at 10° intervals until $\theta = 90°$. Test for symmetry with

* The graphs for the equations in Problems 14S and 15S have to do with radiation patterns for radio stations.

respect to the 90° line and show that there is such symmetry. Then on the same graph plot $r = \sin\theta$. The resulting two graphs are the complete graphs for the two equations.

θ	$(\pi/2)\cos\theta$ $= (90°)(\cos\theta)$	$\cos(\pi/2 \cos\theta)$	$\sin\theta$	r
0°	90.000°	0.00000	0.00000	not defined
1°	89.986°	0.00024	0.01745	0.014
5°	89.657°	0.00597	0.08716	0.069
10°	88.633°	0.0237	0.174	0.136
20°	84.572°	0.0946	0.342	0.277
30°	77.943°	0.209	0.500	0.418

Also plot the corresponding rectangular-coordinate graph, using r as ordinate and θ as abscissa.

15S.* Make a table of values and plot the graphs of the following equations:

(a) $$r = \frac{\cos(2\pi\cos\theta) - 1}{\sin\theta}.$$

(b) $$r = \frac{\cos(\pi\cos\theta) + 1}{\sin\theta}.$$

(c) $$r = \frac{\cos\left(\frac{5\pi}{4}\cos\theta\right) + \cos 45°}{\sin\theta}.$$

6.4 Transformation of Coordinates

Figure 6.9 illustrates two methods of locating a point P: by rectangular coordinates (x, y), and by polar coordinates (r, θ). The student should memorize this figure, for from it he can easily write down relations such as the following: $r^2 = x^2 + y^2$, $\tan\theta = y/x$, $y = r\sin\theta$, and $x = r\cos\theta$.

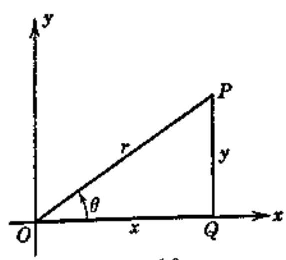


FIG. 6.9

The graph of an equation given in rectangular coordinates sometimes may be sketched more easily from the polar-coordinate equation ($r^2 = \cos 2\theta$ instead of $x^4 + 2x^2y^2 + y^4 + y^2 = x^2$), or conversely. Often the total problem dictates the use of the equation in polar form or in rectangular form. Hence the student must be able to transform from one form to the other with

* The graphs for the equations in Problems 14S and 15S have to do with radiation patterns for radio stations.

ease. The student should observe that the elementary idea of this article is to superimpose a sheet of rectangular-coordinate graph paper upon a sheet of polar-coordinate graph paper and to ask what the relationship is between the rectangular- and polar-coordinate equations for the *same curve* drawn on the two sheets of paper. We illustrate the method in the following examples:

Example 1

Transform $x^2 - y^2 = 4$ to polar coordinates.

Solution. The student should sketch and label the fundamental figure as shown in Fig. 6.9 each time that he transforms from one type of coordinate system to the other. From that figure he should read that $x = r\cos\theta$ and $y = r\sin\theta$. Then the given equation may be written $r^2\cos^2\theta - r^2\sin^2\theta = 4$, or

$$r^2(\cos^2\theta - \sin^2\theta) = 4,$$

or finally $r^2\cos 2\theta = 4$.

Example 2

Transform $r = 3\sin\theta + 4\cos\theta$ to rectangular coordinates.

Solution. From the fundamental triangle we read that $\sin\theta = y/r$ and $\cos\theta = x/r$. The given equation may then be written: $r = 3(y/r) + 4(x/r)$, or $r^2 = 3y + 4x$. But, from the figure, $r^2 = x^2 + y^2$. Hence the final result is $x^2 + y^2 = 3y + 4x$. We notice that this is the equation of a circle and hence that the graph of the polar-coordinate equation is the same circle.

Example 3

Transform $r^2\sin 2\theta = 4$ to rectangular coordinates.

Solution. In transforming from polar to rectangular coordinates the trigonometric functions must be reduced first to trigonometric functions of the angle θ itself. In this problem we may write the given equation in the form

$$r^2(2\sin\theta\cos\theta) = 4,$$

or $r^2\sin\theta\cos\theta = 2$. We draw the fundamental figure and note that $r\sin\theta = y$ and $r\cos\theta = x$. Hence the required equation in rectangular coordinates is $xy = 2$. We observe that the graph of the equation in polar coordinates is therefore an equilateral hyperbola.

PROBLEMS

1. Transform each of the following equations to an equation in rectangular coordinates, and identify the curve:

(*a*) $r = 4\cos\theta$.
(*b*) $r = 5\sin\theta$.
(*c*) $r\cos\theta = 3$.
(*d*) $r\cos\theta + 2r\sin\theta = 7$.
(*e*) $r = 3/(1 + \cos\theta)$.
(*f*) $r = 4/(2 + \sin\theta)$.
(*g*) $\theta = 2\pi/3$.
(*h*) $r = 4$.

2. Transform to polar coordinates:

(a) $xy = 2$.
(b) $x^2 + y^2 = 6x$.
(c) $x^2 - y^2 = 12$.
(d) $y^2 = 4x$.
(e) $x + 3y = 5$.
(f) $x + 4 = 0$.
(g) $y^2 = 4(x + 1)$.
(h) $(x + 4)^2/25 + y^2/9 = 1$.

3. Transform to rectangular coordinates:

(a) $r^2 = 4 \sin 2\theta$.
(b) $r^2 \cos 2\theta = 5$.
(c) $r = a \sec \theta + b$.
(d) $r = 4 \tan \theta$.
(e) $r^3 = 4 \cos \theta$.
(f) $r = 1 + \sin \theta$.
(g) $r = 2 \cos 2\theta$.
(h) $r = 4 \sin^2 (\theta/2)$.
(i) $r = 2 \cos^2 (\theta/2)$.
(j) $\theta = 2 \sin^{-1} (r/2)$.

4. Draw careful graphs on polar-coordinate graph paper of $r = \sin \theta$ and $r = \cos \theta$, and use these graphs to estimate the values of the following (you may check by aid of tables):

(a) $\cos (\pi/5)$.
(b) $\cos 1$.
(c) $\cos (1 + \pi/2)$.
(d) $\sin 5\pi$.
(e) $\sin 1.8$.
(f) $\cos (\pi + 0.5)$.
(g) $\sin (84\pi + 0.6)$.
(h) $\cos (1 - 20\pi)$.

5. Show that the curve $r = 4 + 4 \sin \theta$ (polar coordinates) goes through the point whose rectangular coordinates are $(3\sqrt{3}, 3)$.

6. Does the curve $y = 4x - x^2$ go through the point whose polar coordinates are approximately (3.16, 1.25 radians)?

7. Show that the curve $r = 5 + 2 \sin (\theta/2)$ in polar coordinates *does* go through the point with rectangular coordinates $(2, 2\sqrt{3})$.

8. Plot the graphs of $r = 2 + 2 \sin \theta$ and $2r \sin \theta + 1 = 0$. Then cross-hatch the area outside the first curve and bounded by the two curves.

9. Determine the exact value of the area above $r \sin \theta = 2$ and inside $r = 4 \sin \theta$.

10. Use a sheet of special graph paper that is ruled for simultaneous plotting of polar coordinates and rectangular coordinates. Plot $r = 9/(5 + 4 \sin \theta)$ on the polar rulings; then transform the equation to rectangular coordinates and obtain $25x^2 + 9(y + 4)^2 = 225$; finally, sketch this curve using the rectangular-coordinate rulings. The two graphs must, of course, be the same.

11S. Sketch a graph in polar coordinates of the equation e volts $= 165 \sin 120\pi t$ by using e as the radius vector and $\theta = 120\pi t$ as the angle. Then answer the following questions:

(a) If t is in seconds, how long does it take for the radius vector e to describe a *complete* revolution around the pole?

(b) What is the *total* "area" enclosed by the curve that e describes in a complete revolution (twice the area for a half revolution)?

(c) What is the radius of a circle with center at the pole, whose area is equal to the "area" found in (b)? This last result, in electrical engineering, is the effective or root-mean-square voltage. In this example, it is approximately the amount of the voltage in an ordinary home circuit for lighting purposes.

12S. Transform $r = a/(1 + e \cos \theta)$ to rectangular coordinates. Show that the curve is a parabola if $e = 1$, an ellipse if $0 < e < 1$, and a hyperbola if $e > 1$; what is the graph if $e = 0$?

6.5 Outline of Supplementary Methods for Sketching Polar-Coordinate Curves

The following additional aids in polar-coordinate sketching are included here to complete the discussion. The material of the preceding articles is fundamental and must be learned. The material included in this present article is of somewhat less importance at this stage of the development of the student's mathematical maturity.

I. Addition of radius vectors. To sketch $r = 3 + 2 \sin \theta$, sketch the two component curves $r_1 = 3$ and $r_2 = 2 \sin \theta$, and add radius vectors, being extremely careful to add radius vectors corresponding to the *same* angle.

II. The equivalent of translation of axes in rectangular coordinates. In sketching the graph of $r = 4 \sin (\theta + \pi/6)$, the replacement of $\theta + \pi/6$ by θ' yields the simpler equation $r = 4 \sin \theta'$. The effect of this replacement is *not* to translate axes but to rotate them, so that in this example the polar axis is rotated through an angle of $-30°$. On the other hand, if one replaced $r - 3$ by r' in the equation

$$r = 3 + 2 \sin \theta,$$

the resulting equation would be $r' = 2 \sin \theta$. The graphical effect of this substitution is as follows (compare with the preceding paragraph): What were originally directed lengths measured along radius vectors from the circle $r = 3$ are now directed lengths measured from the pole.

III. Periodicity for polar-coordinate graphs. Suppose the polar-coordinate equation to be $r = f(\theta)$. The curve will repeat if there exists a positive integer q such that $f(\theta + 2\pi q) \equiv f(\theta)$. Sometimes the curve will be periodic for half of the smallest such positive integer q.

IV. Use of the layout or the rectangular-coordinate graph. Suppose that it is required to sketch a graph of $r = f(\theta)$, and that it would be easier to sketch the graph in rectangular coordinates interpreting r as the ordinate and θ as the abscissa instead of as radius vector and angle respectively; that is, it is easier to sketch the graph of $y = f(x)$. The polar-coordinate graph can then be obtained by an easy and rapid mechanical process. For example, the rectangular-coordinate graph of $y = 3 + 3 \sin x$ is easy to sketch. Then the interval from $x = 0$ to $x = 2\pi$ can be divided into an arbitrary number of equal subdivisions, eight, for example. Lay off a radius vector at $\theta = 45°$ on the polar-coordinate graph and use for its directed length the ordinate to the rectangular graph at $x = \pi/4$. Repeat at $\theta = 90°$, $135°$, etc.

V. Outline of special curves in polar coordinates.

A. Straight lines. $\theta = k =$ constant, $r \cos\theta = k$, and $r \sin\theta = k$.

B. Circles. $r =$ constant, $r = a \sin\theta$, and $r = a \cos\theta$.

C. Conics. $r = a/(b + c\cos\theta)$, $r = a/(b + c\sin\theta)$.

D. Cardioid-type curves (heart-shaped). $r = a + b\cos\theta$ and $r = a + b\sin\theta$ (the three types are shown in Fig. 6.10 and are easily distinguished by the relative positions of the four points obtained by the substitution of $\theta = 0°, 90°, 180°$, and $270°$).

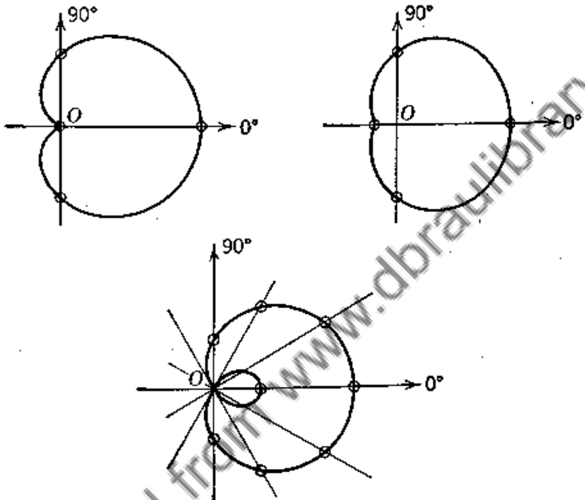


FIG. 6.10

E. Rose-type curves. $r = a \sin n\theta$ and $r = a \cos n\theta$ for n an even integer and for n an odd integer. When n is an even integer the roses have $2n$ petals; when n is odd, they have n petals.

F. Spirals. $r = a\theta$ (the spiral of Archimedes), $r^2\theta = k$ (the lituus), $r = a \ln\theta$ (the logarithmic spiral), $r = ae^{b\theta}$ (the exponential spiral), and $r\theta = c$ (the hyperbolic or reciprocal spiral).

PROBLEMS

1. Sketch each of the following curves:

(*a*) $r \sin\theta = 2$.
(*b*) $r = 4\sin\theta$.
(*c*) $r(2 + 2\cos\theta) = 3$.
(*d*) $r = 2 + \sin\theta$.
(*e*) $r = 2 + 2\sin\theta$.
(*f*) $r = 2 + 4\sin\theta$.
(*g*) $r = 4\sin\theta$ and $r = 4\sin 2\theta$ on the same graph.
(*h*) $r = 4\sin 3\theta$.
(*i*) $r = 4\sin 4\theta$.

2. Make a list (by equation and name) of the polar-coordinate curves listed in the *Encyclopædia Britannica* under the heading *Curves*.

3. Show that $r\cos\theta = k$, $r\sin\theta = k$, and $Ar\cos\theta + Br\sin\theta + C = 0$ are straight lines.

4. Show that $r = a\cos\theta$ is a circle that has its center on the polar axis, that is symmetrical with respect to the polar axis, and that goes through the pole. Show that $r = b\sin\theta$ and $r = c$ are likewise circles.

5. Show that $r = ek/(1 - e\cos\theta)$, $k \neq 0$ (in polar coordinates) and $x^2(1 - e^2) + y^2 - 2ke^2x = k^2e^2$ (in rectangular coordinates) are the same locus. Thus, show that the two equations are equivalent descriptions for conics: parabolas (or two parallel lines) if e = eccentricity = 1, ellipses (or a point locus or no real locus) if $0 \leqq e < 1$, and hyperbolas (or two intersecting lines) if $e > 1$.

6. Sketch the polar-coordinate graph of each of the following curves that are defined by use of the rectangular layout curve (that is defined for $0 < x < 2\pi$ and is understood to repeat with period 2π). Also, give the rectangular- and polar-coordinate equations of the graph on the polar-coordinate paper.

(*a*) The line segment $(0, 2)$ to $(\pi, 4)$ and the line segment $(\pi, 4)$ to $(2\pi, 2)$.

(*b*) The line segment $(0, 2)$ to $(\pi/2, 4)$, the line segment $(\pi/2, 4)$ to $(3\pi/2, 4)$, and the line segment $(3\pi/2, 4)$ to $(2\pi, 2)$.

(*c*) The line segment $(0, 2)$ to $(\pi, 4)$ and the line segment $(\pi, 4)$ to $(2\pi, 4)$.

(*d*) The parabola (with a vertical axis) that goes through $(0, 2)$, $(\pi, 4)$, and $(2\pi, 2)$.

(*e*) The curve $y = a + 2\sin x$ if (1) $a = 2$, (2) $a = 0$, and (3) $a = -1$.

6.6 Locus Derivations in Polar Coordinates

We illustrate the method of locus derivation by the solution of the following examples:

EXAMPLE 1

Derive the polar-coordinate equation of the circle with center at $(a, \pi/2)$ and radius a.

Solution. I. Sketch figure and label the given data (see Fig. 6.11).

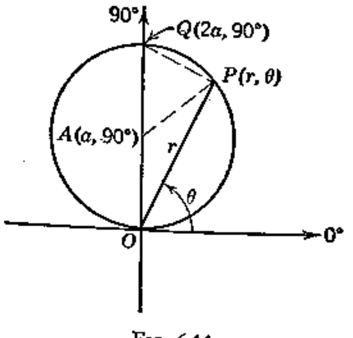


FIG. 6.11

II. Select a general point $P(r, \theta)$ on the circle.

III. Make a geometric statement. Triangle OPQ is a right triangle with right angle at P.

IV. Translate to algebra. $\cos(90° - \theta) = r/(2a)$.

V. Simplify. $r = 2a \sin\theta$.

VI. Check. Use $\theta = 45°$, $r = a\sqrt{2}$ to check the work.

EXERCISE FOR THE STUDENT. Derive the result of this example by use of triangle OPA, an isosceles triangle, and draw a perpendicular from A to OP.

EXAMPLE 2

Derive the equation of the locus of a point that moves so that its distance from the pole divided by its distance from the straight line $r \cos\theta = -k$ is always a constant e.

Solution. I. Sketch figure and label the pole O and the given line.

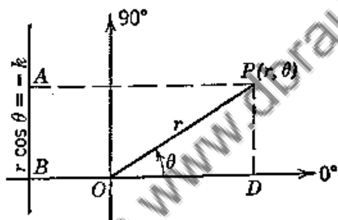


FIG. 6.12

II. Select a general point with coordinates (r, θ), and draw the perpendicular from this point to the given line.

III. Geometric Statement. $\overline{OP}/\overline{AP} = e$, or $\overline{OP} = e(\overline{AP})$.

IV, V. $\overline{OP} = r$, $\overline{AP} = \overline{BO} + \overline{OD} = k + r\cos\theta$. Hence $r = e(k + r\cos\theta)$, or $r = ek/(1 - e\cos\theta)$.

VI. Check. When $\theta = 0$, $r = ek/(1 - e)$. The student should complete this check.

Remark. The equation in this example is the equation for a conic; if $e = 1$, it is a parabola (or two parallel lines); if $e > 1$, it is a hyperbola (or two intersecting lines); if $0 < e < 1$, it is an ellipse (or a point locus or no real locus); and if $e = 0$, it is a point circle.

PROBLEMS

1. Derive by the locus-derivation method the equation of the straight line perpendicular to the polar axis and at a distance of 4 units to the right of the pole.

2. Derive the equation of the circle whose center is at $(a, 0)$ and that goes through the pole.

3. Derive the equation of the locus of a point that moves so that its distance from the pole is always twice its distance from the straight line $r\cos\theta = -2$.

4. Derive the equation of the locus of a point that moves so that its distance from the origin (pole) is always equal to its distance from the line $x = -q$ ($r \cos\theta = -q$). Perform the derivation in rectangular coordinates (or write the equation by inspection) and transform your result to polar coordinates. Then rework the problem directly in polar coordinates.

5. Find, by at least two different methods, the equation in polar coordinates of the ellipse that in rectangular coordinates has its foci at the origin and (6, 0), and its vertices at (−2, 0) and (8, 0). *Hint:* One method would be to use the locus-derivation method in polar coordinates: to find the equation of the locus of a point that moves so that the sum of its distances from the pole and (6, 0°) is always a constant, 10.

6. Derive the equation in polar coordinates of a straight line whose inclination is 30° and whose y-intercept is positive, if the perpendicular from the pole to this line intersects the line at (5, 120°) in polar coordinates.

7. Derive the equation of the locus of a point (r, θ) that moves so that for a given angle θ the radius vector r is, for the same angle, twice the radius vector to the curve $r = a \cos\theta$.

8. Derive the equation of the locus of a point (r, θ) that moves so that for a given angle θ the radius vector r is, for the same angle, always equal to 3 more than 3 times the radius vector to the curve $r = 4 \sin\theta$.

9. Find, both by a direct locus derivation in polar coordinates and by transforming the rectangular-coordinate equation to polar coordinates, the equation of the parabola that has its vertex at (2, 0°) and its focus at the pole.

10. An elliptical cam rotates on an axle through one focus of the ellipse. The ellipse has a major axis of 10 and a minor axis of 6. Find the polar-coordinate equation of the cam when the other focus is on the polar axis to the right of the pole. Then replace θ by $\theta + \omega t$ to obtain the equation at time t.

11. Derive the equation of the locus of a point P that moves so that it satisfies the following condition:

(*a*) The product of its distances from the two fixed points $(a, 0°)$ and $(a, 180°)$ is a^2. This locus is called the *lemniscate*.

(*b*) Consider the circle with a diameter joining the pole O and the point $D(a, 0°)$. Let an arbitrary radius vector intersect the circle at the point A and let it intersect at the point B the line that is tangent to the circle at D. The point P moves so that $\overline{OP} = \overline{AB}$. This locus is called the *cissoid of Diocles*.

(*c*) Let a radius vector intersect the vertical line defined by $r \cos\theta = a$ at the point A. The point P is on this arbitrary radius vector such that the numerical distance $\overline{AP}$ is b. This locus is called the *conchoid of Nicomedes*. (This locus may be used to trisect a given angle. It was developed by Nicomedes, a Greek mathematician of the second century A.D., as an aid for the purpose of constructing the edge of a cube whose volume is twice the volume of a given cube.)

(*d*) Consider the circle with a diameter joining the pole O and the point $A(a, 0°)$. Let an arbitrary radius vector intersect the circle at B. The point P is on this arbitrary radius vector and is such that its numerical distance from B is b. This locus is called the *limaçon of Pascal*.

6.7 Simultaneous Equations in Polar Coordinates

We shall be concerned in this article with the problem of the simultaneous solution of two equations. If the graphs of the equations are easier to draw when we interpret the two variables as radius vector and angle than if we interpret them as ordinate and abscissa, then it will clearly be simpler to utilize polar coordinates in their solution. The problem of solving simultaneously two equations in polar coordinates is the problem of finding all pairs of numbers (r, θ) that satisfy both equations. Usually this will be done most easily by solving both equations for r in terms of θ, equating, and solving the resulting trigonometric equation for θ.

Because of the fact that each point has an infinite number of pairs of polar coordinates, it can happen that the graphs of two polar-coordinate curves will intersect at a point one of whose pairs of coordinates will satisfy one equation but will not satisfy the other equation, while another pair will satisfy the second equation but not the first equation. Such points of graphical intersection may be found approximately by graph.*

EXAMPLE 1

Solve simultaneously $r \cos \theta = 2$ and $r = 6 \sin \theta$.

Algebraic Solution. We eliminate r and obtain $2/\cos \theta = 6 \sin \theta$, or

$$2 = 6 \sin \theta \cos \theta,$$

or $\frac{2}{3} = \sin 2\theta$. From tables of the trigonometric functions we obtain:

$$\begin{array}{lllll} 2\theta \approx 41^\circ 49'; & 138^\circ 11'; & 401^\circ 49'; & 498^\circ 11'; & \text{etc.} \\ \theta \approx 20^\circ 54'; & 69^\circ\ 6'; & 200^\circ 54'; & 249^\circ\ 6'; & \text{etc.} \\ r \approx 2.141; & 5.605; & -2.141; & -5.605; & \text{etc.} \end{array}$$

The corresponding values for r were obtained by substituting the value of θ in the second of the two given equations.

Graphical Solution. The graphs of the two equations are shown in Fig. 6.13. The corresponding pairs of numbers (r, θ) are then read by aid of a protractor and

* To find by algebraic methods *all the points* where two curves intersect, we can solve simultaneously *all* the possible pairs of equations obtained by replacing θ in one equation by $\theta + n\pi$, and in the other equation by $\theta + m\pi$, where m and n are positive or negative integers. Thus, for example, to find all the points where the two curves $r = 2\theta$ and $\theta = \pi/3$ intersect, we can solve simultaneously $r = 2(\theta + n\pi)$ and $\theta + m\pi = \pi/3$ with m and n arbitrary. In this example, there are an infinite number of different points of intersection and these are all given by $(2k\pi + 2\pi/3, \pi/3)$ with $k = 0, \pm 1, \pm 2, \cdots$.

scale. We find (2.1, 21°) and (5.6, 69°) to be two solutions, and the other pairs may be obtained from these two.

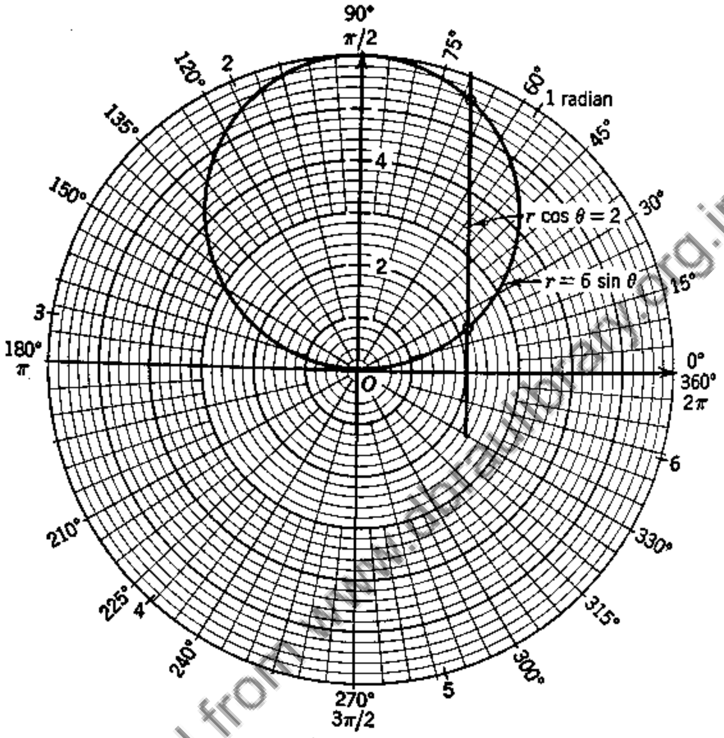


Fig. 6.13

Example 2

Solve simultaneously $r = 1 - \sin\theta$ and $r = \sin\theta - 1$ by algebraic methods, by polar-coordinate graph, and by a rectangular-coordinate graph (draw $y = 1 - \sin x$ and $y = \sin x - 1$).

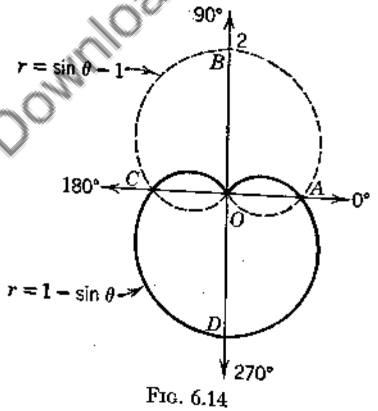


Fig. 6.14

Solution. From $1 - \sin\theta = \sin\theta - 1$ we obtain $\sin\theta = 1$, and hence $\theta = \pi/2$, $5\pi/2$, $9\pi/2$, etc., or $2n\pi + \pi/2$ in general form. The algebraic solutions, then, may all be written in the form $(0, \pi/2 + 2n\pi)$, where n is any positive or negative integer or zero. The polar-coordinate graphs for the two equations are given in Fig. 6.14, and from this graph we read for solutions: $(1, 0)$, $(1, \pi)$, and $(0, \pi/2)$. The first two pairs of numbers do not simultaneously satisfy the two given equations, but

(1, 0) satisfies one equation and $(-1, \pi)$ satisfies the other. Similarly, $(1, \pi)$ satisfies one equation and $(-1, 0)$ satisfies the other. We show the simultaneous solution by aid of the graph in rectangular coordinates in Fig. 6.15 and find again the same solutions that we found by the algebraic method.

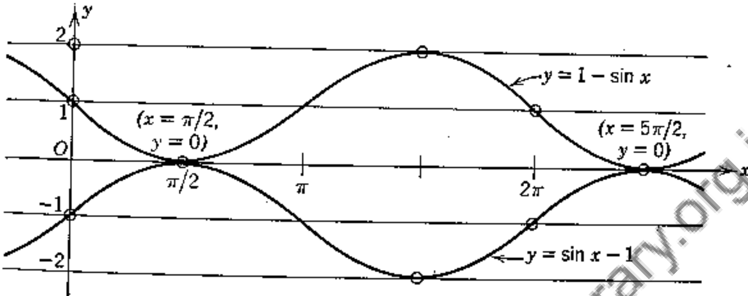


Fig. 6.15

The difficulty encountered in the polar-coordinate graphical solution arises from the fact that each point has an infinite number of different pairs of polar coordinates. The polar-coordinate graphical solution in this example has an interpretation that will be of interest to the student. Suppose that the two curves in polar coordinates represent race tracks, the unit of distance being 800 ft. Suppose that two cars start at the pole, one to travel around the first curve and the other around the second curve. Both are to travel at the same speed and both are to travel in the direction of the curve indicated by increasing angle θ. This means that one car starts at the pole on the unbroken curve ($\theta = 90°$) shown in Fig. 6.14 and travels around that curve in a counterclockwise direction. The second car starts at the pole (again $\theta = 90°$) and goes around the dotted curve in a counterclockwise direction. When car 1 arrives at the point C, car 2 is at A; when car 1 is at D, car 2 is at B; when car 1 is at A, car 2 is at C; etc. Thus the two cars pass points A and C at different times and do not meet there. On the other hand, both cars will arrive back at the pole ($\theta = 450°$) at the same time and will continue to meet at the pole at the end of each complete circuit of their respective tracks.

The student should welcome the material of this article, for it yields an excellent opportunity to review the solution of simultaneous equations and to review basic trigonometry.

PROBLEMS

1. Solve simultaneously by algebraic methods:

(a) $r = 2 + \cos\theta$, $r = 5\cos\theta$.
(b) $r = 4\cos\theta$, $r = 4\sin\theta$.
(c) $r\cos\theta = 2$, $r\sin\theta = 4$.
(d) $r = 1 + \cos\theta$, $r + 1 + \cos\theta = 0$.
(e) $r = \sin\theta$, $r = \cos 2\theta$.
(f) $r = 4/(1 + \sin\theta)$, $r(1 - \sin\theta) = 3$.
(g) $r = 4\sin 2\theta$, $r = 2$.
(h) $r = 4\cos 2\theta$, $r = 4\cos\theta$.
(i) $r^2 = 4\sin 2\theta$, $r = 1$.
(j) $r^2 = 4\cos 2\theta$, $r = 2\sin\theta$.

2. Solve each of the problems in Problem 1 by graphical methods.

3. Determine the points of intersection of $r = 1 + \cos\theta$ and $r = 1$. Then estimate the area outside $r = 1$ and inside $r = 1 + \cos\theta$.

4. Determine the coordinates of the two points (A and B) of intersection of $r = 6\cos\theta$ and $r\cos\theta = 4$. Then find the area of triangle OAB, where O is the pole.

5. Solve simultaneously $r = 4\sin\theta$ and $r(1 + \sin\theta) = 3$. Then estimate the area above the second curve but inside the first.

6. Find the area of the triangle determined by the three graphical points in which the two curves $r = 3\cos\theta$ and $r = 1 + \cos\theta$ intersect.

7. Find, correct to three significant figures, the slope of the line through the points of intersection of the two curves $r = 2\sin\theta$ and $r = 2\sin^2(\theta/2)$.

6.8 Review of Polar Coordinates

The student should review the two chief methods of sketching curves in polar coordinates: point plotting and the discussion method (symmetry, open or closed curve, variation). The following review questions should enable the student to fix the basic ideas of this chapter thoroughly in his mind.

REVIEW QUESTIONS

1. If $r = 4\sin 3\theta$, what is the value of r when $\theta = 5\pi/12$, and in which quadrant does the corresponding point lie? What are four pairs of polar coordinates for the point whose rectangular coordinates are $(-2, 6)$?

2. What are tests for symmetry with respect to the polar axis, the 90° line, and the pole? How does $r = 2 + 2\sin(3\theta/2)$ vary as θ increases from 0° to 60°, and from 60° to 120°?

3. What are two different graphical interpretations for the equation $s = 4\sin t$? Draw them. What is a second equation for a given curve if one equation is $xy = 4$? What is a second equation if one equation is $r\sin\theta = 2$?

4. What are the six steps in any locus derivation? Illustrate them with this problem: Find the equation in polar coordinates of the locus of a point that moves so that the sum of its distances from $(0, 0°)$ and $(8, 0°)$ is always 10.

5. What are four solutions of $r = 4\sin\theta$? What are two simultaneous solutions of $r = 4\sin\theta$ and $r\sin\theta = 1$?

REVIEW PROBLEMS

1. Give four other pairs of coordinates for $(-2, \pi)$.

2. Transform $r^2 \cos\theta = 4 \sin\theta$ to rectangular coordinates.

3. Show that the curve $r^2 = 4 \sin\theta$ goes through $(-2, \pi/2)$. Then determine four other pairs of coordinates for the same point, and determine whether each pair of coordinates satisfies the equation.

4. A cam has for its theoretical curve the equation $r = 2 + 0.8 \sin 2\theta$. Sketch the polar-coordinate graph and also the layout curve, that is, the curve $y = 2 + 0.8 \sin 2x$.

5. Find directly in polar coordinates the equation of the locus of a point that moves so that its distance from the pole is always equal to its (the moving point's) distance from the line $r \cos\theta = -2$. Then obtain the required equation first by finding the equation in rectangular coordinates and then by transforming to polar coordinates.

6. Sketch $r = c + 2 \cos\theta$ for the four cases $c = -1, 0, 2$, and 3. Also sketch $y = c + 2 \cos x$ in rectangular coordinates for the same four cases.

7. Sketch $r + 2r \cos\theta = 3$, $r \cos(\theta - \pi/6) = \sqrt{3}$, and $r \cos(\theta + \pi/6) = \sqrt{3}$ on the same graph. Then transform all three equations to rectangular coordinates and sketch all three again on rectangular-coordinate graph paper (to obtain the same graph). What relationships exist among the three curves on either graph paper?

8. If, in the equation $r = \tan\theta$, we replace θ by $\pi + \theta$, what equation results and what type of symmetry is indicated? If we substitute $-r$ for r and $\pi - \theta$ for θ, what equation results and what type of symmetry is indicated? Finally, make a table of values for $\theta = 0°, 15°, 30°, 45°, 60°, 75°$, and $90°$, and plot the entire curve.

9. Sketch the three curves on the same sheet of graph paper: $r = 4 \sin\theta$, $r = 4 \sin(\theta + 2\pi/3)$, $r = 4 \sin(\theta + 4\pi/3)$. Then sketch the corresponding equations, $y = 4 \sin x$, etc., on the same sheet of rectangular-coordinate graph paper.

10S. Plot on polar-coordinate paper the graphs of

(a) $r = \sin\theta + \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin 5\theta$.
(b) $r = \cos\theta + \frac{1}{9} \cos 3\theta + \frac{1}{25} \cos 5\theta$.

11. Find the equation of the locus of a point that moves so that the product of its distances from $(4, 0°)$ and $(-4, 0°)$ is always 10.

12S. Find coordinates for all *seventeen* points of intersection of $r = \cos(2\pi \cos\theta)$ and $r = \sin(2\pi \cos\theta)$.

CHAPTER 7

Parametric Equations

There are three important forms for equations in two variables that arise in scientific study: rectangular-coordinate equations, polar-coordinate equations, and parametric equations. Of the three, the most natural form to use in any problem in which time is an important factor is the parametric representation. The present chapter concerns the question of curves defined by two equations of the form: $x = f(t)$ and $y = g(t)$, where $f(t)$ is a function of some parameter t, and $g(t)$ is another function of the same parameter, and (x, y) are the usual rectangular coordinates of a point. This parameter, or auxiliary variable, may be the time element, and this is the source of the importance of parametric equations to science and engineering.

We shall again be concerned with the two basic problems of analytic geometry: to draw the graph and to find the equation.

7.1 Plotting by Points

We consider in this article the simplest method whereby we may draw the graph of a curve defined by two parametric equations. The following examples will illustrate the method:

Example 1

Plot the graph of the curve defined by $x = 2 + 3t$, $y = 4 - t$, where t is the parameter.

Solution. We make a table of values for x and y by assigning values to t and computing the corresponding values of x and y.

t	x	y
0	2	4
1	5	3
2	8	2
3	11	1
4	14	0
5	17	−1

We now plot the pairs of values (x, y) just as was done in the preceding chapters of this text. We draw a smooth curve through the plotted points as shown in Fig. 7.1; in this particular example the result seems to be a straight line. We can prove that it is precisely a straight line by eliminating the parameter and thus obtaining the equation that involves only x and y. We do this by solving the

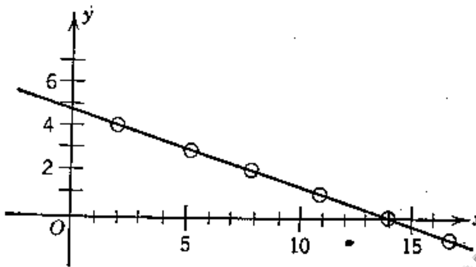


FIG. 7.1

second equation for t in terms of y: $t = 4 - y$; we substitute this result in the first equation and obtain $x = 14 - 3y$, which is, of course, the equation of a straight line. Had we done this in advance, we would have needed only two points to plot the required graph.

EXAMPLE 2

Plot the curve defined by $x = 2 + 4 \cos \phi$, $y = 1 - 2 \sin \phi$, where ϕ is a parameter.

Solution. We assign values to the parameter ϕ and obtain the results shown in the following table. We then plot the corresponding pairs of numbers (x, y) and draw a smooth curve. We should assign enough values to this parameter to assure a neat smooth curve through the points.

ϕ	x	y
0°	6	1
45	4.83	−0.41
90	2	−1
135	−0.83	−0.41
180	−2	1
225	−0.83	2.41
270	2	3
315	4.83	2.41
360	6	1

The student should observe that there are hardly enough points plotted in Fig. 7.2. If he had been doing this problem, he should have computed further pairs of values for (x, y) before drawing the curve.

The resulting curve looks like an ellipse. To prove that it is an ellipse, we may eliminate the parameter ϕ in the following manner: We solve the equations for $\cos\phi$ and $\sin\phi$ respectively and obtain $\cos\phi = (x-2)/4$, $\sin\phi = -(y-1)/2$. We substitute these in the fundamental identity $\sin^2\phi + \cos^2\phi = 1$ and obtain

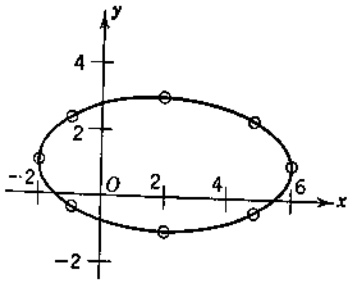


FIG. 7.2

$$\frac{(x-2)^2}{16} + \frac{(y-1)^2}{4} = 1.$$

The student knows this to be an ellipse and can sketch it by translating axes and then by making use of the intercepts on the new axes.

Lest the student suppose that it would be simpler in every problem first to eliminate the parameter, we make the following observations:

1. Often the parameter cannot be eliminated, or, if it can be eliminated, the resulting equation may be cumbersome. For example, from the pair of equations $x = 2(\phi - \sin\phi)$, $y = 2(1 - \cos\phi)$, we obtain

$$x = 2 \operatorname{arc} \cos\left(1 - \frac{y}{2}\right) \pm \sqrt{4y - y^2}.$$

2. Sometimes the parametric equations yield a graph that is only a part of the graph defined by the rectangular-coordinate equation (obtained by eliminating the parameter), and this part of the curve is all that is of importance in the problem in science or engineering. For example: A point on an eccentric cam describes the curve $x = 2\cos\pi t$, $y = 4\sin^2\pi t$, where t is in seconds. The curve is sketched in Fig. 7.3. For several oscillations of the cam the times are shown when the moving point is at each of the points of intersection with the coordinate axes. When we eliminate the parameter t between these two equations, we obtain $y = 4 - x^2$, a parabola, of which the curve shown in the figure is only a part. The parametric equations describe specifically the motion of the point on the cam, and the rectangular-coordinate equation is of little interest.

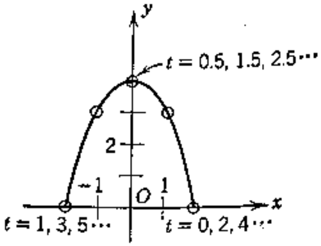


FIG. 7.3

3. In many problems in science and engineering, the position of a point is a function of time, which may therefore be regarded as a parameter. Hence, in any derivation for the position of such a point at any general time t, it would be natural to use a locus-derivation method that would yield equations for x and y, each in terms of t. The following problem, whose solution is given in textbooks on the calculus, illustrates the use of time as a parameter: A gun is fired at an angle of 38° with the horizontal, the initial speed of the shell being 1000 ft./sec. If air resistance is neglected and if the shell is assumed to be a small object, the problem in calculus will be to determine the trajectory (path) of the shell. The solution of that problem leads directly to the two parametric equations $x = 1000\,t\cos 38°$, $y = 1000\,t\sin 38° - gt^2/2$, where g is the gravitational constant (about 32.2 ft./sec.^{2}).

PROBLEMS

1. In Example 1, determine from the original parametric equations the coordinates of the points where the straight line crosses the two axes. Thus, let $x = 0$, solve for t, and then determine y.

2. In Example 2, determine the x- and y-intercepts directly from the parametric equations.

3. Make a table of values for x and y by assigning arbitrary values to the parameter (t or ϕ), and plot the following curves. *Then* eliminate the parameter and identify, if possible, the resulting curves.

(a) $x = 5 - 3t,\ y = 1 + 2t$.
(b) $x = 4\cos\phi,\ y = 2\sin\phi$.
(c) $x = 3\tan\phi,\ y = 3\sec\phi$.
(d) $x = 1 - 2t,\ y = t^2 - 3t$.
(e) $x = 1 + t + t^2,\ y = 3 - t - t^2$.
(f) $x = 3\sin\phi,\ y = 3\cos\phi$.
(g) $x = 400t\cos 30°,\ y = 400t\sin 30° - 16.1t^2$.
(h) $x = 1 + 3\sin 2\phi,\ y = 5 + 3\cos 2\phi$.
(i) $x = 4\cos 2\phi,\ y = 2\sin\phi$.
(j) $x = 2\sin(\phi/2),\ y = 2\cos\phi$.

4. Make a table of values for r and θ by assigning arbitrary values to the parameter t, and plot the following curves in polar coordinates:

(a) $r = 1 + t,\ \theta = 0.2t^2$.
(b) $r = 4t^2(1 - t^2),\ \theta = \arcsin t$.
(c) $r = 20 + 10\sin 120\pi t,\ \theta = 120\pi t$.
(d) $r = 10 + 5\sin 1000\pi t + 2\sin 2000\pi t,\ \theta = 1000\pi t$.

5. Plot each of the following equations from the parametric form. Then eliminate the parameter and sketch in the remainder of the graph of the rectangular equation, showing it in dotted form. (The parameter in each case is to take on only real values.)

(a) $x = 4\cos^2\phi,\ y = 4\sin^2\phi$.
(b) $x = 4 - \cos\phi,\ y = \sin^2\phi$.

(c) $x = a\cos^4\phi$, $y = a\sin^4\phi$, where a is a positive constant.
(d) $x = t^2$, $y = 4t^2 - t^4$.
(e) $x = 2\cos\theta$, $y = 8\sin^2(\theta/2)$.
(f) $x = 4\cos 2\theta$, $y = 2\cos\theta$.
(g) $x = 4\cos 2\phi$, $y = \sin 2\phi$.
(h) $x = 2\cos^2\pi t$, $y = 2 - \sin^2\pi t$.

6. Obtain parametric equations in each of the following problems by making the indicated substitution:

(a) $x^2 + y^2 = 4$, $x = 2\cos\phi$ (with parameter ϕ).
(b) $x^2 + y^2 = 4$, $y = mx$ (with parameter m).
(c) $x^2 + y^2 = 4$, $y = 2 + kx$ (with parameter k).
(d) $x^2 - y^2 = 4$, $y = mx$ (with parameter m).
(e) $x^2 - y^2 = 4$, $y = 2\tan\theta$ (with parameter θ).
(f) $y = 4 - x^2$, $x = 2\cos\theta$ (with parameter θ).

7S. Use the parametric equations $x = 3am/(1 + m^3)$ and $y = 3am^2/(1 + m^3)$, where the parameter is m, and solve the following problems:

(a) Make a table of values for (x, y) by assigning arbitrary values to the parameter m (a is a positive constant). Then plot, using 1 in. = a units on both axes.

