THE FOUNDATIONS OF GEOMETRY

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MATHEMATICAL EXPOSITIONS, No. 1

THE FOUNDATIONS OF GEOMETRY

by

GILBERT DE B. ROBINSON
Associate Professor of Mathematics
University of Toronto

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PREFACE

SPEAKING generally, geometry is the study of those relations which hold between the various elements in terms of which we may describe our concept of space. In *The Foundations of Geometry* we are concerned largely with the "disentangling" of these relations, with their classification and reduction to axiomatic form. The application of the axiomatic method to other branches of mathematics is, perhaps, the most significant development of the last hundred years; in algebra, particularly, the results have been spectacular. Such an abstract point of view leads to broad generalization and to a means of correlating much, if not all, of modern mathematics. The rôle of geometry in such a scheme is to provide a more concrete background in which the intuition may have free rein.

The aim of the present volume is to set out as briefly and clearly as possible the principal stages in the "disentangling" process, with the hope of making these stimulating ideas available to mathematicians and scientists generally.

The book is divided into two parts. In Part I we shall consider the axiomatic foundation of projective geometry and of Euclidean geometry, and endeavour to make clear the relation between the two. Part II is largely concerned with the concept of number, order, and continuity. The gain in postponing a consideration of these more difficult questions to Part II would seem to outweigh the resulting loss of completeness in Part I. Very few specific references are given in the text, but sources for the material in the various sections are indicated in a bibliography at the end of the book.

My acknowledgements are due particularly to my colleagues Professor R. Brauer and Professor H. S. M. Coxeter, who have read the manuscript at various stages of completion and have made many very valuable criticisms and suggestions. My best thanks are also due to the other members of the Editorial Board of this series for their helpful comments and advice, and to Mr. C. E. Helwig for his care in executing the diagrams. In writing the chapter on Philosophical Preliminaries I have benefited greatly from several conversations with Professor E. A. Bott of the Department of Psychology and Professor G. S. Brett of the Department of Philosophy, to the latter of whom are due any virtues which §1.2 may possess. I am also indebted to Mr. A. J. Coleman and to Mr. W. J. R. Crosby for the assistance which they have given me in reading the proof.

In conclusion, I should like to express my appreciation to the staff of the University of Toronto Press for their friendly co-operation at all times.

G. DE B. ROBINSON

The University of Toronto,
November, 1940.

PREFACE TO SECOND EDITION

In this second edition I have taken the opportunity of rewriting chapter I. I still feel that something needs to be said along these lines, but consider this only as a "second approximation." Also, various minor errors have been corrected and one figure redrawn.

G. DE B. R.

PREFACE TO THIRD EDITION

My thanks are due to Dr. B. H. Neumann for drawing my attention to an error in §8.2, which is now corrected.

G. DE B. R.

V. AXIOMS OF EUCLIDEAN GEOMETRY IDEAL ELEMENTS

PART II

VI. NUMBER

VII. COORDINATE SYSTEMS

VIII. ORDER AND CONTINUITY

PART I

CHAPTER I

INTRODUCTION

1.1. Historical Remarks. In the study of geometry one can follow the gradual unfolding of mathematical thought from its earliest beginnings to the present time. There is hardly a branch of modern mathematics which has not at some stage a geometrical interpretation. Perhaps this is natural, since geometrical intuition has played such a large part in the development of the subject. The periods of chief geometrical interest are as follows:

The Greek Period (300 B.C. to 300 A.D.)
  Euclid, Apollonius, Archimedes and Pappus;

Cartesian Geometry and the Calculus (1620 to 1720)
  Descartes, Pascal and Desargues; Newton and Leibniz;

Non-Euclidean, Projective and Algebraic Geometry (1800—)
  Gauss, Bolyai and Lobatschefski, Poncelet, Chasles, Riemann, Cayley, Clebsch, Klein—to mention only a few names;

Foundations (1880—)
  Pasch, F. Schur, Dedekind, Peano, Pieri, Hilbert, Veblen, Moore, Menger and G. Birkhoff.†

Though the boys and girls of today study the same propositions in school as their predecessors have studied for two thousand years, yet the foundations and the superstructure of geometry are very different.

Perhaps the most important achievement of the Greeks is their development of an *abstract* geometry. The realization

†The last two names are added for completeness, though their work is not utilized.

that a "triangle" is a concept to be grasped by the mind, quite independently of any particular triangle drawn on the sand, marked a tremendous advance. But the Greeks accomplished far more than this: for example, they had a fairly complete knowledge of many of the properties of conic sections, though these are more often deduced nowadays by analytical or projective methods. It is also true that some of the fundamental concepts of the calculus were known to them. On the other hand one should guard against attributing to the Greeks too great an understanding of the subtle problems which they encountered. Continuity is a case in point. Also, the true significance of Pappus' famous theorem concerning the intersections of the cross-joins of two triads of collinear points was only discovered by Hilbert about 1900.

Though mathematics was relatively dormant during the Middle Ages, men continued to ponder Euclid's parallel postulate. At long last it began to dawn on Saccheri, Lambert, and Legendre that the assumption cannot be proved and is simply an interpretation of experiments performed in the physical world. Once this idea was grasped it was a short step to the negation of the postulate, either by asserting (a) that there are an infinite number of lines through a given point coplanar with a given line and not meeting it, or (b) that there are no such lines at all. It was Gauss who first saw the significance of these ideas. Almost simultaneously, Bolyai and Lobatschefski published their important investigations of case (a) which we now call *hyperbolic* geometry. Later Riemann studied case (b), called *elliptic* geometry. We shall have much to say regarding projective geometry in the sequel, but algebraic geometry will not concern us at all.

So far we have sketched the progress in one direction; in the other, the discovery of the calculus led to differential geometry and related subjects. One might mention in this connection the theory of sets and topology.

It is significant that the axiomatic approach originated by Euclid has spread like leaven throughout the whole of mathematics, and in studying the Foundations of Geometry one is approaching in perhaps the most natural manner the far deeper problems which lie at the Foundations of Mathematics.

1.2. Our Concept of Space. Our concept of *physical space* is the result of a desire to order our experiences of the external world. This ordering process is accompanied by successive approximations and abstractions which lead to our concept of *mathematical space*. For the physicist the *correspondence* between the data of experience and his concept of physical space is all-important. As the abstracting process continues, this correspondence becomes less significant, so that the mathematician feels free to concentrate upon the logical relations involved. In this and the following section we shall try to suggest in the briefest possible manner why it is that the point and the line play such important roles in geometry.

Our experience of the external world comes to us through our senses, and of these our sense of *vision* and our sense of *touch* seem to be most significant. The process of focussing a bundle of rays of light upon the retina of the eye suggests two fundamental notions in our concept of physical space.

(a) *The point of view.* The continued observation of the world around us, ignoring the *colour* of objects, gradually impresses on our minds the changing shapes or *apparent contours* which accompany a changing point of view.

We attempt to correlate the different images obtained in this manner by seeking for contours which are invariant with respect to *motion*, and thus are tempted to describe the external world as a coherent *continuous* whole.

(b) *The straight line.* The significance of the *invariants* of a transformation was recognized relatively late in the study of geometry, but such invariants have come more and more to play a central role. The most obvious contour which is independent of the point of view, with some exceptions, we may call a *straight line*.

In this coordinating process we become conscious of the relation of *incidence* between (a) and (b). This relationship arises through the fact that we are able to decide whether two suitable objects are *in line*. We probably begin by noticing that from certain specific points of view a small object A obscures another object B, and go on to designate these points of view as *points* of the straight line *determined* by A and B. These special points are in fact the exceptions noted above with regard to a straight line contour through A and B. We have here the origin of Euclid's requirement that a point have position but not size—position in the abstract mathematical space which we are constructing in our minds.

Some such consideration of the origins of our geometrical ideas is instructive since it emphasizes the distinction between *projective* and *metrical* geometry. The sense of touch amplifies our concept of physical space—an object is rough, smooth; soft, hard; it is also near to us or beyond our reach. This last notion of *distance* is associated with a process of comparison of length which is accomplished in a variety of ways, perhaps first of all by touch and coincidences, and later with the aid of binocular vision. Incidentally, it may be worth remarking that many of our units of measurement such as the foot, pace, hand, fathom are associated with the body and come into use through this process of direct comparison.

As suggested above, we are prone to arrive at a notion of continuity through our immediate sense perceptions. This notion proved to be of the greatest importance to the Greeks, if only that it raised all sorts of problems and paradoxes. Even as late as the eighteenth century Leibniz wrote of the "Labyrinth of the Continuum." Since the days of Dedekind and Cantor we have concluded that our sense perceptions are an insufficient guide, and we should turn to that other abstraction from reality—number—for a *definition* of continuity.

1.3. The Choice of a System of Axioms. The final stage in the process of constructing mathematical space is the choice of a set of *undefined* elements, say the point and line, and a set of relations connecting them. Our aim here will be to achieve generality along with simplicity, and this largely because of the aesthetic satisfaction which success will yield. We shall say that a system has *intrinsic simplicity* if the relations we postulate are simple, and that it has *extrinsic simplicity* if the meanings in terms of physical space which we can attribute to our undefined elements are simple.

We feel that a description of our spatial concept in terms of points and lines is simple; a system of geometry built on them would have extrinsic simplicity. On the other hand an object which we may call a sphere also has an *invariability* of contour which resembles that of a straight line. In fact, Euclidean geometry can be constructed by taking the sphere as an undefined element, and we give its axioms in an Appendix. Such a system has, however, neither the intrinsic nor the extrinsic simplicity of the more familiar one.

The assumptions concerning *congruence* form an important part of any system of axioms for geometry; the fact that the Greeks did not appreciate this point lies at the root of the difficulty involved in the method of superposition.

Euclid's initial assumptions for geometry consisted of a number of definitions, followed by a set of postulates and axioms. We shall follow the modern procedure and make the number of undefined elements and relations as small as possible. In terms of these undefined elements and relations we shall state certain unprovable propositions, which we shall call *axioms*. These axioms will be chosen so that no one is a consequence of the others: they will be *independent;* they must also be *consistent*, i.e. lead to no contradiction. We shall distinguish two types of definition, which we shall describe as "explicit" and "implicit." An *explicit definition* is introduced for convenience only: it assigns to a given combination of undefined, or previously defined, elements a suitable name. On the other hand an *implicit definition* is generating or creative. For example, in the introduction of congruence in chapter V we shall assume certain axioms which state properties of congruence; congruence itself is none other than this set of properties.

To a non-mathematician it often comes as a surprise that it is impossible to define explicitly all the terms which are used. This is not a superficial problem but lies at the root of all knowledge; it is necessary to begin somewhere, and to make progress one must clearly state those elements and relations which are undefined and those properties which are taken for granted.

CHAPTER II

PROJECTIVE GEOMETRY AND DESARGUES' THEOREM

2.1. Summary of the Chapter. We have considered some of the philosophical problems which are associated with the foundations of geometry; we now put these difficulties to one side and give all our attention to the logical development of the subject. Our observation of the world around us has led to the concepts of *point* and *line.* We must be clear that these are concepts in our minds, and not entities which we may pick up or handle. "Point" and "line" are objects of thought and our statements concerning them must be exact, not subject to those experimental errors, however small, which are familiar to us in dealing with the external world.

Briefly, we shall proceed as follows. We shall take the point and line as *undefined* elements and shall build up the concept of a *plane.* The relations which we shall postulate between these elements will imply that *any* two lines in a plane have a common point. The resulting projective geometry is a purely ideal conception which is suggested by our visual experience. After developing its more fundamental properties, however, we shall be in a position to discuss the significance of parallelism, congruence, and continuity.

2.2. Axioms of Projective Geometry. Let us consider a *class*† of undefined elements which we shall call *points.* An undefined sub-class of points we shall call a *line.* It is customary to represent points by capital letters A, B, C, . . . and lines by small letters a, b, c . . . If a point belongs to a sub-

†We shall not elaborate the notion of "class"; cf. §6.2.

class which we have called a line, we shall say that it is *on* that line; conversely, we shall say that the line is *on* or *passes through* the point. For formal statements it is convenient to use the word "on" in both senses, though "passes through" is more familiar in the latter connection. We shall say that two lines which have a point in common are *concurrent in*, or *intersect in* that point, and that any three points of a line are *collinear*.

We now make the following assumptions concerning points and lines:

I. *There are at least two distinct points.*

II. *Two distinct points A and B determine one and only one line on both A and B.*

This line we shall call AB (or BA), and speak of it as *joining* A and B. It is not difficult to prove from II that if C and D are distinct points on AB, then A and B are points on CD; also, that *two distinct lines cannot have more than one common point.*

III. *If A and B are distinct points, there is at least one point distinct from A and B on the line AB.*

IV. *If A and B are distinct points, there is at least one point not on the line AB.*

V. *If A, B, C are three non-collinear points, and D is a point on BC distinct from B and C, and E is a point on CA distinct from C and A, then there is a point F on AB such that D, E, F are collinear.*†

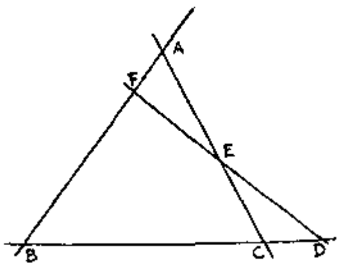


FIG. 2.2A

†Note that this would not necessarily be true in Euclidean geometry; cf. axiom 7 in §5.2. An axiom analogous to V was first used by Pasch in 1882.

The significance of axiom V lies in the fact that it enables us to define a *plane*, which is sometimes taken as a third undefined class of points.

DEFINITION. If A, B, C are three non-collinear points, the *plane* ABC is the class of points lying on lines joining C to the points of the line AB.

From this definition and the axioms I-V it may be proved that the plane ABC is equally well the class of points lying on lines joining $A(B)$ to the points of the line BC (CA); thus we may say that the plane ABC is determined by the three points A, B, C. A more symmetrical, though less convenient definition of a plane would be: ". . . the class of points lying on lines which intersect at least two of the lines AB, BC, CA in points distinct from A, B, C." We shall represent planes by small Greek letters $\alpha, \beta, \gamma, \ldots$ From axiom V it follows that, if P and Q are two distinct points of a plane α, then every point of the line PQ is a point of α; we shall say that the line is *on* or *in* the plane, and conversely, that the plane is *on* or *passes through* the line. Any four points in a plane, or any two lines in a plane, are said to be *coplanar*. The most important consequence of V is that *any two coplanar lines have a common point*, and any given plane is determined by any three non-collinear points on it. Finally, a plane is determined by any two intersecting lines. Though the proofs of these properties of a plane are interesting in themselves and are far from obvious, they are not our chief concern in the present connection.

In order to have a three (or more)-dimensional geometry we must assume that:

VI. *If A, B, C are three non-collinear points, there is at least one point D not on the plane ABC.*

As we defined a plane in terms of three non-collinear points, so we shall define a *three-dimensional space* in terms of four non-coplanar points.

DEFINITION. If A, B, C, D are four non-coplanar points, the *three-dimensional space ABCD* is the class of points lying on lines joining D to the points of the plane ABC.

Again, it follows that the three-dimensional space $ABCD$ is the class of points lying on lines joining any one of these four points to the points of the plane determined by the other three; thus we may say that the three-dimensional space $ABCD$ is determined by the four points A, B, C, D. Every point of the plane determined by three points of the three-dimensional space lies *on* or *in* the space. Any two distinct planes in a three-dimensional space have a line in common; similarly, a plane and a line not in the plane have a point in common, and three distinct planes having no common line have just one common point. Finally, a three-dimensional space is uniquely determined by any four non-coplanar points on it.

It is not long since mathematicians shrank from the consideration of a space of more than three dimensions, believing it to be a meaningless form of language. It was not realized that geometry *exists* in our minds; projective geometry, in particular, has freed itself from that external world which suggested its "terms" and "relations." It is important to remember that the *meaning* which we attribute to our undefined elements is at our own disposal and depends on the purpose for which our logical structure is intended. There is nothing to prevent our assuming the existence of at least one point not in the three-dimensional space $ABCD$. Such an axiom would, logically, be just as significant as either of the axioms IV or VI; whether a *meaning* could be attached to it in terms of the external world, is another matter. Except in the last chapter of this book, we shall limit ourselves to a space of three dimensions for two reasons. In the first place, we do not wish to be carried too far afield, and in the second place, there is a very remarkable difference between spaces of two and three dimensions, while spaces of three and more dimensions are comparatively similar in their properties. To accomplish this limitation we shall make a final assumption:

VII. *Any two distinct planes have a line in common.*

Or, what is the same thing, "Every set of five points lie on the same three-dimensional space." We shall refer to this three-dimensional space as *space*, clearly distinguishing it from our spatial concept of the world around us.

2.3. A Finite Geometry. It is a most significant fact that the axioms of projective geometry which we have given, do not require that the number of points should be "infinite." One cannot help wondering whether a system containing a *finite* number of points might not be constructed, which would satisfy all our assumptions;—remember, a line is an undefined class of points! This question was answered in the affirmative by Fano in 1892, and we shall describe a geometry due to him.

Let us suppose that the fifteen symbols:

(ab), (ac), (ad), (ae), (af), (bc), (bd), (be), (bf), (cd), (ce), (cf), (de), (df), (ef),

where (ij) = (ji), represent *points*. There are thirty-five *lines*, each containing three and only three points. These lines are of two types:

(i) A line of *Type I* contains three points of the form (ab), (bc), (ca), and there are twenty such lines;
(ii) A line of *Type II* contains three points of the form (ab), (cd), (ef), and there are fifteen lines of this type.

Any triad of points, not of one of these two types, determines a *plane*; there are fifteen planes, each containing seven points and seven lines. The accompanying Fig. 2.3A† shows the arrangement of the points and lines in a plane.

†Note that the "circle" represents a line in the finite geometry.

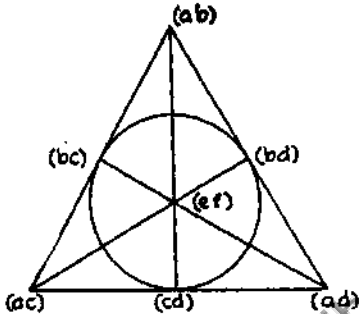


FIG. 2.3A

This system of points, lines, and planes is called a *finite projective geometry*, and is represented in the notation of Veblen and Bussey by the symbol PG(3,2): a notation which will be explained in chapter VII.

The first conclusion which may be drawn from the existence of Fano's finite projective geometry is that the axioms I-VII are consistent with one another.† The second conclusion is that if we are anxious for our geometry to resemble our spatial concept at all, it will be necessary to introduce further axioms.

2.4. Desargues' Theorem. Any three points A, B, C which do not lie on a line are the *vertices* of a *triangle* ABC, and the lines AB, BC, CA are the *sides* of the triangle. We prove now

DESARGUES' THEOREM

If two triangles ABC, $A'B'C'$ are situated in the same plane or in different planes and are such that BC, $B'C'$ meet in L, CA, $C'A'$ meet in M, and AB, $A'B'$ meet in N, where L, M, N are collinear, then AA', BB', CC' are concurrent, and conversely.

†Cf. §8.9.

The full significance of this theorem, which is named after its discoverer Desargues (1593-1662), has only been appreciated in recent years. While it may happen that a vertex of one triangle lies on a side of the other, or the triangles may be further specialized, the accompanying complications are not serious, and we need not go into the different possible cases which arise.

(i) First, let us suppose that the two triangles are in different planes. If we denote the plane containing the triangle ABC by π and the plane containing $A'B'C'$ by π', then the three points L, M, N lie on the line of intersection l of π and π', as in the accompanying Fig. 2.4A.

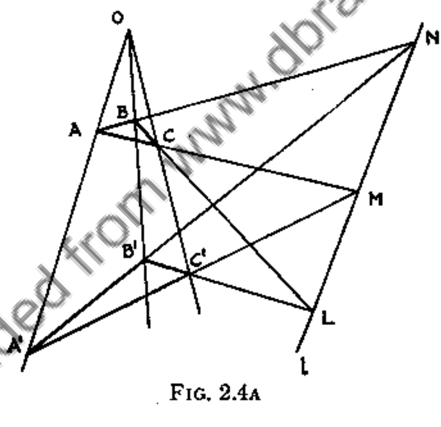


FIG. 2.4A

Evidently, A,A',B,B' are coplanar,† as also are B,B',C,C' and C,C',A,A'. But these three planes must have a point O in common; hence, the three lines of intersection AA', BB', CC' are concurrent, and the two triangles ABC, $A'B'C'$ are said to be *in perspective* from O. The converse theorem follows by reversing the argument.

†The 10 points and lines of this figure are just the points and lines of intersection of 5 planes, 3 through O and 2 through l.

(ii) If the two planes π and π' coincide and L, M, N lie on a line l in π, let us take a plane π_1 distinct from π and passing through l, and take a point P not on π or π_1. If PA', PB', PC' meet π_1 in A_1, B_1, C_1, then A_1, A', B_1, B' are coplanar, as are also B_1, B', C_1, C' and C_1, C', A_1, A'. Thus, since BC, $B'C'$ and B_1C_1 meet in L, CA, $C'A'$ and C_1A_1 meet in M, and AB, $A'B'$ and A_1B_1 meet in N, the two triangles ABC, $A_1B_1C_1$ satisfy the conditions of case (i), and we conclude that AA_1, BB_1, CC_1 are concurrent in some point O'. If we project the plane π_1 from P back on to the plane π, the lines AA_1, BB_1, CC_1 project into the lines AA', BB', CC', and the point O' projects into a point O in π. Thus the triangles ABC, $A'B'C'$ are in perspective from O, which is what we wanted to prove. The converse theorem is equivalent to the direct theorem applied to the triangles $AA'M$, $BB'L$. It is an interesting fact that *without further assumption, no proof is possible if we confine our attention to the plane containing the two triangles.* To justify this statement we shall construct a plane geometry in §8.8 in which Desargues' Theorem is not valid.

Is there a Desargues' Theorem in the finite projective geometry PG(3,2)? Clearly, there must be, for the proof which we have given depends solely on the axioms I-VII. Referring to the preceding section, consider the two triangles (ab)(ac)(ad) and (be)(ce)(de). Clearly:

(ab)(ac) and (be)(ce) meet in the point (bc),
(ac)(ad) and (ce)(de) meet in the point (cd),
(ad)(ab) and (de)(be) meet in the point (db),

and further, (bc), (cd), (db) are collinear. Thus all the conditions of the theorem are satisfied, and it may be verified that the two triangles are in perspective from the point (ae).

2.5. The Principle of Duality. If we interchange the words "point" and "line," "collinear" and "concurrent," "meet in" and "lie on" in a theorem of projective geometry in the plane, we shall obtain what is called the *dual* proposition. The dual of axiom I is that: "There are at least two distinct lines," while the dual of axiom II is that: "Two distinct coplanar lines *a* and *b* determine one and only one point on both *a* and *b*." These two propositions, as well as the duals of III, IV, and V, follow from the axioms I-V. Thus if we are able to prove a theorem in the projective plane from the assumptions I-V, the same reasoning, with suitable changes in wording, will provide a demonstration of the dual theorem on the basis of the duals of I-V. This is known as the *Principle of Duality* in the plane. The plane-dual of Desargues' Theorem in the plane is the converse theorem; but we cannot appeal to the Principle of Duality in the plane for a proof of this converse theorem, since the direct theorem cannot be proved in the plane.

If we include the remaining axioms VI and VII, we may deduce a *Principle of Duality* in space. Points and planes are dual elements, while a line, which is determined by two distinct points or by two distinct planes, is self-dual. The space-dual of Desargues' Theorem in the plane will play an important role in chapter v; the reader will find it an interesting exercise to formulate the space-dual of Desargues' Theorem in space. For the proofs of these dual theorems, we need only invoke the Principle of Duality in space. It should be pointed out that, in Fano's finite geometry, it is not sufficient merely to appeal to a Principle of Duality to deduce the equality of the numbers of points and lines in a plane, and the numbers of points and planes in space.

If we take four coplanar points which we may call *vertices*, no three of which are collinear, and join them in all possible ways, we obtain a *complete quadrangle*; two sides not meeting in a vertex are said to be *opposite*, and the intersections of opposite sides are called the *diagonal points* of the quadrangle. The three diagonal points are the vertices of the *diagonal point* *triangle* of the quadrangle. The plane-dual of a complete quadrangle is a *complete quadrilateral*; two opposite vertices determine a *diagonal line*, and the three diagonal lines form the *diagonal line triangle* of the quadrilateral.

In Fig. 2.3A, the four points (ab), (ac), (ad), (ef) define a complete quadrangle whose diagonal points are (bc), (bd), (cd), and these diagonal points are *collinear*. Dually, the diagonal lines of the complete quadrilateral defined by the four lines (ab)(ac), (ab)(ad), (ac)(ad) and (bc)(cd) are *concurrent* in the point (ef). Surely, in this respect the geometry $PG(3,2)$ is most remarkable, but also it is unsatisfactory for reasons which will soon be apparent.

2.6. The Fourth Harmonic Point. In axiom III we assumed that there are at least three points on every line. If C is a third point on a line AB, and O is a point not on AB, then there is also a third point U on BO and a line CU meeting AO in V. If AU and BV meet in W, $ABUV$ is a complete quadrangle whose diagonal points are O, W, C.

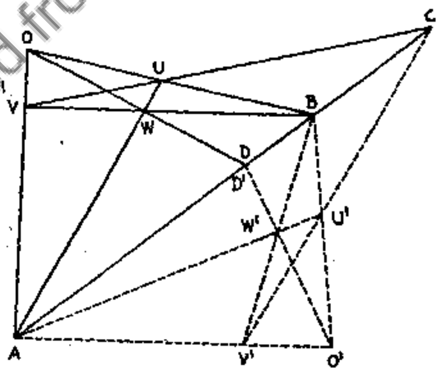


FIG. 2.6A

If O, W, C are not collinear, OW meets AB in a fourth point D, which is called the *harmonic conjugate* of C with respect to A and B.

In this construction the points O and U were arbitrarily chosen on a line through B. Since much of our later theory depends on it, we proceed to prove the uniqueness of the fourth harmonic point D. Let us choose a point O' in the plane OAB, distinct from O and not on AB, and let us repeat the construction as in Fig. 2.6A. In the triangles OUV, $O'U'V'$, the pairs of corresponding sides OU, $O'U'$; UV, $U'V'$; VO, $V'O'$ meet in the collinear points B, C, A respectively. Hence, by Desargues' Theorem in the plane, OO', UU', VV' are concurrent. Similarly, from the triangles UVW, $U'V'W'$, it follows that UU', VV', WW' are concurrent. Thus the three lines OO', UU', WW' are concurrent, and, from the triangles OUW, $O'U'W'$, by the converse of Desargues' Theorem in the plane, it follows that the intersections of OU, $O'U'$; UW, $U'W'$; WO, $W'O'$ are collinear. We conclude that the points D and D' coincide.† If O' does not lie in the plane OAB, the argument is still valid and is based upon Desargues' Theorem in space. Thus:

2.61. *The fourth harmonic point is uniquely determined and is independent of the plane in which the construction is made.*

For convenience, we shall abbreviate the statement that C, D are harmonically conjugate with regard to A, B by writing $H(AB, CD)$. If we had begun with the point D in Fig. 2.6A and chosen first the point U and then O, we should have arrived by the same construction at the point C. Thus C is the harmonic conjugate of D with regard to A, B, or $H(AB, DC)$. In the following chapter we shall prove that the harmonic relationship is completely symmetrical with regard to the two pairs of points.

†It should be pointed out that Desargues' Theorem in the plane is *not* a consequence of the uniqueness of the fourth harmonic point.

Dual to the complete quadrangle $ABUV$ we have a complete quadrilateral *abuv*, and we may define the fourth harmonic line through a point; from the Principle of Duality it follows that this line is unique. There is a very simple relation between an harmonic range of points and an harmonic pencil of lines, namely: *if* $H(AB, CD)$, *and if* O *is any point not on* AB, *then the pencil of lines* OA, OB, OC, OD *or* $O(A, B, C, D)$ *is also harmonic.* To prove this, we need only designate the sides of the complete quadrangle $ABUV$ as in the accompanying Fig. 2.6B.

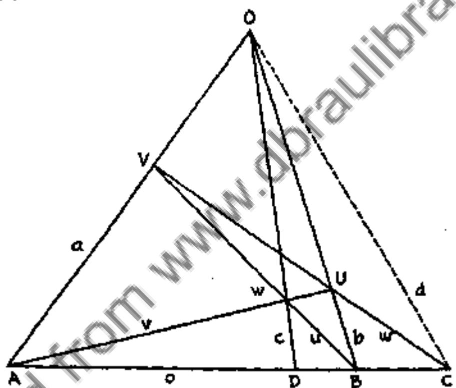


FIG. 2.6B

The complete quadrilateral *abuv* has *o*, *w*, *c* as diagonal lines, and *d* is the harmonic conjugate of *c* with respect to *a*, *b*. We shall write $H(ab, cd)$, or $H(ab, dc)$.

The assumption that the diagonal points of a complete quadrangle are not collinear, implies, with reference to the complete quadrangle $ABUV$ in Fig. 2.6B, that the diagonal lines *o*, *w*, *c* of the complete quadrilateral *abuv* are not concurrent. Such an assumption implies that there are more than three points on a line and more than three lines through a point. To ensure that there are more than a finite number of points on a line, we shall make a stronger assumption in the following section.

2.7. A Harmonic Sequence and Fano's Axiom. The repetition of the construction of the fourth harmonic point leads to what is called a *harmonic sequence* on the line. The conception goes back to Möbius (1827), and it will be fundamental when we come to set up a coordinate system in our geometry. Here, however, we are only interested in constructing further points on the line.

Consider three points P_0, P_1, P_∞ on a line l, and any two points U, V collinear with P_∞, but not on l. If UP_0 meets VP_1 in the point R_1, and if UP_1 and R_1P_∞ meet in R_2, then VR_2

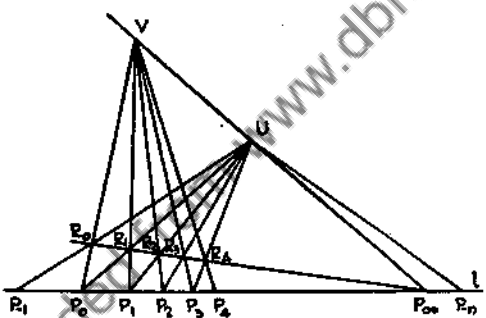


FIG. 2.7A

meets l in a point which we may call P_2. *Assuming that the diagonal points of a complete quadrangle are not collinear,* P_2 is the harmonic conjugate of P_0 with respect to P_1 and P_∞. Again, if UP_2 meets R_2P_∞ in R_3, then VR_3 meets l in the point P_3, and $H(P_2P_\infty, P_1P_3)$. From our assumption it follows that P_3 is distinct from each of P_1, P_2, P_∞; but can we say that P_3 and P_0 are necessarily distinct? If P_3 is distinct from P_0, we may proceed with the construction, joining U and P_3

to meet R_3P_∞ in R_4, while VR_4 meets l in a point P_4; in general, we may construct P_{n+1} such that $H(P_nP_\infty, P_{n-1}P_{n+1})$. In an exactly similar manner, we may construct the harmonic conjugate of P_1 with regard to P_0 and P_∞, obtaining a point P_{-1}; in general, we may construct a point P_{-n-1}, such that $H(P_{-n}P_\infty, P_{-n+1}P_{-n-1})$. The range of points

$$\ldots P_{-2}, P_{-1}, P_0, P_1, P_2, \ldots P_\infty,$$

is the *harmonic sequence* referred to above.

The number of points in an harmonic sequence will be finite if the number of points on the line is finite; for, after some stage, the construction will no longer yield new points, even though the diagonal points of a complete quadrangle are not collinear. Fano describes such a sequence as *re-entrant*, and in order to ensure that the number of points on a line shall be infinite, he introduces the assumption that:

VIII. *Not every harmonic sequence on a line contains a finite number of points.*

The plane-dual of an harmonic sequence of points on a line is an harmonic sequence of lines through a point, and the space-dual is an harmonic sequence of planes through a line. The dual of axiom VIII, or Fano's Axiom, as we shall sometimes call it, follows immediately in each case from I-VIII, and the Principle of Duality remains valid. Fano's Axiom does *not* require that the line be "continuous," according to the exact definition of continuity which we shall introduce in chapter VIII.

