MATRIX AND TENSOR CALCULUS

WITH APPLICATIONS TO MECHANICS, ELASTICITY, and AERONAUTICS

BY

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Luddye Kennerly Michal

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To my wife

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EDITOR'S PREFACE

The editors believe that the reader who has finished the study of this book will see the full justification for including it in a series of volumes dealing with aeronautical subjects.

However, the editor's preface usually is addressed to the reader who starts with the reading of the volume, and therefore a few words on our reasons for including Professor Michal's book on matrices and tensors in the GALCIT series seem to be appropriate.

Since the beginnings of the modern age of the aeronautical sciences a close cooperation has existed between applied mathematics and aeronautics. Engineers at large have always appreciated the help of applied mathematics in furnishing them practical methods for numerical and graphical solutions of algebraic and differential equations. However, aeronautical and also electrical engineers are faced with problems reaching much further into several domains of modern mathematics. As a matter of fact, these branches of engineering science have often exerted an inspiring influence on the development of novel methods in applied mathematics.

One branch of applied mathematics which fits especially the needs of the scientific aeronautical engineer is the matrix and tensor calculus. The matrix operations represent a powerful method for the solution of problems dealing with mechanical systems with a certain number of degrees of freedom. The tensor calculus gives admirable insight into complex problems of the mechanics of continuous media, the mechanics of fluids, and elastic and plastic media.

Professor Michal's course on the subject given in the frame of the war-training program on engineering science and management has found a surprisingly favorable response among engineers of the aeronautical industry in the Southern Californian region. The editors believe that the engineers throughout the country will welcome a book which skillfully unites exact and clear presentation of mathematical statements with fitness for immediate practical applications.

THEODORE VON KÁRMÁN
CLARK B. MILLIKAN

PREFACE

This volume is based on a series of lectures on matrix calculus and tensor calculus, and their applications, given under the sponsorship of the Engineering, Science, and Management War Training (ESMWT) program, from August 1942 to March 1943. The group taking the course included a considerable number of outstanding research engineers and directors of engineering research and development. I am very grateful to these men who welcomed me and by their interest in my lectures encouraged me.

The purpose of this book is to give the reader a working knowledge of the fundamentals of matrix calculus and tensor calculus, which he may apply to his own field. Mathematicians, physicists, meteorologists, and electrical engineers, as well as mechanical and aeronautical engineers, will discover principles applicable to their respective fields. The last group, for instance, will find material on vibrations, aircraft flutter, elasticity, hydrodynamics, and fluid mechanics.

The book is divided into two independent parts, the first dealing with the matrix calculus and its applications, the second with the tensor calculus and its applications. The minimum of mathematical concepts is presented in the introduction to each part, the more advanced mathematical ideas being developed as they are needed in connection with the applications in the later chapters.

The two-part division of the book is primarily due to the fact that matrix and tensor calculus are essentially two distinct mathematical studies. The matrix calculus is a purely analytic and algebraic subject, whereas the tensor calculus is geometric, being connected with transformations of coordinates and other geometric concepts. A careful reading of the first chapter in each part of the book will clarify the meaning of the word "tensor," which is occasionally misused in modern scientific and engineering literature.

I wish to acknowledge with gratitude the kind cooperation of the Douglas Aircraft Company in making available some of its work in connection with the last part of Chapter 7 on aircraft flutter. It is a pleasure to thank several of my students, especially Dr. J. E. Lipp and Messrs. C. H. Putt and Paul Lieber of the Douglas Aircraft Company, for making available the material worked out by Mr. Lieber and his research group. I am also very glad to thank the members of my seminar on applied mathematics at the California Institute for their helpful suggestions. I wish to make special mention of Dr. C. C.

Lin, who not only took an active part in the seminar but who also kindly consented to have his unpublished researches on some dramatic applications of the tensor calculus to boundary-layer theory in aeronautics incorporated in Chapter 18. This furnishes an application of the Riemannian tensor calculus described in Chapter 17. I should like also to thank Dr. W. Z. Chien for his timely help.

I gratefully acknowledge the suggestions of my colleague Professor Clark B. Millikan concerning ways of making the book more useful to aeronautical engineers.

Above all, I am indebted to my distinguished colleague and friend, Professor Theodore von Kármán, director of the Guggenheim Graduate School of Aeronautics at the California Institute, for honoring me by an invitation to put my lecture notes in book form for publication in the GALCIT series. I have also the delightful privilege of expressing my indebtedness to Dr. Kármán for his inspiring conversations and wise counsel on applied mathematics in general and this volume in particular, and for encouraging me to make contacts with the aircraft industry on an advanced mathematical level.

I regret that, in order not to delay unduly the publication of this book, I am unable to include some of my more recent unpublished researches on the applications of the tensor calculus of curved infinite dimensional spaces to the vibrations of elastic beams and other elastic media.

ARISTOTLE D. MICHAL

CALIFORNIA INSTITUTE OF TECHNOLOGY
OCTOBER, 1946

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PART I. MATRIX CALCULUS AND ITS APPLICATIONS

CHAPTER 1

ALGEBRAIC PRELIMINARIES

Introduction.

Although matrices have been investigated by mathematicians for almost a century, their thoroughgoing application to physics,[1]† engineering, and other subjects[2] — such as cryptography, psychology, and educational and other statistical measurements — has taken place only since 1925. In particular, the use of matrices in aeronautical engineering in connection with small oscillations, aircraft flutter, and elastic deformations did not receive much attention before 1935. It is interesting to note that the only book on matrices with systematic chapters on the differential and integral calculus of matrices was written by three aeronautical engineers.‡

Definitions and Notations.

A table of mn numbers, called elements, arranged in a rectangular array of m rows and n columns is called a *matrix*[3] *with m rows and n columns.* If a_j^i is the element in the ith row and jth column, then the matrix can be written down in the following pictorial form with the conventional double bar on each side.

$$\left\| \begin{matrix} a_1^1, & a_2^1, & \cdots, & a_n^1 \\ a_1^2, & a_2^2, & \cdots, & a_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_1^m, & a_2^m, & \cdots, & a_n^m \end{matrix} \right\|.$$

In the expression a_j^i the *index* i is called a *superscript* and the index j a *subscript.* It is to be emphasized that the superscript i in a_j^i is *not* the ith power of a variable a_j.

If the number m of rows is equal to the number n of columns, then

† Superior numbers refer to the notes at the end of the book.

‡ Frazer, Duncan, and Collar, *Elementary Matrices and Some Applications to Dynamics and Differential Equations,* Cambridge University Press, 1938.

the matrix is called a *square matrix.*† The number of rows, or equivalently the number of columns, will be called the *order* of the square matrix. Besides square matrices, two other general types of matrices occur frequently. One is the *row matrix*

$$\| a_1, a_2, \cdots, a_n \| ;$$

the other is the *column matrix*

$$\left\| \begin{matrix} a^1 \\ a^2 \\ \cdot \\ \cdot \\ \cdot \\ a^m \end{matrix} \right\| .$$

It is to be observed that the superscript 1 in the elements of the row matrix was omitted. Similarly the subscript 1 in the elements of the column matrix was also omitted. All this is done in the interest of brevity; the index notation is unnecessary when the index, whether a subscript or superscript, cannot have at least two values.

It is often very convenient to have a more compact notation for matrices than the one just given. This compact notation is as follows: if a^i_j is the element of a matrix in the ith row and jth column we can write simply

$$\| a^i_j \|$$

instead of stringing out all the mn elements of the matrix. In particular, a row matrix with element a_k in the kth column will be written

$$\| a_k \| ,$$

and a column matrix with element a^k in the kth row will be written

$$\| a^k \| .$$

Elementary Operations on Matrices.

Before we can use matrices effectively we must *define* the *addition* of matrices and the *multiplication* of matrices. The definitions are those that have been found most useful in the general theory and in the applications.

Let A and B be *matrices of the same type*, i.e., matrices with the same number m of rows and the same number n of columns. Let

$$A = \| a^i_j \| , \quad B = \| b^i_j \| .$$

Then by the sum $A + B$ of the matrices A and B we shall mean the

† It will occasionally be convenient to write a_{ij} for the element in the ith row and jth column of a square matrix. See Chapter 5 and the following chapters.

uniquely obtainable matrix

$$C = \| c^i_j \|,$$

where

$$c^i_j = a^i_j + b^i_j \qquad (i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n).$$

In other words, to *add two matrices* of the *same type*, calculate the matrix whose elements are precisely the numerical sum of the corresponding elements of the two given matrices. The addition of two matrices of different type has no meaning for us.

To complete the preliminary definitions we must make clear what we mean when we say that *two matrices are equal*. Two matrices $A \equiv \| a^i_j \|$ and $B \equiv \| b^i_j \|$ of the same type are equal, written as $A = B$, if and only if the numerical equalities $a^i_j = b^i_j$ hold for each i and j.

Exercise

$$A = \left\| \begin{matrix} 1, & -1, & \sqrt{2}, & 5 \\ 0, & 0, & 3, & -2 \\ 1.1, & 2, & -4, & 1 \end{matrix} \right\|.$$

$$B = \left\| \begin{matrix} 0, & 0, & -\sqrt{2}, & 1 \\ 0, & 0, & -1, & 3 \\ 1, & 0, & 2, & -4 \end{matrix} \right\|.$$

Then

$$A + B = \left\| \begin{matrix} 1, & -1, & 0, & 6 \\ 0, & 0, & 2, & 1 \\ 2.1, & 2, & -2, & -3 \end{matrix} \right\|.$$

The following results embodied in a theorem show that matric addition has some of the properties of numerical addition.

THEOREM. *If A and B are any two matrices of the same type, then*

$$A + B = B + A.$$

If C is any third matrix of the same type as A and B, then

$$(A + B) + C = A + (B + C).$$

Before we proceed with the definition of *multiplication* of matrices, a word or two must be said about two very important special square matrices. One is the *zero matrix*, i.e., a square matrix all of whose elements are zero,

$$\left\| \begin{matrix} 0, 0, \cdots, 0 \\ 0, 0, \cdots, 0 \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ 0, 0, \cdots, 0 \end{matrix} \right\|.$$

We can denote the zero matrix by the capital letter O. Occasionally we shall use the terminology zero matrix for a non-square matrix with zero elements.

The other is the *unit matrix*, i.e., a matrix

$$I = \| \delta^i_j \| ,$$

where

$$\begin{aligned} \delta^i_j &= 1 \quad \text{if} \quad i = j. \\ &= 0 \quad \text{if} \quad i \neq j. \end{aligned}$$

In the more explicit notation

$$I = \left\| \begin{matrix} 1, 0, 0, \cdots, 0 \\ 0, 1, 0, \cdots, 0 \\ 0, 0, 1, 0, \cdots, 0 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ 0, 0, 0, \cdots, 0, 1 \end{matrix} \right\| .$$

One of the most useful and simplifying conventions in all mathematics is the *summation convention: the repetition of an index once as a subscript and once as a superscript will indicate a summation over the total range of that index.* For example, if the range of the indices is 1 to 5, then

$$a_i b^i \quad \text{means} \quad \sum_{i=1}^{5} a_i b^i \quad \text{or} \quad a_1 b^1 + a_2 b^2 + a_3 b^3 + a_4 b^4 + a_5 b^5.$$

Again we warn the reader that the superscript i in b^i is *not* the ith power of a variable b.

The definition of the multiplication of two matrices can now be given in a neat form with the aid of the summation convention. Let

$$A = \left\| \begin{matrix} a^1_1, a^1_2, \cdots, a^1_m \\ a^2_1, a^2_2, \cdots, a^2_m \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a^n_1, a^n_2, \cdots, a^n_m \end{matrix} \right\| ,$$

$$B = \left\| \begin{matrix} b^1_1, b^1_2, \cdots, b^1_p \\ b^2_1, b^2_2, \cdots, b^2_p \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ b^m_1, b^m_2, \cdots, b^m_p \end{matrix} \right\| .$$

Then, by the *product AB of the two matrices, we shall mean the matrix*

$$C = \| c^i_j \|,$$

where

$$c^i_j = a^i_\alpha b^\alpha_j \qquad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, p).$$

If c^i_j is written out in extenso without the aid of the summation convention, we have

$$c^i_j = a^i_1 b^1_j + a^i_2 b^2_j + \cdots + a^i_m b^m_j.$$

It should be emphasized here that, in order that the product AB of two matrices be well defined, the number of rows in the matrix B must be precisely equal to the number of columns in the matrix A. It follows in particular that, *if A and B are square matrices of the same type, then AB as well as BA is always well defined.* However, it must be emphasized that in general AB is not equal to BA, written as $AB \neq BA$, even if both AB and BA are well defined. In other words, matrix multiplication of matrices, unlike numerical multiplication, is *not* always commutative.

Exercise

The following example illustrates the non-commutativity of matrix multiplication. Take

$$A = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \quad \text{so that} \quad a^1_1 = 0,\ a^1_2 = 1,\ a^2_1 = 1,\ a^2_2 = 0,$$

and

$$B = \begin{Vmatrix} -1 & 0 \\ 0 & 1 \end{Vmatrix} \quad \text{so that} \quad b^1_1 = -1,\ b^1_2 = 0,\ b^2_1 = 0,\ b^2_2 = 1.$$

Now

$$\begin{aligned} c^1_1 &= a^1_\alpha b^\alpha_1 = (0)(-1) + (1)(0) = 0, \\ c^1_2 &= a^1_\alpha b^\alpha_2 = (0)(0) + (1)(1) = 1, \\ c^2_1 &= a^2_\alpha b^\alpha_1 = (1)(-1) + (0)(0) = -1, \\ c^2_2 &= a^2_\alpha b^\alpha_2 = (1)(0) + (0)(1) = 0. \end{aligned}$$

Hence

$$AB = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

Similarly

$$BA = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}.$$

But obviously $AB \neq BA$.

The unit matrix I of order n has the interesting property that it commutes with all square matrices of the same order. In fact, if A is an arbitrary square matrix of order n, then

$$AI = IA = A.$$

The multiplication of row and column matrices with the same number of elements is instructive. Let

$$A = \| \ a_i \ \|$$

be the row matrix and

$$B = \| \ b^i \ \|$$

the column matrix. Then $AB = a_i b^i$, a number, or a matrix with *one* element (the double-bar notation has been omitted).

Exercise

If $A = \| \ 1, 1, 0 \ \|$ and $B = \left\| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\|$, then

$$AB = (1)(0) + (1)(0) + (0)(1) = 0.$$

This example also illustrates the fact that *the product of two matrices can be a zero matrix although neither of the multiplied matrices is a zero matrix.*

The multiplication of a square matrix with a column matrix occurs frequently in the applications. A system of n linear algebraic equations in n unknowns $x^1, x^2, \cdots, x^n$

$$a^i_j x^j = b^i$$

can be written as a single matrix equation

$$AX = B$$

in the unknown column matrix $X = \| \ x^i \ \|$ and the given square matrix $A = \| \ a^i_j \ \|$ and column matrix $B = \| \ b^i \ \|$.