(b) Eliminate the parameter by first showing that $x^3 + y^3 = 27a^3m^3/(1 + m^3)^2$, and then combine this result with the two given equations.

(c) Solve the given pair of equations simultaneously with $x + y + a = 0$ by substituting expressions for x and y each in terms of m. The resulting equation is $(m + 1)^3 = 0$. Since neither x nor y is defined when $m = -1$, the given curve does not intersect this straight line. Plot this line on your graph and tell, if you can, its relation to the given curve.

7.2 Graphical Method for Sketching with Parametric Equations

Instead of making a table of values from two parametric equations, we may draw separate graphs of x in terms of the parameter and of y in terms of the parameter. We may then use these two graphs to read values of x and y that correspond to each value of the parameter, and thus plot the required graph. Thus we may substitute the drawing of two graphs (by any method appropriate) for the computation of a table of values. We use the following illustration to explain the method:

EXAMPLE

Obtain the graph of the curve defined by the parametric equations: $x = 4\cos\theta$, $y = 2\sin\theta$.

Solution. We draw the graphs of x and y each in terms of the parameter as shown in Fig. 7.4. We then read from the two graphs corresponding values for

x and y, and plot the resulting points. Thus, for example, when $\theta = \pi/3$ we read $x = 2$, $y = 1.73$; we then plot (2, 1.73). The final graph is shown as the third curve in Fig. 7.4.

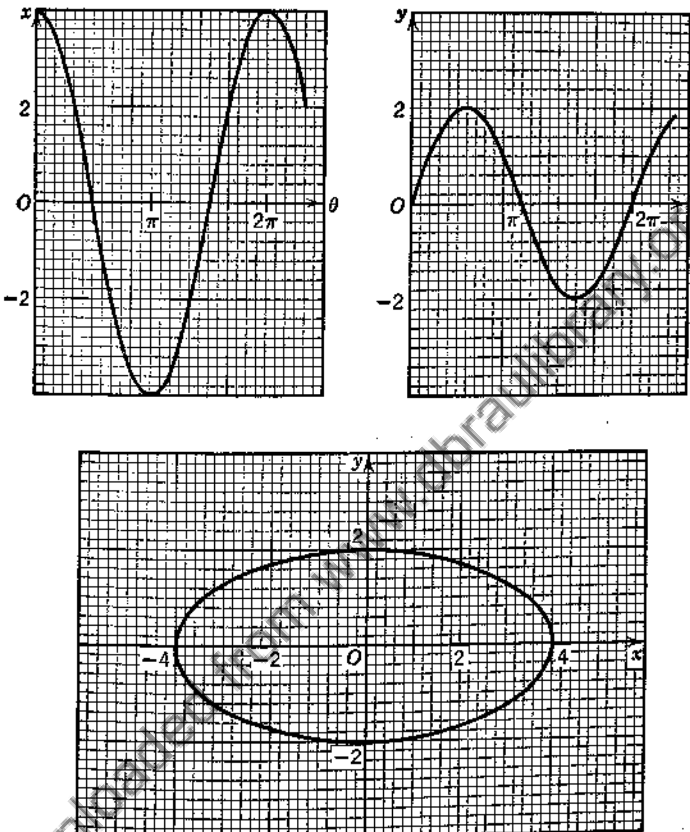


Fig. 7.4

PROBLEMS

1. Plot by the graphical method the graphs of the curves defined by the following pairs of parametric equations:

(a) $y = t^2$ and $x = 2t$ for $-3 < t < 3$.
(b) $x = 4 \sin \theta$, $y = 2 \sin \theta$.
(c) $x = 40 \sin 200\pi t$, $y = 20 \cos 200\pi t$.
(d) $x = 4 \sin 120\pi t$, $y = 2 \cos 60\pi t$.
(e) $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ for $0 < \theta < 4\pi$.

(*f*) $x = 2\cos\theta + \cos 2\theta,\ y = 2\sin\theta - \sin 2\theta.$
(*g*) $x = 2\cos\theta - \cos 2\theta,\ y = 2\sin\theta - \sin 2\theta.$
(*h*) $x = 3\cos\theta - \cos 3\theta,\ y = 3\sin\theta - \sin 3\theta.$

2. Plot the curves in Problem 1 either by making a table of values or by eliminating the parameter.

3. Use 1 in. = 4 units on the x-axis and y-axis, and 1.5 in. = 0.01 sec. on the t-axis. Use the graphical method to determine the graph of y in terms of t, if $y = 0.03(x + 20)^2$ and if x in terms of t is given as follows:

(*a*) $x = -6 + 6\sin 100\pi t.$
(*b*) $x = -8 + 4\sin 100\pi t.$
(*c*) $x = -12 + 8\sin 100\pi t.$
(*d*) $x = -10 + 10\sin 100\pi t.$

4. Use the graphical method to determine the graph of y in terms of the parameter t, if $y = 0.4x$ and if x in terms of the parameter t is given by

(*a*) $x = 4\sin 200\pi t.$
(*b*) $x = 2 + 0.8\cos 500\pi t.$

5. If $y = \sqrt{100 - x^2}$ and if $x = 10\sin 100\pi t$, determine the graph of y in terms of the parameter t. Also determine the equation for y in terms of t.

6. Obtain the graph of each of the following curves:

(*a*) $x = \sin 200\pi t,\ y = 2\cos 200\pi t.$
(*b*) $x = 5\sin 300\pi t,\ y = 3\cos 200\pi t.$
(*c*) $x = 3\sin 100\pi t,\ y = 4\sin(100\pi t - \pi/3).$

7.3 Graph of a Function of a Function

Suppose that $y = f\{g(x)\}$. We may utilize the graphical method described in the preceding article to determine the graph of y in terms

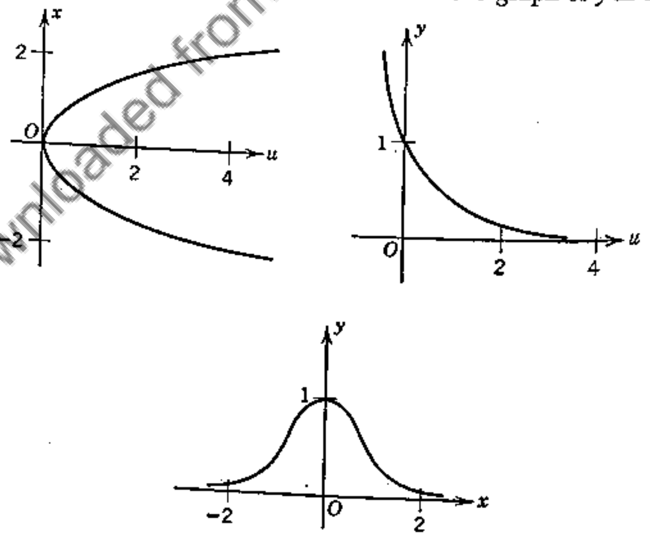


Fig. 7.5

of x. Thus, we may introduce the parameter u by defining $u = g(x)$. Then $y = f(u)$, and the problem becomes that of drawing the rectangular-coordinate graph of the curve defined by the parametric equations $y = f(u)$, $u = g(x)$. We illustrate with an example.

EXAMPLE

Obtain the graph of $y = e^{-x^2}$.

Solution. We define $u = x^2$, from which $y = e^{-u}$, and complete the solution as shown in Fig. 7.5.

PROBLEMS

Introducing a parameter that is appropriate, use the graphical method to sketch the graphs of the following curves:

1. $y = \sin(\pi x^2)$ for $0 < x < 4$.
2. $y = \ln(\sin x)$ for $0 < x < 4\pi$.
3. $y = 10 \sin(\theta + 0.5 \sin 2\theta)$ for $0 < \theta < 2\pi$.
4. $y = 4/(2 + x^2)$ (use $u = x^2$).
5. $y = \sin(\pi/x)$ for $0.1 < x < 10$.
6. $y = 4 \cos(\pi\sqrt{x})$ for $0 < x < 25$.
7. $y = \ln(x^2 + 2)$ (use $u = x^2$).
8. $y = 2e^{0.6 \sin 10\pi t}$.
9. $y^2 = x^2(4 - x^2)$ (take $x = 2 \sin \theta$).

7.4 Locus Derivations with Parametric Equations

We shall illustrate the locus-derivation method in this new connection by the following examples:

EXAMPLE 1

Determine the parametric equations for the locus of a point that moves along a straight line starting at the point (2, 3), if the point always has a speed in the x-direction of 10 ft./sec. and in the y-direction of 20 ft./sec.

Solution. In this simple problem the enumeration of the steps in the locus derivation is omitted. A speed in the x-direction of 10 ft./sec. means that, t sec. after the point starts moving, the abscissa of the point is $10t$ ft. to the right of (2, 3). Hence, at time t sec., the *abscissa* of the moving point is $x = 2 + 10t$. Similarly, at time t sec., the *ordinate* of the moving point is $y = 3 + 20t$ ft. These two equations together constitute the parametric equations for the moving point, and the locus is a straight line.

EXAMPLE 2

In Fig. 7.6 is shown a crank and connecting-rod mechanism that slides the piston B up and down a horizontal slot (not shown). The crank arm (OA) rotates at 60 revolutions per second (60 r.p.s.). If the length of the crank arm is r ft. and the length of the connecting rod (AB) is L ft., determine the positions of the points A and B in terms of the time t.

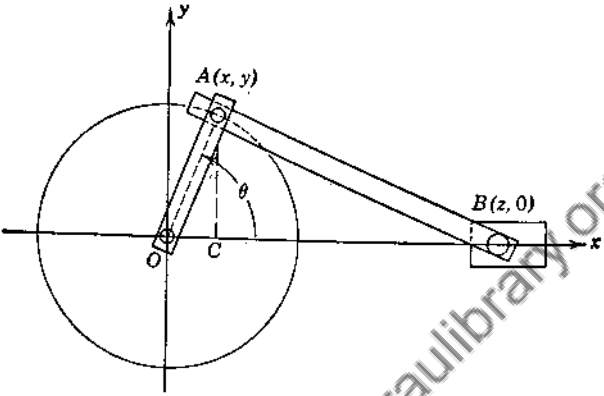


FIG. 7.6

Solution. I. Choose axes with origin at O, so that the equation of the circle will be simple, if that equation is needed.

II. Let the coordinates of the end of the crank arm at A be (x, y) and those of the end of the connecting rod at B be $(z, 0)$.

III. OA is the hypotenuse of a right triangle with ordinate y and abscissa x as the two legs. Also $\overline{OB} = \overline{OC} + \overline{CB}$.

IV and V. $x = \overline{OC} = r\cos\theta,\ y = \overline{CA} = r\sin\theta,$

$$z = \overline{OB} = \overline{OC} + \overline{CB} = r\cos\theta + \sqrt{L^2 - r^2\sin^2\theta}.$$

Since the crank arm rotates at 60 r.p.s., which is equivalent to 120π radians per second, $\theta = 120\pi t$ radians if t is measured from an instant when A is at the right end of the horizontal diameter of the circle. Hence the required equations are

(1) $$x = r\cos 120\pi t, \quad y = r\sin 120\pi t.$$

(2) $$z = r\cos 120\pi t + \sqrt{L^2 - r^2\sin^2 120\pi t}.$$

VI. It will be left to the student to determine the values of x, y, and z when the point A is at the top of the circle ($t = 1/240$ sec.), and to make the numerical check.

PROBLEMS

1. Prove that the graphical construction shown in Fig. 7.7 will yield the graph of an ellipse with semimajor and semiminor axes a and b. The ellipse is the locus

of point $P(x, y)$ as it moves according to the requirements of the construction. In the process obtain parametric equations for the ellipse in terms of the parameter θ.

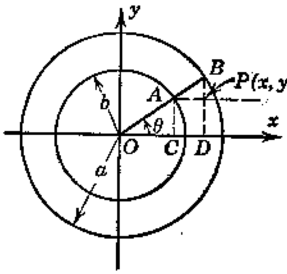


FIG. 7.7

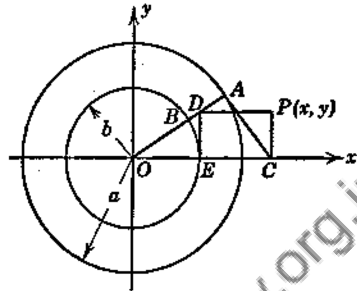


FIG. 7.8

2. Prove that the construction shown in Fig. 7.8 will yield the graph of a hyperbola with semitransverse axis a and semiconjugate axis b. In the process obtain parametric equations for the hyperbola in terms of the parameter θ, where θ is the angle between OC and OA.

3. Derive a pair of parametric equations for a cycloid (see Fig. 7.9). A cycloid is defined as the locus of a point on the rim of a wheel that rolls without slipping along a straight line. *Hint:* Choose axes as indicated. In the figure the wheel has turned so that arc DP is equal to $\overline{OD}$. Then $x = \overline{OA} = \overline{OD} - \overline{PB}$ and $y = \overline{AP} = \overline{DC} - \overline{BC}$.

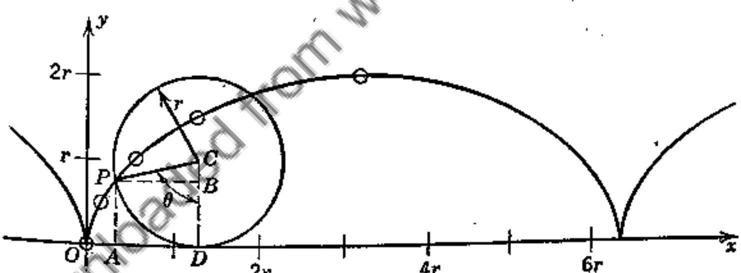


FIG. 7.9

4. A straight line has for its rectangular equation $2x + 5y = 11$. Find a pair of parametric equations for the same straight line, using q as parameter for the directed distance along the line from (3, 1) to the point (x, y). Choose the "upward" direction along the line to be positive.

5. Repeat Problem 4 for each of the following lines and points:

(a) $y = 3x + 7$, (2, 13). (b) $3x - 4y = 10$, (10, 5).
(c) $5x - 8y = 9$, (5, 2). (d) $y = 2x$, (0, 0).

6. The *hypocycloid* is defined as the locus traced by a particular point on the rim of a wheel as this wheel rolls, without slipping, along the inside surface of a second wheel. Derive the parametric equations by aid of Fig. 7.10 and the fol-

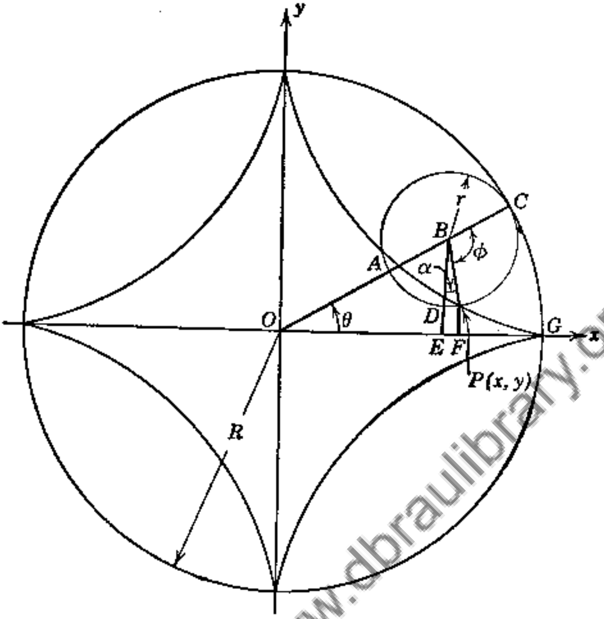


FIG. 7.10

lowing suggestion: $x = \overline{OF} = \overline{OE} + \overline{EF}$, where $\alpha = 90° + \theta - \phi$, and (since arc CG = arc CP) $R\theta = r\phi$. Similarly, $y = \overline{FP} = \overline{EB} - \overline{DB}$. You should obtain the two following parametric equations:

$$x = (R - r)\cos\theta + r\cos\frac{R - r}{r}\theta;$$

$$y = (R - r)\sin\theta - r\sin\frac{R - r}{r}\theta.$$

If the radius of the rolling wheel is one-fourth the radius of the fixed or outer wheel, the resulting curve is a four-cusped hypocycloid and is sometimes called an *astroid*.

7. An *epicycloid* is defined as the locus traced by a particular point on the rim of a wheel as the wheel rolls, without slipping, along the outside surface of a fixed wheel. Derive its parametric equations by aid of Fig. 7.11 and the following suggestion:

$$x = \overline{OF} = \overline{OE} + \overline{EF}, \quad \text{where } \alpha = \theta + \phi - 90° \text{ and } R\theta = r\phi.$$

Similarly $y = \overline{FP} = \overline{EB} - \overline{CB}$. You should obtain the following parametric equations:

$$x = (R + r)\cos\theta - r\cos\frac{R + r}{r}\theta;$$

$$y = (R + r)\sin\theta - r\sin\frac{R + r}{r}\theta.$$

Notice that the results for Problem 6 and this problem are related in that we can substitute (why?) $-r$ for r in either pair of equations and obtain the other pair.

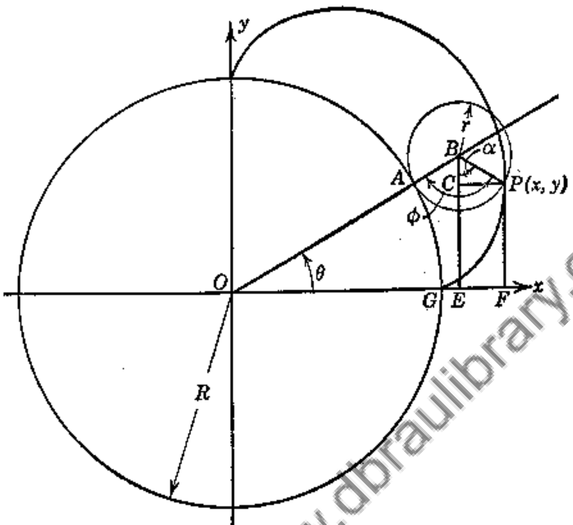


FIG. 7.11

8. A string is wound several times around a wheel and one end is fastened to the wheel. If the string is unwound in such a way that it is always taut, determine the equation of the locus of the end of the string. *Hint:* $x = \overline{OB} = \overline{OA} + \overline{CP}$ and $y = \overline{BP} = \overline{AD} - \overline{CD}$. Notice, too, that the length of $\overline{DP}$ is the same as the length of the arc DE. The locus is called the *involute* of the circle (Fig. 7.12).

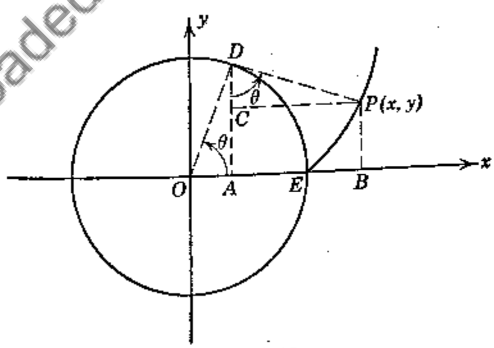


FIG. 7.12

Sometimes the gear teeth in a gear are cut with the edges as involutes of a circle so that when the two gears mesh, the motion will always be at right angles to the point of contact, and hence the amount of wear will be reduced.

9S. Figure 7.13 shows an elliptical cam that rotates so that $\phi = 200\pi t$ or 100 r.p.s. The follower B-B is maintained in a vertical position by the action of the sleeves A and C and presses against the cam because of the action of the compressed spring. Show that the equation for x, the abscissa of the contact point P, is given in terms of the angle ϕ by

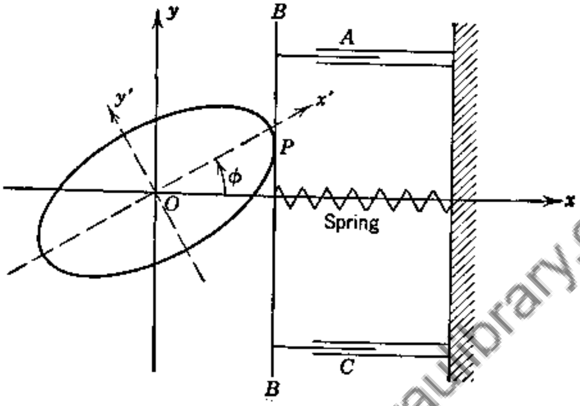


FIG. 7.13

$$x = \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi},$$

where the semiaxes of the ellipse are a and b. *Hint:* Use the parametric representation for an ellipse as given in Problem 1 in terms of an angle θ measured from the x'-axis. Next use the equations of rotation of axes and obtain

$$x = a \cos \theta \cos \phi - b \sin \theta \sin \phi,$$

$$y = a \cos \theta \sin \phi + b \sin \theta \cos \phi.$$

Then reduce the x-expression to the form $x = A \cos (\theta + \alpha)$ and show that

$$A = \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}.$$

CHAPTER 8

Empirical Equations

An important problem in science is to find an equation of a curve that is determined by data secured in a laboratory. The material included in this chapter is simple but is sufficient for the problems that the student will meet in undergraduate courses in science or engineering. With the exception of a few ideas, the material of this chapter will be limited to the basic problem: Starting with a set of data whose graph is approximately a straight line, how can we test these data for linearity, and how can we determine the equation of a straight line that is approximately satisfied by the given data?

8.1 Tests for Linearity of a Given Set of Data

We begin by studying the following example:

EXAMPLE

Examine the following data to determine whether the graph is approximately a straight line:

x	1	3	6	10	12
y	2.34	3.01	3.99	5.30	6.01

Solution. The *graphical test* will be left as an exercise for the student. He should plot these data to a large scale on graph paper and determine whether a straight line can be drawn approximately through the plotted points. The graphical test takes more time than the simple *numerical test* that is illustrated in the following difference table and that compares the slopes of the straight-line segments joining consecutive pairs of points.

x	y	Δx	Δy	$\Delta y/\Delta x$
1	2.34			
		2	0.67	0.335
3	3.01			
		3	0.98	0.327
6	3.99			
		4	1.31	0.328
10	5.30			
		2	0.71	0.355
12	6.01			

In this table Δx denotes *a value of x minus the preceding value;* thus the first number in the Δx-column is obtained from $3 - 1 = 2$. Similarly, the first number in the Δy-column is $\Delta y = 3.01 - 2.34 = 0.67$; the second number is $3.99 - 3.01 = 0.98$. The last column shows the ratios of corresponding values of Δy and Δx, and these ratios are clearly the slopes of the straight-line segments joining consecutive pairs of points. Since these slopes are approximately equal (to two significant figures), a straight-line equation can be determined that will fit the data to approximately three significant figures.

8.2 Numerical Test for Polynomial-Type Curves

The preceding article has indicated a numerical test for a straight line and has introduced the notation Δx and Δy. The present article gives a test for a polynomial-type curve:

$$y = a + bx + cx^2 + \cdots + kx^n,$$

$n \geqq 1$, $k \neq 0$, a test that can be used whenever Δx is constant.

The following difference table illustrates the meaning of first differences Δy, of second differences $\Delta^2 y$, of third differences $\Delta^3 y$, etc. In this table $\Delta x = x_2 - x_1 = x_3 - x_2$, etc., and Δx is to be *constant;* $\Delta y_1 = y_2 - y_1$; $\Delta y_2 = y_3 - y_2$; $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$; $\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$; $\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$; etc.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	etc.
x_1	y_1				
		Δy_1			
x_2	y_2		$\Delta^2 y_1$		
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3		$\Delta^3 y_2$	
x_4	y_4		$\Delta^2 y_3$		
		Δy_4			
x_5	y_5				

EXAMPLE 1

If $y = x^3 + 3x + 4$, compute the values of y that correspond to $x = 0, 2, 4, \cdots,$ 12, and then form a difference table.

Solution.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	4			
		14		
2	18		48	
		62		48
4	80		96	
		158		48
6	238		144	
		302		48
8	540		192	
		494		48
10	1034		240	
		734		
12	1768			

The student should notice in the table in Example 1 that the values of $\Delta^3 y$ (the *third* differences) are constant and that the degree of the original polynomial is *three*. This verifies the following theorem, which we state without proof.

THEOREM. *If a difference table is constructed for y a polynomial, of degree n, in x, and if Δx is constant, then the nth differences will be equal. Conversely, if Δx is constant and if the nth differences are constant, an nth degree polynomial (for y a polynomial in x) can be obtained, which the data will satisfy exactly.*

EXAMPLE 2

Prove that the data shown in the following table satisfy a second-degree polynomial equation, and then determine the equation.

x	y	Δy	$\Delta^2 y$
1	34		
		2	
3	36		−8
		−6	
5	30		−8
		−14	
7	16		−8
		−22	
9	−6		−8
		−30	
11	−36		

Solution. We form the difference table as shown and find that the second differences are constant (note that Δx is constant), and hence we know that an equation of the form $y = ax^2 + bx + c$, whose graph will go through all six of the given points, can be determined.

Since there are three coefficients to be determined, we may require the curve to go through *any three* of the given points. Thus, using (1, 34), (5, 30), and (9, −6), we obtain $a + b + c = 34$, $25a + 5b + c = 30$, and $81a + 9b + c = -6$. The student should solve these simultaneously and obtain for his final result $y = 30 + 5x - x^2$.

8.3 Interpolation

With this difference table idea at our disposal, it is an elementary matter to explain ordinary interpolation and how to do more accurate interpolation. Hence we digress from the problem of the determination of an equation for empirical data and devote this one article to the topic of interpolation.

We use the difference table below, in which an auxiliary variable z has values assigned as shown.

z	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	x_1	y_1			
			Δy_1		
1	x_2	y_2		$\Delta^2 y_1$	
			Δy_2		$\Delta^3 y_1$
2	x_3	y_3		$\Delta^2 y_2$	
			Δy_3		
3	x_4	y_4			

It is assumed that $\Delta x = x_2 - x_1 = x_3 - x_2$ is constant. Therefore a linear relation can be found for z in terms of x, namely

$$z = \frac{x - x_1}{x_2 - x_1}.$$

We consider the following equations:

$$y = y_1 + z\,\Delta y_1;$$

$$y = y_1 + z\,\Delta y_1 + \frac{z(z-1)}{2!}\Delta^2 y_1;$$

$$y = y_1 + z\,\Delta y_1 + \frac{z(z-1)}{2!}\Delta^2 y_1 + \frac{z(z-1)(z-2)}{3!}\Delta^3 y_1.^*$$

* The z-fractions are the coefficients in the expansion by the binomial theorem of $(a + b)^z$.

These are special cases of the Gregory-Newton interpolation formula. The graph of the first equation, considered in terms of the variables y and z, is a straight line that clearly goes through $z = 0$, $y = y_1$. When $z = 1$, $y = y_1 + \Delta y_1 = y_1 + (y_2 - y_1) = y_2$. Hence this straight line goes through the two points defined by the first two pairs of numbers in the difference table.

In a similar manner, it can be seen that the graph of the second equation is a parabola that goes through the first three points, and that the graph of the third equation goes through all four of the given points defined by the four pairs of number (z, y) in the difference table. The first of these three equations is the algebraic statement of ordinary or linear interpolation. (Why?) The second and third equations give successive refinements for ordinary interpolation.

The linear expression for z in terms of x may be substituted in each of these three equations to obtain successive expressions for y in terms of x. Since no requirement has been imposed in the given difference table other than the assumption that Δx is constant, the values of x may increase as we read down the table or as we read up the table. Thus, a table of such values could be obtained from any section of a standard set of tables. In certain cases it would be convenient to invert the table, especially if the values are taken from the end of such a standard table.

If we compare the first two of these three equations, we see that the second-difference correction is

$$\frac{z(z-1)}{2!}\Delta^2 y_1,$$

where z is to assume values between 0 and 1. The graph of the coefficient of $\Delta^2 y_1$, as a function of z, will be a parabola with its vertex at $z = \frac{1}{2}$. Hence the largest numerical correction that we may make by taking account of the second difference is the numerical value of this coefficient at $z = \frac{1}{2}$, and that numerical value is $\frac{1}{8}$. We therefore have established the theorem.

THEOREM. *The numerical error made in using ordinary or linear interpolation, when second-difference interpolation is correct, is not more than one-eighth of the corresponding second difference.*

EXAMPLE

Given the data in the adjoining table.

z	n	$y = n^3$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	4.1	68.921			
			5.167		
1	4.2	74.088		0.252	
			5.419		0.006
2	4.3	79.507		0.258	
			5.677		0.006
3	4.4	85.184		0.264	
			5.941		
4	4.5	91.125			

(*a*) Compute 4.13^3 by ordinary interpolation.

(*b*) Express $y = n^3$ in terms of the auxiliary variable by aid of this difference table, stopping with second differences. Repeat, using all the differences.

(*c*) Compute 4.13^3 using the two results in (*b*).

Solution. (*a*) $4.13^3 \approx 68.921 + (0.3)(5.167) = 70.4711$.

(*b*) $y = 68.921 + 5.167z + [z(z-1)/2!](0.252)$. The student should verify that this equation is satisfied precisely by the first three pairs of numbers (z, y) in the table, and hence that this is the equation of a parabola that goes through those three points. For the second result, we obtain

$$y = 68.921 + 5.167z + \frac{z(z-1)}{2!}(0.252) + \frac{z(z-1)(z-2)}{3!}(0.006).$$

If we were to determine the relation between z and n, namely $n = 4.1 + 0.1z$, substitute this in the preceding equation, and simplify the result, we should obtain $y = n^3$.

(*c*) When $n = 4.13$, $z = 0.3$ (which is the multiplier for ordinary interpolation). We use the two equations in (*b*) and obtain the following results:

$$y = 68.921 + (5.167)(0.3) + (0.3)(-0.7)(0.126) = 70.444{,}64;$$

$$y = 70.444{,}64 + (0.3)(-0.7)(-1.7)(0.001) = 70.444{,}997,$$

and this last result is precisely correct for 4.13^3.

PROBLEMS

1. Prove that the following data satisfy exactly a linear equation, and then determine that equation:

(*a*)

x	2	4	5	7	8
y	7	11	13	17	19

(b)

x	−7	−5	−2	0	3
y	18	14	8	4	−2

(c)

F	−40	5	32	212	392
C	−40	−15	0	100	200

2. Construct a difference table, determine the degree of the polynomial-type equation that will be satisfied exactly by the given data, and then determine that equation:

(a)

x	1	3	5	7	9	11
y	6	18	38	66	102	146

(b)

x	0	1	2	3	4	5
y	0	−4	−2	12	44	100

(c)

x	0	1	2	3	4	5
y	0	3	4	3	0	−5

(d)

x	−1	1	3	5	7
y	2	1	2	5	10

(e)

x	0	2	4	6	8	10
y	0	24	−96	−936	−3456	−9000

3. Use the table below to determine tan 89° 35.4′ by ordinary interpolation, and by interpolating with the second difference included.

z	θ	$\tan\theta$
0	89° 35′	137.51
1	89° 36′	143.24
2	89° 37′	149.47

4. Estimate $e^{5.02}$ by use of the table of data below. Also determine a quadratic approximation, first in terms of z and then in terms of w, that will approximate e^w for values of w between 5.00 and 5.20.

z	w	e^w
0	5.00	148.41
1	5.10	164.02
2	5.20	181.27

5. Compute sin 1.007 by aid of the table below. Also compute sin 1.018 by first reversing the table.

θ	$\sin\theta$
1.00	0.84147
1.01	0.84683
1.02	0.85211

6. Use the data in Problem 2(*a*) to determine the values of y when $x = 2, 4, 6, 8,$ and 10; then check by aid of the equation obtained as the result in that problem.

7. Use a difference table and write rapidly the equation of the parabola whose axis is parallel to the y-axis and that goes through the three points:

(*a*) (0, 1), (2, 7), (4, 21).
(*b*) (1, 2), (2, 0), (3, −4).
(*c*) (−1, 2), (1, 6), (3, 18).
(*d*) (0, 10), (5, 85), (10, 260).

Solution to (*a*):

z	x	y	Δy	$\Delta^2 y$
0	0	1		
			6	
1	2	7		8
			14	
2	4	21		

We use the difference table above and write:

$$y = 1 + 6z + \frac{8}{2!}z(z-1),$$

and, since $x = 2z$,

$$y = 1 + 3x + x(x-2) = x^2 + x + 1.$$

8.4 Fitting a Straight Line by the Method of Selected Points

We shall suppose that the given data have been plotted to a large scale on graph paper, and that a straight line has been drawn which seems *to the eye* to be a reasonable fit for the data. (In drawing such a line a transparent straightedge or a piece of thread can be used to advantage.) We select two points relatively far apart on this straight line and determine the equation of the straight line that goes through them. Notice that these two points may or may not be among the given data.

EXAMPLE

Show that the following data satisfy approximately a straight-line law, and determine the equation:

x	1	3	6	10	12
y	2.34	3.01	3.99	5.30	6.01

Solution. The numerical test for linearity for these data is shown in the example in Art. 8.1. The graphical test is shown in Fig. 8.1. We select (0, 2.00) and (9, 5.00) from the graph and obtain $y = 2.00 + 0.333x$ as the required equation.*

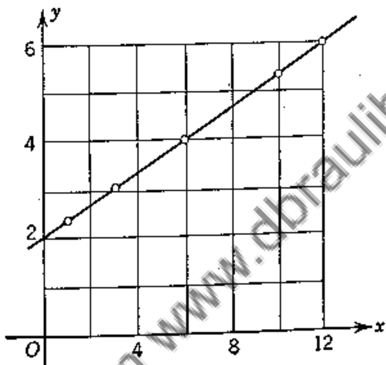


FIG. 8.1

8.5 Fitting a Polynomial-Type Curve by the Method of Selected Points

We shall suppose that we have tested a given set of data by aid of a difference table (Δx being constant) and that we have found the data to satisfy approximately a polynomial-type equation of degree n. We may then find the required coefficients by reading the coordinates of $n + 1$ points on the smooth curve drawn for the given data, substitute these in the type equation ($y = a + bx + cx^2 + \cdots + kx^n$), and solve simultaneously the resulting linear equations in the coefficients. Alternatively, we may select $n + 1$ points from among the given data and proceed in the same manner.

* One may find the equation by either of the basic straight-line formulas: $(y - y_1)/(x - x_1) =$ slope, or $y = mx + b$. In the second case we substitute the coordinates of the two points and obtain two simultaneous equations in m and b.

EXAMPLE

The following data give the specific heat,* s, of water at temperature $\theta°$ C. Show that a straight line can be fitted approximately to the given data, and that a parabola ($s = a + b\theta + c\theta^2$) can be fitted with greater accuracy. Then determine the equation of the parabola by the method of selected points.

Solution.

θ	s	Δs	$\Delta^2 s$
0	1.00664		
		−0.00121	
2	1.00543		0.00013
		−0.00108	
4	1.00435		0.00004
		−0.00104	
6	1.00331		0.00006
		−0.00098	
8	1.00233		0.00014
		−0.00084	
10	1.00149		0.00013
		−0.00071	
12	1.00078		

Since the values of $\Delta\theta$ are constant, we may form a difference table. Since the second differences are irregular but approximately equal in the fourth decimal place (which is the fifth significant figure in the original data), the data may be fitted with high accuracy by a second-degree, polynomial-type curve. The first differences show a definite tendency to decrease. Nevertheless, the first differences are equal to the third decimal or the fourth significant figure of the original data, and a straight line would then furnish a reasonably good fit. To complete the problem we may plot the data, draw a straight line or smooth curve, and read the coordinates of two points on the straight line, or of three points on the curve. In plotting these data it would be appropriate to translate axes by the substitution $s' = s - 1$ and to use a large scale for the s'-axis (it would be unnecessary to show the s-axis). Alternatively, we may select three points from among the given data, keeping them widely spaced, and proceed with these. If we use the points corresponding to $\theta = 0$, 6, and 12, we obtain the three simultaneous equations (we substitute in $s = a + b\theta + c\theta^2$): $a = 1.00664$, $a + 6b + 36c = 1.00331$, and $a + 12b + 144c = 1.00078$. We solve these three equations simultaneously and obtain $a = 1.00664$, $b = -0.000,621,667$, and $c = 0.000,011,111$. The final result, then, is $s = 1.00664 - 0.000,622\theta + 0.000,011,1\theta^2$, where we have rounded the coefficients with the idea that this equation will be used for values of θ between 0° and 12°.

* Specific heat is defined in the metric system as the quantity of heat in calories required to raise the temperature of 1 gram of the substance 1 degree centigrade.

PROBLEMS

1. Plot the data of the example in Art. 8.5 after transforming by $s' = s - 1$, and fit a straight line by the method of selected points.

2. Show that each of the following sets of data is linear. Then determine the equation by the method of selected points.

(a)

x	−5	−1	3	7	11
y	8.20	12.80	17.00	21.50	26.00

(b)

x	2	4	8	10
y	20.00	16.40	10.00	6.20

(c)

w	20	30	40	50
F	9.5	13.2	17.0	20.7

(d)

T	10	20	30	40
R	38.9	40.5	42.0	43.5

3. A portion of a stress-strain curve, showing the results of a tension test of a mild steel bar, is given by the following data; s is unit stress in pounds per square inch, and ϵ is unit strain in inches per inch:

s	60,000	124,000	190,000	244,000	306,000	362,000
ϵ	0.002	0.004	0.006	0.008	0.010	0.012

Show that the data satisfy approximately a straight-line law, and find the approximate equation in the form $s = a + b\epsilon$. What is the slope of this line (the slope is called the modulus of elasticity in engineering and, for steel, is approximately 30,000,000 lb./sq. in.)?

4. The following data were taken on a compression test of a concrete cylinder of diameter 6 in. and height 12 in.; s is the load per square inch of cross-sectional area in pounds per square inch, and ϵ is the decrease in height per inch of the original height in inches per inch:

s	0	430	1250	1700	2030	2460	2840	3090
$10^4\epsilon$	0	1.10	3.58	5.00	6.25	8.10	10.6	15.0

(a) Fit a straight line to the first four points of the data.

(b) Fit a parabola ($s = b\epsilon + c\epsilon^2$) to these data by plotting s/ϵ in terms of ϵ on ordinary graph paper. (Why should this be expected to lead to a straight-line graph?) Then determine the vertex of this parabola and hence determine an estimate for the maximum load (per square inch of cross-sectional area) that the concrete cylinder will withstand.

5. It is found by experiment that the number of grams (W) of potassium iodide that will dissolve in 100 grams of water at a temperature (T° C.) is given by the following table:

T	10	15	20	25	30	40	50
W	136	140	144	148	152	160	168

Prove that these data satisfy approximately a linear law, and determine the equation ($W = a + bT$).

Caution. The student is warned not to use a resultant equation, such as the one obtained in this problem, for computation *outside* the range of the given data. This process, which is called *extrapolation,* is not valid without further information showing that the curve continues in the same manner for a greater range of values of the independent variable.

6. The following data were obtained in a test of white pine columns. $P/A = y$ is the load per unit area (cross-sectional) causing failure of the column. $L/r = x$ is the "slenderness ratio" of the column, where L is the length of the column and r measures a geometric property of the shape of the cross section. Determine a straight-line equation for P/A in terms of L/r for the straight portion of that curve.

L/r	96.4	82.5	68.8	55.1	41.3	27.5	20.6	13.8	6.9	2.5
P/A	1250	1860	2750	3500	4270	5060	5360	5860	5880	5900

7. The following data give the electrical resistance (R ohms) of a certain copper wire at various temperatures (T° F.):

T	19.1	25.0	30.1	36.0	40.0	45.1	50.0
R	76.30	77.80	79.75	80.80	82.35	83.90	85.10

(*a*) Prove, by use of slopes, that R is approximately a linear function of T.

(*b*) Determine the values of a and b in $R = a + bT$ by the method of selected points.

(*c*) What geometric and physical interpretations can you give for a and b, assuming that extrapolation is allowable.

8. Fit a parabola, $y = ax^2 + bx + c$, to the following data by the method of selected points:

(*a*)

x	0	2	4	6	8
y	1.43	3.14	6.00	10.00	15.14

(*b*)

x	−2	0	2	4	6	8
y	1.47	0.53	0.13	0.27	0.93	2.13

9. In Problem 8, estimate the value of c by reading the value of y when $x = 0$. Then translate axes by $y' = y - c$ and plot y'/x in terms of x. Why should this

be expected to lead to a straight-line graph? Complete the problem by the method of selected points to find the equation of the straight line for y'/x in terms of x, and then write out the solution to the original problem.

10. Use the following table of values and plot a graph of y/x in terms of x^2. Then find the equation for y/x in terms of x^2.

x	0.25	0.50	0.75	1.00	1.25
y	0.247	0.479	0.682	0.841	0.950

Actually this table gives the values for $y = \sin x$ with x in radians. Use your resulting equation to approximate the value of sin (1.1 radians).

8.6 Fitting a Power-Law Curve by the Method of Selected Points

The student should review the three possible shapes for the graphs of the power-law curve $y = ax^n$, assuming that a is positive. He should recall especially the behavior of the curves in the neighborhood of the origin if n is positive (to which axis the curve is tangent), and the shape if n is negative.

If a graph is plotted for a given set of data, and if the resulting curve has the shape of one of these three possibilities, then the curve may be a power-law curve. The next problem, then, is to determine a manner of checking to see whether this guess is a correct one. Since this is not a polynomial-type equation, the difference-table test cannot be used. Instead, we shall show how to test for this guess by the use of special coordinate graph paper.

If we equate common logarithms of the two sides of the equation $y = ax^n$, we obtain $\log y = \log a + n \log x$. If we make the substitutions $Y = \log y$, $X = \log x$, and (for convenience) $A = \log a$, this equation becomes $Y = A + nX$. Hence we may test for a power-law curve as follows: First look up the logarithms of the given data (x, y) and denote $Y = \log y$ and $X = \log x$. Then plot the resulting transformed data (X, Y) on ordinary graph paper. If the resulting graph is a straight line, the given data satisfy a power-law equation.

This procedure is equivalent to plotting the *original* data on loglog paper, i.e., graph paper that has logarithmic scales on both axes. This statement follows, since the method of constructing loglog paper is to lay off logarithmic scales on both axes according to the law $Y = \log y$ and $X = \log x$. Hence we have established the following test:

TEST FOR POWER-LAW EQUATIONS. Plot the original data on log-log paper; if the resulting graph is a straight line, the given data satisfy a power-law equation (this statement is correct to the accuracy of a graphical solution).

The conclusion could, of course, be checked by transforming the data and comparing the slopes of consecutive line segments on the graph of the transformed data.

The problem of determining the equation, once we know it to be a power-law type, is simply to read off the coordinates of two points (relatively far apart) on the loglog plot, to substitute these in the given equation $y = ax^n$, and to solve simultaneously the two resulting equations for a and n.

EXAMPLE

The following data were obtained in a mechanical engineering laboratory course. p is the pressure, in pounds per square inch, of a gas in a compression chamber (piston, for example), and v is the volume of gas in cubic feet. Determine the equation relating p and v.

p	20	30	40	50	60	70	80	90
v	19.8	13.5	10.3	8.35	7.04	6.09	5.37	4.81

Solution. The graph of p in terms of v is shown in Fig. 8.2, and since both axes may be asymptotes the curve may be a power-law curve with negative exponent.

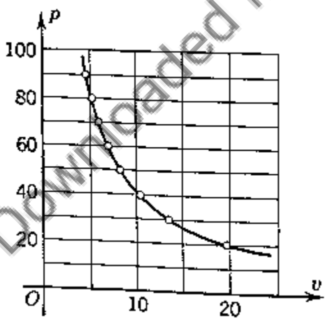


FIG. 8.2

The original data are then plotted on loglog paper as indicated in Fig. 8.3. In this problem it is desirable to use two-cycle paper, since the span of the v-data overlaps the range from 1 to 10 and the range from 10 to 100. Thus, the position of the number 1 on the v-scale is a matter of choice, and that particular number might not even appear on the sheet of paper. What is essential is that the v-numbers to the right and to the left of the position chosen for $v = 1$ are to be labeled as shown in this figure. The same statement is true for the other axis.