CHAPTER III

PROJECTIVE GEOMETRY AND PAPPUS' THEOREM

3.1. Summary of the Chapter. We referred in chapter I to the ordinary integers of arithmetic—how they are derived from the process called counting. This process is sufficiently familiar to us, but to a primitive man it might even be unknown. To answer the question, "Are there as many apples in one pile as oranges in another pile?" merely requires a "pairing off" of apples against oranges. Our primitive man might have no conception of the number of apples or oranges, yet, by setting up such a *correspondence*,† he would be able to answer the question. Thus correspondence is an even more fundamental concept than number, and its role in mathematics, especially in geometry, is most important.

In this chapter we shall investigate the properties of a *projective correspondence*‡ between the points of two lines, which may coincide; in particular, we shall define an *involution* on a line. That such a projective correspondence is uniquely determined by assigning three pairs of corresponding points, is called the Fundamental Theorem of projective geometry. The proof which we shall give is due to F. Schur, and is based upon the assumption of Pappus' Theorem in axiom IX. At the end of the chapter we shall define a *conic*, proving Pascal's Theorem and a second theorem due to Desargues.

It should be mentioned at this point that a projective correspondence is not the only type of correspondence which can be established between the points of two lines. A discussion of the problem of determining the possible types will be given in chapter IX.

†Throughout this book the word "correspondence" will mean a "(1,1) correspondence," no other type being considered. Cf. §6.2.

‡Cf. §3.2.

3.2. Related Ranges of Points. A set of points $P_1, Q_1, R_1, \ldots$ on a line l_1 is called a *range* of points on l_1. If O_1 is any point not lying on l_1, and l_2 is any line in the plane determined by l_1 and O_1, not passing through O_1, then l_2 will meet the *pencil* of lines $O_1(P_1, Q_1, R_1, \ldots)$ in a range of points $P_2, Q_2, R_2, \ldots$ These two ranges are said to be *in perspective* from O_1, and we write

$$(P_1, Q_1, R_1, \ldots) \overset{O_1}{\overline{\overline{\wedge}}} (P_2, Q_2, R_2, \ldots).$$

Such a *perspectivity* establishes a correspondence between the points of l_1 and l_2. From another point O_2, not lying on l_2, we may *project* the points of l_2 into the points of another line l_3, and so on indefinitely. Such a *chain* of perspectivities

$$(P_1, Q_1, R_1, \ldots) \overset{O_1}{\overline{\overline{\wedge}}} (P_2, Q_2, R_2, \ldots) \overset{O_2}{\overline{\overline{\wedge}}} (P_3, Q_3, R_3, \ldots) \ldots$$

$$\ldots \overset{O_{n-1}}{\overline{\overline{\wedge}}} (P_n, Q_n, R_n, \ldots)$$

determines a *projective* correspondence, or a *projectivity*

$$(P_1, Q_1, R_1, \ldots) \,\overline{\wedge}\, (P_n, Q_n, R_n, \ldots);$$

and we shall say that the range of points on l_1 is *projectively* or *homographically related*, or simply *related* to the range of points on l_n. The construction need not be confined to one plane. It is only necessary that successive lines l_r, l_{r+1} and their *centre of perspectivity* O_r should be coplanar.

The plane-dual of a range of points on a line is a pencil of lines through a point. If the intersections of corresponding lines of two related pencils lie on a line, the pencils are *in perspective*, and we shall call this line the *axis of perspectivity*. The space-dual of a range of points on a line is a pencil of planes through a line, and we may have related pencils of planes in space. We shall not attempt to apply the Principle of Duality in every case but we shall draw attention to the dual form of a theorem if it is of special interest.

There is a fundamental property of a projective correspondence which is of the greatest importance in the sequel, namely:

3.21. *The harmonic property is invariant under a projectivity.*

To prove 3.21, consider four distinct points P_1, Q_1, R_1, S_1 on a line l_1 and P_2, Q_2, R_2, S_2 on a line l_2, such that

$$(P_1, Q_1, R_1, S_1) \overset{O_1}{\underset{\wedge}{=}} (P_2, Q_2, R_2, S_2).$$

If $H(P_1Q_1, R_1S_1)$, we may suppose the fourth harmonic point to be defined by a complete quadrangle in a plane π_1 through l_1, not passing through O_1. This complete quadrangle in π_1 will project from O_1 into a complete quadrangle in a plane π_2 through l_2, and we have $H(P_2Q_2, R_2S_2)$. Since any projectivity is a chain of perspectivities, four harmonic points must correspond to four harmonic points under a projective correspondence.

On the other hand, it is not difficult to show that

3.22. *Any three distinct points P_1, Q_1, R_1 on a line l_1 may be related to any three distinct points P_3, Q_3, R_3 on a line l_3, distinct from l_1, by at most two perspectivities.*

We may choose a line l_2 through P_1, distinct from l_1, which intersects l_3 in a point distinct from P_3,—say the line P_1Q_3,—and let $P_1 = P_2$ and $Q_2 = Q_3$. Taking any point O_1 on Q_1Q_3 let O_1R_1 meet l_2 in R_2; if P_2P_3 and R_2R_3 meet in O_2, as in Fig. 3.2A, then

$$(P_1, Q_1, R_1) \overset{O_1}{\underset{\wedge}{=}} (P_2, Q_2, R_2) \overset{O_2}{\underset{\wedge}{=}} (P_3, Q_3, R_3).$$

It is unnecessary that l_1, l_3 be coplanar; our choice of l_2 is always possible, and is by no means unique. Clearly, 3.22 is a "best possible" result, since it would be impossible to relate four

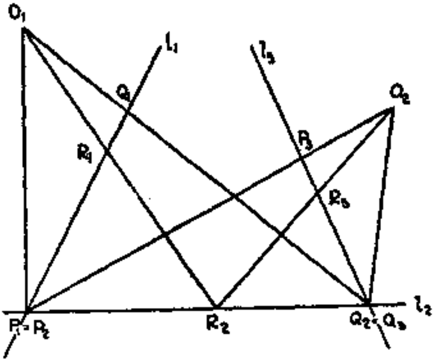


FIG. 3.2A

distinct points on l_1 to *any* four points on l_3, in view of 3.21. If l_1 and l_3 should coincide, one additional perspectivity would be required. In particular, for any three collinear points A, B, C,

3.23. $$(A, B, C) \barwedge (A, C, B) \barwedge (B, A, C) \barwedge (B, C, A) \barwedge (C, A, B) \barwedge (C, B, A).$$

While it is impossible to relate any four points A, B, C, D on a line l to *every* permutation of these four points, yet, for those permutations which effect a double interchange of the four letters, we have:

3.24. $$(A,B,C,D) \barwedge (B,A,D,C) \barwedge (C,D,A,B) \barwedge (D,C,B,A).$$

It will be sufficient to construct the first of these projectivities. Choosing any line l' through A distinct from l, and any point O in the plane determined by l and l', but not on either of these two lines, we may project the points B, C, D from O into B', C', D' on l', as in Fig. 3.2B. If CD' meets BB' in B'', then

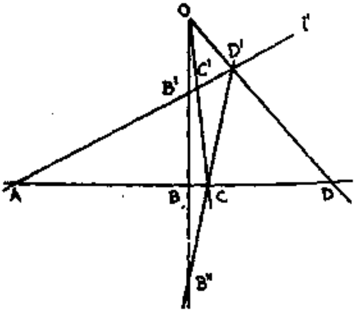


FIG. 3.2B

$$(A, B, C, D) \overset{O}{\overline{\wedge}} (A, B', C', D') \overset{C}{\overline{\wedge}} (B, B', O, B'') \overset{D'}{\overline{\wedge}} (B, A, D, C),$$

which is the required result. From 3.24 it follows that

3.24′. $(A, B, D, C) \,\overline{\wedge}\, (B, A, C, D) \,\overline{\wedge}\, (C, D, B, A) \,\overline{\wedge}\, (D, C, A, B).$

If we assume that $H(AB, CD)$, then $H(AB, DC)$, and in consequence of 3.21 it follows from 3.24 and 3.24′ that the harmonic relationship is completely symmetrical and *the points of each pair are harmonic conjugates of one another with regard to the other pair.*

In order to obtain an example of a line set into correspondence with itself, let E, F be any two points on l, and O, U any two points on a line through E, distinct from l. If G is the harmonic conjugate of E with regard to O, U, join FG.

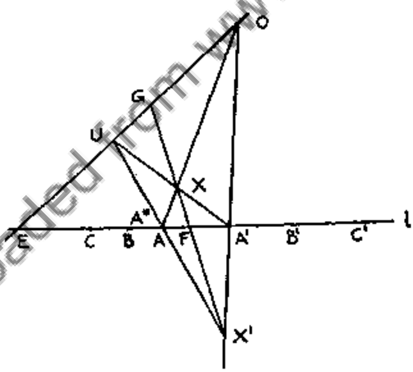


FIG. 3.2c

For any point A on l, OA meets FG in X and UX meets l in A', which corresponds to A under the chain of perspectivities

$$(A, B, C, \ldots) \overset{O}{\overline{\wedge}} (X, Y, Z, \ldots) \overset{U}{\overline{\wedge}} (A', B', C', \ldots).$$

Since $H(OU, EG)$, it follows that $H(EF, AA')$, and OA', UA intersect in a point X' on FG. Thus

$$(A', B', C', \ldots) \overset{O}{\overline{\wedge}} (X', Y', Z', \ldots) \overset{U}{\overline{\wedge}} (A, B, C, \ldots);$$

from which we conclude that the direct correspondence and the inverse correspondence coincide. A projectivity having this property† is called an *involution*. The two self-corresponding points E, F are called the *double points* of the involution, while the two corresponding points A, A' are called a *point pair*, or simply a *pair* of the involution.

3.3. The Reduction of a Projectivity to Two Perspectivities. In 3.22 we showed that any three points of one line may be related to any three points of another line by at most *two* perspectivities. Our problem in this section is to prove that if a range of points on a line l_1 is related by a chain of n perspectivities to a range of points on a line $l_{n+1} \neq l_1$, then this projectivity is equivalent to at most *two* perspectivities. Two preliminary theorems are necessary.

Consider a sequence of two perspectivities between the points of l_1, l_2, l_3; we shall call l_2 the *intermediary line*. If l_1, l_2 intersect in a point L_{12} and l_2, l_3 intersect in a point L_{23}, the two points L_{12} and L_{23} may or may not coincide. In the former case:

3.31. *If l_1, l_2, l_3 are concurrent, the sequence of two perspectivities between the points of l_1 and l_3 is equivalent to a single perspectivity.*

The proof is immediate. For consider any two triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$, as in Fig. 3.3A.

†Cf. 3.52.

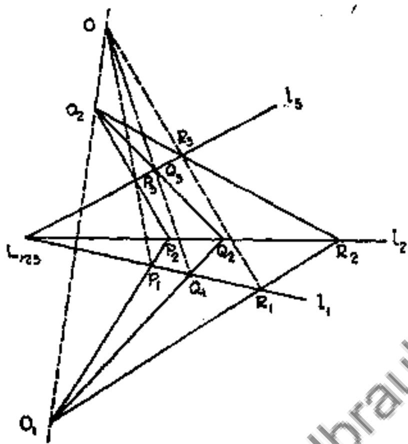


FIG. 3.3A

If P_1P_2, Q_1Q_2 meet in O_1, and P_2P_3, Q_2Q_3 meet in O_2, then it follows from Desargues' Theorem that P_1P_3, Q_1Q_3 meet in a point O on O_1O_2. Similarly, taking any other triad of corresponding points $R_1R_2R_3$, P_1P_3 and R_1R_3 meet on O_1O_2. Hence, all such lines are concurrent in the point O, and

$$(P_1, Q_1, R_1, \ldots) \overset{O}{\overline{\wedge}} (P_3, Q_3, R_3, \ldots).$$

In the latter case, where L_{12} and L_{23} are distinct:

3.32. *In the sequence of two perspectivities between the points of l_1 and l_3, the intermediary line l_2 may be replaced by any other line l_2^*, not joining a pair of corresponding points on l_1 and l_3, nor passing through the common point of l_1 and l_3 if these two lines intersect.*

The proof involves a double application of 3.31 and is valid whether l_1, l_2, l_3 are coplanar or not. The construction is similar to that in Fig. 3.2A.

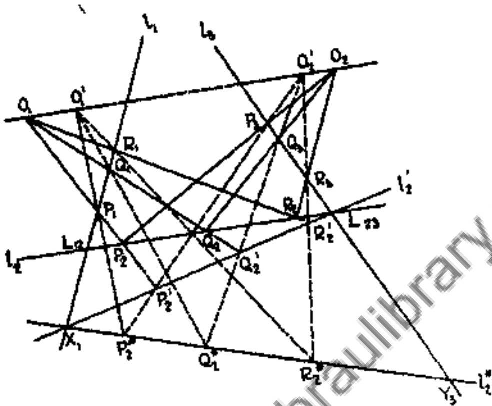


FIG. 3.3B

Let l_2' be any line through L_{23} distinct from l_2 and l_3 and not passing through O_1, which meets l_1 in X_1. Such a line fulfils the condition of the theorem, in that X_1 and L_{23} cannot be corresponding points. If we project the range $P_2, Q_2, R_2, \ldots$ on l_2 from O_1 into the range $P_2', Q_2', R_2', \ldots$ on l_2', then, from 3.31,

$$(P_1, Q_1, R_1, \ldots) \overset{O_1}{\overline{\overline{\wedge}}} (P_2', Q_2', R_2', \ldots) \overset{O_2'}{\overline{\overline{\wedge}}} (P_3, Q_3, R_3, \ldots),$$

where O_2' lies on O_1O_2. Similarly, we may choose any line l_2^* through X_1, distinct from l_1 and l_2' and not passing through O_2', which meets l_3 in Y_3. If we project the range P_2', Q_2', R_2', $\ldots$ on l_2' from O_2' into the range $P_2^*, Q_2^*, R_2^*, \ldots$ on l_2^*, then again from 3.31,

$$(P_1, Q_1, R_1, \ldots) \overset{O_1'}{\overline{\overline{\wedge}}} (P_2^*, Q_2^*, R_2^*, \ldots) \overset{O_2'}{\overline{\overline{\wedge}}} (P_3, Q_3, R_3, \ldots),$$

where O_1' lies on O_1O_2', i.e. on O_1O_2. The line l_2^* is the line in question.

We are now in a position to prove the important theorem to which this section is devoted, namely that†

3.33. *Every projectivity between two distinct lines is equivalent to at most two perspectivities.*

It will be sufficient if we show that any sequence of three perspectivities can always be replaced by at most two perspectivities. Consider the chain

$$(P_1, Q_1, R_1, \ldots) \overset{O_1}{\underset{\wedge}{=}} (P_2, Q_2, R_2, \ldots) \overset{O_2}{\underset{\wedge}{=}} (P_3, Q_3, R_3, \ldots)$$
$$\overset{O_3}{\underset{\wedge}{=}} (P_4, Q_4, R_4, \ldots),$$

relating points on the four lines, l_1, l_2, l_3, l_4. We may suppose that l_1, l_2, l_3 are not concurrent, for by 3.31 we could then reduce the number of perspectivities by one. Similarly, we may suppose that l_2, l_3, l_4 are not concurrent. *Let us assume that l_1, l_3, l_4 are not concurrent.* By 3.32, we may replace l_2 by a line l_2^* without affecting the perspectivity between l_3 and l_4. In particular, we may arrange that l_2^* passes through the point L_{34} on l_3, since we have assumed that this point does not lie on l_1. But now l_2^*, l_3, l_4 are concurrent, and we may reduce the number of perspectivities by one. *If l_1, l_3, l_4 are concurrent,* then l_1, l_2, l_4 are *not* concurrent, or all the lines would be concurrent, contrary to supposition. Again, we may replace l_3 by l_3^* which passes through L_{12}, without affecting the perspectivity between l_1 and l_2. The lines l_1, l_2, l_3^* are now concurrent, and once more we may reduce the number of perspectivities by one. Thus we conclude that such a reduction is always possible, and by a continued application we prove 3.33.

†Actually, 3.33 is an immediate consequence of 3.22 if we assume the Fundamental Theorem (cf. §3.5).

3.4. Pappus' Theorem. What is the necessary and sufficient condition that a projectivity should be equivalent to a single perspectivity? In the first place, a necessary condition is that l_1 and l_3 should intersect in a point L_{13}; further, this point must be a self-corresponding point in the projectivity. Can we say conversely, that if L_{13} is self-corresponding then l_1 and l_3 are in perspective? Certainly this will follow if l_2 passes through L_{13}, as in 3.31. If l_2 does not pass through L_{13}, this point will be self-corresponding if and only if it lies on O_1O_2. Does this condition imply that l_1 and l_3 are in perspective?

Let us consider the question with reference to Fig. 3.4A.

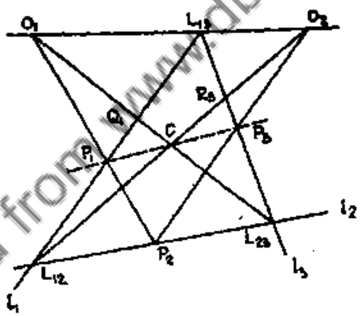


FIG. 3.4A

If O_1L_{23} meets l_1 in Q_1, and O_2L_{12} meets l_3 in R_3, then the points L_{23}, R_3 on l_3 correspond respectively to Q_1, L_{12} on l_1. Clearly, if l_1 and l_3 are to be in perspective, the centre of perspectivity O must be the point of intersection of Q_1L_{23} and $L_{12}R_3$. Taking any point P_1 on l_1, O_1P_1 meets l_2 in P_2 and O_2P_2 meets l_3 in P_3. *If l_1 and l_3 are in perspective, P_1P_3 must pass through O.* Subject to this assumption, we have the following important theorem:

3.41. *A projectivity between two distinct coplanar lines is equivalent to a perspectivity if and only if their point of intersection is a self-corresponding point.*

If instead of O_1, L_{13}, O_2 we write A, B, C and instead of L_{12}, P_2, L_{23} we write A', B', C', and represent P_1 by $(AB', A'B)$, etc., then 3.41 is dependent upon the assumption of axiom:

IX. *If A, B, C are any three distinct points on a line l, and A', B', C' any three distinct points on a line l' intersecting l, then the three points $(BC', B'C)$, $(CA', C'A)$, $(AB', A'B)$ are collinear* (Pappus' Theorem).

Pappus of Alexandria (circ. 340 A.D.) gave a proof of the theorem which we have associated with his name, under the assumptions of Euclidean geometry (cf. chapter v); but *no proof is possible on the basis of axioms* I-VIII. In chapter VIII we shall show that the assumption of axiom IX implies the validity of Desargues' Theorem in the plane, so that in a non-Desarguesian plane geometry Pappus' Theorem is not valid. Sometimes Pappus' Theorem is called after Pascal (1623-1662), the great contemporary of Desargues; but we shall reserve the name "Pascal's Theorem" for a more general result. If we think of $(BC', B'C)$, $(CA', C'A)$, $(AB', A'B)$ as the points of intersection of pairs of opposite sides of the hexagon $AB'CA'BC'$, the line containing these three points is known as the *Pappus line* of the hexagon. Of a number of theorems equivalent to Pappus' Theorem, we mention only one: "If each of three lines a, b, c in space, no two of which intersect, is met by each of three other lines a', b', c', then every transversal of the first set is met by every transversal of the second set."

The dual of Pappus' Theorem in the plane is that, "If a, b, c are any three distinct lines through a point L, and a', b', c' any three distinct lines through a point L', then the three lines $(bc', b'c)$, $(ca', c'a)$, $(ab', a'b)$ are concurrent."

If the direct theorem is valid it is an easy matter to prove the dual theorem; for we may take L, (a, b'), (a, c') on a and L', (a', c), (a', b) on a', and an application of the direct theorem proves that the required lines are concurrent. We leave the dual theorem in space for the reader to formulate and prove.

3.5. The Fundamental Theorem of Projective Geometry. In 3.22 we saw that any three points of one line may be related to any three points of another line by at most two perspectivities. We also saw that such a statement could not be made for more than three points. Certainly, these two perspectivities determine a projectivity between the points of the two lines; but we have also seen that the intermediary line is to a large extent arbitrary. It is natural to ask, is the projectivity so determined unique? The answer to this important question is contained in

THE FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY

A projectivity between the points of two lines, which may coincide, is uniquely determined when three pairs of corresponding points are given.

Let us suppose that three distinct points P_1, Q_1, R_1 on l_1 correspond respectively to three distinct points P_3, Q_3, R_3 on l_3, and that l_1 and l_3 are distinct; the argument which follows is valid whether l_1 and l_3 intersect or not. We may choose P_1Q_3 to be the intermediary line l_2, so that $P_1 = P_2$ and $Q_2 = Q_3$, as in Fig. 3.5A (cf. Fig. 3.2A).

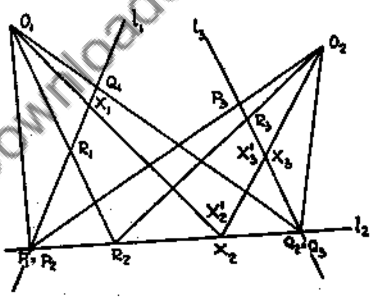


FIG. 3.5A

From any point X_1 on l_1 we arrive at X_3 on l_3 by a sequence of perspectivities

$$(P_1, Q_1, R_1, X_1) \overset{O_1}{\overline{\wedge}} (P_2, Q_2, R_2, X_2) \overset{O_2}{\overline{\wedge}} (P_3, Q_3, R_3, X_3).$$

If, by a different sequence of perspectivities, we have

$$(P_1, Q_1, R_1, X_1) \overline{\wedge} (P_3, Q_3, R_3, X_3'),$$

then, by projecting X_3' from O_2 we obtain a point X_2' on l_2, and

$$(P_1, Q_1, R_1, X_1) \overline{\wedge} (P_2, Q_2, R_2, X_2').$$

In this latter projectivity, however, $P_1(=P_2)$ is a self-corresponding point, and X_2' will coincide with X_2 if and only if 3.41 is valid. Clearly, if X_2' coincides with X_2, then X_3' will coincide with X_3, which is what we wished to prove. Conversely, 3.41 or Pappus' Theorem is a consequence of the assumption of the Fundamental Theorem; thus Pappus' Theorem and the Fundamental Theorem of projective geometry are equivalent to one another.

If the lines l_1 and l_3 coincide, we may project the points of l_3 on to a line l_3', and apply the preceding argument to l_1 and l_3'. In particular, *if a line is set into projective correspondence with itself so that three distinct points are self-corresponding, then every point is self-corresponding and the projectivity is the identity.*

If l_1 and l_3 intersect, and we choose two corresponding points, say P_3 and P_1, as centres of perspectivity, the correspondence between the lines of the two pencils $P_3(P_1, Q_1, R_1, \ldots)$ and $P_1(P_3, Q_3, R_3, \ldots)$ is completely determined by the dual of the Fundamental Theorem. But these two related pencils have a self-corresponding ray P_1P_3, so that they are in perspective; if we denote (P_1Q_3, P_3Q_1), (P_1R_3, P_3R_1), ... by Q_2, R_2, ... as in Fig. 3.5B, the axis of perspective is the line Q_2R_2 and

$$(P_1, Q_1, R_1, \ldots) \overset{P_3}{\overline{\wedge}} (P_2, Q_2, R_2, \ldots) \overset{P_1}{\overline{\wedge}} (P_3, Q_3, R_3, \ldots),$$

where Q_2R_2 meets P_1P_3 in P_2. The line Q_2R_2 is thus the intermediary line l_2; but it is also the Pappus line of the hexagon $P_1Q_3R_1P_3Q_1R_3$, and any two corresponding points would serve as centres of perspectivity. If l_2 meets l_1, l_3 in S_1, T_3, it follows that S_3 and T_1 coincide in the point of intersection of l_1, l_3. If l_2 passes through the point of intersection of l_1, l_3, then this point is self-corresponding and l_1, l_3 are in perspective by 3.31. An application of Desargues' Theorem enables us to prove Pappus' Theorem in this special case.

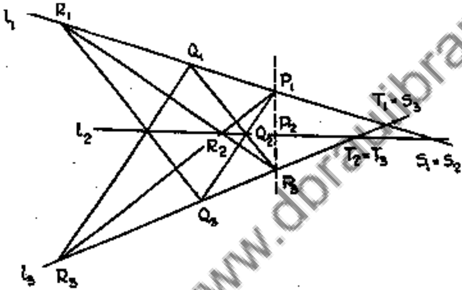


FIG. 3.5B

There are two applications of the Fundamental Theorem which are suggested by our definition of an involution at the end of §3.2. We prove first that

3.51. *A necessary and sufficient condition that $H(EF, AA')$ is that*

$$(E, F, A, A') \overline{\wedge} (E, F, A', A).$$

Referring to Fig. 3.2C, the necessity of the condition follows from the projectivity

$$(E, F, A, A') \overset{O}{\overline{\wedge}} (G, F, X, X') \overset{U}{\overline{\wedge}} (E, F, A', A).$$

To show that the condition is sufficient, choose any two points O, U on a line through E distinct from l. Reconstructing Fig. 3.2C, if OA, UA' intersect in a point X, let FX meet OU

in G and OA' in X'. Hence, if UX' meets EF in A^*, we have

$$(E, F, A, A') \overset{O}{\overline{\wedge}} (G, F, X, X') \overset{U}{\overline{\wedge}} (E, F, A', A^*),$$

and it follows from the Fundamental Theorem that A and A^* must coincide. We conclude from the complete quadrangle $OUAA'$ that $H(EF, AA')$.

The significance of an involution is thrown into sharp relief by the following theorem:

3.52. *A projectivity on a line l, in which a point A' corresponds to a point $A(\neq A')$, is an involution if and only if A corresponds to A'.*

The necessity of the condition follows from the definition of an involution. To prove that it is sufficient, let us suppose that B is any point on l distinct from A and that A', B' correspond to A, B under a projectivity π. Thus

$$(A, A', B) \overline{\wedge} (A', A, B'),$$

and π is completely determined by the Fundamental Theorem. Now we know, from 3.24, that there is a projectivity such that

$$(A, A', B, B') \overline{\wedge} (A', A, B', B);$$

hence this projectivity must coincide with π, and B corresponds to B'. Since B may be any point on l, π is by definition an involution.

3.6. The Conic. Although our aim in this chapter is not the development of projective geometry, but the laying of its foundations, it would seem unfair to the reader not to give some indication of the beautiful theorems which lie beyond. The more so, since the proofs of most of them are singularly elegant and easily understood.

The plane-dual of the Fundamental Theorem of projective geometry is given by: "A projectivity between the lines of two pencils, whose vertices L_1, L_2 may coincide, is uniquely determined when three pairs of corresponding lines are given." If to the three lines p_1, q_1, r_1 through L_1 there correspond respectively p_2, q_2, r_2 through L_2, then to any other line s_1 through L_1 there corresponds a unique line s_2 through L_2, and conversely. *The locus of the intersections of corresponding lines of these two related pencils* is called a *conic*, which we shall denote by $\mathfrak{C}$. The conic passes through the vertices of the defining pencils, and is determined by the *five* points L_1, L_2, $P=(p_1, p_2)$, $Q=(q_1, q_2)$, $R=(r_1, r_2)$; it is not difficult to show that *L_1, L_2 may be any two points on $\mathfrak{C}$.* If P, Q, R lie on a line l, the projectivity reduces to a perspectivity and the locus degenerates into a pair of lines, l and L_1L_2. It is an important fact that the plane-dual of a conic is also a conic.

To the line L_1L_2, thought of as belonging to the pencil with vertex L_1, there corresponds a line L_2L, which we shall call the *tangent* to $\mathfrak{C}$ at L_2; L_2L and $\mathfrak{C}$ have no other points in common, and we shall say that L_2L *touches* $\mathfrak{C}$ at L_2. Similarly, we may define the tangent at L_1. If, as in Fig. 3.6A, a line through L meets $\mathfrak{C}$ in U, V and meets the line l, joining L_1, L_2, in the point M, then†

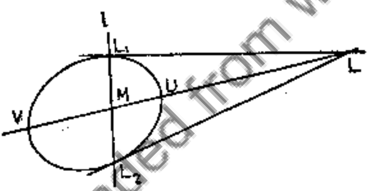


FIG. 3.6A

$$L_1(U, V, L_1, L_2) \barwedge L_2(U, V, L_1, L_2).$$

It follows that

$$(U, V, L, M) \barwedge (U, V, M, L),$$

and we conclude from 3.51 that $H(UV, LM)$. The point L is called the *pole* of l, and l the *polar* of L with regard to $\mathfrak{C}$. Using this property, we may define the polar of L as the locus of the harmonic conjugates of L with regard to $\mathfrak{C}$; this defini-

†Denoting the lines L_1L and L_2L by L_1L_1 and L_2L_2.

tion determines a unique line, even when no tangents can be drawn from L to the conic. Conversely, every line has a unique pole, which is the intersection of the polars of any two points on it.

If we consider a conic as the generalization of a pair of intersecting lines, we should expect to obtain a generalization of Pappus' Theorem. The generalization is known as

PASCAL'S THEOREM

If a hexagon is inscribed in a conic, the intersections of the three pairs of opposite sides are collinear.

Taking the hexagon to be $AB'CA'BC'$, as in Fig. 3.6B, and A, C to be the vertices of the pencils of lines defining

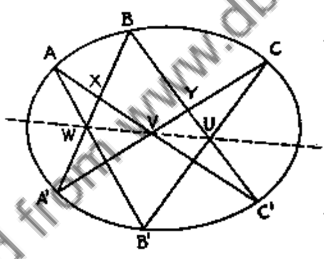


FIG. 3.6B

the conic $\mathfrak{C}$, then

$$A(B, A', B', C') \barwedge C(B, A', B', C').$$

If $U=(BC', B'C)$, $V=(CA', C'A)$, $W=(AB', A'B)$, and if AC' meets BA' in X and $A'C$ meets BC' in Y, we have on BA' and BC',

$$(B, A', W, X) \barwedge (B, Y, U, C').$$

Since B is a self-corresponding point, we conclude that $A'Y$, WU, XC' are concurrent, and U, V, W are collinear. The plane-dual of Pascal's Theorem is known as "Brianchon's Theorem."

We shall close this chapter by proving a second theorem due to Desargues. Since it takes five points to determine a conic, a "singly infinite" system or *pencil* of conics may be drawn through any four coplanar points of general position.

DESARGUES' INVOLUTION THEOREM

Any line of general position meets the conics of a pencil—three of the conics being pairs of lines—in point pairs of an involution.

If the four fixed points are A, B, C, D, any line l meets the pairs of opposite sides of the complete quadrangle $ABCD$ in P, P'; Q, Q'; R, R', and meets any conic $\mathfrak{C}$ of the pencil in S, S', as in Fig. 3.6c.

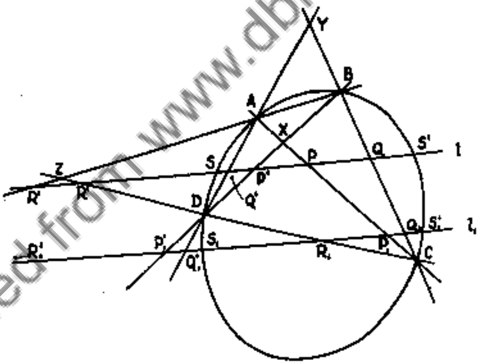


FIG. 3.6c

Clearly,
$$C(S, S', A, B) \;\overline{\wedge}\; D(S, S', A, B);$$
and on the line l we have, by 3.24,
$$(S, S', P, Q) \;\overline{\wedge}\; (S, S', Q', P') \;\overline{\wedge}\; (S', S, P', Q').$$
By 3.52, SS', PP', QQ' are point pairs of an involution on l. Similarly, by taking B and C as vertices, SS', PP', RR' are point pairs of an involution on l. But an involution is uniquely determined by any two of its pairs, so that the two involutions

are identical. The plane-dual of Desargues' Involution Theorem is known as "Sturm's Involution Theorem."

If l passes through the diagonal point Z of the complete quadrangle $ABCD$, then R and R' coincide in Z, which is a double point of the involution on l. From the harmonic property of the complete quadrangle, the other double point is the point of intersection of l with the opposite side XY of the diagonal point triangle, and XY is the polar of Z with regard to every conic of the pencil. Thus Desargues' Theorem provides a simple construction for the polar of any point with regard to a given conic.

The less general result that

3.61. *Any transversal meets the pairs of opposite sides of a complete quadrangle in point pairs of an involution,*

follows from the projectivity

$$(P, P', Q, R) \overset{C}{\underset{\wedge}{=}} (X, P', B, D) \overset{A}{\underset{\wedge}{=}} (P, P', R', Q') \;\overline{\wedge}\; (P', P, Q', R'),$$

again by 3.24. The points PP', QQ', RR' are sometimes called a *quadrangular set* on l.