A system of first-order differential equations

$$\frac{dx^i}{dt} = a^i_j x^j$$

can be written as one matric differential equation

$$\frac{dX}{dt} = AX.$$

Finally a system of second-order differential equations occurring in the theory of small oscillations

$$\frac{d^2x^i}{dt^2} = a^i_j x^j$$

can be written as one matric second-order differential equation

$$\frac{d^2X}{dt^2} = AX.$$

The above illustrations suffice to show the compactness and simplicity of matric equations when use is made of matrix multiplication.

Exercises

1. Compute the matrix AB when

$$A = \begin{Vmatrix} 1, 0 \\ 0, 1 \\ 1, 0 \end{Vmatrix} \quad \text{and} \quad B = \begin{Vmatrix} 0, 0, 2 \\ -1, 0, 0 \end{Vmatrix}.$$

Is BA defined? Explain.

2. Compute the matrix AX when

$$A = \begin{Vmatrix} 1, 3, 0 \\ -1, 2, 1 \\ 0, 0, 2 \end{Vmatrix} \quad \text{and} \quad X = \begin{Vmatrix} 1 \\ 0 \\ -2 \end{Vmatrix}.$$

Is XA defined? Explain.

CHAPTER 2

ALGEBRAIC PRELIMINARIES (Continued)

Inverse of a Matrix and the Solution of Linear Equations.[1]

The inverse a^{-1}, or reciprocal, of a real number a is well defined if $a \neq 0$. There is an analogous operation for square matrices. *If A is a square matrix*

$$A = \| a_j^i \|$$

of order n and if the determinant $| a_j^i | \neq 0$, or in more extended notation

$$\begin{vmatrix} a_1^1, & a_2^1, & \cdots, & a_n^1 \\ a_1^2, & a_2^2, & \cdots, & a_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_1^n, & a_2^n, & \cdots, & a_n^n \end{vmatrix} \neq 0,$$

then there exists a unique matrix, written A^{-1} in analogy to the inverse of a number, with the important properties

$$(2 \cdot 1) \qquad \begin{cases} AA^{-1} = I \\ A^{-1}A = I \end{cases} \qquad (I \text{ is the unit matrix.})$$

The matrix A^{-1}, if it exists, is called the *inverse matrix of A.*

In fact, the following more extensive result holds good. *A necessary and sufficient condition that a matrix $A = \| a_j^i \|$ have an inverse is that the associated determinant $| a_j^i | \neq 0$.*

From now on we shall refer to the determinant $a = | a_j^i |$ as the *determinant a* of the matrix A. *Occasionally we shall write $| A |$ for the determinant of A.*

The general form of the inverse of a matrix can be given with the aid of a few results from the theory of determinants. Let $a = | a_j^i |$ be a determinant, not necessarily different from zero. Let α_j^i be the cofactor † of a_i^j in the determinant a; note that the indices i and j are interchanged in α_j^i as compared with a_i^j. Then the following results

† The $(n-1)$-rowed determinant obtained from the determinant a by striking out the jth row and ith column in a, and then multiplying the result by $(-1)^{i+j}$.

come from the properties of determinants:

$$a^i_j\alpha^j_k = a\,\delta^i_k \qquad \text{(expansion by elements of } i\text{th row);}$$
$$\alpha^i_j a^j_k = a\,\delta^i_k \qquad \text{(expansion by elements of } k\text{th column).}$$

If then the determinant $a \neq 0$, we obtain the following relations,

$$(2\cdot 2) \qquad \begin{cases} a^i_j\beta^j_k = \delta^i_k, \\ \beta^i_j a^j_k = \delta^i_k \end{cases}$$

on defining

$$\beta^i_j = \frac{\alpha^i_j}{a}.$$

Let $A = \| a^i_j \|$, $B = \| \beta^i_j \|$; then relations $2\cdot 2$ state, in terms of matrix multiplication, that

$$AB = I, \quad BA = I.$$

In other words, the matrix B is precisely the inverse matrix A^{-1} of A.

To summarize, we have the following computational result: *if the determinant a of a square matrix $A = \| a^i_j \|$ is different from zero, then the inverse matrix A^{-1} of A exists and is given by*

$$A^{-1} = \| \beta^i_j \|,$$

where $\beta^i_j = \dfrac{\alpha^i_j}{a}$ *and* α^i_j *is the cofactor of* a^j_i *in the determinant* a *of the matrix* A.

These results on the inverse of a matrix have a simple application to the solution of n non-homogeneous linear (algebraic) equations in n unknowns $x^1, x^2, \cdots, x^n$. Let the n equations be

$$a^i_j x^j = b^i$$

(the n^2 numbers a^i_j are given and the n numbers b^i are given). On defining the matrices

$$A = \| a^i_j \|, X = \| x^i \|, B = \| b^i \|,$$

we can, as in the first chapter, write the n linear equations as one matric equation

$$AX = B$$

in the unknown column matrix X. If we now assume that the determinant a of the matrix A is not zero, the inverse matrix A^{-1} will exist and we shall have by matrix multiplication

$$A^{-1}(AX) = A^{-1}B.$$

Since $A^{-1}A = I$ and $IX = X$, we obtain *the solution*

$$X = A^{-1}B$$

of the equation $AX = B$. In other words, if α^i_j is the cofactor of a^j_i in the determinant a of A, then $x^i = \alpha^i_j b^j/a$ *is the solution of the system* *of n equations* $a^i_j x^j = b^i$ *under the condition* $a \neq 0$. This is equivalent to Cramer's rule[2] for the solution of non-homogeneous linear equations as ratios of determinants. It is *more explicit than Cramer's rule* in that the determinants in the numerator of the solution expressions are *expanded* in terms of the given right-hand sides $b^1, b^2, \cdots, b^n$ of the linear equations. It is sometimes possible to solve the equations $a^i_j x^j = b^i$ readily and obtain $x^i = \lambda^i_j b^j$. The inverse matrix A^{-1} to $A = \| a^i_j \|$ can then be read off by inspection — in fact, $A^{-1} = \| \lambda^i_j \|$.

Practical methods, including approximate methods, for the calculation of the inverse (sometimes called *reciprocal*) of a matrix are given in Chapter IV of the book on matrices by Frazer, Duncan, and Collar. A method based on the Cayley-Hamilton theorem will be presented at the end of the chapter.

A simple example on the inverse of a matrix would be instructive at this point.

Exercise

Consider the two-rowed matrix

$$A = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

According to our notations

$$a^1_1 = 0, a^1_2 = 1, a^2_1 = -1, a^2_2 = 0.$$

Hence the cofactors α^i_j of A will be

$$\alpha^1_1 = (\text{cofactor of } a^1_1) = 0, \quad \alpha^1_2 = (\text{cofactor of } a^2_1) = -1,$$
$$\alpha^2_1 = (\text{cofactor of } a^1_2) = 1, \quad \alpha^2_2 = (\text{cofactor of } a^2_2) = 0.$$

Now $A^{-1} = \| \beta^i_j \|$, where $\beta^i_j = \alpha^i_j / a$. But the determinant of A is $a = 1$. This gives us immediately $\beta^1_1 = 0$, $\beta^1_2 = -1$, $\beta^2_1 = 1$, $\beta^2_2 = 0$. In other words,

$$A^{-1} = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}.$$

Approximate numerical examples abound in the study of airplane-wing oscillations. For example, if

$$A = \begin{Vmatrix} 0.0176, & 0.000128, & 0.00289 \\ 0.000128, & 0.00000824, & 0.0000413 \\ 0.00289, & 0.0000413, & 0.000725 \end{Vmatrix},$$

then approximately

$$A^{-1} = \begin{Vmatrix} 170.9, & 1{,}063., & -741.7 \\ 1063., & 176{,}500., & -14{,}290. \\ -741.7, & -14{,}290., & 5{,}150. \end{Vmatrix}.$$

See exercise 2 at the end of Chapter 7.

From the rule for the product of two determinants,[3] the following result is immediate on observing closely the definition of the product of two matrices:

If A and B are two square matrices with determinants a and b respectively, then the determinant c of the matric product $C = AB$ is given by the numerical multiplication of the two numbers a and b, i.e., $c = ab$.

This result enables us to calculate immediately the determinant of the inverse of a matrix. Since $AA^{-1} = I$, and since the determinant of the unit matrix I is 1, the above result shows that the *determinant of* A^{-1} is $1/a$, *where a is the determinant of A.*

From the associativity of the operation of multiplication of square matrices and the properties of inverses of matrices, the usual index laws for powers of numbers hold good for powers of matrices even though matric multiplication is not commutative. By the *associativity* of the operation of matric multiplication we mean that, if A, B, C are any three square matrices of the same order, then †

$$A(BC) = (AB)C.$$

If then A is a square matrix, there is a unique matrix $AA \cdots A$ with s factors for any given positive integer s. We shall write this matrix as A^s and call it the sth power of the matrix A. Now if we *define* $A^0 = I$, the unit matrix, then the following *index laws hold for all positive integral and zero indices r and s:*

$$A^rA^s = A^sA^r = A^{s+r}$$
$$(A^r)^s = (A^s)^r = A^{rs}.$$

Furthermore, these index laws hold for all integral r and s, positive or negative, whenever A^{-1} exists. This is with the understanding that *negative powers of matrices are defined as positive powers of their inverses,* i.e., A^{-r} is defined for any positive integer r by

$$A^{-r} = (A^{-1})^r.$$

Multiplication of Matrices by Numbers, and Matric Polynomials.

Besides the operations on matrices that have been discussed up to this section, there is still another one that is of great importance. If $A = \| a^i_j \|$ is a matrix, not necessarily a square matrix, and α is a number, real or complex, then by αA *we shall mean the matrix* $\| \alpha a^i_j \|$. This operation of multiplication by numbers enables us to consider matrix polynomials of type

$$(2\cdot3) \qquad \alpha_0A^n + \alpha_1A^{n-1} + \alpha_2A^{n-2} + \cdots + \alpha_{n-1}A + \alpha_nI.$$

† Similarly, if the two square matrices A and B and the column matrix X have the same number of rows, then $(AB)X = A(BX)$.

In expression 2·3, $\alpha_0, \alpha_1, \cdots, \alpha_n$ are numbers, A is a square matrix, and I is the unit matrix of the same order as A. In a given matric polynomial, the α_i's are given numbers, and A is a variable square matrix.

Characteristic Equation of a Matrix and the Cayley-Hamilton Theorem. We are now in a position to discuss some results whose importance cannot be overestimated in the study of vibrations of all sorts (see Chapter 6).

If $A = \| a^i_j \|$ is a given square matrix of order n, one can form the matrix $\lambda I - A$, called the *characteristic matrix* of A. The determinant of this matrix, considered as a function of λ, is a (numerical) polynomial of degree n in λ, called the *characteristic function of A*. More explicitly, let $f(\lambda) = | \lambda I - A |$; then $f(\lambda)$ has the form $f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$. Since $a_n = f(0)$, we see that $a_n = | -A |$; i.e., *a_n is $(-1)^n$ times the determinant of the matrix A*. The algebraic equation of degree n for λ,

$$f(\lambda) = 0$$

is called the *characteristic equation of the matrix A*, and the roots of the equation are called the *characteristic roots of A*.

We shall close this chapter with what is, perhaps, the most famous theorem in the algebra of matrices.

THE CAYLEY-HAMILTON THEOREM. *Let*

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

be the characteristic function of a matrix A, and let I and O be the unit matrix and zero matrix respectively with an order equal to that of A. Then the matric polynomial equation

$$X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_nI = O$$

is satisfied by X = A.

Example

Take $A = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$; then $f(\lambda) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1$. Here $n = 2$, and $a_1 = 0$, $a_2 = -1$. But $A^2 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$. Hence $A^2 - I = O$.

The Cayley-Hamilton theorem is often laconically stated in the form "A matrix satisfies its own characteristic equation." In symbols, if $f(\lambda)$ is the characteristic function for a matrix A, then $f(A) = O$. Such statements are, of course, nonsensical if taken literally at their face value. However, such mnemonics are useful to those who thoroughly understand the statement of the Cayley-Hamilton theorem.

A knowledge of the characteristic function of a matrix enables one to compute the inverse of a matrix, if it exists, with the aid of the Cayley-Hamilton theorem. In fact, let A be an n-rowed square matrix with an inverse A^{-1}. This implies that the determinant a of A is not zero. Since $0 \neq a_n = (-1)^n a$, we find with the aid of the Cayley-Hamilton theorem that A satisfies the matric equation

$$I = -\frac{1}{a_n}[A^n + a_1 A^{n-1} + \cdots + a_{n-2}A^2 + a_{n-1}A].$$

Multiplying both sides by A^{-1}, we see that *the inverse matrix A^{-1} can be computed by the following formula:*

$$(2\cdot4) \qquad A^{-1} = \frac{-1}{a_n}[A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-2}A + a_{n-1}I].$$

To compute A^{-1} by formula 2·4 one has to know the coefficients a_1. $a_2, \cdots, a_{n-1}, a_n$ in the characteristic function of the given matrix A. Let $A = \| a^i_j \|$; then the *trace* of the matrix A, *written* tr (A), is defined by tr $(A) = a^i_i$, the sum of the n diagonal elements $a^1_1, a^2_2, \cdots, a^n_n$. Define the numbers[4] $s_1, s_2, \cdots, s_n$ by

$$(2\cdot5) \quad s_1 = \text{tr}\,(A), \; s_2 = \text{tr}\,(A^2), \; \cdots, \; s_r = \text{tr}\,(A^r), \; \cdots, \; s_n = \text{tr}\,(A^n)$$

so that s_r is the trace of the rth power of the given matrix A. It can be shown[5] by a long algebraic argument that the numbers $a_1, \cdots, a_n$ can be computed successively by the following recurrence formulas:

$$(2\cdot6) \quad \begin{cases} a_1 = -s_1 \\ a_2 = -\frac{1}{2}(a_1 s_1 + s_2) \\ a_3 = -\frac{1}{3}(a_2 s_1 + a_1 s_2 + s_3) \\ \cdot \\ \cdot \\ \cdot \\ a_n = -\frac{1}{n}(a_{n-1}s_1 + a_{n-2}s_2 + \cdots + a_1 s_{n-1} + s_n). \end{cases}$$

We can summarize our results in the following rule for the calculation of the inverse matrix A^{-1} to a given matrix A.

A Rule for Calculation of the Inverse Matrix A^{-1}. First compute the first $n-1$ powers $A, A^2, \cdots, A^{n-1}$ of the given n-rowed matrix A. Then compute the diagonal elements only of A^n. Next compute the n numbers $s_1, s_2, \cdots, s_n$ as defined in 2·5. Insert these values for the s_i in formula 2·6, and calculate $a_1, a_2, \cdots, a_n$ successively by means of 2·6. Finally by formula 2·4 one can calculate A^{-1} from the knowledge of $a_1, \cdots, a_n$, and the matrices $A, A^2, \cdots, A^{n-1}$. Notice that the whole A^n is not needed in the calculation but merely $s_n = \text{tr}\,(A^n)$, the trace of A^n.