The next step in the solution of the problem is to read the coordinates of two widely spaced points on the straight line. We read: $v = 30$, $p = 13.0$; $v = 2$, $p = 228$. We substitute these in $p = av^n$ and obtain $228 = a(2^n)$, $13.0 = a(30^n)$. The student can solve these simultaneously in a number of different ways. They can be solved rapidly by aid of a loglog slide rule, in which case the first step would probably be to divide one equation by the other: $228/13.0 = 2^n/30^n$ or $17.5 = 15^{-n}$.

The value of n may then be determined by aid of the loglog scales and then the value of a determined by aid of one of the original equations (actually the student should solve for a from both equations as a check on his work). Another method of solution would be to start in the same manner, but to use a four-place table of logarithms for the division, thence to find n, and finally to find a (the student could use logarithms at each step of the computations). Still a different mode of

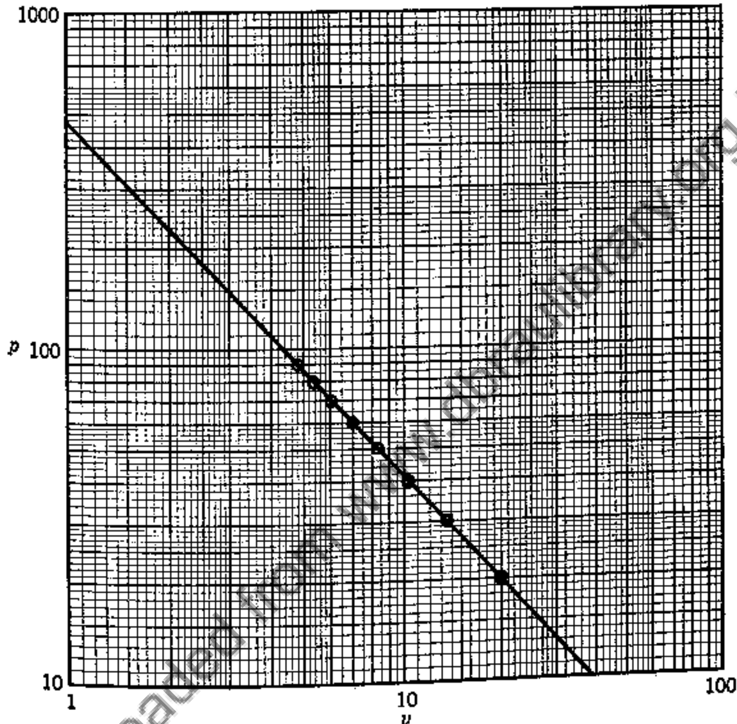


FIG. 8.3. Loglog (two-cycle) paper.

solution is as follows: We take logarithms of both sides of the two original equations and obtain $\log 228 = \log a + n \log 2$, $\log 13.0 = \log a + n \log 30$. We substitute the values of the logarithms from a four-place table and obtain $2.3579 = \log a + 0.3010n$, $1.1139 = \log a + 1.4771n$. The student should solve these simultaneously and obtain $n = -1.058$, $\log a = 2.6763$, $a = 474.6$, whence the required equation is $p = 475v^{-1.06}$, or $pv^{1.06} = 475$.

EXERCISE FOR THE STUDENT. When we can read the value of the dependent variable that corresponds to the value of *one* for the independent variable, the ensuing arithmetic is easier. (Why?) Solve this illustrative example by reading from Fig. 8.3: $v = 1$, $p = 475$; $v = 10$, $p = 41.0$.

8.7 Fitting Exponential Curves by the Method of Selected Points

The graph of $y = ae^{bx}$ (assuming $a > 0$) has the x-axis as an asymptote and crosses the y-axis above the origin. We may change this equation to straight-line form by equating logarithms (to the base 10, but the use of logarithms to the base e would be just as convenient) of the two sides of this equation and obtain $\log y = \log a + bx \log e$. If we make the substitutions $Y = \log y$, $A = \log a$, and $B = b \log e$, this equation becomes $Y = A + Bx$. Hence we may test for an exponential curve by looking up the logarithms $Y = \log y$ of the original data and then by plotting these new data (x, Y) on ordinary graph paper; if the resulting graph is a straight line, then we know that the original data satisfy to graphical accuracy an exponential equation.

Instead of plotting the transformed data (x, Y) on ordinary graph paper, we may plot the original data (x, y) on semilog paper. Semilog paper has a uniform scale on the horizontal axis and a logarithm scale on the vertical axis so that the plotting of the given data on this paper is precisely equivalent to plotting the transformed data (x, Y) on ordinary graph paper. Hence we have established the following test:

TEST. To test a given set of data to determine whether it is of exponential type, plot the original data on semilog paper. If the resulting graph is a straight line, the data satisfy to graphical accuracy an exponential-type equation of the form $y = ae^{bx}$, which may, of course, be written in the alternative forms $y = a(10^{cx})$ or $y = a(d^x)$.

To fit an exponential curve, $y = ae^{bx}$, by the method of selected points, we read the coordinates of two widely spaced points on the semilog graph, substitute their coordinates (x, y) in the given equation, and solve the resulting equations simultaneously for a and b. Those equations may be solved simultaneously by aid of a slide rule or by aid of a table of logarithms.

EXAMPLE

The following data give the excess of temperature ($\theta°$ C.) of a body over the surrounding temperature at various times (t min.) since the beginning of an experiment. Show that the data satisfy an exponential law, and find the equation.

t	0	4	8	12	16	20	24
θ	31.2	29.4	27.6	26.0	24.5	23.1	21.8

Solution. The data are plotted on ordinary graph paper as shown in Fig. 8.4. The curve is not quite straight and the shape suggests an exponential curve, so we replot the data on semilog paper as shown in Fig. 8.5. In this case we have used a sheet of one-cycle semilog paper, but actually we would gain better accuracy by plotting the logarithms of θ in terms of t, since the spread of the θ-data is only a small portion of a single cycle.

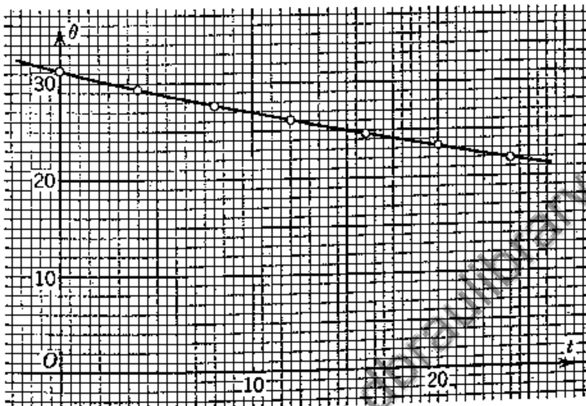


FIG. 8.4

From the semilog graph we read: $\theta = 31$ when $t = 0$ (why should we use this particular value for t if it is on the graph?), and $\theta = 27$ when $t = 10$. We substitute these in $\theta = ae^{bt}$, and obtain $31 = a$, $27 = ae^{10b}$. Hence $27 = 31e^{10b}$ or $\log 27 = \log 31 + 10b \log e$, whence $b = -0.014$. If we use logarithms to the base e instead of common logarithms, we have $\ln 27 = \ln 31 + 10b$, from which we obtain the same result with less arithmetic. The final result is $\theta = 31e^{-0.014t}$.

The use of loglog and semilog paper in science and engineering is not restricted to the determination of empirical curves. Such paper is used whenever a graph is desired in which one or both variables have a large range. For example, if we wished to plot a graph of volumes (at constant pressure) for water as both liquid and steam in terms of temperatures, a logarithmic scale for temperatures would be appropriate. An ordinary or uniform scale corresponds to decimal accuracy and a logarithmic scale corresponds to significant-figure accuracy.

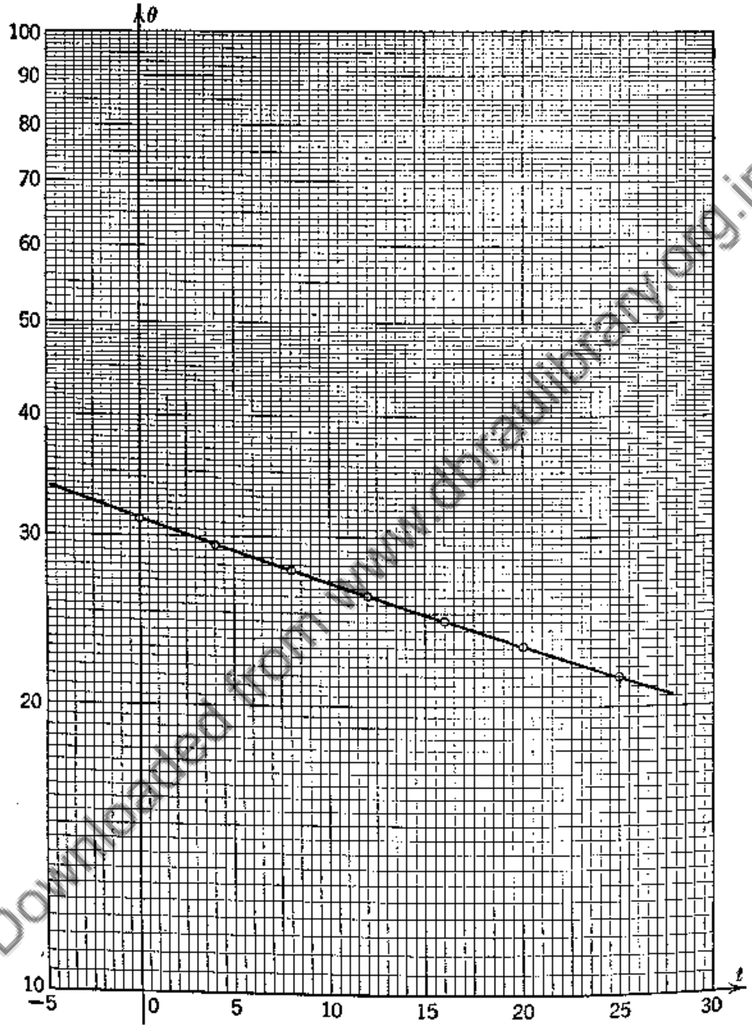


FIG. 8.5. Semilog (one-cycle) paper.

PROBLEMS

1. Find the values of a and n, each correct to three significant figures, if the graph of $y = ax^n$ goes through the two given points (check by computing y for the two given values of x or by sketching the resulting locus):

(a) (1, 2) and (4, 8). (b) (2, 10) and (10, 4).
(c) (2, 4) and (8, 5). (d) (2, 1.2) and (4, 9.6).

2. Find the values of a and b, each correct to three significant figures, if the graph of $y = ae^{bx}$ goes through the two given points (check by computing y for the two given values of x or by sketching the resulting locus):

(a) (0, 8) and (4, 1). (b) (−2, 4) and (4, 0.5).
(c) (−3, 2) and (3, ⅛). (d) (−3, 1) and (6, 5).

3. Determine an equation of the form $y = ax^n$ that will fit the following data. Plot the data on ordinary graph paper and again on loglog paper, and use the method of selected points.

(a)

x	2	4	6	8	10	12
y	0.854	1.21	1.48	1.71	1.91	2.09

(b)

x	2	4	6	8
y	2.52	3.16	3.64	4.04

(c)

m	3	6	9	12
L	3.61	4.51	5.20	5.73

(d)

t	2	4	6	8	9
s	64.3	258	580	1030	1300

(e)

h	200	500	1000	2000	10,000
v	113	180	253	359	802

(f)

x	0.2	0.4	0.8	1.5	3.0	6.0	10.0
y	0.42	0.33	0.23	0.16	0.11	0.086	0.063

4. Determine by the method of selected points an equation of the form $y = ae^{bx}$ to fit the following data. Plot the data on ordinary graph paper and again on semilog paper.

(a)

x	−2	2	6	10
y	1.70	2.47	3.57	5.15

(b)

x	0.4	0.8	1.2	1.6
y	2.45	2.98	3.65	4.44

(c)

x	10	20	30	40	50
y	64.72	58.03	52.00	46.54	41.75

(d)

n	−1	1	3	5
v	0.071	0.286	1.14	4.57

(e)

t	−100	100	300	500	700	900
h	4.65	2.88	1.78	1.10	0.680	0.422

(f)

x	−1.1	−0.5	0.5	1.5	2.5
y	6.50	3.60	1.15	0.41	0.14

5. The relative distances (d) of the planets from the sun and their periods of revolution (t years) are given below. Deduce the law which relates d to t (Kepler's law in astronomy).

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
d	0.387	0.723	1.00	1.52	5.20	9.54	19.2	30.1
t	0.240	0.615	1.00	1.88	11.9	29.5	84.0	165

6. (a) For what value of x can we read the value of a in $y = ax^n$ directly from a sheet of loglog paper?

(b) For what value of x can we read the value of a in $y = ae^{bx}$ directly from a sheet of semilog paper?

(c) What is the geometrical meaning of n for the graph of the power-law equation on loglog paper?

7. The following data give the capacity (A acre-feet) of water in a reservoir corresponding to various heights of water at the dam (H ft.). Assume that the equation $A = aH^b$ can be fitted to the data, and determine the values of a and b. Then compute A when $H = 70$ ft., and compare with the graphical or interpolated result.

H	25	35	45	55	65	75	85	95
$A/1000$	1.4	3.2	6.1	11	17	26	36	49

8. Plot a graph of each of the following curves on semilog paper:

(a) $y = 10^x$. (b) $y = e^x$. (c) $y = 3(2^x)$. (d) $y = 5(7^{-x})$.

9. Transform as indicated and give the values of a, b, c, and d, each correct to three significant figures:

(a) $y = 8(3^x) = 8(10^{ax}) = 8e^{bx} = 10^{ax+c} = e^{bx+d}$.
(b) $y = 25(\frac{2}{3})^x = 25(10^{ax}) = 25(e^{bx}) = 10^{ax+c} = e^{bx+d}$.

10. Plot the graph of each of the following curves on loglog paper:

(a) $y = x^2$. (b) $y = x^3$. (c) $y = 3x^{1/2}$. (d) $y = 2x^{1.4}$.

(e) $y = 45/x^3$. (f) $s = gt^2/2$ and $v = \sqrt{2gh}$ where $g = 32.2$ ft./sec.²

11. The following data give the quantity of water (Q cu. ft./sec.) that will flow each second over a triangular weir when the head of water is H feet. Determine the equation for Q in terms of H.

H	1.2	1.4	1.6	1.8	2.0	2.4
Q	4.2	6.1	8.5	11.5	14.9	23.5

12. The following table gives the intercollegiate track records for various distances (t sec., d yd.) as recorded a few years ago:

d	100	220	440	880	1760	3520
t	9.6	20.9	47.0	111.0	254.4	562.0

(a) Assume that $t = ad^n$, and determine a and n.

(b) What record, assuming your result in (a) to be valid for the given range of data, would probably be broken first? *Hint:* Compute the residuals. A residual is the observed value less the value computed from the equation.

13. The following data (x, y) satisfy an equation of the form $y = ax^n$:

x	y	$X = \log x$	$Y = \log y$
0.4	0.503	−0.398	−0.298
0.6	1.13	−0.222	+0.053
1	3.14	0	0.497
3	28.3	0.477	1.452
5	78.6	0.699	1.895

(a) Plot the original data on three-cycle loglog paper, and determine the values of a and n by the method of selected points.

(b) Plot the (X, Y) data on ordinary graph paper and find the equation of the resulting straight line by the method of selected points. Then transform your result back to (x, y) form.

14. The volume of a certain gas (V cc.) under constant pressure and at temperatures (T° C.) was measured with the following results:

T	20	25	30	35	40
V	80.8	82.2	83.5	85.0	86.4

(a) Determine an equation which relates V and T.

(b) Extrapolate to determine the T-intercept of your resulting equation; give, if you can, the physical interpretation of this result.

15. Carbon tetrachloride has the following vapor pressures (p mm.) at temperatures ($T°$ C.):

T	0	20	40	60
p	33.1	89.5	210.9	439.0

What is the equation of the straight line obtained by plotting the logarithm of the vapor pressure in terms of the reciprocal of the absolute temperature (where absolute temperature $= K = T + 273$)? Extrapolate by use of your resulting equation to estimate the temperature at which carbon tetrachloride boils; i.e., estimate the temperature at which the pressure becomes 760 mm.

16. The following data were obtained from tensile elongation curves for steel at a temperature of 1000° F. Obtain the equation for load per unit of cross-sectional area (s) in terms of the total elongation or creep (C in per cent) in the form $C = ke^{bs}$. Also determine the value of s that corresponds to $C = 1$ (a value used in design specifications).

s	1900	2200	2700	3000	3400	3800	4200
C	10	15	35	45	95	165	330

17. The following data were obtained in a pressure-volume test in a mechanical engineering laboratory course; determine the equation that relates p lb./sq. in. to v cu. ft.:

v	4.00	7.00	10.0	14.0	20.0
p	11.6	5.86	3.86	2.56	1.66

18. Compute k and n if $V^{1/2} = kt + n$ for the following data (plot a straight-line graph as a part of your solution):

V	225	390	590	870	1222
t	1.50	2.00	2.50	3.00	3.50

19. Introduce new variables to reduce each of the following equations to straight-line form; do not use a or b in defining the new variables:

(*a*) $1/y = a + b/x$.
(*b*) $y = a \sin x + b$.
(*c*) $y = ae^{-bx^2}$.
(*d*) $y = a/(x + b)$.
(*e*) $y = a + bx^2$.
(*f*) $y = x^2 - 4x + 7$.

20S. Show that the graph of $y = 10^x$ is exactly the same on ordinary graph paper and on loglog graph paper.

8.8 Curve Fitting by the Method of Averages

The method of averages takes more time than the method of selected points but usually yields more accurate results. *This method can be* *used only on polynomial-type curves.* However, we have already indicated how data for a power-law equation and data for an exponential-type equation may each be transformed to straight-line data. We proceed to illustrate the method with the data of the examples of the last several articles.

EXAMPLE 1

Use the method of averages to fit a straight line to the data of the example in Art. 8.4.

Solution.

x	1	3	6	10	12
y	2.34	3.01	3.99	5.30	6.01

Suppose that the straight line $y = a + bx$ goes through all five of the given points (a physical impossibility since the data do not precisely satisfy a linear law); then we would have

$$a + b = 2.34, \qquad a + 6b = 3.99,$$

$$a + 3b = 3.01, \qquad a + 10b = 5.30,$$

$$a + 12b = 6.01.$$

These are five equations in two unknowns a and b, and they cannot be satisfied simultaneously. In the *method of averages* we form two *new* equations (there are two unknowns) by adding the groups of these equations together. In this example, the first of the required equations is obtained by adding the first two of the original equations; the second such equation is obtained by adding the last three equations. We obtain

$$2a + 4b = 5.35,$$

$$3a + 28b = 15.30.$$

The student should solve these simultaneously and obtain $a \approx 2.014$ and $b \approx 0.3307$; and the final result is $y = 0.331x + 2.01$, where we have rounded the coefficients to the accuracy of the given data.*

* In this and the succeeding examples we shall suppose that the required equation is the *final* result. In an actual problem, the equation, which we have found as a result of using the method of averages, would be an intermediate result. That equation might be used to compute values of y for x within the range of the given data; or the equation might be used in other ways.

If this equation is to be considered as an intermediate result, the correct answer is $y = 0.3307x + 2.014$. Then, for example, if we wish to compute y for $x = 5$, we obtain $y = 1.6535 + 2.014 = 3.6675$, or, rounding to the accuracy of the given data, $y = 3.67$.

EXAMPLE 2

Fit an equation of the form $s = a + b\theta + c\theta^2$ to the data given below (see the example in Art. 8.5) by the method of averages:

θ	0	2	4	6	8	10	12
s	1.00664	1.00543	1.00435	1.00331	1.00233	1.00149	1.00078

Solution. We substitute the given data in the given equation and obtain the following seven equations:

$$\begin{aligned} a \qquad\quad &= 1.00664 \\ a + 2b + 4c &= 1.00543 \\ \hline 2a + 2b + 4c &= 2.01207, \end{aligned} \qquad \begin{aligned} a + 4b + 16c &= 1.00435 \\ a + 6b + 36c &= 1.00331 \\ \hline 2a + 10b + 52c &= 2.00766, \end{aligned}$$

$$\begin{aligned} a + 8b + 64c &= 1.00233 \\ a + 10b + 100c &= 1.00149 \\ a + 12b + 144c &= 1.00078 \\ \hline 3a + 30b + 308c &= 3.00460. \end{aligned}$$

We combine these equations by addition (as shown above) and solve the resulting three equations simultaneously; we obtain

$$s = 1.00663 - 0.000{,}610\theta + 0.000{,}009{,}8\theta^2.*$$

EXAMPLE 3

Fit a power-law equation to the following data (see the example in Art. 8.6) and use the method of averages:

p	v	$y = \log p$	$x = \log v$
20	19.8	1.3010	1.2967
30	13.5	1.4771	1.1303
40	10.3	1.6021	1.0128
50	8.35	1.6990	0.9217
60	7.04	1.7782	0.8476
70	6.09	1.8451	0.7846
80	5.37	1.9031	0.7300
90	4.81	1.9542	0.6821

* If this equation is presumed to be an intermediate result, the correct answer could be

$$s = 1.006{,}626 - 0.000{,}610{,}34\theta + 0.000{,}009{,}848\theta^2.$$

Thus, if θ is any value between 0 and 12, and if s is to be computed and the result given to five decimals, then a little thought will indicate to what accuracy each coefficient in this result should be given.

Solution. We transform the data as shown in the table above. From the equation $p = av^n$ we obtain, by equating logarithms, $\log p = \log a + n \log v$. If we substitute $y = \log p$, $x = \log v$, and $A = \log a$, we obtain $y = A + nx$. We substitute the data (x, y) in this equation and obtain eight equations in the two unknowns A and n. We then add the first four of these equations together and the last four equations together, and obtain

$$4A + 4.3615n = 6.0792,$$

$$4A + 3.0443n = 7.4806.$$

We then solve these simultaneously and obtain $A = \log a \approx 2.6799$ and $n \approx -1.0639$. Hence $a \approx 478.6$, and the final result is

$$pv^{1.06} = 479.*$$

EXAMPLE 4

Use the method of averages and fit an exponential-type equation to the following data (see the example in Art. 8.7):

t	θ	$y = \log \theta$
0	31.2	1.4942
4	29.4	1.4683
8	27.6	1.4409
12	26.0	1.4150
16	24.5	1.3892
20	23.1	1.3636
24	21.8	1.3385

Solution. We start with the equation $\theta = ae^{bt}$, take logarithms, and obtain $\log \theta = \log a + bt \log e$. We make the substitutions $y = \log \theta$, $A = \log a$, and $B = b \log e$; the resulting equation is $y = A + Bt$, which is the equation of a straight line. The student should substitute the transformed data in this equation, obtain seven equations, add the first three and the last four, and obtain $3A + 12B = 4.4034$, $4A + 72B = 5.5063$. The student should solve these simultaneously and obtain, successively:

$B = -0.06516 = b \log e = b(0.43429)$, whence $b = -0.01500$;

$A = 1.4939 = \log a$, whence $a = 31.18$. The final result is $\theta = 31.2e^{-0.0150t}$.

We conclude this article by restating the fact that the method of averages is to be used only for data that satisfy an equation of polynomial type (what fundamental property of exponents would make the method of averages of no value if applied to the original data in either Example 3 or 4?).

* If this is to be regarded as an intermediate result, the answer should be $pv^{1.064} = 478.6$.

There are other methods of curve fitting than the two discussed in this book. The method of least squares will usually yield more accurate results than either of those explained, but it requires more arithmetic. This method will not be explained in this book since an adequate explanation requires an understanding of the calculus.

PROBLEMS

1–8. Use the method of averages to work Problems 1–8 at the end of Art. 8.5.

9. Solve Example 4 of Art. 8.8 by the method of averages, but use logarithms to the base e.

10. Solve Example 2 of Art. 8.2 by the method of averages, and show that the equation that results is the same as the result obtained in that example.

11–26. Solve all of Problems 1–16 at the end of Art. 8.7 for which the method of averages is appropriate.

CHAPTER 9

Planes and Lines in Solid Analytic Geometry

The remainder of this book will be devoted to a study of analytic geometry in three dimensions. Since many of the topics are generalizations of previous material in this course, the good student will review the related information with each new assignment.

Many of the problems in this chapter will be solved by the use of analytical methods, but these same problems could be solved by graphical methods as taught in the course in descriptive geometry. The accuracy of solution by graphical methods is limited, whereas a solution by algebraic methods can be carried to any desired degree of accuracy.

9.1 Cartesian Coordinates in Space

In order to locate a point in space we need three coordinates instead of the two coordinates that we used in the first portion of this course. If we locate a point in the xy-plane by coordinates x and y, we may locate a point in space by giving the directed distance z from the point in the xy-plane and along a line perpendicular to that plane. We shall write these coordinates in alphabetical order in the form (x, y, z), and the axes will usually be chosen as indicated in Fig. 9.1.

We locate the point P with coordinates $(3, 3, -1)$ (see Fig. 9.1) by moving along the positive x-axis 3 units, thence parallel to the positive y-axis 3 units, and finally 1 unit in the negative z-direction. In this figure the x- and z-axes are in the plane of the sheet of paper and the y-axis is supposed to be at right angles to each of the other two axes. This figure is a typical oblique projection (as it is called in mechanical drawing) and is drawn so that the angle between the negative x-axis and the positive y-axis appears to be between 30°

and 60°. The closer that angle is to 60°, the more it will seem that one is looking downward at the space figure. For an undistorted

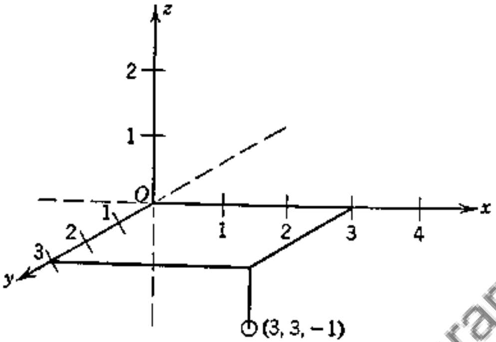


FIG. 9.1

pictorial sketch, the unit on the y-axis should be about half the unit length on the x- or z-axis.

Figure 9.2 shows the location of the point $P(x, y, z)$ as one corner of a rectangular solid with three of its edges along the three axes. It

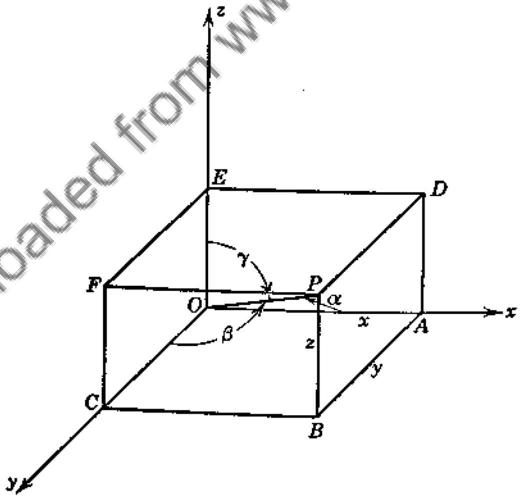


FIG. 9.2

should be clear from the figure that $x = \overline{OA} = \overline{CB} = \overline{ED} = \overline{FP}$, $y = \overline{OC} = \overline{AB} = \overline{EF} = \overline{DP}$, $z = \overline{OE} = \overline{AD} = \overline{CF} = \overline{BP}$ (positive if read in the directions indicated by the arrows on the three axes.)

The three coordinate planes divide space into eight parts or *octants*. The *first octant* is the totality of points all of whose three coordinates are positive. Thus, the rectangular solid of Fig. 9.2 is in the first octant.*

9.2 Right-Hand and Left-Hand Systems of Axes

The system of axes shown in Figs. 9.1 and 9.2 forms what is known as a left-hand system; the system of axes shown in Fig. 9.3 is a right-hand system. Thus, imagine in what follows that a right-handed screw is placed with its head in the xy-plane and its tip along the positive z-axis. If the positive x-axis is rotated toward the positive y-axis, and if the set of axes is right-handed, the screw will move in the positive z-direction; if the set of axes is left-handed, the screw will unscrew. The actual orientation of the axes and the choice for the positive directions and variables for the three axes will depend on the particular problem. The relationships, which we shall derive and apply, will be true for any orientation of axes.

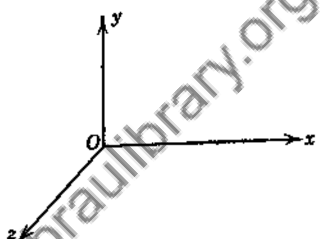


FIG. 9.3

9.3 Elementary Formulas

Figure 9.4 shows a radius vector $\overline{OP} = r$, the radius vector of a point P being the line segment directed *from the origin to the point*, and the point P having coordinates (x, y, z). Since the triangle OAB is a right triangle with $\overline{OB}$ as hypotenuse, we see that $\overline{OB}^2 = x^2 + y^2$. Also, since triangle OPB is a right triangle with right angle at B and $\overline{OP}$ as hypotenuse, $\overline{OP}^2 = \overline{OB}^2 + z^2$, or

$$r^2 = x^2 + y^2 + z^2.$$

Therefore the length of a radius vector from the origin to a point with coordinates (x, y, z) is equal to the square root of the sum of the squares of the three coordinates.

In order to determine the direction of a line we shall use three angles α, β, γ as shown in Fig. 9.4. The cosine of each angle would be

* See Problem 10 in Art. 9.3 for the method of numbering the other octants.

a convenient function to use since the range of principal values for the inverse cosine function is from 0° to 180°. Thus, if the angle is acute, its cosine is positive; if the angle is obtuse, its cosine is negative.

In Fig. 9.4 triangle OAP is a right triangle with right angle at A. Let α, $0° \leqq \alpha \leqq 180°$, be the angle between the positive x-axis and the radius vector $\overline{OP}$. Then, in this figure, $\cos\alpha = x/r$. Similarly, we can use triangles OCP and OEP to obtain the further relations

$$x = r\cos\alpha,$$

$$y = r\cos\beta,$$

$$z = r\cos\gamma.$$

The three angles α, β, γ are called the *direction angles* of the radius vector $\overline{OP}$. The cosines of these three angles are called the *direction*

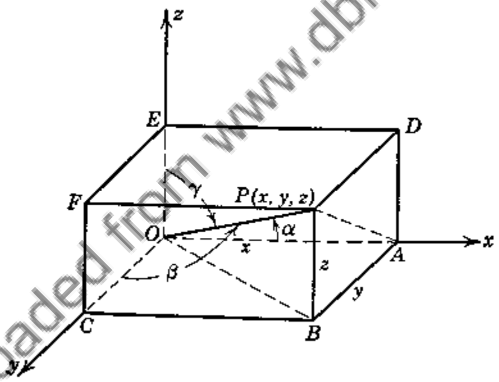


FIG. 9.4

cosines of the radius vector. Three numbers that are proportional to the direction cosines are called *direction numbers* for the radius vector $\overline{OP}$.

We observe that, if we square the three preceding equations and add, we obtain

$$x^2 + y^2 + z^2 = r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma).$$

But $x^2 + y^2 + z^2 = r^2$, and therefore the *sum of the squares of the direction cosines is equal to +1.*

EXAMPLE

Plot the point $A(4, 3, 2)$ and draw the line segment joining the origin to the point A. Determine the length of $\overline{OA}$, the direction cosines for $\overline{OA}$, and the direction angles (assuming that γ is acute and hence that $\cos\gamma$ is positive). Label these angles on your figure.

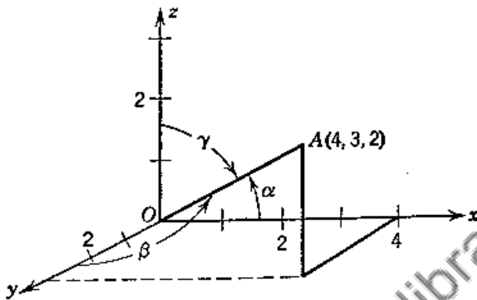


FIG. 9.5

Solution.

$$\overline{OA} = r = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}.$$

$$\cos\alpha = \frac{4}{\sqrt{29}} \approx 0.743, \alpha \approx 42.0°;$$

$$\cos\beta = \frac{3}{\sqrt{29}} \approx 0.557, \beta \approx 56.1°;$$

$$\cos\gamma = \frac{2}{\sqrt{29}} \approx 0.371, \gamma \approx 68.2°.$$

Three illustrative sets of direction numbers for $\overline{OA}$ are the following:

$$\{4, 3, 2\}; \quad \{8, 6, 4\}; \quad \{1, \tfrac{3}{4}, \tfrac{1}{2}\}.$$

The student should notice that, if we know a set of direction numbers for a line segment joining the origin O to a point P, then the coordinates of the point P are proportional to the three direction numbers. On the other hand, the three direction numbers are also the coordinates of a point Q that necessarily is on the line through O and P. From this fact it follows that we may determine the direction cosines of a line from the direction numbers by dividing the direction numbers by the square root of the sum of the squares of the direction numbers. We choose the positive or negative sign for the square root according to whether some one of the three angles is to be acute or obtuse.

PROBLEMS

1. Plot each of the following points. Find the length of the radius vector from the origin to the point, the three direction cosines, and the corresponding direction angles for the radius vector; label the direction angles on a figure using left-hand axes. Also, give the values for the projected directed lengths $(OP)_x$, $(OP)_y$, and $(OP)_z$, where O is the origin and P is the point.

(*a*) (3, 2, 4); (*b*) (1, 2, −2); (*c*) (6, −3, 2); (*d*) (3, −4, −5).

2. Sketch and label the figures for Problem 1, using right-hand axes.

3. A force $F = 140$ lb. acts along the radius vector $\overline{OP}$ that joins the origin and the point $P(6, -2, 3)$. Determine the components of this force F_x, F_y, and F_z, by multiplying F, in turn, by each of the three direction cosines of $\overline{OP}$. Use γ acute since F acts from O toward P.

4. Work Problem 3 if the force $F = 200$ lb. and the point P has coordinates (4, 5, 6).

5. An air shaft in a coal mine has the bottom end 50 ft. south, 40 ft. east, and 200 ft. down beginning from the mouth of the shaft. Determine the angles that the shaft makes with the three directions (east, south, and down), and sketch a figure.

6. Draw the straight line through the origin with direction numbers {1, 2, 3} by plotting the point $P(1, 2, 3)$. Also draw the line by use of the point Q with coordinates (2, 4, 6). Then draw a line through the origin with direction numbers {4, 8, 12}.

7. The boom of a derrick is 70 ft. long and has its foot on the ground. The boom makes an angle of 60° with the east direction and an angle of 60° with the south direction. Determine the angle that the boom makes with the upward direction. Also find the vertical distance from the ground to the top of the boom.

8. Work Problem 7 if both angles are 70° instead of 60°.

9. The top of a taut guy wire is 30 ft. east, 20 ft. south, and 30 ft. up, all measured from the lower end of the wire. Determine the length of the wire (assume it to be straight) and the angles that the wire makes with the three directions.

10. The following table gives eight points, one in each octant, together with the numbers of the octants in which they lie; locate each point, and number the associated octant:

x	4	−3	−3	5	4	−3	−3	5
y	2	2	−1	−1	2	2	−1	−1
z	1	2	4	4	−1	−2	−4	−4
Octant	I	II	III	IV	V	VI	VII	VIII

11. Plot the rectangle defined by each of the following sets of four points. Then sketch the ellipse that is tangent to each side of the rectangle at its mid-point.

(*a*) (0, 0, 0), (6, 0, 0), (6, 4, 0), (0, 4, 0).
(*b*) (0, 4, 2), (0, 4, −2), (0, −4, 2), (0, −4, −2).
(*c*) (5, 3, 2), (5, −3, 2), (−5, 3, 2), (−5, −3, 2).

12. A hyperbola has one asymptote that goes through the origin and (6, 3, 0). A second asymptote goes through the origin and (6, −3, 0), and one branch of the hyperbola goes through (4, 0, 0). Draw the hyperbola.

13. A parabola has its vertex at (2, 0, 4), has its focus at (2, 1, 4), and goes through (2, 4, 0). Draw the parabola (draw the parabola at its vertex so that it looks tangent to a line perpendicular to the axis of symmetry of the parabola).

9.4 Distance between Two Points

Before studying this article, the student should review the formulas for directed lengths parallel to one of the axes and the general length formula in plane analytic geometry.

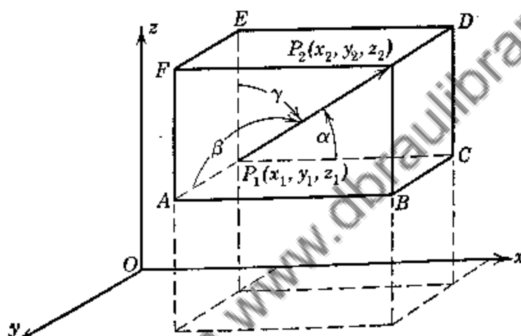


FIG. 9.6

If the line segment joining two points is parallel to the x-axis, the two points will necessarily have the same y- and z-coordinates. In this case the *directed* distance between the two points is equal to the x-value of the point *to which* the measurement is made *minus* the x-value of the point *from which* the measurement is made. Similar statements hold for line segments parallel to the y- or z-axis.

To obtain the general distance formula for the distance between any two points, such as $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, we proceed in the same manner as in plane analytic geometry. Figure 9.6 shows the two points, P_1 and P_2, located as opposite vertices of a rectangular solid. We first notice that triangle P_1CB is a right triangle with $\overline{P_1B}$ as hypotenuse. Then

$$\overline{P_1C} = (P_1P_2)_x = x_2 - x_1, \quad \overline{CB} = (P_1P_2)_y = y_2 - y_1,$$

$$(P_1B)^2 = (P_1C)^2 + (CB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Also triangle P_1BP_2 is a right triangle with right angle at B. Then

$$\overline{BP_2} = (P_1P_2)_z = z_2 - z_1$$

and

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Since we are seeking the numerical distance between the two points, we take the positive square root and obtain

$$L = \overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2};$$

or (in words), *the numerical value of the distance between two points is equal to the square root of the sum of the squares of the differences of the x's, y's, and z's.*

EXAMPLE

Find the distance between $A(4, -3, 5)$ and $B(-1, 2, 8)$.

Solution. $L = \sqrt{5^2 + (-5)^2 + (-3)^2} = \sqrt{59}$.

9.5 The Direction Angles for a General Directed Line

In order to define the direction angles for a general directed line we make use of the direction angles for a line through the origin as in the following definitions.

DEFINITIONS. *The direction angles for a general line with a specified direction are defined as the direction angles of a line through the origin that is parallel to the general line and has the same direction. An equivalent statement is that if the axes are translated to some point on the general line, then the direction angles to this line are the direction angles measured from the translated axes.*

In Fig. 9.6, if the required direction angles are α, β, and γ for the line directed from P_1 toward P_2, and if L is the length of $\overline{P_1P_2}$, then

$$\cos\alpha = \frac{(P_1P_2)_x}{L} = \frac{x_2 - x_1}{L},$$

$$\cos\beta = \frac{(P_1P_2)_y}{L} = \frac{y_2 - y_1}{L},$$

$$\cos\gamma = \frac{(P_1P_2)_z}{L} = \frac{z_2 - z_1}{L}.$$

If this last equation were to give the supplement of the desired angle for γ, in order to obtain γ we could take the direction of all the projections in the opposite sense (which would be equivalent to dividing by $-L$ instead of by $+L$).

We recall that the sum of the squares of the direction cosines is $+1$ and that a set of direction numbers is a set of numbers that are proportional to the direction cosines. Thus, if $\{2/3, -2/3, 1/3\}$ are a set of direction cosines, then several sets of direction numbers are as follows:

$$\{2, -2, 1\}; \quad \{10, -10, 5\}; \quad \{-4, 4, -2\}; \quad \text{and} \quad \{1, -1, 1/2\}.$$

Also notice that $(2/3)^2 + (-2/3)^2 + (1/3)^2 = 1$.

If a set of direction numbers is given as $\{a, b, c\}$, then the direction cosines are a particular multiple of this set of numbers, and we may denote the direction cosines by $\{ka, kb, kc\}$. But

$$(ka)^2 + (kb)^2 + (kc)^2 = 1,$$

whence $k^2 = 1/(a^2 + b^2 + c^2)$. This determines two values for k. The proper value will be the positive square root if c is positive and if γ is to be acute (i.e., if $\cos\gamma$ is to be positive); if γ is to be obtuse and if c is positive, the proper value will be the negative square root. Hence, if a line has direction numbers $\{a, b, c\}$, its direction cosines are given by

$$\frac{a}{\pm\sqrt{a^2 + b^2 + c^2}}, \quad \frac{b}{\pm\sqrt{a^2 + b^2 + c^2}}, \quad \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}}.$$

We observe that a set of direction numbers for the line segment joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the set of three projections on the coordinate axes, all being taken in the same order; thus $\{(P_1P_2)_x, (P_1P_2)_y, (P_1P_2)_z\}$ is a set of direction numbers for the line.

We observe from this last fact that, if the line through the two points has direction numbers $\{a, b, c\}$, we may draw a line through the origin parallel to the given line by drawing the line through the origin and the point with coordinates (a, b, c).

EXAMPLE 1

Given the points $A(1, 3, 2)$ and $B(4, 1, 6)$. Determine $(AB)_x$, $(AB)_y$, $(AB)_z$; the length of $\overline{AB}$; the direction cosines if the line is directed so that the angle γ is acute; and the direction angles (each correct to the nearest minute). Plot the two points; draw the line segment $\overline{AB}$; draw a line through the origin with the same direction numbers that $\overline{AB}$ has; and label the angles.

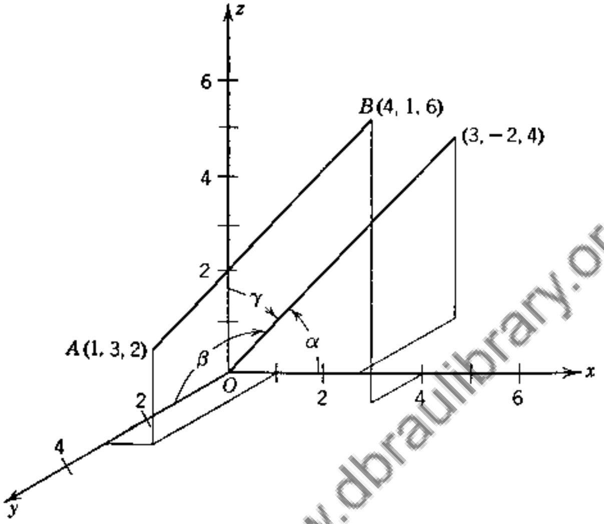


FIG. 9.7

Solution. (See Fig. 9.7.)

$$(AB)_x = 3, \quad (AB)_y = -2, \quad (AB)_z = 4; \quad L = \sqrt{29} \approx 5.38516.$$

$$\cos \alpha = \frac{3}{\sqrt{29}} \approx 0.55709, \quad \alpha \approx 56° \, 9'.$$

$$\cos \beta = \frac{-2}{\sqrt{29}} \approx -0.37139, \quad \beta \approx 111° \, 48'.$$

$$\cos \gamma = \frac{4}{\sqrt{29}} \approx 0.74278, \quad \gamma \approx 42° \, 2'.$$

EXAMPLE 2

A force $F = 300$ lb. acts along the line segment *from* the point $P_1(2, 4, 5)$ *toward* the point $P_2(4, 2, 6)$. Determine the components of this force in directions parallel to the three axes.