In this discussion of the foundations of projective geometry we have avoided entirely the introduction of the relation of *order* between the points on a line. It should be mentioned at this stage, however, that such a relation enables us to distinguish two types of involution, according as any two pairs P, P' and Q, Q' do or do not "separate" one another. In the former case, the involution is said to be *elliptic* and there are no double points; in the latter, *hyperbolic*, and there are two double points. We have an example of a hyperbolic involution on the line l in Fig. 3.6C, and of an elliptic involution on the line l_1. This distinction will be considered in detail in chapters VIII and IX, though it will be necessary to refer to it again in chapter IV.

CHAPTER IV

AFFINE AND EUCLIDEAN GEOMETRY

4.1. Summary of the Chapter. Looking out on the world around us, we have recognized the line as a significant contour; but in supposing that any two coplanar lines of projective geometry have a point in common, we have adopted a simplification which is suggested by the visual process. Two "coplanar" lines in physical space may be so situated that, though they appear to "approach" one another, yet no point of intersection can be found. It was this property of lines in physical space which was transferred by the Greeks to the geometry which they constructed in their minds: two such lines were called *parallel*. Our first purpose in this chapter is to modify projective geometry by introducing an analogue of parallelism. The resulting modification of projective geometry is called *affine geometry*.

There is another property of physical space which we have not taken into consideration, namely, that we intuitively ascribe a certain invariance with respect to motion to the objects about us. One such invariant property we call *length*, and the equality of length we call *congruence*; area and volume are dependent on length and share its invariance. With the introduction of a suitable measure of length and of angle into affine geometry, we obtain *Euclidean geometry*. In this chapter we shall take only the first step upon the road. By considering properties which may be discussed without the use of a coordinate system,† we shall try to make it clear that Euclidean geometry is a special case of affine geometry.

†Cf. chap. VII.

Such an attempt to indicate the passage from projective geometry to Euclidean geometry is necessarily incomplete. But it is desirable to say something along these lines, in order to pave the way for the axioms of Euclidean geometry which we shall give in the following chapter. We use the term "Euclidean geometry" somewhat loosely here, for we say nothing about order or continuity.

4.2. Affine Geometry in the Plane. Confining our attention to a projective plane embedded in projective space, let us choose an *arbitrary* line in this plane which we shall call the *line at infinity* of the plane, denoting it by l_∞. The body of theorems which results from such a specialization is known as *affine geometry* in the plane. Any other line in the plane meets l_∞ in a single point, called the *point at infinity* of the line. Two lines which meet on l_∞ are said to be *parallel*. Through a given point of the plane, not on l_∞, there can be drawn one and only one line parallel to a given line.

It is possible to introduce a restricted form of congruence into affine geometry by defining the pairs of opposite "sides" of a parallelogram to be *congruent* to one another *by "translation."* If three lines AB, BC, CA meet l_∞ in C_∞, A_∞, B_∞ respectively, and if $H(AB, C'C_\infty)$ and $A_\infty C'$ meets AC in B', as in Fig. 4.2A, then

$$(A, B, C', C_\infty) \overset{A_\infty}{\overline{\overline{\wedge}}} (A, C, B', B_\infty),$$

and therefore $H(AC, B'B_\infty)$. Similarly, if $B_\infty C'$ meets BC in A', then $H(BC, A'A_\infty)$. The segment† BA' is congruent to $C'B'$, from the parallelogram $BA'B'C'$, and $C'B'$ is congruent to $A'C$, from the parallelogram $A'CB'C'$. It is natural to extend our definition by saying that BA' is also *congruent by "translation"* to $A'C$, and we may speak of A' as the *mid-point*

†For the definition of a segment cf. §8.3.

of BC. By reconstructing the figure, it is easy to see that the harmonic condition $H(BC, A'A_\infty)$ is not only necessary but

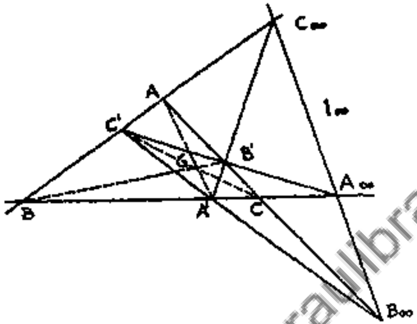


FIG. 4.2A

also sufficient for A' to be the mid-point of BC; from this we conclude that B' is the mid-point of AC and C' is the mid-point of AB. Otherwise stated,

4.21. *The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half of it.*

It is natural to speak of AA', BB', CC' in Fig. 4.2A as the *medians* of the triangle ABC. If AA', BB' meet in G, then, from the complete quadrangle $BA'B'A$, CG must intersect AB in the harmonic conjugate of C_∞ with regard to A, B; i.e. CG must pass through C'. Thus AA', BB', CC' are concurrent in G, which we may call the *centroid* of the triangle ABC.

Consider a conic $\mathfrak{C}$ in the plane. The two related pencils of lines which define $\mathfrak{C}$ intersect any line l of the plane in two related ranges of points. In chapter IX we shall prove that in such a correspondence there are two, one, or no self-corresponding points. A self-corresponding point on l will necessarily lie on $\mathfrak{C}$; so that $\mathfrak{C}$ and l intersect in two distinct points, in one point, or have no point in common. In particular, $\mathfrak{C}$ is an *hyperbola* if it meets the line at infinity in two distinct points, a *parabola* if it meets it in a single point, and an *ellipse* if $\mathfrak{C}$ and l_∞ do not intersect. The pole of the line at infinity is called the *centre* of the conic; in particular, the centre of a parabola is at its point of contact with the line at infinity. In the case of an hyperbola, the tangents from the centre to the curve are called the *asymptotes*.

If U, V are any two points on a conic $\mathfrak{C}$, the segment UV is called a *chord* of $\mathfrak{C}$; if the line UV passes through the centre C of $\mathfrak{C}$, the chord UV is called a *diameter* of $\mathfrak{C}$. Clearly, the centre is the mid-point of every diameter of an hyperbola or an ellipse. If we generalize the notion of a diameter to include any line through the centre of a conic,

4.22. *The locus of the mid-points of a system of chords of a conic parallel to a given diameter d is a diameter d'.*

If d meets l_∞ in D_∞, then, since the polar of C passes through D_∞, the polar d' of D_∞ passes through C. If a chord UV is parallel to d, then UV passes through D_∞ and the locus of the mid-points of the chords UV is the diameter d'. The two diameters d and d' are said to be *conjugate* diameters of the conic $\mathfrak{C}$.

4.3. Euclidean Geometry in the Plane. In order to obtain a comparison of pairs of points or of "segments" upon lines which are not parallel to one another, and a comparison of angles, let us consider an arbitrarily chosen elliptic† involution $A_\infty A'_\infty, B_\infty B'_\infty, C_\infty C'_\infty, \ldots$ on l_∞. The choice of such an *absolute involution* gives rise to Euclidean geometry. By introducing a coordinate system at this stage we could arrive

†Proceeding similarly with a hyperbolic involution we would obtain the two-dimensional case of Minkowski's geometry of space-time, which was used by Einstein (in the four-dimensional case) for his special theory of relativity.

at the familiar formulae of elementary analytical geometry; we shall not do this, however, for in order to appreciate the significance of these formulae it is necessary to investigate other possible "metrics" and the corresponding "non-Euclidean" geometries.† This problem of the introduction of a measure of length and of angle is of great importance and deserves a more detailed treatment than it would be possible to give here. We shall return to the matter again at the end of chapter IX.

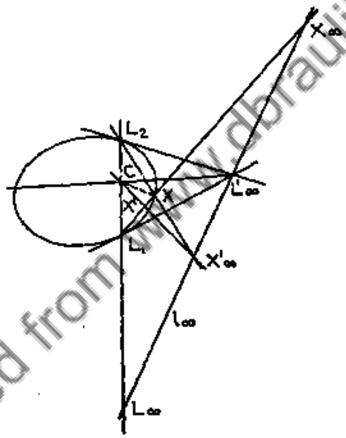


FIG. 4.3A

If L_1 and L_2 are any two distinct points not lying on l_∞, as in Fig. 4.3A, and if the line $L_1 L_2$ meets l_∞ in L_∞, then

$$L_1(A_\infty, A'_\infty, B_\infty, B'_\infty, \ldots L_\infty, L'_\infty, \ldots X_\infty, X'_\infty, \ldots)$$
$$\overline{\wedge}\, L_2(A'_\infty, A_\infty, B'_\infty, B_\infty, \ldots L'_\infty, L_\infty, \ldots X'_\infty, X_\infty, \ldots).$$

The locus of the points of intersection of pairs of corresponding lines of these two related pencils is a conic $\mathfrak{C}$, which we shall call a *circle*. Moreover, $L_1L'_\infty$ and $L_2L'_\infty$ are the tangents‡

†Cf. *Non-Euclidean Geometry*, in this series. ‡Cf. §3.6.

to the circle at L_1 and L_2, and these tangents are parallel to one another. Thus the centre C of $\mathfrak{C}$ is the mid-point of the diameter L_1L_2, and $H(L_1L_2, CL_\infty)$. From the fact that the absolute involution has no double points, it follows that $\mathfrak{C}$ and l_∞ do not intersect. If CX'_∞ meets L_1X in X', then, since

$$(L_1, L_2, C, L_\infty) \overset{X'_\infty}{\overline{\overline{\wedge}}} (L_1, X, X', X_\infty),$$

we conclude that $H(L_1X, X'X_\infty)$, and the polar of X_∞ is CX'_∞. Similarly, the polar of X'_∞ is CX_∞, and the points of any pair of the absolute involution are *conjugate* with regard to the circle $\mathfrak{C}$. It follows that the tangents at the extremities M_1, M_2 of any other diameter of $\mathfrak{C}$ are parallel to one another, and that M_1, M_2 could replace L_1, L_2 in the definition of $\mathfrak{C}$.

We are now in a position to complete our introduction of congruence by defining CL_1 to be *congruent* to CX *by "rotation."* We shall call the segment CX the *radius* of the circle $\mathfrak{C}$. In general, two segments AB and XY will be *congruent* if there exist two other segments CL and CL', such that CL is congruent to AB by translation and CL' is congruent to XY by translation, and CL is congruent to CL' by rotation. The reader should compare this introduction of congruence with that given in the Appendix.

Closely connected with the definition of a circle and the introduction of congruence is the notion of "perpendicularity." If P is any point not lying on l_∞, we shall define PX_∞ to be *perpendicular to* PX'_∞, for every pair X_∞, X'_∞ of the absolute involution. It follows from this definition that L_1X is perpendicular to L_2X in Fig. 4.3A; or, in other words, that

4.31. *The angle in a semi-circle is a right angle.*

It also follows that CX'_∞ is perpendicular to L_1X; and, since X' is the mid-point of the segment L_1X,

4.32. *The right bisector of any chord of a circle passes through the centre.*

Again, CL_∞ is perpendicular to CL'_∞, and every pair of conjugate diameters of a circle are mutually perpendicular; conversely, if every pair of conjugate diameters of a conic are perpendicular the conic must be a circle.

In order to define the orthocentre of a triangle, let us consider three lines AB, BC, CA which meet l_∞ in C_∞, A_∞, B_∞ respectively, as in Fig. 4.3B. If A'_∞ is the point corresponding to A_∞ in the absolute involution on l_∞, then AA'_∞ is perpendicular to BC, meeting it in A''. If AA'' and BB'' meet in O, the four points A, B, C, O define a complete quadrangle, whose two pairs of opposite sides BC, AO and AC, BO meet l_∞ in A_∞, A'_∞ and B_∞, B'_∞ respectively. It follows from 3.61 that the remaining pair of opposite sides, namely AB, CO, meet l_∞ in a point pair C_∞, C'_∞ of the involution determined by A_∞, A'_∞ and B_∞, B'_∞. Hence the pair C_∞, C'_∞ belong to the absolute involution on l_∞, and CO is perpendicular to AB. The point O is the *orthocentre* of the triangle ABC.

If A', B', C' are the mid-points of the sides of the triangle ABC in Fig. 4.3B (cf. Fig. 4.2A), then $A'A'_\infty$, $B'B'_\infty$, $C'C'_\infty$ are the right bisectors of these sides. Moreover, $A'A'_\infty$, $B'B'_\infty$, $C'C'_\infty$ are respectively perpendicular to $B'C'$, $C'A'$, $A'B'$, so that they are concurrent in the orthocentre O' of the triangle $A'B'C'$; by 4.32, O' is the *circumcentre* of the triangle ABC.

We shall close this chapter by proving that O' lies on the line OG, which is called the *Euler line* of the triangle ABC. It is only necessary to remark that the two triangles $AA'A'_\infty$, $BB'B'_\infty$ are in perspective from the point C_∞, so that it follows from Desargues' Theorem that the three points $(AA', BB') = G$, $(AA'_\infty, BB'_\infty) = O$, $(A'A'_\infty, B'B'_\infty) = O'$ are collinear.†

†After giving the above proof (cf. (8), p. 91), Schur states that the result is independent of the absolute polarity and remains true in any non-Euclidean geometry. H. S. M. Coxeter remarks that this statement is incorrect, and that the argument is valid only in the Euclidean case.

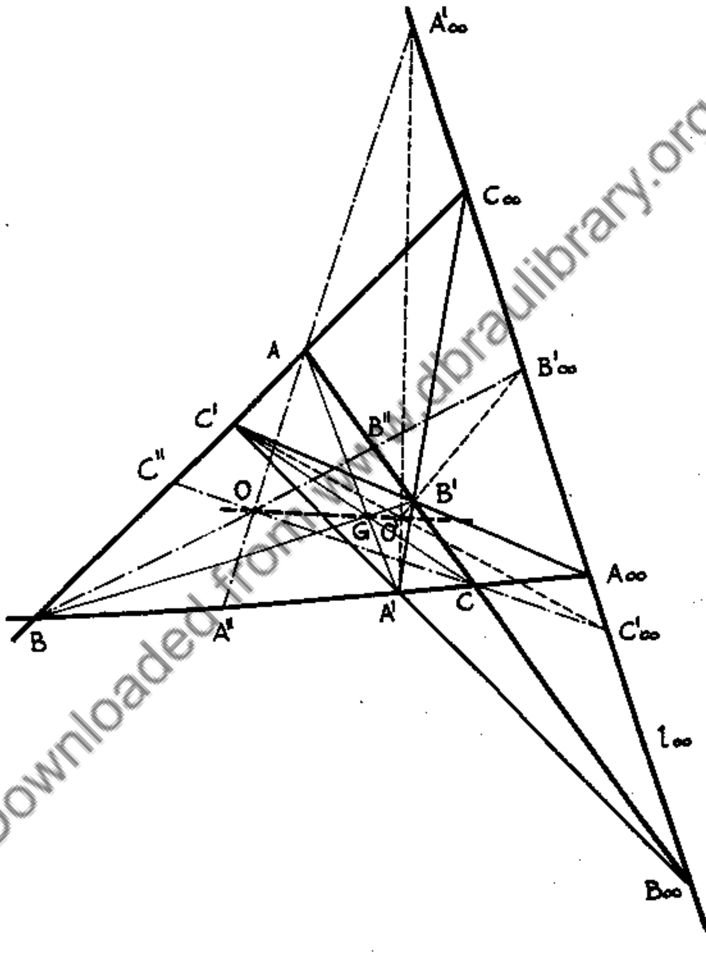


FIG. 4.3B

CHAPTER V

AXIOMS OF EUCLIDEAN GEOMETRY
IDEAL ELEMENTS

5.1. Summary of the Chapter. Would it not be possible to reverse the argument of the preceding chapter and obtain projective geometry as a generalization of Euclidean geometry? Preliminary to carrying out this programme, we shall give an axiomatic foundation of Euclidean geometry in §5.2. In order to replace the statement that two lines do not intersect (are parallel), by the statement that these two lines have an ideal point (point at infinity) in common, we shall follow an argument which is due to F. Schur. This argument is so elegant that we have given it in its most general form in §§5.3, 5.4 and 5.5, though, in its complete generality, it is not necessary for our purposes. The results have an important application in the study of non-Euclidean geometries and, in §5.6, in the making of a construction within a limited area or upon a given sheet of paper.

Much may be said in favour of obtaining projective geometry as a generalization of Euclidean geometry in the manner of this chapter. Historically, this was the course of discovery, but it was a long and painful one. As we shall see, all the difficulties are presented to us at once—congruence, parallelism, and continuity. Not till the end of the nineteenth century were these ideas sorted out and their independence established. In this book we have followed the opposite course, and this chapter is to be regarded as an interlude in the general development.

5.2. Axioms of Euclidean Geometry. A proper set of initial assumptions for Euclidean geometry was first given by Pasch in 1882. Following Pasch, Peano gave a system of axioms in 1889 which were based upon an undefined entity called a "point," and an undefined relation of "betweenness." In 1904, Veblen modified Peano's system by replacing the notion of "betweenness" by a three point relation of "order." A somewhat different approach was given by Hilbert in 1899, and another by Pieri in the same year. The axioms which we shall adopt are a combination of those of Veblen with the axioms of congruence and continuity of Hilbert.

Following Veblen, then, let us take a *point* as an undefined element, and *order* as an undefined relation, such that three points A, B, C are in the *order* ABC. We have the following axioms of Euclidean geometry.

1. *There are at least two distinct points.*

2. *If A, B are any two distinct points there is a point C such that A, B, C are in the order ABC.*

3. *If points A, B, C are in the order ABC, they are distinct.*

4. *If points A, B, C are in the order ABC, they are not in the order BCA.*

DEFINITIONS. If A, B are any two distinct points, the *line* AB consists of A and B and all points X in one or other of the possible orders ABX, AXB, XAB. The points X in the order AXB constitute the *segment* AB, and are said to lie *between* A and B. A and B are the *end-points* of the segment AB.

5. *If two distinct points C, D lie on the line AB, then A lies on the line CD.*

It follows from axioms 1-5 that, if A,B,C are in the order ABC, they are also in the order CBA and not in any of the orders CAB, BAC, ACB, BCA.

6. *If A, B are two distinct points, there is at least one point C not on the line AB.*

7. *If A, B, C are three non-collinear points and D, E are two points in the order BCD and CEA, then there is a point F in the order AFB such that D, E, F are collinear.*

It follows without difficulty that *D*, *E*, *F* are in the order *DEF* (cf. Fig. 2.2A).

DEFINITIONS. Three non-collinear points *A*, *B*, *C* are the *vertices* of a *triangle ABC* whose *sides* are the segments *AB*, *BC*, *CA*. The class of points collinear with two points of the sides of the triangle *ABC* is called the *plane ABC*. A point *O* is *in the interior* of a triangle if it lies on a segment whose end-points are points of different sides of the triangle; the totality of such points *O* is the *interior* of the triangle.

8. *If A, B, C are three non-collinear points, there is at least one point D not on the plane ABC.*

DEFINITIONS. Four non-coplanar points *A*, *B*, *C*, *D* are the *vertices* of the *tetrahedron ABCD*, whose *edges* are the six segments *AB*, *AC*, *AD*, *BC*, *BD*, *CD*, and whose *faces* are the interiors of the four triangles *ABC*, *ABD*, *ACD*, *BCD*. The class of points collinear with two points of the faces of the tetrahedron *ABCD* is called the *space ABCD*.

9. *Two planes which have a point in common, have a line in common.*

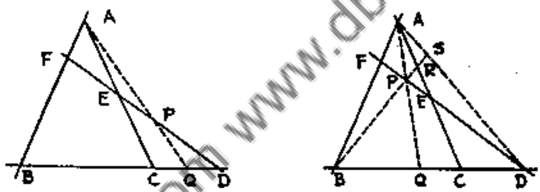
It is worth remarking that the introduction of a "three point" relation of *order* in Euclidean geometry is in agreement with our concept of physical space. Our definition of a segment is not greatly different from Euclid's definition—"A straight line is that which lies evenly between its extreme points,"—if we concentrate our attention upon the undefined word "between." In projective geometry, however, it is necessary to consider a "four point" relation of *order*, as will appear in chapter VIII where the whole question will be discussed in detail.

Axiom 2 implies that there are an infinite number of points on a line. To prove that there is a point between any two given points A, B, let us suppose that E is a point not on the line AB, as in Fig. 5.2A. It follows from axiom 2 that there is a point C in the order AEC, and a point D in the order BCD. Hence, from axiom 7, there is a point F in the order AFB.

The following theorem is complementary to axiom 7.

5.21. *If A, B, C are three non-collinear points and D, F are two points in the orders BCD and AFB, then there is a point E in the order CEA such that D, E, F are collinear.*

If P is a point of the segment DF, then from axiom 7 applied to the triangle DBF it follows that AP meets BD in a point Q; two cases arise, according as Q is in the segment DC or CB.



Case (i) Case (ii)

FIG. 5.2A

(i) If Q lies in the segment DC, we may apply axiom 7 to the triangle ACQ and conclude that DP meets AC in a point E, between A and C.

(ii) If Q lies in the segment CB, we may apply axiom 7 to the triangle ACQ and deduce that BP meets AC in a point R; similarly, from the triangle ADC, BR meets AD in S. From the triangle PDS, AR meets PD in E; moreover E lies between A and C.

DEFINITIONS. The *ray* AB consists of all points X in the order AXB, the point B itself, and all points X in the order ABX. All points of the ray AB are on the *same side* of A as the point B. If B' is a point of the ray AB, then the ray

AB' coincides with the ray AB. All points X in the order XAB belong to the ray *complementary* to the ray AB and are on the *opposite side* of A from the point B.

If l is any line in a plane α and A any point of α not lying on l, then there are points B in α, not lying on l, such that the segment AB does not contain a point of l; such points lie on the *same side* of l as the point A. The remaining points B in α are such that the segment AB does contain a point of l; such points lie on the *opposite side* of l from the point A. Similarly, any plane α divides the points of space not lying on α, into two classes on *opposite sides* of α.

The *angle* BAC ($\angle BAC$) consists of the point A (the *vertex* of the angle) and the two rays AB, AC (the *sides* of the angle). The three angles BAC, CBA, ACB are called the *angles* of the triangle ABC. The angle BAC is said to be *included* between the sides AB, AC of the triangle ABC.

With Hilbert, let us take the relation of "congruence" to to be defined implicitly by the following assumptions:

AXIOMS OF CONGRUENCE

10. *If A, B are any two distinct points on a line l, and A' is a point on a line l', then there are two and only two points B', B'' on l', where B', B'' are on opposite sides of A', such that the segments AB and $A'B'$ are congruent to one another,*† *and AB and $A'B''$ are congruent to one another; in symbols:*

$$AB \equiv A'B' \text{ and } AB \equiv A'B''.$$

Every segment is congruent to itself.‡

11. *Two segments congruent to the same segment are congruent to one another.*||

†The relation is *symmetric.*

‡The relation is *reflexive.*

||The relation is *transitive.*

12. *If the points A, B, C are in the order ABC, and if A', B', C' are in the order $A'B'C'$, and moreover, if*

$$AB \equiv A'B' \text{ and } BC \equiv B'C', \text{ then } AC \equiv A'C'.$$

13. *If BAC is an angle whose sides do not lie in the same line and A', B' are two distinct points, then there are two and only two distinct rays $A'C'$, $A'C''$ from A', where C', C'' are on opposite sides of $A'B'$, such that the two angles BAC and $B'A'C'$ are congruent to one another, and BAC and $B'A'C''$ are congruent to one another; in symbols:*

$$\angle BAC \equiv \angle B'A'C' \text{ and } \angle BAC \equiv \angle B'A'C''.$$

Every angle is congruent to itself.

14. *Two angles congruent to the same angle are congruent to one another.*

15. *If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the remaining angles of the first triangle are congruent to the corresponding angles of the second triangle.*

We shall not attempt to deduce the familiar properties of Euclidean geometry which follow from these assumptions. It is important to notice, however, that in axiom 15 we explicitly assume the theorem which Euclid proved by the method of superposition. The difficulty with this method is that it implies that the triangle is a material object which can be moved about. Secondly, it implies that this material object is *rigid*. Now there is no perfectly rigid body in nature; but in any case, what do we mean by rigid? We cannot define the term without reference to congruence. If we had taken *motion* as a fundamental concept, we could have defined congruence explicitly and proved axiom 15.

With the help of these fifteen axioms we are in a position to prove an important "existence theorem."

5.22. *If A is any point and l any line not passing through A, there is at least one line through A coplanar with l and not meeting it.*

Hilbert's proof is simple and direct. If B, C are any two distinct points on l, construct $\angle BAD$ congruent to $\angle ABC$, where D, C are on opposite sides of the line AB.

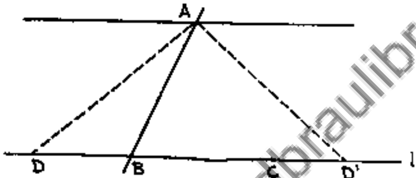


FIG. 5.2B

Two possibilities arise according as AD does or does not intersect l. If AD does intersect l, let the point of intersection be D, as in Fig. 5.2B. If D, D' are on opposite sides of B and if $BD' \equiv AD$, then, in the two triangles ABD and BAD', $\angle ABD \equiv \angle BAD'$ by axiom 15. It is not difficult to prove that, since $\angle ABD$ and $\angle ABD'$ are *supplementary*, then $\angle BAD'$ and $\angle BAD$ are also supplementary, and D, A, D' are collinear. Since this is impossible, AD does not meet l.

In order to limit the number of lines through A not intersecting l, we assume the

AXIOM OF PARALLELISM

16. *If A is any point and l any line not passing through A, there is not more than one line through A coplanar with l and not meeting it.*

Two non-intersecting coplanar lines are said to be *parallel*, and from the properties of parallel lines Euclid proved that "the sum of the angles of a triangle is two right angles." Conversely, if we assume this theorem it follows that there is not more than one line through a given point coplanar with a given line and not meeting it. If there is more than one such line, then the sum of the angles of a triangle is less than two right angles and conversely, and the resulting geometry is known as *hyperbolic geometry*. By taking an undefined order relation of the type which we shall consider in chapter VIII, our proof of 5.22 is no longer valid, and there may be no line through a given point coplanar with a given line and not meeting it; in this case, the sum of the angles of a triangle is greater than two right angles, and we have *elliptic geometry*.

The introduction of continuity is an even more subtle problem than the introduction of congruence. The Greeks were able to take the initial step in its solution by the recognition of the necessity of assuming Archimedes' Axiom. The concluding step required the mathematical maturity which was to come from the development of the Theory of Functions in the nineteenth century. A systematic discussion of *number* then became possible, which brought with it a deeper understanding of continuity and its relation to geometry. This subject will occupy our attention in the following chapters; here we shall only state Hilbert's

AXIOMS OF CONTINUITY

17. *Let A_1 be any point between two arbitrarily chosen points A, B. Take the points A_2, A_3, A_4, . . . such that A_1 lies between A and A_2, A_2 between A_1 and A_3, etc., and such that the segments AA_1, A_1A_2, A_2A_3, . . . are congruent to one another. Then there exists a point A_n such that B lies between A and A_n* (Archimedes' Axiom).

18. *The points of a line form a system of points such that no new points can be added to the space and assigned to the line without violating one of the other axioms* (Axiom of Completeness).

5.3. Duality in Euclidean Geometry. An important difference between projective and Euclidean geometry is that in the latter we cannot formulate a Principle of Duality, either in the plane or in space. The idea of speaking of two parallel lines as having a *point at infinity* in common, is at least as old as Desargues. From this point of view the theorem that *if pairs of corresponding sides of two triangles in a plane are parallel to each other, then the joins of corresponding vertices are concurrent or are parallel to one another, and conversely*, may be recognized as a special case of Desargues' Theorem. The proof follows immediately from the properties of triangles. Again, the theorem that *if A, B, C are any three points of a line l and A', B', C' any three points of a line l' intersecting l, such that BC' is parallel to $B'C$, and CA' is parallel to $C'A$, then AB' is parallel to $A'B$*, is a special case of Pappus' Theorem. The following simple proof is due to Hilbert: If l and l' intersect in O, as in Fig. 5.3A, construct through the point B a line BD' making $\angle OD'B = \angle OCA' = \angle OAC'$, since CA' is parallel to $C'A$.

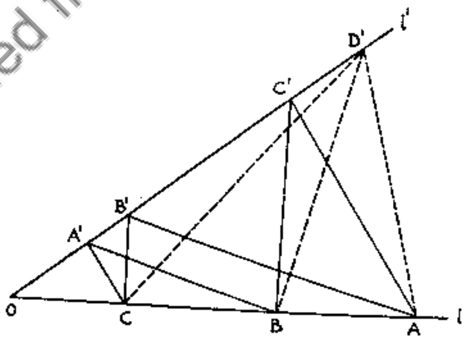


FIG. 5.3A

Clearly, B, C, D', A' are concyclic, and hence $\angle OBA' \equiv \angle OD'C$. Again, A, B, D', C' are concyclic, and hence $\angle OAD' \equiv$

$\angle OC'B \equiv \angle OB'C$ since BC' is parallel to $B'C$. It follows that A, C, D', B' are concyclic, and we conclude that $\angle OBA' \equiv \angle OD'C \equiv \angle OAB'$, and AB' is parallel to $A'B$.

If O is any point in space, the totality of lines through O is called a *bundle* having O as *centre*. Any two lines through O determine a plane through O, and any two planes through O have a line in common. Hence, *there is a Principle of Duality for a bundle*, not only in projective geometry but also in Euclidean geometry. In projective geometry a bundle of lines and planes is the space-dual of the lines and points in a plane; dual to a triangle in the plane, we have a *trihedron* with vertex O. Any three lines a, b, c through O determine such a trihedron abc, whose *faces* are the planes defined by the *edges* a, b, c taken two at a time.

The space-dual of Desargues' Theorem in the projective plane is a theorem concerning two trihedra having a common vertex. In view of what we have said, we should expect it to be valid for a bundle in Euclidean geometry.

5.31. *If the planes determined by pairs of corresponding edges of two trihedra abc and a′b′c′, having the same vertex O, meet in a line l, then the lines of intersection of corresponding faces lie in a plane, and conversely.*

In proving Desargues' Theorem in the projective plane, we found it necessary to make use of the properties of projective space. It is not surprising, then, that we must give an independent proof of 5.31 in Euclidean space; this proof is based upon axioms 1-9 alone.

By hypothesis, the three lines a, a', l in Fig. 5.3B are coplanar and pass through O. If A is a point on a distinct from O, we may choose A' on a' in such a manner that AA' meets l in a point L, lying *between* A and A'. Since b, b', l are coplanar, we may similarly choose a point B' on b' so that LB' meets b in a point B lying between L and B'. Again, we may choose† a point C on c and not in the plane ABL, so that LC meets c' in a point C' lying between L and C. Thus the two planes ABC and $A'B'C'$ are distinct. Applying 5.21 to the triangle LBC, we conclude that BC, $B'C'$ intersect in a point A''. Similarly, by applying axiom 7 to the two triangles LAC and $LA'B'$, we conclude that AC, $A'C'$ and AB, $A'B'$ intersect respectively in the points B'' and C''. But these

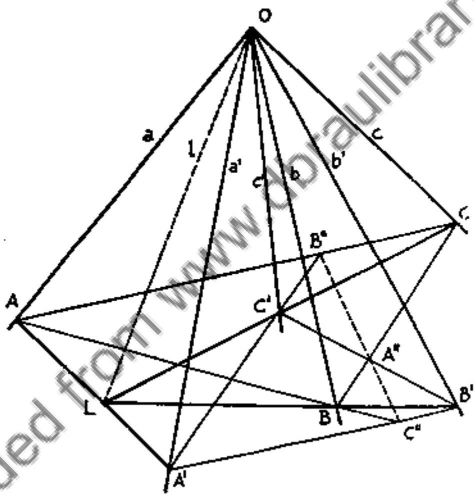


FIG. 5.3B

three points A'', B'', C'' lie on each of the two planes ABC and $A'B'C'$, so they must be collinear, and OA'', OB'', OC'' are coplanar as we desired to show.