Punched-card methods can be used to calculate the powers of the matrix A. The rest of the calculations are easily made by standard calculating machines. Hence one method of getting *numerical solutions of a system of n linear equations in the n unknowns* x^i

$$a^i_j x^j = b^i \qquad (|\ a^i_j\ | \neq 0)$$

is to compute A^{-1} of $A = \|\ a^i_j\ \|$ by the above rule with the aid of punched-card methods and then to compute $A^{-1}B$, where $B = \|\ b^i\ \|$, by punched-card methods. The solution column matrix $X = \|\ x^i\ \|$ is given by $X = A^{-1}B$.

Exercises

1. Calculate the inverse matrix to $A = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|$ by the last method of this chapter.

Solution.

$$A^2 = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, \ s_1 = 0, \ s_2 = 2, \ a_1 = 0, \ a_2 = -1.$$

Now $A^{-1} = -\frac{1}{a_2}[A + a_1 I] = A$. Hence

$$A^{-1} = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|.$$

2. See the exercise given in M. D. Bingham's paper. See the bibliography.

3. Calculate A^{-1} by the above rule when

$$A = \left\| \begin{matrix} 15 & 11 & 6 & -9 & -15 \\ 1 & 3 & 9 & -3 & -8 \\ 7 & 6 & 6 & -3 & -11 \\ 7 & 7 & 5 & -3 & -11 \\ 17 & 12 & 5 & -10 & -16 \end{matrix} \right\|.$$

After calculating A^2, A^3, A^4, and the diagonal elements of A^5, calculate $s_1 = 5$, $s_2 = -41$, $s_3 = -217$, $s_4 = -17$, $s_5 = 3185$. Inserting these values in 2·6, find

$$a_1 = -5, \ a_2 = 33, \ a_3 = -51, \ a_4 = 135, \ a_5 = 225.$$

Incidentally the characteristic equation of A is

$$\begin{aligned} f(\lambda) &= \lambda^5 - 5\lambda^4 + 33\lambda^3 - 51\lambda^2 + 135\lambda + 225 \\ &= (\lambda + 1)(\lambda^2 - 3\lambda + 15)^2 = 0. \end{aligned}$$

Finally, using formula 2·4, find

$$A^{-1} = -\frac{1}{225}\left\| \begin{matrix} -207 & 64 & -124 & 111 & 171 \\ -315 & 30 & 195 & -180 & 270 \\ -315 & 30 & -30 & 45 & 270 \\ -225 & 75 & -75 & 0 & 225 \\ -414 & 53 & 52 & -3 & 342 \end{matrix} \right\|.$$

CHAPTER 3

DIFFERENTIAL AND INTEGRAL CALCULUS OF MATRICES

Power Series in Matrices.

Before we discuss the subject of power series, it is convenient to make a few introductory remarks on general series in matrices. Let $A_0, A_1, A_2, A_3 \cdots$ be an infinite sequence of matrices of the same type (i.e., same number of rows and columns) and let $S_p = A_0 + A_1 + A_2 + \cdots + A_p$ be the matric sum of the matrices $A_0, A_1, A_2, \cdots$, and A_p. If every element in the matrix S_p converges (in the ordinary numerical sense) as p tends to infinity, then by $S = \lim_{p \to \infty} S_p$ we shall mean the matrix S of the limiting elements. If then the matrix $S = \lim_{p\to\infty} S_p$ *exists in the above sense,* we shall say, *by definition, that the matric infinite series* $\sum_{r=0}^{\infty} A_r$ *converges to the matrix* S.

Example

Take $A_0 = I$, $A_1 = I$, $A_2 = \frac{1}{2!}I$, $A_3 = \frac{1}{3!}I$, $\cdots$, $A_i = \frac{1}{i!}I$, $\cdots$. Then

$$S_p = A_0 + A_1 + A_2 + \cdots + A_p = \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{p!}\right)I.$$

Hence, on recalling the expansion for the exponential e, we find that $\lim_{p\to\infty} S_p = eI$. In other words, $\sum_{r=0}^{\infty} A_r = eI$.

If A is a square matrix and the $a_1, a_2, \cdots$ are numbers, one can consider matric *power series* in A

$$\sum_{r=0}^{\infty} a_r A^r.$$

In other words, matric *power* series are particular matric series in which each matrix A_r is of special type † $A_r = a_r A^r$, where A^r is the rth power of a square matrix A. ($A^0 = I$ is the identity matrix.) Clearly matric polynomials (see Chapter 2) are special matric power series in which *all* the numbers a_i after a certain value of i are zero.

An important example of a matric power series is the *matric exponential function* e^A defined by the following matric power series:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots + \cdots.$$

† The index r is not summed.

The following properties of the matrix exponential have been used frequently in investigations on the matrix calculus:

1. The matric power series expansion for e^A is convergent[1] for all square matrices A.

2. $e^A e^B = e^B e^A = e^{A+B}$ whenever A and B are commutative matrices, i.e., whenever $AB = BA$.

3. $e^A e^{-A} = e^{-A} e^A = I$. (These relations express the fact that e^{-A} is the inverse matrix of e^A.)

Every numerical power series has its matric analogue. However, the corresponding matric power series have more complicated properties — for example, e^A. Other examples are, say, the matric sine, sin A, and the matric cosine, cos A, defined by

$$\sin A = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \cdots$$

$$\cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots.$$

The usual trigonometric identities are not always satisfied by sin A and cos A for arbitrary matrices.

Differentiation and Integration of Matrices Depending on a Numerical Variable.

Let $A(t)$ be a matrix depending on a numerical variable t so that the elements of $A(t)$ are numerical functions of t.

$$A(t) = \left\| \begin{matrix} a_1^1(t), & a_2^1(t), & \cdots, & a_n^1(t) \\ a_1^2(t), & a_2^2(t), & \cdots, & a_n^2(t) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_1^m(t), & a_2^m(t), & \cdots, & a_n^m(t) \end{matrix} \right\| .$$

Then we *define the derivative of* $A(t)$, and write it $\frac{dA(t)}{dt}$, *by*

$$\frac{dA(t)}{dt} = \left\| \begin{matrix} \frac{da_1^1(t)}{dt}, & \frac{da_2^1(t)}{dt}, & \cdots, & \frac{da_n^1(t)}{dt} \\ \frac{da_1^2(t)}{dt}, & \frac{da_2^2(t)}{dt}, & \cdots, & \frac{da_n^2(t)}{dt} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{da_1^m(t)}{dt}, & \frac{da_2^m(t)}{dt}, & \cdots, & \frac{da_n^m(t)}{dt} \end{matrix} \right\| .$$

Similarly we *define the integral of* $A(t)$ *by*

$$\int A(t)\,dt = \begin{Vmatrix} \int a_1^1(t)\,dt, & \int a_2^1(t)\,dt, & \cdots, & \int a_n^1(t)\,dt \\ \int a_1^2(t)\,dt, & \int a_2^2(t)\,dt, & \cdots, & \int a_n^2(t)\,dt \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \int a_1^m(t)\,dt, & \int a_2^m(t)\,dt, & \cdots, & \int a_n^m(t)\,dt \end{Vmatrix}$$

It is no mathematical feat to show that differentiation of matrices has the following properties:

(3·1) $$\frac{d[A(t) + B(t)]}{dt} = \frac{dA(t)}{dt} + \frac{dB(t)}{dt}$$

(3·2) $$\frac{d[A(t)B(t)]}{dt} = \frac{dA(t)}{dt}B(t) + A(t)\frac{dB(t)}{dt}$$

(3·3) $$\frac{d}{dt}[A(t)B(t)C(t)] = \frac{dA(t)}{dt}B(t)C(t) + A(t)\frac{dB(t)}{dt}C(t) + A(t)B(t)\frac{dC(t)}{dt},$$

etc.

There are important immediate consequences of properties 3·2 and 3·3. For example, from 3·2 and $A^{-1}(t)A(t) = I$, we see that

(3·4) $$\frac{dA^{-1}(t)}{dt} = -A^{-1}(t)\frac{dA(t)}{dt}A^{-1}(t).$$

Also, from 3·3, we obtain

(3·5) $$\frac{dA^3(t)}{dt} = \frac{dA(t)}{dt}A^2(t) + A(t)\frac{dA(t)}{dt}A(t) + A^2(t)\frac{dA(t)}{dt}.$$

There are similar formulas for the derivative of any positive integral power of $A(t)$.

If t is a real variable and A a constant square matrix, then one obtains

$$\frac{d(t^r A)}{dt} = rt^{r-1}A.$$

Then, with the usual term-by-term differentiation of the numerical exponential, the following differentiation can be justified:

(3·6) $$\begin{cases} \dfrac{d(e^{tA})}{dt} = A + tA^2 + \dfrac{t^2}{2!}A^3 + \dfrac{t^3}{3!}A^4 + \cdots + \cdots \\ \qquad\quad = Ae^{tA} = e^{tA}A. \end{cases}$$

There is an important theorem in the matrix calculus that turns up in the mathematical theory of *aircraft flutter* (see Chapter 7). The proof, into which we can not enter here, makes use of the modern theory of *functionals*.

THEOREM. *If $F(\lambda)$ is a power series that converges for all λ, then the matric power series $F(A)$ can be computed by the expansion*[2]

$$F(A) = \sum_{i=1}^{n} F(\lambda_i) G_i, \tag{3·7}$$

where A is an n-rowed square matrix with n distinct characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$, and $G_1, G_2, \cdots, G_n$ are n matrices defined by[3]

$$G_i = \frac{1}{\prod_{j \neq i} (\lambda_j - \lambda_i)} \prod_{j \neq i} (\lambda_j I - A). \tag{3·8}$$

There are a few matters that must be kept in mind in order to have a clear understanding of the meaning of this result. In the first place the matric power series $F(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \cdots$ whenever $F(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots + \cdots$. In other words $\lambda^0 = 1$ is "replaced" by $A^0 = I$, the unit matrix, in the transition from $F(\lambda)$ to $F(A)$. Secondly to avoid ambiguities we must write explicitly the compact products occurring in equation 3·8.

$$\prod_{j \neq i} (\lambda_j - \lambda_i) = (\lambda_1 - \lambda_i)(\lambda_2 - \lambda_i) \cdots (\lambda_{i-1} - \lambda_i)(\lambda_{i+1} - \lambda_i) \cdots (\lambda_n - \lambda_i),$$

$$\prod_{j \neq i} (\lambda_j I - A) = (\lambda_1 I - A)(\lambda_2 I - A) \cdots (\lambda_{i-1} I - A)(\lambda_{i+1} I - A) \cdots (\lambda_n I - A).$$

There are special cases of particular interest in vibration theory (see Chapters 6 and 7). They correspond to the power of a matrix A^r and the matrix exponential e^A. The expansion 3·7 yields immediately

$$A^r = \sum_{i=1}^{n} \lambda_i^r G_i \tag{3·9}$$

and

$$e^A = \sum_{i=1}^{n} e^{\lambda_i} G_i, \tag{3·10}$$

where the matrices G_i have the same meaning as in 3·8.

Exercise

Calculate the matrix e^A when A is the matrix $A = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$. Check the result by calculating e^A directly.

Solution. The characteristic roots are $\lambda_1 = 1$, $\lambda_2 = -1$. Hence the matrices G_1 and G_2 are as follows:

$$G_1 = \frac{\lambda_2 I - A}{\lambda_2 - \lambda_1} = \frac{1}{2}(I + A) = \frac{1}{2}\left\|\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}\right\|,$$

$$G_2 = \frac{\lambda_1 I - A}{\lambda_1 - \lambda_2} = \frac{1}{2}(I - A) = \frac{1}{2}\left\|\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right\|.$$

Now

$$e^A = \sum_{i=1}^{2} e^{\lambda_i} G_i = \frac{e}{2}\left\|\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}\right\| + \frac{e^{-1}}{2}\left\|\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right\|.$$

Hence

$$e^A = \left\|\begin{matrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{matrix}\right\|.$$

CHAPTER 4

DIFFERENTIAL AND INTEGRAL CALCULUS OF MATRICES (Continued)

Systems of Linear Differential Equations with Constant Coefficients.

The matric exponential has important applications to the solution of systems of n linear differential equations in n unknown functions $x^1(t)$, $x^2(t)$, $\cdots$, $x^n(t)$ and with n^2 constant coefficients a^i_j. The variable t is usually the time in physical and engineering problems. Without defining the derivative $\frac{dX}{dt}$, we merely mentioned in the first chapter that we can write such a system of equations as *one* matric equation

$$\frac{dX(t)}{dt} = AX(t). \tag{4·1}$$

Having defined the matric derivative, we are enabled to view this equation with complete understanding.

From formula 3·6 of the previous chapter we find that

$$\frac{d}{dt}e^{(t-t_0)A} = Ae^{(t-t_0)A}, \tag{4·2}$$

where t_0 is an arbitrarily given value of t. But this result is equivalent to saying that $X(t) = [e^{(t-t_0)A}]X_0$ is a solution of the matric differential equation 4·1 for an arbitrary column matrix X_0. A glance at the expansion for the matric exponential $e^{(t-t_0)A}$ shows that the solution $X(t)$ has the property

$$X(t_0) = X_0.$$

In summary, we have the result[1] that

$$X(t) = [e^{(t-t_0)A}]X_0 \tag{4·3}$$

is a solution[2] *of* 4·1 *with the property that* $X(t_0) = X_0$ *for any preassigned constant column matrix* X_0.

Example

$$\frac{dx^1(t)}{dt} = x^2, \quad \frac{dx^2(t)}{dt} = x^1$$

so that

$$\frac{dX}{dt} = AX,$$

where

$$A = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\| \quad \text{and} \quad X = \left\| \begin{matrix} x^1 \\ x^2 \end{matrix} \right\|.$$

Now $\lambda_1 = 1$, $\lambda_2 = -1$, and we saw in the last exercise of the previous chapter that

$$G_1 = \frac{1}{2}\left\| \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \right\|, \quad G_2 = \frac{1}{2}\left\| \begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix} \right\|.$$

Hence

$$e^{(t-t_0)A} = \sum_{i=1}^{2} e^{(t-t_0)\lambda_i} G_i = \left\| \begin{matrix} \cosh (t - t_0) & \sinh (t - t_0) \\ \sinh (t - t_0) & \cosh (t - t_0) \end{matrix} \right\|.$$

Therefore the unique solution of the differential system

$$\frac{dX}{dt} = AX, \quad X(t_0) = X_0 = \left\| \begin{matrix} x_0^1 \\ x_0^2 \end{matrix} \right\|$$

is

$$X(t) = \left\| \begin{matrix} \cosh (t - t_0) & \sinh (t - t_0) \\ \sinh (t - t_0) & \cosh (t - t_0) \end{matrix} \right\|.$$

This means that the unique solution of the differential system

$$\frac{dx^1}{dt} = x^2, \quad \frac{dx^2}{dt} = x^1, \quad x^1(t_0) = x_0^1, \quad x^2(t_0) = x_0^2$$

is

$$\begin{cases} x^1(t) = [\cosh (t - t_0)]x_0^1 + [\sinh (t - t_0)]x_0^2 \\ x^2(t) = [\sinh (t - t_0)]x_0^1 + [\cosh (t - t_0)]x_0^2. \end{cases}$$

Systems of Linear Differential Equations with Variable Coefficients.