Solution. The components of the force are given by $F_x = 300 \cos \alpha$, $F_y = 300 \cos \beta$, and $F_z = 300 \cos \gamma$, where α, β, and γ are the direction angles for the line $\overline{P_1P_2}$. A set of direction numbers for this line is given by $\{2, -2, 1\}$; the corresponding direction cosines are given by $\{\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\}$, and not by $\{-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\}$ since the force is directed from P_1 toward P_2. We find that

$$F_x = 200 \text{ lb.}, \quad F_y = -200 \text{ lb.}, \quad \text{and} \quad F_z = 100 \text{ lb.}$$

PROBLEMS

1. For each of the following pairs of points A and B, determine $(AB)_x$, $(AB)_y$, $(AB)_z$; the length of $\overline{AB}$; the direction cosines if the angle γ is acute; and the direction angles (each correct to the nearest tenth of a degree). Using a left-hand system of axes, plot the points; draw the line segment $\overline{AB}$; draw a line through the origin with the same direction numbers that $\overline{AB}$ has; and label the direction angles.

(*a*) $A(2, 1, 1)$, $B(6, 5, 3)$.
(*b*) $A(-2, 1, 2)$, $B(4, 3, 5)$.
(*c*) $A(-4, 2, -1)$, $B(1, 2, 3)$.
(*d*) $A(3, 2, 1)$, $B(6, 4, 2)$.
(*e*) $A(4, 0, 0)$, $B(0, 4, 2)$.
(*f*) $A(6, 6, 0)$, $B(0, 0, 6)$.

2. Redraw the figures for Problem 1, using a right-hand system of axes.

In the next two problems assume the east direction to be the positive x-direction, the south direction to be the positive y-direction, and the upward direction to be the positive z-direction.

3. The ends of a guy wire are at $A(0, 0, 30)$ and $B(20, 10, 4)$, and the dimensions are given in feet. Determine the length of the taut wire and the angles that it makes with the east, south, and upward directions.

4. The vertex opposite one corner of a room is 12 ft. east, 10 ft. south, and 8 ft. up from the first corner. Determine the length of the diagonal joining two opposite vertices. Also sketch a figure and give the coordinates of all eight vertices of the room.

5. The post of a crane is the line segment from the origin to $C(0, 0, 20)$, where 1 unit is 1 ft. Guy wires are connected from C to $E(-20, 15, 0)$, and from C to $D(-20, -10, 0)$. The boom for the crane joins $G(0, 0, 6)$ to $H(20, 0, 6)$. A guy wire connects H to $F(0, 0, 16)$. Plot a figure that shows the crane, guy wires, and boom, and determine the lengths of all the members. What angles do the guy wires $\overrightarrow{EC}$ and $\overrightarrow{DC}$ make with the post?

6. A round, three-legged table has the center of the top at $A(0, 0, 3)$. Its legs are attached symmetrically, and one of the legs joins $B(2, 0, 0)$ to $C(1.5, 0, 3)$. Find the coordinates of the ends of the other two legs.

7. The mouth of a straight tunnel is at $(0, 0, 30)$; the other end of the tunnel is 120 ft. east, 40 ft. south, and 30 ft. down from the mouth. Determine the coordinates of the bottom of the tunnel, the length of the tunnel, and the acute angles that the tunnel makes with the three given directions (east, south, and down). First work this problem by methods of analytic geometry; then give the results required in this problem.

8. A force $F = 100$ lb. acts *from* $A(1, 2, 3)$ *toward* $B(5, -4, 6)$. Determine the components of this force in the directions of the three coordinate axes.

9. Determine which of the following sets of three points are collinear, and sketch a figure for each case:

(*a*) $(2, -3, 4)$, $(8, 0, -2)$, $(12, 2, -6)$.
(*b*) $(1, -2, -5)$, $(5, 6, 3)$, $(8, 12, 9)$.
(*c*) $(3, 0, 2)$, $(0, 4, 4)$, $(-3, 8, 7)$.
(*d*) $(4, -5, 3)$, $(5, -2, 6)$, $(6, 0, 9)$.
(*e*) $(7, 0, -4)$, $(5, 3, -1)$, $(-1, 9, 5)$.

10. Show that the triangle with vertices at $(7, 3, 4)$, $(1, 0, 6)$, and $(4, 5, -2)$ is a right triangle, and find its area.

11. A right pyramid with a square base has its base in the xy-plane and its vertex on the z-axis. If one lateral edge joins the two points (6, 0, 0) and (0, 0, 8), determine the coordinates of the other corners of the base. Then find the volume and the lateral surface area of the pyramid, stating both results in exact form.

12. Use a figure such as Fig. 9.6 and prove that the coordinates of the mid-point of a line segment that joins $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}, \quad \frac{z_1 + z_2}{2}\right).$$

13. Use the theorem established in Problem 12 to aid in finding the coordinates of the mid-points of the line segments $\overline{AB}$ defined as follows:

(a) $A(4, 6, -8)$, $B(2, 2, 4)$.
(b) $A(0, 0, 4)$, $B(6, 8, 0)$.
(c) $A(0, -2, 8)$, $B(4, 2, -8)$.
(d) $A(-3, -5, 7)$, $B(2, -4, 6)$.

14. Figure 9.8 shows a solid with a square base $OAFE$. The plane BCD is parallel to the xy-plane, and the lines $\overline{AB}$ and $\overline{ED}$ are perpendicular to that xy-plane.

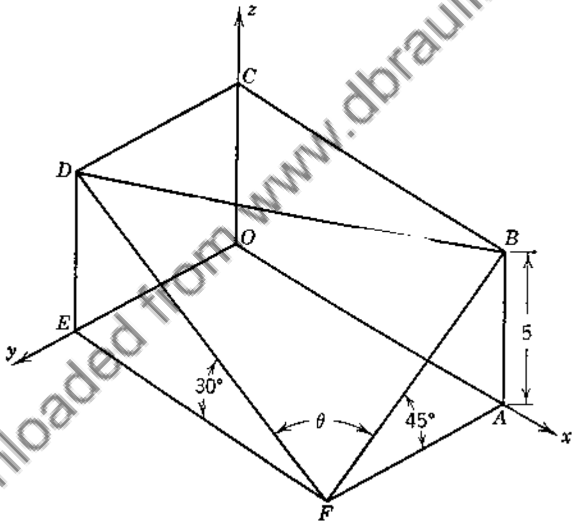


Fig. 9.8

Use the data from the figure and solve the following problems:

(a) Find direction angles for lines $\overline{FB}$ and $\overline{FD}$, assuming that γ is acute in each case.

(b) Find the corresponding direction cosines for lines $\overline{FB}$ and $\overline{FD}$.

(c) On a new set of three axes, draw lines through the origin that have the direction cosines obtained in (b); give the coordinates of one point on each line (in addition to the origin).

(d) Determine coordinates for points B and D, and find the length of $\overline{BD}$.

(e) Find angle θ in triangle FBD by aid of the law of cosines.

15. A pyramid has its vertices at $A(0, 0, 12)$, $B(16, 0, 0)$, $C(-2, -6, 0)$, and $D(-8, 12, 0)$. First find the coordinates of the mid-points of the sides of the base in the xy-plane; then find the volume of the pyramid whose base is the triangle joining the preceding mid-points and whose altitude is the same as the altitude of the original pyramid. What relationship exists between the volumes of this pyramid and the original pyramid?

16. A triangle has its vertices at $A(6, 0, 0)$, $B(0, -3, 2)$, and $C(3, 2, -6)$. Find the coordinates of the mid-points of the three sides. Then find the lengths of the line segments that join pairs of these mid-points. Finally, find the area of the triangle determined by the three mid-points.

9.6 Angle between Two Lines. Parallel and Perpendicular Lines

We shall be concerned in this article with properties similar to those developed in the first chapter of plane analytic geometry: the angle between two lines, parallel lines, and perpendicular lines.

DEFINITION. *The angle between two directed lines is defined as the angle θ ($0 \leqq \theta \leqq 180°$) between the positive directions of two other lines that pass through the origin and that are parallel to and have the same directions as the two given lines.* If the problem does not specify the positive direction for a line, we shall suppose that γ is acute (if $\gamma = 90°$, then we shall suppose that β is acute; if both $\gamma = 90°$ and $\beta = 90°$, then we shall suppose that $\alpha = 0$).

Figure 9.9 shows two line segments $\overline{OP_1}$ and $\overline{OP_2}$, which make an angle θ with each other. We seek an equation for some trigonometric

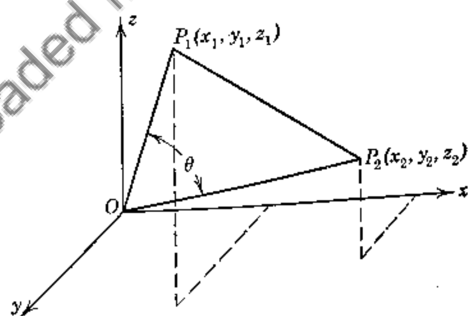


FIG. 9.9

function of θ, preferably one that will distinguish whether θ is acute or obtuse. We proceed to derive an expression for $\cos\theta$, and we shall express $\cos\theta$ in terms of trigonometric functions of the direction angles for the two line segments. According to the method of the locus derivation, we must make some geometric statement from the figure that characterizes the angle θ. Accordingly, we observe that triangle OP_1P_2 has θ as one vertex angle and that we can find the lengths of the three sides of that triangle. Therefore we may use the law of cosines for Step III of the locus-derivation method. Thus,

$$(P_1P_2)^2 = (OP_1)^2 + (OP_2)^2 - 2(OP_1)(OP_2)\cos\theta.$$

We substitute

$$(OP_1)^2 = x_1^2 + y_1^2 + z_1^2,$$
$$(OP_2)^2 = x_2^2 + y_2^2 + z_2^2,$$
$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

and obtain

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}.$$

The student should perform the requisite algebra to obtain this result and should notice that the positive square roots of the two quantities have been used.

Because of the fact that a line that goes through the origin and has direction numbers $\{a, b, c\}$ will necessarily go through the point with coordinates (a, b, c), we may restate the preceding result as follows:

THEOREM. *Let two directed lines have direction numbers* $\{a_1, b_1, c_1\}$ *and* $\{a_2, b_2, c_2\}$. *The angle* θ *between the two lines is given by*

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and $0 \leqq \theta \leqq 180°$.

If we utilize direction cosines instead of direction numbers, the denominator of the preceding expression is $+1$ and we may restate the theorem as follows:

THEOREM. *The angle* θ *between two lines with direction angles* $\alpha_1, \beta_1, \gamma_1$, *and* $\alpha_2, \beta_2, \gamma_2$ *is given by*

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2,$$

and $0 \leqq \theta \leqq 180°$.

A direct consequence of the preceding theorem is a test for perpendicular lines, since in this case $\cos\theta = 0$.

COROLLARY. Two lines with direction numbers $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$ are *perpendicular* if, and only if,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Another consequence of the definitions of direction numbers and direction angles is the fact that, if two lines are *parallel*, the sets of direction numbers for the two lines are proportional.

The three essential facts to be learned in this article are the following: the use of the law of cosines to find the angle between two lines, the test for parallel lines, and the test for perpendicular lines.

EXAMPLE 1

Two line segments $\overline{AB}$ and $\overline{CD}$ have the following data: $A(1, 2, 3)$, $B(2, 4, 1)$, $C(1, -1, 7)$, $D(-5, 2, 5)$.

(*a*) Compute the angle between the two lines correct to the nearest minute, assuming that the required angle is acute.

(*b*) Draw the two line segments.

(*c*) Draw two line segments that go through the origin and that are parallel to the two given line segments, and label the angle θ.

Solution. Direction numbers for $\overline{AB}$ are $\{-1, -2, 2\}$, for $\overline{CD}$ are $\{6, -3, 2\}$. We locate the points $E(-1, -2, 2)$ and $F(6, -3, 2)$, and determine the lengths of the three sides of the triangle OEF: $\overline{OE} = 3$, $\overline{OF} = 7$, and $\overline{EF} = \sqrt{50}$. Then, from the law of cosines, $50 = 9 + 49 - 2(3)(7)\cos\theta$, whence $\cos\theta \approx +0.19048$. (If this were negative it would indicate that we had chosen positive directions incorrectly and the angle we seek would be the supplement of the angle determined by this equation.) The required acute angle is $79° 1'$. The student should complete this exercise first by drawing a figure for the original two line segments, and then by drawing the triangle OEF and labeling the angle θ.

EXAMPLE 2

A line segment has direction numbers $\{2, -5, 3\}$.

(*a*) A second line has direction numbers $\{7, p, q\}$ and is parallel to the first line. Determine the values of p and q.

(*b*) A second line has direction numbers $\{3, g, -4\}$ and is perpendicular to the first line. Find the value of g.

Solution. (*a*) If two lines are parallel, their direction numbers must be proportional. Hence $7/2 = p/(-5) = q/3$, from which $p = -17.5$ and $q = 10.5$.

(*b*) The requirement that two lines be perpendicular in this case means that $(2)(3) + (-5)(g) + (3)(-4) = 0$, whence $g = -1.2$.

EXAMPLE 3

The vertices of a triangle are $A(3, 5, -1)$, $B(2, 4, 3)$, and $C(5, 1, 4)$. Find the interior angle at C correct to the nearest tenth of a degree.

First Solution. We find the lengths of the three sides to be $\overline{AB} = \sqrt{18}$, $\overline{AC} = \sqrt{45}$, $\overline{BC} = \sqrt{19}$. We use the law of cosines and find that

$$\cos C = \frac{23}{\sqrt{(45)(19)}} \approx 0.7866, \quad C \approx 38.1°.$$

Second Solution. Direction numbers for the line segment *from C toward A* are $\{-2, 4, -5\}$, and direction numbers for the line *from C toward B* are $\{-3, 3, -1\}$. We use the triangle with vertices at the origin O, the point $D(-2, 4, -5)$, and the point $E(-3, 3, -1)$. The lengths of the sides are: $\overline{OD} = \sqrt{45}$, $\overline{OE} = \sqrt{19}$, and $\overline{DE} = \sqrt{18}$. The remainder of the solution is the same as in the first solution.

EXERCISE FOR THE STUDENT. Draw the figure for Example 3 and show both the triangle ABC and the triangle ODE. Then label the angle C in both triangles.

PROBLEMS

1. Use direction numbers to show that the triangle with vertices $E(7, 3, 3)$, $F(1, 0, 5)$, and $G(4, 5, -3)$ is a right triangle. Then determine the other two vertex angles, each correct to the nearest tenth of a degree.

2. Direction numbers of two lines are respectively $\{2, -3, 2\}$ and $\{4, 2, -1\}$. Are the two lines perpendicular? Why?

3. Show that the triangle with vertices at $A(2, 6, -3)$, $B(-2, 7, 2)$, and $C(-4, 3, -3)$ is an isosceles triangle. Then find the interior angle at A by aid of the law of cosines, and check by right-triangle methods.

4. Which of the pairs of lines defined by the following data are parallel, and which are perpendicular?

(*a*) The lines have direction numbers $\{2, 3, 3\}$ and $\{3, 4, -6\}$.

(*b*) One line goes through $(7, 6, 9)$ and $(4, 3, 5)$; the other line goes through $(-1, -1, -1)$ and $(5, 5, 7)$.

(*c*) One line goes through $(2, 6, 9)$ and $(4, 3, 5)$; the other line goes through $(0, 10, -1)$ and $(-6, 2, 2)$.

(*d*) One line has direction numbers $\{4, 4, 0\}$; the other line has $\alpha = 45°$ and $\beta = 45°$ for two of the three direction angles.

(*e*) One line has direction angles: $\alpha = 60°$, $\beta = 120°$, and $\cos\gamma < 0$; the other line goes through the origin and the point $(8, -4, 2\sqrt{2})$.

5. Prove that the points $(2, 1, 1)$, $(0, 1, 0)$, $(1, 3, -2)$, and $(3, 3, -1)$ are the vertices of a rectangle, and draw it.

6. Show that the three points $(5, 0, -4)$, $(-1, -3, 2)$, and $(9, 2, -8)$ are collinear. Then determine the coordinates of two more points on this line: one has 5 for its z-coordinate, the other is in the yz-plane.

7. Determine the interior angles of the triangle with vertices as given below; draw the triangle and label the angles:

(*a*) $A(4, 2, 3)$, $B(2, 1, 0)$, $C(3, 0, 6)$.
(*b*) $A(1, 2, 3)$, $B(6, 5, 4)$, $C(-1, 0, -2)$.
(*c*) $A(4, 0, 0)$, $B(0, 5, 0)$, $C(0, 0, -3)$.
(*d*) $A(6, 0, 0)$, $B(0, 6, 0)$, $C(0, 0, 6)$.
(*e*) $A(4, 0, 0)$, $B(0, -3, 0)$, $C(0, 0, -12)$.

8. Determine the exact length of the projection of the line segment joining $(1, 1, 2)$ to $(2, -1, 4)$ upon the line that goes through $(2, 1, -2)$ and $(4, -5, 1)$.

9. Determine the acute angle between the line through $A(1, 5, 2)$ and $B(4, -1, 0)$ and the line through $C(4, 4, 2)$ and $D(-4, -2, 6)$.

10. Two forces act upward from the origin. One force is $S = 70$ lb.; it acts along a line with direction numbers $\{2, 3, 6\}$. The other force is $T = 30$ lb.; it acts along a line with direction numbers $\{2, 1, 2\}$.

(*a*) Find the sum of the components of the two forces in the x-direction (add the sum of the products of the forces by their respective direction cosines in the x-direction); find the sum of the components in the y-direction; and find the sum of the components in the z-direction. Then find the magnitude of the resultant of these two forces that is the positive square root of the sum of the squares of these component sums. Also find the direction angles for this resultant; that is, find the direction angles for a line that goes through the origin and the point whose coordinates are these component sums.

(*b*) Find the angle between the directions of the two given forces.

(*c*) Find the magnitude of the resultant of these two forces by aid of the law of cosines and your result in (*b*).

11. One pipe runs along the floor of a room and joins the points $(12, 10, 0)$ and $(0, 5, 0)$. A second pipe is to connect with the first pipe and to join $(0, 5, 0)$ and $(0, 0, 8)$. What is the angle at the fitting where the two pipes join?

9.7 The General Equation of a Plane

In plane analytic geometry we learned that the graph of a linear equation in two variables is always a straight line. In this article we shall learn that the graph of a linear equation in three variables is always a plane.

THEOREM. *Let a line through the origin have direction numbers $\{a, b, c\}$. The equation of the plane that goes through the point (a, b, c) and that is perpendicular to the line is*

$$ax + by + cz = a^2 + b^2 + c^2.$$

Proof. We use the general locus-derivation process. Let the point with coordinates (a, b, c) in Fig. 9.10 be A. Let $P(x, y, z)$ be any general point in the required plane. Then $\overline{AP}$ is perpendicular to $\overline{OA}$.

But a set of direction numbers for $\overline{OA}$ is $\{a, b, c\}$, for $\overline{AP}$ is $\{x - a, y - b, z - c\}$. Then

$$a(x - a) + b(y - b) + c(z - c) = 0.$$

If we expand this last equation, we will obtain the result stated in the theorem.

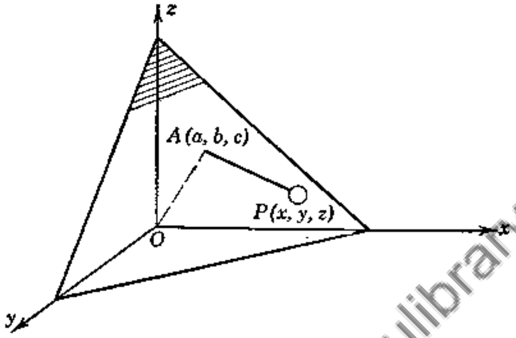


Fig. 9.10

We consider next an equation that is linear in the three variables but that has zero for its constant term.

Theorem. *Let a line go through the origin and have direction numbers $\{a, b, c\}$. The equation of a plane that goes through the origin and that is perpendicular to the line is*

$$ax + by + cz = 0.$$

Proof. Let a general point P in the plane have coordinates (x, y, z). Direction numbers for the line $\overline{OP}$ will be $\{x, y, z\}$, and this line will be perpendicular to the given line with direction numbers $\{a, b, c\}$. Therefore $ax + by + cz = 0$.

We may use these two theorems and prove the following theorem:

Theorem. *The locus of every linear equation in three variables x, y, z is a plane.*

Proof. A general linear equation in the three variables x, y, z is

$$Ax + By + Cz + D = 0,$$

where at least one of A, B, and C is not zero. If $D \neq 0$, we divide both sides of this equation by k, where k is chosen so that

$$-\frac{D}{k} = \left(\frac{A}{k}\right)^2 + \left(\frac{B}{k}\right)^2 + \left(\frac{C}{k}\right)^2.$$

The resulting equation is of the form $ax + by + cz = a^2 + b^2 + c^2$; the locus is a plane by the first theorem of this article. If $D = 0$, the equation $Ax + By + Cz = 0$ satisfies the conditions of the second theorem of this article and the locus is again a plane. Therefore the theorem is established.

MENTAL EXERCISES FOR THE STUDENT. Show that the following statements are correct:

1. $2x + 3y + 4z = 17$ is the equation of a plane. Any line perpendicular to this plane has direction numbers proportional to $\{2, 3, 4\}$.
2. $3x + 4y = 11$ is the equation of a *plane*. Any line perpendicular to this plane has direction numbers proportional to $\{3, 4, 0\}$.
3. The equation of the xy-plane is $z = 0$, since zero is the z-coordinate of every point in that plane.

We consider next how to determine the perpendicular distance from an oblique plane to a point. That the required formula is a direct generalization of the corresponding formula in plane analytic geometry should be evident at once.

THEOREM. *The numerical value of the perpendicular distance from the plane $ax + by + cz + d = 0$ to the point $P(X, Y, Z)$ is given by*

$$L = \left| \frac{aX + bY + cZ + d}{\sqrt{a^2 + b^2 + c^2}} \right|.$$

Proof. This theorem is trivial and of little interest if any two of a, b, and c are zero. (Why?) Hence we shall suppose that at least two of these three numbers are not zero. For the proof that we shall give we shall suppose that $a \neq 0$ and leave to the student the task of completing the proof by showing that the theorem is correct for the special case: $a = 0, b \neq 0, c \neq 0$.

We use Fig. 9.11, which shows the given plane, the point P, the required distance L, and the perpendicular $\overline{OA}$ from the origin to the plane. Then $L =$ (sum of projections of $\overline{OB}$, $\overline{BC}$, and $\overline{CP}$ on line $\overline{OA}$) $- \overline{OA}$. But line $\overline{OA}$ has direction numbers $\{a, b, c\}$ and direction angles α, β, γ, where $\cos \alpha = a/\sqrt{a^2 + b^2 + c^2}$, etc. Since the length of the projection of one line segment on a line is given by the length of the line segment multiplied by the cosine of the angle between the line segment and the line, the length of the projection of $\overline{OB}$, for example,

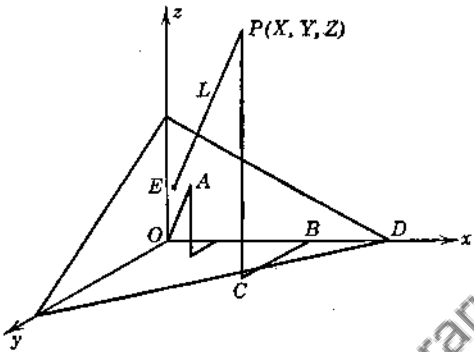


Fig. 9.11

upon $\overline{OA}$ is given by $(\overline{OB})$ $(\cos \alpha) = X \cos \alpha$. Therefore the length of L may be written in the form

$$L = X \cos \alpha + Y \cos \beta + Z \cos \gamma - \overline{OA}.$$

Now $\overline{OA} = \overline{OD} \cos \alpha$ and $\overline{OD}$ is equal to the x-intercept for the plane and is therefore given by $-d/a$. We substitute for the direction cosines and $\overline{OA}$, we simplify this, and we obtain the result stated in the theorem.

9.8 Drawing of Planes

The preceding article has shown that the graph of a general linear equation, $ax + by + cz + d = 0$, where at least one of a, b, and c is not zero, is a plane. To draw a plane, we need three points in the plane and not all on a straight line, a line and a point not on the line, two intersecting lines, or two parallel lines. The following examples illustrate the methods to be used to sketch planes.

Rule. The x-intercept of a plane is obtained by taking both $y = 0$ and $z = 0$, the y-intercept by taking $x = 0$ and $z = 0$, and the z-intercept by taking $x = 0$ and $y = 0$.

Example 1

Sketch $2x + 3y + 4z = 7$.

Solution. The x-intercept is $7/2$, the y-intercept is $7/3$, and the z-intercept is $7/4$. We locate the points corresponding to these three intercepts, and connect them by straight lines. Some shading will help to show the required plane. The student must realize that Fig. 9.12 shows only a portion of the plane.

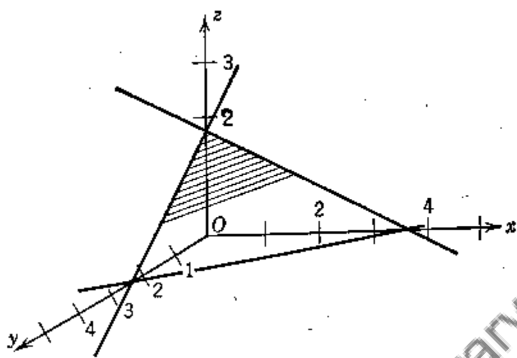


FIG. 9.12

EXAMPLE 2

Sketch $x + 2y = 3z$.

Solution. The plane crosses all three axes at the origin. We obtain the graph in the yz-plane by placing $x = 0$ (every point in the yz-plane has zero for its x-coordinate). This yields $2y = 3z$, which together with $x = 0$ is a straight line in the yz-plane. Similarly, the graph in the xz-plane is the straight line given by $y = 0$

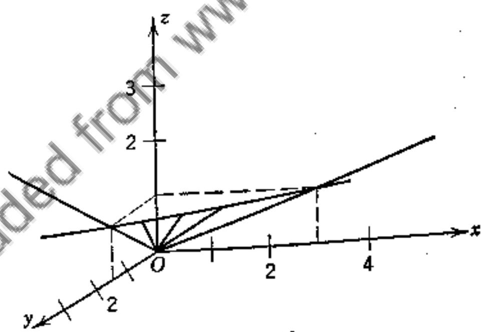


FIG. 9.13

and $x = 3z$. We sketch these two straight lines (Fig. 9.13) and connect some point on one line with a point on the other line. Some shading will help to make the figure clearer.

DEFINITION. *The intersection of the graph of an equation in three variables with the graph of a plane is called the trace in that plane.* Thus, we used the traces in the yz-plane and the xz-plane to sketch the required plane in this last example.

EXAMPLE 3

Sketch $x + 2z = 4$.

Solution. Since this equation is a special case of the equation

$$ax + by + cz + d = 0,$$

the graph is a *plane*. The x-intercept is 4, the z-intercept is 2, and there is no y-intercept, i.e., the graph does not cross the y-axis. Therefore the plane is parallel to the y-axis.

To draw the plane, we locate the two points corresponding to the x- and z-intercepts, and join these by a straight line. We also draw lines through each of these two points parallel to the y-axis. The graph will be clearer if additional lines are drawn as shown in Fig. 9.14.

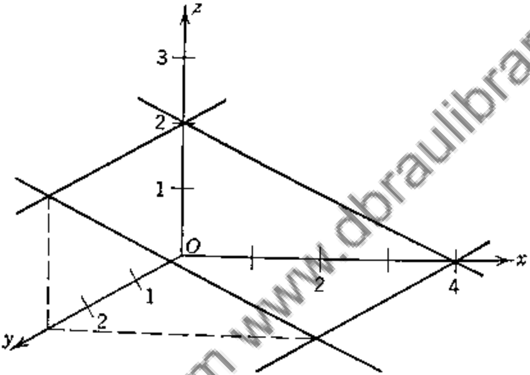


FIG. 9.14

Note that direction numbers of a line perpendicular to this plane are $\{1, 0, 2\}$, whence $\cos\beta = 0$ and $\beta = 90°$; therefore a line perpendicular to this plane is perpendicular to the y-axis; hence the plane is parallel to the y-axis.

EXAMPLE 4

Sketch the graph of $z = 4$.

Solution. This equation is a special case of the general linear equation in three variables, and therefore the graph is a plane. Since every point in the required plane has 4 for its z-coordinate, the graph of this equation is a plane that is parallel to the xy-plane and is 4 units above it. The sketch will be left as an exercise for the student.

The *angle between two planes* is, by definition, the angle between two directed lines that are perpendicular respectively to the two planes. Thus, for example, the acute angle between the two planes

$$2x - 3y + 4z = 8 \quad \text{and} \quad x + 5y - z = 10$$

is the acute angle between two lines with direction numbers $\{2, -3, 4\}$ and $\{1, 5, -1\}$. This angle may be found by the methods of Art. 9.6.

9.9 The Equation of a Plane That Satisfies Three Conditions

The methods to be used to find the equation of a plane that satisfies three given conditions, such as being required to go through three given points, are illustrated in the following examples:

EXAMPLE 1

Find the equation of a plane that goes through the three points $(2, 0, 0)$, $(0, -3, 0)$, and $(0, 0, 4)$.

Solution. The general equation of a plane is $ax + by + cz = k$. We substitute the three sets of coordinates, one set at a time, and obtain the three equations: $2a = k$, $-3b = k$, and $4c = k$. We solve for a, b, and c in terms of k, we substitute the results in the original equation, and we obtain

$$\frac{k}{2}x - \frac{k}{3}y + \frac{k}{4}z = k.$$

We next divide by k, multiply by 12, and obtain

$$6x - 4y + 3z = 12.$$

The student should verify, as a check, that this plane does go through the three given points.

EXAMPLE 2

Determine the equation of the plane that goes through the three points $(1, 4, 2)$, $(2, 5, 3)$, and $(3, -3, 1)$.

Solution. We substitute the three sets of coordinates in $ax + by + cz = k$ and obtain

$$a + 4b + 2c = k,$$

$$2a + 5b + 3c = k,$$

$$3a - 3b + c = k.$$

We have three linear *homogeneous* equations in four unknowns. Unless there is something very special about the three given points, we can solve for *some three* of the unknowns in terms of the fourth, substitute as we did in the preceding example, and finally simplify to the required equation. If we try to solve, in this particular example, for a, b, and c in terms of k, we shall arrive at some difficulties, owing to the fact that $k = 0$ in this example. If we were to use determinants to solve for a, b, and c in terms of k, the denominator determinant would be zero and so also would all three of the numerator determinants. In such cases, we should back up and solve for another set of three in terms of a fourth unknown.

EXERCISE FOR THE STUDENT. Solve the preceding three equations for a, b, and k in terms of c, and show that $a = -2c/3$, $b = -c/3$, and $k = 0$ (the method of addition-subtraction would be a proper one to use). Then show that the equation of the plane in this example reduces to $2x + y - 3z = 0$.

EXAMPLE 3

Find the equation of a plane that goes through (2, 1, 3) and (1, 2, 4) and that is perpendicular to the plane $5x + 2y + 16z = 11$.

Solution. We substitute the two sets of coordinates in the equation $ax + by + cz = k$ and obtain

$$2a + b + 3c = k,$$

$$a + 2b + 4c = k.$$

We need a third equation that we may obtain by the following reasoning: Direction numbers for a line perpendicular to the general plane are $\{a, b, c\}$, and direction numbers for a line perpendicular to the given plane are $\{5, 2, 16\}$. If the two planes are perpendicular, then these two lines will be perpendicular. But the condition that these two lines shall be perpendicular yields the necessary third equation

$$5a + 2b + 16c = 0.$$

The student should solve the three equations simultaneously for a, b, and c in terms of k and obtain $a = k/2$, $b = 3k/4$, and $c = -k/4$. He should then substitute these in the general equation for the plane, simplify, and obtain $2x + 3y - z = 4$, the equation for the plane required in this example.

PROBLEMS

1. Sketch each of the following, and give direction numbers of a line perpendicular to the plane:

(*a*) $3x + 5y + 4z = 12$.
(*b*) $2x + y - z = 4$.
(*c*) $x + y = 4$.
(*d*) $x + y + z = 3$.
(*e*) $y = 2$.
(*f*) $z = 3 - y$.
(*g*) $x + y = 2z$.
(*h*) $x + 3y + 6z = 12$.
(*i*) $x + 2y = 4$.
(*j*) $y = 2x$.

2. Determine the volume below $x + y + z = 6$ and in the first octant.

3. Determine the equation of the plane that goes through each of the following sets of points:

(*a*) (1, 2, 1), (6, −1, 2), (1, 0, 5).
(*b*) (0, 4, 0), (−2, 0, 0), (0, 0, 3).
(*c*) (1, 2, 0), (0, 2, 3), (1, 0, 4).
(*d*) (1, 1, −1), (−2, −2, 2), (1, −1, 2).
(*e*) (2, 0, 0), (1, 2, 4), (3, −2, 3).

4. Find the equation of the plane that goes through the point (2, 3, 4) and that is perpendicular to a line with direction numbers $\{4, 7, 2\}$.

5. Prove that the graphs of the two equations $2x + 3y + 4z = 7$ and $6x + 4y - 6z = 11$ are planes perpendicular to each other.

6. Find the equation of a plane that goes through (2, 1, −1) and (1, 1, 2), and that is perpendicular to the plane $7x + 4y - 4z = 20$.

7. Draw a graph of $s_g = s_f + (L/T)$, an equation which arises in thermodynamics. Use variables $(s_f, L/T, s_g)$.

8. A thin vein of coal lies approximately in the plane $2x + y + 2z = 40$, where the unit is 1 ft. and the xy-plane is on the ground level. Draw the plane, and show the trace of the vein of coal at a level 40 ft. below the surface of the earth (i.e., at $z = -40$).

9. Determine the area of that portion of the plane $6x + 5y + 6z = 30$ that lies in the first octant.

10. A right pyramid with square base has its vertex at (0, 0, 10) and the four corners of its base at (2, 2, 0), (2, −2, 0), (−2, 2, 0), and (−2, −2, 0). Determine the equations of all five faces of the pyramid.

11. Find the direction angles for a line through the origin that is perpendicular to the plane $2x + 3y + 4z = 10$ (assuming that α is acute). Draw the plane and the line, and label the angles.

12. Find the equation of the plane that is parallel to the given plane and that goes through the given point:

(a) $2x + 5y + 7z = 8$, (1, 2, 5). (b) $4x - 3y - 7z = 11$, (5, 1, 0).
(c) $2x + 3y = 4$, (4, 7, 5).

13. Find the dihedral angle (assume it to be acute) between the planes $2x + 5y + 2z = 10$ and $x + y + 4z = 4$.

14. Find the equation of the plane that goes through (4, 2, 1) and that is perpendicular to each of the planes $2x + y + 2z = 7$ and $4x - y = 3$.

15. Prove that the following three planes are mutually perpendicular and then find the coordinates of their point of intersection:

$$3x + 2y + 4z = 1, \quad 2x + 5y - 4z = 25, \quad 28x - 20y - 11z = -10.$$

16. Find the numerical value for the perpendicular distance from the given plane to the given point:

(a) $2x + 2y + z = 11$, (4, 7, 5). (b) $6x - 2y - 3z = 8$, (3, 4, 5).
(c) $2x + 5y + 11z = 30$, (3, −2, 6).

17S. Find the perpendicular distance from the origin to the plane

$$2x + 3y + 6z = 12:$$

by finding the volume between this plane and the three coordinate planes (use the base in the xy-plane and altitude along the z-axis), by finding the area of that part of this plane that lies in the first octant and then finding an expression for the preceding volume in terms of the required perpendicular distance, and finally by equating these two volumes. Check your solution by finding the numerical value of the perpendicular distance by aid of the last theorem in Art. 9.7.

18S. Draw and find the volume in the first octant bounded by $z = 0$, $y = 0$, $2x + 3y + 4z = 12$, and $2x = 3y$.

9.10 A Straight Line as the Intersection of Two Planes

In solid geometry it was shown that two non-parallel and non-coincident planes intersect in a straight line. Therefore we shall need the equations of **two planes** to determine a line in space.

To draw a straight line in space, we can draw two planes that intersect in the line, and, from the graphs of these two planes, we can locate the required line. We shall illustrate by drawing the two planes $2x + 3y + 6z = 12$ and $6x + 3y + z = 6$, and locating the line of intersection by several methods.

DEFINITION. *The piercing point of a line in a plane is the point where the line passes through that plane.*

The piercing point in the xy-plane of the line we are seeking to draw is at the point of intersection of the traces of the two planes in the xy-plane and is easily located in the figure. (See Fig. 9.15.) Similarly,

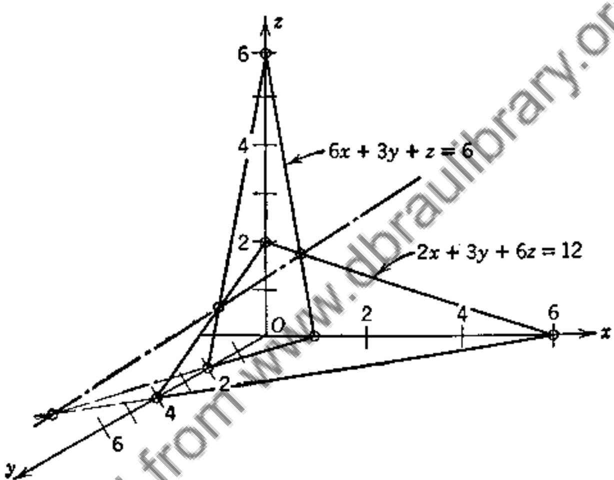


FIG. 9.15

the piercing points in the yz-plane and in the xz-plane are located. The three piercing points all lie on the required line, and we show these three points and the line in Fig. 9.15.

We can readily determine the coordinates of a point on this line by assigning an arbitrary value to one of the three variables and then solving the equations simultaneously for the corresponding values of the other two variables. (When we assign a value to one of the three variables we are really writing the equation of a very simple plane, and three planes ordinarily determine one point.) In particular, we can use this method to determine algebraically the coordinates of the piercing points in the three coordinate planes. Thus, in this example, to determine the coordinates of the piercing point in the xz-plane, we first substitute $y = 0$ in the two given equations and obtain $2x + 6z = 12$, $6x + z = 6$, which yield $x = 12/17$, $z = 30/17$. Hence the xz-piercing point is at $(12/17, 0, 30/17)$.

EXERCISE FOR THE STUDENT. Show that the coordinates of the piercing points in the xy-plane and in the yz-plane are at $(-3/2, 5, 0)$ and $(0, 8/5, 6/5)$, and check by the figure.

The student could, if he wished, determine the coordinates of two points on the required line, plot those two points, and then draw the line. Thus, in the example, we may find the piercing point of the line in the plane $x = 1$ to be $(1, -2/3, 2)$. Similarly, we may find the piercing point in the plane $x = 6$ to be $(6, -12, 6)$. We could then plot these two points and draw the same line as that shown in Fig. 9.15.

The next information that we seek for this line is a set of direction numbers for it. These can, of course, be found from the coordinates of any two points on this line, and this is both a straightforward method and an easy one to remember. An alternative method has the following sequence of ideas. Let the required direction numbers be $\{a, b, c\}$. Now the line of intersection is in the plane $2x+3y+6z=12$, and any line perpendicular to this plane is perpendicular to the line of intersection. But a line perpendicular to this plane has direction numbers $\{2, 3, 6\}$; hence, if these two lines are to be perpendicular, $2a + 3b + 6c = 0$. Similarly, the line of intersection is in the other plane $6x + 3y + z = 6$, and hence is perpendicular to a line with direction numbers $\{6, 3, 1\}$. Therefore $6a + 3b + c = 0$. We solve these two equations simultaneously for two of the unknowns in terms of the third and obtain, for example, $a = 5c/4$, $b = -17c/6$. Hence one way of writing the required direction numbers would be in the form $\{5c/4, -17c/6, c\}$. Since any other set of three numbers proportional to these three forms a set of direction numbers, we may simplify these to $\{15, -34, 12\}$.

An alternative and shorter method for finding direction numbers of a line determined by two planes is given by the following theorem:

THEOREM. *Direction numbers $\{a, b, c\}$ for the line of intersection of the two planes*

$$a_1x + b_1y + c_1z = k_1,$$

and

$$a_2x + b_2y + c_2z = k_2$$

*are given by the values of the following determinants:**

$$a = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad b = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad c = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

* Note that if all three determinants are zero the two planes are either parallel or coincident, and hence there is no line of intersection.

One proof of this theorem makes use of the same reasoning as that given in the preceding paragraph. Thus, any line perpendicular to the first plane has direction numbers $\{a_1, b_1, c_1\}$ and is perpendicular to the line of intersection, which has direction numbers $\{a, b, c\}$. Hence

$$a_1a + b_1b + c_1c = 0.$$

Similarly,

$$a_2a + b_2b + c_2c = 0.$$

That the stated determinantal expressions for a, b, and c form a solution of these two simultaneous equations can be shown either by substituting them in both equations and making use of the properties of determinants or by substituting the expanded expressions in both equations and showing that both reduce to $0 = 0$. The student should complete this proof by substituting the following expanded expressions for a, b and c (obtained by expanding the three determinants) in the two preceding equations.

$$a = b_1c_2 - b_2c_1, \quad b = -a_1c_2 + a_2c_1, \quad c = a_1b_2 - a_2b_1.$$

In the present example we may apply the theorem and obtain

$$a = \begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} = -15, \quad b = -\begin{vmatrix} 2 & 6 \\ 6 & 1 \end{vmatrix} = 34, \quad c = \begin{vmatrix} 2 & 3 \\ 6 & 3 \end{vmatrix} = -12.$$

Exercise for the Student. Determine a set of direction numbers for the line of this example by using the coordinates of two of the three piercing points previously determined.

The direction cosines and angles, assuming that γ is acute, are as follows:

$$\cos\alpha \approx 0.384, \quad \cos\beta \approx -0.870, \quad \cos\gamma \approx 0.307,$$

$$\alpha \approx 67.4°, \qquad \beta \approx 150.5°, \qquad \gamma \approx 72.1°.$$

We continue a study of this example and introduce the idea of a projecting plane.

Definition. *A projecting plane, for a line determined by two given planes, is a plane that contains the given line and that is perpendicular to one of the three coordinate planes.*

If a plane is to be perpendicular to one of the three coordinate planes, it will necessarily be parallel to the third axis. Then the angle between that axis and the perpendicular to the plane will be 90°, and the corresponding direction cosine and direction number will be *zero*. Hence a projecting plane upon the xy-plane, for example, will have no z-term, or the coefficient of z will be zero.

We again use the line of intersection of the two planes

$$2x + 3y + 6z = 12 \quad \text{and} \quad 6x + 3y + z = 6$$

and determine the equations for the three projecting planes. For the projecting plane upon the xy-plane we seek a plane (a single linear equation) in which the z-term is absent. This suggests that we eliminate z between the two equations. We do this and obtain

$$34x + 15y = 24.$$

The graph of this plane is easily drawn. We determine the x-intercept to be $^{12}/_{17}$ and the y-intercept to be $^{8}/_{5}$. We draw the line joining the two points corresponding to these two intercepts, and then draw lines passing through this line and parallel to the z-axis.

We may obtain this projecting plane graphically, if we have already drawn the two planes to locate the line, as was done in Fig. 9.15. We project the yz-piercing point upon the xy-plane; we also project the xz-piercing point upon the xy-plane. The projections upon the xy-plane of these two piercing points, together with the xy-piercing point, lie on a straight line that is the intersection of the projecting plane with the xy-plane (Fig. 9.16). We can then draw a few lines through this line and parallel to the z-axis to make evident the required projecting plane.