The converse of 5.31 is the dual theorem in the bundle, but we cannot appeal to the Principle of Duality in the bundle

†The point O, and hence the line c, cannot lie in the plane ABL, for then the faces of the two trihedra determined by a, b and a', b' would coincide.

for proof, since the proof of the direct theorem depends on properties of space. Representing the plane determined by the two lines a, a' by (a, a'), if we assume that OA'', OB'', OC'' are coplanar and that (a, a') and (b, b') intersect in a line l, we may suppose that (l, c) and (a', c') intersect in a line c_1. The two trihedra abc, $a'b'c_1$ satisfy the conditions of the theorem. Hence, the lines of intersection of the three pairs of planes, (b, c) and (b', c_1), (a, c) and $(a', c_1) = (a', c')$, (a, b) and (a', b') are coplanar. It follows that (b', c_1) must coincide with (b', c'), and hence that c_1 must coincide with c'.

How can we justify the statement that two non-intersecting lines (parallel lines, if we assume axiom 16) have an "ideal point" (point at infinity) in common? To do so, it is necessary to define such "points" so that they will be completely equivalent to the ordinary points of the plane. The method which we shall adopt is to utilize the duality which we have seen to hold between the lines and planes of a bundle, basing our argument in the following section upon 5.31.

5.4. Ideal Elements in the Plane. If we are to relate the points and lines of a plane ω to the lines and planes of a bundle with centre O, not lying on ω, the choice of the point O should not be material to the argument. If O_1 is another point not on ω, we shall show that it is possible, simultaneously and in the same manner, to relate the points and lines of ω to the lines and planes of a bundle with centre O_1. In effect, we shall be setting up a correspondence between the lines and planes of the two bundles. As in the previous section, we shall base our argument upon axioms 1-9.

Any point P of ω determines two lines OP and O_1P; we shall say that these are *corresponding lines* of the bundles with centres O and O_1. Any line a in ω determines a plane α through O, and a plane α_1 through O_1; we shall say that these are *corresponding planes* of the bundles with centres O and O_1.

Clearly, if P lies on a, the line corresponding to OP in α is a line O_1P in α_1. If a, b are any two lines in ω, the planes α, β joining a, b to O intersect in a line l through O; similarly, the planes α_1, β_1 joining a, b to O_1 intersect in a line l_1 through O_1. Any point C of ω determines a plane γ through l, which meets ω in a line c through C. If a, b intersect in a point P, then l, l_1 and c all pass through P, and there is a plane γ_1 through l_1 and c corresponding to γ through l and c. The lines l and l_1 are corresponding lines of the two bundles with centres O and O_1. But, if a, b do *not* intersect, does it still follow that l_1 and c are coplanar? To establish this fact we must leave the intersection of a, b, if it exists, entirely out of consideration.

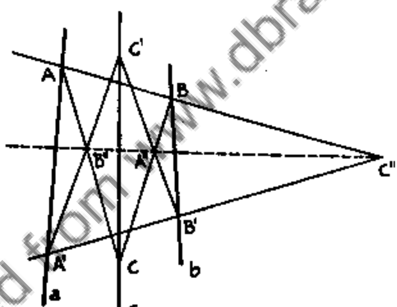


FIG. 5.4A

(i) If A, A' are any two points of the line a on the same side of each of the lines b, c, as in Fig. 5.4A, we may choose a point C'' such that $C''A$, $C''A'$ meet b in B, B' between C'' and A, A' respectively. Again, we may choose a point C' on c such that C, C' are on the same side of each of a, b and such that B', C' are on opposite sides of the line BC. If BC and $B'C'$ intersect in A'', we deduce from axiom 7 applied to the triangle $A'B'C'$ that $C''A''$ meets $C'A'$ in a point B''. Assuming that the three planes α, β, γ through a, b, c pass through l, it follows from 5.31 that AC passes through B''. On the other hand, if O_1 be any point not on ω and if the planes α_1, β_1 through O_1 containing a, b intersect in a line l_1, then, since O_1A'', O_1B'', O_1C'' are coplanar, it follows from the converse of 5.31 that the plane γ_1 through O_1, containing c, passes through l_1. Thus, we may say that *l and l_1 are corresponding lines of the bundles with centres O and O_1, whether a, b intersect or not.*

If a, b do not intersect, we are now justified in postulating the existence of an *ideal point* common to a, b. The logical equivalence of an ideal point with an ordinary point is based upon the equivalence of the lines of the bundle with centre O.

(ii) Instead of two lines in the plane ω, let us consider two points in ω, or more generally, two lines a'', b'' through O. These two lines a'', b'' determine a plane λ through O, and the corresponding lines a_1'', b_1'' through O_1 determine a plane λ_1 through O_1. If c'' is any line† through O in λ, does it follow that the corresponding line c_1'' through O_1 lies in λ_1? Certainly this is true if λ meets ω, but we must prove it without taking this intersection into consideration. We shall reconstruct Fig. 5.3B, taking OA'', OB'', OC'' to be the lines a'', b'', c'', respectively. In any plane through b'' choose two lines a, c through O, meeting ω in A, C, and denote the line of intersection of the two planes (a, c'') and (c, a'') by b. Similarly, in any plane through c'' choose two lines a', b' through O meeting ω in A', B', and denote the line of intersection of the two planes (a', b'') and (b', a'') by c'. From the converse of 5.31 we conclude that the three planes (a, a'), (b, b'), (c, c') intersect in a line l.

If O_1 does not lie on λ, and we join O_1 to A, C in ω, we determine a_1, c_1 corresponding to a, c through O; the planes (a_1, c_1''), (c_1, a_1'') determine the line b_1 through O_1 corresponding

†In particular, we may think of the line c'' as the intersection with the plane λ of a plane through O determined by an arbitrary line in ω.

to b through O. As before, we may determine the trihedron $a_1'b_1'c_1'$ with vertex O_1. Now we have proved in (i) that if the three planes (a, a'), (b, b'), (c, c') meet in a line l, then the corresponding planes (a_1, a_1'), (b_1, b_1'), (c_1, c_1') meet in a line l_1. Hence, we conclude from 5.31 that a_1'', b_1'', c_1'' are coplanar, or that c_1'' lies in the plane λ_1.

If O_1 does lie on λ, we must show that the two planes λ and λ_1 coincide. This follows immediately if λ intersects ω, for the line of intersection may be determined by two points of ω, and λ and λ_1 are determined by these same two points. If λ does not meet ω, it follows that O and O_1 are on the same

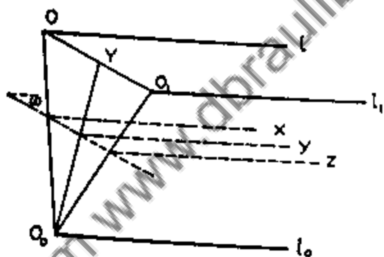


FIG. 5.4B

side of ω. If O_0 is a point on the opposite side of ω, as in Fig. 5.4B, then for any line l through O there is a corresponding line l_0 through O_0. If Y is a point of the segment OO_1 and the planes through l_0 determined by O, Y, O_1 meet ω in the lines x, y, z respectively, we have proved in (i) that the three planes through O containing x, y, z meet in the line l; similarly, the three planes through O_1 containing x, y, z meet in l_1. But with reference to the plane η through l_0 and Y, l and l_1 are corresponding lines of the bundles with centres O and O_1, determined by l_0 and y. Since O and O_1 are on opposite sides of η, a plane λ containing O and O_1 will intersect η; hence, we conclude as before than λ and λ_1 coincide.

We sum up what we have proved in (i) and (ii) in the following theorem:

5.41. *To every line through O there corresponds a definite line through O_1, and conversely. To any three coplanar lines through O there correspond three coplanar lines through O_1—the two planes coinciding if the former plane passes through O_1.*

The significance of this result should be apparent. *Irrespective of the choice of the point O*, any line through O determines a *point* of ω—*ordinary* if the line meets ω, *ideal* if it does not meet ω. Any two lines through O determine a plane through O which may or may not meet ω. In the former case, two points on ω, one or both of which may be ideal, determine an *ordinary line* of ω. In the latter case, the two lines through O determine two ideal points of ω, and we shall say that the plane through O determines an *ideal line* of ω. Clearly, every point of an ideal line must be an ideal point, by axiom 9. The logical equivalence of an ideal line with an ordinary line is based upon the equivalence of the corresponding planes of the bundle. As two ordinary lines in ω intersect in an ordinary or an ideal point, so an ordinary line and an ideal line, or two ideal lines, have an ideal point in common.

It follows from these considerations that a point, whether ordinary or ideal, may be considered as a bundle of lines. Similarly, a line, whether ordinary or ideal, may be considered as a pencil of planes. Desargues' Theorem in the plane follows from 5.31, if the two triangles ABC, $A'B'C'$ in Fig. 5.3B are coplanar. If these two triangles are not coplanar, we obtain Desargues' Theorem in space.

5.5. Ideal Elements in Space. With the introduction of ideal elements in the plane we have recovered the projective plane of chapter II, and it becomes necessary to consider the definition of a plane with reference to these ideal elements.

If P is an ordinary point and l an ordinary line not passing through P, then the class of points lying on lines joining P to the points of l, whether ordinary or ideal, constitutes the *ordinary plane* determined by P and l. If P is an ordinary point and l an ideal line, or if P is an ideal point and l an ordinary line, it follows in a similar manner that P and l determine an ordinary plane.

If both P and l are ideal, the situation is more complicated. In order to speak of the class of points π on the lines joining P to the points on l as a *plane*, it is necessary to show that π meets an arbitrary plane in a line. To this end consider three points A, B, C on l, which are necessarily ideal, and two ordinary points A_1, B_1 on an ordinary line through C. The ordinary point A_1 and the ideal line l determine an ordinary plane ω; let us suppose that AB_1, A_1B intersect in a point C_1 in ω, which may be ordinary or ideal.

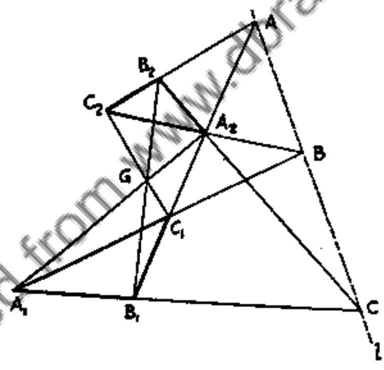


FIG. 5.5A

If C_2 is an ordinary point of ω, we may choose an ordinary point A_2 on C_2B such that A_1 and A_2 are on opposite sides of the ordinary line C_1C_2, as in Fig. 5.5A. If A_1A_2 meets C_1C_2 in G, and A_2C meets C_2A in B_2, then the ordinary line B_1B_2 passes through G, in virtue of Desargues' Theorem in the plane ω.

If ω' is an ordinary plane distinct from ω and not passing through P (i.e. not belonging to the bundle defining P), ω' intersects the three lines PA, PB, PC in three points

A', B', C' of π, which may be ordinary or ideal; we must show that A', B', C' are collinear. Aside from l, all the lines in Fig. 5.5A are ordinary lines which determine ordinary planes through P. These ordinary planes intersect ω' in ordinary or ideal lines, defining two triangles $A_1'B_1'C_1'$, $A_2'B_2'C_2'$ which are in perspective from a point G'. Corresponding sides of these two triangles intersect in A', B', C', so we conclude from Desargues' Theorem in ω' that A', B', C' lie on a line l', which may be ordinary or ideal. It follows immediately that if two points of a line, whether ordinary or ideal, lie in π, then every point of the line lies in π, and any two such lines in π have a point in common.

Two possibilities arise, according as π does or does not contain at least one ordinary point. In the former case, π must be an ordinary plane. In the latter case, we shall call π an *ideal plane*. Since an ideal plane contains no ordinary points, it contains no ordinary lines. Clearly, an ideal line which does not lie in an ordinary plane intersects this ordinary plane in an ideal point. Similarly, an ideal line which does not lie in an ideal plane intersects this ideal plane in an ideal point. Thus, two ideal planes have an ideal line in common, which is determined by the points of intersection of two ideal lines in one plane with the other plane. Finally, three planes have a point (or line) in common, which may be ordinary or ideal; if one of the three planes is ideal, then this point (or line) must also be ideal. Thus, the points, lines and planes in space, ordinary and ideal, satisfy the axioms I-VII of projective geometry.

5.6. Ideal Elements in Euclidean Geometry. With the assumption of the Axiom of Parallelism the situation is much simplified. If a line l is parallel to a line a, and if l is also parallel to a line b, then it follows that a, b are parallel to one another, and this is so whether the three lines l, a, b are coplanar or not. If l does not lie in the plane ω containing a, b then any point C in ω determines a plane through l meeting ω in a line c through C and parallel to each of l, a, b; l is said to be *parallel* to the plane ω. Conversely, if we take any point O not on ω, the planes through O determined by any three parallel lines a, b, c in ω, intersect in a line l through O parallel to ω. Two lines through O parallel to ω determine a plane parallel to ω, and there is one and only one such plane through each point O.

These properties of parallelism in Euclidean geometry enable us to formulate a theorem analogous to 5.41, and to speak of ideal points, lines and planes. In this case, however, there is only one ideal point on a given line, only one ideal line in a given plane, and only one ideal plane in space, which we may call the *point at infinity* of the line, the *line at infinity* of the plane, and the *plane at infinity* in space. In particular then, Euclidean geometry, when modified by the adjunction of these elements "at infinity," satisfies the axioms I-VII of projective geometry. From Pappus' Theorem, proved† at the beginning of §5.3, the Fundamental Theorem of projective geometry follows immediately.

In making a geometrical construction one is often faced with the difficulty that the point of intersection of two lines does not lie on the given sheet of paper; such a point may be said to be *inaccessible*, in contrast to a point on the paper which is *accessible*. With the aid of Desargues' Theorem in the plane it is possible to construct a line c through a given accessible point C and the inaccessible point common to two lines a, b. Referring to Fig. 5.4A, if C'' is any accessible point on the same side of a, b, and two lines through C'' meet a, b in the accessible points A, A' and B, B' respectively,

†The proof of Pappus' Theorem when the Pappus line is the line at infinity is sufficient to prove it in general, since any line can be projected into a line at infinity.

then, if A'' is any point between B and C, $C''A''$ will meet AC in a point B'' between A and C. If $A'B''$ and $B'A''$ meet in the accessible point C', CC' is the required line. *It is unnecessary to distinguish whether or not a, b are parallel.*

Two inaccesible points determine a line, part of which may very well lie on the sheet of paper. Let us suppose that the two pairs of lines a, a' and b, b' determine two inaccessible points (a, a') and (b, b'), and that (a, b) and (a', b') are accessible points. If O is any accessible point on the line joining (a, b) and (a', b'), and p and q are two lines through O meeting a, b, a', b' in accessible points, denote the lines joining their points of intersection, taken in pairs, by c, d, c', d', as in Fig. 5.6A.

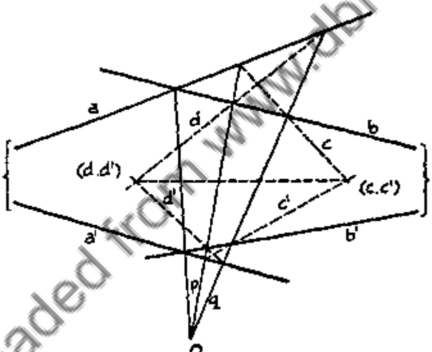


FIG. 5.6A

The triangles abc, $a'b'c'$ and also the triangles abd, $a'b'd'$ are in perspective from O. Hence, by Desargues' Theorem in the plane, (c, c') and (d, d') lie on, and determine the line joining (a, a') and (b, b').

PART II

CHAPTER VI

NUMBER

6.1. Summary of the Chapter. We have already referred to the fact that a proper understanding of continuity is best obtained through a study of number. In turn, the integers themselves are intimately connected with the logical process. In §6.2 we shall refer to this connection and state the laws of addition and multiplication of integers. Extending this concept of number, we shall define *integral numbers* (positive and negative integers) in §6.3, *rational numbers* in §6.4, *real numbers* in §6.5 and finally *complex numbers* in §6.6. The remaining two sections of the chapter will be devoted to the introduction of the general concepts of *ring* and *field*. While our remarks in all cases will be brief, we shall try to emphasize those ideas which are so remarkably significant in geometry.

6.2. Number. The introduction of the ordinary integers of arithmetic from a purely logical point of view, is a difficult and delicate piece of work. The method adopted by Peano is based upon the principle of mathematical induction. The concept of a *series* of objects, whether physical or psychological, in which each object has a determinate successor and, except for the first, a determinate predecessor, plays a fundamental rôle. From such series are abstracted the integers

$$1, 2, 3, \ldots .$$

A second method, which is due to Russell, is based upon the concept of a *class*. Two classes are *similar*, if it is possible to set up a correspondence between the terms of the classes. For example, the class of points on a given line in a projective space is similar to the class of points on any other line in that space. This property enables us to define the number of terms in a class as "the class of classes similar to the given class." A number is thus only a symbol for a class; in particular, the number 1 is the symbol for the class of single terms.

It is beyond the scope of this book to discuss either of these methods in detail. Suffice it to say that the two operations of *addition* (+) and *multiplication* (.), which can be performed upon the integers $a, b, c, \ldots$, may be defined in a purely logical manner. Representing the identity of two integers by the symbol $=$, it follows that

$$a+b=b+a,$$

which is known as the *commutative law of addition*; again,

$$a+(b+c)=(a+b)+c,$$

which is known as the *associative law of addition*. Similarly,

$$a.b=b.a,$$

which is the *commutative law of multiplication*, and

$$a.(b.c)=(a.b).c,$$

which is the *associative law of multiplication*. The relation between addition and multiplication is given by

$$a.b=a+a+a+\ldots\ (b \text{ times});$$

from which we deduce that

$$a.(b+c)=a.b+a.c,$$

which is the *distributive law of multiplication*.

A consideration of these operations of addition and multiplication raises two questions: is there a number x such that

6.21. $$a+x=b,$$

and is there a number x such that

6.22. $$a.x=b,$$

for any pair of numbers a, b? We can answer these questions only when we have introduced an *order relation* amongst the integers. From Peano's point of view such a relation is implicit in the definition of an integer: n is *less than* ($<$) $n+1$, and $n+1$ is *greater than* ($>$) n. This order relation is transitive, and

$$1<2<3< \dots .$$

In Russell's treatment, if any two finite† classes A, B are similar to one another, the number of terms in A is equal to the number of terms in B; while if A is similar to a proper part of B, the number of terms in A is less than the number of terms in B.

From such an order relation we conclude that there is an integer x satisfying 6.21, if and only if $b>a$; and for only certain $b>a$ is there an x satisfying 6.22. In the following section we shall extend our concept of number to include the solutions of 6.21, making a further extension in §6.4 to include the solutions of 6.22.

6.3. Integral Numbers. Let us associate with any *two* integers a, b the symbol (a, b), or more familiarly $a-b$, which shall satisfy the following relations:

(1) $(a, b)=(a', b')$ if and only if $a+b'=b+a'$;

(2) $(a, b)+(a', b')=(a+a', b+b')$;

(3) $(a, b) \cdot (a', b')=(aa'+bb', ab'+ba')$.‡

Clearly, these symbols obey the commutative and the associative laws of addition, and the commutative, associative and distributive laws of multiplication. It is for this reason that we call them *numbers*, ordering them by requiring that

(4) $(a, b) \lesseqgtr (a', b')$ according as $a+b' \lesseqgtr b+a'$.

†The advantage of Russell's method is that finite and "infinite" classes, or numbers, are treated on the same footing.

‡Each term in (2) and (3) may be replaced by an equal term, in the sense of (1).

In particular, since $(a+b, b)=(a+b', b')$ from (1), these rules of combination yield the same relations between the symbols $(a+b, b)$ as hold between the integers a; thus we may identify $(a+b, b)$ with a, writing

(5) $$(a+b, b)=a.$$

Again, since all the numbers (a, a) are equal, let us write

$$(a, a)=0,$$

which we shall call the number *zero*. If $(a, b)=A$, it follows from (2) that

$$A+0=A,$$

and from (3) that

$$A.0=0.$$

Let us write $(b, a)=-A$, which we shall call *minus* A. Clearly,

$$-(-A)=-(b, a)=(a, b)=A,$$

and

$$A+(-A)=(a, b)+(b, a)=(a+b, a+b)=0.$$

If the integer a is greater than the integer b, then $A=(a,b)>0$, and we shall say that A is *positive*; also, $-A=(b, a)<0$, and we shall say that $-A$ is *negative*. If $A=(a, b)$ and $A'=(a', b')$ are both positive, it follows from (2) and (3) that $A+A'$ and $A.A'$ are also positive. Our purpose in avoiding the more familiar notation has been to emphasize the fact that the rules of combination (1)-(5) are matters of definition.

We are now in a position to solve 6.21 for all a, b; but, more generally, we may solve the equation

6.31. $$(a, b)+x=(a', b').$$

The solution is evidently given by $x=(b+a', a+b')$, which enables us to speak of the *subtraction* of any two integral numbers.

6.4. Rational Numbers. As we have already remarked, an integral solution of 6.22 is possible in certain cases provided $b>a$. Our present concern is to extend the concept of number so that a solution will be possible in every case.

Representing an integral number, for the time being, by a small Greek letter, let us associate with any two such numbers, say α and β, the symbol $\{\alpha, \beta\}$ which is more familiarly written α/β. *We shall assume that* $\beta > 0$, and that these symbols satisfy the following relations:

(1′) $\{\alpha, \beta\} = \{\alpha', \beta'\}$, if and only if $\alpha\beta' = \beta\alpha'$;
(2′) $\{\alpha, \beta\} + \{\alpha', \beta'\} = \{\alpha\beta' + \beta\alpha', \beta\beta'\}$;
(3′) $\{\alpha, \beta\} \cdot \{\alpha', \beta'\} = \{\alpha\alpha', \beta\beta'\}$;
(4′) $\{\alpha, \beta\} \lesseqgtr \{\alpha', \beta'\}$, according as $\alpha\beta' \lesseqgtr \beta\alpha'$.

If β and β' are both greater than zero, then $\beta\beta' > 0$ in (2′) and (3′). It may be verified at once that these symbols obey the commutative and associative laws of addition, and the commutative, associative and distributive laws of multiplication. We shall call them *rational numbers*. Corresponding to (5), we may identify the rational number $\{\alpha\beta, \beta\} = \{\alpha, 1\}$ with the integral number α, writing:

(5′) $$\{\alpha, 1\} = \alpha.$$

The assumption that $\beta > 0$ is allowable since

$$\{\alpha, \beta\} = \{-\alpha, -\beta\},$$

from (1′). If we set $\{\alpha, \beta\} = A$, and $\{-\alpha, \beta\} = -A$, then

$$-(-A) = -\{-\alpha, \beta\} = \{\alpha, \beta\} = A,$$

and

$$A + (-A) = \{\alpha, \beta\} + \{-\alpha, \beta\} = \{0, 1\} = 0.$$

Also,

$$A = \{\alpha, \beta\} > \{0, 1\} = 0, \text{ or } A = \{\alpha, \beta\} < \{0, 1\} = 0,$$

according as $\alpha > 0$ or $\alpha < 0$; it is natural to say that $\{\alpha, \beta\}$ is *positive* in the former case and *negative* in the latter. If A and A' are both positive it follows from (2′) and (3′) that $A + A'$ and $A.A'$ are also positive. It should be remarked that this ordering of the rational numbers preserves our former ordering of the integral numbers.

In order to define the *subtraction* of any two rational numbers, we note that the solution of the equation

6.41. $$\{\alpha, \beta\} + x = \{\alpha', \beta'\},$$

is given by $x = \{\beta\alpha' - \alpha\beta', \beta\beta'\}$. We are now in a position to solve the equation 6.22, but we may generalize it, considering instead

6.42. $$\{\alpha, \beta\} . x = \{\alpha', \beta'\}.$$

If $\alpha \neq 0$, the solution of 6.42 is given by $x = \{\beta\alpha', \alpha\beta'\}$, which we shall call the *quotient* of $\{\alpha', \beta'\}$ by $\{\alpha, \beta\}$, or the *inverse* of $\{\alpha, \beta\}$ if $\{\alpha', \beta'\} = 1$.

6.5. Real Numbers. Greek mathematics recognized the importance of *ratio* in representing a number, but a clear conception of numbers which cannot be thus represented, i.e. those which are *irrational*, did not come until the latter part of the nineteenth century. At the risk of being tedious, we shall give Cantor's theory in some detail, since the ideas involved will be fundamental in a later chapter. Afterwards, we shall deduce the Dedekind property of a real number from Cantor's definition.

Consider an unending sequence of rational numbers

$$a_1, a_2, a_3, \ldots,$$

which we may represent by $\{a_i\}$. We shall say that $\{a_i\}$ is *convergent* if, for every rational number $\epsilon > 0$, there is an integer n_ϵ, such that for $n > n_\epsilon$,

$$|a_n - a_{n+m}| < \epsilon,$$

for all $m>0$. As usual, we understand by $|x|$ the *absolute value* of x, or that one of x and $-x$ which is positive. In §6.3, any two integers defined an integral number, and in §6.4, any two integral numbers defined a rational number. Following Cantor, let the convergent sequence of rational numbers $\{a_i\}$ define the symbol $[a_i]$, which we shall call a *real number*.

To justify such a definition, we must show that it is possible to define addition and multiplication of the symbols $[a_i]$, and that these operations satisfy the familiar laws. First, let us say that $[a_i]$ is *equal* to $[b_i]$, if, for every rational number $\epsilon>0$, there is an integer n_0, such that for $n>n_0$,

$$|a_n-b_n|<\epsilon.$$

Without going into the proofs, we know that if $\{a_i\}$, $\{b_i\}$ are two convergent sequences of rational numbers, then

$$\{a_i+b_i\} \text{ and } \{a_i.b_i\},$$

are also convergent. Let us define the *sum* of $[a_i]$ and $[b_i]$ to be $[a_i+b_i]$, and the *product* of $[a_i]$ and $[b_i]$ to be $[a_i.b_i]$, writing:

$$[a_i]+[b_i]=[a_i+b_i] \text{ and } [a_i]\,.\,[b_i]=[a_i\,.\,b_i].$$

Since addition of rational numbers is commutative and associative, so also is addition of real numbers. Similarly, multiplication of real numbers is commutative, associative and distributive. Moreover, if $[a_i]=[a_i']$ and $[b_i]=[b_i']$, then $[a_i]+[b_i]=[a_i']+[b_i']$ and $[a_i]\,.\,[b_i]=[a_i']\,.\,[b_i']$.

If there exists a rational number $\delta>0$ and an integer n_0, such that for $n>n_0$,

$$-(a_n-b_n)>\delta>0,$$

we shall say that $[a_i]$ is *less than* $[b_i]$. On the other hand, if, under the same circumstances,

$$+(a_n-b_n)>\delta>0,$$

we shall say that $[a_i]$ is *greater than* $[b_i]$. If there is no such rational number $\delta > 0$, we conclude that $[a_i] = [b_i]$.

If a is any rational number, the sequence

$$a, a, a, \ldots,$$

or $\{a\}$, is convergent and defines the real number $[a]$. Since the rules of combination yield the same relations between the symbols $[a]$, $[b]$, ... as hold between the rational numbers $a, b, \ldots$, we may identify the real number $[a]$ with the rational number a, writing:

$$[a] = a.$$

In particular, the sequences $\{0\}$ and $\{1\}$ define the real numbers $[0] = 0$ and $[1] = 1$. Evidently,

$$[a_i] + [0] = [a_i], \quad [a_i] \,.\, [0] = [0], \quad [a_i] \,.\, [1] = [a_i].$$

If $[a_i] > [0]$, we shall say that $[a_i]$ is *positive*; and if $[a_i] < [0]$, we shall say that $[a_i]$ is *negative*. It follows from our definitions that the sum and the product of two positive real numbers is a positive real number.†

Assuming that $\{a_i\}$, $\{b_i\}$ are two convergent sequences of rational numbers, we know that the sequence

$$\{a_i - b_i\}$$

is also convergent. Thus we are able to define the *subtraction* of two real numbers, writing

$$[a_i] - [b_i] = [a_i - b_i],$$

which ensures that the equation

6.51. $$[a_i] + x = [b_i],$$

has as solution $x = [b_i - a_i]$, in every case. Again, we know that the sequence

$$\{a_i/b_i\}, \text{ where } b_i \neq 0,$$

†Could a convergent sequence of real numbers be taken to represent some new type of number, of a still more general character? This question is of great importance in geometry. We shall consider the matter in §8.2, showing that the answer must be in the negative.

is convergent, *provided* there is a rational number $\delta > 0$, and an integer n_0, such that for $n > n_0$, $|b_n| > \delta$. Clearly, this condition is equivalent to requiring that $[b_i] \neq [0]$, and, with this proviso, we define the *quotient* of $[a_i]$ and $[b_i]$ to be $[a_i/b_i]$, writing

$$[a_i]/[b_i] = [a_i/b_i].$$

Thus the equation

6.52. $$[a_i] \cdot x = [b_i]$$

has a solution $x = [b_i/a_i]$, provided $[a_i] \neq [0]$.

Dedekind's theory of real numbers is of quite a different character and utilizes the notion of a *cut* in the aggregate of rational numbers. We think of this aggregate as being divided into two classes R_1 and R_2, according to some rule, such that every number in R_1 is less than every number in R_2. Any rational number N would give rise to such a *cut* (R_1, R_2), where now R_1 contains all rational numbers less than N and R_2 contains all rational numbers greater than N; we may say that N belongs either to R_1 or to R_2. To say that all rational numbers whose cube is less than 2 belong to R_1 and all rational numbers whose cube is greater than 2 belong to R_2, defines a cut (R_1, R_2), but this case is different from the former in that there is no greatest number in R_1 and no least number in R_2. Dedekind's theory is based upon the following definition:

Every cut (R_1, R_2) *in the aggregate* R *of rational numbers, such that every number of* R *belongs to one or other of the two classes* R_1, R_2 *and every number in* R_1 *is less than every number in* R_2, *defines a REAL NUMBER.*

In case either R_1 *contains a greatest number* x, *or* R_2 *a least number* x, *the cut* (R_1, R_2) *defines a real number which we shall identify with the rational number* x.

In case neither R_1 *contains a greatest number nor* R_2 *a least number, we shall speak of the real number defined by the cut* (R_1, R_2) *as an "irrational number."*

In order to deduce the Dedekind property from Cantor's definition, let r be any rational number and $[x_i]$ a definite real number. If $[x_i] \neq [r]$, there exists a rational number $\delta > 0$ and an integer n_0, such that for $n > n_0$,

$$[x_i] \lessgtr [r], \text{ according as } \mp(x_n - r) > \delta > 0.$$

Let us place every rational number r for which $-(x_n - r) > 0$ in the class R_1, and every rational number r for which $+(x_n - r) > 0$ in the class R_2. If there exists a rational number r such that $[x_i] = [r]$, we may place r either in R_1 or in R_2. *Thus the real number $[x_i]$ determines a unique cut (R_1, R_2) in the aggregate of rational numbers.* Conversely, it is possible to deduce the Cantor property from Dedekind's definition, proving that the two points of view are logically equivalent. We have favoured Cantor's approach, largely because the rules of combination of real numbers are more conveniently expressed in terms of sequences than in terms of cuts. It may also be argued that the notion of a convergent sequence of rational numbers is simpler and less open to question on logical grounds than the notion of a cut in the aggregate of rational numbers, but we shall not enter upon this discussion.

There is a familiar but most important property of real numbers which has far-reaching consequences in both algebra and geometry, namely:

6.53. *For any real number $a > 0$ we can always find an integer N, such that $N > a$.*

Consider this "Archimedean" property first with reference to a rational number a. Assuming that $a = a_1/a_2$, where a_1, a_2 are positive integers, it is sufficient to choose $N = a_1 + 1$. If $\{a_i\}$ is a convergent sequence of rational numbers defining the real number $[a_i]$, and if $n > n_\epsilon$, we have

$$\begin{aligned} |a_{n+m}| &= |a_n - (a_n - a_{n+m})| \\ &\leqslant |a_n| + |a_n - a_{n+m}| \\ &< |a_n| + \epsilon. \end{aligned}$$

It follows from this inequality and the Archimedean property of rational numbers that there is an integer A such that $A > |a_i|$, for every i. Thus, for any integer $N > A$, we conclude that

$$N = [N] > [a_i].$$

In spite of many similarities, there is one great difference between the aggregate of rational numbers and the aggregate of real numbers. This difference lies in the fact that the rational numbers may be set into correspondence with the positive integers, i.e. they are "countable," while the real numbers will not admit such a correspondence. Referring to the aggregate of real numbers as the *arithmetic continuum*, we shall say that x is *continuous* if it may be set equal to any real number; these numbers are called the "values" of the continuous "variable" x. There are many unsolved problems here,† and it is not surprising that this notion of continuity should have provoked so much philosophical discussion.