Although the matric exponential is not applicable to the solution of a system of linear differential equations with variable coefficients $a_j^i(t)$, there are some analogous matric expansions that enter into the solution of such a system. The system of differential equations

(4·4) $$\frac{dx^i(t)}{dt} = a_j^i(t)x^j(t)$$

is written as one matric differential equation

(4·5) $$\frac{dX(t)}{dt} = A(t)X(t)$$

where $A(t) = \| a_j^i(t) \|$ and $X(t)$ is the column matrix of the n unknown functions $x^i(t)$.

On integrating both sides of 4·5 between t_0 and t we obtain the equivalent matric equation

(4·6) $$X(t) = X(t_0) + \int_{t_0}^{t} A(s)X(s)\, ds.$$

By the method of successive substitutions, we are led to *consider* the following expansion as a solution of 4·6:

$$(4\cdot 7)\quad X(t) = \left[I + \int_{t_0}^{t} A(s)\,ds + \int_{t_0}^{t} A(s)\,ds \int_{t_0}^{s} A(s_1)\,ds_1 + \cdots + \cdots\right] X(t_0).$$

Now the method of successive substitutions for equation 4·6 can be described as follows. In the integral term in 4·6 substitute for $X(s)$ its equivalent as given by formula 4·6 itself. This yields

$$X(t) = X(t_0) + \left[\int_{t_0}^{t} A(s)\,ds\right] X(t_0) + \int_{t_0}^{t} A(s)\,ds \int_{t_0}^{s} A(s_1) X(s_1)\,ds_1.$$

Again substituting for $X(s_1)$ its equal as given by 4·6 we are led to a new expansion for $X(t)$. Continuing indefinitely this way we are led to the matric infinite series 4·7.

If we define the matrix

$$(4\cdot 8)\quad \Omega_{t_0}^{t}(A) = I + \int_{t_0}^{t} A(s)\,ds + \int_{t_0}^{t} A(s)\,ds \int_{t_0}^{s} A(s_1)\,ds_1 + \int_{t_0}^{t} A(s)\,ds \int_{t_0}^{s} A(s_1)\,ds_1 \int_{t_0}^{s_1} A(s_2)\,ds_2 + \cdots + \cdots,$$

then it can be proved that, for $a_j^i(t)$ continuous in $t_0 \leqq t \leqq t_1$,

$$(4\cdot 9)\quad X(t) = \Omega_{t_0}^{t}(A) X_0$$

is the unique solution of the matric differential equation 4·5 *that takes on the arbitrarily given constant matric value* X_0 for $t = t_0$. It is often simpler to carry out the matrix multiplications first in 4·8 and 4·9 before carrying out the successive integrations. If the matrix is independent of t, then, by an evident calculation, solution 4·9 reduces precisely to the matrix exponential type 4·3.

For approximate numerical calculations, a few terms in the expansion for $\Omega_{t_0}^{t}(A)$ may suffice in 4·9 to give a good approximation to the solution of the matric differential equation 4·5.

Exercises

1. Integrate by matrix methods the second-order differential equation

$$\frac{d^2x(t)}{dt^2} - x(t) = 0$$

subject to the initial conditions $x(t_0) = x_0$, $\left(\frac{dx}{dt}\right)_{t=t_0} = y_0$.

(*Hint.* Write the differential equation as a system of two first-order equations

$$\frac{dx^1}{dt} = x^2, \quad \frac{dx^2}{dt} = x^1$$

with initial conditions $x^1(t_0) = x_0$, $x^2(t_0) = y_0$, and use the results of the example illustrating formula 4·3.)

2. Integrate by matrix methods the second-order differential system for harmonic oscillations with frequency ω

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = 0, \quad x(t_0) = x_0, \quad \left(\frac{dx}{dt}\right)_{t=t_0} = y_0.$$

(*Hint.* Write the equation as a system of two first-order linear equations.)

3. Discuss the solutions of the differential equation

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

for free damped oscillations by matrix methods with the restriction that $\beta \neq 2\sqrt{km}$. m = mass, β = damping factor, and k = elastic constant, so that all three m, β, k are positive constants. Clearly the restriction rules out the critical damping case.

(*Hint.* Write the differential equation as a first-order matric differential equation

$$\frac{dX}{dt} = AX,$$

where

$$A = \left\| \begin{matrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{\beta}{m} \end{matrix} \right\|,$$

and notice that the characteristic equation of this *matrix* is the "characteristic equation" of the given second-order differential equation in the usual elementary sense.)

4. Integrate by matrix methods the second-order differential equation

$$\frac{d^2x(t)}{dt^2} - tx(t) = 0$$

subject to the initial conditions $x(0) = x_0$, $\left(\frac{dx}{dt}\right)_{t=t_0} = y_0$.

CHAPTER 5

MATRIX METHODS IN PROBLEMS OF SMALL OSCILLATIONS

Differential Equations of Motion.[1]

The problem of small oscillations[2] (of conservative dynamical systems) about an equilibrium position concerns itself with the solution of the Lagrangian differential equations of motion in which the kinetic and potential energies are homogeneous quadratic forms, in the velocities and coordinates respectively, with constant coefficients. The theory is approximate in that the constancy of the coefficients in the kinetic energy and the quadratic type of the potential energy are due to approximations in the actual form of the kinetic and potential energies respectively. If, without loss of generality, we take all the coordinates of the equilibrium position to be zero, these approximations are due to the assumed smallness of the coordinates and velocities about the equilibrium position.

Let

$$T = \tfrac{1}{2}a_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt} \qquad (a_{ij} = a_{ji})$$

and

$$V = \tfrac{1}{2}b_{ij}q^iq^j \qquad (b_{ij} = b_{ji})$$

be the kinetic and potential energies respectively of our oscillating system with n degrees of freedom. In view of what we have already said, the a_{ij} and b_{ij} are constants. We shall consider the case in which the equilibrium point is *stable*, i.e., the potential energy V has a *minimum* at $q^i = 0$. Now it can be proved that the *positive definiteness* of V is a necessary and sufficient condition that $(0, 0, \cdots, 0)$ be a *stable* equilibrium point. V is, by definition, positive definite if $V \geq 0$ for all q^i and $V = 0$ if and only if $q^i = 0$. Clearly the kinetic energy T is positive definite in the velocities $\dfrac{dq^i}{dt}$.

Lagrange's equations of motion for our oscillating system are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) = -\frac{\partial V}{\partial q^i} \quad (i = 1, 2, \cdots, n)$$

on using the notation $\dot{q}^i = \dfrac{dq^i}{dt}$. If we use the explicit form for the kinetic and potential energies, Lagrange's equations reduce to the system of n second-order differential equations

$$(5\cdot 1) \qquad a_{ij}\ddot{q}^i = -b_{ij}q^i,$$

where we have used the notation $\ddot{q}^i = \dfrac{d^2q^i}{dt^2}$. If we define two square matrices †

$$A = \| \, a_{ij} \, \|, \qquad B = \| \, b_{ij} \, \|,$$

and the unknown column matrix

$$Q(t) = \| \, q^i(t) \, \|,$$

then we can write our differential equations $5\cdot 1$ of motion as the one matric differential equation

$$(5\cdot 2) \qquad A\,\frac{d^2Q(t)}{dt^2} = -BQ(t).$$

Since the kinetic energy is positive definite, it can be proved that the determinant $| \, A \, | \neq 0$. Hence it follows from our discussion in Chapter 2 that the inverse matrix A^{-1} exists. On multiplying both sides of equation $5\cdot 2$ on the left by A^{-1} and remembering that $A^{-1}A = I$, the unit matrix, we obtain the following equivalent matric differential equation

$$(5\cdot 3) \qquad \frac{d^2Q(t)}{dt^2} = -CQ(t),$$

where C is the (constant) square matrix $C = A^{-1}B$.

To summarize, we have the following result. *If A and B are the constant square matrices of the coefficients of the kinetic and potential energies respectively, then the motion of our oscillatory system is governed by the matric differential equation* $5\cdot 3$.

Illustrative Example

Two equal masses, each of mass m, are connected by a spring with elastic constant k while each mass is connected to a fixed wall by a spring with elastic constant k. The kinetic and potential energies of this two-degree-of-freedom problem are

$$T = \frac{m}{2}\left[\left(\frac{dq^1}{dt}\right)^2 + \left(\frac{dq^2}{dt}\right)^2\right]$$

$$V = \frac{k}{2}[(q^1)^2 + (q^2)^2 + (q^1 - q^2)^2],$$

† Recall the notations for matrices given in Chapter 1.

where q^1 and q^2 are the respective displacements of the centers of the two masses "parallel" to the springs and are measured from the equilibrium position in which all three springs are unstressed. Hence by a direct calculation from the kinetic and potential energies we find that the matric equations of motion are

$$\frac{d^2Q(t)}{dt^2} = -CQ(t), \tag{5·4}$$

where

$$C = \left\| \begin{matrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{matrix} \right\|.$$

If we define the column matrix $R(t) = \left\| \begin{matrix} r^1(t) \\ r^2(t) \end{matrix} \right\|$ by $\frac{dQ(t)}{dt} = R(t)$, we can write the second-order matric differential equation 5·4 as a first-order matric differential equation

$$\frac{dS(t)}{dt} = US(t), \tag{5·5}$$

where

$$S(t) = \left\| \begin{matrix} q^1(t) \\ q^2(t) \\ r^1(t) \\ r^2(t) \end{matrix} \right\|$$

and

$$U = \left\| \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & \frac{-2k}{m} & 0 & 0 \end{matrix} \right\|. \tag{5·6}$$

The characteristic equation of the matrix U turns out to be

$$\lambda^4 + \frac{4k}{m}\lambda^2 + \frac{3k^2}{m^2} = 0.$$

Now $\frac{k}{m} > 0$, so that there are four distinct pure imaginary characteristic roots of U given by

$$\lambda_1 = \sqrt{\frac{-3k}{m}},\ \lambda_2 = -\sqrt{\frac{-3k}{m}},\ \lambda_3 = \sqrt{\frac{-k}{m}},\ \lambda_4 = -\sqrt{\frac{-k}{m}}. \tag{5·7}$$

Exercise

A shaft of length $2l$, fixed at one end, carries one disk at the free end and another in the middle. If μ is the moment of inertia of each disk, and q^1, q^2 are the respective angular deflections of the two disks, then the kinetic and potential energies are

$$T = \frac{\mu}{2}\left[\left(\frac{dq^1}{dt}\right)^2 + \left(\frac{dq^2}{dt}\right)^2\right]$$

$$V = \frac{\tau}{2l}\left[(q^1)^2 + (q^2 - q^1)^2\right]$$

under the assumption that the shaft has a uniform torsional stiffness τ. Find the matric differential equation of motion. Write this equation as a system of two first-order matric equations. Discuss the solutions of this system and then the motion of the disks.

CHAPTER 6

MATRIX METHODS IN PROBLEMS OF SMALL OSCILLATIONS (Continued)

Calculation of Frequencies and Amplitudes.

Let us inquire into the pure harmonic solutions of our differential equation of motion

(6·1) $$\frac{d^2Q(t)}{dt^2} = -CQ(t),$$

where $C = A^{-1}B$. We thus seek solutions of 6·1 of type

(6·2) $$Q(t) = \sin(\omega t + \psi)\Gamma,$$

where ω is an angular frequency, ψ an arbitrary phase angle, and Γ a column matrix of amplitudes. On substituting 6·2 in 6·1 we obtain

$$-\omega^2 \sin(\omega t + \psi)\Gamma = -\sin(\omega t + \psi)C\Gamma.$$

Hence a necessary and sufficient condition that 6·2 be a solution of 6·1 is that the frequency ω and the corresponding column matrix Γ of amplitudes satisfy the matrix equation

(6·3) $$(\omega^2 I - C)\Gamma = 0.$$

In order that there exist a solution matrix $\Gamma \neq 0$ of 6·3, it is clear from the theory of systems of linear homogeneous algebraic equations that ω^2 *must be a characteristic root of the matrix* C. Since $C = A^{-1}B$, we verify immediately the statement that

(6·4) $$\omega^2 I - C = A^{-1}(\omega^2 A - B).$$

On recalling that the determinant of the product of two matrices is equal to the product of their determinants, we see that the determinant

$$| A^{-1}(\omega^2 A - B) | = | A^{-1} | \; | \omega^2 A - B |$$

and hence, by 6·4,

$$| \omega^2 I - C | = | A^{-1} | \; | \omega^2 A - B |.$$

But $| A^{-1} | \neq 0$, so that the characteristic roots of the matrix C are identical with the roots of the "frequency" equation

(6·5) $$| \lambda A - B | = 0.$$

Since the kinetic and potential energies are positive definite quadratic forms, it can be proved (see any book on dynamics such as Whittaker's) *that all the roots of the frequency equation are positive.* Hence *all the characteristic roots of the matrix* $C = A^{-1}B$ *are positive.*

Since the potential energy is a positive definite quadratic form, it follows that $|B| > 0$ and hence that B^{-1} exists. Therefore C^{-1} exists and is given by $C^{-1} = B^{-1}A$. On multiplying both sides of equation 6·1 by C^{-1}, we obtain the matric differential equation

$$Q(t) = -D\frac{d^2Q(t)}{dt^2}, \tag{6·6}$$

where $D = C^{-1} = B^{-1}A$. This equation is obviously equivalent to equation 6·1. If we now proceed with 6·6 as we did with 6·1, we are led to the equation

$$\left(\frac{1}{\omega^2}I - D\right)\Gamma = 0. \tag{6·7}$$

This equation can also be derived by operating directly with equation 6·3. It is clear from equations 6·3 and 6·7 that *the characteristic roots of the matrix $D = C^{-1}$ are the corresponding reciprocals of the characteristic roots of the matrix C.* We shall call D the *dynamical matrix.*

Let us write 6·7 in the equivalent form

$$\Gamma = \omega^2 D\Gamma. \tag{6·8}$$

The classical method of finding the frequencies and amplitudes of our oscillating system consists in first finding the frequencies by solving the frequency equation 6·5, or equivalently in finding the characteristic roots of the matrix $C = A^{-1}B$, and then in determining the amplitudes by solving the system of linear homogeneous equations that corresponds to the matrix equation 6·3. Such a direct way of calculating the frequencies and amplitudes often involves laborious calculations. For approximate numerical calculations, the *method of successive approximations* when applied to equation 6·8 greatly reduces the laborious calculations. This is especially true when only the fundamental frequency (lowest frequency) and the corresponding amplitudes are desired.[1] *We shall assume now that all the frequencies of our oscillating system are distinct.* The method of successive approximations for equation 6·8 is as follows. Let Γ_0 be an arbitrarily given column matrix. Define

$$\Gamma_1 = \omega^2 D\Gamma_0$$
$$\Gamma_2 = \omega^2 D\Gamma_1$$
$$\cdot$$
$$\cdot$$
$$\cdot$$

and in general

$$\Gamma_r = \omega^2 D\Gamma_{r-1}.$$

By successive use of this recurrence relation we can express Γ_r in terms of Γ_0. In fact, we have

(6·9) $$\Gamma_r = (\omega^2)^r D^r \Gamma_0,$$

where D^r is the rth power of the dynamical matrix D. Now it can be shown that for large r the *ratio of the elements of the column matrix* $D^{r-1}\Gamma_0$ *to the corresponding elements of the column matrix* $D^r\Gamma_0$ *is approximately a constant equal to* ω_1^2, *the square of the fundamental frequency* ω_1 *of our fundamental mode of oscillation with distinct frequencies. The matrix* Γ_0 *is only restricted by the non-vanishing of* R (*see equality* 6·11 *below*).