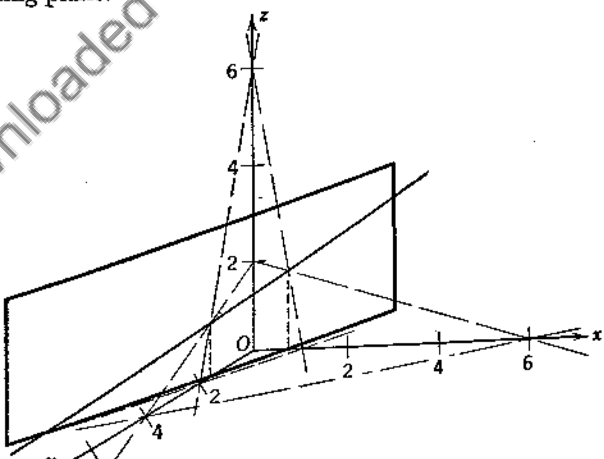


FIG. 9.16

EXERCISE FOR THE STUDENT. Show that the equation of the projecting plane upon the xz-plane of the line of intersection of these two planes is $4x - 5z + 6 = 0$, and draw this projecting plane. Then show that the projecting plane upon the yz-plane is $6y+17z=30$, and draw it on the same figure. Locate the line of intersection of these two projecting planes (which should be the same line as shown in Fig. 9.15).

9.11 Equations of a Straight Line

In plane analytic geometry we have usually found the equation of a straight line by use of one point on that line and the slope of the line. In solid analytic geometry we may follow a very similar method. Suppose that we seek the equations of a straight line (i.e., the equations of two planes that intersect in the required line) and that we have as given information the coordinates of one point $P_1(x_1, y_1, z_1)$ on the line and a set of direction numbers $\{a, b, c\}$ for the line. If $P(x, y, z)$ is any other point on the line, we may use for direction numbers for the line the corresponding differences of the x's, y's, and z's, i.e., $\{x - x_1, y - y_1, z - z_1\}$. Since the two sets of direction numbers are for the same line, they must be proportional. Hence *

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

(The good student will not try to memorize this result but will prefer, instead, to use similar reasoning whenever he desires to obtain equations for a line.)

It is easy to see that we have obtained the equations of three planes that are the projecting planes for the given line.

EXERCISE FOR THE STUDENT. Show, by eliminating the parameter t between pairs of the three equations $x = x_1 + at$, $y = y_1 + bt$, $z = z_1 + ct$, that these three equations are parametric equations for a straight line. Show that the line goes through the point (x_1, y_1, z_1) and that the line has direction numbers $\{a, b, c\}$.

* We have assumed implicitly that all three of a, b, and c are not zero. If, for example, $c = 0$ and $a \neq 0$, $b \neq 0$, we would write

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}, \quad z - z_1 = 0.$$

PROBLEMS

1. For each of the following lines draw the line, locate the piercing points in the three coordinate planes, and determine their coordinates by algebraic methods:

(a) $x + 2y + 2z = 4,\ 2x + 2y + z = 6.$

(b) $x + 2z = 3,\ z - y = 2.$

(c) $x + 2y + z = 6,\ 3x + 4y - z = 8.$

(d) $x + 2y + 2z = 6,\ 2x - 2y + z = 4.$

(e) $x + 2y + z = 4,\ 3x + y - z = 0.$

(f) $x + 2y = 4,\ z = 3.$

2. Determine a set of direction numbers and the direction angles for each line in Problem 1. Label the angles on a figure.

3. Determine the equations for the projecting planes for each line in Problem 1.

4. Obtain the equations of two different planes both of which go through (1, 2, 3) and (4, 6, −4), and thus obtain one pair of equations of the straight line through these two points. Work this problem by assigning arbitrarily the coordinates of a third point so that the three points will determine one plane; then repeat with a second arbitrary point.

5. Use the method of the locus derivation as illustrated in the context of Art. 9.11 to find a pair of equations for the line through each of the following pairs of points:

(a) (1, 4, 6), (3, 1, 8).

(b) (2, 3, 4), (4, 6, 8).

(c) (−1, 3, −2), (2, −1, 4).

(d) (0, 0, 2), (3, 0, 0).

(e) (1, 3, 4), (2, 7, 11).

(f) $(0, 0, 0), (x_1, y_1, z_1)$.

6. Use the *three* equations

$$\frac{x-2}{4} = \frac{y+1}{3} = \frac{z-3}{7}$$

to solve the following problems:

(a) Write the three equations separately and draw the graph of each one. Show by aid of your graph that these three planes intersect in a line.

(b) Equate each of the three preceding fractions to a parameter t and obtain $x = 2 + 4t,\ y = -1 + 3t,\ z = 3 + 7t$. Draw the line by finding the coordinates of two points on the line: assign two different values to the parameter t and determine (x, y, z).

7. Determine equations for the line that goes through the point (2, 5, 7) and that is perpendicular to the plane $2x - 3y + 4z = 11$.

8. Show that $x = 1 + 2t,\ y = 3 + t,\ z = 4 + t$ are parametric equations for a straight line. (*Suggestion:* Eliminate t between pairs of equations and obtain the equations of two planes.) Does the line go through the point (1, 3, 4) and does the line have direction numbers {2, 1, 1}?

9. Solve the following problems about the line determined by $4x + 4y + z = 17$ and $2x + 4y - z = 3$:

(a) Draw the line by use of the graphs of the two planes.

(b) Determine the coordinates of the piercing point of the line in the plane $x = 2$.

(c) Determine the coordinates of the piercing point of this line in the plane $z = 1$.

(d) Use your results from (b) and (c) and determine the direction angles for the line, each correct to the nearest tenth of a degree, assuming that $\gamma > 0$. Then check by finding the direction numbers by a different method.

(e) Draw the line by aid of the points determined in (b) and (c) and label the direction angles on your figure.

(f) Find the equations of the projecting planes.

10. Repeat Problem 9 for the line determined by

(a) $3x - y + z = 10$ and $4x + 2y - z = 3$.

(b) $x + y - 5z + 12 = 0$ and $x - y = 1$.

9.12 Review Problems

The following problems are intended to serve as a review of the material in this chapter.

1. Figure 9.17 shows a rectangular box. H is the mid-point of $\overline{GA}$.

(a) Find direction angles for line $\overline{AC}$, assuming that γ is obtuse.

(b) Find direction angles for line $\overline{HF}$, assuming that α is obtuse.

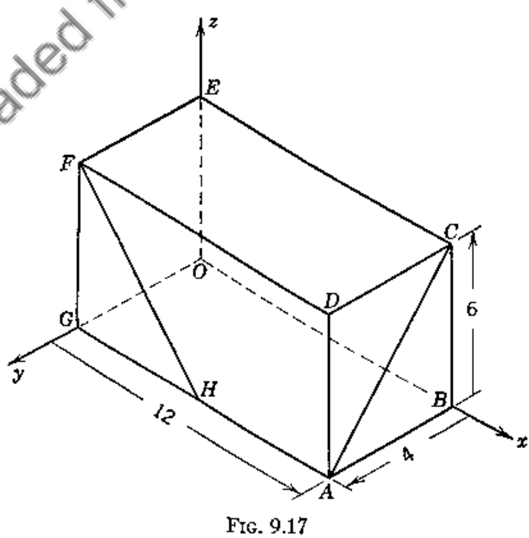


Fig. 9.17

(*c*) Draw, on a new set of axes, two lines that go through the origin and that are parallel respectively to $\overline{AC}$ and $\overline{HF}$; give the coordinates of one point (other than the origin) on each line.

(*d*) Find the acute angle between $\overline{AC}$ and $\overline{HF}$.

2. Using the *one point* $A(4, -1, 2)$, find the length of the radius vector from the origin to this point. Also find the direction angles for this radius vector. Then draw a figure and label the direction angles.

3. Using the *two points* $A(4, -1, 2)$ and $B(-1, 4, 7)$:

(*a*) Find the length of $\overline{AB}$ and the direction angles for $\overline{AB}$, assuming that α is acute. Draw a figure and label the direction angles.

(*b*) Draw a line through the origin that is parallel to $\overline{AB}$ and check by eye.

(*c*) Find the angle at the vertex O of the triangle with vertices at A, B, and the origin O.

(*d*) Find a pair of equations for the straight line through A and B.

4. Using the *three points* $A(4, 4, 0)$, $B(2, -1, -1)$, and $C(0, 4, 3)$:

(*a*) Find the interior angle at C in triangle ABC.

(*b*) Find the area of the triangle.

(*c*) Find the equation of the plane that goes through the three points.

(*d*) Find a set of direction numbers for the line that goes through the origin and that is perpendicular to the plane in (*c*).

(*e*) Use the direction numbers found in (*d*) to draw the line, and draw the plane found in (*c*). Find the coordinates of the point where the line pierces the plane. *Hint:* First write the equations of two planes that determine the line, and then solve these two equations simultaneously with the equation of the plane.

(*f*) Find the length of the perpendicular from the origin to the plane in (*c*) and check by aid of your results in (*e*).

5. Using the *four points* $A(4, 4, 0)$, $B(2, -1, -1)$, $C(0, 4, 3)$, and $D(1, 1, 6)$, and your results from Problem 3:

(*a*) Find the length of the altitude of the pyramid from the plane of A, B, and C to the fourth vertex D.

(*b*) Find the volume of the pyramid.

6. Using the *single plane* $x + 2y + 3z = 6$:

(*a*) Give five solutions for this equation.

(*b*) Draw the plane.

(*c*) Determine the direction angles for a line perpendicular to this plane. Draw a line through the origin perpendicular to this plane and label the direction angles.

(*d*) What are the equations of the trace of this plane in the xy-plane?

(*e*) Find the perpendicular distance from the plane to the origin by use of the oblique-distance formula. Then check by aid of the volume of the pyramid between the coordinate planes and this plane; find the volume by use of a base in the xy-plane and then by use of the base on the oblique plane.

(*f*) Determine the equations of the line that goes through the origin and that is perpendicular to this plane. Then find the coordinates of the piercing point of this line in the given plane. Check by finding the distance from the origin to this point.

7. Using the *two planes* $x + 2y + 3z = 6$ and $2x - 2y - z = 0$:

(*a*) Draw the two planes and the line of intersection.
(*b*) Find the coordinates of the three piercing points of this line in the three coordinate planes.
(*c*) Find the equations of the projecting planes.
(*d*) Find the coordinates of the point where this line pierces the plane $x = 4$.
(*e*) Find the direction angles for the line assuming that α is obtuse.
(*f*) Find the equation of a plane that is perpendicular to this line and that goes through the point (4, 5, 6).

8. Determine the nature of the intersection of each of the following sets of *three planes*:

(*a*) $x + 2y + 3z = 6$, $2x - 2y - z = 0$, $x + y = 4$.
(*b*) $x + 2y + 3z = 6$, $2x - z = 2y$, $5x - 2y + z = 6$.
(*c*) $x + 2y + 3z = 6$, $2x + 4y + 6z = 15$, $3x + 6y + 9z + 7 = 0$.

9. Find the volume in the first octant above $x + y + 3z = 6$ and below $2x + 2y + z = 6$.

10. A line that goes through the origin O and that makes equal angles with the three coordinate axes pierces the plane $3x + 2y + 5z = 30$ at the point P. Find the length of $\overline{OP}$.

11. Draw two different graphical interpretations for the simultaneous equations: $x + 2z = 2$ and $y + 3z = 6$. *Hint:* One of these is in solid analytics; the other is to treat z as a parameter and to draw the graph in plane analytics.

CHAPTER 10

Surfaces in Solid Analytic Geometry

In this chapter we shall discuss the simpler steps that may be used to sketch a given surface. The objective of this study is to develop the ability to identify and to sketch rapidly those surfaces to be met in later mathematics and science courses. Those surfaces are cylinders, cones, and the quadric surfaces, which are equations of degree two in the three variables x, y, and z.

10.1 Introduction

DEFINITION. *A surface is the locus of points whose coordinates satisfy a single equation in the three variables x, y, and z.*

We have already seen that a plane, which is a special surface, is the locus of points whose coordinates satisfy a linear equation in the three variables. We have also seen that we need the equations of two planes to determine a line. Hence it should not be surprising to the student to learn that a *curve is the intersection of two surfaces*, and hence *is the locus of points whose coordinates satisfy simultaneously two equations in the three variables x, y, and z.*

For the graph of an equation to be a surface, it is not necessary that all three variables appear explicitly in the equation. Thus, in the preceding chapter we saw that the equation $2x + 3y = 7$ represents a *plane*, which happens to be perpendicular to the xy-plane (or parallel to the z-axis).

The first step in sketching any surface is usually to make use of the **intercepts,** that is, to determine the directed distances to the points where the surface crosses the three axes. These are obtained by equating two of the three variables to zero and by solving for the third variable.

EXERCISE FOR THE STUDENT. Show that the intercepts for the surface $2x^2 + 3xz + y^2 = 8$ are: $x = \pm 2$, $y = \pm\sqrt{8}$, and there are no z-intercepts (i.e., the surface does not cross the z-axis).

10.2 Symmetry

The second step in sketching a given surface is to observe any possible symmetry with respect to any of the three coordinate planes, with any of the three coordinate axes, or with the origin. Figure 10.1 shows a point $P(x, y, z)$ in the first octant. If there is to be symmetry with respect to the xz-plane, the point $A(x, -y, z)$ must also be on the surface. Hence, to test for symmetry with respect to the xz-plane,

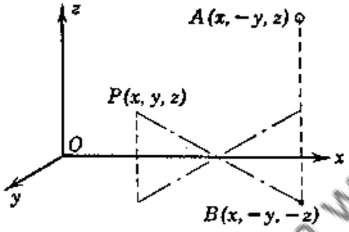


FIG. 10.1

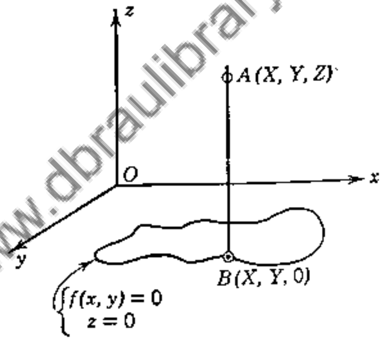


FIG. 10.2

replace y by $-y$; if the resulting equation is equivalent to the given equation, the surface is symmetrical with respect to the xz-plane.

If there is to be symmetry with respect to the x-axis, the point $B(x, -y, -z)$ must be on the surface. Hence, to test for symmetry with respect to the x-axis, replace y by $-y$ and z by $-z$. The test for symmetry with respect to the origin is to replace all three variables by their negatives.

These three tests and others of a similar nature are easily summarized and learned by the following rules:

If an equation is unchanged when one variable is replaced by its negative, the surface is symmetrical with respect to the plane of the other two variables. For example, if the equation is unchanged when x is replaced by $-x$, the surface is symmetrical with respect to the yz-plane.

If an equation is unchanged when two variables are replaced by their negatives, the surface is symmetrical with respect to the axis of the third (or unchanged) *variable.* For example, if the equation is unchanged when x and z are replaced by their negatives, the surface is symmetrical with respect to the y-axis.

If an equation is unchanged when all three variables are replaced by their negatives, the surface is symmetrical with respect to the origin.

EXERCISE FOR THE STUDENT. Verify the following statements concerning symmetry:

1. The surface $3x^2 + 4xy + z^2 = 3$ is symmetrical with respect to the xy-plane, and with respect to the z-axis.
2. The surface $x = 4 - 2y^2 - 3z^2$ is symmetrical with respect to the xz-plane, the xy-plane, and the x-axis.
3. The surface $xy + xz + yz = 4$ is symmetrical with respect to the origin.

10.3 Cylinders

The student's idea of a cylinder is probably that of a right circular cylinder, though he may be familiar with an elliptic cylinder. Such cylinders have the properties of the following definition.

DEFINITION. *A cylinder is the locus of all the points on a line that moves so that it is always parallel to a fixed straight line and that always intersects a fixed plane curve (called the directrix curve). The moving line in any one of its positions is called an element of the surface.*

We shall be concerned in this article only with those cylinders generated by a line moving parallel to one of the three coordinate axes and passing through a plane curve in the plane of the other two variables.

Suppose now that the cylinder is generated by a line moving parallel to the z-axis and passing through the directrix curve whose equation in the xy-plane is $f(x, y) = 0$ as shown in Fig. 10.2. Consider the locus of the equation $f(x, y) = 0$ as an equation in the three variables x, y, and z. We observe first that it is a surface. Let $B(X, Y, 0)$ be the coordinates of any point on the given plane curve in the xy-plane whose equation in the xy-plane is $f(x, y) = 0$; therefore $f(X, Y) = 0$. Then the coordinates of the point $A(X, Y, Z)$ will satisfy the equation $f(x, y) = 0$ irrespective of the value of Z, and therefore all points on the line parallel to the z-axis and passing through $(X, Y, 0)$ will satisfy the requirement. Since this is true for every point on the given plane curve, the locus of $f(x, y) = 0$ is a cylinder.

Whereas these statements have been made on the assumptions that the plane curve was in the xy-plane, and that the elements of the cylinder (the moving line) were parallel to the z-axis, it should be clear that similar statements could be made if the plane curve were in one of the other two coordinate planes and if the elements were parallel to the third-variable axis. Hence we have established the following theorem:

THEOREM. *The locus of an equation in which one of the variables is missing is a cylinder whose elements are parallel to the axis of the missing variable.*

The problem of sketching a cylinder thus reduces to drawing the directrix curve and then showing some elements of the cylinder parallel to the axis of the missing variable.

EXAMPLE

Identify and sketch in three dimensions: $x^2 = 2z$.

Solution. Since y is missing, the locus is a parabolic cylinder with elements parallel to the y-axis. We first sketch the trace in the xz-plane as given by the two

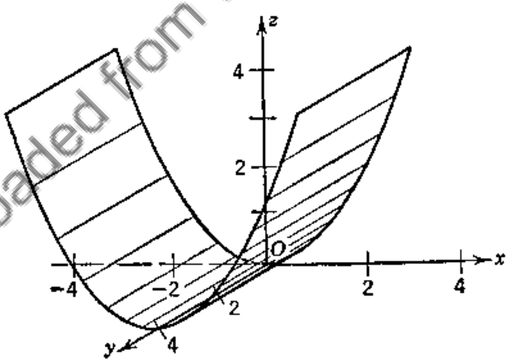


FIG. 10.3

equations: $x^2 = 2z$, $y = 0$. We then sketch the same curve in the arbitrarily chosen plane $y = 4$, and use some of the elements of the cylinder to make the figure clear (see Fig. 10.3).

The student must realize that this sketch shows only a part of the cylinder. The elements (straight lines) that are used to shade the figure are actually of unlimited length.

PROBLEMS

1. Show that each of the following surfaces is a cylinder and assign an appropriate adjective to describe the particular cylinder. Draw the cylinders.

(a) $x^2 + 4y^2 = 9$.
(b) $y^2 = 4x$.
(c) $y = 2e^{-0.25x}$.
(d) $y = 2 \ln x$.
(e) $y = 4 - x^2$.
(f) $z = 2 - y^2$.
(g) $z = \sin 0.5\pi x$ (corrugated roofing).
(h) $x^2 + z^2 = 4$.
(i) $y = \sqrt{4 - z^2}$.
(j) $x + 2y = 4$.
(k) $xz = 4$.
(l) $y = 2x^{1/4}$.
(m) $x^2 - 4y^2 = 4$.
(n) $z = 2 \cos 0.4\pi x$.
(o) $x^2 + 2xy - 2y^2 + 4x - 3y = 7$.

2. Test each of the cylinders in Problem 1 for symmetry.

3. Determine the volume inside $x^2 + y^2 = 4$, below $z = 6$, and above $z = 0$.

4. Tell what symmetry each of the following surfaces possesses:

(a) $x^2 + y^2 - 4z^2 = 8$.
(b) $x^2 + z^2 = 4y$.
(c) $xy = z^2$.
(d) $z = 4 - x^2 - y^2$.
(e) $x^3 + y^3 + z^3 = 4xyz$.
(f) $xy + xz + yz = 4$.

5. Draw the solid in the first octant that is inside $y^2 = 4x$ and below $x + z = 4$.

6. Draw the solid that is bounded by $x^2 + y^2 = 4$, $z = 0$, and $x + z = 2$.

7. Draw the solid in the first octant that is inside $x^2 + y^2 = 16$ and inside $z^2 = 4x$.

8. Draw the surface whose parametric equations are the following and show that the surface is a cylinder:

(a) $x = 4 \sin \theta$, $y = 2 \cos \theta$.
(b) $y = 2 \sin^2 \theta$, $z = \cos 2\theta$.
(c) $x = \theta - \sin \theta$, $z = 1 - \cos \theta$.

9. Draw the trace of $y^2 = 4x$ in the plane $z = 2$.

10. Draw the trace of $x^2 + z^2 = 4$ in the plane $y = 2$.

11. Draw the trace of $x^2 + z^2 = 4$ in the plane $x = 1$.

10.4 Cones

The student's understanding of a cone is probably that of a right circular cone. Let us generalize that idea with the following definition:

DEFINITION. *A cone is the locus of the points of a line that moves so that it always passes through a fixed point (called the vertex of the cone) and through a fixed plane curve (called the directrix of the cone). An element of the cone is the moving line in any one of its positions.*

The student should notice that, since the generating lines are of unlimited length, the cones will necessarily be double. We shall be concerned in this article only with those cones that have their vertices at the origin and the fixed plane curve in a plane parallel to one of the coordinate planes.

We proceed to derive the general equation for a cone whose vertex is at the origin and whose directrix is a plane curve in the plane $z = k$; the plane curve is defined by the two equations $f(x, y) = 0$ and $z = k$. Let $A(X, Y, k)$ be any point on this plane curve in the plane $z = k$ as shown in Fig. 10.4. Join the point A to the origin O by a straight line, and let $P(x, y, z)$ be the coordinates of any point on this line.

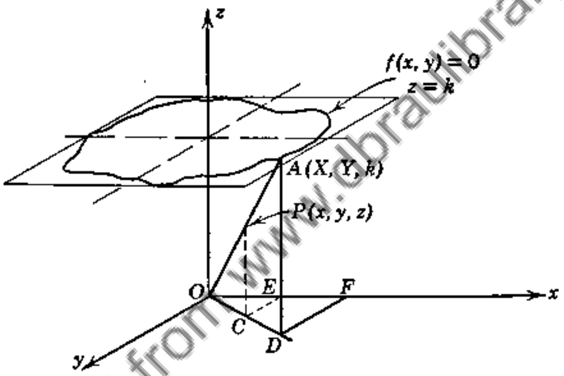


FIG. 10.4

We seek an equation relating x, y, and z. This single equation will be a surface and, in particular, will be the required equation of the cone. Since we must make use of the given data to obtain this required equation, we examine the figure to see what evident geometric statements can be made about the point P. Probably the simplest statement is that triangles OPC and OAD are similar and likewise that triangles OCE and ODF are similar. From these similar triangles we obtain the following equations of proportion: $x/X = y/Y = z/k$. Hence $X = kx/z$ and $Y = ky/z$. But $f(X, Y) = 0$, and we substitute and obtain the equation

$$f\left(\frac{kx}{z}, \frac{ky}{z}\right) = 0.$$

Since this equation relates the coordinates (x, y, z) of any point on the line $\overline{OA}$, and since A was any point on the given plane curve, this last equation is satisfied by the coordinates of every point on the required cone.

In algebra a "polynomial" equation is *homogeneous* if every term is of the same degree. But if every term of an equation in three variables is of the same degree, that equation may be expressed in the form of the preceding equation. Hence we have established the theorem.

THEOREM. *The locus of a homogeneous equation in the three variables x, y, and z is a cone with vertex at the origin (or else is a point locus, or is a line locus, or there is no locus).*

An equivalent condition that the equation represent a cone is that the equation can be rewritten so that each term is dimensionless, assuming that x, y, and z all have the same dimensions and that the constant coefficients in the equation are themselves dimensionless.

Once we have identified a surface to be a cone, we can obtain its graph by constructing the trace of the cone in some convenient plane parallel to one of the coordinate planes, and then joining points on this plane curve to the origin by straight lines.

Examples of equations that represent cones (according to the definition) are the following:

$$x_v + 4y^2 = z^2; \quad 2x^2 + 3xy = z^2; \quad x^3 - y^3 + z^3 = 0;$$

$x + y = z$ satisfies the definition of a cone, but it should be called a plane; $xy + yz + xz = 0$; $y/z = \ln(x/z)$.

EXAMPLE

Identify and sketch the surface $x^2 + 4y^2 = z^2$.

Solution: Since every term of this equation is of degree two, this surface is a cone. (Also we can show that it is a cone by writing the equation in the form $(x/z)^2 + 4(y/z)^2 = 1$, an equation whose terms are dimensionless if x, y, and z each have the same dimensions.) Since it is easier to sketch in space an ellipse than a hyperbola, we arbitrarily choose $z = 4$ as the plane section (instead of x = constant or y = constant). The equations of the curve of intersection then are $z = 4$ and $x^2 + 4y^2 = 16$. We sketch this ellipse (sketching first the circumscribing rectangle as an aid in sketching the curve) in the plane $z = 4$. We also sketch the trace of the surface in the plane $z = -2$, which is the ellipse defined by $x^2 + 4y^2 = 4$ together with $z = -2$. We then join points on the first ellipse to the origin and continue the straight lines on to the lower ellipse.

This cone was constructed with the plane curve $z = 4$, $x^2 + 4y^2 = 16$ as ist directrix. It could have been drawn with the plane curve $x = 2$, $4 + 4y^2 = z^2$ as the directrix, or with $y = 1$, $x^2 + 4 = z^2$ as the directrix.

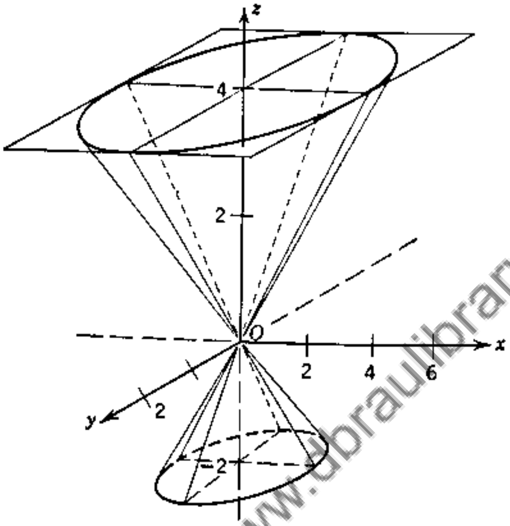


FIG. 10.5

10.5 Spheres

DEFINITION. *A sphere is the locus of a point that moves so that its distance (radius) from a fixed point (the center) is always a constant.*

The student should show, by a locus-derivation process, that the general equation of a sphere with center at (h, j, k) and with radius r is

$$(x - h)^2 + (y - j)^2 + (z - k)^2 = r^2.$$

If the center is at the origin, or if the axes are translated so that the center is at the new origin, the equation of the sphere is $x^2 + y^2 + z^2 = r^2$. The student should see that this last equation satisfies the requirements for symmetry with respect to all three axes, with respect to all three coordinate planes, and with respect to the origin; and that the traces in the three coordinate planes are all circles.

It should be evident to the student that he can reduce (by completing the squares) any equation of the form

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0$$

to the equivalent form $(x - h)^2 + (y - j)^2 + (z - k)^2 = r^2$, and hence that the locus of the equation may be a sphere. If $r = 0$, the locus is called a point sphere; if $r^2 < 0$, the locus is imaginary or there is no locus; if $r^2 > 0$, the locus is a real sphere with radius r.

10.6 Surfaces of Revolution

In addition to cylinders, spheres, and cones, there is another group of simple surfaces that merit special attention. These are the *surfaces of revolution*. Such a surface is the locus described by a plane curve as it revolves about a straight line in its plane (called the axis of the surface). Clearly each trace of such a surface in a plane perpendicular to the straight line will be a circle. And, conversely, any surface is a surface of revolution if it has the property that its traces in all planes parallel to one of the coordinate planes are circles whose centers are all on a line perpendicular to the planes of the traces. We shall illustrate these two ideas in the following examples:

EXAMPLE 1

Find the equation of the surface generated by revolving about the y-axis the plane curve whose equation in the xy-plane is $y^2 = 2x$.

Solution. In Fig. 10.6 we show the plane curve and a general point $P(x, y, z)$ on the required surface. From the statement of the problem we see that the point A

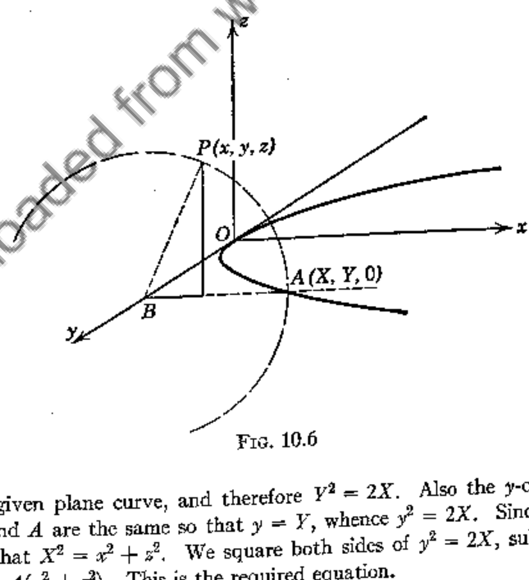


FIG. 10.6

is on the given plane curve, and therefore $Y^2 = 2X$. Also the y-coordinates o points P and A are the same so that $y = Y$, whence $y^2 = 2X$. Since $\overline{BP} = \overline{BA}$ it follows that $X^2 = x^2 + z^2$. We square both sides of $y^2 = 2X$, substitute, and obtain $y^4 = 4(x^2 + z^2)$. This is the required equation.

Example 2

Discuss the surface $x^2 + y^2 = 4z$ with reference to its being a surface of revolution.

Solution. Since the trace in each plane z = constant is a circle, this is a surface of revolution. Moreover, it is easy to see that the surface could be generated by revolving the plane curve $x^2 = 4z$, $y = 0$ about the z-axis.

PROBLEMS

1. Identify and sketch each of the following curves:

(*a*) $x^2 + y^2 + z^2 = 9, x = 2.$
(*b*) $x^2 + y^2 = z^2, z = 2.$
(*c*) $y^2 = 4x, y = 2.$
(*d*) $x^2 + y^2 = z^2, y = 2.$

2. Identify and sketch each of the following surfaces:

(*a*) $x^2 + 4y^2 = 9.$
(*b*) $x^2 + y^2 = z^2.$
(*c*) $x^2 + 4z^2 = y^2.$
(*d*) $xy = 3.$
(*e*) $x^2 + y^2 + z^2 = 10.$
(*f*) $x^2 - y^2 = z^2.$
(*g*) $x^2 + y^2 + z^2 = 4x.$
(*h*) $y = 2 \sin (\pi x/4).$

3. Show that the plane $y = 2x$ satisfies both the definition for a cylinder and the definition for a cone.

4. Show that the surface $x^2 + y^2 = (z - 2)^2$ is a circular cone with vertex at (0, 0, 2) and sketch it.

5. Determine the coordinates of the center and radius of each of the following spheres:

(*a*) $x^2 + y^2 + z^2 + 4x - 2y + 6z = 2.$
(*b*) $2x^2 + 2y^2 + 2z^2 + 3x - 5y + 7z = 9.$
(*c*) $x^2 + y^2 + (z - 3)^2 = 16 + 4x.$
(*d*) $x^2 + y^2 + z^2 = 12x + 4y + 6z.$
(*e*) $2x^2 + 2y^2 + 2z^2 + 4x + 6y - 8z = 3.$

6. Determine the volume above $z = 0$ and inside $x^2 + y^2 + z^2 = 16.$

7. Show that the graph of the "surface" $x^2 + y^2 = 0$ is the z-axis, and notice that the "surface" satisfies the requirements for both a cylinder and a cone.

8. Sketch the following pairs of surfaces on the same figure:

(*a*) $x^2 + y^2 = (z - 4)^2, y^2 + (x - 2)^2 = 4.$
(*b*) $x^2 + y^2 = z^2, z = 4.$
(*c*) $y = 4/(x^2 + 1), x + y = 1.$

9. Find the equation of the sphere that is determined by the following:

(*a*) It has (4, −3, 2) and (8, −5, 6) as the ends of a diameter.
(*b*) It has its center at (1, −2, 3) and it is tangent to the plane

$$2x + 3y + 6z = 63$$

at the point (3, 1, 9).

(c) It has its center at (2, 1, 0) and it is tangent to the plane $z = 4$.

(d) It is inscribed in the pyramid in the first octant whose faces are the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 6$.

(e) It has its center at $(2, -1, 1)$ and it is tangent to $x + 2y + 3z = 6$.

10. Find the equation of the surface generated by revolving each of the following plane curves about the indicated axis:

(a) $x^2 + y^2 = 4$, $z = 0$, axis the x-axis.
(b) $x^2 + (y - 1)^2 = 9$, $z = 0$, axis the y-axis.
(c) $y^2 = 4x$, $z = 0$, axis the x-axis.
(d) $xz = 2$, $y = 0$, axis the z-axis.
(e) $x^2 + y^2 = 4$, $z = 0$, axis the line of intersection of $x = 6$ and $z = 0$.
(f) $y = 2x$, $z = 0$, axis the y-axis.
(g) $z = \ln x$, $y = 0$, axis the z-axis.

11. Determine the volume inside $x^2 + y^2 = z^2$, above $z = 0$, and below $z = 3$.

10.7 Steps in Sketching Surfaces

The important steps in the construction of a given surface have been illustrated in the preceding articles of this chapter. These steps are:

1. Try to identify the given surface (cylinder, cone, sphere, etc.).
2. Determine the intercepts.
3. Test for symmetry with respect to any of the coordinate planes, coordinate axes, or origin.
4. Sketch the traces in the coordinate planes.
5. Sketch traces in planes parallel to the coordinate planes, usually in planes parallel to one of the coordinate planes.

The additional illustrations in the next articles will be special quadric surfaces that are defined as follows:

DEFINITION. *The locus of the general second-degree equation*

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

is called a quadric surface, and "quadric" refers to the fact that this equation is of the second degree.

Besides the special cases of cylinders, cones, spheres, pairs of planes, and other degenerate cases, there are five distinct types of such surfaces, which will be discussed in succeeding articles. Because the intersection of such a quadric surface with a plane is a conic section, quadric surfaces are also called *conicoids*.

10.8 Identification of Quadric Surfaces

The student has already learned to identify planes, cylinders, cones, and spheres. We shall now show how to identify other quadric surfaces.

For the following quadric surfaces (each of which possesses a maximum amount of symmetry with respect to the coordinate planes and axes), consider the *types* of curves obtained as traces in the three coordinate planes. Of the three possible traces, two will be the same type of curve and the third will be (in general) a different type of curve. The corresponding name for the surface can then be given by making the single type of trace into an adjective (by adding the suffix "-ic") and the double type of trace into a noun (by adding the suffix "-oid"). The only exception to this terminology occurs if one trace is a circle: the surface is then circular and is called a surface of revolution. Sometimes "elliptical" is used as a variant of "elliptic." We give some examples of this method of identification.

1. $x^2 + 3y^2 + 7z^2 = 43$. All three traces in the coordinate planes are ellipses and the name of the surface is *ellipsoid* (elliptic ellipsoid would be redundant).

2. $x^2 + 4y^2 = 7z$. The traces are: ellipse, parabola, parabola. Hence we derive the adjective from "ellipse" and the noun from "parabola." The surface is an *elliptic paraboloid.*

3. $4x^2 - 3y^2 = 20z$. The three traces are: hyperbola, parabola, parabola. Hence the name of the surface is *hyperbolic paraboloid.*

PROBLEMS

Determine the names of the traces in the three coordinate planes and verify the names of the following surfaces:

1. $x^2 + 2y^2 - 3z^2 = 12$. *Elliptic hyperboloid.* Since the ellipse in the xy-plane is a real ellipse, the surface is in one piece, or in one sheet, as it is called in mathematics.

2. $x^2 + 5y^2 + 2z = 0$. *Elliptic paraboloid.*

3. $x^2 - y^2 - 3z^2 = 4$. *Elliptic hyperboloid of two sheets* (since the trace in the plane $x = 0$ is an imaginary ellipse, the yz-plane does not intersect the surface in any real trace and the surface is in two pieces or two sheets).

4. $x^2 + 4y^2 + z^2 = 11$. Circular ellipsoid or *ellipsoid of revolution.*

5. $3x^2 + 3z^2 = 4y$. *Paraboloid of revolution.*

6. $x^2 - 5y^2 - 5z^2 = 17$. *Hyperboloid of revolution of two sheets.*

10.9 Graphs of the Quadric Surfaces

We proceed to draw the graphs of the quadric surfaces on the next few pages. The student is expected to identify and to sketch rapidly any of these surfaces. He should memorize the approximate shapes of these various quadric surfaces but not necessarily their corresponding equations. He can always identify such a quadric surface by the rule of the preceding article and can then follow through the succeeding four steps of sketching surfaces. The foreknowledge of what a surface looks like will make these four steps relatively easy.

I. *The Ellipsoid.* $Ax^2 + By^2 + Cz^2 = D$ with all four of A, B, C, and D positive numbers. This equation may be written in the equivalent form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Step 1. Identification. All the traces in the coordinate planes are ellipses. Hence the name of the surface is *ellipsoid.*

Step 2. Intercepts. $x = \pm a, y = \pm b, z = \pm c.$

Step 3. Symmetry. The surface is symmetrical with respect to all three coordinate planes, with respect to all three coordinate axes, and with respect to the origin.

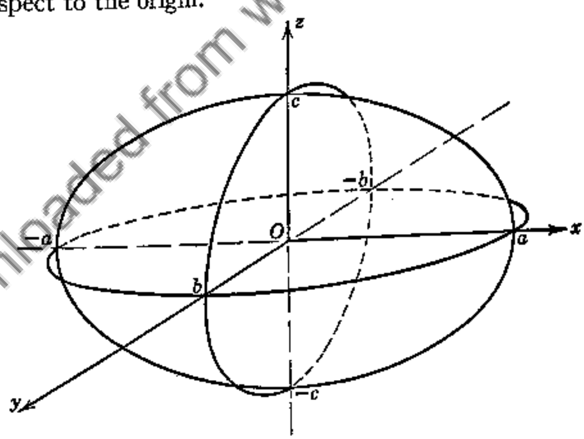


FIG. 10.7

Step 4. Traces in the coordinate planes.

$$x = 0, \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$y = 0, \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1.$$

$$z = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

These three ellipses are shown in Fig. 10.7. The student will find it helpful to draw the circumscribing rectangles for each ellipse with fine light lines.

Step 5. Traces in planes parallel to the coordinate planes.

$$x = k, \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}\text{; etc.}$$

These ellipses, not shown in the figure, may be sketched for arbitrary values of k and may be used in the present example for shading purposes.

II. *The Elliptic Hyperboloid of One Sheet.* $Ax^2 + By^2 + Cz^2 = D$ with some two of A, B, and C positive, the other negative, and D positive. An equivalent form for the case $C < 0$ may be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Step 1. Identification. The names of the traces in the coordinate planes are: ellipse, hyperbola, hyperbola. Since the trace in the xy-plane is a real ellipse, the hyperboloid is in one piece or of one sheet and the given name follows.

Step 2. Intercepts. $x = \pm a$, $y = \pm b$, no z-intercepts.

Step 3. Symmetry. The surface is symmetrical with respect to all three coordinate planes, with respect to all three coordinate axes, and with respect to the origin.

Step 4. Traces in the coordinate planes.

$$x = 0, \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

$$y = 0, \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1.$$

$$z = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

These traces are drawn in Fig. 10.8. Again the student will find it helpful to draw the appropriate guide lines; these can help him to draw the curves so that they are tangent to the guide lines at the proper points. For example, the trace in the yz-plane should be tangent to the guide line at $(0, b, 0)$.

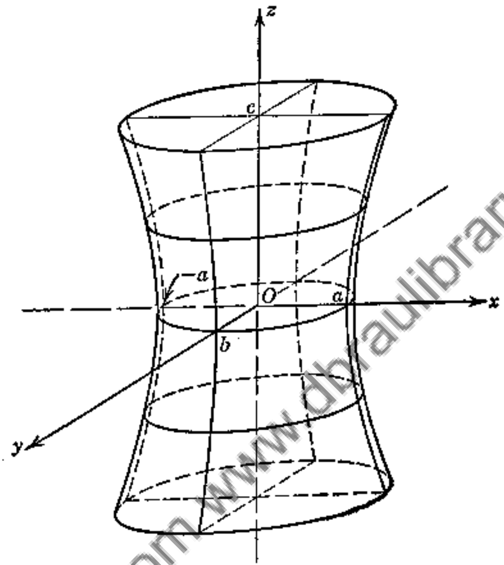


FIG. 10.8

Step 5. A few of the traces in planes parallel to the xy-plane are shown in the figure.

III. *The Elliptic Hyperboloid of Two Sheets.* The equation for a surface of this type has the form $Ax^2 + By^2 + Cz^2 = D$ with D positive, and two of A, B, and C negative and the third positive. An alternative form for this equation, for example, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Step 1. Identification. The names of the traces in the coordinate planes are: hyperbola, hyperbola, and ellipse. Since the trace in the yz-plane is an imaginary ellipse this hyperboloid is in two pieces and the given name follows.

Steps 2, 3, 4, and 5 are left to the student as an exercise. The surface is shown in Fig. 10.9.

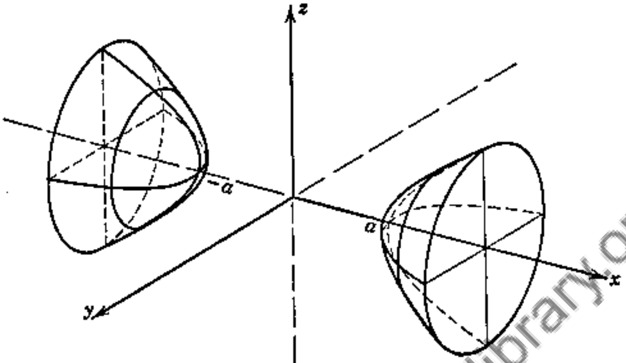


FIG. 10.9

IV. *The Elliptic Paraboloid.* The typical equation for this type of surface is of the form $Ax^2 + By^2 = cz$ with both A and B positive and c not zero. An alternative form for this equation, then, would be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz.$$

The graph of this surface is shown in Fig. 10.10, and the discussion will be omitted.

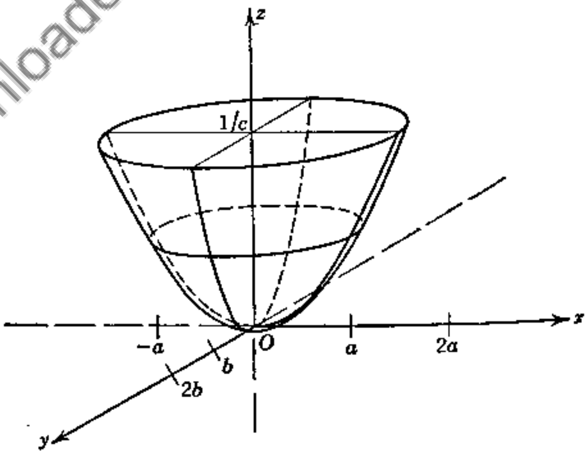


FIG. 10.10

V. *The Hyperbolic Paraboloid* (*Saddle Surface*). The typical equation for this surface is of the form $Ax^2 + By^2 = cz$ with A and B of opposite sign and c not zero. An alternative form, then, would be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz.$$

The graph for this surface is shown in Fig. 10.11. We shall study an example of this surface in the next article.

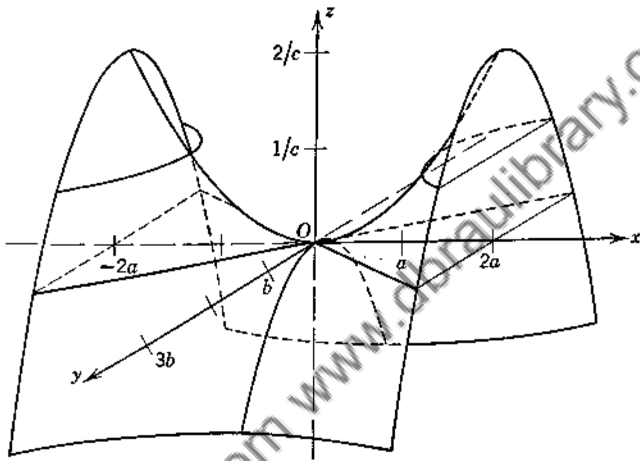


FIG. 10.11

PROBLEMS

1. Sketch the circumscribing rectangle and the ellipse:

(a) $x = 0, x^2 + 4y^2 + 3z^2 = 12.$
(b) $z = 2, x^2 + 4y^2 = 2z.$
(c) $x = 2, x^2 + 4y^2 + z^2 = 12.$

2. Sketch the asymptotes and the hyperbola:

(a) $x = 0, 4x^2 - y^2 + 4z^2 = 4.$
(b) $z = 2, 4x^2 - y^2 + 4z^2 = 4.$
(c) $y = 2. 4x^2 - y^2 - z^2 = 4.$

3. Identify each of the following surfaces; give the intercepts; discuss symmetry. Then sketch the surface and show at least one trace in a plane parallel to a coordinate plane.