6.6. Complex Numbers. In seeking a solution of certain simple algebraic equations, e.g. $x^3 = 2$, we were led to the introduction of irrational numbers. If we pursue this quest, it is clear that we must further extend our number system to include solutions of equations of the form $x^2 + 1 = 0$.

Analogously to the introduction of negative and fractional numbers, let us suppose that a pair of real numbers a, b defines a symbol $[a, b]$, and that these symbols shall satisfy the following relations:

(1″) $[a, b] = [a', b']$, if and only if $a = a'$, $b = b'$;

(2″) $[a, b] + [a', b'] = [a + a', b + b']$;

(3″) $[a, b] \cdot [a', b'] = [aa' - bb', ab' + ba']$;

(4″) $[a, 0] = a$.

It is significant that we do not order these *complex numbers*

†Cf. *The Infinite in Mathematics*, in this series.

$[a, b]$. It follows from (2″) and (3″) that complex numbers satisfy the commutative and associative laws of addition, and the commutative, associative and distributive laws of multiplication. A more familiar notation is to write

$$[a, b] = a+ib,$$

where $i=[0, 1]$ so that $i^2=[-1, 0]=-1$, but this is only a matter of convenience.

It is a remarkable fact that any algebraic equation with real or complex coefficients may be solved in terms of complex numbers, and the number of solutions is equal to the degree of the equation. This theorem is known as the Fundamental Theorem of algebra. The simplicity of expression thus introduced into algebra has its counterpart in geometry. The recognition of the significance of *imaginary elements* by Poncelet (1788-1867) paved the way for what may be called "modern" geometry. These imaginary elements were accepted with reluctance, however, and von Staudt constructed a theory in which such an element is represented by certain real elements. We shall return to this question in chapter IX.

6.7. Rings and Fields. A very interesting generalization of complex numbers was given by Sir W. R. Hamilton, and to this generalized system he gave the name *quaternions*. If the *basis elements* are 1, i, j, k, where

$$i^2=j^2=k^2=ijk=-1,$$

implying

$$jk=i=-kj, \quad ki=j=-ik, \quad ij=k=-ji,$$

a *quaternion* is an expression

$$a+bi+cj+dk,$$

where a, b, c, d are any real numbers. As in the case of complex numbers, this is only a convenient way of representing a symbol $[a, b, c, d]$, which we assume shall satisfy the following relations:

(1‴) $[a, b, c, d] = [a', b', c', d']$, if and only if $a = a'$, $b = b'$, $c = c'$, $d = d'$;

(2‴) $[a, b, c, d] + [a', b', c', d'] = [a+a', b+b', c+c', d+d']$;

(3‴) $[a, b, c, d] \, . \, [a', b', c', d'] = [aa' - bb' - cc' - dd', ab' + ba' + cd' - dc', ac' + ca' + db' - bd', ad' + da' + bc' - cb']$;

(4‴) $[a, 0, 0, 0] = a$.

Clearly, addition is commutative and associative, while multiplication is associative and distributive but *not* commutative. The invention of quaternions marked a great step forward, not for the intrinsic merit of the system, but because of the possibility of having a system at all in which the commutative law of multiplication was not satisfied.

Let us consider two *undefined* operations of addition $(+)$ and multiplication $(.)$ as applicable to a set of abstract elements

$$a, b, c, \ldots,$$

such that from any two elements a, b we may form the *sum* $(a+b)$ and the *product* $(a.b$ or $ab)$, *which themselves are elements of the set.* These operations shall be subject to the following formal laws:

I. LAWS OF ADDITION

(i) *Commutative Law*: $a+b=b+a$;
(ii) *Associative Law*: $a+(b+c)=(a+b)+c$;
(iii) *Solvability of the Equation* $a+x=b$, for all a, b, where x belongs to the set.

II. LAWS OF MULTIPLICATION

(i) *Associative Law*: $a.bc = ab.c$;
(ii) *Distributive Laws*: $a.(b+c) = ab+ac$;
$(b+c).a = ba+ca$.

Such a set of elements is called a *ring*. It is an interesting exercise in manipulation to show that the laws of addition require the existence of an element *zero* (0) in the set, such that $a+0=a$ for all a, and an element $-a$ associated with each a, such that $-(-a)=a$ and $a+(-a)=0$. From the laws of multiplication it follows that $a.0=0$. As an obvious example of a ring, we have the system of integral numbers.

If we require the elements of a ring to satisfy also:

II. (iii) *Solvability of the Equations* $ax=b$, $ya=b$, for all $a(\neq 0)$, b, where x, y belong to the set,

the system is called a *division ring*. Exactly the same argument which would show the existence of a zero element, would show, from II.(iii), the existence of a *unit* element 1 belonging to the set, such that $a.1=1.a=a$. By continued addition of this unit element we may define the *integral elements* of the division ring. The solution of the equation $ax=1$ is called the *right inverse* of a, and the solution of $ya=1$ is called the *left inverse* of a; combining the two ideas, it may be shown that the right inverse is equal to the left inverse, which is called the *inverse* of a and written $1/a$. It follows that if $a.b=0$, then one of a, b must be 0; this is not necessarily true for a ring. The system of quaternions is a division ring; for, referring to their law of multiplication, $[a, b, c, d].[a, -b, -c, -d]=[a^2+b^2+c^2+d^2, 0, 0, 0]$, where $N=a^2+b^2+c^2+d^2$ is a real number. The unit element of the division ring is the quaternion [1, 0, 0, 0], and the inverse of $[a, b, c, d]$ is $[a/N, -b/N, -c/N, -d/N]$.

Finally, if the elements of a division ring satisfy:

II. (iv) *Commutative Law*: $a.b=b.a$, for all a, b,

we shall say that we have a *field*. In this case the two distributive laws II.(ii) are equivalent. The two equations in II.(iii) are also equivalent, and their solution is b/a. This assumption of the commutative law of multiplication introduces great simplification—we have in fact a generalization of number. It is obvious that *the rational numbers form a field*; similarly, we have the *field of real numbers* and the *field of complex numbers*.

6.8. Finite Fields. In defining a division ring in the preceding section, we did not limit the number of elements in any way. We assume now that *the number of elements is finite*, from which it follows that multiplication is commutative, and we have a *finite field*. This elegant result was proved by Wedderburn in 1905.

If the number of elements, or *marks* as they are sometimes called, in our finite field is m, let us denote them by

$$a_0, a_1, a_2, \ldots, a_{m-1};$$

we shall call m the *order* of the finite field. If a_0 and a_1 are the zero and unit elements of the field, we may obtain the integral elements by continued addition, writing:

$$a_{(0)} = a_0,\ a_{(1)} = a_1,\ a_{(2)} = a_1 + a_1, \ldots,$$
$$a_{(p-1)} = a_1 + a_1 + \ldots (p-1) \text{ times.}$$

Since the field is finite, we cannot obtain new elements in this way indefinitely, and there will be a least integer p such that

$$a_{(p)} = a_{(0)}.$$

Clearly, each of the elements $a_{(0)}, a_{(1)}, a_{(2)}, \ldots, a_{(p-1)}$ is distinct, and

$$a_{(p+1)} = a_{(0)} + a_1 = a_{(1)},\ a_{(p+2)} = a_{(1)} + a_1 = a_{(2)}, \text{ etc.}$$

Thus the integral elements of a finite field are related by addition and multiplication, as are the ordinary integers taken modulo p. In particular, we have

$$a_{(k)} \cdot a_{(l)} = a_{(kl)}.$$

6.81. *The integer p is a prime.*

For suppose that $p = p_1 p_2$. Then

$$a_{(p_1)} \cdot a_{(p_2)} = a_{(p_1 p_2)} = a_{(p)} = a_{(0)},$$

and one of $a_{(p_1)}$, $a_{(p_2)}$, say $a_{(p_1)}$, is equal to $a_{(0)}$. Thus $p_1 = p$, and p is a prime.

6.82. *The order m is a power of p, i.e. $m = p^n$.*

Take any element $b_1 \neq a_0$, then

$$b_1 a_{(r_1)}, \; (r_1 = 0, 1, 2, \ldots, p-1),$$

are p distinct elements of the field. If the set is not exhausted, take another element b_2, and

$$b_1 a_{(r_1)} + b_2 a_{(r_2)},$$

are p^2 distinct elements of the field, and so on. Finally, since the process must terminate, we may represent any element of the field in the form

$$b_1 a_{(r_1)} + b_2 a_{(r_2)} + \ldots + b_n a_{(r_n)},$$

and their number is $m = p^n$.

If x is an element of the finite field, the smallest number k for which

$$x^k = x,$$

is called the *order* of the element x. An element whose order is p^n is called a *primitive* element of the field. We shall not attempt to show that such elements always exist, i.e. that

6.83. *Every element of a finite field may be written as the power of a given primitive element of the field.*

Consider two fields σ and σ', which may be finite or infinite, such that a correspondence may be established between the elements $x, y, u, v, \ldots$ of σ and the elements $x', y', u', v', \ldots$ of σ'. If when

$$x+y=u \text{ and } x.y=v,$$

we have

$$x'+y'=u' \text{ and } x'.y'=v',$$

for every pair of elements x, y of σ, then the two fields σ and σ' are said to be *isomorphic*. We state without proof the following important theorem:

6.84. *Any two finite fields of the same order are isomorphic.*

Thus we may speak of *the* finite field of order p^n, which is called a *Galois* field and denoted $GF(p^n)$. The simplest Galois field is obtained by putting $n=1$. In this case the only elements of the field are the integral elements, which combine as do the integers

$$0, 1, 2, \ldots, (p-1),$$

taken modulo p.

CHAPTER VII

COORDINATE SYSTEMS

7.1. Summary of the Chapter. Modern mathematics may be said to have begun with Descartes. The representation of a point by a set of numbers so greatly increased the power of the Greek geometry that a new subject was born. On what is the validity of this representation based? What is the real connection of a system of numbers, or more generally, of an algebraic field, with geometry? The theorems of Desargues and Pappus are of the utmost importance in this connection. By a geometrical construction we may define a point which we may designate as the *sum* of two other points; similarly, we may define a point which is the *product* of two points. The resulting "algebra of points" on a line is a division ring. From axiom IX, it follows that multiplication is commutative, and we have a field. Since these constructions for the sum and product of two points are invariant under projection, the fields of points on different lines are isomorphic. If we take a new field σ, isomorphic with each of these fields, we may call the element of σ associated with a point X on a given line the *non-homogeneous coordinate* of X on that line. The definition of the *cross ratio* $\{A_0A_1, XI\}$ of four points on a line in §7.4, leads immediately to its identification with the coordinate of the point X.

To extend the coordinate system in a line to a plane, and eventually to space, is not a difficult matter. The problem arises of deducing the "equation" of a line in the plane, or of a plane in space. In the solution of this problem we shall follow a method originally due to Sturm and to W. Fiedler, which leads to the desired conclusions with a minimum of effort.

Homogeneous coordinates, which follow naturally from non-homogeneous coordinates, add to the symmetry and elegance of what is usually called *analytical projective geometry*.

In such a brief account of the foundations of the subject, it is not possible to do justice to the principle of duality. The machinery having been set in motion, however, there is no difficulty in completing the discussion from this point of view. In the last section of the chapter we shall consider the case where σ is a finite field, obtaining the more important properties of the associated finite geometry.

7.2. Addition and Multiplication of Points on a Line. If A_0, A_1 are any two distinct points on a line l, consider two other distinct points X, Y on l. If I'' and A' are two points collinear with A_1 but not on l, let us suppose that $I''X$ meets A_0A' in a point X'. If A_1X' meets $A'Y$ in U'', we shall denote the intersection of $I''U''$ and l by U, as in the accompanying Fig. 7.2A.

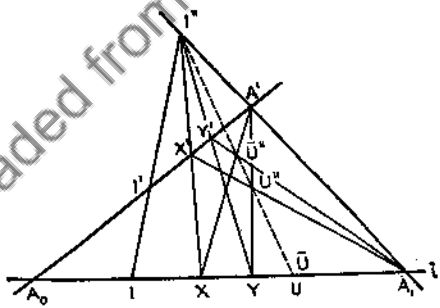


FIG. 7.2A

Let us define U to be the *sum* of the two points X, Y with reference to A_0, A_1, and write

$$X + Y = U.$$

With reference to the complete quadrangle $I''U''A'X'$, it follows from 3.61 that X, Y; A_0, U are point pairs of an involution on l, of which A_1 is a double point. If instead of joining I'' to X we join it to Y, and join A' to X, we may similarly define a point $\overline{U}$ such that

$$Y+X=\overline{U}.$$

But again X, Y; A_0, $\overline{U}$ are point pairs of an involution of which A_1 is a double point. Hence, from the uniqueness of the fourth harmonic point, U and $\overline{U}$ must coincide, i.e.

$$X+Y=Y+X,$$

and addition is commutative. If Z is another point on l, we may construct $Y+Z$, and by a similar argument we may show that

$$(X+Y)+Z=X+(Y+Z).$$

Thus addition is also associative.

The inverse operation of addition is subtraction, and by reversing the construction in Fig. 7.2A we may define a point Y which is the *difference* between U and X. We shall write

$$U-X=Y.$$

In particular, we may construct a point Y which is the difference between A_0 and X; i.e. we may suppose that A_0 and U coincide. In this case, both A_0 and A_1 are double points of the involution, and $H(A_0A_1, XY)$. Let us write $-X$ for Y, leading to the equations

$$A_0-X=-X, \text{ and } X-X=A_0.$$

To define the multiplication of points on l, choose a third point I on l distinct from A_0, A_1. It should be emphasized that *addition is independent of the choice of this third point on l.*

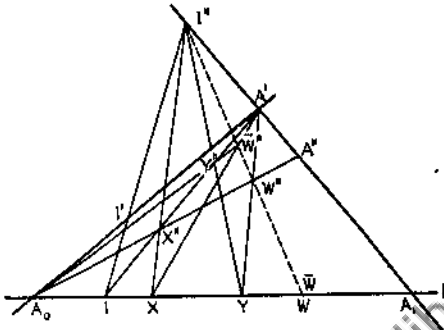


FIG. 7.2B

If $A'I$ and $I''X$ meet in X'', and A_0X'' and $A'Y$ meet in W'', then, denoting the point of intersection of $I''W''$ and l by W, we shall write

$$X \, . \, Y = W.$$

The point W is called the *product* of X and Y. Since the line l is a transversal of the complete quadrangle $I''W''A'X''$, we conclude that A_0, A_1; X, Y; I, W are point pairs of an involution on l. If we interchange the roles of X and Y, and instead of joining I'' to X join I'' to Y, $A'I$ will meet $I''Y$ in a point Y'', and A_0Y'' will meet $A'X$ in $\overline{W}''$. Let $I''\overline{W}''$ meet l in $\overline{W}$. Clearly, the condition that W and $\overline{W}$ shall coincide is that $\overline{W}''$, W'', I'' be collinear; but these three points are the intersections of the cross joins of the two triads of collinear points X'', Y'', A' and Y, X, A_0. Thus *the condition is equivalent to the assumption of Pappus' Theorem.* It follows from axiom IX that

$$X . Y = Y . X,$$

and multiplication is commutative, the product of two points being uniquely defined with reference to A_0, A_1, I. As in the case of addition, if Z is any further point on l, it follows from Desargues' Theorem that

$$X.(Y.Z)=(X.Y).Z,$$

and multiplication is associative. Another consequence of Desargues' Theorem is that

$$X.(Y+Z)=X.Y+X.Z,$$

and multiplication is also distributive.

The inverse of multiplication is division, and by reversing the construction in Fig. 7.2B we may pass from the two points W and X to their *quotient* Y, writing

$$W/X=Y.$$

If, in particular, W and I coincide in a double point of the involution, Y is called the *inverse* of X, and is written I/X. Under these circumstances, the other double point of the involution will be $-I$, and X, I/X are harmonically conjugate with regard to I, $-I$. It is not difficult to see that

$$A_0/X=A_0, \text{ and } X/A_0=A_1.$$

The history of the constructions which we have given in this section goes back to von Staudt (1798-1867). To Hilbert, however, is due the elegant proof that commutativity of multiplication is equivalent to the assumption of Pappus' Theorem. It is hard to overemphasize the significance of these results, for they provide a geometrical illustration of those abstract laws of combination which we considered in such detail in the preceding chapter. If we leave the point A_1 out of consideration, the operations of addition and multiplication, which we have defined with reference to the points on l, satisfy all the conditions for a division ring; if we assume Pappus' Theorem to be valid, they satisfy the conditions for a field. Modern algebra does not seem quite so terrifying when expressed in these geometrical terms!

7.3. Coordinates on a Line. Let us construct the harmonic sequence on l defined by the three points A_0, A_1, I, as in the accompanying Fig. 7.3A, and let us identify the points with those in Fig. 2.7A. By this identification we set up a corres-

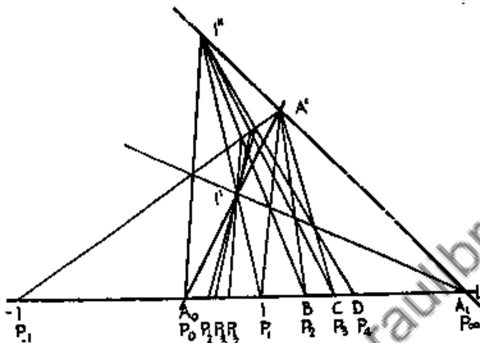


FIG. 7.3A

pondence between the points of the harmonic sequence, and the positive and negative integers of arithmetic. The point P_1 being identified with the point I, it follows that

$$I+I=B=P_2, \quad I+B=C=P_3, \quad I+C=D=P_4, \text{ etc.}$$

Applying our constructions for addition, subtraction, multiplication and division to the points of this harmonic sequence, let us write

$$P_x+P_y=P_{x+y} \text{ and } P_x-P_y=P_{x-y},$$
$$P_x \,.\, P_y=P_{xy} \text{ and } P_x/P_y=P_{x/y}.$$

The harmonic conjugates of P_2, P_3, P_4, ... with regard to P_1 and P_{-1}, are $P_{1/2}$, $P_{1/3}$, $P_{1/4}$, ... The three points P_0, P_∞, $P_{1/n}$ determine the harmonic sequence:

$$\ldots P_{-3/n}, P_{-2/n}, P_{-1/n}, P_0, P_{1/n}, P_{2/n}, P_{3/n}, \ldots$$

Thus, by successive stages, we may arrive at a point $P_{m/n}$, where m, n are any two positive or negative integers. The totality of such points is called the *harmonic net* or *net of rationality*, determined by P_0, P_∞, P_1. We shall designate this net by the symbol $\boldsymbol{R}(P_0P_\infty P_1)$, and we shall speak of x as the *non-homogeneous coordinate* of P_x in the *scale* determined by P_0, P_∞, P_1, or with reference to $\boldsymbol{R}(P_0P_\infty P_1)$. We shall call P_0 the *origin* of coordinates and P_∞ the *point at infinity* on l.

Essentially, what we have done is to set up a correspondence between the field of points $\boldsymbol{R}(A_0A_1I)$, exclusive of A_1, and the field of rational numbers. Our geometrical definitions of addition, subtraction, multiplication and division have the important property that geometrical sums and products correspond to algebraic sums and products; in other words, we have an *isomorphism*. To remove the exceptional character of the point A_1, let us replace x by the *ratio* of two rational numbers:

$$x = \frac{x_1}{x_0},$$

and represent the point $X = P_x$ by the *homogeneous coordinates* (x_0, x_1). Clearly, x_0 and x_1 are determined only to a constant factor different from zero. With this convention, the homogeneous coordinates of A_0 and A_1 may be taken to be $(1, 0)$ and $(0, 1)$, while those of X may be taken to be $(1, x)$.

But what of the other points on the line l? If we should introduce an axiom of closure requiring that there be no other points on l, we should obtain what we may call a "rational" geometry. Such an assumption is quite legitimate, since the rational numbers form a field, but is it desirable? Would such a "rational" geometry give a satisfactory description of our concept of *space*? In the preceding chapter we were led to the introduction of irrational numbers, and thence to a definition of continuity. Could we not use the real number system to give meaning to continuity as applied to geometry? This problem we shall leave to the following chapter. For our present purposes, however, let us represent the field of points $X, Y, \ldots$ on l, exclusive of A_1, by Σ. *We make no assumption* *concerning* Σ, *which may be finite or infinite.* Consider a projectivity between the points of l and the points of another line l', such that

7.31. $(A_0, A_1, I, \ldots X, Y, \ldots) \,\overline{\wedge}\, (A_0', A_1', I', \ldots X', Y', \ldots)$,

and let us represent the field of points $X', Y', \ldots$ on l', exclusive of A_1', by Σ'. If we require that the points A_0, A_1, I shall correspond respectively to A_0', A_1', I', it follows from the invariance of the harmonic property and the Fundamental Theorem of projective geometry, that sums and products are preserved and the two fields Σ and Σ' are isomorphic. Since the l' may be any line whatsoever, we are confronted with the problem of choosing a representative field from the many isomorphic fields $\Sigma, \Sigma', \ldots$ To avoid this arbitrary choice, let us take a new field σ, isomorphic with each of $\Sigma, \Sigma', \ldots$ If the element of σ associated with X is x, we shall say that x is the *non-homogeneous coordinate* of X with reference to $\boldsymbol{R}(A_0A_1I)$. Similarly, we shall say that x is the non-homogeneous coordinate of X' with reference to $\boldsymbol{R}(A_0'A_1'I')$, etc. *Homogeneous coordinates* are defined as before, x_0 and x_1 being elements of σ, but not both the zero element.

7.4. Cross Ratio. Having defined a coordinate system on a line, we are in a position to construct the function $\{XY, ZT\}$, known as the *cross ratio* of the four collinear points $X(x_0, x_1)$, $Y(y_0, y_1)$, $Z(z_0, z_1)$, $T(t_0, t_1)$. Let us write, in homogeneous coordinates:

7.41. $$\{XY, ZT\} = \frac{(x_1z_0 - x_0z_1)(y_1t_0 - y_0t_1)}{(x_1t_0 - x_0t_1)(y_1z_0 - y_0z_1)},$$

and, if $x_0y_0z_0t_0 \neq 0$, in non-homogeneous coordinates

7.41′. $$\{XY, ZT\} = \frac{(x-z)(y-t)}{(x-t)(y-z)}.$$

It is important to observe that our definition does not depend on any concept of length. In particular, if we choose X, Y, T to be the points of reference $A_0(1, 0)$, $A_1(0, 1)$, $I(1, 1)$ respectively, we obtain:

7.42.
$$\{A_0A_1, XI\} = \frac{(0.x_0-1.x_1)(1.1-0.1)}{(0.1-1.1)(1.x_0-0.x_1)},$$
$$= \frac{x_1}{x_0} = x,$$

for any point X on l. This identification of the cross ratio with the non-homogeneous coordinate x of X, with reference to $R(A_0A_1I)$, corresponds to the original procedure of von Staudt and leads to the addition and multiplication of cross ratios, as in the constructions of §7.2. Since the same coordinate x is associated with the point X' with reference to $R(A_0'A_1'I')$, under the projectivity 7.31, it follows that

7.43. *The cross ratio is invariant under a projectivity.*

Consider the *homogeneous linear transformation*

7.44.
$$sx_0' = a_{00}x_0 + a_{01}x_1,$$
$$sx_1' = a_{10}x_0 + a_{11}x_1,$$

where the a_{ij} are elements of the field σ, subject to the condition that $a_{00}a_{11} - a_{01}a_{10} \neq 0$, and where s is an arbitrary non-zero element of σ. In terms of non-homogeneous coordinates, 7.44 may be written as a *fractional linear transformation*

7.44′.
$$x' = \frac{a_{10} + a_{11}x}{a_{00} + a_{01}x}.$$

Since $a_{00}a_{11} - a_{01}a_{10} \neq 0$, we may solve 7.44′ backwards to obtain the inverse transformation

7.45.
$$x = \frac{a_{10} - a_{00}x'}{-a_{11} + a_{01}x'}.$$

Clearly, a transformation such as 7.44 or 7.44′ sets up a correspondence between the points $X(x_0, x_1)$, $X'(x_0', x_1')$ on l;

if we represent the transformation by the letter T, no confusion will result if we represent the correspondence also by T. Though the calculation is somewhat lengthy, it is not difficult to show that *the cross ratio is invariant under a linear transformation.*†

The significance of these conclusions will appear from the following theorem:

7.46. *Any projectivity* Π *in a line l may be expressed as a homogeneous linear transformation*

$$T: \begin{array}{l} sx_0' = a_{00}x_0 + a_{01}x_1 \\ sx_1' = a_{10}x_0 + a_{11}x_1 \end{array}.$$

Conversely, the most general transformation of this form determines a projective correspondence between the points on l.

The Fundamental Theorem of projective geometry states that the projectivity Π is completely determined by assigning three distinct points A_0', A_1', I' on l to correspond respectively to A_0, A_1, I. If we require that these be corresponding points under T, we obtain three equations from which we may determine the ratios of the four constants a_{ij}, and completely determine T. In the correspondence which is set up between the points of l by applying first Π and then the inverse T^{-1} of T, it is clear that the points A_0, A_1, I will remain fixed. Moreover, since the cross ratio $\{A_0A_1, XI\}$ remains unaltered under Π and also under T^{-1}, it remains unaltered under their product, which we may write $\Pi . T^{-1}$. From 7.42, it follows that every point on l remains fixed under $\Pi . T^{-1}$; thus the correspondence must be the identity, and $\Pi = T$. The converse theorem is a consequence of the fact that *at most* three of the constants a_{ij} may be assigned arbitrarily.

Turning to the projectivities which we established between

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†We note, in passing, that a linear transformation does not preserve *ratio* but only the ratio of ratios, i.e. cross ratio.

four collinear points in chapter III, we conclude from 3.24 and 7.43 that

7.47. $\{AB, CD\} = \{BA, DC\} = \{CD, AB\} = \{DC, BA\}$,

as is also evident from 7.41 or 7.41′. Similarly, from 3.24′

7.47′. $\{AB, DC\} = \{BA, CD\} = \{CD, BA\} = \{DC, AB\}$.

If we permute the four letters A, B, C, D in all possible ways, it readily appears that the 24 cross ratios fall into six sets of four equal cross ratios, two of these sets being 7.47 and 7.47′. We may take as typical of these sets the following:

$$\{AB, CD\} = \lambda, \quad \{AC, BD\} = 1-\lambda, \quad \{AD, BC\} = (\lambda-1)/\lambda,$$
$$\{AB, DC\} = 1/\lambda, \quad \{AC, DB\} = 1/(1-\lambda), \quad \{AD, CB\} = \lambda/(\lambda-1).$$

In particular, if $\boldsymbol{H}(AB, CD)$, then

$$(A, B, C, D) \overline{\wedge} (A_0, A_1, -I, I),$$

and *the cross ratio of any four harmonic points is* -1; in this case, $\lambda = -1 = 1/\lambda$, and each of the cross ratios in 7.47 and 7.47′ is -1.

If X, Y, U, V, W are any five distinct points on l, it follows immediately from 7.41 or 7.41′ that

7.48. $\{XY, UV\}.\{XY, VW\}.\{XY, WU\} = 1.$

Multiplying each side of 7.48 by the inverse of $\{XY, WU\}$, namely $\{XY, UW\}$, we obtain

7.49. $\{XY, UV\}.\{XY, VW\} = \{XY, UW\}.$

These two relations, 7.48 and 7.49, will be fundamental in the following section.

7.5. Coordinates in a Plane and in Space. To extend our coordinate system from a line to a plane π, let us choose three non-collinear points A_0, A_1, A_2 in π, denoting the lines A_1A_2, A_2A_0, A_0A_1 by l_0, l_1, l_2, as in the accompanying Fig. 7.5A.

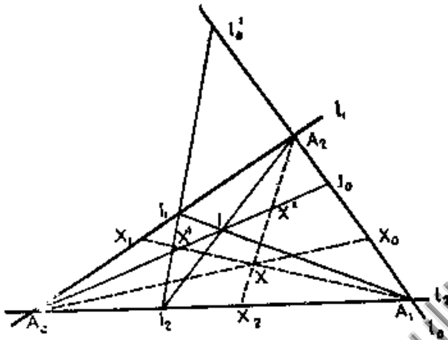

FIG. 7.5A

We shall call $A_0A_1A_2$ the *triangle of reference* in π. To determine a unit point on each of l_0, l_1, l_2, choose a fourth point I, not lying on l_0, l_1, or l_2, and let A_0I, A_1I, A_2I meet l_0, l_1, l_2 in I_0, I_1, I_2 respectively. If X is any point in the plane π, not lying on l_0, l_1, or l_2, let A_0X, A_1X, A_2X meet l_0, l_1, l_2 in X_0, X_1, X_2 respectively. If A_1X and A_2X meet A_0I in X' and X'', as in Fig. 7.5A, then it follows from 7.48 that

$$\{A_0I_0, X'X''\}.\{A_0I_0, X''I\}.\{A_0I_0, IX'\}=1.$$

Projecting these cross ratios from X, A_2, A_1 in turn, we obtain, on rearrangement,

7.51. $$\{A_1A_2, X_0I_0\}.\{A_2A_0, X_1I_1\}.\{A_0A_1, X_2I_2\}=1.$$

Hence there exist elements x_0, x_1, x_2 of σ, such that

$$\{A_1A_2, X_0I_0\}=x_2/x_1, \quad \{A_2A_0, X_1I_1\}=x_0/x_2,$$
$$\{A_0A_1, X_2I_2\}=x_1/x_0.$$

These elements (x_0, x_1, x_2), which are determined only to a non-zero factor, are called the *homogeneous coordinates* of the point X. If X lies on A_1A_2, we have

$$\{A_2A_0, X_1I_1\}=\{A_2A_0, A_2I_1\}=0,$$

and therefore $x_0=0$. Similarly, x_1 or x_2 vanishes when X

lies on A_2A_0 or A_0A_1, and the coordinates of X_0, X_1, X_2 are $(0, x_1, x_2)$, $(x_0, 0, x_2)$, $(x_0, x_1, 0)$ respectively. The coordinates of A_0 are naturally taken to be $(1, 0, 0)$. Similarly, the coordinates of A_1 and A_2 are $(0, 1, 0)$ and $(0, 0, 1)$. Thus we may assign three homogeneous coordinates to any point in the plane; and, conversely, any three elements of σ, not all zero, define a unique point.

The major problem which confronts us in setting up a coordinate system in a plane is to characterize, in terms of their coordinates, those points which lie on a given line in the plane —in other words, to find the "equation" of the line. To this end let us return to our construction for the sum of two points on a line, redrawing Fig. 7.2A, as in Fig. 7.5B.

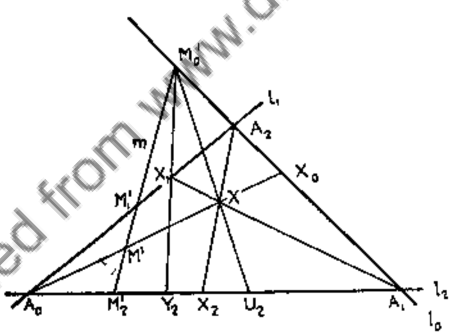

FIG. 7.5B

If we take the given line to be m, meeting l_0, l_1, l_2 respectively in M_0', M_1', M_2', then, if these three points are *distinct*, it follows from 7.42 that

$$\{A_0A_1, X_2M_2'\} + \{A_0A_1, Y_2M_2'\} = \{A_0A_1, U_2M_2'\}.$$

Taking M_2' to be the unit point on l_2 is quite legitimate, since addition does not depend on the choice of the unit point. Projecting from M_0', we obtain

$$\{A_0A_1, X_2M_2'\} + \{A_0A_2, X_1M_1'\} = \{A_0X_0, XM'\}.$$

The point X will lie on m if, and only if, X and M' coincide, when

7.52. $$\{A_0A_1, X_2M_2'\} + \{A_0A_2, X_1M_1'\} = 1.$$

Again, if M_0, M_1, M_2 are the harmonic conjugates of M_0', M_1', M_2', with regard to A_1, A_2; A_2, A_0; A_0, A_1, an application of Desargues' Theorem tells us that the triangle $M_0M_1M_2$ is in perspective with the triangle $A_0A_1A_2$ from a unique point M, as in Fig. 7.5c (cf. Fig. 4.2A).