The proof of this result is a little involved and makes use of the Cayley-Hamilton theorem, a theorem † of Sylvester, and a few other theorems on matrices.[2] These theorems are instrumental in showing that, for r large enough, the following approximate equality holds:

(6·10) $$D^r\Gamma_0 = \frac{\lambda_1^r R}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)\cdots(\lambda_n - \lambda_1)} \begin{vmatrix} \alpha^1 \\ \alpha^2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha^n \end{vmatrix},$$

where (*a*) $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ are the characteristic roots of D, and hence $\lambda_1 = \dfrac{1}{\omega_1^2}$ in terms of the fundamental frequency ω_1; (*b*) the numbers $\alpha^1, \alpha^2, \cdots, \alpha^n$ are proportional to the amplitudes of the fundamental mode of oscillation; and (*c*)

(6·11) $$R = A_i\gamma_0^i.$$

In 6·11, the γ_0^i are the elements of the *arbitrarily chosen* column matrix Γ_0 and the $A_1, \cdots, A_n$ are n constants that are themselves obtainable by a successive approximation method.

Clearly, $D^r\Gamma_0$ is a column matrix. Hence the approximate formula 6·10 shows that *for* r *large enough, and for arbitrarily chosen* Γ_0, such that $R \neq 0$, *the column matrix* $D^r\Gamma_0$ *has elements proportional to the amplitudes of the fundamental mode of oscillation. All this is subject to the restriction that all the frequencies are distinct.*

On using equation 6·3 instead of 6·7, one can similarly obtain the

† $F(A) = \sum_{r=1}^{n} F(\lambda_r)G_r$, where $G_r = \dfrac{\prod_{s\neq r}(\lambda_s I - A)}{\prod_{s\neq r}(\lambda_s - \lambda_r)}$ and $\lambda_1, \cdots, \lambda_n$ are characteristic roots of A. See the discussion in Chapter 3.

greatest frequency and corresponding amplitudes of our oscillating system. The intermediate overtones and corresponding amplitudes can be obtained by the above successive approximation methods on reducing the number of degrees of freedom successively by one. Any one interested in these topics will find the following paper by W. J. Duncan and A. R. Collar very useful: "A Method for the Solution of Oscillation Problems by Matrices," *Philosophical Magazine and Journal of Science*, vol. 17 (1934), pp. 865–909. By *approximating oscillating continuous systems*, such as *beams*, by oscillating systems with a large but *finite* number of degrees of freedom, the Duncan-Collar paper shows how the methods of this chapter are applicable in solving oscillation problems for continuous systems.

CHAPTER 7

MATRIX METHODS IN THE MATHEMATICAL THEORY OF AIRCRAFT FLUTTER

In recent years a group of phenomena known under the caption "flutter" has engaged the attention of aeronautical engineers. The vibrations taking place in flutter phenomena can often lead to loss of control or even to structural failure in such aircraft parts as wing, aileron, and tail. Such dangerous situations may arise when the airplane is flown at a high speed. It is of the greatest practical importance therefore so to design the plane as to have the maximum operating speed less than the critical speed at which flutter occurs. Unfortunately experiments in wind tunnels are idealized and difficult, and actual flight testing is obviously highly dangerous. It is here that mathematics enters the stage at a most opportune moment. Although the results of mathematical theories of flutter are now being applied in the design of aircraft, the need for an adequate mathematical theory is becoming critical. There is no time in this brief set of mathematical lectures to deal adequately with the present simplified mathematical theories of the mechanism of flutter. We shall only give the matric form for the equations of motion and say a few words about the approximate solutions with the aid of matrix iteration methods.

The vibrations of an airplane wing and aileron can be considered as those of a mechanical system with *three* degrees of freedom: the bending and twisting of the wing accounts for two degrees of freedom, and the relative deflection of the aileron gives rise to the third degree of freedom. Not only is the system non-conservative, but there is the additional complication of damping forces leading to terms depending on the velocities in the equations of motion. The differential equations of motion are of type

$$a_{ij}\frac{d^2q^j(t)}{dt^2} + c_{ij}\frac{dq^j(t)}{dt} + b_{ij}q^j(t) = 0, \tag{7.1}$$

a system of three linear differential equations in the three unknowns $q^1(t)$, $q^2(t)$, $q^3(t)$. Since there are three degrees of freedom, all indices have the range 1 to 3. The constant coefficients a_{ij}, b_{ij}, c_{ij} are computed from a large number of aerodynamic constants of our aircraft structure; see T. Theodorsen's "General Theory of Aerodynamic Instability and the Mechanism of Flutter," *N. A. C. A. Report* 496, for 1934, pp. 413–433, especially pp. 419–420. Now the general structure of equations 7·1 differs from the equations of motion of the preceding two chapters in that $b_{ij} \neq b_{ji}$ (giving rise to a non-conservative system) and in the presence of the linear damping terms $c_{ij}\frac{dq^j(t)}{dt}$. If we define $A = \| a_{ij} \|$, $B = \| b_{ij} \|$, $C = \| c_{ij} \|$, $Q(t) = \| q^i(t) \|$, then the equations of motion 7·1 can be written as the one matric differential equation

$$(7\cdot 2) \qquad A\frac{d^2Q(t)}{dt^2} + C\frac{dQ(t)}{dt} + BQ(t) = 0$$

in terms of the three known constant matrices A, B, C, and the unknown column matrix $Q(t)$. As we are interested in small oscillations around an unstable point of equilibrium, it is to be expected that complex *imaginary frequencies* will play a role in the work.

Since A arises from the kinetic energy, A^{-1} exists and hence 7.2 is equivalent to

$$(7\cdot 3) \qquad \frac{d^2Q(t)}{dt^2} = -A^{-1}C\frac{dQ(t)}{dt} - A^{-1}BQ(t).$$

We can replace the one second-order differential equation 7·3 by an equivalent pair of *two first*-order differential equations with the column matrices $Q(t)$ and $R(t)$ as unknowns

$$(7\cdot 4) \qquad \begin{cases} \dfrac{dQ(t)}{dt} = R(t) \\ \dfrac{dR(t)}{dt} = -A^{-1}BQ(t) - A^{-1}CR(t). \end{cases}$$

Define the column matrix of six elements

$$S(t) = \begin{Vmatrix} Q(t) \\ R(t) \end{Vmatrix}$$

and the constant square matrix of six rows

$$(7\cdot 5) \qquad U = \begin{Vmatrix} O, & I \\ -A^{-1}B, & -A^{-1}C \end{Vmatrix},$$

where O and I are the three-rowed *zero* and *unit* matrices respectively. Then equations 7·4 can be written as the one first-order matric differential equation

$$(7\cdot 6) \qquad \frac{dS(t)}{dt} = US(t).$$

We are led therefore to consider solutions

$$S(t) = e^{\lambda t}\Delta \qquad (\Delta, \text{ a constant column matrix of six elements})$$

of 7·6. This obviously leads us to the equation

$$(7\cdot7) \qquad (\lambda I - U)\Delta = O,$$

where I is the six-row unit matrix. To solve the problem we must get good approximations to the values of λ and the matrix Δ that will satisfy 7·7. The matrix iteration method for small oscillations of conservative systems can now be applied with some modifications made necessary by the fact that the possible values of λ in 7·7 are in general complex imaginary. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the characteristic roots of the matrix U lexicographically arranged so that their moduli † are in descending order, i.e., $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$, then the characteristic root λ_1 with the largest modulus can be obtained by the methods of the previous chapter. Some further aids in computation of the real and imaginary parts of complex characteristic roots are given on pp. 148–150 and 327–331 of the Frazer-Duncan-Collar book on matrices. A more readable and self-contained account is given in the paper by W. J. Duncan and A. R. Collar entitled "Matrices Applied to the Motions of Damped Systems," *Phil. Mag.*, vol. 19 (1935), pp. 197–219. An illuminating discussion of a specialized flutter problem with *two degrees of freedom* is given in a 1940 book by Kármán and Biot, *Mathematical Methods in Engineering*, pp. 220–228. It would be interesting and instructive to solve such specialized flutter problems with the aid of the matrix calculus.

Another useful method of solving flutter problems is the combination of matrix methods and Laplace transform methods. The Laplace transform of a function $x(t)$ is a function $\bar{x}(p)$ defined by

$$\bar{x}(p) = \int_0^\infty e^{-pt}x(t)\,dt.$$

If one is willing to omit the *proofs* of one or two theorems, the whole Laplace transform theory needed does not require one to be conversant with the residue theory of complex variable theory. For such an elementary treatment of Laplace transforms see *Operational Calculus in Applied Mathematics* by Carslaw and Jaeger, Chapters I–III. The methods given there can be immediately extended in the obvious way to apply directly to the matric differential equations 7·3 for flutter problems. A good table of Laplace transforms together with mechanical or electric methods can cut down the labor of flutter calculations

† The modulus of a complex number $z = x + \sqrt{-1}\,y$ is denoted by $|z|$ and is defined by $|z| = \sqrt{x^2 + y^2}$.

materially. Unfortunately there is no time to take up these matters in detail in our brief introductory treatment.

Exercises

1. Show that the matric differential equation 7·6 for flutter can be written as

$$S(t) = U^{-1}\frac{dS(t)}{dt},$$

where U^{-1}, the inverse matrix of U, is given by the six-row square matrix

$$U^{-1} = \left\| \begin{matrix} -B^{-1}C & -B^{-1}A \\ I & O \end{matrix} \right\|.$$

2. A model airplane wing is placed at a small angle of incidence in a uniform air stream. The three degrees of freedom are the wing bending, wing twist, and aileron angle measured relative to the wing chord at the wing tip. When the wind speed is 12 feet per second, the matrices for the differential equations of flutter are as follows. The data are obtained from R. A. Frazer and W. J. Duncan, "The Flutter of Aeroplane Wings," *Reports and Memoranda of the Aeronautical Research Committee*, No. 1155, August, 1928.

$$A = \left\| \begin{matrix} 17.6 & 0.128 & 2.89 \\ 0.128 & 0.00824125 & 0.0413 \\ 2.89 & 0.0413 & 0.725 \end{matrix} \right\|$$

$$B = \left\| \begin{matrix} 121.042 & 1.89 & 15.9497 \\ 0 & 0.027 & 0.0145 \\ 11.9097 & 0.364 & 15.4722 \end{matrix} \right\|$$

$$C = \left\| \begin{matrix} 7.65833 & 0.245 & 2.10 \\ 0.023 & 0.0104 & 0.0223 \\ 0.60 & 0.0756 & 0.658333 \end{matrix} \right\|$$

= matrix of damping coefficients.

Show that

$$A^{-1} = \left\| \begin{matrix} 0.170883 & 1.06301 & -0.741731 \\ 1.06301 & 176.433 & -14.2880 \\ -0.741731 & -14.2880 & 5.14994 \end{matrix} \right\|$$

and that the matric differential equation for flutter is

$$\frac{dS(t)}{dt} = US(t),$$

where the "flutter matrix" U is given by

$$U = \left\| \begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -11.8502 & -0.08168 & 8.73526 & -0.888089 & 0.003153 & 0.105747 \\ 41.4969 & -1.57195 & 201.554 & -3.62604 & -1.01517 & 3.23949 \\ 28.4464 & -0.086931 & -67.6433 & 2.91908 & -0.059016 & -1.51412 \end{matrix} \right\|.$$

For the lengthy details of the calculations of flutter frequencies and amplitudes see the *Phil. Mag.* 1935 paper by Duncan and Collar.

More recent developments in aircraft design require an extension of the flutter theory to handle four-degree- (or more) of-freedom problems in which the motion of the *tab* defines the fourth degree † of freedom with generalized coordinate q^4. The aerodynamic forces and moments are obtained theoretically in accordance with T. Theodorsen's and I. E. Garrick's investigations and not from wind-tunnel data. It is for this reason that the coefficients in the differential equations of motion will be in general complex. See Fig. 7·1. The four differential equations of motion are of the form ‡

$$a_{ij}\frac{d^2q^j}{dt^2} + c_{ij}\frac{dq^j}{dt} + b_{ij}q^j = 0, \tag{7·8}$$

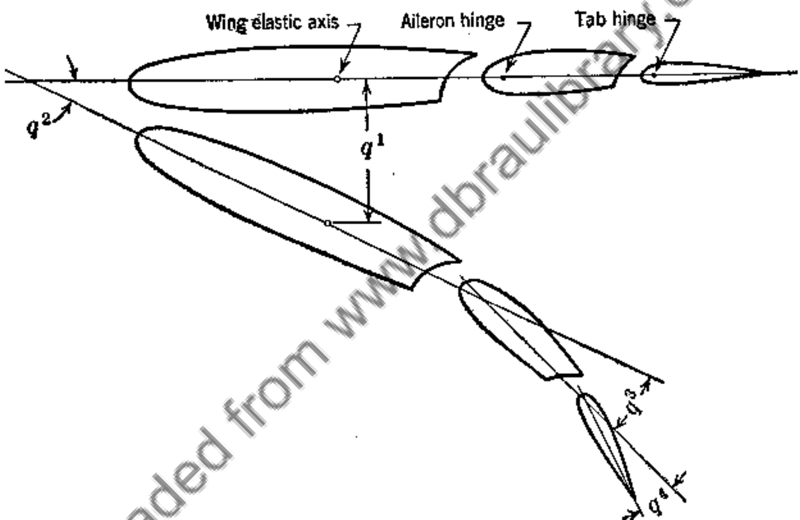


FIG. 7·1.

where all the indices, in contradistinction to 7·1, have the range 1 to 4 and the coefficients a_{ij}, b_{ij}, and c_{ij} are in general complex.

The flutter velocity v appears in general in the coefficients c_{ij} and b_{ij} and is replaced by the quantity $\frac{b\omega}{k}$, where b, ω, and k are the airfoil semi-chord, flutter frequency, and the flutter parameter $k = \frac{v}{b_\omega}$ respectively. This yields a system of four linear differential equations 7·8

† In accordance with the flutter notation used in this country, $q^1 = h$, $q^2 = \alpha$, $q^3 = \beta$, $q^4 = \gamma$. See Fig. 7·1.