(a) $x^2 = 4z.$
(b) $x^2 + z^2 = 9.$
(c) $x^2 + y^2 = 4z^2.$
(d) $x^2 + y^2 + z^2 = 9.$

(e) $4x^2 + 4y^2 = z$.
(f) $x^2 + 3y^2 + 5z^2 = 20$.
(g) $x^2 + y^2 - z^2 = 4$.
(h) $x^2 - 2y^2 - 3z^2 = 12$.
(i) $x^2 - y^2 = 4z$.
(j) $2x = y + 4$.
(k) $x^2 + 4z^2 = 4y$.
(l) $x^2 - 4x + y^2 = 0$.
(m) $x^2 - y^2 - z^2 = 4$.
(n) $x^2 + 2y^2 + 2z^2 = 4$.
(o) $(x - 1)^2 + (y + 2)^2 = 4(5 - z)$.
(p) $(x - 2y)^2 = 4$.
(q) $[(x - 1)^2/4] - [(y + 2)^2/9] - [(z - 3)^2/16] = 1$.

4. Sketch each of the following surfaces according to the given directions. Show the circumscribing rectangle for each elliptical trace, and show only that portion of the surface which lies in the first octant.

(a) $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$; let $a = 2$ in., $b = 1$ in., and $c = 1.5$ in.
(b) $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$; let $a = 1$ in., $b = 0.75$ in., and $c = 2$ in.
(c) $x^2/a^2 - y^2/b^2 - z^2/c^2 = 1$; let $a = 2$ in., $b = 1$ in., and $c = 1$ in.
(d) $x^2/a^2 + y^2/b^2 = cz$; let $a = 1$ in., let $b = 0.75$ in., and let $1/c$ units on the z-axis correspond to 1 in.
(e) $x^2/a^2 - y^2/b^2 = cz$; let $a = 1$ in., $b = 0.6$ in., $1/c = 1$ in.

5. A building has the shape of $x^2 + y^2 = 10{,}000$ from $z = 0$ to $z = 40$, and above $z = 40$ it has the shape given by $x^2 + y^2 = 200(90 - z)$. Draw the building.

6. In Problem 5, if the inside of the dome has a finish that is light reflecting, where would be an appropriate position to place an indirect light of large power, or where would be an appropriate position to place a loudspeaker if the sound is to reflect from the dome?

7. The surface of a water tank is given by $x^2 + y^2 + (z - 40)^2 = 400$ from $z = 20$ to $z = 40$; by $x^2 + y^2 = 400$ from $z = 40$ to $z = 60$; by $x^2 + y^2 = 4(z - 70)^2$ from $z = 60$ to $z = 70$. Draw the tank.

8. Sketch the portion of the surface $x^2 + 4y^2 = 4z$ between the planes $z = 0$ and $z = 9$, and then work the following problems:

(a) Show on your figure the trace in an arbitrary plane z units above the xy-plane (z is any value between 0 and 9).
(b) What are the lengths of the semiaxes of this elliptical trace in terms of z?
(c) What is the area of this ellipse as a function of z, if the area of an ellipse is πab, where a and b are the semiaxes of the ellipse?
(d) Sketch the graph of this area A in terms of z (a plane analytic geometry graph) for z from 0 to 9. Then determine the *area* of the triangle bounded by the graph of A in terms of z, the z-axis, $z = 0$, and $z = 9$. This "area" is equivalent to the volume of the original paraboloid.

9. Sketch the first-quadrant portion of both

$$x^2 + y^2 + z^2 = r^2 \quad \text{and} \quad x^2 + 2y^2 + 3z^2 = 4r^2.$$

Then imagine the first surface to be coated with bristles that extend to the second surface along radial lines (lines that go through the origin). What would be the lengths of the longest and shortest bristles?

10. Draw a graph of the surface $x^2 + y^2 + z^2 = 100$ according to the following directions:

(a) Draw the circle in the xy-plane with a center at the origin and radius 10 (use graph paper and make a plane analytic geometry graph), and use 2.5 in. = 10 units. Mark this circle with the notation: $z = 0$. (This is the trace of the given surface in the xy-plane.)

(b) Draw on the same sheet of graph paper the circle with center at the origin and radius $\sqrt{99}$, and mark this second circle with the notation: $z = 1$. (This is the projection upon the xy-plane of the trace of the original surface in the plane $z = 1$.)

(c) Repeat step (b) for the projections on the xy-plane of the traces of the given surface in the planes $z = 2, 3, \cdots, 10$ and mark each circle with the corresponding z-value.

The resulting graph is a *contour* graph of the upper half of the given surface. Contour graphs are used in map making to show, for example, the cross sections of a mountain at various altitudes. They also occur in some science and engineering courses.

11. Use the method of Problem 10 to draw contour graphs of the following surfaces:

(a) $x^2 + y^2 = 2z$ for the range from $z = 0$ to $z = 10$.
(b) $x^2 - y^2 = 2z$ for the range from $z = -4$ to $z = 8$.
(c) $z = xy$ for the range from $z = -5$ to $z = 10$.
(d) $x^2 + y^2 - 4z^2 = 16$ for the range from $z = 0$ to $z = 8$.

10.10 The Surface for the General Gas Law

In this article we shall apply the methods of the preceding articles to sketch the surface that corresponds to the general gas law: $pv = kT$, where p is pressure, v is volume, T is absolute temperature, and k is a positive constant.

We observe that the graph is a quadric surface, since the degree of this equation is two. Since p, v, and T are each restricted to positive values because of the physical meaning of the equation, we shall draw only that portion of the surface that lies in the first octant. We proceed to apply the five steps enumerated in the preceding articles.

Step 1. The rules for identifying quadric surfaces do not apply to a surface with a product term and therefore we cannot identify the surface at this stage.

Step 2. When $p = 0$ and $v = 0$, we must have $T = 0$. Hence the surface crosses the T-axis at the origin. When $p = 0$ and $T = 0$, the equation becomes $0 = 0$ irrespective of the value of v. Hence the surface crosses the v-axis all along that axis; that is, the v-axis is on the surface. Similarly, the p-axis is on the surface.

Step 3. Since we can substitute $-p$ for p and $-v$ for v and obtain an equivalent equation, the entire surface is symmetrical with respect to the T-axis. Similarly, the surface is symmetrical with respect to the p-axis and with respect to the v-axis.

Step 4. The trace in the pv-plane is $pv = 0$, i.e., the p-axis and the v-axis. The trace in the pT-plane is the p-axis, and the trace in the vT-plane is the v-axis.

Step 5. Let $T = c/k$ = constant, and the trace is defined by the two equations $pv = c$ and $T = c/k$. Thus, traces in planes parallel to the pv-plane are equilateral hyperbolas. Two of these are shown in Fig. 10.12. Let $p = h$ = constant, and the trace becomes $hv = kT$,

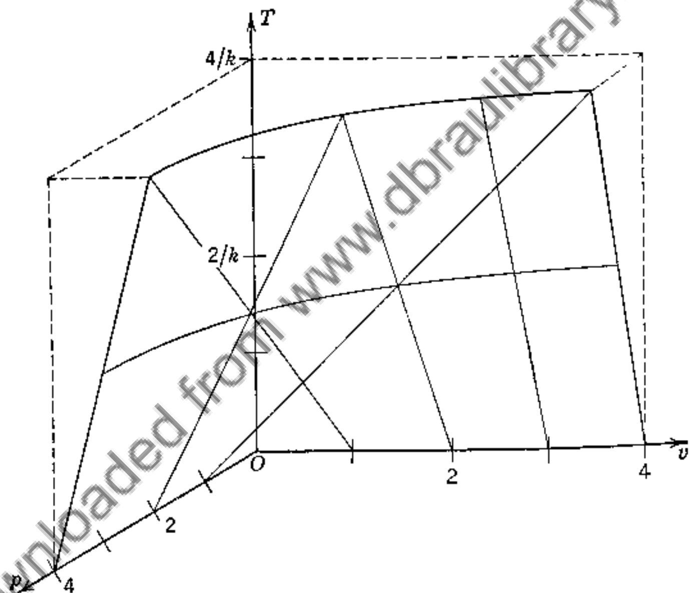


FIG. 10.12

which is a straight line passing through the origin in the translated Tv-plane. Similarly, if we set $v = c$ = constant, the trace becomes $cp = kT$, which is a straight line. Several of these straight-line traces are shown in the figure.

We can correlate these traces with Boyle's and Charles's laws for gases as follows: Boyle's law in chemistry and physics states that, if the temperature of a given quantity of gas is constant, the pressure varies inversely as the volume. The traces in the figure that are equilateral hyperbolas correspond to this law. Charles's law states that, if the pressure is held constant, the volume of a given quantity of gas varies directly as the absolute temperature. A second part of his law states that, if the volume is constant, the pressure varies directly as the absolute temperature. These two statements correspond to the two sets of straight-line traces.

In problems requiring the drawing of complicated surfaces, it might be convenient to translate axes or to rotate some pair of the axes about the third axis or even to rotate all three axes according to some law. In the present example, suppose that we rotate the p-axis and v-axis through an angle of 45° about the T-axis. Let the new or rotated axes be designated by x and y. Then the equations of rotation may be written as $p = (x - y)/\sqrt{2}$ and $v = (x + y)/\sqrt{2}$, and the equation $pv = kT$ becomes $x^2 - y^2 = 2kT$. We recognize this to be a hyperbolic paraboloid or "saddle" surface.

The student of science or of some phase of engineering may realize that the same type of equation for this general gas law applies to many other problems. For example, the same surface would be obtained for a graph of Ohm's law in electricity: $E = Ir$; and for a graph of the equation in mechanics $W = Fd$ (work is force in the direction of motion multiplied by the distance); and so on.

10.11 Curves of Intersection of Two Surfaces

This topic has already been discussed in connection with the line of intersection of two planes and traces in planes parallel to the coordinate planes. We shall introduce certain definitions and ideas in connection with the solid in the first octant below $x + y + z = 4$ and above $4x^2 + 9y^2 = 36z$.

The required solid is shown in Fig. 10.13. It is a solid that lies below a plane and inside the elliptic paraboloid. Two points on the curve of intersection of the plane and paraboloid are easily determined as the intersection of the traces in the yz-plane and in the xz-plane, and the curve is drawn free-hand through these two points. It is drawn so that it appears to be in the given plane.

This curve is, of course, defined by the intersection of the two given surfaces. We may, if we wish, find the coordinates of points on the curve by assigning arbitrary values to some one of the three variables and solving for the other two. If, for example, we assign the value $x = 2$, then we have three simultaneous equations: $x + y + z = 4$, $4x^2 + 9y^2 = 36z$, and $x = 2$; and the simultaneous solution is the point (or points) of intersection of the given plane, the paraboloid, and the second plane.

This example illustrates the idea that two equations are required to define a curve in solid analytic geometry, and three equations are required to determine a point.

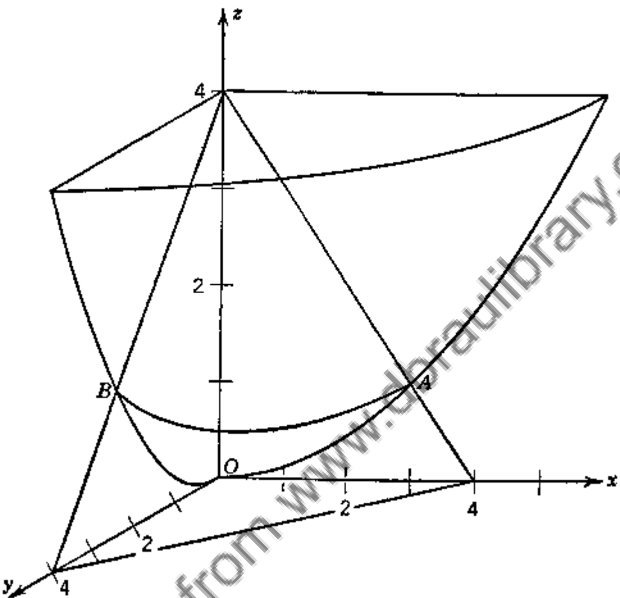


FIG. 10.13

Associated with the curve in this example are three significant *projecting cylinders*. For example, the projecting cylinder upon the xy-plane is a cylinder that has its elements parallel to the z-axis, that is perpendicular to the xy-plane, and that goes through the given curve. This cylinder, then, will have the variable z missing in its equation.

In the example, we may find the equation of the projecting cylinder upon the xy-plane by eliminating z between the two given equations, and we obtain $4x^2 + 9y^2 = 36(4 - x - y)$. We transpose, complete the squares, and obtain

$$4(x + 4.5)^2 + 9(y + 2)^2 = 261.$$

This is an elliptic cylinder with elements parallel to the z-axis. The axis of the cylinder is at the intersection of the two *planes* $x = -4.5$ and $y = -2$.

The *piercing point* of the curve in the xz-plane is the point of intersection of the given curve and the xz-plane. This statement implies that we are to combine the new equation $y = 0$ with the two given equations to yield three simultaneous equations in three variables. The coordinates of the required point (point A in Fig. 10.13) are easily found to be (3, 0, 1).

EXERCISE FOR THE STUDENT. Show that the equation of the projecting cylinder upon the xz-plane for the curve of this example is

$$4x^2 + 9(4 - x - z)^2 = 36z$$

or

$$13x^2 + 18xz + 9z^2 - 72x - 108z + 144 = 0.$$

Show, by aid of the indicator test of plane analytic geometry, that this is an elliptic cylinder. Then start afresh and show that the coordinates of the piercing point (in the first octant) in the yz-plane are (0, 2.472, 1.528) approximately.

10.12 Cylindrical Coordinates

Many problems in science that require the concepts of solid analytic geometry are easier to work if the analysis is made in terms of *cylindrical* coordinates or in terms of *spherical* coordinates instead of in rectangular coordinates. If the surface has symmetry with respect to a line, then cylindrical coordinates would be likely to simplify the work of the problem. If the surface has symmetry with respect to a point, then the use of spherical coordinates could be expected to provide a simpler solution. These two types of coordinate systems are the subject of the study in this and the next article.

Instead of using the three coordinates (x, y, z) to locate a point in space, we could use three coordinates, two of which would be polar coordinates in one of the three coordinate planes and the third of which would be the distance along the rectangular-coordinate axis perpendicular to the plane in which the polar coordinates are used. For example, we could use polar coordinates in the xy-plane and z for the third coordinate, and the coordinates could then be written (r, θ, z). Figure 10.14 illustrates the meaning of these coordinates.

The transformation from these cylindrical coordinates to rectangular coordinates, or from rectangular coordinates to cylindrical coordinates, may be performed by aid of this figure. Note that this is essentially the same process that was used in the chapter on polar coordinates to allow for transforming from polar to rectangular coordinates.

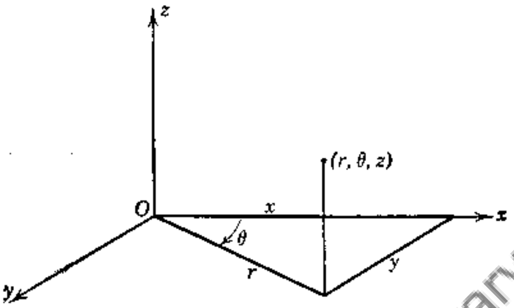


FIG. 10.14

The name *cylindrical coordinates* arises from the fact that the equation of a right circular cylinder with axis the z-axis is given by the simple equation $r = R$, where R is the radius of the cylinder.

EXAMPLE

Transform to rectangular coordinates and identify the surface whose equation is $2r \cos \theta + 3r \sin \theta = 4z$.

Solution. We use Fig. 10.14 and observe that $r \cos \theta = x$ and $r \sin \theta = y$. Then the given equation may be written as $2x + 3y = 4z$, and the surface is a plane.

10.13 Spherical Coordinates

Spherical coordinates require the use of two angles and a distance measured from the origin. In Fig. 10.15 we show two angles θ and ϕ and a radius vector quantity ρ. In order to transform from spherical

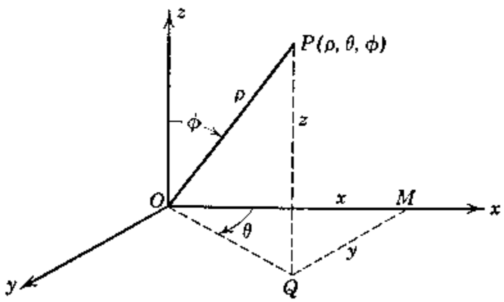


FIG. 10.15

coordinates to rectangular coordinates, we may use the figure to obtain relations such as the following:

$$z = \overline{QP} = \rho \sin (90° - \phi) = \rho \cos \phi,$$

$$\overline{OQ} = \rho \cos (90° - \phi) = \rho \sin \phi,$$

$$x = \overline{OM} = \overline{OQ} \cos \theta = \rho \sin \phi \cos \theta,$$

$$y = \overline{MQ} = \overline{OQ} \sin \theta = \rho \sin \phi \sin \theta.$$

It is easy to show from these equations that $x^2 + y^2 + z^2$ reduces to ρ^2. This means that, if we use spherical coordinates, the equation of a sphere with center at the origin and with radius R is given by the very simple equation $\rho = R$.

In terms of the position of a point on the earth, θ is the longitude, ϕ is the colatitude, and ρ is the radius of the earth (assuming the earth to be a perfect sphere).

PROBLEMS

1. Sketch that part of both surfaces that lies in the first octant, and show the curve of intersection:

(a) $x^2 + y^2 + z = 9$, $2x + y + 2 = 2z$.
(b) $x^2 + 2y^2 = 6$, $z + x = 3$.
(c) $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 = 2$.
(d) $x^2 + y^2 = 16$, $x^2 + z^2 = 16$.
(e) $x^2 + y^2 + z^2 = 4$, $x + y + 2z = 4$.
(f) $x^2 + y^2 = 2z$, $x + z = 4$.

2. Use the two equations $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 = 9$, and work the following problems:

(a) Sketch both surfaces in the first octant and sketch the curve of intersection.
(b) Find the equations of the three projecting cylinders.
(c) Transform both equations to cylindrical coordinates.
(d) Transform both equations to spherical coordinates.

3. Sketch the two surfaces $x^2 + y^2 = z^2$ and $y + z = 2$ and the curve of intersection. Then find the equations of the three projecting cylinders and the coordinates of the piercing points of this curve in the three coordinate planes.

4. Repeat Problem 3 for the two surfaces $x^2 + y^2 = z^2$ and $x + y = 2$.

5. Repeat Problem 3 for the two surfaces $x^2 + y^2 = z^2$ and $y + 2z = 3$.

6. Find the area bounded by the plane curve defined by $x^2 + y^2 + z^2 = 36$, $x^2 + y^2 = 3z^2$, $z > 0$.

7. Find the length of the curve of intersection of $x^2 + y^2 + z^2 = 36$ and $x^2 + y^2 = 5z$.

8. When an ordinary pencil with a hexagonal cross section is sharpened, identical curves are formed on the ends of the six sides. Why are all these curves arcs of hyperbolas?

9. Find the area on the surface $x^2 + y^2 + z^2 = 16$ that is above $z = 0$ and that is inside $x^2 + y^2 = z^2$. *Hint:* From solid geometry the surface area of a zone is $2\pi rh$, where r is the radius of the small circle or base of the zone and h is the altitude of the zone.

10. Transform each of the following equations to cylindrical coordinates and to spherical coordinates:

(*a*) $x^2 + y^2 = 4$.
(*b*) $x^2 + y^2 + z^2 = 4$.
(*c*) $y^2 = 3x$.
(*d*) $x^2 + y^2 + z = 4$.
(*e*) $xy = 2$.
(*f*) $x^2 + y^2 - z^2 = 9$.

11. Transform from cylindrical coordinates to rectangular coordinates, and sketch the surface:

(*a*) $r = 2$.
(*b*) $r = z$.
(*c*) $r^2 = 4 \cos 2\theta$.
(*d*) $z^2 + r^2 = 6$.

12. Transform from spherical coordinates to rectangular coordinates, and sketch the surface:

(*a*) $\rho = 2$.
(*b*) $\rho^2 + 3\rho^2 \cos^2 \phi = 4$.
(*c*) $\rho^2 - \rho^2 \sin^2 \phi \cos^2 \theta = 4$.
(*d*) $\rho^2 - 3\rho + 2 = 0$.
(*e*) $\phi = 1$ radian.
(*f*) $\theta = \pi/3$.

13. Sketch the curve of intersection:

(*a*) $\rho \sin \phi = 2$ and $\rho = 4$ in spherical coordinates.
(*b*) $r = 2$ and $z = 3$ in cylindrical coordinates.
(*c*) $z = 4 - r$ and $z = 1$ in cylindrical coordinates.

14. Identify and sketch the following surfaces, which are given in cylindrical coordinates:

(*a*) $r = 4$.
(*b*) $r \cos \theta = 2$.
(*c*) $r = z/2$.
(*d*) $r^2 + z^2 = 4$.
(*e*) $r^2 = 2z$.
(*f*) $r^2 + 2z^2 = 4$.

15. Identify and sketch the following surfaces, which are given in spherical coordinates:

(*a*) $\rho = 2$.
(*b*) $\rho \sin \phi = 2$.
(*c*) $\rho \cos \phi = 3$.
(*d*) $\rho^2 - \rho^2 \cos^2 \phi = 4$.
(*e*) $\rho \sin \phi \sin \theta = 5$.
(*f*) $\rho^2 + 2\rho \cos \phi = 5$.

16. Introduce polar coordinates in the yz-plane (measure θ in a direction from the positive y-axis toward the positive z-axis). Locate the point whose rectangular coordinates are (4, 3, 2) and give the corresponding cylindrical coordinates for this point.

10.14 Ruled Surfaces

In this article we shall use some of the ideas of the preceding articles of this chapter as we study a special group of surfaces known as ruled surfaces.

DEFINITION. *A ruled surface is a surface that is the locus of a line as it moves according to some law.*

From this definition we observe that planes, cylinders, and cones are special cases of ruled surfaces. We saw in Art. 10.10 that a hyperbolic paraboloid (saddle surface), when written as $pv = kT$, is a ruled surface; in fact we saw that the straight lines or rulings are the graphical expression of the two forms of Charles's law for gases.

Models of ruled surfaces may be constructed by aid of strings to simulate the positions of the moving line.

We shall now show that a hyperboloid of one sheet is a ruled surface. We use the following general equation for that hyperboloid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Let a general point on the trace in the xy-plane (see Fig. 10.16) be $A(a \cos\theta,\ b \sin\theta,\ 0)$, where θ is a parameter. Let an arbitrary line

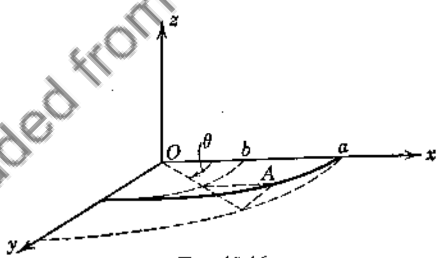


FIG. 10.16

through this point have direction numbers $\{la, mb, nc\}$, where the multipliers a, b, and c are used to simplify the ensuing algebra. We shall see that we can determine these direction numbers so that the line lies on the given surface. To this end, we write the equations of the line in parametric form (see Art. 9.11):

$$x = a\cos\theta + lat,$$

$$y = b\sin\theta + mbt,$$

$$z = nct.$$

(For each value assigned to t, and for a fixed value of θ, we obtain a point on the line.) We substitute these three expressions in the equation of the hyperboloid, simplify, and obtain

$$t^2(l^2 + m^2 - n^2) + 2t(l \cos\theta + m \sin\theta) = 0.$$

If we choose l, m, and n so that

$$l^2 + m^2 - n^2 = 0,$$

$$l \cos\theta + m \sin\theta = 0,$$

then the "quadratic" equation in t will reduce to $0 = 0$ and hence will have every value of t as a solution. Hence every point on the line

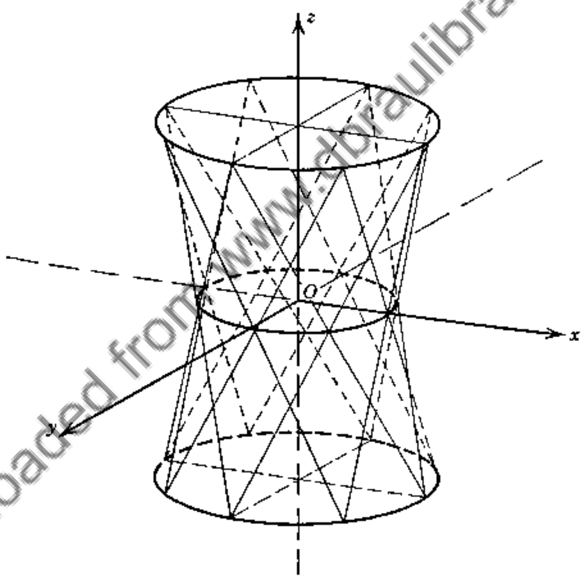


FIG. 10.17

will lie on the hyperboloid. We solve for l and n in terms of m, and obtain for direction numbers of two lines: $\{-a \sin\theta, b \cos\theta, \pm c\}$. Thus there are two lines that go through each point on the xy-trace of the hyperboloid and that lie on this surface. The hyperboloid is shown in Fig. 10.17.

PROBLEMS

1. Sketch the following ruled surfaces and show several rulings on each:

(a) $x^2 + 4y^2 = z^2$.
(b) $x^2 + 4y^2 = 4$.
(c) $y^2 = 4x$.
(d) $x + 2z = 4$.
(e) $9x^2 + 9y^2 - z^2 = 9$.
(f) $y = 2 \sin \pi x$.
(g) $x^2 + 4y^2 = 4(z - 2)^2$.
(h) $xy = 2z$.

2. Write the equation for a hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$$

in the form

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = cz.$$

Let $x/a - y/b = k$, whence $k(x/a + y/b) = cz$. How do these two resulting equations establish the fact that a hyperbolic paraboloid is a ruled surface?

3. Draw the ruled surface to act as a connecting pipe to join two cylinders: one cylinder is $x^2 + y^2 = 36$ from $z = -10$ to $z = 0$; the other cylinder is $x^2 + y^2 = 16$ from $z = 3$ to $z = 12$. Then show that an equation of this ruled surface may be of the form

$$x^2 + y^2 = a(z - b)^2,$$

and find the values of a and b.

4. Draw a ruled surface to act as a connecting pipe to join the ellipse $x^2 + 4y^2 = 16$, $z = 0$ and the circle $x^2 + y^2 = 4$, $z = 3$. Locate a point on the ellipse by the method illustrated in Fig. 10.16 with coordinates $A(4 \cos \theta, 2 \sin \theta, 0)$. For the same angle θ locate the point $B(2 \cos \theta, 2 \sin \theta, 3)$ on the circle. Show that $y = 2 \sin \theta$, $3x + 2z \cos \theta = 12 \cos \theta$ are equations for the line $\overline{AB}$. Hence, as angle θ varies, this line will generate one ruled surface for this connecting pipe. Draw this line in several positions.

5. Let a general point on the trace in the xy-plane of a hyperboloid of one sheet be $A(a \cos \theta, b \sin \theta, 0)$ as shown in Fig. 10.16. Let a second point on the trace in the plane $z = c$ be $B(a\sqrt{2} \cos \phi, b\sqrt{2} \sin \phi, c)$. Show first that the line $\overline{AB}$ is determined by the two planes:

$$x = a \cos \theta + \left(\frac{az}{c}\right)(\sqrt{2} \cos \phi - \cos \theta),$$

$$y = b \sin \theta + \left(\frac{bz}{c}\right)(\sqrt{2} \sin \phi - \sin \theta).$$

Next substitute these in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

simplify by aid of the rules of trigonometry, and obtain

$$z^2[1 - \sqrt{2} \cos (\phi - \theta)] - cz[1 - \sqrt{2} \cos (\phi - \theta)] = 0.$$

Finally, show that this equation will be satisfied by every value for z if ϕ is such that

$$\phi = \theta \pm 45°.$$

You have thus shown that, for each point A on the trace in the xy-plane of the hyperboloid of one sheet, there are two points B_1 and B_2 on the trace of this surface in the plane $z = c$, and that these two points have coordinates:

$$B_1(a\sqrt{2}\cos(\theta + 45°),\quad b\sqrt{2}\sin(\theta + 45°),\quad c),$$

$$B_2(a\sqrt{2}\cos(\theta - 45°),\quad b\sqrt{2}\sin(\theta - 45°),\quad c).$$

Moreover, these two lines $\overline{AB_1}$ and $\overline{AB_2}$ both lie on the hyperboloid for each value of θ, and hence the hyperboloid is a ruled surface.

6. Use the results of the preceding problem to draw eight of the rulings for the hyperboloid of one sheet:

(a) $x^2 + y^2 - z^2/4 = 1$. (b) $x^2 + y^2 - z^2 = 4$.
(c) $x^2/4 + y^2/9 - z^2/16 = 1$.

7. Find the equation of the locus of a line that moves so that it is always parallel to the yz-plane, so that it intersects the line $x + 2z = 4$, $y = 0$, and so that it intersects the line $y = 4$, $z = 0$. Draw the part of the locus that lies in the first octant.

8. Find the equation of the locus of a line that moves so that it is always parallel to the yz-plane, so that it intersects the circle $x^2 + y^2 = 9$, $z = 0$, and so that it intersects the line $z = 4$, $y = 0$. Draw that part of the locus that lies in the first octant and that is below $z = 4$.

9. Eliminate the parameter t between the two equations

$$x - 2y = 4t \quad \text{and} \quad xt + 2yt = 8z,$$

and identify the resulting locus. Then plot several of the lines defined by the two given equations (by assigning values to the parameter t).

10. Sketch the ruled surfaces defined by the following equations in cylindrical coordinates, and show several rulings:

(a) $r = 2\cos\theta$. (b) $r = 4$.
(c) $r = z$. (d) $r = 2 - 2z$.

11. Sketch the ruled surfaces defined by the following equations in spherical coordinates, and show several rulings:

(a) $\phi = 1$. (b) $\theta = 2$.
(c) $\rho\cos\phi = 2$. (d) $\rho\sin\phi = 3$.

General Review

As the student prepares for a comprehensive final examination in any course, his first step should be to outline carefully the material in the text and to associate some simple idea or "catch phrase" with each important fact. We illustrate this idea in the first portion of the following outline. The student will find it to be of value if he will go through the remainder of the outline in a similar manner.

REVIEW OF PLANE ANALYTIC GEOMETRY

I. Elementary formulas.

A. Directed length of a line segment parallel to one of the coordinate axes: associate the "abscissa to which measurement is made minus abscissa from which the measurement is made," etc.

B. Length of an oblique or inclined line segment: associate the Pythagorean theorem.

C. Mid-point of a line segment: associate the "average of the x's" and the "average of the y's."

D. Slope of an inclined line segment: associate the definition of the tangent of a general angle: the ordinate divided by the abscissa.

E. Inclination: associate the principal value of the angle whose tangent is the slope.

F. Angle between two lines: use the inclinations of the two lines or else make use of the formula from trigonometry for $\tan (A - B)$.

G. Parallel lines have the same slope; perpendicular lines have slopes that are negative reciprocals of each other.

H. Area of a triangle or polygon: associate the column scheme.

II. Curve sketching in rectangular coordinates.

- *A*. General methods.
  1. Discussion method.
     - *a*. Intercepts.
     - *b*. Symmetry.
     - *c*. Asymptotes.
     - *d*. Excluded regions.
  2. Combination of ordinates.
     - *a*. Addition.
     - *b*. Multiplication (boundary curves).
     - *c*. Division, etc.
  3. Transformation of variable.
     - *a*. Translation of axes.
     - *b*. Rotation of axes.
     - *c*. Change of scale.
  4. Comparison with basic curves.
- *B*. Important special classes of curves.
  1. Straight line.
  2. Power-law curves (3 types).
  3. Conics—how (given the equation) to draw circles, parabolas, ellipses, hyperbolas.
  4. Transcendental curves: sine and cosine waves, exponential function, logarithmic function.

III. Finding the equation in rectangular coordinates.

- *A*. General methods.
  1. Locus derivation (6 steps).
  2. Use of type-equation and the fundamental principle of analytic geometry.
     - *a*. Straight line obtained from a point and the slope.
     - *b*. Straight line obtained from slope and y-intercept.
     - *c*. Circle obtained from center and radius.
     - *d*. Circle through three points.
     - *e*. Parabola with axis and vertex given which passes through a point.
     - *f*. Parabola with vertical axis which goes through three points.
     - *g*. Ellipses.
     - *h*. Hyperbolas, including hyperbolas with axes as asymptotes.
  3. Determination of an equation from a set of data.
     - *a*. Straight lines.
     - *b*. Polynomials.
     - *c*. Power-law curves.
     - *d*. Exponential curves.

IV. Polar coordinates.

- *A*. Curve sketching.
  1. Point plotting.
  2. Discussion method.
     - *a*. Symmetry.
     - *b*. Open or closed curve.
     - *c*. Variation.
     - *d*. Tangency at pole.
  3. Important classes of curves.
     - *a*. Straight lines.
     - *b*. Circles.
     - *c*. Cardioid curves.
  4. Transformation of coordinates.
- *B*. Curve equation by locus-derivation methods.

V. Parametric equations.

- *A*. Plotting curves from a table of values.
- *B*. Eliminating parameter.
- *C*. Finding equation by method of locus derivation.

REVIEW OF SOLID ANALYTIC GEOMETRY

I. Elementary formulas.

   A. Lengths.
      1. Of a line segment parallel to a coordinate axis.
      2. Of a general line segment.
   B. Direction.
      1. Direction numbers.
      2. Direction cosines.
      3. Direction angles.
      4. Angle between two lines.
      5. Parallel and perpendicular lines.

II. Surface sketching.

   A. General method.
      1. Identification.
      2. Intercepts.
      3. Symmetry.
      4. Traces in coordinate planes.
      5. Traces in a plane parallel to one of the coordinate planes.
   B. Important special classes of surfaces.
      1. Planes ($ax + by + cz + d = 0$; what is the graphical meaning of a, b, c?).
      2. Cylinders.
      3. Cones.
      4. Surfaces of revolution.
      5. Other quadric surfaces.
         a. Ellipsoid.
         b. Elliptic paraboloid.
         c. Hyperboloid of one sheet.
         d. Hyperboloid of two sheets.
         e. Hyperbolic paraboloid.

III. A curve as the intersection of two surfaces.

   A. A straight line as the intersection of two planes.
   B. Projecting cylinders for a curve.

IV. Other coordinate systems.

   A. Cylindrical coordinates.
   B. Spherical coordinates.

Review of Basic Material in Prerequisite Courses

Geometry

TRIANGLE. Area = ½(base)(height) = $\sqrt{s(s-a)(s-b)(s-c)}$, where a, b, and c are the three sides and s is the semiperimeter.

TRAPEZOID. Area = ½(sum of lengths of parallel bases)(height).

CIRCLE. Area = π(radius)^2; circumference = 2π(radius).

SECTOR OF CIRCLE. Area = ½(radius)(arc length); length of arc = (radius)(central angle in radians).

(Note the similarity of the formulas for areas for a triangle and sector.)

SIMILAR PLANE FIGURES. Corresponding linear dimensions of similar plane figures are proportional. The areas of two plane figures that are similar are to each other as the squares of corresponding linear dimensions.

PRISM. Volume = (area of base)(height); lateral surface area of a prism = sum of areas of faces.

PYRAMID AND CONE. Volume = ⅓(area of base)(height). The lateral surface area of a pyramid is the sum of the areas of the triangular faces. The lateral surface area of a right circular cone is the same as the area when unrolled, and that unrolled area forms a sector of a circle with radius equal to the slant height of the cone and with arc length equal to the circumference of the base of the cone.

FRUSTUM OF PYRAMID OR CONE. The volume of a frustum of a pyramid (or cone) is equal to the difference in the volumes of two pyramids (or cones).

SPHERE. Volume = $(4\pi/3)(\text{radius})^3$; surface area = $4\pi(\text{radius})^2$.

SIMILAR SOLIDS. Areas of similar solids are to each other as the squares of corresponding linear dimensions. The volumes of similar solids are to each other as the cubes of corresponding linear dimensions.

Algebra

LAWS AND DEFINITIONS FOR EXPONENTS.

$$a^m \cdot a^n = a^{m+n}.$$

$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}, \quad a \neq 0.$$

$$(a^m)^n = a^{mn}.$$

$$a^m \cdot b^m = (a \cdot b)^m.$$

$$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m, \quad b \neq 0.$$

$$a^0 = 1, \quad a \neq 0.$$

$$a^{-n} = \frac{1}{a^n}\left(\text{hence } a^{-n} + b^{-n} = \frac{1}{a^n} + \frac{1}{b^n}\right).$$

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a}\,)^m, \quad a > 0.$$

Notice that the laws of exponents have to do with multiplication and division. There are *no* corresponding laws for addition and subtraction, save that unlike quantities are added by indicating the addition with the plus or addition sign.

LAWS AND DEFINITIONS FOR RADICALS. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$, provided, *if n is even*, that both a and b are positive. If, for example, we wish to multiply $\sqrt{-3}$ by $\sqrt{-12}$, we may write

$$\sqrt{-3} = \sqrt{-1} \cdot \sqrt{3} = i\sqrt{3}, \quad \sqrt{-12} = i\sqrt{12},$$

and we obtain

$$(\sqrt{-3})(\sqrt{-12}) = (i\sqrt{3})(i\sqrt{12}) = i^2\sqrt{36} = (-1)(6) = -6.$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \; b \neq 0,$$

provided, as above, that if n is even both a and b are to be positive.

$\sqrt[n]{a}$, if a is positive, is a positive number. If a is negative and n is *odd*, the (real) nth root is a negative number.

Laws of Logarithms. (M and N are to denote positive numbers.) If a is a positive number and $a \neq 1$, and if $a^L = N$, then $L = \log_a N$.

$$\log (M \cdot N) = \log M + \log N.$$

$$\log \frac{M}{N} = \log M - \log N.$$

$$\log M^n = n \log M.$$

$$\log \sqrt[n]{M} = \frac{1}{n} \log M.$$

$$\log_a 1 = 0, \quad \log_a a = 1.$$

$\log_a 0$ does not exist.

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

The Quadratic Formula. The two roots of the general quadratic equation $ax^2 + bx + c = 0$ are given by

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant $= b^2 - 4ac$. If a, b, and c are real, and

if $b^2 - 4ac < 0$, the roots are imaginary;

if $b^2 - 4ac = 0$, the roots are real and equal;

if $b^2 - 4ac > 0$, the roots are real and unequal.

The sum of the two roots $= x_1 + x_2 = -b/a$.
The product of the two roots $= x_1 \cdot x_2 = c/a$.

Binomial Theorem.

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3}b^3 + \cdots + b^n,$$

where n is a positive integer and $n!$ (read "n factorial") is given by

$$n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1).$$

DETERMINANTS.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

Textbooks on algebra give special rules for evaluating second- and third-order determinants. Higher-order determinants are evaluated by the use of minors (or cofactors) and the rules for determinants.

SIMULTANEOUS EQUATIONS. A solution of an equation in three variables, for example, is a set of three numbers that reduce the equation to a numerical identity when these three numbers are substituted for the unknowns or variables.

If we have n non-homogeneous linear equations in n unknowns, we may use either the method of addition-subtraction or the method of determinants (or substitution or comparison) to proceed with the solution.

For simultaneous quadratic equations in two unknowns, and for other groups of simultaneous equations, we may generally use the method of substitution.

EQUIVALENT EQUATIONS. Two equations are said to be equivalent when every solution of one equation is a solution of the other equation, and vice versa.

If we add (or subtract) the same quantity to (or from) both sides of a given equation, the resulting equation is equivalent to the original equation. If we multiply both sides of an equation by any number not zero (or divide by any number other than zero), the resulting equation is equivalent to the given equation. If, however, we multiply (or divide) both sides of an equation by a quantity containing the variable, the resulting equation is not necessarily equivalent to the given equation. Squaring both sides of an equation may likewise lead to a non-equivalent equation.

THEORY OF EQUATIONS. The *remainder theorem* states that, if a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

The *factor theorem* states that, if for a polynomial $P(x)$ it is true that $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

Synthetic division. Suppose that we wish to divide $2x^3 - 5x^2 + 8$ by $x - 2$ (it is essential that the coefficient of x in the divisor be unity). The solution is as follows:

$$\begin{array}{rrrr|r} 2 & -5 & 0 & 8 & +2 \\ & 4 & -2 & -4 & \\ \hline 2 & -1 & -2 & 4 & \end{array}$$

Hence

$$\frac{2x^3 - 5x^2 + 8}{x - 2} = 2x^2 - x - 2 + \frac{4}{x - 2}.$$

Rational roots of an equation. If the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0,$$

has all its coefficients $a_0, a_1, \cdots, a_n$ integers (some may be zero), then any rational root of this equation has the form

$$x = \frac{\text{Factor of } a_n}{\text{Factor of } a_0}.$$

Trigonometry

DEFINITIONS (see Fig. A.1.).

$$\sin\theta = \frac{\text{ordinate}}{\text{distance}} = \frac{y}{r} = \frac{1}{\csc\theta},$$

$$\cos\theta = \frac{\text{abscissa}}{\text{distance}} = \frac{x}{r} = \frac{1}{\sec\theta},$$

$$\tan\theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x} = \frac{1}{\cot\theta}.$$

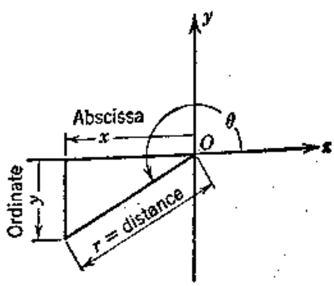


FIG. A.1

If θ is an acute angle, then the right-triangle definitions may also be used (see Fig. A.2):

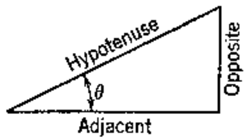


FIG. A.2

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}},$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}},$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

SPECIAL ANGLES. The functions of 30°, 45°, and 60° (and multiples of these angles by aid of reduction formulas) may be read from the right triangles in Fig. A.3:

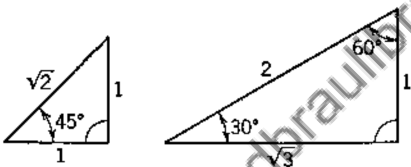


FIG. A.3

SIGNS OF FUNCTIONS. The indicated functions are positive and all others are negative as shown in Fig. A.4.

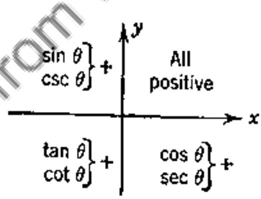


FIG. A.4

RADIAN MEASURE. One radian is a *central* angle of a circle that subtends an arc whose length is equal to the radius. π radians = 180°.

FUNDAMENTAL IDENTITIES.

$$\sin\theta = \frac{1}{\csc\theta}, \quad \cos\theta = \frac{1}{\sec\theta}, \quad \tan\theta = \frac{1}{\cot\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$$

Notice that the last two identities are easy to derive from the basic identity $\sin^2 \theta + \cos^2 \theta = 1$ by dividing either by $\sin^2 \theta$ or $\cos^2 \theta$.