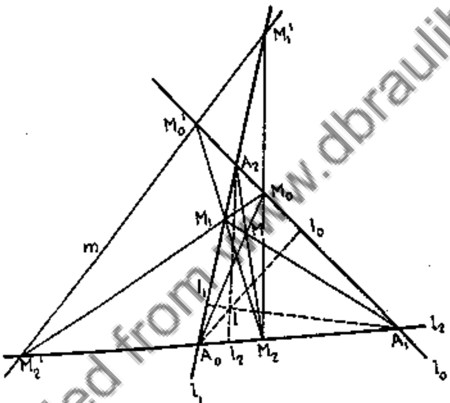

FIG. 7.5C

It follows from 7.49 that we may write 7.52 in the form

$$\{A_0A_1, X_2I_2\} . \{A_0A_1, I_2M_2\} . \{A_0A_1, M_2M_2'\}$$
$$+ \{A_0A_2, X_1I_1\} . \{A_0A_2, I_1M_1\} . \{A_0A_2, M_1M_1'\} = 1.$$

Since $\{A_0A_1, M_2M_2'\} = \{A_0A_2, M_1M_1'\} = -1$, we have

$$1 + \frac{x_1}{x_0} \cdot \frac{m_0}{m_1} + \frac{x_2}{x_0} \cdot \frac{m_0}{m_2} = 0,$$

or

7.53. $$m_1m_2x_0 + m_2m_0x_1 + m_0m_1x_2 = 0,$$

where (m_0, m_1, m_2) are the coordinates of the point M. This is known as the *homogeneous equation* of the line m. If $m_0m_1m_2 \neq 0$, 7.53 may be written in the simpler form

7.54. $$\frac{1}{m_0}x_0+\frac{1}{m_1}x_1+\frac{1}{m_2}x_2=0.$$

It is not a difficult matter to obtain the equations of lines which are specially related to the triangle of reference. For example, if M_0, M_0' and M_1, M_1' should coincide in A_2, the equation of the line m would become

$$m_1x_0+m_0x_1=0,$$

and similarly if m should pass through A_0 or A_1. The equations of the sides l_0, l_1, l_2 of the triangle of reference are $x_0=0$, $x_1=0$, $x_2=0$, respectively.

So far, in this section, we have not taken into account the Principle of Duality. Should there not be a parallel discussion treating the line, instead of the point, as fundamental? To this end, we should start with the three sides instead of with the three vertices of the triangle of reference and, instead of considering ranges of points upon l_0, l_1, l_2, we should consider pencils of lines through A_0, A_1, A_2. Let us rewrite equation 7.54 in the form

7.55. $$u_0x_0+u_1x_1+u_2x_2=0,$$

where $u_i=1/m_i(i=0, 1, 2)$; and let us designate (u_0, u_1, u_2) as the *homogeneous coordinates* of the line m. If we think of $X(x_0, x_1, x_2)$ as being a fixed point, this equation gives the condition that the varying line $m(u_0, u_1, u_2)$ shall pass through X, and we may speak of 7.55 as the *line equation* of the point X. A complete discussion of these interesting questions would necessarily be lengthy. We pass over them thus lightly, with reluctance.

To extend our coordinate system to space, let us choose a *tetrahedron of reference* $A_0A_1A_2A_3$ and a unit point I not on any plane face of the tetrahedron. If we designate the plane face opposite the vertex A_i by π_i, joining A_i to I we obtain a point I_i in each of π_i $(i=0, 1, 2, 3)$, as in Fig. 7.5D.

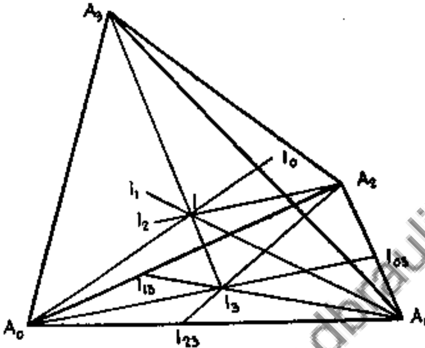

FIG. 7.5D

In the face π_3, for example, if A_2I_3 meets A_0A_1 in I_{23}, it is not difficult to see that in the face π_2, A_3I_2 will also meet A_0A_1 in I_{23}; thus $I_{23}=I_{32}$. In a similar fashion, we may determine each of the other five points I_{01}, I_{02}, I_{03}, I_{12}, I_{13} on the other five edges of the tetrahedron of reference. Setting up a coordinate system in each of π_1, π_2, π_3, we may define the homogeneous coordinates (x_0, x_1, x_2, x_3) of a point X as follows. The three relations analogous to 7.51 are clearly:

$$\{A_2A_3, X_{01}I_{01}\}\,.\,\{A_3A_0, X_{21}I_{21}\}\,.\,\{A_0A_2, X_{31}I_{31}\}=1,$$
$$\{A_3A_1, X_{02}I_{02}\}\,.\,\{A_1A_0, X_{32}I_{32}\}\,.\,\{A_0A_3, X_{12}I_{12}\}=1,$$
$$\{A_1A_2, X_{03}I_{03}\}\,.\,\{A_2A_0, X_{13}I_{13}\}\,.\,\{A_0A_1, X_{23}I_{23}\}=1;$$

the first of which is associated with π_1, the second with π_2, and the third with π_3. By multiplication, we obtain a fourth relation which is associated with the face π_0:

$$\{A_2A_3, X_{01}I_{01}\}\,.\,\{A_3A_1, X_{02}I_{02}\}\,.\,\{A_1A_2, X_{03}I_{03}\}=1.$$

Hence there exist elements x_0, x_1, x_2, x_3 of σ such that

$$\{A_iA_j, X_{kl}\, I_{kl}\} = \frac{x_j}{x_i},$$

where i, j, k, l are 0, 1, 2, 3 in any order. If ν is a plane meeting A_0A_1, A_0A_2, A_0A_3 in N_{23}', N_{13}', N_{12}' respectively, then by the same argument as before, the condition that a point $X(x_0, x_1, x_2, x_3)$ shall lie on ν is given by

7.56. $\{A_0A_1, X_{23}N_{23}'\} + \{A_0A_2, X_{31}N_{31}'\} + \{A_0A_3, X_{12}N_{12}'\} = 1.$

As the line m determined a unique point M in the plane, so the plane ν determines a unique point $N(n_0, n_1, n_2, n_3)$ in space, such that $\{A_0A_1, N_{23}N_{23}'\} = \{A_0A_2, N_{31}N_{31}'\} = \{A_0A_3, N_{12}N_{12}'\} = -1$. Thus we obtain the equation of ν in homogeneous coordinates in the form

7.57. $$n_1n_2n_3x_0 + n_2n_3n_0x_1 + n_3n_0n_1x_2 + n_0n_1n_2x_3 = 0.$$

Since a line is the intersection of two planes in space, so a line will be defined analytically by *two* linear equations. We shall carry the matter no further.

7.6. Finite Geometries. If the coordinate field σ is a finite field, the number of points on a line is finite, and we have a *finite geometry*. Clearly, the number of points in an harmonic sequence is also finite and Fano's axiom VIII is not satisfied. If the order of σ is s, then the number of points on a line will be $s+1$; from axiom III, $s \geqslant 2$. By setting up perspectivities between the different lines in a plane, it follows that every line contains the same number of points.

Consider a line l and a point P not on l. If we join P to each of the $s+1$ points on l, we obtain $s+1$ lines through P, on each of which there are s distinct points in addition to P. Thus the number of points in the plane determined by P and l is

$$s(s+1)+1 = s^2+s+1.$$

From the correspondence which we established in the preceding section between the points and lines in a plane, it follows that the number of lines in the plane is equal to the number of points in the plane. Again, by setting up perspectivities between planes, we conclude that every plane contains the same number of points, and the number of points in space is evidently

$$s(s^2+s+1)+1=s^3+s^2+s+1.$$

Moreover, the number of planes in space is equal to the number of points in space. In general, a finite geometry of k dimensions contains

$$s^k+s^{k-1}+\ldots+s+1=(s^{k+1}-1)/(s-1)$$

points, which is in accord with the number of ways of choosing $k+1$ homogeneous coordinates from the field σ, not all of which are zero. Various finite plane geometries are known, in which Desargues' Theorem is not valid. Clearly, in such a geometry, it would be impossible to set up a coordinate system. On the other hand, the assumption of Desargues' Theorem in the plane implies that the algebra of points on a line is a division ring, and being finite, it must be a finite field. It follows that, in this case, *Pappus' Theorem is a consequence of the axioms I-VII.*

From 6.82 we conclude that s is a power of a prime. If $s=p^n$, the field σ is the Galois field $GF(p^n)$ and the finite projective geometry on a line is represented, in the notation of Veblen and Bussey, by the symbol $PG(1, p^n)$. In general, the finite projective geometry of k dimensions associated with $GF(p^n)$ is represented by the symbol $PG(k, p^n)$.

As an illustrative example, we have set up a coordinate system in the seven point geometry $PG(2, 2)$, of Fig. 2.3A. The equations of the lines are to be interpreted here as congruences, taken modulo 2. We could readily extend this coordinate system to $PG(3, 2)$, in which each of the fifteen points would be represented by four homogeneous coordinates (0 or 1, but not all zero). The reader should compare the accompanying Fig. 7.6A with Fig. 7.5C.

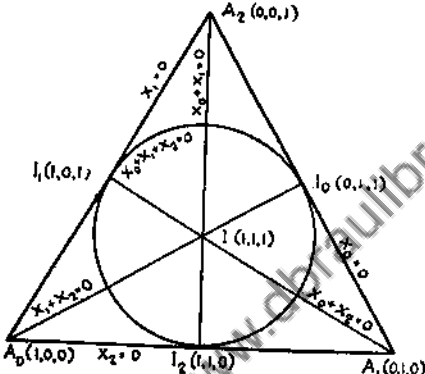

FIG. 7.6A

CHAPTER VIII

ORDER AND CONTINUITY

8.1. Summary of the Chapter. It is at this point in our discussion of the Foundations of Geometry that the philosophical approach is most significant. True, philosophers have not concerned themselves with the notion of *order*, but to make up for this lack they have thought and written much on *continuity*. "Unity in multiplicity" summarizes the philosophical problem. Continuity, to Cantor and to all mathematicians since his time, must be *defined*; it has no sufficiently precise meaning when thought of as an intuitive concept.

To arrive at such a definition of continuity in geometry we shall first introduce the notion of order, proving the important properties of ordered fields in §8.2 and giving their geometrical analogues in §§8.3, 8.4 and 8.5. With this introduction, the axioms of continuity in §8.6 appear in their proper setting. §8.7 is devoted to von Staudt's continuity proof of the Fundamental Theorem of projective geometry, while §8.8 deals with Pappus' Theorem and Desargues' Theorem in the light of these further assumptions. In the last section of the chapter we shall make some general remarks concerning the consistency and "categoricalness" of a system of axioms.

8.2. Ordered Fields. We first encountered the notion of *order* in our discussion of the integers, where it seemed natural and inevitable. Appealing to this order amongst the integers, we established an order amongst the integral numbers, the rational numbers, and eventually amongst the real numbers, Complex numbers were not ordered. It is to this question of order as applied to a field that we now turn our attention. Let us take the following definition:

A field σ is ORDERED if it is possible to divide the non-zero elements of σ into two distinct classes σ_p and σ_n such that

(i) *σ_p and σ_n have no elements in common;*
(ii) *if a is in σ_n, then $-a$ is in σ_p;*
(iii) *if a and b are in σ_p, then $a+b$ and $a.b$ are also in σ_p.*

The elements of σ_p are called the *positive* elements of σ and the elements of σ_n the *negative* elements of σ; in symbols,

$$s>0, \text{ or } s<0,$$

according as s is contained in σ_p or in σ_n. If $s_1-s_2>0$, we shall say that s_1 is *greater than* s_2, while if $s_1-s_2<0$, that s_1 is *less than* s_2, writing

$$s_1>s_2 \text{ or } s_1<s_2.$$

It follows from (i) that one and only one of the three relations

$$s_1>s_2, \quad s_1=s_2, \quad s_1<s_2,$$

can hold between any two elements s_1, s_2 of σ, and from (iii) that these relations are transitive.

An immediate consequence of this definition of an ordered field is that:

8.21. *A finite field cannot be ordered.*

For consider the finite field $\sigma=GF(p^n)$. If σ is an ordered field, the unit element 1 is certainly in σ_p; for if not, then -1 is in σ_p by (ii), but $-1.-1=1$, contrary to (iii). Thus

$$1+1+1+\ldots (p \text{ times})>0,$$

again by (iii); but this is contrary to the definition of the integer p. Moreover,

8.22. *Every ordered field contains a sub-field isomorphic to the field of rational numbers.*

The integral elements of σ generate such a sub-field, and, if we identify each integral element with the corresponding integral number, we may identify this sub-field with the field of rational numbers ρ, writing†

$$\rho \subseteq \sigma.$$

In 6.53 we proved the Archimedean property of the field of real numbers, having first deduced it for the field of rational numbers. It is natural to assume that σ is Archimedean ordered according to the following definition:

An ordered field σ is ARCHIMEDEAN ORDERED if, for any positive element s of σ, it is always possible to find an integer N such that $N > s$.

Consider an unending sequence of elements $s_1, s_2, s_3, \ldots,$ contained in σ. This sequence $\{s_i\}$ is *convergent*, if for any $\epsilon > 0$ contained in σ, there is an integer n_ϵ, such that for $n > n_\epsilon$,

$$|s_n - s_{n+m}| < \epsilon,$$

for all m. As in §6.5, we shall suppose that a convergent sequence $\{s_i\}$ defines a symbol $[s_i]$. To define the addition and multiplication of these symbols, we proceed exactly as before, the only difference being that we are concerned here with elements of σ instead of ρ. It follows that the symbols $[s_i]$ are elements of a field σ', known as the *derived field* of σ. If we identify $[s]$ of σ' with s of σ, as before, we may say that σ is a sub-field of σ'; i.e.

8.23. $$\rho \subseteq \sigma \subset \sigma', \text{ and } \rho' \subset \sigma'.$$

By definition in §6.5, the derived field ρ' of the field of rational numbers ρ is the field of real numbers. Moreover, σ' is an ordered field according to our definition.

It follows therefore that we can always find a rational

†The symbol $\subset$ indicates inclusion in the broad sense, where ρ and σ may coincide.

number r such that $s > r > 0$. Applying the argument used in proving 6.53 to σ instead of to ρ, we conclude that

8.24. *The derived field σ' of an Archimedean ordered field σ is also Archimedean ordered.*

This brings us to the crucial result in the theory of ordered fields:

8.25. *Every Archimedean ordered field is isomorphic to a subfield of the field of real numbers.*

To prove 8.25 it will be sufficient to show that $\sigma' \subset \rho'$, in virtue of 8.23. Consider an element $[s_i] > 0$ of σ'. By definition, there exists an element $\delta > 0$ of σ and an integer n_0, such that for $n > n_0$,

$$s_n > \delta > 0.$$

Since σ is Archimedean ordered, we may assume that δ is a rational number. There is no loss of generality if we assume also that $s_n > \delta > 0$, for *all* n. Since σ is Archimedean ordered, there exists an integer greater than $2^n . s_n$, and we choose k_n to be the smallest such integer. Consequently,

$$k_n - 1 \leqslant 2^n . s_n < k_n,$$
$$\text{or } 0 \leqslant 2^n . s_n - (k_n - 1) < 1,$$
$$\text{or } 0 \leqslant s_n - r_n < 2^{-n},$$

where $r_n = 2^{-n}(k_n - 1)$ is a rational number. Since the sequence $\{s_i\}$ is convergent, so also is the sequence $\{r_i\}$, and

$$[s_i] = [r_i].$$

If $[s_i] < 0$, then $[s_i] = -[r_i] = [-r_i]$. Thus every element of σ' is contained in ρ', and we have proved 8.25.

8.3. Order in Projective Geometry. Let us suppose, as in the preceding chapter, that σ is the field of non-homogeneous coordinates of points on a line l, and let us temporarily make the following assumption:

8.3A. *The coordinate field σ is an ordered field.*

If A, B, C, D are four distinct points on l and $\{AB, CD\} = \lambda$, then $\{AB, DC\} = 1/\lambda$, where λ is an element of σ; in particular, if $H(AB, CD)$, then $\lambda = 1/\lambda = -1$. Let us say that A, B *separate* C, D, writing $AB||CD$, if and only if

$$\{AB, CD\} = \lambda < 0.$$

Since $1/\lambda$ is also negative, it follows from 7.47 and 7.47′ that this relation of separation is completely symmetrical with regard to the two pairs of points. If $\lambda < 0$, then

$$\{AC, BD\} = (1-\lambda) > 0 \text{ and } \{AD, BC\} = (\lambda - 1)/\lambda > 0,$$

and A, C do *not* separate B, D, neither do A, D separate B, C. The resulting order amongst the points on l is a "cyclic" or "four point" order and is best represented as in the accompanying Fig. 8.3A or 8.3B. Similarly, if $(1-\lambda) < 0$, then

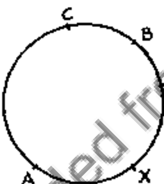

FIG. 8.3A

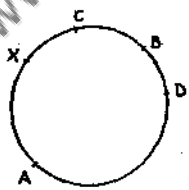

FIG. 8.3B

$\lambda > 0$ and $(\lambda - 1)/\lambda > 0$; and if $(\lambda - 1)/\lambda < 0$, then $\lambda > 0$ and $(1-\lambda) > 0$. Having assumed that A, B, C, D are all distinct, we conclude that at least one of λ, $(1-\lambda)$, $(\lambda-1)/\lambda$ is negative, and

8.31. $$AB||CD \text{ or } AC||BD \text{ or } AD||BC.$$

Since σ is an ordered field, we may divide the points X on l which are distinct from A, B into two classes, according as

$$\{AB, CX\} \lessgtr 0.$$

We shall say that every point X for which $\{AB, CX\}<0$, as in Fig. 8.3A, lies in the *segment* AB/C. Clearly, AB/C does not contain C, since $\{AB, CC\}=1$, and we may speak of AB/C as "the segment AB remote from C." If $\{AB, CD\}<0$, it follows from the relation

8.32. $$\{AB, CD\}.\{AB, DX\}.\{AB, XC\}=1,$$

that $\{AB, DX\}>0$. If $\{AB, CX\}>0$, as in Fig. 8.3B, then $\{AB, DX\}<0$ and X lies in the segment AB/D, i.e. in the segment AB remote from D.

The fundamental property of the cross ratio is its invariance under projection. Thus our definitions of separation and of segment are both invariant under projection. In particular, the assumption 8.3A makes it possible to distinguish two types of projectivity T on a given line, where

$$T: x' = \frac{a_{10}+a_{11}x}{a_{00}+a_{01}x}.$$

If $(a_{00}a_{11}-a_{01}a_{10})>0$, we shall say that the projectivity T is *direct*, while if $(a_{00}a_{11}-a_{01}a_{10})<0$, that T is *opposite*. If T is determined by the correspondence

$$(A, B, C) \barwedge (A', B', C'),$$

it is natural to interpret this distinction in terms of the *alternating function* $(\mathbf{a}-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-\mathbf{a})$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are the non-homogeneous coordinates of A, B, C respectively. It may easily be verified that

8.33. $$(\mathbf{a}'-\mathbf{b}')(\mathbf{b}'-\mathbf{c}')(\mathbf{c}'-\mathbf{a}') = \frac{(a_{00}a_{11}-a_{01}a_{10})^3}{(a_{00}+a_{01}\mathbf{a})^2\,(a_{00}+a_{01}\mathbf{b})^2\,(a_{00}+a_{01}\mathbf{c})^2}(\mathbf{a}-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-\mathbf{a}).$$

Thus $(\mathbf{a}-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-\mathbf{a})$ and $(\mathbf{a}'-\mathbf{b}')(\mathbf{b}'-\mathbf{c}')(\mathbf{c}'-\mathbf{a}')$ *have the same sign if and only if T is direct.* Under these circumstances, we shall say that ABC and $A'B'C'$ belong to the same *sense class*, or have the same *sense*. Any triad of points ABC on l determines such a sense class, which we may denote by $\mathbf{S}(ABC)$. If T is direct, $\mathbf{S}(ABC)$ and $\mathbf{S}(A'B'C')$ are identical, and we shall write

$$\mathbf{S}(ABC)=\mathbf{S}(A'B'C').$$

On the other hand, $(\mathbf{a}-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-\mathbf{a})$ *and* $(\mathbf{a}'-\mathbf{b}')(\mathbf{b}'-\mathbf{c}')(\mathbf{c}'-\mathbf{a}')$ *have opposite signs if and only if* T *is opposite.* For such a T,

$$\mathbf{S}(ABC)\neq\mathbf{S}(A'B'C'),$$

and the triads ABC, $A'B'C'$ have opposite senses. From the form of the alternating function, it follows immediately that

8.34.
$$\begin{gathered}\mathbf{S}(ABC)=\mathbf{S}(BCA)=\mathbf{S}(CAB),\\ \mathbf{S}(ACB)=\mathbf{S}(BAC)=\mathbf{S}(CBA),\\ \text{and } \mathbf{S}(ABC)\neq\mathbf{S}(ACB).\end{gathered}$$

Separation and *sense* in a line are not independent concepts, for

8.35.
$$\{AB,\ CD\}=\frac{(\mathbf{a}-\mathbf{b})(\mathbf{b}-\mathbf{d})(\mathbf{d}-\mathbf{a})}{(\mathbf{a}-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-\mathbf{a})}\cdot\frac{(\mathbf{c}-\mathbf{a})^2}{(\mathbf{d}-\mathbf{a})^2}.$$

Whence,

8.36.
$$\begin{gathered}\{AB,\ CD\}>0 \text{ if } \mathbf{S}(ABC)=\mathbf{S}(ABD),\\ \{AB,\ CD\}<0 \text{ if } \mathbf{S}(ABC)\neq\mathbf{S}(ABD),\end{gathered}$$

and conversely. Thus we may define separation in terms of sense; likewise, the segment AB/C is made up of all those points X such that $\mathbf{S}(ABX)=\mathbf{S}(ABD)$, where $\mathbf{S}(ABD)\neq\mathbf{S}(ABC)$. Conversely, we may define sense in terms of separation. Equality of sense, like separation, is invariant under projection. It is essentially a property of a single line; for it would be impossible to compare the sense of a triad of points on a line l with the sense of a triad of points on another line l', since the coordinate system on l' is determined only to a projectivity T which may be direct or opposite.

8.4. Order in Affine and Euclidean Geometry. The characteristic feature of our axiomatic foundation of Euclidean geometry in chapter v is the assumption of an undefined relation of "betweenness." This relation was independent of congruence. Thus in seeking the relation of order in projective geometry to order in Euclidean geometry, we shall begin by considering order in affine geometry.

Affine geometry is obtained from projective geometry by specializing a point in a line, a line in a plane, or a plane in space, as in chapter IV. Actually, we may think of these elements "at infinity" as being removed. If A, B are any two distinct points on a projective line l, and if P is the harmonic conjugate of P_∞ with regard to A, B, the segment AB/P_∞ is unaffected by the removal of P_∞, and we may speak of it as *the segment* AB. The point P was called the "mid-point" of AB in §4.2. Let us say that a point C is *between* A, B or that the three points are in the *order* ACB, if and only if C lies in the segment AB. All the assumptions which were made in chapter v concerning such a relation are evidently satisfied.

Pasch's Axiom, as formulated in §5.2, may be deduced from the axioms of projective geometry, along with Desargues' Theorem in the plane and the assumption 8.3A. It is only necessary to apply the relation 7.51 to the figure 7.5c, to show that

8.41. $\{A_1A_2, M_0I_0\} . \{A_2A_0, M_1I_1\} . \{A_0A_1, M_2I_2\} = 1.$

If we replace $\{A_1A_2, M_0I_0\}$ by the product $\{A_1A_2, M_0M_0'\} . \{A_1A_2, M_0'I_0\}$, and similarly for each of the other two cross ratios in 8.41, we obtain

8.42. $\{A_1A_2, M_0'I_0\} . \{A_2A_0, M_1'I_1\} . \{A_0A_1, M_2'I_2\} = -1,$

since $\{A_1A_2, M_0M_0'\} = \{A_2A_0, M_1M_1'\} = \{A_0A_1, M_2M_2'\} = -1.$ Thus, either (i) all the cross ratios in 8.42 are negative, or

(ii) one is negative and the other two are positive. If I_0, I_1, I_2 are the mid-points of the segments A_1A_2, A_2A_0, A_0A_1, as in §4.3, i.e. if I is the centroid of the triangle $A_0A_1A_2$, then either (i) all three of the points M_0', M_1', M_2' lie outside the segments A_1A_2, A_2A_0, A_0A_1 respectively, or (ii) one lies outside and two inside. Case (ii) yields Pasch's Axiom.

The removal of a point P_∞ from the projective line l limits the projectivities which map l upon itself to those which leave P_∞ fixed. Such a restriction implies that $a_{01}=0$, and 7.44′ becomes

8.43. $$x'=a_{10}+a_{11}x,$$

taking $a_{00}=1$. If $a_{11}=1$, 8.43 is called a *translation*. Just as the succession of two general linear transformations is a linear transformation, so the succession of two translations is a translation. Thus the translations in a line constitute a "sub-group" of the group† of affine projectivities in the line, which is itself a sub-group of the full group of projectivities in the line.

Hilbert introduced continuity into Euclidean geometry by means of two axioms: Archimedes Axiom, which involved the notion of congruence, and the Axiom of Completeness. We may obtain the Archimedean property in affine geometry by assuming that:

8.4A. *The coordinate field σ is an Archimedean ordered field.*

From 8.25 it follows that σ must be a sub-field of the field of real numbers. To insure that σ is precisely the field of real numbers, we assume that:

8.4B. *The coordinate field σ coincides with its derived field σ'.*

†The set of abstract elements considered in §6.6 forms a *group* with regard to multiplication, if II(i) and II(iii) are satisfied. This is called the *multiplicative group* of the division ring. The multiplicative group of a field is commutative or "abelian."

Consider the simplest case in which σ is the field of rational numbers ρ. If a, b are rational, all the numbers which may be written in the form $a+b\sqrt{2}$ form a field $(\sqrt{2})$, which has been obtained from $\sigma=\rho$ by the *adjunction* of $\sqrt{2}$. While $(\sqrt{2})$ is a sub-field of the field of real numbers ρ', yet it is impossible to obtain ρ' by a finite or even a countably infinite number of such adjunctions. The assumption 8.4B is equivalent to the requirement that no such adjunctions are possible, which is Hilbert's Axiom of Completeness.

From a continuous affine geometry we may obtain Euclidean geometry by the introduction of an absolute involution, as was suggested in chapter IV. Such an involution leads immediately to the definition of length and the notion of congruence.

If we confine our attention to a single line it is not difficult to construct a geometry† which is ordered but *not* Archimedean ordered. Consider an unending sequence of parallel lines which lie in a Euclidean plane π and are evenly spaced, as in Fig. 8.4A.

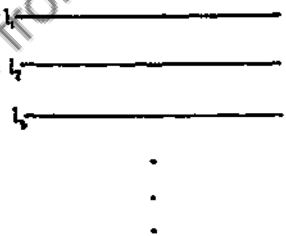

FIG. 8.4A

We may order the totality l of points on l_1, l_2, l_3, ... by ordering the points on each line from left to right, and saying that a point A on l_i *precedes* or *follows* a point B on l_j, according

†An example which is more satisfactory from the point of view of the field may be found in Hilbert (6), §33.

as $i \lessgtr j$. A point which follows A and precedes B, or follows B and precedes A, lies in the *segment* AB. Two segments on l are *congruent* if we can superimpose their end points by means of a parallel translation† in the plane π. Clearly, no integral multiple of a segment both of whose end points lie on l_1 is greater than a segment one of whose end points lies on l_1 while the other lies on l_2. Thus Archimedes Axiom is not satisfied.

8.5. Axioms of Order in Projective Geometry. The assumption 8.3A is essentially algebraic. Could we not replace it by something of a more geometrical character? Two courses are open to us: on the one hand, we may associate with any *four* distinct points an undefined relation of *separation*; while on the other, we may associate with any *three* distinct points an undefined relation of *sense*. Following the former method, let us assume an *undefined* relation of separation, which shall be subject to the following Axioms of Order, due to Vailati‡.

X. *For any five distinct collinear points A, B, C, D and X:*

(i) *If $AB||CD$, then $AB||DC$;*

(ii) *If $AB||CD$, then A, C do not separate B, D;*

(iii) $AB||CD$ *or* $AC||BD$ *or* $AD||BC$;

(iv) *If $AB||CD$ and $AC||BX$, then $AB||DX$;*

(v) *If $AB||CD$ and $(A, B, C, D) \barwedge (A', B', C', D')$, then $A'B'||C'D'$.*

---

†Cf. chapter IX.

‡The analogy with our former definition of separation in §8.3 is:

X(i) and X(ii) correspond to the symmetry of a harmonic range, since X(v) and 3.24 imply that $CD||AB$ etc.;

X(iii) corresponds to 8.31;

X(iv) corresponds to 8.32, since if $\{AC, BX\} < 0$ then $\{AB, XC\} > 0$;

X(v) corresponds to 7.43.

By interchanging B, C and D, X in X(iv), it follows from X(i) that $AC||DX$.

As in §8.3, we shall say that every point X such that $AB||CX$ lies in the *segment* AB/C. If A, B do not separate C, X, it follows from X(iii) that (a) $AC||BX$, or (b) $AX||BC$.

(a) If $AB||CD$ and $AC||BX$, we conclude from X(iv) that $AB||DX$ as in Fig. 8.3B, and the point X lies in the segment AB/D by definition.

(b) If $AB||CD$ and $AX||BC$, as in the accompanying Fig. 8.5A, the situation is a little more complicated. From the symmetry of the relation of separation (cf. footnote on p. 119), the hypothesis becomes $BA||CD$ and $BC||AX$. By interchanging A, B in X(iv), it follows that $BA||DX$. We conclude that $AB||DX$, and X lies in the segment AB/D. Thus:

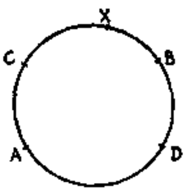

FIG. 8.5A

8.51. *If $AB||CD$, then every point on the line AB, exclusive of A and B, lies in one or other of the two segments AB/C, AB/D.*

A and B are said to be the *end points* of the two segments.

If $H(AB, CD)$, we know from 3.53 that

$$(A, B, C, D) \barwedge (A, B, D, C).$$

In view of X(v), and X(i) and X(ii), such an interchange of C, D would be permissible if and only if $AB||CD$. This association of the harmonic property with separation implies that *the field of points on a line is an ordered field*, according to the definition in §8.2; i.e. axiom X is equivalent to the assumption 8.3A. For, if A_0, A_1, I are three collinear points and $H(A_0A_1, II')$, we may say that every point in the segment A_0A_1/I' is *positive* and every point in the segment A_0A_1/I is *negative.* Clearly, these two segments have no points in common. The relation $\boldsymbol{H}(A_0A_1, XX')$ implies that X' is positive if X is negative, and we may write $X' = -X$. If X, Y are both positive, then $U = X + Y$ and $W = X.Y$ are also positive. In the case of addition, this statement is a consequence of the projectivity

$$(-X, A_0, Y, A_1) \,\overline{\wedge}\, (A_0, X, U, A_1),$$

considered with reference to Fig. 7.2A; and in the case of multiplication, of the projectivity

$$(A_0, I, Y, A_1) \,\overline{\wedge}\, (A_0, X, W, A_1),$$

considered with reference to Fig. 7.2B. Both conclusions are based on the invariance of separation under projection. According as $X - Y$ is positive or negative, we shall say that X *follows* or *precedes* Y with reference to A_0, A_1, I.

8.6. Axioms of Continuity in Projective Geometry. In the preceding section we showed that axiom X is the geometrical equivalent of 8.3A. Along with X, the following assumption is the geometrical equivalent of 8.4A.

XI. *For any point X of the segment $A_0A_1/-I$ there exists a point N, which is integral with reference to $\mathbf{R}(A_0A_1I)$, and which follows X with reference to A_0, A_1, I.*

From the association of the harmonic property with separation, this assumption is automatically satisfied for any point X belonging to the harmonic net $\boldsymbol{R}(A_0A_1I)$; but, for those points which do not belong to $\boldsymbol{R}(A_0A_1I)$, the Archimedean property must be assumed. It is sufficient to make the assumption for a single segment, in virtue of X(v).