‡ The differential equations of motion and the contributions of the aerodynamic forces and moments to the coefficients of the differential equations for the four-degree-of-freedom problem are given in the Douglas reports.

in which c_{ij} and b_{ij} are expressed as functions of the flutter frequency ω and not of the flutter velocity v. If one then considers solutions of the form

$$q^i = q_0^i e^{i\omega t} \qquad (i = \sqrt{-1})$$

of the differential equations, one is led to a system of four linear algebraic equations with complex coefficients, some of which are functions of ω and the structural damping coefficients g_{ij}. The damping coefficients g_{ij} are defined by

$$F_i = \sqrt{-1}\; g_{ij} K_{ij} q^j \quad (g_{ij} = 0 \quad \text{if} \quad i \neq j),$$

where F_i is the damping force in the q^ith direction and the K_{ij} are the spring constants. Upon obtaining the characteristic roots of the matrix of the linear algebraic equations by matric iteration methods, one ultimately finds the flutter velocity v as a function of the flutter frequency ω and the structural damping coefficients g_{ij}. Flutter is likely to occur if the structural damping coefficients g_{ij} from the pure imaginary part of the characteristic root exceeds 0.03, provided that no extraneous damping devices are used. The algebraic equations are so arranged — and this constitutes an important aspect of the development in that it lends itself to matric iteration procedures — that the characteristic roots are of the form

$$z = \frac{c}{\omega^2} + ig \qquad (i = \sqrt{-1}),$$

where c is some constant, ω is the flutter frequency, and g is a structural damping coefficient.

The flutter analyst is attempting to approximate the actual flutter characteristics of the airplane by representing them by as small a number of degrees of freedom as possible. The design of faster and larger aircraft requires the consideration of a larger number of degrees of freedom so as to make the flutter analysis an *adequate approximation.* When many degrees of freedom are required to represent the flutter characteristics of an airplane, the need for matrix methods becomes acute. Matrix methods † also serve to improve the theory of the mechanism of flutter.

† Matrix methods are also used in treating other phenomena related to flutter. See Douglas reports.

CHAPTER 8

MATRIX METHODS IN ELASTIC DEFORMATION THEORY

Although the *tensor calculus* is the most natural and powerful mathematical method in the treatment of the fundamentals of elastic deformation of bodies, the matrix calculus can also be used to advantage in furnishing a short and neat treatment. This chapter is purely introductory and suggestive.

Let a medium be acted on by deforming forces. The position of the medium before and after deformation will be called the initial and final state of the medium respectively. Let a^1, a^2, a^3 be the rectangular cartesian coordinates of a representative particle of the medium in the initial state, and x^1, x^2, x^3 the rectangular cartesian coordinates of the corresponding particle in the final state. Then the elastic deformation is represented by particle-to-particle transformations

$$x^i = f^i(a^1, a^2, a^3). \tag{8·1}$$

Hence by the ordinary differential calculus

$$dx^i = f^i_j \, da^j, \tag{8·2}$$

where

$$f^i_j = \frac{\partial f^i(a^1, a^2, a^3)}{\partial a^j}.$$

The classical theory of elastic bodies assumes that the deformations 8·1 are "infinitesimal." Such crude approximations have been found inadequate in some investigations on thin plates and shells.† As a result the finite deformation theory is beginning to be used in engineering problems. In what we shall have to say we shall make the restrictive assumptions of the classical infinitesimal theory only toward the end of the chapter.

Let A and X be defined as the column matrices of three elements:

$$A = \| a^i \| \qquad \text{and} \qquad X = \| x^i \| .$$

Similarly the differential matrices are $dA = \| da^i \|$, $dX = \| dx^i \|$. Define the square matrix F by

$$F = \| f^i_j \| ,$$

† See the *Kármán Anniversary Volume*, California Institute of Technology, 1941, for various papers and other references.

i.e., the matrix of the partial derivatives of the deformation 8·1. Then the differential relations 8·2 can be written in matric form as

(8·3) $$dX = F\,dA.$$

DEFINITION. The adjoint M^* of a matrix M is the matrix obtained from M by interchanging the rows and columns of M.

Thus A^* and X^* are row matrices while

$$F^* = \left\| \begin{matrix} \frac{\partial x^1}{\partial a^1}, & \frac{\partial x^2}{\partial a^1}, & \frac{\partial x^3}{\partial a^1} \\ \frac{\partial x^1}{\partial a^2}, & \frac{\partial x^2}{\partial a^2}, & \frac{\partial x^3}{\partial a^2} \\ \frac{\partial x^1}{\partial a^3}, & \frac{\partial x^2}{\partial a^3}, & \frac{\partial x^3}{\partial a^3} \end{matrix} \right\|.$$

It can be proved by a routine procedure that $(M_1M_2)^* = M_2^*M_1^*$. In other words, the adjoint of the product of two matrices is the product of their adjoints in the *reverse order*. For example,

$$dX^* = dA^*F^*.$$

Hence the square of the differential line element in the final state of the medium will be

(8·4) $$ds_X^2 = dA^*F^*F\,dA$$

since

$$ds_X^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dX^*\,dX.$$

By a direct computation it can be shown that the matrix

(8·5) $$F^*F = \|\ \psi_{ij}\ \|,$$

where

(8·6) $$\psi_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial a^i}\frac{\partial x^k}{\partial a^j}.$$

Note that $\psi_{ij} = \psi_{ji}$. This is expressed by saying that F^*F is a *symmetric matrix*.

The square of the differential line element in the initial state is

(8·7) $ds_A^2 = dA^*\,dA$ $\quad$ ($= dA^*I\,dA$, where I is the unit matrix)

and hence with the aid of 8·4 we find the formula

(8·8) $$ds_X^2 - ds_A^2 = dA^*(F^*F - I)\,dA.$$

On defining the matrix, called the *deformation* or *strain matrix*,

(8·9) $$H = \tfrac{1}{2}(F^*F - I),$$

we find

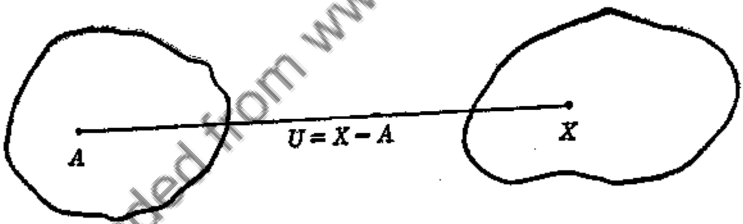
(8·10) $$ds_X^2 - ds_A^2 = 2\,dA^*H\,dA.$$

Now, if $ds_X^2 = ds_A^2$ for all particles of the initial state and for all dA, we have, by definition, a *rigid displacement* of the medium from the initial state to the final state. A glance at 8·10 shows that *a necessary and sufficient condition that the change of the medium from the initial to the final state be a rigid displacement is that the strain matrix H be a zero matrix.* In other words, when $H = O$, the medium is *not* deformed or strained but is merely transported to a different position by a rigid displacement. This property then justifies the terminology "strain matrix H" since H measures, in a sense, the amount of strain or deformation undergone by the medium. It is clear from definition 8·9 that the *strain matrix H is a symmetric matrix.* Let η_{ij} be the elements (more commonly called components in elasticity theory) of the strain matrix H, i.e., $H = \| \eta_{ij} \|$. On using result 8·5 and definition 8·9, we see that

$$\eta_{ij} = \frac{1}{2}\left(\sum_{k=1}^{3} \frac{\partial x^k}{\partial a^i}\frac{\partial x^k}{\partial a^j} - \delta_{ij}\right),$$

where

$$\delta_{ij} = 1 \quad \text{if} \quad i = j, \qquad = 0 \quad \text{if} \quad i \neq j.$$



Initial state *Final state*

FIG. 8·1.

Let $u^i = x^i - a^i$, then

$$\frac{\partial x^i}{\partial a^j} = \frac{\partial u^i}{\partial a^j} + \delta^i_j.$$

Define the matrix

$$V = \left\| \frac{\partial u^i}{\partial a^j} \right\|,$$

and obtain the relation

$$F = V + I.$$

Hence, from definition 8·9, for the strain matrix H we obtain

$$H = \tfrac{1}{2}[(V^* + I)(V + I) - I].$$

Expanding the right-hand side gives

$$H = \tfrac{1}{2}[V^*V + V^* + V].$$

In the classical infinitesimal theory of elasticity only first-degree terms in $\dfrac{\partial u^i}{\partial a^j}$ are kept. Hence, to the *degree of approximation contemplated by infinitesimal theory of elastic deformations, the strain matrix is given by*

$$H = \tfrac{1}{2}(V^* + V)$$

in rectangular cartesian coordinates. In other words, the components η_{ij} of H are given by the familiar

$$\eta_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial a^j} + \frac{\partial u^j}{\partial a^i}\right).$$

From the symmetry of η_{ij} there are thus in general five distinct components of the strain matrix in three dimensions while there are three components for plane elastic problems.

PART II. TENSOR CALCULUS AND ITS APPLICATIONS

CHAPTER 9

SPACE LINE ELEMENT IN CURVILINEAR COORDINATES

Introductory Remarks.

The vague beginnings of the tensor calculus, or absolute differential calculus as it is sometimes called, can be traced back more than a century to Gauss's researches on curved surfaces. The systematic investigation of tensor calculi by a considerable number of mathematicians has taken place since 1920. With few exceptions, the applications of tensor calculus were confined to the *general theory of relativity*. The result was an undue emphasis on the tensor calculus of *curved* spaces as distinguished from the tensor calculus of Euclidean spaces. The subjects of elasticity[1] and hydrodynamics,[2] as studied and used by aeronautical engineers, are developed and have their being in plane and solid Euclidean space. It is for this reason that we shall be primarily concerned with Euclidean tensor calculus in this book. We shall, however, devote two chapters to curved tensor calculus in connection with the fundamentals of classical mechanics[3] and fluid mechanics.

It is worthy of notice that the tensor calculus is a generalization of the widely studied differential calculus of freshman and sophomore fame. In fact, as we shall see, a detailed study of the classical differential calculus along a certain direction demands the introduction of the tensor calculus.

Notation and Summation Convention.

Before we begin the study of tensor calculus, we must embark on some formal preliminaries including some matters of notation.

Consider a linear function in the n real variables $x, y, z, \cdots, w$

$$\alpha x + \beta y + \gamma z + \cdots + \lambda w. \tag{9·1}$$

Define

$$\begin{aligned} a_1 = \alpha,\ a_2 = \beta,\ a_3 = \gamma,\ \cdots,\ a_n = \lambda, \\ x^1 = x,\ x^2 = y,\ x^3 = z,\ \cdots,\ x^n = w. \end{aligned} \tag{9·2}$$

We emphasize once for all that $x^1, x^2, \cdots, x^n$ are n independent variables and *not* the first n powers of one variable x. In terms of the notations of 9·2 we can rewrite 9·1 in the form

$$a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n, \tag{9·3}$$

or as

$$\sum_{i=1}^{n} a_ix^i. \tag{9·4}$$

The set of n integer values 1 to n is called the *range* of the index i in 9·4. A lower index i as in a_i will be called a *subscript*, and an upper index i as in x^i will be called a *superscript*. Throughout our work we shall adopt the following useful *summation convention:*

The repetition of an index in a term once as a subscript and once as a superscript will denote a summation with respect to that index over its range. An index that is not summed out will be called a *free index.*

In accordance with this convention then we shall write the sum 9·4 simply as

$$a_ix^i. \tag{9·5}$$

A summation index as i in 9·5 is called a *dummy* or an *umbral*, since it is immaterial what symbol is used for the index. For example, a_jx^j is the same sum as 9·5. All this is analogous to the (umbral) variable of integration x in an integral

$$\int_a^b f(x)\, dx.$$

Any other letter, say y, could be used in the place of x. Thus

$$\int_a^b f(y)\, dy = \int_a^b f(x)\, dx.$$

Aside from compactness, the subscript and superscript notation together with the summation convention has advantages that will become evident later.

As a further illustration of the summation convention, consider the square of the line element

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{9·6}$$

in a three-dimensional Euclidean space with rectangular cartesian coordinates x, y, and z. Define

$$y^1 = x,\ y^2 = y,\ y^3 = z \tag{9·7}$$

and

$$\begin{cases} \delta_{11} = \delta_{22} = \delta_{33} = 1, \\ \delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0. \end{cases} \tag{9·8}$$

Then 9·6 can be rewritten

(9·9) $$ds^2 = \sum_{i=1}^{3} (dy^i)^2,$$

or again

(9·10) $$ds^2 = \delta_{ij}\, dy^i\, dy^j$$

with the understanding that the range of the indices i and j is 1 to 3. Note that there are two summations in 9·10 one over the index i and one over the index j.

Let $f(x^1, x^2, \cdots, x^n)$ be a function of n numerical variables $x^1, x^2, \cdots, x^n$; then its differential can be written

$$df = \frac{\partial f}{\partial x^i}\, dx^i$$

with the *understanding that the summation convention has been extended so as to apply to repeated superscripts in differentiation formulas.* We shall adhere to this extension of the summation convention.

It is worth while at this early stage to give an example of a tensor and show the fundamental nature of such a concept even for elementary portions of the usual differential and integral calculus. This will dispel, I hope, any illusions common among educated laymen that the tensor calculus is a very "highbrow" and esoteric subject and that its main applications are to the physical speculations of relativistic cosmology.

Euclidean Metric Tensor.[4]

In the following example, free as well as umbral indices will have the range 1 to 3 as we shall deal with a three-dimensional Euclidean space. Let

(9·11) $$x^i = f^i(y^1, y^2, y^3)$$

be a transformation of coordinates from the rectangular cartesian coordinates y^1, y^2, y^3 to some general coordinates x^1, x^2, x^3 not necessarily rectangular cartesian coordinates; for example, they may be spherical coordinates. The inverse transformation of coordinates to 9·11 is the transformation of coordinates that takes one from the coordinates x^1, x^2, x^3 to the rectangular cartesian coordinates y^1, y^2, y^3. Let

(9·12) $$y^i = g^i(x^1, x^2, x^3)$$

be the inverse transformation of coordinates to 9·11.

Example

Let y^1, y^2, y^3 be rectangular cartesian coordinates and x^1, x^2, x^3 polar spherical coordinates. The transformation of coordinates from rectangular cartesian to polar spherical coordinates is clearly

$$x^1 = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$$

$$x^2 = \cos^{-1}\left(\frac{y^3}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}\right)$$

$$x^3 = \tan^{-1}\left(\frac{y^2}{y^1}\right).$$

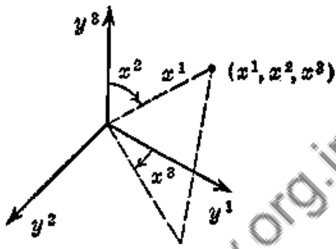


FIG. 9·1.