REDUCTION FORMULAS.

$$\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = +\cos\theta,\ \tan(-\theta) = -\tan\theta.$$

$$\text{Any function of } (\text{even}\cdot 90^\circ \pm \theta) = \pm \text{ same function of } \theta,$$

$$\text{any function of } (\text{odd}\cdot 90^\circ \pm \theta) = \pm \text{ cofunction of } \theta,$$

and in both cases the sign of the right-hand side is determined by the sign of the original function in the quadrant in which the original angle terminates. (For the purpose of these rules we assume that θ is an acute angle. Whether θ is acute or not, the rule is still true if it is applied as though θ were acute. For example: $\sin(180^\circ + \theta) = -\sin\theta$, whether θ is acute or not.)

MULTIPLE ANGLE FORMULAS.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

From these basic identities we may easily derive the following:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha,$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$

The result for $\cos 2\alpha$ may be written in two other forms by applying the identity $\sin^2\alpha + \cos^2\alpha = 1$:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

We may change each of these last two results and obtain

$$\sin^2\alpha = \tfrac{1}{2}(1 - \cos 2\alpha), \quad \cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha).$$

Hence

$$\sin \tfrac{1}{2}\theta = \pm \sqrt{\frac{1 - \cos\theta}{2}}, \quad \cos \tfrac{1}{2}\theta = \pm \sqrt{\frac{1 + \cos\theta}{2}},$$

$$\tan \tfrac{1}{2}\theta = \pm \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}},$$

where in all three cases the sign in front of the radical is to be *chosen* on the basis of the sign of the function on the left side of the equation in the quadrant in which $\theta/2$ terminates.

To combine, for example, $\sin A + \sin B$, let $x + y = A$ and $x - y = B$. Solve these for x and y in terms of A and B. Then expand

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y,$$

and add to obtain

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y.$$

Finally, replace x and y by their values in terms of A and B.

To write, for example, $\cos 11x \cos 3x$ as a sum or difference, expand $\cos(11x + 3x)$ and $\cos(11x - 3x)$ and combine the two results.

Formulas for any Triangle.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, \quad \text{the law of sines.}$$

$$c^2 = a^2 + b^2 - 2ab\cos C, \quad \text{the law of cosines.}$$

The Greek Alphabet

Α	α	alpha	Ι	ι	iota	Ρ	ρ	rho
Β	β	beta	Κ	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	Τ	τ	tau
Δ	δ	delta	Μ	μ	mu	Υ	υ	upsilon
Ε	ε	epsilon	Ν	ν	nu	Φ	φ	phi
Ζ	ζ	zeta	Ξ	ξ	xi	Χ	χ	chi
Η	η	eta	Ο	ο	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Table 1. Mantissas of Common Logarithms, base 10

N	0	1	2	3	4	5	6	7	8	9
100	.0000	.0004	.0009	.0013	.0017	.0022	.0026	.0030	.0035	.0039
101	.0043	.0048	.0052	.0056	.0060	.0065	.0069	.0073	.0077	.0082
102	.0086	.0090	.0095	.0099	.0103	.0107	.0111	.0116	.0120	.0124
103	.0128	.0133	.0137	.0141	.0145	.0149	.0154	.0158	.0162	.0166
104	.0170	.0175	.0179	.0183	.0187	.0191	.0195	.0199	.0204	.0208
105	.0212	.0216	.0220	.0224	.0228	.0233	.0237	.0241	.0245	.0249
106	.0253	.0257	.0261	.0265	.0269	.0273	.0278	.0282	.0286	.0290
107	.0294	.0298	.0302	.0306	.0310	.0314	.0318	.0322	.0326	.0330
108	.0334	.0338	.0342	.0346	.0350	.0354	.0358	.0362	.0366	.0370
109	.0374	.0378	.0382	.0386	.0390	.0394	.0398	.0402	.0406	.0410
110	.0414	.0418	.0422	.0426	.0430	.0434	.0438	.0441	.0445	.0449
111	.0453	.0457	.0461	.0465	.0469	.0473	.0477	.0481	.0484	.0488
112	.0492	.0496	.0500	.0504	.0508	.0512	.0515	.0519	.0523	.0527
113	.0531	.0535	.0538	.0542	.0546	.0550	.0554	.0558	.0561	.0565
114	.0569	.0573	.0577	.0580	.0584	.0588	.0592	.0596	.0599	.0603
115	.0607	.0611	.0615	.0618	.0622	.0626	.0630	.0633	.0637	.0641
116	.0645	.0648	.0652	.0656	.0660	.0663	.0667	.0671	.0674	.0678
117	.0682	.0686	.0689	.0693	.0697	.0700	.0704	.0708	.0711	.0715
118	.0719	.0722	.0726	.0730	.0734	.0737	.0741	.0745	.0748	.0752
119	.0755	.0759	.0763	.0766	.0770	.0774	.0777	.0781	.0785	.0788
120	.0792	.0795	.0799	.0803	.0806	.0810	.0813	.0817	.0821	.0824
121	.0828	.0831	.0835	.0839	.0842	.0846	.0849	.0853	.0856	.0860
122	.0864	.0867	.0871	.0874	.0878	.0881	.0885	.0888	.0892	.0896
123	.0899	.0903	.0906	.0910	.0913	.0917	.0920	.0924	.0927	.0931
124	.0934	.0938	.0941	.0945	.0948	.0952	.0955	.0959	.0962	.0966
125	.0969	.0973	.0976	.0980	.0983	.0986	.0990	.0993	.0997	.1000
126	.1004	.1007	.1011	.1014	.1017	.1021	.1024	.1028	.1031	.1035
127	.1038	.1041	.1045	.1048	.1052	.1055	.1059	.1062	.1065	.1069
128	.1072	.1075	.1079	.1082	.1086	.1089	.1092	.1096	.1099	.1103
129	.1106	.1109	.1113	.1116	.1119	.1123	.1126	.1129	.1133	.1136
130	.1139	.1143	.1146	.1149	.1153	.1156	.1159	.1163	.1166	.1169
131	.1173	.1176	.1179	.1183	.1186	.1189	.1193	.1196	.1199	.1202
132	.1206	.1209	.1212	.1216	.1219	.1222	.1225	.1229	.1232	.1235
133	.1239	.1242	.1245	.1248	.1252	.1255	.1258	.1261	.1265	.1268
134	.1271	.1274	.1278	.1281	.1284	.1287	.1290	.1294	.1297	.1300
135	.1303	.1307	.1310	.1313	.1316	.1319	.1323	.1326	.1329	.1332
136	.1335	.1339	.1342	.1345	.1348	.1351	.1355	.1358	.1361	.1364
137	.1367	.1370	.1374	.1377	.1380	.1383	.1386	.1389	.1392	.1396
138	.1399	.1402	.1405	.1408	.1411	.1414	.1418	.1421	.1424	.1427
139	.1430	.1433	.1436	.1440	.1443	.1446	.1449	.1452	.1455	.1458
140	.1461	.1464	.1467	.1471	.1474	.1477	.1480	.1483	.1486	.1489
141	.1492	.1495	.1498	.1501	.1504	.1508	.1511	.1514	.1517	.1520
142	.1523	.1526	.1529	.1532	.1535	.1538	.1541	.1544	.1547	.1550
143	.1553	.1556	.1559	.1562	.1565	.1569	.1572	.1575	.1578	.1581
144	.1584	.1587	.1590	.1593	.1596	.1599	.1602	.1605	.1608	.1611
145	.1614	.1617	.1620	.1623	.1626	.1629	.1632	.1635	.1638	.1641
146	.1644	.1647	.1649	.1652	.1655	.1658	.1661	.1664	.1667	.1670
147	.1673	.1676	.1679	.1682	.1685	.1688	.1691	.1694	.1697	.1700
148	.1703	.1706	.1708	.1711	.1714	.1717	.1720	.1723	.1726	.1729
149	.1732	.1735	.1738	.1741	.1744	.1746	.1749	.1752	.1755	.1758
150	.1761	.1764	.1767	.1770	.1772	.1775	.1778	.1781	.1784	.1787
N	0	1	2	3	4	5	6	7	8	9

Table 1. Mantissas of Common Logarithms, base 10 (Continued)

N	0	1	2	3	4	5	6	7	8	9
150	.1761	.1764	.1767	.1770	.1772	.1775	.1778	.1781	.1784	.1787
151	.1790	.1793	.1796	.1798	.1801	.1804	.1807	.1810	.1813	.1816
152	.1818	.1821	.1824	.1827	.1830	.1833	.1836	.1838	.1841	.1844
153	.1847	.1850	.1853	.1855	.1858	.1861	.1864	.1867	.1870	.1872
154	.1875	.1878	.1881	.1884	.1886	.1889	.1892	.1895	.1898	.1901
155	.1903	.1906	.1909	.1912	.1915	.1917	.1920	.1923	.1926	.1928
156	.1931	.1934	.1937	.1940	.1942	.1945	.1948	.1951	.1953	.1956
157	.1959	.1962	.1965	.1967	.1970	.1973	.1976	.1978	.1981	.1984
158	.1987	.1989	.1992	.1995	.1998	.2000	.2003	.2006	.2009	.2011
159	.2014	.2017	.2019	.2022	.2025	.2028	.2030	.2033	.2036	.2038
160	.2041	.2044	.2047	.2049	.2052	.2055	.2057	.2060	.2063	.2066
161	.2068	.2071	.2074	.2076	.2079	.2082	.2084	.2087	.2090	.2092
162	.2095	.2098	.2101	.2103	.2106	.2109	.2111	.2114	.2117	.2119
163	.2122	.2125	.2127	.2130	.2133	.2135	.2138	.2140	.2143	.2146
164	.2148	.2151	.2154	.2156	.2159	.2162	.2164	.2167	.2170	.2172
165	.2175	.2177	.2180	.2183	.2185	.2188	.2191	.2193	.2196	.2198
166	.2201	.2204	.2206	.2209	.2212	.2214	.2217	.2219	.2222	.2225
167	.2227	.2230	.2232	.2235	.2238	.2240	.2243	.2245	.2248	.2251
168	.2253	.2256	.2258	.2261	.2263	.2266	.2269	.2271	.2274	.2276
169	.2279	.2281	.2284	.2287	.2289	.2292	.2294	.2297	.2299	.2302
170	.2304	.2307	.2310	.2312	.2315	.2317	.2320	.2322	.2325	.2327
171	.2330	.2333	.2335	.2338	.2340	.2343	.2345	.2348	.2350	.2353
172	.2355	.2358	.2360	.2363	.2365	.2368	.2370	.2373	.2375	.2378
173	.2380	.2383	.2385	.2388	.2390	.2393	.2395	.2398	.2400	.2403
174	.2405	.2408	.2410	.2413	.2415	.2418	.2420	.2423	.2425	.2428
175	.2430	.2433	.2435	.2438	.2440	.2443	.2445	.2448	.2450	.2453
176	.2455	.2458	.2460	.2463	.2465	.2467	.2470	.2472	.2475	.2477
177	.2480	.2482	.2485	.2487	.2490	.2492	.2494	.2497	.2499	.2502
178	.2504	.2507	.2509	.2512	.2514	.2516	.2519	.2521	.2524	.2526
179	.2529	.2531	.2533	.2536	.2538	.2541	.2543	.2545	.2548	.2550
180	.2553	.2555	.2558	.2560	.2562	.2565	.2567	.2570	.2572	.2574
181	.2577	.2579	.2582	.2584	.2586	.2589	.2591	.2594	.2596	.2598
182	.2601	.2603	.2605	.2608	.2610	.2613	.2615	.2617	.2620	.2622
183	.2625	.2627	.2629	.2632	.2634	.2636	.2639	.2641	.2643	.2646
184	.2648	.2651	.2653	.2655	.2658	.2660	.2662	.2665	.2667	.2669
185	.2672	.2674	.2676	.2679	.2681	.2683	.2686	.2688	.2690	.2693
186	.2695	.2697	.2700	.2702	.2704	.2707	.2709	.2711	.2714	.2716
187	.2718	.2721	.2723	.2725	.2728	.2730	.2732	.2735	.2737	.2739
188	.2742	.2744	.2746	.2749	.2751	.2753	.2755	.2758	.2760	.2762
189	.2765	.2767	.2769	.2772	.2774	.2776	.2778	.2781	.2783	.2785
190	.2788	.2790	.2792	.2794	.2797	.2799	.2801	.2804	.2806	.2808
191	.2810	.2813	.2815	.2817	.2819	.2822	.2824	.2826	.2828	.2831
192	.2833	.2835	.2838	.2840	.2842	.2844	.2847	.2849	.2851	.2853
193	.2856	.2858	.2860	.2862	.2865	.2867	.2869	.2871	.2874	.2876
194	.2878	.2880	.2882	.2885	.2887	.2889	.2891	.2894	.2896	.2898
195	.2900	.2903	.2905	.2907	.2909	.2911	.2914	.2916	.2918	.2920
196	.2923	.2925	.2927	.2929	.2931	.2934	.2936	.2938	.2940	.2942
197	.2945	.2947	.2949	.2951	.2953	.2956	.2958	.2960	.2962	.2964
198	.2967	.2969	.2971	.2973	.2975	.2978	.2980	.2982	.2984	.2986
199	.2989	.2991	.2993	.2995	.2997	.2999	.3002	.3004	.3006	.3008
200	.3010	.3012	.3015	.3017	.3019	.3021	.3023	.3025	.3028	.3030
N	0	1	2	3	4	5	6	7	8	9

Table 2. Natural (Napierian) Logarithms. Base e = 2.71828

N	0	1	2	3	4	5	6	7	8	9	Prop. parts 1	2	3	4	5
1.0	0.0000	0.0100	0.0198	0.0296	0.0392	0.0488	0.0583	0.0677	0.0770	0.0862	Interpolate				
1.1	0.0953	0.1044	0.1133	0.1222	0.1310	0.1398	0.1484	0.1570	0.1655	0.1740					
1.2	0.1823	0.1906	0.1989	0.2070	0.2151	0.2231	0.2311	0.2390	0.2469	0.2546					
1.3	0.2624	0.2700	0.2776	0.2852	0.2927	0.3001	0.3075	0.3148	0.3221	0.3293					
1.4	0.3365	0.3436	0.3507	0.3577	0.3646	0.3716	0.3784	0.3853	0.3920	0.3988					
1.5	0.4055	0.4121	0.4187	0.4253	0.4318	0.4383	0.4447	0.4511	0.4574	0.4637					
1.6	0.4700	0.4762	0.4824	0.4886	0.4947	0.5008	0.5068	0.5128	0.5188	0.5247					
1.7	0.5306	0.5365	0.5423	0.5481	0.5539	0.5596	0.5653	0.5710	0.5766	0.5822					
1.8	0.5878	0.5933	0.5988	0.6043	0.6098	0.6152	0.6206	0.6259	0.6313	0.6366					
1.9	0.6419	0.6471	0.6523	0.6575	0.6627	0.6678	0.6729	0.6780	0.6831	0.6881					
2.0	0.6931	0.6981	0.7031	0.7080	0.7129	0.7178	0.7227	0.7275	0.7324	0.7372					
2.1	0.7419	0.7467	0.7514	0.7561	0.7608	0.7655	0.7701	0.7747	0.7793	0.7839	5	9	14	19	23
2.2	0.7885	0.7930	0.7975	0.8020	0.8065	0.8109	0.8154	0.8198	0.8242	0.8286	4	9	13	18	22
2.3	0.8329	0.8372	0.8416	0.8459	0.8502	0.8544	0.8587	0.8629	0.8671	0.8713	4	9	13	17	21
2.4	0.8755	0.8796	0.8838	0.8879	0.8920	0.8961	0.9002	0.9042	0.9083	0.9123	4	8	12	16	20
2.5	0.9163	0.9203	0.9243	0.9282	0.9322	0.9361	0.9400	0.9439	0.9478	0.9517	4	8	12	16	20
2.6	0.9555	0.9594	0.9632	0.9670	0.9708	0.9746	0.9783	0.9821	0.9858	0.9895	4	8	11	15	19
2.7	0.9933	0.9969	1.0006	1.0043	1.0080	1.0116	1.0152	1.0188	1.0225	1.0260	4	7	11	15	18
2.8	1.0296	1.0332	1.0367	1.0403	1.0438	1.0473	1.0508	1.0543	1.0578	1.0613	4	7	11	14	18
2.9	1.0647	1.0682	1.0716	1.0750	1.0784	1.0818	1.0852	1.0886	1.0919	1.0953	3	7	10	14	17
3.0	1.0986	1.1019	1.1053	1.1086	1.1119	1.1151	1.1184	1.1217	1.1249	1.1282	3	7	10	13	16
3.1	1.1314	1.1346	1.1378	1.1410	1.1442	1.1474	1.1506	1.1537	1.1569	1.1600	3	6	10	13	16
3.2	1.1632	1.1663	1.1694	1.1725	1.1756	1.1787	1.1817	1.1848	1.1878	1.1909	3	6	9	12	15
3.3	1.1939	1.1969	1.2000	1.2030	1.2060	1.2090	1.2119	1.2149	1.2179	1.2208	3	6	9	12	15
3.4	1.2238	1.2267	1.2296	1.2326	1.2355	1.2384	1.2413	1.2442	1.2470	1.2499	3	6	9	12	14
3.5	1.2528	1.2556	1.2585	1.2613	1.2641	1.2669	1.2698	1.2726	1.2754	1.2782	3	6	8	11	14
3.6	1.2809	1.2837	1.2865	1.2892	1.2920	1.2947	1.2975	1.3002	1.3029	1.3056	3	5	8	11	14
3.7	1.3083	1.3110	1.3137	1.3164	1.3191	1.3218	1.3244	1.3271	1.3297	1.3324	3	5	8	11	13
3.8	1.3350	1.3376	1.3403	1.3429	1.3455	1.3481	1.3507	1.3533	1.3558	1.3584	3	5	8	10	13
3.9	1.3610	1.3635	1.3661	1.3686	1.3712	1.3737	1.3762	1.3788	1.3813	1.3838	3	5	8	10	13
4.0	1.3863	1.3888	1.3913	1.3938	1.3962	1.3987	1.4012	1.4036	1.4061	1.4085	2	5	7	10	12
4.1	1.4110	1.4134	1.4159	1.4183	1.4207	1.4231	1.4255	1.4279	1.4303	1.4327	2	5	7	10	12
4.2	1.4351	1.4375	1.4398	1.4422	1.4446	1.4469	1.4493	1.4516	1.4540	1.4563	2	5	7	9	12
4.3	1.4586	1.4609	1.4633	1.4656	1.4679	1.4702	1.4725	1.4748	1.4770	1.4793	2	5	7	9	11
4.4	1.4816	1.4839	1.4861	1.4884	1.4907	1.4929	1.4951	1.4974	1.4996	1.5019	2	4	7	9	11
4.5	1.5041	1.5063	1.5085	1.5107	1.5129	1.5151	1.5173	1.5195	1.5217	1.5239	2	4	7	9	11
4.6	1.5261	1.5282	1.5304	1.5326	1.5347	1.5369	1.5390	1.5412	1.5433	1.5454	2	4	6	9	11
4.7	1.5476	1.5497	1.5518	1.5539	1.5560	1.5581	1.5602	1.5623	1.5644	1.5665	2	4	6	8	11
4.8	1.5686	1.5707	1.5728	1.5748	1.5769	1.5790	1.5810	1.5831	1.5851	1.5872	2	4	6	8	10
4.9	1.5892	1.5913	1.5933	1.5953	1.5974	1.5994	1.6014	1.6034	1.6054	1.6074	2	4	6	8	10
5.0	1.6094	1.6114	1.6134	1.6154	1.6174	1.6194	1.6214	1.6233	1.6253	1.6273	2	4	6	8	10
5.1	1.6292	1.6312	1.6332	1.6351	1.6371	1.6390	1.6409	1.6429	1.6448	1.6467	2	4	6	8	10
5.2	1.6487	1.6506	1.6525	1.6544	1.6563	1.6582	1.6601	1.6620	1.6639	1.6658	2	4	6	8	10
5.3	1.6677	1.6696	1.6715	1.6734	1.6752	1.6771	1.6790	1.6808	1.6827	1.6845	2	4	6	7	9
5.4	1.6864	1.6882	1.6901	1.6919	1.6938	1.6956	1.6974	1.6993	1.7011	1.7029	2	4	6	7	9
5.5	1.7047	1.7066	1.7084	1.7102	1.7120	1.7138	1.7156	1.7174	1.7192	1.7210	2	4	5	7	9
5.6	1.7228	1.7246	1.7263	1.7281	1.7299	1.7317	1.7334	1.7352	1.7370	1.7387	2	4	5	7	9
5.7	1.7405	1.7422	1.7440	1.7457	1.7475	1.7492	1.7509	1.7527	1.7544	1.7561	2	3	5	7	9
5.8	1.7579	1.7596	1.7613	1.7630	1.7647	1.7664	1.7681	1.7699	1.7716	1.7733	2	3	5	7	9
5.9	1.7750	1.7766	1.7783	1.7800	1.7817	1.7834	1.7851	1.7867	1.7884	1.7901	2	3	5	7	8
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
											Prop. parts				

Table 2. Natural Logarithms (*Continued*)

N	0	1	2	3	4	5	6	7	8	9	Prop. parts 1	2	3	4	5
6.0	1.7918	1.7934	1.7951	1.7967	1.7984	1.8001	1.8017	1.8034	1.8050	1.8066	2	3	5	7	8
6.1	1.8083	1.8099	1.8116	1.8132	1.8148	1.8165	1.8181	1.8197	1.8213	1.8229	2	3	5	7	8
6.2	1.8245	1.8262	1.8278	1.8294	1.8310	1.8326	1.8342	1.8358	1.8374	1.8390	2	3	5	6	8
6.3	1.8405	1.8421	1.8437	1.8453	1.8469	1.8485	1.8500	1.8516	1.8532	1.8547	2	3	5	6	8
6.4	1.8563	1.8579	1.8594	1.8610	1.8625	1.8641	1.8656	1.8672	1.8687	1.8703	2	3	5	6	8
6.5	1.8718	1.8733	1.8749	1.8764	1.8779	1.8795	1.8810	1.8825	1.8840	1.8856	2	3	5	6	8
6.6	1.8871	1.8886	1.8901	1.8916	1.8931	1.8946	1.8961	1.8976	1.8991	1.9006	2	3	5	6	8
6.7	1.9021	1.9036	1.9051	1.9066	1.9081	1.9095	1.9110	1.9125	1.9140	1.9155	1	3	4	6	7
6.8	1.9169	1.9184	1.9199	1.9213	1.9228	1.9242	1.9257	1.9272	1.9286	1.9301	1	3	4	6	7
6.9	1.9315	1.9330	1.9344	1.9359	1.9373	1.9387	1.9402	1.9416	1.9430	1.9445	1	3	4	6	7
7.0	1.9459	1.9473	1.9488	1.9502	1.9516	1.9530	1.9544	1.9559	1.9573	1.9587	1	3	4	6	7
7.1	1.9601	1.9615	1.9629	1.9643	1.9657	1.9671	1.9685	1.9699	1.9713	1.9727	1	3	4	6	7
7.2	1.9741	1.9755	1.9769	1.9782	1.9796	1.9810	1.9824	1.9838	1.9851	1.9865	1	3	4	6	7
7.3	1.9879	1.9892	1.9906	1.9920	1.9933	1.9947	1.9961	1.9974	1.9988	2.0001	1	3	4	5	7
7.4	2.0015	2.0028	2.0042	2.0055	2.0069	2.0082	2.0096	2.0109	2.0122	2.0136	1	3	4	5	7
7.5	2.0149	2.0162	2.0176	2.0189	2.0202	2.0215	2.0229	2.0242	2.0255	2.0268	1	3	4	5	7
7.6	2.0281	2.0295	2.0308	2.0321	2.0334	2.0347	2.0360	2.0373	2.0386	2.0399	1	3	4	5	7
7.7	2.0412	2.0425	2.0438	2.0451	2.0464	2.0477	2.0490	2.0503	2.0516	2.0528	1	3	4	5	6
7.8	2.0541	2.0554	2.0567	2.0580	2.0592	2.0605	2.0618	2.0631	2.0643	2.0656	1	3	4	5	6
7.9	2.0669	2.0681	2.0694	2.0707	2.0719	2.0732	2.0744	2.0757	2.0769	2.0782	1	3	4	5	6
8.0	2.0794	2.0807	2.0819	2.0832	2.0844	2.0857	2.0869	2.0882	2.0894	2.0906	1	2	4	5	6
8.1	2.0919	2.0931	2.0943	2.0956	2.0968	2.0980	2.0992	2.1005	2.1017	2.1029	1	2	4	5	6
8.2	2.1041	2.1054	2.1066	2.1078	2.1090	2.1102	2.1114	2.1126	2.1138	2.1150	1	2	4	5	6
8.3	2.1163	2.1175	2.1187	2.1199	2.1211	2.1223	2.1235	2.1247	2.1258	2.1270	1	2	4	5	6
8.4	2.1282	2.1294	2.1306	2.1318	2.1330	2.1342	2.1353	2.1365	2.1377	2.1389	1	2	4	5	6
8.5	2.1401	2.1412	2.1424	2.1436	2.1448	2.1459	2.1471	2.1483	2.1494	2.1506	1	2	4	5	6
8.6	2.1518	2.1529	2.1541	2.1552	2.1564	2.1576	2.1587	2.1599	2.1610	2.1622	1	2	3	5	6
8.7	2.1633	2.1645	2.1656	2.1668	2.1679	2.1691	2.1702	2.1713	2.1725	2.1736	1	2	3	5	6
8.8	2.1748	2.1759	2.1770	2.1782	2.1793	2.1804	2.1815	2.1827	2.1838	2.1849	1	2	3	5	6
8.9	2.1861	2.1872	2.1883	2.1894	2.1905	2.1917	2.1928	2.1939	2.1950	2.1961	1	2	3	4	6
9.0	2.1972	2.1983	2.1994	2.2006	2.2017	2.2028	2.2039	2.2050	2.2061	2.2072	1	2	3	4	6
9.1	2.2083	2.2094	2.2105	2.2116	2.2127	2.2138	2.2148	2.2159	2.2170	2.2181	1	2	3	4	5
9.2	2.2192	2.2203	2.2214	2.2225	2.2235	2.2246	2.2257	2.2268	2.2279	2.2289	1	2	3	4	5
9.3	2.2300	2.2311	2.2322	2.2332	2.2343	2.2354	2.2364	2.2375	2.2386	2.2396	1	2	3	4	5
9.4	2.2407	2.2418	2.2428	2.2439	2.2450	2.2460	2.2471	2.2481	2.2492	2.2502	1	2	3	4	5
9.5	2.2513	2.2523	2.2534	2.2544	2.2555	2.2565	2.2576	2.2586	2.2597	2.2607	1	2	3	4	5
9.6	2.2618	2.2628	2.2638	2.2649	2.2659	2.2670	2.2680	2.2690	2.2701	2.2711	1	2	3	4	5
9.7	2.2721	2.2732	2.2742	2.2752	2.2762	2.2773	2.2783	2.2793	2.2803	2.2814	1	2	3	4	5
9.8	2.2824	2.2834	2.2844	2.2854	2.2865	2.2875	2.2885	2.2895	2.2905	2.2915	1	2	3	4	5
9.9	2.2925	2.2935	2.2946	2.2956	2.2966	2.2976	2.2986	2.2996	2.3006	2.3016	1	2	3	4	5

Table 3. Natural (Napierian) Logarithms of Powers of 10

u	Nap. log. 10^u	u	Nap. log. 10^u	u	Nap. log. 10^u	u	Nap. log. 10^u
0	0.000 000	2.5	5.756 463	5.0	11.512 925	7.5	17.269 388
0.5	1.151 293	3.0	6.907 755	5.5	12.664 218	8.0	18.420 681
1.0	2.302 585	3.5	8.059 048	6.0	13.815 511	8.5	19.571 973
1.5	3.453 878	4.0	9.210 340	6.5	14.966 803	9.0	20.723 266
2.0	4.605 170	4.5	10.361 633	7.0	16.118 096	9.5	21.874 558

Table 4. Natural Sines and Cosines

Ang., deg.	Sine 0′	10′	20′	30′	40′	50′	60′		Prop. parts 1′	2′	3′	4′	5′
0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89	3	6	9	12	15
1	0.0175	0.0204	0.0233	0.0262	0.0291	0.0320	0.0349	88	3	6	9	12	15
2	0.0349	0.0378	0.0407	0.0436	0.0465	0.0494	0.0523	87	3	6	9	12	15
3	0.0523	0.0552	0.0581	0.0610	0.0640	0.0669	0.0698	86	3	6	9	12	15
4	0.0698	0.0727	0.0756	0.0785	0.0814	0.0843	0.0872	85	3	6	9	12	14
5	0.0872	0.0901	0.0929	0.0958	0.0987	0.1016	0.1045	84	3	6	9	12	14
6	0.1045	0.1074	0.1103	0.1132	0.1161	0.1190	0.1219	83	3	6	9	12	14
7	0.1219	0.1248	0.1276	0.1305	0.1334	0.1363	0.1392	82	3	6	9	12	14
8	0.1392	0.1421	0.1449	0.1478	0.1507	0.1536	0.1564	81	3	6	9	12	14
9	0.1564	0.1593	0.1622	0.1650	0.1679	0.1708	0.1736	80	3	6	9	11	14
10	0.1736	0.1765	0.1794	0.1822	0.1851	0.1880	0.1908	79	3	6	9	11	14
11	0.1908	0.1937	0.1965	0.1994	0.2022	0.2051	0.2079	78	3	6	9	11	14
12	0.2079	0.2108	0.2136	0.2164	0.2193	0.2221	0.2250	77	3	6	9	11	14
13	0.2250	0.2278	0.2306	0.2334	0.2363	0.2391	0.2419	76	3	6	8	11	14
14	0.2419	0.2447	0.2476	0.2504	0.2532	0.2560	0.2588	75	3	6	8	11	14
15	0.2588	0.2616	0.2644	0.2672	0.2700	0.2728	0.2756	74	3	6	8	11	14
16	0.2756	0.2784	0.2812	0.2840	0.2868	0.2896	0.2924	73	3	6	8	11	14
17	0.2924	0.2952	0.2979	0.3007	0.3035	0.3062	0.3090	72	3	6	8	11	14
18	0.3090	0.3118	0.3145	0.3173	0.3201	0.3228	0.3256	71	3	6	8	11	14
19	0.3256	0.3283	0.3311	0.3338	0.3365	0.3393	0.3420	70	3	5	8	11	14
20	0.3420	0.3448	0.3475	0.3502	0.3529	0.3557	0.3584	69	3	5	8	11	14
21	0.3584	0.3611	0.3638	0.3665	0.3692	0.3719	0.3746	68	3	5	8	11	14
22	0.3746	0.3773	0.3800	0.3827	0.3854	0.3881	0.3907	67	3	5	8	11	13
23	0.3907	0.3934	0.3961	0.3987	0.4014	0.4041	0.4067	66	3	5	8	11	13
24	0.4067	0.4094	0.4120	0.4147	0.4173	0.4200	0.4226	65	3	5	8	11	13
25	0.4226	0.4253	0.4279	0.4305	0.4331	0.4358	0.4384	64	3	5	8	11	13
26	0.4384	0.4410	0.4436	0.4462	0.4488	0.4514	0.4540	63	3	5	8	10	13
27	0.4540	0.4566	0.4592	0.4617	0.4643	0.4669	0.4695	62	3	5	8	10	13
28	0.4695	0.4720	0.4746	0.4772	0.4797	0.4823	0.4848	61	3	5	8	10	13
29	0.4848	0.4874	0.4899	0.4924	0.4950	0.4975	0.5000	60	3	5	8	10	13
30	0.5000	0.5025	0.5050	0.5075	0.5100	0.5125	0.5150	59	3	5	8	10	13
31	0.5150	0.5175	0.5200	0.5225	0.5250	0.5275	0.5299	58	2	5	7	10	12
32	0.5299	0.5324	0.5348	0.5373	0.5398	0.5422	0.5446	57	2	5	7	10	12
33	0.5446	0.5471	0.5495	0.5519	0.5544	0.5568	0.5592	56	2	5	7	10	12
34	0.5592	0.5616	0.5640	0.5664	0.5688	0.5712	0.5736	55	2	5	7	10	12
35	0.5736	0.5760	0.5783	0.5807	0.5831	0.5854	0.5878	54	2	5	7	9	12
36	0.5878	0.5901	0.5925	0.5948	0.5972	0.5995	0.6018	53	2	5	7	9	12
37	0.6018	0.6041	0.6065	0.6088	0.6111	0.6134	0.6157	52	2	5	7	9	12
38	0.6157	0.6180	0.6202	0.6225	0.6248	0.6271	0.6293	51	2	5	7	9	11
39	0.6293	0.6316	0.6338	0.6361	0.6383	0.6406	0.6428	50	2	4	7	9	11
40	0.6428	0.6450	0.6472	0.6494	0.6517	0.6539	0.6561	49	2	4	7	9	11
41	0.6561	0.6583	0.6604	0.6626	0.6648	0.6670	0.6691	48	2	4	7	9	11
42	0.6691	0.6713	0.6734	0.6756	0.6777	0.6799	0.6820	47	2	4	6	9	11
43	0.6820	0.6841	0.6862	0.6884	0.6905	0.6926	0.6947	46	2	4	6	8	11
44	0.6947	0.6967	0.6988	0.7009	0.7030	0.7050	0.7071	45	2	4	6	8	10
	60′	50′	40′	30′	20′	10′	0′	Ang., deg.	1′	2′	3′	4′	5′
	Cosine								Prop. parts				

Table 4. Natural Sines and Cosines (*Continued*)

Ang., deg.	Sine 0′	10′	20′	30′	40′	50′	60′		Prop. parts 1′	2′	3′	4′	5′
45	0.7071	0.7092	0.7112	0.7133	0.7153	0.7173	0.7193	44	2	4	6	8	10
46	0.7193	0.7214	0.7234	0.7254	0.7274	0.7294	0.7314	43	2	4	6	8	10
47	0.7314	0.7333	0.7353	0.7373	0.7392	0.7412	0.7431	42	2	4	6	8	10
48	0.7431	0.7451	0.7470	0.7490	0.7509	0.7528	0.7547	41	2	4	6	8	10
49	0.7547	0.7566	0.7585	0.7604	0.7623	0.7642	0.7660	40	2	4	6	8	9
50	0.7660	0.7679	0.7698	0.7716	0.7735	0.7753	0.7771	39	2	4	6	7	9
51	0.7771	0.7790	0.7808	0.7826	0.7844	0.7862	0.7880	38	2	4	5	7	9
52	0.7880	0.7898	0.7916	0.7934	0.7951	0.7969	0.7986	37	2	4	5	7	9
53	0.7986	0.8004	0.8021	0.8039	0.8056	0.8073	0.8090	36	2	3	5	7	9
54	0.8090	0.8107	0.8124	0.8141	0.8158	0.8175	0.8192	35	2	3	5	7	8
55	0.8192	0.8208	0.8225	0.8241	0.8258	0.8274	0.8290	34	2	3	5	7	8
56	0.8290	0.8307	0.8323	0.8339	0.8355	0.8371	0.8387	33	2	3	5	6	8
57	0.8387	0.8403	0.8418	0.8434	0.8450	0.8465	0.8480	32	2	3	5	6	8
58	0.8480	0.8496	0.8511	0.8526	0.8542	0.8557	0.8572	31	2	3	5	6	8
59	0.8572	0.8587	0.8601	0.8616	0.8631	0.8646	0.8660	30	1	3	4	6	7
60	0.8660	0.8675	0.8689	0.8704	0.8718	0.8732	0.8746	29	1	3	4	6	7
61	0.8746	0.8760	0.8774	0.8788	0.8802	0.8816	0.8829	28	1	3	4	6	7
62	0.8829	0.8843	0.8857	0.8870	0.8884	0.8897	0.8910	27	1	3	4	5	7
63	0.8910	0.8923	0.8936	0.8949	0.8962	0.8975	0.8988	26	1	3	4	5	6
64	0.8988	0.9001	0.9013	0.9026	0.9038	0.9051	0.9063	25	1	3	4	5	6
65	0.9063	0.9075	0.9088	0.9100	0.9112	0.9124	0.9135	24	1	2	4	5	6
66	0.9135	0.9147	0.9159	0.9171	0.9182	0.9194	0.9205	23	1	2	3	5	6
67	0.9205	0.9216	0.9228	0.9239	0.9250	0.9261	0.9272	22	1	2	3	4	6
68	0.9272	0.9283	0.9293	0.9304	0.9315	0.9325	0.9336	21	1	2	3	4	5
69	0.9336	0.9346	0.9356	0.9367	0.9377	0.9387	0.9397	20	1	2	3	4	5
70	0.9397	0.9407	0.9417	0.9426	0.9436	0.9446	0.9455	19	1	2	3	4	5
71	0.9455	0.9465	0.9474	0.9483	0.9492	0.9502	0.9511	18	1	2	3	4	5
72	0.9511	0.9520	0.9528	0.9537	0.9546	0.9555	0.9563	17	1	2	3	3	4
73	0.9563	0.9572	0.9580	0.9588	0.9596	0.9605	0.9613	16	1	2	2	3	4
74	0.9613	0.9621	0.9628	0.9636	0.9644	0.9652	0.9659	15	1	2	2	3	4
75	0.9659	0.9667	0.9674	0.9681	0.9689	0.9696	0.9703	14	1	1	2	3	4
76	0.9703	0.9710	0.9717	0.9724	0.9730	0.9737	0.9744	13	1	1	2	3	3
77	0.9744	0.9750	0.9757	0.9763	0.9769	0.9775	0.9781	12	1	1	2	3	3
78	0.9781	0.9787	0.9793	0.9799	0.9805	0.9811	0.9816	11	1	1	2	2	3
79	0.9816	0.9822	0.9827	0.9833	0.9838	0.9843	0.9848	10	1	1	2	2	3
80	0.9848	0.9853	0.9858	0.9863	0.9868	0.9872	0.9877	9	0	1	1	2	2
81	0.9877	0.9881	0.9886	0.9890	0.9894	0.9899	0.9903	8	0	1	1	2	2
82	0.9903	0.9907	0.9911	0.9914	0.9918	0.9922	0.9925	7	0	1	1	2	2
83	0.9925	0.9929	0.9932	0.9936	0.9939	0.9942	0.9945	6	0	1	1	1	2
84	0.9945	0.9948	0.9951	0.9954	0.9957	0.9959	0.9962	5	0	1	1	1	1
85	0.9962	0.9964	0.9967	0.9969	0.9971	0.9974	0.9976	4	0	0	1	1	1
86	0.9976	0.9978	0.9980	0.9981	0.9983	0.9985	0.9986	3	0	0	1	1	1
87	0.9986	0.9988	0.9989	0.9990	0.9992	0.9993	0.9994	2	0	0	0	1	1
88	0.9994	0.9995	0.9996	0.9997	0.9997	0.9998	0.9998	1	0	0	0	0	0
89	0.9998	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0
	60′	50′	40′	30′	20′	10′	0′	Ang., deg.	1′	2′	3′	4′	5′
	Cosine								Prop. parts				

Table 5. Natural Tangents and Cotangents

Ang., deg.	Tangent 0′	10′	20′	30′	40′	50′	60′		Prop. parts 1′	2′	3′	4′	5′
0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89	3	6	9	12	15
1	0.0175	0.0204	0.0233	0.0262	0.0291	0.0320	0.0349	88	3	6	9	12	15
2	0.0349	0.0378	0.0407	0.0437	0.0466	0.0495	0.0524	87	3	6	9	12	15
3	0.0524	0.0553	0.0582	0.0612	0.0641	0.0670	0.0699	86	3	6	9	12	15
4	0.0699	0.0729	0.0758	0.0787	0.0816	0.0846	0.0875	85	3	6	9	12	15
5	0.0875	0.0904	0.0934	0.0963	0.0992	0.1022	0.1051	84	3	6	9	12	15
6	0.1051	0.1080	0.1110	0.1139	0.1169	0.1198	0.1228	83	3	6	9	12	15
7	0.1228	0.1257	0.1287	0.1317	0.1346	0.1376	0.1405	82	3	6	9	12	15
8	0.1405	0.1435	0.1465	0.1495	0.1524	0.1554	0.1584	81	3	6	9	12	15
9	0.1584	0.1614	0.1644	0.1673	0.1703	0.1733	0.1763	80	3	6	9	12	15
10	0.1763	0.1793	0.1823	0.1853	0.1883	0.1914	0.1944	79	3	6	9	12	15
11	0.1944	0.1974	0.2004	0.2035	0.2065	0.2095	0.2126	78	3	6	9	12	15
12	0.2126	0.2156	0.2186	0.2217	0.2247	0.2278	0.2309	77	3	6	9	12	15
13	0.2309	0.2339	0.2370	0.2401	0.2432	0.2462	0.2493	76	3	6	9	12	15
14	0.2493	0.2524	0.2555	0.2586	0.2617	0.2648	0.2679	75	3	6	9	12	16
15	0.2679	0.2711	0.2742	0.2773	0.2805	0.2836	0.2867	74	3	6	9	13	16
16	0.2867	0.2899	0.2931	0.2962	0.2994	0.3026	0.3057	73	3	6	9	13	16
17	0.3057	0.3089	0.3121	0.3153	0.3185	0.3217	0.3249	72	3	6	10	13	16
18	0.3249	0.3281	0.3314	0.3346	0.3378	0.3411	0.3443	71	3	6	10	13	16
19	0.3443	0.3476	0.3508	0.3541	0.3574	0.3607	0.3640	70	3	7	10	13	16
20	0.3640	0.3673	0.3706	0.3739	0.3772	0.3805	0.3839	69	3	7	10	13	17
21	0.3839	0.3872	0.3906	0.3939	0.3973	0.4006	0.4040	68	3	7	10	13	17
22	0.4040	0.4074	0.4108	0.4142	0.4176	0.4210	0.4245	67	3	7	10	14	17
23	0.4245	0.4279	0.4314	0.4348	0.4383	0.4417	0.4452	66	3	7	10	14	17
24	0.4452	0.4487	0.4522	0.4557	0.4592	0.4628	0.4663	65	4	7	11	14	18
25	0.4663	0.4699	0.4734	0.4770	0.4806	0.4841	0.4877	64	4	7	11	14	18
26	0.4877	0.4913	0.4950	0.4986	0.5022	0.5059	0.5095	63	4	7	11	15	18
27	0.5095	0.5132	0.5169	0.5206	0.5243	0.5280	0.5317	62	4	7	11	15	18
28	0.5317	0.5354	0.5392	0.5430	0.5467	0.5505	0.5543	61	4	8	11	15	19
29	0.5543	0.5581	0.5619	0.5658	0.5696	0.5735	0.5774	60	4	8	12	15	19
30	0.5774	0.5812	0.5851	0.5890	0.5930	0.5969	0.6009	59	4	8	12	16	19
31	0.6009	0.6048	0.6088	0.6128	0.6168	0.6208	0.6249	58	4	8	12	16	20
32	0.6249	0.6289	0.6330	0.6371	0.6412	0.6453	0.6494	57	4	8	12	16	20
33	0.6494	0.6536	0.6577	0.6619	0.6661	0.6703	0.6745	56	4	8	13	17	21
34	0.6745	0.6787	0.6830	0.6873	0.6916	0.6959	0.7002	55	4	9	13	17	21
35	0.7002	0.7046	0.7089	0.7133	0.7177	0.7221	0.7265	54	4	9	13	18	22
36	0.7265	0.7310	0.7355	0.7400	0.7445	0.7490	0.7536	53	5	9	14	18	23
37	0.7536	0.7581	0.7627	0.7673	0.7720	0.7766	0.7813	52	5	9	14	18	23
38	0.7813	0.7860	0.7907	0.7954	0.8002	0.8050	0.8098	51	5	10	14	19	24
39	0.8098	0.8146	0.8195	0.8243	0.8292	0.8342	0.8391	50	5	10	15	20	24
40	0.8391	0.8441	0.8491	0.8541	0.8591	0.8642	0.8693	49	5	10	15	20	25
41	0.8693	0.8744	0.8796	0.8847	0.8899	0.8952	0.9004	48	5	10	16	21	26
42	0.9004	0.9057	0.9110	0.9163	0.9217	0.9271	0.9325	47	5	11	16	21	27
43	0.9325	0.9380	0.9435	0.9490	0.9545	0.9601	0.9657	46	6	11	17	22	28
44	0.9657	0.9713	0.9770	0.9827	0.9884	0.9942	1.0000	45	6	11	17	23	29
	60′	50′	40′	30′	20′	10′	0′	Ang., deg.	1′	2′	3′	4′	5′
	Cotangent								Prop. parts				