It will simplify what we have to say if we revert to the notation of §7.3, and denote by P_x the point of $\boldsymbol{R}(A_0A_1I)$ whose rational non-homogeneous coordinate is x. Consider an unending sequence of points $\{P_{r_i}\}$. We shall say that this sequence is *convergent*, if for any point P_ϵ of the segment $A_0A_1/-I$, or P_0P_∞/P_{-1}, there is an integer n_ϵ, such that for $n>n_\epsilon$,

$$|P_{r_n}-P_{r_{n+m}}| \text{ precedes } P_\epsilon,$$

for all m. In other words, *the sequence* $\{P_{r_i}\}$ *is convergent if and only if the sequence* $\{r_i\}$ *is convergent.* The assumption 8.4B is equivalent to the assumption that every convergent sequence $\{P_{r_i}\}$ has a unique *limit point* $P_r=[P_{r_i}]$ on l, where

$$|P_r-P_{r_n}| \text{ precedes } P_\epsilon,$$

for $n>n_\epsilon$. Thus 8.4B is equivalent to the assumption that every point on l either belongs to $R(A_0A_1I)$, or is the limit point of a convergent sequence of points of $R(A_0A_1I)$, by 8.25. From the invariance of order under projection the exceptional character of the point A_1 is not significant. We assume then:

XII. *Every convergent sequence of points of* $R(A_0A_1I)$ *has a unique limit point.*

It is well to emphasize that there is no geometrical distinction between points which are rational and points which are irrational, since the choice of A_0, A_1, I is quite arbitrary.

Following Hilbert, the axiom XII could equally well be phrased: *It is impossible to add to the class of points in such a manner that the system thus generalized shall form a new geometry which shall satisfy all the other assumptions.* According to F. Schur, a convergent sequence *defines* its limit point, and axiom XII is superfluous. This agrees with our *definition* of the field σ', but the statement that σ and σ' coincide, or that $[P_{r_i}]$ is a point of l, would seem to require justification.

There is an alternative method of introducing continuity into projective geometry, which corresponds to Dedekind's introduction of real numbers; namely, by assuming that:

For every division of the points of a segment $\alpha = A_0A_1/-I$ *into two classes*, R_1, R_2, *such that:*

(i) *Every point of* α *belongs either to* R_1 *or to* R_2*;*

(ii) *Every point of* R_1 *precedes every point of* R_2*;*

there exists a point P in α *which may belong to* R_1 *or to* R_2, *such that every point of* α *which precedes P belongs to* R_1 *and every point of* α *which follows P belongs to* R_2.

It cannot be denied that this assumption is, superficially at least, less involved than XI and XII. The difficulty lies, as in chapter VI, in the discussion of the field properties.

It is important to note that the Principle of Duality is still valid in our *ordered* and *continuous* projective geometry. Order and continuity are properties of a line, and their invariance under projection enables us to define the corresponding properties of lines through a point, or of planes through a line.

In conclusion, we remark that our axioms I-XII completely determine the geometry. As a matter of fact, there is a certain redundancy, since, as we shall see in the following section, Pappus' Theorem or Axiom IX is a consequence of X-XII. Moreover the assumption of order in X rules out the finite geometries. All that is required is to assume the existence of a point D such that $AB \| CD$ for any three collinear points A, B, C. Such an assumption can then replace Fano's Axiom VIII.

8.7. von Staudt's Continuity Proof of the Fundamental Theorem of Projective Geometry. By assuming Pappus' Theorem in chapter III we were able to give a proof of the Fundamental Theorem of projective geometry, namely, that a projectivity between two lines l, l' is completely determined by assigning three distinct pairs of corresponding points. It follows from the invariance of the harmonic property that the projectivity

8.71. $$(A_0, A_1, I) \barwedge (A_0', A_1', I')$$

sets up a correspondence between the points of $\boldsymbol{R}(A_0A_1I)$ on l and the points of $\boldsymbol{R}(A_0'A_1'I')$ on l'. The uniqueness of this correspondence depends only on the uniqueness of the fourth harmonic point, i.e. on Desargues' Theorem, and *not* on axiom IX.

Consider a convergent sequence of points $\{P_{r_t}\}$ belonging to $\boldsymbol{R}(A_0A_1I)$. From axiom XII, this sequence determines a limit point $P_r = [P_{r_t}]$ on l, which may or may not belong to $\boldsymbol{R}(A_0A_1I)$. To each point P_{r_t} there corresponds a point P_{r_t}' belonging to $\boldsymbol{R}(A_0'A_1'I')$; if the sequence $\{P_{r_t}\}$ is convergent, then so also is the sequence $\{P_{r_t}'\}$, since relations of order are invariant under projection. Thus the limit point P_r' must correspond to the limit point P_r, and a projective correspondence is completely established between the points of l and the points of l'. This is von Staudt's proof of the Fundamental Theorem of projective geometry, from which Pappus' Theorem follows immediately.

8.8. Desargues' Theorem in the Plane. In chapter II we emphasized the fact that Desargues' Theorem in the plane cannot be proved from the axioms I-V alone. With the assumption of Pappus' Theorem in axiom IX, however, the situation is changed. In this section we shall give a proof of Desargues' Theorem in the plane, which is due to Hessenberg. While this proof might have been given in chapter III, it is of more significance here. To complete the picture, we shall construct a geometry in which neither Pappus' Theorem nor Desargues' Theorem is valid.

Let two triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ be in perspective from a point L_{123}, as in Fig. 8.8A (cf. Fig. 3.3A). If P_1P_2 and

Q_1Q_2 meet in O_{12} $(=O_1)$, P_2P_3 and Q_2Q_3 meet in O_{23} $(=O_2)$, and if $O_{12}O_{23}$ meets l_1, l_2, l_3 in M_1, M_2, M_3 respectively, then

8.81. $$(P_1, Q_1, M_1) \overset{O_{12}}{\overline{\overline{\wedge}}} (P_2, Q_2, M_2) \overset{O_{23}}{\overline{\overline{\wedge}}} (P_3, Q_3, M_3).$$

Let us denote the perspectivity with centre O_{ij} between l_i and l_j by Π_{ij}, and the sequence of two perspectivities Π_{ij}, Π_{jk} by $\Pi_{ij}.\Pi_{jk}$. Then if P_1P_3 and Q_1Q_3 meet in O_{13} $(=O)$, Desargues' Theorem amounts to the statement that $\Pi_{12}.\Pi_{23}=\Pi_{13}$,

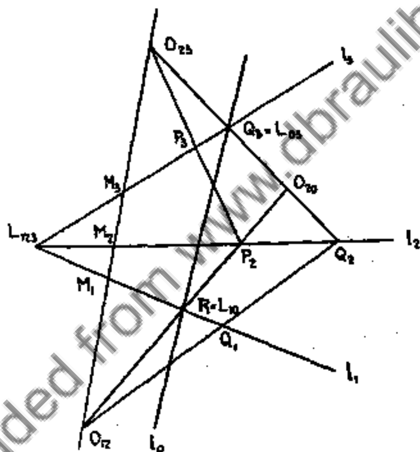

FIG. 8.8A

which is equivalent to 3.31; we must substitute a demonstration of 3.31 based upon Pappus' Theorem, or upon 3.41. To this end, let us designate the line joining P_1 and Q_3 by l_0, as in Fig. 8.8A. The point of intersection O_{20} of P_1P_2 and Q_2Q_3 cannot lie on l_0; let us denote the perspectivity with centre O_{20} mapping the points of l_2 upon l_0 by Π_{20}, and the perspectivity mapping the points of l_0 upon l_2 by Π_{02}. It follows that

8.82. $$\Pi_{12}.\Pi_{23}=\Pi_{12}.\Pi_{20}.\Pi_{02}.\Pi_{23}.$$

The projectivity $\Pi_{12}.\Pi_{20}$ between l_1 and l_0 is such that their common point L_{10} is self-corresponding; hence, from 3.41, $\Pi_{12}.\Pi_{20}=\Pi_{10}$. Similarly, the point L_{03} is self-corresponding and $\Pi_{02}.\Pi_{23}=\Pi_{03}$. Hence,

8.83. $$\Pi_{12}.\Pi_{23}=\Pi=\Pi_{10}.\Pi_{03}.$$

From the left-hand side of 8.83, it follows that L_{123} is a self-corresponding point of the projectivity Π between l_1 and l_3; from the right-hand side and 3.41, we conclude that Π is equivalent to a single perspectivity Π_{13}.

von Staudt's proof of the Fundamental Theorem is based upon the uniqueness of the fourth harmonic point. Thus the Fundamental Theorem is a consequence of (i) the assumption of Pappus' Theorem, or (ii) the assumption of Desargues' Theorem in the plane along with suitable axioms of order and continuity. To show that it is impossible to prove the Fundamental Theorem without some such assumption, it will be sufficient to construct a geometry in which Desargues' Theorem, and hence Pappus' Theorem, is not valid. Various such *non-Desarguesian* geometries have been given; the one which we shall describe is due to F. R. Moulton.

Consider a Euclidean plane π and in it two rectangular axes of coordinates OX, OY. All loci of the form

$$y=m(x-a)f(y,m),$$

where the function f is defined as follows:

(i) if $m\leqslant 0$, $f(y,m)=1$;

(ii) if $m>0$ and $y\leqslant 0$, $f(y,m)=1$;

(iii) if $m>0$ and $y>0$, $f(y,m)=\frac{1}{2}$;

will be called *modified lines*. A modified line is identical with an ordinary line of the plane π, provided $m\leqslant 0$, as in case (i); if $m>0$, a modified line is made up of two "half-lines," LM determined by (ii) and MN determined by (iii), as in Fig. 8.8B.

Certainly, two points P, Q both above or both below OX will determine one and only one modified line. If P is above OX and Q below OX and we obtain a point P' by doubling the ordinate at P and Q' by halving the ordinate at Q, then by elementary proportion PQ' and $P'Q$ intersect in M on OX. The two points P, Q uniquely determine the modified line

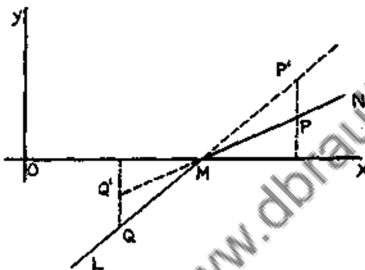

FIG. 8.8B

QMP. Two modified lines are *parallel* if their corresponding half-lines above or below OX are parallel in the ordinary sense. It is easy to see that in this new plane geometry Desargues' Theorem is not valid.

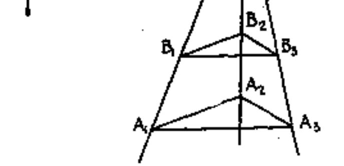

FIG. 8.8C

For example, in Fig. 8.8C, if A_1A_2, A_2A_3, A_3A_1 are respectively parallel to B_1B_2, B_2B_3, B_3B_1 in the lower half-plane, then the corresponding modified lines will also be parallel. If $m \leqslant 0$ for A_2B_2, A_3B_3 while $m > 0$ for A_1B_1, it follows that the modified line A_1B_1 cannot pass through the point of intersection S of the modified lines A_2B_2, A_3B_3; though, clearly, the ordinary line A_1B_1 does pass through S. If we make Euclidean geometry projective by adding "points at infinity," then the points of π along with the modified lines of π and the "line at infinity" satisfy axioms I-V.

This non-Desarguesian geometry is interesting for another reason. In chapter V, we saw that it was necessary to make an explicit assumption concerning congruent triangles in axiom 15. To introduce congruence into Moulton's geometry, it is sufficient to refer to congruence in Euclidean geometry. With regard to segments PQ, the only difficulty arises when P, Q lie on opposite sides of OX, as in Fig. 8.8B; but, obviously, we may say that $PQ \equiv PM + MQ$. Angles are said to be congruent in the Euclidean sense, with the convention that $\angle LMN$ in Fig. 8.8B is a straight angle and $\angle NMX = \angle OML$. With these assumptions, if $AB \equiv A'B'$, $BC \equiv B'C'$ and $\angle ABC \equiv \angle A'B'C'$, as in Fig. 8.8D, it is clear that $AC \not\equiv A'C'$.

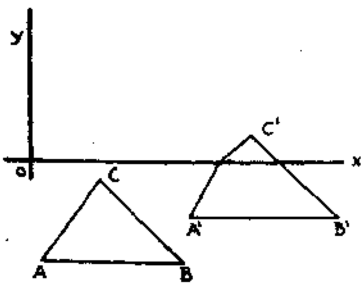

FIG. 8.8D

Moulton's geometry is certainly continuous though neither Pappus' Theorem nor the Fundamental Theorem holds.

8.9. Consistency and Categoricalness. With the completion of our system of axioms for "real" geometry, let us glance back over the course which we have travelled, having in mind, first, the *consistency* of our assumptions. Without considering the possibility of an "internal" logical test, the only means at our disposal of deciding the question is by the construction of a model in which *meanings* are attached to the undefined elements called *points* and *lines*. We have one such "realization" of axioms I-VII in Fano's finite geometry, and since, in this case, it is possible to verify that every "point" and "line" actually does satisfy the axioms, we conclude that these axioms are consistent with one another.

In the preceding chapter we saw how a coordinate system may be introduced into projective geometry: how a point in a plane may be represented by three homogeneous coordinates and a line in the plane may be represented by a homogeneous linear equation in three variables, and so on. If we include the axioms of order and continuity of the present chapter, it may be verified, conversely, that these arithmetical *meanings* for the undefined elements called points and lines satisfy all our assumptions and provide a model of "real" geometry. The situation is more complicated than in the case of the finite geometry, however, and all that we can say is that our axioms are consistent if the system of real numbers is consistent. Ultimately, then, the consistency of "real" geometry is based upon the consistency of the ordinary integers of arithmetic, which last is taken for granted.

If we have two distinct models M_1, M_2, which satisfy a given system of axioms, it may be possible to set up the *same* coordinate system in each model. Under such circumstances, this coordinate system establishes a correspondence between the "points" and "lines" in M_1, and the "points" and "lines" in M_2, such that, if three points are collinear in M_1, then the three corresponding points are collinear in M_2, and so on. The two models M_1, M_2 are isomorphic in this generalized sense, and any theorem which is true in M_1 is also true in M_2. If such an isomorphism holds between any two models, the given system of axioms is said to be *categorical*. We have an example of a categorical system in the axioms of "real" geometry; certainly, axioms I-VII are not categorical, for the coordinate field σ is not uniquely defined.

Addendum to the proof of 8.51 on page 120.

It remains to show that the three relations $AB\|CD$, $AB\|CX$, $AB\|DX$ cannot hold simultaneously. By Axiom X (iii), A is in just one of the segments CD/X, DX/C, XC/D, say in CD/X. By Axiom X (iv), the relations $XC\|AB$ and $XA\|CD$ imply $XC\|BD$, while $XD\|AB$ and $XA\|DC$ imply $XD\|BC$. Finally, $XD\|CB$ and $XC\|DB$ imply $XD\|BB$, which is contrary to the assumption that the four points be distinct in any separation relation.

CHAPTER IX

CORRESPONDENCES AND IMAGINARY ELEMENTS IN GEOMETRY

9.1. Summary of the Chapter. In chapter III there were proved a number of important theorems concerning a projective correspondence between the points of two lines. It is impossible to appreciate the significance of these theorems without considering their generalizations in terms of a projective correspondence, first between the points of two planes and then between the points of two general n-dimensional projective spaces. A classification of the different types of these more general correspondences is interesting from various points of view and plays a fundamental role in the development of projective geometry. Such a classification is best accomplished through a discussion of the invariant factors of the matrix of the correspondence; but, as such, it is beyond the scope of this book. In §9.2, when considering correspondences in a plane, we shall discuss, in particular, an involutory correspondence; such a correspondence in space is of special interest, as we shall see in §9.3.

Imaginary elements may be introduced into geometry in a number of different ways; each of them, however, is based on the analytical approach and, ultimately, on the notion of a complex number. The situation is parallel to the introduction of continuity into geometry, which was based on the notion of a real number. While the significance of imaginary elements had been appreciated earlier, it remained for von Staudt and his successors, notably Klein and Lüroth, to put the matter on a proper geometrical basis. In §§9.4 and 9.5 we shall give a brief account of von Staudt's theory as it appears in his *Beiträge zur Geometrie der Lage* (1856). The theory rests upon the properties of an involution and is not inherently difficult. Instead of making use of an involution, i.e. of a correspondence of period two, Klein utilizes a correspondence of period three. While this latter method has certain geometrical advantages, it is less natural from the analytical point of view.

In §9.6 we shall define a *collineation*, proving that the only collineation in real geometry is a projectivity. In complex geometry a collineation may be either a projectivity or an antiprojectivity. After defining a *correlation* in §9.7 we shall classify correlations of period two and briefly consider the analytical definition of congruence and the introduction of length into geometry.

9.2. A Projectivity Between Two Planes. The discussion of a projective correspondence between the points of two planes is much the same as of that between the points of two lines; the Fundamental Theorem of projective geometry is all that is required. The argument extends, indeed, to the discussion of a projective correspondence between the points of two n-dimensional projective spaces, and we shall give the general form of each theorem in a foot-note.

Consider two planes π_1, π_2, and a point O_1 not in either plane. If A_1, B_1, C_1, D_1, . . . are points of π_1, and if OA_1, OB_1, OC_1, OD_1, . . . meet π_2 in A_2, B_2, C_2, D_2, . . . respectively, we shall say that π_1, π_2 are *in perspective* from O_1 and that A_1, A_2; B_1, B_2; . . . are pairs of corresponding points; in particular, every point of the line of intersection of π_1, π_2 is self-corresponding. A chain of such perspectivities will be called a *projectivity* as before. Clearly, a line corresponds to a line in a projective correspondence between two planes, and corresponding lines intersect in corresponding points. We begin by proving the analogue of 3.22:

9.21. *Any four points of one plane may be related to any four points of another plane by at most three perspectivities, provided no three of the four points in either plane lie on a line.*†

If A_1, B_1, C_1, D_1 in π_1 correspond respectively to A_4, B_4, C_4, D_4 in π_4, let A_1B_1 meet C_1D_1 in P_1 and A_4B_4 meet C_4D_4 in P_4. From any point O_1 on P_1P_4, project the plane π_1 into a plane π_2 passing through P_4; the points A_1, B_1, C_1, D_1 will project into A_2, B_2, C_2, D_2, as in Fig. 9.2A. If A_2A_4, B_2B_4 meet in O_2,

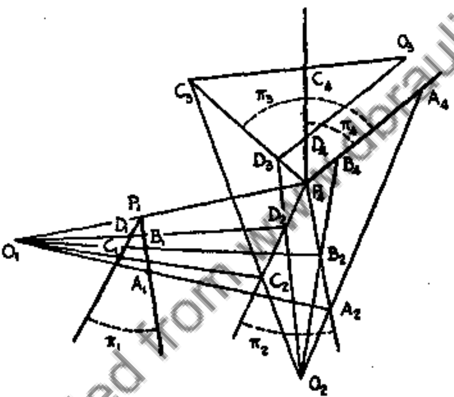

FIG. 9.2A

project the plane π_2 into a plane π_3 passing through the line A_4B_4; under such a projection the point P_4 will be self-corresponding, and C_2D_2 will project into C_3D_3 which passes through P_4. Finally, if C_3C_4 and D_3D_4 meet in O_3, project π_3 into π_4, every point on A_4B_4 remaining fixed. We sum up the construction thus:

†Any $n+2$ points of one n-dimensional space may be related to any $n+2$ points of another n-dimensional space by at most $n+1$ perspectivities, provided no $n+1$ points of either space lie in an $(n-1)$-dimensional space.

$$((A_1, B_1, C_1, D_1)) \overset{O_1}{\underset{\wedge}{=}} ((A_2, B_2, C_2, D_2)) \overset{O_2}{\underset{\wedge}{=}} ((A_4, B_4, C_3, D_3))$$
$$\overset{O_3}{\underset{\wedge}{=}} ((A_4, B_4, C_4, D_4)),$$

where the double brackets indicate that the points lie in a plane but not on a line.

When the line of intersection p of two planes π, π_1, which are projectively related, is made up entirely of self-corresponding points, the situation is particularly simple. If A, B, C are three non-collinear points of π, no one of which lies on p, which correspond respectively to A_1, B_1, C_1 in π_1, then BC meets p in a self-corresponding point L which must lie on B_1C_1 in π_1. By a similar argument, the pairs of corresponding lines CA, C_1A_1 and AB, A_1B_1 also intersect on p in, say, M and N. By Desargues' Theorem AA_1, BB_1, CC_1 are concurrent in some point O. It follows from 3.41 that each of the three pairs of corresponding lines BC, B_1C_1; CA, C_1A_1; AB, A_1B_1 are in perspective from O. But C may be any point in π, and hence:

9.22. *If two planes are projectively related and every point of their line of intersection is self-corresponding, then the two planes are in perspective from some point O.*†

Let us project the plane π_1 on to the plane π from a point U of general position. If UA_1, UB_1, . . . meet π in A', B', . . ., the line $A'B'$ will correspond to the line AB, and AB, $A'B'$ will intersect in a self-corresponding point N on p. Moreover, AA' and BB' will intersect in a self-corresponding point V, which is the projection of the point O from U. Such a correspondence between the points and lines of π is called an

†If two n-dimensional sub-spaces of an $(n+1)$-dimensional space are projectively related and every point of their $(n-1)$-dimensional intersection is self-corresponding, then the two n-dimensional spaces are in perspective from some point O.

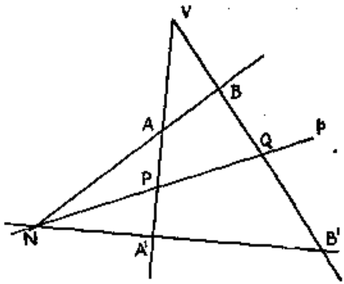

FIG. 9.2B

homology, of which V is the *centre* and p the *axis*, as in Fig. 9.2B. If AA', BB' meet p in P, Q respectively, then

$$\{VP, AA'\} = \{VQ, BB'\};$$

this cross ratio is a constant for every choice of the point A, and is known as the *cross ratio of the homology*. In particular, if $\{VP, AA'\} = -1$, the correspondence is an *harmonic homology*, and if A' corresponds to A then A corresponds to A'. Any line AA' through V is self-corresponding, and A, A' is a point pair of an involution of which V and P are the double points.

Conversely, if a plane π is set into projective correspondence with itself such that to each point A in π there corresponds a point A' and A corresponds to A', then, if A and A' are distinct, the line AA' is a self-corresponding line, and two such lines AA', BB' must intersect in a self-corresponding point V. If AB, $A'B'$ intersect in a point N, then it follows that N is a self-corresponding point; similarly, the intersection R of AB' and $A'B$ is a self-corresponding point. Clearly, the line NR is self-corresponding, and if NR meets AA', BB' in P, Q, as in Fig. 9.2C, then P, Q, are self-corresponding points.

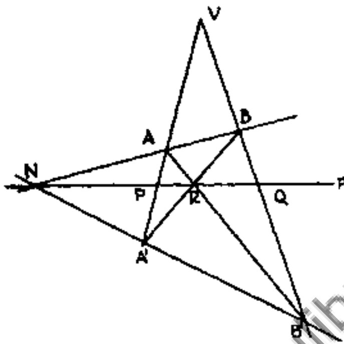

FIG. 9.2C

We conclude that every point on NR is self-corresponding, and, since $\{VP, AA'\} = -1$, the correspondence must be an harmonic homology, provided it is not the identical correspondence.

The two-dimensional analogue of the Fundamental Theorem of projective geometry is contained in the following theorem:

9.23. *A projectivity between two planes, which may coincide, is uniquely determined when four pairs of corresponding points are given, provided no three of the four points in either plane lie on a line.*†

Let us take A, B, C, D in π and A_1, B_1, C_1, D_1 in π_1, subject to the condition of the theorem. By 9.21, we can set up a projectivity between π and π_1, such that

$$((A, B, C, D)) \barwedge ((A_1, B_1, C_1, D_1));$$

the question is as to whether it is unique. If AB, CD meet

†A projectivity between two n-dimensional spaces is uniquely determined when $n+2$ pairs of corresponding points are given, provided no $n+1$ points in either space lie in an $(n-1)$-dimensional sub-space.

in P and A_1B_1, C_1D_1 meet in P_1, then in such a correspondence:

9.24.
$$A(P, C, D, \ldots) \bar{\wedge} A_1(P_1, C_1, D_1, \ldots),$$
$$B(P, C, D, \ldots) \bar{\wedge} B_1(P_1, C_1, D_1, \ldots),$$
$$\text{and } (P, A, B, \ldots) \bar{\wedge} (P_1, A_1, B_1, \ldots).$$

Any point Q in π, not on AB, is the intersection of two lines AQ, BQ; the corresponding point Q_1 is the intersection of the corresponding lines A_1Q_1, B_1Q_1 in π_1. If X, Y, Z lie on a line in π, then X_1, Y_1, Z_1 lie on the corresponding line in π_1; for the defining pencils of lines are in perspective in π, and hence also in π_1. Clearly, the correspondence between the points and lines of π and π_1 is uniquely determined, provided the projectivities in 9.24 are uniquely determined; but this is ensured by the Fundamental Theorem, if A_1, B_1, C_1, D_1 in π_1 correspond respectively to A, B, C, D in π.

From 9.21 and 9.23, we conclude that *a projectivity between two distinct planes is equivalent to at most three perspectivities*,† which is the generalization of 3.33. One further perspectivity is necessary if a plane is set into projective correspondence with itself. The analytical expression of such a projectivity in a plane is given by the homogeneous linear transformation‡

9.25.
$$sx_0' = a_{00}x_0 + a_{01}x_1 + a_{02}x_2,$$
$$sx_1' = a_{10}x_0 + a_{11}x_1 + a_{12}x_2,$$
$$sx_2' = a_{20}x_0 + a_{21}x_1 + a_{22}x_2,$$

where s is an arbitrary non-zero element of the coordinate field σ. The proof of 7.46 generalizes immediately, and there is no need of repeating the argument. There are eight inde-

†A projectivity between two n-dimensional spaces is equivalent to at most $n+1$ perspectivities.

‡The analytical expression of a projectivity in an n-dimensional space is given by the homogeneous linear transformation

$$sx_i' = \Sigma a_{ij}x_j, \ (i, j = 0, 1, 2, \ldots, n).$$

pendent constants in 9.25, which are determined by the eight linear equations to which the four pairs of corresponding points give rise. In order to find the fixed points of the correspondence, it is necessary to solve a cubic equation for s, obtained by setting $x_0'=x_0$, $x_1'=x_1$, $x_2'=x_2$ in 9.25. In the case of the homology, two of the roots of this equation will be equal, and hence all three must be real. After the introduction of imaginary elements in §§9.4 and 9.5, we shall be able to say that every general correspondence in the plane leaves three points fixed, of which two may be imaginary.

9.3. Involutory Correspondence in Space. Just as we passed from 9.22 to the notion of an involutory correspondence in a plane, so we may pass from the 3-dimensional analogue of 9.22 to the notion of an involutory correspondence in space. If A, A' are any two corresponding points, then, if A is distinct from A', the line AA' is a self-corresponding line. From the Fundamental Theorem there cannot be more than two self-corresponding points on AA', unless every point is self-corresponding. We conclude that the correspondence on AA' is an involution *induced by* the correspondence in space. *If two self-corresponding lines intersect in a point* V, then V is a self-corresponding point, and the correspondence induced in the self-corresponding plane through V, determined by the two self-corresponding lines, is an harmonic homology whose axis p will not in general pass through V. By considering lines through V which do not lie in this self-corresponding plane, it is not difficult to see that there must be at least one line q through V which is self-corresponding and does not intersect p. Two cases arise, according as (i) every point on q is self-corresponding, or (ii) the correspondence induced on q is an involution of which V, Q are the self-corresponding points.

(i) The first case is the more interesting of the two and is known as a *bi-axial harmonic homography*. Every plane through p (or q) is a self-corresponding plane, and the correspondence induced in such a plane is an harmonic homology having p (or q) as axis and the point of intersection with q (or p) as centre. If l, m, n are any three lines meeting p, q,

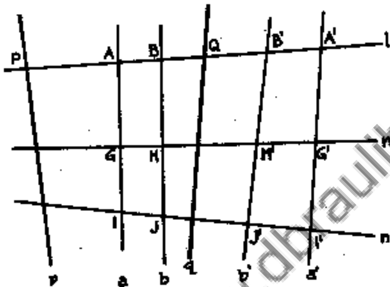

FIG. 9.3A

as in Fig. 9.3A, they are skew to one another, since p, q do not intersect. Through any point A on l may be drawn one and only one transversal a of m and n, and the totality of such transversals is called a *regulus* $\mathfrak{R}$. Each of l, m, n is a self-corresponding line, and if AA', BB', . . . are point pairs of the involution induced on l which determine aa', bb', . . . of $\mathfrak{R}$, then

$$(A, A', B, B', \ldots) \barwedge (G, G', H, H', \ldots),$$

where aa', bb', . . . meet m in GG', HH', . . . respectively. We note, in passing, that a regulus may also be defined as the totality of lines joining pairs of corresponding points of two related ranges on two skew lines. Similarly, any three lines a, b, c of $\mathfrak{R}$ determine a regulus of transversals $\mathfrak{S}$ containing l, m, n. The points of $\mathfrak{R}$ and $\mathfrak{S}$ constitute a *quadric surface*, of which the lines of $\mathfrak{R}$ and $\mathfrak{S}$ are *generators of opposite systems*. The bi-axial harmonic homography induces a correspondence between the lines of $\mathfrak{R}$, so that we may speak of p, q as the self-corresponding lines of the *involution system of lines*

aa', bb', ... Every line of $\mathfrak{S}$ is a self-corresponding line, meeting p, q in the self-corresponding points and pairs of corresponding lines of $\mathfrak{R}$ in pairs of corresponding points of an involution.

(ii) In the second case, where only two points on q are self-corresponding, the plane ω determined by Q and p is a self-corresponding plane. Clearly, any plane through q is a self-corresponding plane in which the induced correspondence is an harmonic homology, whose axis passes through Q and intersects p. Thus every point of ω is a self-corresponding point, and the correspondence is known as an *harmonic homology* in space, V being the *centre* and ω the *axial plane* of the homology.

If no two self-corresponding lines have a common point the situation is quite different, for no point in space can then be self-corresponding. All that can be said is that two corresponding planes intersect in a self-corresponding line l, and the involution induced on l has *no self-corresponding points*. If AA', BB' are two point pairs of this involution on l, we may suppose that GG', HH' are two point pairs of the involution induced on a second self-corresponding line m, distinct from l, such that $\{AA', BB'\} = \{GG', HH'\}$, or

$$(A, A', B, B') \barwedge (G, G', H, H').$$

This correspondence between the points of l and m defines a regulus $\mathfrak{R}$, whose lines correspond in pairs. *No line of this involution system of lines is self-corresponding.* Such a correspondence in space is fundamental in von Staudt's introduction of imaginary elements, which we shall describe in the following §§9.4 and 9.5.

9.4. Imaginary Points on a Line. Consider the general projective transformation given by

9.41. $$x' = \frac{ax+b}{cx+d},$$

where, for convenience, we have written $a_{11}=a$, $a_{10}=b$, $a_{01}=c$, $a_{00}=d$ in 7.44′. If this transformation is applied to the points of a line l, the condition under which a point $X(x)$ is self-corresponding is that x must satisfy the equation

9.42. $$cx^2+(d-a)x-b=0,$$

obtained by setting $x'=x$ in 9.41. The discriminant of 9.42 is given by

$$\Delta=(d-a)^2+4bc=(d+a)^2-4(ad-bc)$$
$$=\Delta_1{}^2-4\Delta_2,$$

where $\Delta_1=d+a$ and $\Delta_2=ad-bc$, and the roots x_1, x_2 of 9.42 will be real and different, equal, or conjugate imaginary, according as $\Delta \gtreqless 0$. If x is distinct from x_1 and x_2, a straightforward calculation shows that

9.43. $$\{x_1x_2,\ xx'\}=\frac{\Delta_1+\sqrt{\Delta}}{\Delta_1-\sqrt{\Delta}},$$

and *this is independent of the choice of* x.