The inverse transformation of coordinates is given by

$$y^1 = x^1 \sin x^2 \cos x^3$$
$$y^2 = x^1 \sin x^2 \sin x^3$$
$$y^3 = x^1 \cos x^2.$$

The differentials of the transformation functions in 9·12 may be written

(9·13) $$dy^i = \frac{\partial y^i}{\partial x^\alpha}\, dx^\alpha.$$

On using 9·13 we obtain, after an evident rearrangement, the formula

(9·14) $$ds^2 = \sum_{i=1}^{3} \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^i}{\partial x^\beta}\, dx^\alpha\, dx^\beta.$$

If we define the functions $g_{\alpha\beta}(x^1, x^2, x^3)$ of the three independent variables x^1, x^2, x^3 by

(9·15) $$g_{\alpha\beta}(x^1, x^2, x^3) = \sum_{i=1}^{3} \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^i}{\partial x^\beta},$$

we see that *the square of the line element in the general x^1, x^2, x^3 coordinates takes the form*

(9·16) $$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

This is a homogeneous quadratic polynomial, called *quadratic differential form* in the three independent variables dx^1, dx^2, dx^3.

Caution: Once an index has been used in one summation of a series of repeated summations, it cannot be used again in *another* summation of the same series. For example, $g_{\alpha\alpha}\, dx^\alpha\, dx^\alpha$ has a meaning and is equal to $g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$, but that is not what one gets by carrying out the double repeated summation in 9·16. Expanded in extenso, 9·16 stands for

$$(9\cdot 17)\qquad \begin{cases} ds^2 = g_{11}(dx^1)^2 + 2g_{12}\,dx^1\,dx^2 + 2g_{13}\,dx^1\,dx^3 \\ \qquad + g_{22}(dx^2)^2 + 2g_{23}\,dx^2\,dx^3 + g_{33}(dx^3)^2. \end{cases}$$

The factor 2 in three of the terms in 9·17 comes from combining terms due to the fact that $g_{\alpha\beta}$ is symmetric in α and β; a glance at the definition 9·15 shows that

$$g_{\alpha\beta} = g_{\beta\alpha} \quad \text{for each } \alpha \text{ and } \beta$$

and hence

$$g_{12} = g_{21},\ g_{13} = g_{31},\ g_{23} = g_{32}.$$

Now let $\bar{x}^1, \bar{x}^2, \bar{x}^3$ be any chosen general coordinates, not necessarily distinct from the general coordinates x^1, x^2, x^3. Let

$$(9\cdot 18)\qquad x^i = F^i(\bar{x}^1, \bar{x}^2, \bar{x}^3)$$

be the transformation of coordinates from the general coordinates $\bar{x}^1, \bar{x}^2, \bar{x}^3$ to the general coordinates x^1, x^2, x^3. Clearly the differentials dx^α have the form

$$(9\cdot 19)\qquad dx^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^\gamma}\,d\bar{x}^\gamma.$$

Define the functions $\bar{g}_{\gamma\delta}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ of the three variables $\bar{x}^1, \bar{x}^2, \bar{x}^3$ by

$$(9\cdot 20)\qquad \bar{g}_{\gamma\delta}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = g_{\alpha\beta}(x^1, x^2, x^3)\frac{\partial x^\alpha}{\partial \bar{x}^\gamma}\frac{\partial x^\beta}{\partial \bar{x}^\delta}.$$

Then, if we use 9·19 in 9·16, we obtain, with the aid of the definition 9·20, the formula

$$(9\cdot 21)\qquad ds^2 = \bar{g}_{\gamma\delta}(\bar{x}^1, \bar{x}^2, \bar{x}^3)\,d\bar{x}^\gamma\,d\bar{x}^\delta,$$

which gives the square of the line element in the $\bar{x}^1, \bar{x}^2, \bar{x}^3$ coordinates. We have thus arrived at the following result:

If x^1, x^2, x^3 and $\bar{x}^1, \bar{x}^2, \bar{x}^3$ are two arbitrarily chosen sets of general coordinates, and if the transformation of coordinates 9·18 from the $\bar{x}$'s to the x's has suitable differentiability properties, then the coefficients $g_{\alpha\beta}(x^1, x^2, x^3)$ of the square of the line element 9·16 in the x^1, x^2, x^3 coordinates are related to the coefficients $\bar{g}_{\alpha\beta}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ of the square of the line element 9·21 in the coordinates $\bar{x}^1, \bar{x}^2, \bar{x}^3$ by means of the law of transformation 9·20.

In each coordinate system with coordinates x^1, x^2, x^3, we have a set of functions $g_{\alpha\beta}(x^1, x^2, x^3)$, called the components of the Euclidean metric tensor (field), and the components of the Euclidean metric tensor in any two coordinate systems with coordinates x^1, x^2, x^3 and $\bar{x}^1, \bar{x}^2, \bar{x}^3$ respectively are related by means of the *characteristic rule* 9·20.

An analogous discussion can obviously be given for the line element and Euclidean metric tensor of the plane (a two-dimensional Euclidean space). We now have two coordinates instead of three so that the *range of the indices* is 1 to 2. For example, the line elements ds in rectangular coordinates (y^1, y^2)

$$ds^2 = (dy^1)^2 + (dy^2)^2$$

will become

$$ds^2 = g_{\alpha\beta}(x^1, x^2)\, dx^\alpha\, dx^\beta$$

in general coordinates (x^1, x^2), and the components of the plane Euclidean metric tensor $g_{\alpha\beta}(x^1, x^2)$ will undergo the transformation

$$\bar{g}_{\gamma\delta}(\bar{x}^1, \bar{x}^2) = g_{\alpha\beta}(x^1, x^2)\frac{\partial x^\alpha}{\partial \bar{x}^\gamma}\frac{\partial x^\beta}{\partial \bar{x}^\delta}.$$

Exercises

1. Find the components of the plane Euclidean metric tensor in polar coordinates (x^1, x^2) and the corresponding expression for the line element.

$$\begin{cases} y^1 = x^1 \cos x^2, \\ y^2 = x^1 \sin x^2, \end{cases} \text{ and } \begin{cases} x^1 = \sqrt{(y^1)^2 + (y^2)^2}, \\ x^2 = \sin^{-1}\left(\dfrac{y^2}{\sqrt{(y^1)^2 + (y^2)^2}}\right). \end{cases}$$

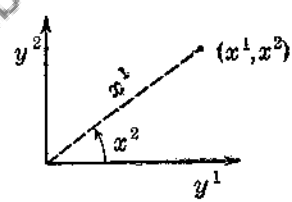


FIG. 9·2.

Hence

$$g_{ij}(x^1, x^2) = \sum_{\alpha=1}^{2}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\alpha}{\partial x^j} = \frac{\partial y^1}{\partial x^i}\frac{\partial y^1}{\partial x^j} + \frac{\partial y^2}{\partial x^i}\frac{\partial y^2}{\partial x^j}.$$

Therefore

$$\begin{aligned} g_{11}(x^1, x^2) &= \cos^2 x^2 + \sin^2 x^2 = 1, \\ g_{12}(x^1, x^2) &= (\cos x^2)(-x^1 \sin x^2) + (\sin x^2)(x^1 \cos x^2) = 0 = g_{21}(x^1, x^2), \\ g_{22}(x^1, x^2) &= (-x^1 \sin x^2)^2 + (x^1 \cos x^2)^2 = (x^1)^2. \end{aligned}$$

The line element $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ in polar coordinates (x^1, x^2) is then

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2,$$

which in the usual notation is written

$$ds^2 = dr^2 + r^2\, d\theta^2.$$

2. Find the components of the (space) Euclidean metric tensor and the expression for the line element in polar spherical coordinates.

Answer.

$$g_{11}(x^1, x^2, x^3) = 1,\; g_{22}(x^1, x^2, x^3) = (x^1)^2,\; g_{33}(x^1, x^2, x^3) = (x^1)^2(\sin x^2)^2$$

and all other $g_{ij}(x^1, x^2, x^3) = 0$, so that

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1)^2(\sin x^2)^2(dx^3)^2.$$

CHAPTER 10

VECTOR FIELDS, TENSOR FIELDS, AND EUCLIDEAN CHRISTOFFEL SYMBOLS

The Strain Tensor.

Another interesting example of a tensor is to be found in elasticity. Let a^1, a^2, a^3 be the curvilinear coordinates of a representative particle in an elastic medium, and let x^1, x^2, x^3 be the coordinates of the representative particle *after* an elastic deformation of the medium.

FIG. 10·1.

Let

(10·1) $$ds_0^2 = c_{\alpha\beta}\, da^\alpha\, da^\beta$$

be the square of the line element in the medium, and let

(10·2) $$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

be the corresponding square of the line element in the deformed medium induced by the elastic deformation whose equations are

(10·3) $$x^i = f^i(a^1, a^2, a^3).$$

In terms of the coordinates x^1, x^2, x^3 the line element 10·1 can be written

(10·4) $$ds_0^2 = h_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

where

(10·5) $$h_{\alpha\beta} = c_{\gamma\delta} \frac{\partial a^\gamma}{\partial x^\alpha} \frac{\partial a^\delta}{\partial x^\beta}.$$

On subtracting corresponding sides of 10·2 and 10·4 we find

(10·6) $$ds^2 - ds_0^2 = 2\epsilon_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

if we define $\epsilon_{\alpha\beta}$ by

$$(10\cdot 7) \qquad \epsilon_{\alpha\beta} = \tfrac{1}{2}(g_{\alpha\beta} - h_{\alpha\beta}).$$

Now $\epsilon_{\alpha\beta}$ are functions of the coordinates x^1, x^2, x^3. If then we calculate $ds^2 - ds_0^2$ in any other curvilinear coordinates $\bar{x}^1, \bar{x}^2, \bar{x}^3$, we would obtain by the method of the preceding chapter

$$ds^2 - ds_0^2 = 2\bar{\epsilon}_{\alpha\beta}\, d\bar{x}^\alpha\, d\bar{x}^\beta,$$

where

$$(10\cdot 8) \qquad \bar{\epsilon}_{\alpha\beta} = \epsilon_{\gamma\delta} \frac{\partial x^\gamma}{\partial \bar{x}^\alpha} \frac{\partial x^\delta}{\partial \bar{x}^\beta}.$$

Because of the characteristic law 10·8, $\epsilon_{\alpha\beta}$ are the components of a tensor (field), and because $\epsilon_{\alpha\beta}$ in 10·6 is a measure of the strain of the elastic medium, $\epsilon_{\alpha\beta}$ are the components of a *strain tensor*. We shall have a good deal to say about the strain tensor in some of the later chapters.

Scalars, Contravariant Vectors, and Covariant Vectors.

We shall now begin the subject of the tensor calculus by defining the simplest types of tensors. An object is called a *scalar (field)* if in each coordinate system there corresponds a function, called a component, such that the relationship between the components in (x^1, x^2, x^3) coordinates and $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ coordinates respectively is

$$(10\cdot 9) \qquad s(x^1, x^2, x^3) = \bar{s}(\bar{x}^1, \bar{x}^2, \bar{x}^3).$$

An object is called a *contravariant vector field* (an equivalent terminology is *contravariant tensor field of rank one*) if in each coordinate system there corresponds a set of three functions, called components, such that the relationship between the components in any two coordinate systems is given by the characteristic law

$$(10\cdot 10) \qquad \bar{\xi}^i(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \xi^\alpha(x^1, x^2, x^3) \frac{\partial \bar{x}^i}{\partial x^\alpha}.$$

An object is called a *covariant vector field* (an equivalent terminology is *covariant tensor field of rank one*) if in each coordinate system there corresponds a set of three functions, called components, such that the relationship between the components in any two coordinate systems is given by the characteristic law

$$(10\cdot 11) \qquad \bar{\eta}_i(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \eta_\alpha(x^1, x^2, x^3) \frac{\partial x^\alpha}{\partial \bar{x}^i}.$$

It is to be noticed at this point that the laws of transformation 10·10 and 10·11 are in general distinct so that there is a difference[1] between the notions of contravariant vector field and covariant vector field. However, if *only rectangular cartesian coordinates* are considered, this distinction disappears. It is for this reason that the notions of contravariant as distinguished from covariant vector fields are not introduced in elementary vector analysis. That the characteristic laws 10·10 and 10·11 are identical in rectangular cartesian coordinates follows from some calculations leading to the result

$$\frac{\partial \bar{x}^i}{\partial x^\alpha} = \frac{\partial x^\alpha}{\partial \bar{x}^i} \tag{10·12}$$

between rectangular coordinates (x^1, x^2, x^3) and any other rectangular coordinates $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$.

Tensor Fields of Rank Two.

In all three objects — scalar fields, contravariant vector fields, and covariant vector fields — there are components in any two coordinate systems, and the components in any two coordinate systems are related by characteristic transformation laws. We have to consider other objects, called tensor fields (of various sorts), whose components in any two coordinate systems are related by a characteristic transformation law. To shorten the statements of the following definitions we shall merely give the characteristic transformation law of components.

Covariant Tensor Field of Rank Two.

$$\bar{t}_{\alpha\beta}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = t_{\lambda\mu}(x^1, x^2, x^3)\,\frac{\partial x^\lambda}{\partial \bar{x}^\alpha}\frac{\partial x^\mu}{\partial \bar{x}^\beta}. \tag{10·13}$$

Contravariant Tensor Field of Rank Two.

$$\bar{t}^{\alpha\beta}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = t^{\lambda\mu}(x^1, x^2, x^3)\,\frac{\partial \bar{x}^\alpha}{\partial x^\lambda}\frac{\partial \bar{x}^\beta}{\partial x^\mu}. \tag{10·14}$$

Mixed Tensor Field of Rank Two.

$$\bar{t}^\alpha_\beta(\bar{x}^1, \bar{x}^2, \bar{x}^3) = t^\lambda_\mu(x^1, x^2, x^3)\,\frac{\partial \bar{x}^\alpha}{\partial x^\lambda}\frac{\partial x^\mu}{\partial \bar{x}^\beta}. \tag{10·15}$$

Again because of relation 10·12, the difference between the above three types of tensor fields is non-existent as long as one considers *only rectangular cartesian coordinates.*

It is worthy of notice at this point that the indices in the various characteristic transformation laws tell a story which depends on whether the index is a superscript, called *contravariant index*, or a subscript, called *covariant index.*

Before proceeding any further with the development of our subject, it would be illuminating to have some examples of vector fields and other tensor fields. Perhaps the most important example of a contravariant vector field is a *velocity field*. Suppose that the motion of a particle is governed by the differential equations

$$\frac{dx^i}{dt} = \xi^i(x^1, x^2, x^3),$$

where t is the time variable. If $\bar{x}^i = f^i(x^1, x^2, x^3)$ is a transformation of coordinates to new coordinates $\bar{x}^i$, then

$$\frac{d\bar{x}^i}{dt} = \frac{\partial \bar{x}^i}{\partial x^j}\frac{dx^j}{dt} = \frac{\partial \bar{x}^i}{\partial x^j}\xi^j(x^1, x^2, x^3).$$

Thus the components $\bar{\xi}^i(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ of the velocity field in the $\bar{x}^i$ coordinates are related to the components $\xi^i(x^1, x^2, x^3)$ in the x^i coordinates by the rule

$$\bar{\xi}^i(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \frac{\partial \bar{x}^i}{\partial x^j}\xi^j(x^1, x^2, x^3),$$

which is precisely the contravariant vector field rule.