Table 5. Natural Tangents and Cotangents (*Continued*)

Ang., deg.	Tangent 0′	10′	20′	30′	40′	50′	60′		Prop. parts 1′	2′	3′	4′	5′
45	1.000	1.006	1.012	1.018	1.024	1.030	1.036	44	1	1	2	2	3
46	1.036	1.042	1.048	1.054	1.060	1.066	1.072	43	1	1	2	2	3
47	1.072	1.079	1.085	1.091	1.098	1.104	1.111	42	1	1	2	3	3
48	1.111	1.117	1.124	1.130	1.137	1.144	1.150	41	1	1	2	3	3
49	1.150	1.157	1.164	1.171	1.178	1.185	1.192	40	1	1	2	3	3
50	1.192	1.199	1.206	1.213	1.220	1.228	1.235	39	1	1	2	3	4
51	1.235	1.242	1.250	1.257	1.265	1.272	1.280	38	1	1	2	3	4
52	1.280	1.288	1.295	1.303	1.311	1.319	1.327	37	1	2	2	3	4
53	1.327	1.335	1.343	1.351	1.360	1.368	1.376	36	1	2	2	3	4
54	1.376	1.385	1.393	1.402	1.411	1.419	1.428	35	1	2	3	3	4
55	1.428	1.437	1.446	1.455	1.464	1.473	1.483	34	1	2	3	4	5
56	1.483	1.492	1.501	1.511	1.520	1.530	1.540	33	1	2	3	4	5
57	1.540	1.550	1.560	1.570	1.580	1.590	1.600	32	1	2	3	4	5
58	1.600	1.611	1.621	1.632	1.643	1.653	1.664	31	1	2	3	4	5
59	1.664	1.675	1.686	1.698	1.709	1.720	1.732	30	1	2	3	5	6
60	1.732	1.744	1.756	1.767	1.780	1.792	1.804	29	1	2	4	5	6
61	1.804	1.816	1.829	1.842	1.855	1.868	1.881	28	1	3	4	5	6
62	1.881	1.894	1.907	1.921	1.935	1.949	1.963	27	1	3	4	5	7
63	1.963	1.977	1.991	2.006	2.020	2.035	2.050	26	1	3	4	6	7
64	2.050	2.066	2.081	2.097	2.112	2.128	2.145	25	2	3	5	6	8
65	2.145	2.161	2.177	2.194	2.211	2.229	2.246	24	2	3	5	7	8
66	2.246	2.264	2.282	2.300	2.318	2.337	2.356	23	2	4	5	7	9
67	2.356	2.375	2.394	2.414	2.434	2.455	2.475	22	2	4	6	8	10
68	2.475	2.496	2.517	2.539	2.560	2.583	2.605	21	2	4	6	9	11
69	2.605	2.628	2.651	2.675	2.699	2.723	2.747	20	2	5	7	10	12
70	2.747	2.773	2.798	2.824	2.850	2.877	2.904	19	3	5	8	11	13
71	2.904	2.932	2.960	2.989	3.018	3.047	3.078	18	3	6	9	12	14
72	3.078	3.108	3.140	3.172	3.204	3.237	3.271	17	3	6	10	13	16
73	3.271	3.305	3.340	3.376	3.412	3.450	3.487	16	4	7	11	14	18
74	3.487	3.526	3.566	3.606	3.647	3.689	3.732	15	Interpolate				
75	3.732	3.776	3.821	3.867	3.914	3.962	4.011	14					
76	4.011	4.061	4.113	4.165	4.219	4.275	4.331	13					
77	4.331	4.390	4.449	4.511	4.574	4.638	4.705	12					
78	4.705	4.773	4.843	4.915	4.989	5.066	5.145	11					
79	5.145	5.226	5.309	5.396	5.485	5.576	5.671	10					
80	5.671	5.769	5.871	5.976	6.084	6.197	6.314	9	Do not interpolate here				
81	6.314	6.435	6.561	6.691	6.827	6.968	7.115	8					
82	7.115	7.269	7.429	7.596	7.770	7.953	8.144	7					
83	8.144	8.345	8.556	8.777	9.010	9.255	9.514	6					
84	9.514	9.788	10.08	10.39	10.71	11.06	11.43	5					
85	11.43	11.83	12.25	12.71	13.20	13.73	14.30	4					
86	14.30	14.92	15.60	16.35	17.17	18.07	19.08	3					
87	19.08	20.21	21.47	22.90	24.54	26.43	28.64	2					
88	28.64	31.24	34.37	38.19	42.96	49.10	57.29	1					
89	57.29	68.75	85.94	114.6	171.9	343.8	∞	0					
	60′	50′	40′	30′	20′	10′	0′	Ang., deg.	1′	2′	3′	4′	5′
	Cotangent								Prop. parts				

Table 6. Exponentials e^u and e^{-u}

u	e^u	e^{-u}	u	e^u	e^{-u}	u	e^u	e^{-u}
0.00	1.000	1.0000	0.50	1.649	0.6065	1.0	2.718	0.3679
0.01	1.010	0.9900	0.51	1.665	0.6005	1.1	3.004	0.3329
0.02	1.020	0.9802	0.52	1.682	0.5945	1.2	3.320	0.3012
0.03	1.030	0.9704	0.53	1.699	0.5886	1.3	3.669	0.2725
0.04	1.041	0.9608	0.54	1.716	0.5827	1.4	4.055	0.2466
0.05	1.051	0.9512	0.55	1.733	0.5769	1.5	4.482	0.2231
0.06	1.062	0.9418	0.56	1.751	0.5712	1.6	4.953	0.2019
0.07	1.073	0.9324	0.57	1.768	0.5655	1.7	5.474	0.1827
0.08	1.083	0.9231	0.58	1.786	0.5599	1.8	6.050	0.1653
0.09	1.094	0.9139	0.59	1.804	0.5543	1.9	6.686	0.1496
0.10	1.105	0.9048	0.60	1.822	0.5488	2.0	7.389	0.1353
0.11	1.116	0.8958	0.61	1.840	0.5434	2.1	8.166	0.1225
0.12	1.127	0.8869	0.62	1.859	0.5379	2.2	9.025	0.1108
0.13	1.139	0.8781	0.63	1.878	0.5326	2.3	9.974	0.1003
0.14	1.150	0.8694	0.64	1.896	0.5273	2.4	11.02	0.09072
0.15	1.162	0.8607	0.65	1.916	0.5220	2.5	12.18	0.08209
0.16	1.174	0.8521	0.66	1.935	0.5169	2.6	13.46	0.07427
0.17	1.185	0.8437	0.67	1.954	0.5117	2.7	14.88	0.06721
0.18	1.197	0.8353	0.68	1.974	0.5066	2.8	16.44	0.06081
0.19	1.209	0.8270	0.69	1.994	0.5016	2.9	18.17	0.05502
0.20	1.221	0.8187	0.70	2.014	0.4966	3.0	20.09	0.04979
0.21	1.234	0.8106	0.71	2.034	0.4916	3.1	22.20	0.04505
0.22	1.246	0.8025	0.72	2.054	0.4868	3.2	24.53	0.04076
0.23	1.259	0.7945	0.73	2.075	0.4819	3.3	27.11	0.03688
0.24	1.271	0.7866	0.74	2.096	0.4771	3.4	29.96	0.03337
0.25	1.284	0.7788	0.75	2.117	0.4724	3.5	33.12	0.03020
0.26	1.297	0.7711	0.76	2.138	0.4677	3.6	36.60	0.02732
0.27	1.310	0.7634	0.77	2.160	0.4630	3.7	40.45	0.02472
0.28	1.323	0.7558	0.78	2.181	0.4584	3.8	44.70	0.02237
0.29	1.336	0.7483	0.79	2.203	0.4538	3.9	49.40	0.02024
0.30	1.350	0.7408	0.80	2.226	0.4493	4.0	54.60	0.01832
0.31	1.363	0.7334	0.81	2.248	0.4449	4.5	90.02	0.01111
0.32	1.377	0.7261	0.82	2.271	0.4404	5.0	148.4	0.00674
0.33	1.391	0.7189	0.83	2.293	0.4360	5.5	244.7	0.00409
0.34	1.405	0.7118	0.84	2.316	0.4317	6.0	403.4	0.00248
0.35	1.419	0.7047	0.85	2.340	0.4274	6.5	665.1	0.00150
0.36	1.433	0.6977	0.86	2.363	0.4232	7.0	1097	0.00091
0.37	1.448	0.6907	0.87	2.387	0.4190	8.0	2981	0.00034
0.38	1.462	0.6839	0.88	2.411	0.4148	9.0	8103	0.00012
0.39	1.477	0.6771	0.89	2.435	0.4107	10.0	22026	0.00005
0.40	1.492	0.6703	0.90	2.460	0.4066	$\pi/4$	2.193	0.45594
0.41	1.507	0.6637	0.91	2.484	0.4025	$2\pi/4$	4.811	0.20788
0.42	1.522	0.6570	0.92	2.509	0.3985	$3\pi/4$	10.55	0.09478
0.43	1.537	0.6505	0.93	2.535	0.3946	$4\pi/4$	23.14	0.04321
0.44	1.553	0.6440	0.94	2.560	0.3906	$5\pi/4$	50.75	0.01970
0.45	1.568	0.6376	0.95	2.586	0.3867	$6\pi/4$	111.3	0.00898
0.46	1.584	0.6313	0.96	2.612	0.3829	$7\pi/4$	244.2	0.00410
0.47	1.600	0.6250	0.97	2.638	0.3791	$8\pi/4$	535.5	0.00187
0.48	1.616	0.6188	0.98	2.664	0.3753	$9\pi/4$	1175	0.00085
0.49	1.632	0.6126	0.99	2.691	0.3716	$10\pi/4$	2576	0.00039

Answers

Art. 1.4, pages 7–8

1. $\overline{CB} \approx 7.62$.
2. (a) 12. (c) 30. (e) 33.12. (g) 20.37.
3. 33.1.
5. (a) 17. (c) 9. (e) $12a^2$.
7. (a) −6, 5, 5, 8.25. (c) −11, −7, 12.08, 2.
8. (a) 7.81, 8.94. (c) 13.04, 11.18.
9. $x^2 + y^2 - 6x + 4y = 12$.
11. 5.22.

Art. 1.5, pages 10–12

1. 3.81.
2. (a) (5, 2), (7, 3.5), (5, 3.5). (c) (2.00, 5.67), (−0.46, 6.23), (0.10, 4.22).
3. (2, −1), 6.71.
4. (c) $H(\frac{8}{3}, 1)$.
5. (a) $x_A = 14$, $x_B = 4$. (c) $y = \frac{28}{9}$.
7. (5.8, 8.2).
9. (a) (4, 1), (−6, −5), (8, −3). (c) (0, 6), (4, 0), (4, 6).
11. (a) (−4, 7). (b) (−2.5, 5.5).
13. (a) (3, 3.75).
15. 0.30200.
17. (a) (4, 4). (c) (1, 2) and (3, 1).

Art. 1.7, pages 17–18

1. (a) −1.25. (c) 0. (e) 2.49. (g) −1.40.

Art. 1.7 (cont.), pages 17–18

2. (a) −51.3°. (c) 0°. (e) 68.1°. (g) −54.5°.
3. 5.
*7. Mid-point of diagonals is at (3, 3).
11. 26.6°, 63.4°, 90°.
13. 0.4.
14. (a) 37.9°, 81.9°, 60.3°. (c) 35.8°, 94.4°, 49.8°. (e) 90.0°, 63.4°, 26.6°.
15. $y = -\frac{10}{3}$.
17. (c) 105.9°, 90°, 111.8°, 52.3°.

Art. 1.9, pages 22–23

1. (a) 9. (c) 29.9. (e) 15.
3. (3, 0) and (10.2, 0).

Art. 1.10, pages 24–25

1. (a) 6.32, −6. (b) $-\frac{1}{3}$, 0. (c) −76.0°, 90°. (d) 57.5°. (g) 34.
2. m = 30,000,000; 89° 59′ 59.993″.
3. 1.8.
5. (c) 1100 sq. ft.

Art. 2.1, pages 28–29

3. $a = \frac{1}{3}$.
*5. Goes through (4, 2).
*7. Goes through (8, −10).
9. 3.32.
*11. Goes through (π, −1.5).
*13. Goes through (−1, −5).
*15. Equation should be satisfied by the three given sets of values for (x, y, z).

* Partial answer.

ART. 2.2, PAGES 34–36

1. (a) $x = 3$.
*(c) Locus goes through (1, 5.46).
(e) $16x^2 + 25y^2 = 400$.
*(g) Locus crosses x-axis at $x = \pm 12$.
*(i) Locus goes through (6, 2) and (2, 2). (k) $2xy = 1$.

*3. Locus crosses x-axis at $x = 5$.

*5. Locus goes through (4, 15). Note that the locus does not include (2, 3) and (0, 3) which satisfy the final equation but do not satisfy the first equation.

6. (a) $y = 2x + 3$.
*(c) Locus goes through (0, −2).
*(e) Locus goes through (3, 9).
(g) $y - y_1 = m(x - x_1)$.
*(i) Line goes through the two given points.

11. $x^2 + y^2 - 4x - 4 = 0$ if $x > 0$, $x^2 + y^2 + 4x - 4 = 0$ if $x < 0$.

13. (a) $x = r\cos\theta$, $y = r\sin\theta$.
(c) $L^2 = r^2 + z^2 - 2xz$.

ART. 2.6, PAGES 40–42

1. (a) $y = 4x - 11$.
*(c) Goes through (8, −12).
(e) $y \approx 2.53x - 12.1$.
(g) Abscissa = 3.
*(i) Line goes through the two given points.
*(k) The x-intercept is 10.5.
*(m) y-intercept is 25.

2. (a) 63.4°. (c) −71.6°.
(e) −14.0°. (g) −59.0°.
(i) 24.9°.

3.*(a) y-intercept is −6.93.
(c) $x + 3y + 4 = 0$.
(e) $y \approx \pm 3.87x - 2$.
(g) Ordinate = −2.
(i) $y \approx 5710x - 5.59$.

5. $A = 0.8$.

7. $A = 23.2$ sq. in.

9. (a) $3A + C = 0$. (c) $A = 0$.
(e) $A = 0$.

* Partial answer.

ART. 2.6 (CONT.), PAGES 40–42

10. (a) 63.4°. (c) 31.3°. (e) 90°.

11. $F = C = -40°$.

13. (a) $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

14. (a) $(-7/24, 5/12)$.
(c) (1, −2).

ART. 2.7, PAGES 45–47

1. (a) 6.6. (c) 4.2. (e) −3.
(g) −2.8.

2. (a) 7.42. (c) 10. (e) −21.2.

3. (0, 6), (0, −8/3).

4. (a) 5.14. (c) 2.06.

*5. y-intercepts of locus are 2 and −6.

7. (a) $x - y = 4$, $7x + 7y = 20$.
(c) $7x + 12 = 4y$, $4x + 7y = 21$.

8. (a) $x + 10 = 2y$, $9x + 2y = 0$, $x + 4y = 17$.
(e) $r = 3.5$.

9. (a) $x + y = 0$, $x = 3y$, $2x = y$.

10. (a) Area = 6.5 sq. units.

11. 4, −1.6.

ART. 2.8, PAGES 49–50

1. (a) $y - 5 = m(x - 2)$.
(c) $y = m(x - 9)$.
(e) $3x + 2y = k$.

3. $4x + 5y + 6 = 0$.

5. (a) $2x + 3y = 5$.
(c) $2x - y = 9$.
*(e) Locus goes through (8, 5).
(g) $x = 4$, $15x = 8y + 68$.

9. (a) $2x + 10 = 43y$.
(c) $x + 1.95y \approx 4.38$, $x - 1.20y \approx 3.12$.

ART. 2.9, PAGES 51–53

1. 300 in.-lb.

3. $h = 106{,}000 - 500T$.

4. (e) 73° 44′.
(i) $y = 2$, $4x + 3y = 5$, $4x + 7 = 3y$.
(k) $y = 2$, $9x + 2 = 4y$, $9x + 4y = 14$.
(m) Area = 12.

Art. 2.9 (cont.), pages 51–53

*5. Goes through (2, 3).
7. 84.6°, −3.1°.
9. $T = 10 + 7.5x$.
11. (b) $D(3, -4)$.
(c) $E(1, -\frac{4}{3})$.
(d) $F(0, 0)$.
(e) $G(1, -2)$.
(f) $R = 5$. (g) $r = 2$.
(i) 53° 8′, 90°, 36° 52′.
13. 10.5.

Art. 3.7, pages 65–68

1. (a) $x = \pm 2$. (c) $x = \pm 4$.
2. (a) y-axis. (c) Origin. (e) y-axis.
5. (a) $x = 2, y = 4$.
(c) $x = 3, y = 0$.
(e) $t = 1, 2, s = 1$.
11. $A = 2X(4 - X^2)$.

Art. 3.8, pages 70–71

6. (c) 67.6.
11. (1.3, 2.2).
12. (a) 1.3. (c) 1.4. (e) 2.2.

Art. 3.9, pages 73–74

13. $A = 2\pi r^2 + 160/r$.
15. $x \approx 0.5$.

Art. 3.10, pages 76–77

1. $2x' + 3y' = 0$.
5. $2x' + y' = 0, 5x' + 3y' = 0$.
6. (a) $y' + 3 = 4 \cos 2x'$.
(c) $y' = 5 \log x'$.
9. $x'^4 + x'^2y'^2 = y'^2$.

Art. 3.13, pages 82–83

1. (a) $y = 4$.
(b) (1) Origin. (3) y-axis.
(e) $x = 2, x = -2, y = 0$.
11. Three.
13. Approx. 1000.

Art. 4.1, pages 87–89

1. (a) $(-3, 2), r = 4$.
(c) $(\frac{1}{2}, -\frac{1}{4}), r = \frac{1}{4}$.

* Partial answer.

Art. 4.1 (cont.), pages 87–89

(e) $(1.27, 1.64), r \approx 2.45$.
(g) $(p = 10, v = 25), r \approx 39.9$.
(i) $(s = -2, t = -\frac{1}{2}), r \approx 3.04$.
2. (a) $x^2 + y^2 - 4x + 6y = 12$.
*(c) Crosses $x = 2$ at $y \approx 6.83$ and $y \approx 1.17$.
*(e) y-intercepts are $y = 1, 3$.
*(g) Goes through (−3, 2).
*(i) One y-intercept is 3.46.
3. (a) $x^2 + y^2 + 6x - 20y + 9 = 0$.
*(c) y-intercepts are 2, 7.
(e) $x^2 + y^2 - 8.61x - 3.25y \approx -10.95$.
*5. Goes through $(0, 1\frac{1}{3})$.
7. (a) $x^2 + y^2 + 4y = 3x$.
*(c) One x-intercept is $x = 6$.
8. (a) $(x - 1)^2 + (y + 2)^2 = 25$.
*(c) Center is at (2, 2).
*9. Goes through (7, 2).
10. (a) 8π. (c) 32π.
*11. Goes through (−2, 2).
15. 10 in.
17. $x = (b^2 - a^2)/r$.
18. Error is less than 0.005%.

Art. 4.2, pages 93–94

1. (a) $(x - 2)^2 + (y + 3)^2 = r^2$.
(c) $(x - h)^2 + y^2 = h^2 + 4$.
(e) $(x - h)^2 + (y - 2h)^2 = (2.2h - 2)^2$.
2. (a) $5x^2 + 5y^2 = 24x + 3.5y$.
(c) $(x - 24)(x - 2) + (y - 5)(y - 1) = 0$.
*(e) One y-intercept is $-\frac{23}{7}$.
3. (a) $(i, i), (-i, -i)$.
*(c) Other x-intercept is $x = 2$.
4. (a) $4x + 5y = 9$, common chord.
(c) $8x + 12y = 21$, radical axis.
(e) $21x + 34y = 21$, common chord.
5. 3.
*7. $x + 7y = 3$, circles are tangent.

Art. 4.3, pages 98–101

4. (a) $y^2 = 6.25x$.
*(c) Goes through (−8, 3).

ART. 4.3 (CONT.), PAGES 98–101

*(e) Goes through (12, 12).
(g) $y^2 = -12x$.
5. 20.8 ft.
7. $s = 16t^2$.
8. (a) $y = 3 + 2x - x^2$.
*(c) Goes through (−2, 2).
(e) Linear data.
9. 21 approx.
10. (a) $V(3, -2)$, $F(4.5, -2)$, directrix: $x = 1.5$.
(c) $V(-1, 1)$, $F(-3, 1)$, directrix: $x = 1$.
11. $2p$.
12. (a) 6.93.
13. (a) 0.25 ft. (b) 0.25 ft. approx.
(c) 0.5 ft.
14. (a) $x = 2y^2 - 4y + 2$.
*(c) Goes through the three given points.
15. (b) $R = (v^2/g) \sin 2\theta$.
(c) $(v^2/2g) \sin^2 \theta$.
(d) $y = v^2/(2g)$.
17. 42.5 ft.
19. (a) (1, −4). (c) $(-\frac{3}{2}, -\frac{5}{8})$.
21. $0 < N < 500$, $R = 4N$; $500 < N$, $R = 6.5N - 0.005N^2$; R is largest when $N = 650$.
23. (a) $(t/4)(8 + 2y - y^2)$.
(b) $(t/32)(16y + 12y^2 - y^4)$.

ART. 4.4, PAGES 106–108

2. (a) $64x^2 + y^2 = 1600$.
*(c) Check with given points.
(e) $7x^2 + 16y^2 = 448$.
*(g) Ends of minor axis at (7.46, −1) and (0.54, −1).
*(i) Eccentricity is approximately 0.45.
*(k) One vertex is at (7.07, −1).
5. $16x^2 + 25y^2 = 256 + 96x$.
7. (a) $25x^2 + 9y^2 = 90y$.
(b) $y = 2\sqrt{100 - x^2}$.
(c) $5x^2 + 9y^2 = 100 + 40x$.
9. 4.90.
11. 2.28.

* Partial answer.

ART. 4.4 (CONT.), PAGES 106–108

12. (a) $2a = 10$; $2b = 6$; $F: (\pm 4, 0)$; $V: (\pm 5, 0)$.
(c) $2a = 5$; $2b = 4$; $F: (\frac{5}{2}, -2)$, $(-\frac{1}{2}, -2)$; $V: (\frac{7}{2}, -2)$, $(-\frac{3}{2}, -2)$.
(e) $2a = 8$; $2b = 4$; $V: (6, -1)$, $(-2, -1)$; $F: (5.46, -1)$, $(-1.46, -1)$.
13. $-\frac{8}{5}$.
15. (a) $a = (\frac{1}{2}) - (z/288)$, $b = (\frac{3}{8}) - (z/288)$.
(b) Area $= k(144 - z)(108 - z)$, $k = \pi/82{,}944$.
19. (0.707, 1.32), etc.

ART. 4.6, PAGES 115–117

3. (a) $a = \frac{3}{2}$, $b = 2$, $c = \frac{5}{2}$.
(b) $C: (1, 2)$; $V: (1, 3.5), (1, 0.5)$; $F: (1, 4.5), (1, -0.5)$.
(c) $3x - 4y + 5 = 0$, $3x + 4y = 11$.
4. (a) $C: (-1, 1)$; $V: (1, 1), (-3, 1)$; $F: (1.83, 1), (-3.83, 1)$; $x + y = 0$, $x - y + 2 = 0$.
(c) $V: (2, 2), (-2, -2)$; $F: (\sqrt{8}, \sqrt{8}), (-\sqrt{8}, -\sqrt{8})$; $y = 0$, $x = 0$.
5. (a) $49y^2 = 25x^2 + 1225$.
*(c) Goes through the given vertices.
(e) $8x^2 - 9y^2 = 119$.
*(g) Goes through $(\frac{20}{3}, 8)$.
*(i) Goes through (1, −14).
*(k) Goes through (5, 6).
7. $x = \sqrt{(4y^2 + 27)/12}$.
9. 5.41.
11. $y = 250 + (\frac{3}{4})\sqrt{x^2 + 40{,}000}$.

ART. 4.7, PAGE 120

1. (a) $9(x - \frac{16}{3})^2 + 12y^2 = 64$.
(c) $9(x + \frac{4}{3})^2 - 3y^2 = 64$.
2. (a) Vertex (0, 0); focus (0, 2); directrix $y = -2$; eccentricity $= 1$.

ART. 4.7 (CONT.), PAGE 120

(c) Center (0, 0); vertices (4, 0), (−4, 0); foci $(2\sqrt{5}, 0)$, $(-2\sqrt{5}, 0)$; directrices $x = \pm 8/\sqrt{5}$; asymptotes $2y = \pm x$; eccentricity $= \frac{\sqrt{5}}{2}$.

(e) Center (0, 0); vertices $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$; foci (2, 2), (−2, −2); eccentricity $= \sqrt{2}$; directrices $x + y = 2$, $x + y = -2$; asymptotes $x = 0$, $y = 0$.

(g) Center (1, −1); vertices (1, 4), (1, −6); foci (1, 3), (1, −5); eccentricity = $\frac{4}{5}$; directrices $y = \frac{21}{4}$, $y = -\frac{29}{4}$.

(i) Center (1, −1); vertices (1, 2), (1, −4); foci (1, 4), (1, −6); eccentricity = $\frac{5}{3}$; directrices $y = \frac{4}{5}$, $y = -\frac{14}{5}$; asymptotes $3x - 4y = 7$, $3x + 4y + 1 = 0$.

ART. 4.8, PAGES 122–123

1. (a) Parabola. (c) Ellipse.
   (e) Ellipse. (g) Two lines.
   (i) Hyperbola. (k) Hyperbola.
2. (a) Ellipse. (c) Parabola.
   (e) Straight line. (g) Hyperbola.
   (i) Hyperbola. (k) Hyperbola.
   (m) Hyperbola.
3. $3x + 1 = 2y$, $x + 2y + 3 = 0$.

ART. 4.9, PAGES 124–125

1. $x' = x \cos\theta + y \sin\theta$,
   $y' = -x \sin\theta + y \cos\theta$.
3. $5x' = 6$.
4. $C: x' = y' \approx 4.12$.
5. $5x'^2 + 2y'^2 = 10$.
7. (1.36, 5.21).

* Partial answer.

ART. 4.10, PAGES 129–130

1. $4x''^2 - y''^2 = 16$.
2. (a) $x''^2 + 3y''^2 = 72$.
   (c) $4x'^2 - y'^2 = 4$.
   (e) $4x''^2 - y''^2 = 4$.
   (g) $2x''^2 + y''^2 = 6$.
3. (a) $x''^2 + 5.83y''^2 \approx 17.28$.
   (c) $x''^2 - 10.91y''^2 \approx 122.2$.
5. $-30°$ or $+60°$.
6. (a) $5x = 4x'' + 3y'' + 10$, $5y = -3x'' + 4y'' + 5$.
   (c) $2x + 11y = 15$, $2x + y = 5$.
7. (4.4, −0.8).
9. $y = 4x$, $x = 0$.

ART. 5.3, PAGES 141–143

6. (a) $y = 50 \sin(3x + \phi)$, where $\phi \approx 36.9° \approx 0.643$ rad.
   (c) $F = 100 \sin(100\pi t - \phi)$, where $\phi \approx 53.1° \approx 0.928$ rad.
10. (a) 4. (c) 0.13.

ART. 5.4, PAGE 146

3. 8.

ART. 5.6, PAGES 151–153

9. $T = -460°$ F., $L \approx 39.8$ in.
11. (a) $a = 100$, $b \approx -0.266$.
    (c) $a \approx 0.877$, $b \approx -0.334$.
13. (b) 40 ft.

ART. 5.7, PAGES 155–156

5. (b) $|A/B| = |B/C| = e^{0.4}$.
   (c) $|A/B| = e^{b\pi/a}$.

ART. 5.8, PAGES 158–159

1. (a) $x = 0, 1.66, -1.66$.
   (c) $x \approx -1.13, 4.43$, etc.
   (e) $x \approx 2.10$.
   (g) $x \approx 2.3, 2.8, 4.1, 4.9, 6.1, 7.0$, etc.
3. 3.5.
5. (a) 6.1. (c) 0.8. (e) 1, 4, 4.
7. −2.4, 1.

ART. 6.3, PAGES 171–173

7. About 2 sq. units.
9. $\theta = 45°$, $m = -1$.

Art. 6.4, pages 174–175

1. (a) $x^2 + y^2 = 4x$.
 *(c) Locus goes through $x = 3$, $y = 17$.
 *(e) Parabola with focus at (0, 0).
 (g) $y = -x\sqrt{3}$.
2. (a) $r^2 \sin 2\theta = 4$.
 (c) $r^2 \cos 2\theta = 12$.
 *(e) Locus goes through $r = 5$, $\theta = 0°$ and $r = 5/3$, $\theta = \pi/2$.
 (g) $r = 2/(1 - \cos\theta)$ or $r = -2/(1 + \cos\theta)$.
3. (a) $(x^2 + y^2)^2 = 8xy$.
 (c) $b^2x^2 = (x - a)^2(x^2 + y^2)$.
 (e) $(x^2 + y^2)^2 = 4x$.
 (g) $(x^2 + y^2)^3 = 4(x^2 - y^2)^2$.
 (i) $(x^2 + y^2 - x)^2 = x^2 + y^2$.
9. 2π.
11. (a) $1/60$ sec. (c) 117.

Art. 6.5, pages 177–178

6. (a) $0 \leqq \theta \leqq \pi$, $r = 2 + (2/\pi)\theta$, $x^2 + y^2 = (2 + 2/\pi \text{ arc tan } y/x)^2$; $\pi \leqq \theta \leqq 2\pi$, $r = 6 - (2/\pi)\theta$, $x^2 + y^2 = (6 - 2/\pi \text{ arc tan } y/x)^2$.
 (c) $0 < \theta \leqq \pi$, $r = 2 + (2/\pi)\theta$, $\pi \leqq \theta < 2\pi$, $r = 4$.
 (e) $r = a + 2\sin\theta$, $(x^2 + y^2 - 2y)^2 = a^2(x^2 + y^2)$.

Art. 6.6, pages 179–180

1. $r\cos\theta = 4$.
*3. Goes through $r = 4/3$, $\theta = \pi$.
*5. Goes through (8, 0°) and (2, 180°).
7. $r = 2a\cos\theta$.
9. $r = 4/(1 + \cos\theta)$.
11. (a) $r^2 = 2a^2\cos 2\theta$.
 (c) $(r - a\sec\theta)^2 = b^2$.

Art. 6.7, page 184

1. (a) (2.5, 60°), (2.5, −60°), etc.
 (c) (4.47, 63.4°), (−4.47, 243.4°), etc.
 (e) (0.5, 30°), (0.5, 150°), (−1, 270°), etc.

Art. 6.7 (cont.), page 184

 (g) (2, 15°), (2, 75°), etc.
 (i) (1, 7.2°), (1, 82.8°), etc.
3. Area ≈ 2.8.
5. (2, 30°), (2, 150°), area ≈ 9 sq. units.
7. −1.33.

Art. 6.8, page 185

1. (2, 0), (2, 2π), etc.
5. $r = 2/(1 - \cos\theta)$.
11. $(r^2 + 16)^2 - 64r^2\cos^2\theta = 100$.

Art. 7.1, pages 189–190

1. $x = 14$, $y = 14/3$.
3. (a) $2x + 3y = 13$.
 (c) $y^2 - x^2 = 9$.
 (e) $x + y = 4$.
 (g) $y \approx 0.577x - 0.000{,}134x^2$.
 (i) $x + 2y^2 = 4$.
5. (a) $x + y = 4$.
 (c) $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
 (e) $2x + y = 4$.
 (g) $x^2 + 16y^2 = 16$.
6. (a) $x = 2\cos\phi$, $y = 2\sin\phi$.
 (c) $x = -4k/(1 + k^2)$, $y = (2 - 2k^2)/(1 + k^2)$.
 (e) $x = 2\sec\theta$, $y = 2\tan\theta$.

Art. 7.2, page 192

5. $y = 10\,|\cos 100\pi t|$.

Art. 7.4, pages 194–198

1. $x = a\cos\theta$, $y = b\sin\theta$.
3. $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.
5. (a) $x \approx 2 + 0.316q$, $y \approx 13 + 0.949q$.
 (c) $x \approx 5 + 0.848q$, $y \approx 2 + 0.530q$.

Art. 8.3, pages 204–206

1. (a) $y = 2x + 3$.
 (c) $9C = 5(F - 32)$.
2. (a) $y = x^2 + 2x + 3$.
 (c) $y = 4x - x^2$.
 (e) $y = x^2(10 - x^2)$.
3. 139.80, 139.74.

* Partial answer.

ART. 8.3 (CONT.), PAGES 204–206

5. $\sin 1.007 \approx 0.84523$,
$\sin 1.018 \approx 0.85106$.

*7. Goes through the three given points.

ART. 8.5, PAGES 209–211

2. (a) $y \approx 13.8 + 1.10x$.
(c) $F \approx 2.0 + 0.38w$.

3. $s \approx 30{,}000{,}000$.

5. $W \approx 128 + 0.80T$.

7. $R \approx 70 + 0.30T$.

8. (a) $y \approx 1.43 + 0.57x + 0.14x^2$.

ART. 8.7, PAGES 217–220

1. (a) $y = 2x$. (c) $y \approx 3.58x^{0.160}$.

2. (a) $y \approx 8.0e^{-0.52x}$.
(c) $y \approx 0.50e^{-0.46x}$.

3. (a) $y \approx 0.60x^{0.50}$.
(c) $L \approx 2.5m^{0.34}$.
(e) $v \approx 8.0h^{0.50}$.

4. (a) $y \approx 2.05e^{0.092x}$.
(c) $y \approx 72e^{-0.011x}$.
(e) $h \approx 3.66e^{-0.0024t}$.

5. $d^3 = t^2$.

7. $A \approx 0.18H^{2.75}$, 21,000 acre-ft.

9. (a) $a \approx 0.477$, $b \approx 1.10$,
$c \approx 0.903$, $d \approx 2.08$.

11. $Q \approx 2.65H^{2.5}$.

13. $y \approx 3.14x^2$.

15. $\log p \approx 7.765 - 1700/(T + 273)$.

17. $pv^{1.2} \approx 60$.

19. (a) $Y = 1/y$, $X = 1/x$.
(c) $Y = \ln y$, $X = x^2$.
(e) $X = x^2$, $Y = y$.

ART. 9.3, PAGES 230–231

1. (a) $\alpha \approx 56.2°$, $\beta \approx 68.2°$,
$\gamma \approx 42.0°$, $(OP)_x = 3$,
$(OP)_y = 2$, $(OP)_z = 4$.
(c) $\alpha \approx 31.0°$, $\beta \approx 115.4°$,
$\gamma \approx 73.4°$, $(OP)_x = 6$,
$(OP)_y = -3$, $(OP)_z = 2$.

3. $F_x = 120$ lb., $F_y = -40$ lb.,
$F_z = 60$ lb.

* Partial answer.

ART. 9.3 (CONT.), PAGES 230–231

5. $\theta_{\text{east}} \approx 79.0°$, $\theta_{\text{south}} \approx 76.2°$,
$\theta_{\text{down}} \approx 17.8°$.

7. 45°, 49.5 ft.

9. 47.0 ft.; 50.2°, 64.8°, 50.2°.

ART. 9.5, PAGES 235–237

1. (a) $\alpha \approx 48.2°$, $\beta \approx 48.2°$,
$\gamma \approx 70.5°$.
(c) $\alpha \approx 38.7°$, $\beta = 90°$,
$\gamma \approx 51.3°$.
(e) $\alpha \approx 131.8°$, $\beta \approx 48.2°$,
$\gamma \approx 70.5°$.

3. $L \approx 34.4$ ft., $\alpha \approx 125.7°$,
$\beta \approx 107.0°$, $\gamma \approx 40.7°$.

5. $\theta_{EC} \approx 51.3°$, $\theta_{DC} \approx 48.2°$.

7. $\theta_{\text{east}} \approx 22.8°$, $\theta_{\text{south}} \approx 72.1°$,
$\theta_{\text{down}} \approx 76.7°$.

9. (a) Are collinear.
(c) Not collinear.
(e) Not collinear.

11. $V = 192$ cu. units.
L.S.A. $= 24\sqrt{41}$ sq. units.

13. (a) $(3, 4, -2)$. (c) $(2, 0, 0)$.

14. (a) $\overline{FB}$: 90°, 135°, 45°.

(c) $FB: \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(e) 69° 17′.

15. 180 cu. units. ¼.

ART. 9.6, PAGES 239–241

1. 45°.

3. 61.1°.

4. (a) Perpendicular.
(c) Perpendicular. (e) Neither.

7. (a) 110.9°, 34.5°, 34.5°.
(c) 60.0°, 48.0°, 72.0°.
(e) 75.4°, 81.6°, 23.0°.

9. 87.0°.

10. (a) 98.0 lb., 65.9°, 65.9°, 35.3°.

11. 101.6°.

ART. 9.9, PAGES 248–249

1. (a) $\{3, 5, 4\}$. (c) $\{1, 1, 0\}$.
(e) $\{0, 1, 0\}$. (g) $\{1, 1, -2\}$.
(i) $\{1, 2, 0\}$.

ART. 9.9 (CONT.), PAGES 248–249

3. (a) $x + 2y + z = 6$.
*(c) Goes through (1, 1, 2).
*(e) x-intercept is 2, y-intercept is 4.
9. 24.6 sq. units.
11. 68.2°, 56.1°, 42.0°.
12. (a) $2x + 5y + 7z = 47$.
*(c) Goes through (7, 5, 6).
13. 52.0°.
15. (1, 3, −2).
16. (a) 5.33. (c) 2.61.
17. $^{12}/_7$.

ART. 9.11, PAGES 255–256

1. (a) $(\frac{8}{3}, 0, \frac{2}{3})$, (2, 1, 0), (0, 4, −2).
(c) $(-4, 5, 0)$, $(0, \frac{7}{3}, \frac{4}{3})$, $(\frac{7}{2}, 0, \frac{5}{2})$.
(e) $(0, \frac{4}{3}, \frac{4}{3})$, (1, 0, 3), $(-\frac{4}{5}, \frac{12}{5}, 0)$.
2. (a) 61.0°, 136.7°, 61.0°.
(c) 36.7°, 122.3°, 74.5°.
(e) 64.9°, 124.4°, 45.0°.
3. (a) $x - z = 2$, $3x + 2y = 8$, $2y + 3z = 2$.
(c) $2x + 3y = 7$, $x - 3z = -4$, $y + 2z = 5$.
(e) $4x + 3y = 4$, $5x - 3z = -4$, $5y + 4z = 12$.
5. (a) $\dfrac{x-1}{2} = \dfrac{y-4}{-3}$, $\dfrac{x-1}{2} = \dfrac{z-6}{2}$.
*(c) Both planes go through (5, −5, 10).
*(e) Both planes go through (0, −1, −3).
*7. Both planes go through (6, −1, 15).
9. (b) (2, 1, 5). (c) (6, −2, 1).
(d) 128.7°, 62.1°, 51.3°.
*(f) Check by use of results in (b).

* Partial answer.

ART. 9.12, PAGES 256–258

1. (a) 90°, 56.3°, 146.3°.
(b) 135°, 90°, 45°. (d) 54.0°.
2. 29.2°, 102.6°, 64.1°.
3. (a) 54.7°, 125.3°, 125.3°.
(c) 80.7°.
4. (a) 53.4°.
(b) 13.5 sq. units.
*(c) Goes through the three given points.
(e) $(\frac{12}{29}, -\frac{8}{29}, \frac{16}{29})$.
5. (a) 3.90.
6. (c) 74.5°, 57.7°, 36.7°.
(d) $z = 0$, $x + 2y = 6$.
(e) 1.60.
(f) $(\frac{3}{7}, \frac{6}{7}, \frac{9}{7})$.
7. (c) $3x + 2z = 6$, $7x - 4y = 6$, $6y + 7z = 12$.
(e) 113.5°, 134.1°, 53.3°.
8. (a) Point.
(c) No intersection.
9. 3.84.
10. 5.20.

ART. 10.3, PAGE 263

1. (a) Elliptical cylinder.
(c) Exponential cylinder.
(e) Parabolic cylinder.
(g) Sinusoidal cylinder.
(i) Semicircular cylinder.
(k) Hyperbolic cylinder.
(m) Hyperbolic cylinder.
3. 75.4 cu. units.
4. (a) All three coordinate planes, all three axes, origin.
(c) xy-plane, z-axis, origin.
(e) Origin.

ART. 10.6, PAGES 268–269

1. (a) Circle. (c) Straight line.
2. (a) Elliptical cylinder.
(c) Elliptical cone. (e) Sphere.
(g) Sphere.
5. (a) (−2, 1, −3), $r = 4$.
(c) (2, 0, 3), $r \approx 4.47$.
(e) $(-1, -\frac{3}{2}, 2)$, $r \approx 2.96$.
9. *(a) Locus goes through (7, −6, 6).

ART. 10.6 (CONT.), PAGES 268–269

*(c) x-intercepts are approximately 5.87 and -1.87.

*(e) Locus goes through $(2.80, -1, 1)$ approximately.

10. (a) $x^2 + y^2 + z^2 = 4$.

(c) $y^2 + z^2 = 4x$.

(e) $[(x-6)^2 + y^2 + z^2 + 32]^2 = 144\,[(x-6)^2 + z^2]$.

(g) $x^2 + y^2 = e^{2z}$.

11. 28.3 cu. units.

ART. 10.9, PAGES 275–277

3. (a) Parabolic cylinder.

(c) Circular cone.

(e) Paraboloid of revolution.

(g) Hyperboloid of revolution of one sheet.

(i) Hyperbolic paraboloid.

(k) Elliptic paraboloid.

(m) Hyperboloid of revolution of two sheets.

(o) Paraboloid of revolution.

(q) Elliptic hyperboloid of two sheets.

8. (c) Area $= 2\pi z$.

9. r, $0.155r$.

* Partial answer.

ART. 10.13, PAGES 283–284

2. (c) $r^2 + z^2 = 16$, $r = 3$.

3. $(0, 1, 1)$, $(2, 0, 2)$, $(-2, 0, 2)$.

5. $(0, 1, 1)$, $(0, -3, 3)$, $(1.5, 0, 1.5)$, $(-1.5, 0, 1.5)$.

7. 28.1 units.

9. $16\pi(\sqrt{2} - 1)$.

10. (a) $r = 2$, $\rho \sin \phi = 2$.

(c) $r \sin \theta \tan \theta = 3$, $\rho \sin \phi \sin \theta \tan \theta = 3$.

(e) $r^2 \sin 2\theta = 4$, $\rho^2(1 - \cos 2\phi)(\sin 2\theta) = 8$.

11. (a) $x^2 + y^2 = 4$.

(c) $(x^2 + y^2)^2 = 4(x^2 - y^2)$.

12. (a) $x^2 + y^2 + z^2 = 4$.

(c) $y^2 + z^2 = 4$.

(e) $x^2 + y^2 \approx 2.43z^2$.

14. (a) Circular cylinder.

(c) Cone.

(e) Paraboloid of revolution.

15. (a) Sphere. (c) Plane.

(e) Plane.

ART. 10.14, PAGES 287–288

3. $b = 9$, $a = 4/9$.

7. $(y - 4)(x - 4) = 8z$.

9. $x^2 - 4y^2 = 32z$.

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