By a comparison of 7.44′ and 7.45, we see that the condition for an involution is that $d=-a$, or $\Delta_1=0$; 9.41 becomes

9.44. $$x'=\frac{ax+b}{cx-a},$$

and the roots of 9.42 are, in this case, given by

9.45. $$x=(a\pm\sqrt{a^2+bc})/c.$$

These roots are real and different, equal, or conjugate imaginary, according as $\Delta=-4\Delta_2 \gtreqless 0$; the cross ratio 9.43 is -1 in every case. If XX', YY' are two distinct point pairs of the involution 9.44, the condition that $\{XX',\ YY'\}=-1$ turns out to be that

9.46. $$y=\frac{(a\pm\sqrt{-(a^2+bc)})x+b}{cx-(a\mp\sqrt{-(a^2+bc)})},$$

where the choice of sign corresponds to the interchange of **y** and **y**′ subject to 9.44. In general,

$$\{XX', YY'\} \gtrless 0, \text{ according as } \Delta = -4\Delta_2 = 4(a^2+bc) \gtrless 0.$$

In the former case, two point pairs do not separate one another, and the involution is *direct*, since $S(XX'Y)=S(XX'Y')$, or *hyperbolic*; the roots in 9.45 are real and different and define the two self-corresponding points of the involution. In the latter case, two point pairs do separate one another, and the involution is *opposite*, since $S(XX'Y)\neq S(XX'Y')$, or *elliptic*; the roots in 9.45 are conjugate imaginary numbers, and there are no (real) self-corresponding points. We remark in passing that the self-corresponding points of 9.41 are also the self-corresponding points of the involution 9.44, in which a is replaced by $-\frac{1}{2}(d-a)$.

In order to set up a correspondence between the elliptic involutions on a line and complex numbers, defined as in chapter VI by a pair of real numbers, let us choose four distinct points A, A', B, B' on l such that $AA'||BB'$. If the coordinates of B' and A' are taken to be 0 and ∞ respectively, after the choice of a unit point I, the coordinates of A and B may be taken to be a and $-b/a$ respectively, where $a<-b/a$, since $AA'||BB'$. The two pairs of corresponding points AA', BB' determine the elliptic involution

9.44′. $$x' = \frac{ax+b}{x-a},$$

there being no loss of generality in taking c to be 1; since, if $c=0$, $\Delta>0$, contrary to supposition. If $A'(\infty)$ and $B'(0)$ remain fixed, every possible involution 9.44′ will be obtained by choosing every possible pair of points $A(a)\neq A'(\infty)$ and $B(-b/a)$, such that $a<-b/a$. Every involution 9.44′ determines *two* conjugate imaginary numbers

9.45′. $$x = a\pm\sqrt{a^2+b} = a\pm i\sqrt{-(a^2+b)},$$

and every imaginary number may be written in one or other of these two forms after a suitable choice of a, b. In order to make the correspondence (1, 1), it is sufficient to associate with a given elliptic involution 9.44′ either one of the two senses $S(AA'B)$ or $S(AA'B')$ in the line. The association is arbitrary, but when once made in a single case, it is definite. In the degenerate case, where $A(a)=B(-b/a)$ and $a^2+b=0$, the two imaginary numbers 9.45′ coincide in the real number a; the distinction with regard to sense is inoperative, and the two double points of the involution 9.44′ coincide in the point $A(a)$. To sum up: *every non-degenerate elliptic involution on a line l determines a pair of real numbers* $[a, +\sqrt{-(a^2+b)}]$, *or a pair of real numbers* $[a, -\sqrt{-(a^2+b)}]$, *according to the associated sense in l; every degenerate involution on l determines a pair of real numbers* $[a, 0]$, *and conversely.*

Let us make the following assumption:

XIII (i). *Every non-degenerate elliptic involution* $AA', BB', \ldots$ *on a line l, determines two self-corresponding IMAGINARY POINTS,* $[AA', BB']$ *associated with* $S(AA'B)$, *and* $[AA', B'B]$ *associated with* $S(AA'B')$, *on l.*

If EE', FF' are any other two point pairs of the involution AA', BB', . . ., then the point $[AA', BB']$ would be equally well represented by $[EE', FF']$, provided $S(AA'B)=S(EE'F)$. We may, in fact, arrange that $\{EE', FF'\}$ has any desired negative value, in particular -1, subject to 9.46.

In virtue of XIII(i), *every* involution on l has two self-corresponding or double points, which may coincide. More generally, every projectivity on l has two self-corresponding points, which may coincide. In order to discuss the field properties of the totality of real and imaginary points on a line, it is necessary to carry through the constructions in §7.2 for the sum and product of any two points. For this, we must show that our extended class of points satisfies axioms I-VII.

9.5. Complex Geometry. If $AA' \| BB'$ on a line l, and L is any real point not on l, we shall denote LA, LA', LB, LB' by a, a', b, b' respectively, and say that *aa' separate bb'*, writing $aa' \| bb'$, in virtue of X(v). Corresponding to the two senses $S(AA'B)$, $S(AA'B')$ on l, we may distinguish two senses $S(aa'b)$, $S(aa'b')$ in the elliptic involution pencil of lines aa', bb', . . . through L. The plane dual of XIII(i) is:

XIII (ii). *Every non-degenerate elliptic involution pencil of lines aa', bb', . . . through the real point L, determines two self-corresponding IMAGINARY LINES OF THE FIRST KIND $[aa', bb']$ and $[aa', b'b]$ through L. If l is any real line meeting aa', bb', . . . in AA', BB', . . . respectively, then $[aa', bb']$ passes through the imaginary point $[AA', BB']$ on l.*

Thus an imaginary line of the first kind has one, and only one, real point on it.

The space dual of XIII(i) is obtained by considering two pairs of planes $\alpha\alpha'$, $\beta\beta'$ through a real line m. If any transversal l meets α, α', β, β' in A, A', B, B' respectively, we shall say that $\alpha\alpha' \| \beta\beta'$, if and only if $AA' \| BB'$, and we shall associate the senses $S(\alpha\alpha'\beta)$, $S(\alpha\alpha'\beta')$ with $S(AA'B)$, $S(AA'B')$. Thus:

XIII (iii). *Every non-degenerate elliptic involution pencil of planes $\alpha\alpha'$, $\beta\beta'$, . . . through the real line m, determines two self-corresponding IMAGINARY PLANES $[\alpha\alpha', \beta\beta']$ and $[\alpha\alpha', \beta'\beta]$ through m. If l is any real line meeting $\alpha\alpha'$, $\beta\beta'$, . . . in AA', BB', . . . respectively, then $[\alpha\alpha', \beta\beta']$ passes through the imaginary point $[AA', BB']$ on l.*

An imaginary plane has one, and only one, real line on it and is met by any real plane, not passing through the real line, in an imaginary line of the first kind.

If l, m are any two real lines which intersect in the point A, then an imaginary point $[AA', BB']$ on l and an imaginary point $[AG', HH']$ on m determine an imaginary line of the first kind, which may be constructed in the following manner. Subject to 9.46, we assume that

$$\{AA', BB'\} = \{AG', HH'\} = -1.$$

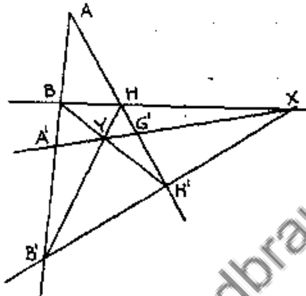

FIG. 9.5A

Thus BH, $A'G'$, $B'H'$ are concurrent, and the two involutions AA', BB', . . . and AG', HH', . . . are in perspective from a point X, as in Fig. 9.5A. If we denote XA, XA', XB, XB' by a, a', b, b' respectively, then the imaginary line of the first kind determined by $[AA', BB']$ and $[AG', HH']$ is given by $[aa', bb']$; similarly, the imaginary line of the first kind determined by $[AA', B'B]$ and $[AG', H'H]$ is given by $[aa', b'b]$. Again, BH', $A'G'$, $B'H$ are concurrent in a point Y, and if we denote YA, YG', YH, YH' by g, g', h, h' respectively, then the imaginary line of the first kind determined by $[AA', B'B]$ and $[AG', HH']$ is given by $[gg', hh']$; similarly, the imaginary line of the first kind determined by $[AA', BB']$ and $[AG', H'H]$ is given by $[gg', h'h]$. Dualizing in the plane, it is clear that two elliptic involution pencils of lines in a real plane are in perspective from two real lines x, y on which they determine two pairs of conjugate imaginary points. The association of a sense with each involution pencil, distinguishes the two perspectivities. We leave the space dual for the reader to formulate.

Consider three real lines l, m, n such that l, m intersect and m, n intersect, as in the accompanying Fig. 9.5B.

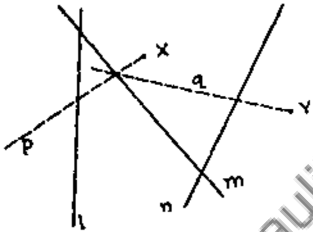

FIG. 9.5B

An imaginary point $[AA', BB']$ on l and an imaginary point $[GG', HH']$ on m, determine an imaginary line of the first kind p through a real point X. Similarly, $[GG', HH']$ on m and $[II', JJ']$ on n, determine an imaginary line of the first kind q through a real point Y. The two imaginary lines p, q, or the three imaginary points, determine a plane through the real line XY, which is real or imaginary, according as XY does, or does not, meet l, m, n; i.e. according as l, m, n are, or are not coplanar. Again, the imaginary line of the first kind p and a real point R determine a real or imaginary plane, according as RX does, or does not meet l, m. Finally, one imaginary point $[AA', BB']$ and two real points R, S determine a real or an imaginary plane, according as RS does, or does not meet l. Dually, we may obtain the condition that three real or imaginary planes determine a real or imaginary point in space.

If two real lines l, m do *not* intersect, an imaginary point $[AA', BB']$ on l and an imaginary point $[GG', HH']$ on m will *not* determine an imaginary line of the first kind. The elliptic involutions AA', BB', . . . on l and GG', HH', . . . on m

will, however, determine an elliptic involution system of lines and a bi-axial harmonic homography in space, *in which no point is self-corresponding.* Assuming that $\{AA', BB'\} = \{GG', HH'\}$, if we require that

$$(A, A', B, B', \ldots) \;\overline{\wedge}\; (G, G', H, H', \ldots),$$

then the lines AG, $A'G'$, BH, $B'H'$, ..., or a, a', b, b', ... (cf. Fig. 9.3A, in which the involutions are *hyperbolic*), belong to a regulus $\mathfrak{R}$. If n is any line belonging to the regulus $\mathfrak{S}$ of transversals of $\mathfrak{R}$, and if we suppose that the points of intersection II', JJ', ... of n with aa', bb', ... are point pairs of an involution on n, then it is not difficult to see that we have completely determined the correspondence in space. If we associate with the two senses $S(AA'B)$ and $S(AA'B')$ on l, the two senses $S(aa'b)$ and $S(aa'b')$ in the involution system of lines, it is natural to make the following assumption:

XIII (iv). *Every non-degenerate elliptic involution system of lines, aa', bb', . . . determines two self-corresponding IMAGINARY LINES OF THE SECOND KIND $[aa', bb']$ and $[aa', b'b]$. If l is any line of the regulus of transversals meeting aa', bb', . . . in AA', BB', . . . respectively, then $[aa', bb']$ passes through the imaginary point $[AA', BB']$ on l.*

An imaginary line of the second kind has no real points on it. If we denote the planes through l and aa', bb', ... by $\alpha\alpha'$, $\beta\beta'$, . . . and the planes through m and aa', bb', . . . by $\gamma\gamma'$, $\delta\delta'$, . . ., then the imaginary line of the second kind $[aa', bb']$ is the line of intersection of the two imaginary planes $[\alpha\alpha', \beta\beta']$ and $[\gamma\gamma', \delta\delta']$.

To these four assumptions XIII(i)-(iv) is added:

XIV. *Every degenerate involution determines its self-corresponding real element,*

which corresponds to the identification of the complex number $[a, 0]$ with the real number a in (4″) of §6.6. Analogously, let us speak of a real or imaginary point as a *complex point,* of a real or an imaginary line of the first or second kind as a *complex line*, and of a real or imaginary plane as a *complex plane*. These complex elements being considered in a *complex space*, we have *complex projective geometry*.

Desargues' Theorem remains valid in complex geometry, since its proof depends only on the incidence relations between points, lines, and planes. The Principle of Duality is also valid. *The Fundamental Theorem in complex projective geometry is a consequence of the Fundamental Theorem in real projective geometry.* Certainly, if three distinct real points A_1, B_1, C_1 on a line l_1 are related to three distinct real points A_2, B_2, C_2 on a line l_2, then to each set of four points P_1, P_1', Q_1, Q_1' on l_1, such that $P_1P_1' || Q_1Q_1'$, there corresponds a set of four points P_2, P_2', Q_2, Q_2' on l_2, such that $P_2P_2' || Q_2Q_2'$. The involutions determined by these two sets of points will determine two corresponding imaginary points on l_1 and l_2, according to the associated sense in l_1 and l_2. In addition to this *real projectivity*, various other cases arise as one or more of the three points on each of l_1 and l_2 are imaginary. Without considering the general case in which all the points are imaginary, let us prove the theorem when the real points A_1, C_1 on l_1 are related to the real points A_2, C_2 on l_2 and the real point B_1 on l_1 is related to the imaginary point $[A_2A_2', C_2C_2']$ on l_2, as in Fig. 9.5c.

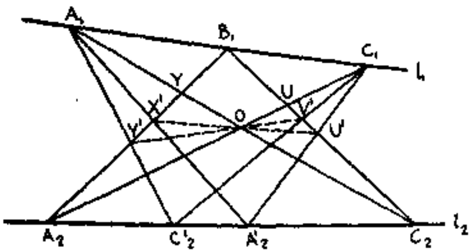

FIG. 9.5C

The imaginary line of the first kind $A_1[A_2A_2', C_2C_2']$ intersects the real line B_1A_2 in the imaginary point $[A_2X', YY']$, and $C_1[A_2A_2', C_2C_2']$ intersects B_1C_2 in $[UU', C_2V']$. From Pappus' Theorem in the real geometry, $X'U'$, $Y'V'$ both pass through the point of intersection O of A_1C_2 and C_1A_2. Thus $[A_2X', YY']$, O, $[UU', C_2V']$ are collinear, and the intermediary or Pappus' line is an imaginary line of the first kind through O.

From the validity of Desargues' Theorem and Pappus' Theorem, it follows that the sum and the product of two complex points is uniquely defined; multiplication is commutative, and the points on a line form a field which is isomorphic with the field of complex numbers. As the simplest set of axioms for real projective geometry is obtained by taking I-VII, along with the assumption that the coordinate field σ is the field of real numbers, so the simplest set of axioms for complex projective geometry is obtained by taking I-VII, along with the assumption that σ is the field of complex numbers. A more significant procedure is to obtain complex projective geometry from real projective geometry by the method of this section, or its equivalent. The axioms I-VIII are valid in complex geometry without modification, but those axioms which involve the concept of order, namely X-XII, apply only to real geometry. With this proviso, the adjunction of XIII and XIV yields a set of independent assumptions for complex projective geometry.

9.6. Collineations. As we remarked in a foot-note at the end of §9.2, the most general projectivity in a space of n dimensions, defined with reference to a field σ, is given by

9.61. $$sx_i' = \Sigma a_{ij}x_j, \quad (i, j = 0, 1, 2, \ldots, n),$$

where s is any non-zero element of σ.

If a point P is transformed into a point P', and if an $(n-1)$-dimensional sub-space $[n-1]$ of an n-dimensional projective space $[n]$ is transformed into an $(n-1)$-dimensional sub-space $[n-1]'$ of $[n]$ in such a manner that P' lies in $[n-1]'$, if and only if P lies in $[n-1]$, then every $[k]$ is transformed into a $[k]'$, for $0 \leqslant k \leqslant n$. Such a transformation is known as a *collineation* in $[n]$. Now, while we shall prove that every collineation in a real projective space is a projectivity 9.61, this is untrue in a complex space. For simplicity consider the case of a line l; the most general projectivity in l may be written

9.62. $$x' = \frac{ax+b}{cx+d},$$

where a, b, c, d are complex numbers. Clearly, the transformation

9.62′. $$x' = \bar{x},$$

where the bar indicates the conjugate complex quantity, is a collineation and sets up a correspondence between the points of l under which *every real point is self-corresponding*, yet the transformation is not the identity. In contrast to 9.62, 9.62′ is called an *anti-projectivity*, and the most general anti-projectivity in l is obtained by combining 9.62 and 9.62′ to yield

9.63. $$x' = \frac{a\bar{x}+b}{c\bar{x}+d}.$$

Consider two sets of four collinear points A, B, C, D and G, H, I, J in $[n]$, such that

$$(A, B, C, D) \barwedge (G, H, I, J),$$

where $\{AB,\ CD\} = \{GH,\ IJ\} = \lambda$ is an element of σ. A collineation C in $[n]$ transforms A, B, C, D and G, H, I, J into A', B', C', D' and G', H', I', J'; from the definition of C, a perspectivity is transformed into a perspectivity, and

$$(A', B', C', D') \barwedge (G', H', I', J').$$

9.69. *The most general collineation in* $[n]$ *is obtained by combining* 9.68 *with* 9.61 *to yield*†

$$sx_i' = \Sigma a_{ij}\varphi(x_j), \quad (i, j = 0, 1, 2, \ldots, n),$$

for every automorphism $\varphi(\lambda)$ *of the field* σ.

The problem of determining all collineations in a projective space defined with reference to a field σ, is thus reduced to the problem of determining all possible automorphisms of σ. No general solution is possible, and we shall confine our attention to the three cases where σ is (i) the field of real numbers, (ii) the field of complex numbers, and (iii) a finite field.

(i) Since $\varphi(0) = 0$ and $\varphi(1) = 1$, it follows that in any automorphism of the field of real numbers a rational number is mapped upon itself. Also, since $\varphi(\lambda^2) = \varphi(\lambda) . \varphi(\lambda)$ for any real number λ, it follows that positive numbers are mapped upon positive numbers. We conclude that relations of order are preserved, any real number must be mapped upon itself, and the only automorphism of the field of real numbers is the identity. Hence, *the only collineation in real projective geometry is a projectivity.*

(ii) If σ is the field of complex numbers, it follows as before that any rational number is mapped upon itself. But for a real number to be mapped upon itself, it is necessary to make some assumption concerning the function $\varphi(\lambda)$, say that $\varphi(\lambda)$ is continuous in λ. Since $i^2 = -1$, $\varphi(i) . \varphi(i) = \varphi(-1) = -1$ and $\varphi(i) = \pm i$. If we take the positive sign, $a + ib$ is mapped upon $a + ib$ and the automorphism is the identity. If $\varphi(i) = -i$, $a + ib$ is mapped upon $a - ib$. Thus, *the only continuous collineations in complex projective geometry are the projectivity and anti-projectivity.*

†Sometimes called a *semi-linear* transformation.

(iii) With regard to a finite field $GF(p^n)$, we know that any element may be written as the power of a given primitive element a of the field. Thus we may suppose that $\varphi(a) = a^m$, and

$$\varphi(a^k) = [\varphi(a)]^k = a^{mk} = (a^k)^m.$$

But an integral element must be mapped upon itself; so, for such an a, $\varphi(a) = a^m = a (\mathrm{mod}\ p)$. We conclude that $m = p^l$, and *the only collineations in a finite geometry are given by* 9.69 *where*

$$\varphi(x_j) = x_j^{p^l},$$

for $l = 0, 1, 2, \ldots, (n-1)$.

9.7. Correlations. By an extension of the argument of §7.5, it may readily be seen that the coordinates of any point $P(x_0, x_1, \ldots, x_n)$ of an $(n-1)$-dimensional sub-space of $[n]$ satisfy a linear equation

9.71. $$X_0x_0 + X_1x_1 + \ldots + X_nx_n = 0,$$

where the coefficients $X_0, X_1, \ldots, X_n$ are elements of σ, not all being zero. Clearly, we may represent such a *hyperplane* by the $n+1$ *homogeneous coordinates* $(X_0, X_1, \ldots, X_n)$. A correspondence between the points and hyperplanes of $[n]$, such that to each point P there corresponds an unique hyperplane p' and to each hyperplane p there corresponds an unique point P', is called a *correlation* or *reciprocity*, provided p' passes through P', if and only if p passes through P. If the coordinates of P' are $(x_0', x_1', \ldots, x_n')$ and the coordinates of p' are $(X_0', X_1', \ldots, X_n')$, this condition is equivalent to

9.72. $$X_0'x_0' + X_1'x_1' + \ldots + X_n'x_n' = 0.$$

We obtain a special correlation by writing

9.73. $$X_i' = x_i. \quad (i = 0, 1, 2, \ldots, n).$$

Comparing 9.71 and 9.72, it follows that

9.73′. $$x_i' = X_i. \quad (i = 0, 1, 2, \ldots, n).$$

Clearly, the inverse of a correlation is a correlation, and the product of two correlations is a collineation. Hence:

9.74. *The most general correlation is obtained by combining* 9.73 *and* 9.73′ *with* 9.69, *to yield*

$$sX_i' = \Sigma a_{ij}\varphi(x_j), \quad sx_i' = \Sigma A_{ij}\varphi(X_j).$$

The dual form is again a consequence of the comparison of 9.71 and 9.72. The matrices (a_{ij}) and (A_{ij}) are said to be *contragredient.*

The class of correlations of period two is of particular importance; we state the following theorem without proof.

9.75. *A correlation of period two associates with a point* $P(x_0, x_1, \ldots, x_n)$ *an hyperplane whose coordinates are given by:*

(i) $X_i' = \Sigma a_{ij}x_j$, *with* $a_{ij} = a_{ji}$ *(polarity);*

or by (ii) $X_i' = \Sigma a_{ij}x_j$, *with* $a_{ij} = -a_{ji}$ *(null system, for n odd);*

or by (iii) $X_i' = \Sigma a_{ij}\varphi(x_j)$, *with* $a_{ij} = \varphi(a_{ji})$,

where $\lambda \to \varphi(\lambda)$ *is an automorphism of* σ *of period two.*

Since the only automorphism of the real field is the identity, the only correlation of period two in real projective geometry is a polarity, or, if the number of dimensions is odd, a null system. The reason for the restriction on n in case (ii) is that a skew-symmetrical determinant of odd order is identically zero. In complex projective geometry, putting $\varphi(\lambda) = \bar{\lambda}$, case (iii) yields a correspondence which we may call an *anti-polarity.*

In order to obtain the point equation of the hyperplane corresponding to P, it is necessary to substitute in 9.72. From the condition $a_{ij} = -a_{ji}$, it follows that *in a null system every point lies on its corresponding hyperplane.* In a polarity, the locus of points lying on their corresponding hyperplanes is a hypersurface of the second order given by

9.76. $$\Sigma a_{ij}x_ix_j = 0, \text{ or } \Sigma A_{ij}X_iX_j = 0.$$

If $n=1$, the polarity reduces to a correspondence between the points of a line, and it is readily seen from 9.72 that this correspondence is an involution. The double points of the involution are given by 9.76, the two equations being identical in virtue of 9.71.

If $n=2$, 9.76 represents a conic $\mathfrak{C}$ in the plane, the first equation being written in point coordinates and the second in line coordinates. By a suitable choice of the triangle of reference, these two equations become

9.76′. $$\frac{1}{\epsilon}x_0^2+x_1^2+x_2^2=0, \text{ or } \epsilon X_0^2+X_1^2+X_2^2=0.$$

There are three possibilities, according as $\epsilon<0$, $\epsilon>0$ or $\epsilon=0$. If $\epsilon<0$, $\mathfrak{C}$ is a real curve, while if $\epsilon>0$, there are no real points on $\mathfrak{C}$. If $\epsilon=0$, $\mathfrak{C}$ degenerates into a pair of points given by

9.76″. $$x_1^2+x_2^2=0=x_0, \text{ or } X_1^2+X_2^2=0.$$

In this case, the polarity degenerates into an elliptic involution on the line at infinity $x_0=0$. Could we not take this as the absolute involution by means of which we defined Euclidean Geometry in chapter v?

As we have said before, it is not possible for us to give a complete account of the introduction of a measure of length into geometry, but we shall indicate briefly how this may be done in the Euclidean case. In the first place, those collineations which leave 9.76″ unaltered may be written in point coordinates in the form

9.77. $$\begin{aligned} sx_0' &= a_{00}x_0, \\ sx_1' &= a_{10}x_0+a_{11}x_1-a_{12}x_2, \\ sx_2' &= a_{20}x_0+a_{12}x_1+a_{11}x_2, \end{aligned}$$

or in the form

9.78. $$\begin{aligned} sx_0' &= a_{00}x_0, \\ sx_1' &= a_{10}x_0+a_{11}x_1+a_{12}x_2, \\ sx_2' &= a_{20}x_0+a_{12}x_1-a_{11}x_2. \end{aligned}$$

If we change to non-homogeneous coordinates, writing $x = x_1/x_0$ and $y = x_2/x_0$, and if $a_{00}^2 = a_{11}^2 + a_{12}^2$, 9.77 and 9.78 become

9.77′.
$$x' = \cos\theta . x - \sin\theta . y + m,$$
$$y' = \sin\theta . x + \cos\theta . y + n,$$

and

9.78′.
$$x' = \cos\theta . x + \sin\theta . y + m,$$
$$y' = \sin\theta . x - \cos\theta . y + n.$$

When $m = n = 0$, the transformation 9.77′ is called a *rotation* about the origin of coordinates; when $\theta = 0$, it is called a *translation*, or a *parallel displacement.* If both these conditions are satisfied, 9.77′ is the identity while 9.78′ is a *reflection* in the x-axis. If two points A, B are transformed by 9.77′ or 9.78′ into A', B', the segment AB is said to be *congruent* to the segment $A'B'$. The *distance* between $A(x_1, y_1)$ and $B(x_2, y_2)$ is defined by the function

$$d(AB) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

which is clearly invariant under 9.77′ and 9.78′.

If $\epsilon > 0$, we may similarly define a measure of length which is invariant under those collineations which leave 9.76′ unaltered. The corresponding geometry is called *elliptic*, and is analogous to the geometry on the surface of a sphere. If $\epsilon < 0$, we obtain *hyperbolic* geometry. By dualizing the definition of length, we obtain the definition of *angle* between two intersecting lines; and, while this duality does not hold in Euclidean geometry, it is not difficult to see that *Euclidean geometry is a limiting case between elliptic and hyperbolic geometry.*

APPENDIX

Taking a *sphere* as an undefined element and *inclusion* as an undefined relation, Huntington has given a system of axioms for Euclidean geometry (*Math. Ann.* 73 (1913), p. 522; *Scripta Math.* 5 (1938), p. 149). These axioms are here rearranged to conform with the method of presentation in this book.

1. *If a sphere A includes a sphere B and B includes C, then A includes C.*
2. *If a sphere A includes a sphere B, then A and B are distinct.*

Definition of a Point. A sphere which does not include any other sphere is called a "point-sphere" or a *point*.

3. *There are at least two distinct points.*
4. *If the class of spheres which include the point A is the same as the class of spheres which include the point B, then A and B coincide. If the class of points included by the sphere S is the same as that included by the sphere T, then S and T coincide.*

Definition of a Line. If A and B are two points the *segment* AB, or $[AB]$, is the class of points X such that every sphere which includes A and B also includes X. A and B are the *end-points* or the *boundary* of $[AB]$. The *extension of* $[AB]$ *beyond* A is the class of points X such that $[BX]$ contains A; similarly, the extension of $[AB]$ beyond B is the class of points X such that $[AX]$ contains B. The *ray* AB is the class of points belonging to $[AB]$ or to the extension of $[AB]$ beyond B. The *line* AB is the class of points belonging to $[AB]$ or to one of its two extensions.

5. *If X is a point of the segment AB, then AB is made up of two non-overlapping*† *segments AX and BX.*
6. *If two lines have two points in common they coincide.*
7. *If A, B are two distinct points, there is a point C not on the line AB.*

Definition of a Plane. If A, B, C are three non-collinear points the *triangle* ABC, or $[ABC]$, is the class of points X such that every sphere which includes A, B, C also includes X. The *boundary* of $[ABC]$ consists of the three vertices A, B, C and the three edges

†*Non-overlapping* in that the boundary point X is counted only once; cf. axioms 8 and 11.

[AB], [BC], [CA]. The *extension of* [ABC] *beyond* A is the class of points X such that [BCX] contains A. The *extension of* [ABC] *beyond* [AB] is the class of points X such that [CX] intersects [AB]. The *plane* ABC is the class of points belonging to the triangle [ABC] or to one of its six extensions.

8. *If X is a point of the triangle ABC, then ABC is made up of the three non-overlapping triangles ABX, BCX, CAX.*

9. *If two planes have three non-collinear points in common they coincide.*

10. *If A, B, C are three non-collinear points, there is a point D not in the plane ABC.*

DEFINITION OF A SPACE. If A, B, C, D are four non-coplanar points the *tetrahedron* $ABCD$, or [$ABCD$], is the class of points X such that every sphere which includes A, B, C, D also includes X. The *boundary* of [$ABCD$] consists of the four vertices A, B, C, D, the six edges [AB], [AC], . . . and the four triangular faces [ABC], [ABD]. . . . The *extension of* [$ABCD$] *beyond* A is the class of points X such that [$BCDX$] contains A. The *extension of* [$ABCD$] *beyond* [AB] is the class of points X such that [CDX] intersects [AB]. The *extension of* [$ABCD$] *beyond* [ABC] is the class of points X such that [DX] intersects [ABC]. The *space* $ABCD$ is defined to be the class of points belonging to [$ABCD$] or to one of its fourteen extensions.

11. *If X is a point of the tetrahedron $ABCD$, then $ABCD$ is made up of the four non-overlapping tetrahedra $ABCX$, $BCDX$, $CDAX$, $DABX$.*

12. *If A, B, C, D are four non-coplanar points, every point belongs to the space $ABCD$.*

AXIOMS OF PARALLELISM

Two coplanar lines, a line and a plane, or two planes which have no point in common are said to be *parallel*.

13. *If two lines are parallel to a third line they are either parallel or coincident.*

14. *If C is a point not on a line AB, there exists a point D such that CD is parallel to AB.*

15. *If AB and CD are parallel lines, no one of the four points A, B, C, D lies within the triangle formed by the other three.*

16. *If the line AB is parallel to the plane CDE, no one of the five points A, B, C, D, E lies within the tetrahedron formed by the other four.*

AXIOMS OF CONGRUENCE

If AB is parallel to CD and BC to DA, then the four points are said to form a *parallelogram* $ABCD$ of which $[AC]$ and $[BD]$ are the *diagonals*. From 15 and the definition of a plane, it follows that these diagonals intersect in a point M called the *mid-point* of each diagonal. In order to show that the same mid-point M of the segment AC would be obtained from any other parallelogram $AB'CD'$ it is sufficient to make the following assumption:

17. *Suppose four points determine one set of six lines and four other points determine another set of six lines. If the first five lines of one set are parallel to or coincident with the first five lines of the other set, then the remaining line of the first set will be parallel to or coincident with the remaining line of the other set.*

This assumption 17 is a particular case of the theorem that the sixth point of a quadrangular set is uniquely determined when the other five are given.

If a sphere S includes two points A and B but does not include any point belonging to either of the extensions of $[AB]$, then A and B lie on the *surface* of S and $[AB]$ is a *chord* of S. Again, if S includes a point O such that every chord through O has O as its mid-point, then O is the *centre* of S.

18. *Every sphere has a centre, provided it is not itself a point.*

Two segments AB and CD which lie in the same or parallel lines are said to be *congruent by "translation"* if the mid-point of AD is the same as the mid-point of BC, or if the mid-point of AC is the same as the mid-point of BD. Two segments OA and OB having a common end-point O are said to be *congruent by "rotation"* if A and B lie on the surface of a sphere S with centre O. We may call OA the *radius* of S. In general, two segments AB and CD are said to be *congruent* if there exist two other segments OX and OY such that OX is congruent to AB by translation and OY is congruent to CD by translation, and OX is congruent to OY by rotation. We shall assume that:

19. *If a segment AB is congruent to a segment CD and CD is congruent to EF, then AB is congruent to EF.*

20. *If AB is any segment, then on any ray OP there is a point X such that the segment OX is congruent to the segment AB.*

21. *The portions of two radii intercepted between the surfaces of two concentric spheres are congruent.*

This last axiom ensures that the "sum" or "difference" of congruent segments shall also be congruent.

Congruence of angles, and the familiar properties of perpendicular lines and of congruent triangles follow from the assumption that:

22. *Suppose A, B, C, X are four points (A, B, C collinear) and A′, B′, C′, X′ another set of four points (A′, B′, C′ collinear). If the segments AB, AC, BC, AX, and BX are congruent to the corresponding segments of the other set, then also CX is congruent to C′X′.*

AXIOM OF CONTINUITY

23. *If S′, S″, . . . is an infinite sequence of spheres each of which is included by the preceding one, then there exists a point X which is included by them all.*

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