If $s(x^1, x^2, x^3)$ is a scalar field, then the "gradient" $\frac{\partial s}{\partial x^i}$ are the components of a covariant vector field. An important example of a scalar field is the potential energy of a moving particle.

We gave two examples of a covariant tensor field of rank two: the Euclidean metric tensor and the strain tensor. A little later in the chapter we shall give an example of a contravariant tensor field of rank two, the $g^{\alpha\beta}$ associated with the Euclidean metric tensor $g_{\alpha\beta}$.

As an example of a mixed tensor field of rank two, we have the mixed tensor field with constant components

$$\begin{aligned} \delta^\alpha_\beta &= 0 \quad \text{if} \quad \alpha \neq \beta \\ &= 1 \quad \text{if} \quad \alpha = \beta \end{aligned}$$

in the x^i coordinates. But

$$\begin{aligned} \bar{\delta}^\alpha_\beta(\bar{x}^1, \bar{x}^2, \bar{x}^3) &= \delta^\lambda_\mu \frac{\partial x^\mu}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\alpha}{\partial x^\lambda} \\ &= \frac{\partial x^\lambda}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\alpha}{\partial x^\lambda}. \end{aligned}$$

Hence

$$\begin{aligned} \bar{\delta}^\alpha_\beta(\bar{x}^1, \bar{x}^2, \bar{x}^3) &= 0 \quad \text{if} \quad \alpha \neq \beta \\ &= 1 \quad \text{if} \quad \alpha = \beta. \end{aligned}$$

In other words, not only are the components constant throughout space, but they are also the same constants in all coordinates.

One of the first fundamental problems in the tensor calculus is to extend the notion of partial derivative to the notion of *covariant derivative* in such a manner that the covariant derivative of a tensor field is also some tensor field. It is true that, if one restricts his work only to cartesian coordinates (oblique axes), then the partial derivatives of any tensor field behave like the components of a tensor field under a transformation of cartesian coordinates to cartesian coordinates. For example, suppose that (x^1, x^2, x^3) are cartesian coordinates and $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ are any other cartesian coordinates; then it can be shown that for suitable constants a^i_j and a^i

$$\bar{x}^i = a^i_j x^j + a^i \tag{10·16}$$

is the transformation of coordinates taking the cartesian coordinates x^i to the cartesian coordinates $\bar{x}^i$. Hence

$$\frac{\partial \bar{x}^i}{\partial x^j} = a^i_j, \tag{10·17}$$

a set of constants, and so

$$\frac{\partial^2 \bar{x}^i}{\partial x^j\, \partial x^k} = 0. \tag{10·18}$$

Now, if $\xi^i(x^1, x^2, x^3)$ are the components of a contravariant tensor field, then

$$\bar{\xi}^i(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \xi^\alpha(x^1, x^2, x^3)\, \frac{\partial \bar{x}^i}{\partial x^\alpha}. \tag{10·19}$$

On differentiating corresponding sides of 10·19, we obtain

$$\frac{\partial \bar{\xi}^i}{\partial \bar{x}^j} = \frac{\partial \xi^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^\alpha} + \xi^\alpha \frac{\partial^2 \bar{x}^i}{\partial x^\alpha\, \partial x^\beta}\frac{\partial x^\beta}{\partial \bar{x}^j}. \tag{10·20}$$

But both x^i and $\bar{x}^i$ are cartesian coordinates. Hence on using 10·18 in 10·20 we obtain

$$\frac{\partial \bar{\xi}^i}{\partial \bar{x}^j} = \frac{\partial \xi^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial \bar{x}^j}\frac{\partial \bar{x}^i}{\partial x^\alpha}, \tag{10·21}$$

which states that the partial derivatives $\dfrac{\partial \xi^\alpha}{\partial x^\beta}$ *behave as though they were the components of a mixed tensor field of rank two and this under a transformation from cartesian coordinates to cartesian coordinates.*

The presence of the second derivative terms in 10·20 in curvilinear coordinates x^i shows that the $\dfrac{\partial \xi^\alpha}{\partial x^\beta}$ are not really the components of a tensor field. So the fundamental question arises whether it is possible to add corrective terms C^α_β (all zero in cartesian coordinates) to the partial derivatives $\dfrac{\partial \xi^\alpha}{\partial x^\beta}$ so as to make

$$\frac{\partial \xi^\alpha}{\partial x^\beta} + C^\alpha_\beta \tag{10·22}$$

the components of a mixed tensor field of rank two for all contravariant fields ξ^α. The answer to this question is in the affirmative, and the possibility of the corrective terms depends on the existence of the Euclidean Christoffel symbols.[2]

Euclidean Christoffel Symbols.

We saw in the previous chapter that the element of arc length squared in general coordinates takes the form

$$ds^2 = g_{\alpha\beta}(x^1, x^2, x^3)\, dx^\alpha\, dx^\beta, \tag{10·23}$$

where, in the present terminology, $g_{\alpha\beta}$ are the components of a covariant tensor field of rank two, called the Euclidean metric tensor. Now it can be proved that the determinant[3]

$$g = \begin{vmatrix} g_{11}, & g_{12}, & g_{13} \\ g_{21}, & g_{22}, & g_{23} \\ g_{31}, & g_{32}, & g_{33} \end{vmatrix} \neq 0. \tag{10·24}$$

Define

$$g^{\alpha\beta} = \frac{\text{Cofactor of } g_{\beta\alpha} \text{ in } g}{g}. \tag{10·25}$$

As the notation indicates, it can be proved that the *functions $g^{\alpha\beta}$ are the components of a contravariant tensor field of rank two with the following properties:*

$$\begin{aligned} &g^{\alpha\beta} = g^{\beta\alpha} \\ &g^{\alpha\sigma} g_{\sigma\beta} = \delta^\alpha_\beta \qquad \text{(equals 0 if } \alpha \neq \beta \text{ and 1 if } \alpha = \beta). \end{aligned} \tag{10·26}$$

Define the Euclidean Christoffel symbols $\Gamma^i_{\alpha\beta}(x^1, x^2, x^3)$ as follows:

$$\Gamma^i_{\alpha\beta}(x^1, x^2, x^3) = \tfrac{1}{2} g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right). \tag{10·27}$$

Since the law of transformation of the components $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are known, one can calculate the law of transformation of the $\Gamma^i_{\alpha\beta}(x^1, x^2, x^3)$ under a general transformation of coordinates $\bar{x}^i = f^i(x^1, x^2, x^3)$.

Let $\bar{g}_{\alpha\beta}$ and $\bar{g}^{\alpha\beta}$ be the components of the Euclidean metric tensor and its associated contravariant tensor respectively in the $\bar{x}^i$ coordinates. Then if we define

$$\bar{\Gamma}^i_{\alpha\beta}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \tfrac{1}{2} \bar{g}^{i\sigma}\left(\frac{\partial \bar{g}_{\sigma\beta}}{\partial \bar{x}^\alpha} + \frac{\partial \bar{g}_{\alpha\sigma}}{\partial \bar{x}^\beta} - \frac{\partial \bar{g}_{\alpha\beta}}{\partial \bar{x}^\sigma}\right), \tag{10·28}$$

we can prove by a long but straightforward calculation that *the $\Gamma^i_{\alpha\beta}(x^1, x^2, x^3)$ are related to the $\bar{\Gamma}^i_{\alpha\beta}(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ by the following famous transformation law:*[4]

$$\bar{\Gamma}^i_{\alpha\beta}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \Gamma^\lambda_{\mu\nu}(x^1, x^2, x^3)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^i}{\partial x^\lambda} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^\alpha\, \partial \bar{x}^\beta}\frac{\partial \bar{x}^i}{\partial x^\lambda}. \tag{10·29}$$

In the previous chapter we saw that

$$g_{\alpha\beta}(x^1, x^2, x^3) = \sum_{i=1}^{3} \frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^i}{\partial x^\beta}, \tag{10·30}$$

where the y's are rectangular cartesian coordinates. Consequently, if the x's are *cartesian* coordinates, it follows that all the components $g_{\alpha\beta}(x^1, x^2, x^3)$ are *constants*. In other words, $\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = 0$ in cartesian coordinates x^i. We have then immediately the important result that the *Euclidean Christoffel symbols* $\Gamma^i_{\alpha\beta}(x^1, x^2, x^3)$ *are identically zero in cartesian coordinates.*

If the y^i are cartesian coordinates and the x^i are general coordinates, one can calculate the Euclidean Christoffel symbols $\Gamma^i_{\alpha\beta}(x^1, x^2, x^3)$ directly in terms of the derivatives of the transformation functions in the transformation of coordinates

$$x^i = f^i(y^1, y^2, y^3)$$

and in the inverse transformation of coordinates

$$y^i = \phi^i(x^1, x^2, x^3).$$

Since all the Euclidean Christoffel symbols are zero when they are evaluated in cartesian coordinates y^i, it follows immediately from the transformation law 10·29 that *the Euclidean Christoffel symbols in general coordinates x^i are given by the simple formula*

$$\Gamma^i_{\alpha\beta}(x^1, x^2, x^3) = \frac{\partial^2 y^\lambda}{\partial x^\alpha\, \partial x^\beta}\frac{\partial x^i}{\partial y^\lambda}. \tag{10·31}$$

This formula is often found to be more convenient in computations than in the defining formula 10·27.

Caution: The Christoffel symbols are *not* the components of a tensor field so that i, α, and β are not tensor indices; i.e., i is not a contravariant index and α, β are not covariant indices.

The concepts of tensor fields and Euclidean Christoffel symbols can, by the obvious changes, be studied in plane geometry — two-dimensional Euclidean space. Since we have two coordinates for a point in the plane, all components of tensors and the Christoffel symbols will depend on two variables, and the range of the indices will be from 1 to 2 instead of 1 to 3. Thus the *Euclidean Christoffel symbols for the plane will be*

$$\Gamma^i_{\alpha\beta}(x^1, x^2) = \tfrac{1}{2}g^{i\sigma}(x^1, x^2)\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right) \tag{10·32}$$

in terms of the Euclidean metric tensor $g_{\alpha\beta}(x^1, x^2)$ for the plane. The alternative expression in terms of the derivatives of the transformation functions from rectangular coordinates (y^1, y^2) to general coordinates (x^1, x^2) and of the inverse transformation functions will be

$$\Gamma^i_{\alpha\beta}(x^1, x^2) = \frac{\partial^2 y^\lambda}{\partial x^\alpha\, \partial x^\beta} \frac{\partial x^i}{\partial y^\lambda}. \tag{10·33}$$

Exercise

Compute from the definition 10·32 the Euclidean Christoffel symbols for the plane in polar coordinates. Then check the results by computing them from 10·33.

Hint: Use results of exercise 1 of Chapter 9 and find $g^{11} = 1$, $g^{12} = g^{21} = 0$, $g^{22} = \dfrac{1}{(x^1)^2}$.

Answer.

$\Gamma^1_{22} = -x^1$, $\Gamma^2_{12} = \Gamma^2_{21} = \dfrac{1}{x^1}$, $\Gamma^1_{11} = 0$, $\Gamma^2_{11} = 0$, $\Gamma^1_{12} = \Gamma^1_{21} = 0$, $\Gamma^2_{22} = 0$.

CHAPTER 11

TENSOR ANALYSIS

Covariant Differentiation of Vector Fields.

Having shown the existence of the Euclidean Christoffel symbols, we are now in a position to give a complete answer to the fundamental question — enunciated in the previous chapter — on the extension of the notion of partial differentiation. We shall now prove that *the functions*

$$\frac{\partial \xi^i(x^1, x^2, x^3)}{\partial x^\alpha} + \Gamma^i_{\sigma\alpha}(x^1, x^2, x^3)\, \xi^\sigma(x^1, x^2, x^3) \tag{11·1}$$

are the components of a mixed tensor field of rank two, *called the covariant derivative of* ξ^i. This result holds for every differentiable contravariant vector field with the understanding that the $\Gamma^i_{\sigma\alpha}(x^1, x^2, x^3)$ are the Euclidean Christoffel symbols. We shall use the *notation* $\xi^i_{,\alpha}$ *for the covariant derivative of* ξ^i. By hypothesis we have

$$\bar{\xi}^i(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \xi^\lambda(x^1, x^2, x^3)\, \frac{\partial \bar{x}^i}{\partial x^\lambda}. \tag{11·2}$$

If we then differentiate corresponding sides of equation **11·2** we obtain, by the well-known rules of partial differentiation,

$$\frac{\partial \bar{\xi}^i}{\partial \bar{x}^\alpha} = \frac{\partial \xi^\lambda}{\partial x^\mu}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^i}{\partial x^\lambda} + \xi^\lambda \frac{\partial^2 \bar{x}^i}{\partial x^\lambda\, \partial x^\mu}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}. \tag{11·3}$$

We also have

$$\bar{\Gamma}^i_{j\alpha} = \Gamma^\lambda_{s\mu}\frac{\partial x^s}{\partial \bar{x}^j}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^i}{\partial x^\lambda} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^j\, \partial \bar{x}^\alpha}\frac{\partial \bar{x}^i}{\partial x^\lambda}. \tag{11·4}$$

If we multiply corresponding sides of 11·4 by $\bar{\xi}^j$ and sum on j we obtain

$$\bar{\Gamma}^i_{j\alpha}\bar{\xi}^j = \Gamma^\lambda_{s\mu}\xi^s\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^i}{\partial x^\lambda} + \xi^\mu\frac{\partial \bar{x}^j}{\partial x^\mu}\frac{\partial^2 x^\lambda}{\partial \bar{x}^j\, \partial \bar{x}^\alpha}\frac{\partial \bar{x}^i}{\partial x^\lambda} \tag{11·5}$$

on using

$$\xi^s = \bar{\xi}^j\frac{\partial x^s}{\partial \bar{x}^j}$$

in the first set of terms and

$$\bar{\xi}^j = \xi^\mu\frac{\partial \bar{x}^j}{\partial x^\mu}$$

in the second set of terms. Since λ and μ are summation indices, we can interchange λ and μ in the second derivative terms of 11·5. On carrying out the renaming of these umbral indices, we can add corresponding sides of 11.3 and 11.5 and obtain

$$(11\cdot 6)\quad \begin{cases} \dfrac{\partial \bar{\xi}^i}{\partial \bar{x}^\alpha} + \bar{\Gamma}^i_{j\alpha}\bar{\xi}^j = \left(\dfrac{\partial \xi^\lambda}{\partial x^\mu} + \Gamma^\lambda_{s\mu}\xi^s\right)\dfrac{\partial x^\mu}{\partial \bar{x}^\alpha}\dfrac{\partial \bar{x}^i}{\partial x^\lambda} \\ \qquad + \xi^\lambda\left(\dfrac{\partial^2 \bar{x}^i}{\partial x^\lambda\, \partial x^\mu}\dfrac{\partial x^\mu}{\partial \bar{x}^\alpha} + \dfrac{\partial \bar{x}^j}{\partial x^\lambda}\dfrac{\partial^2 x^\mu}{\partial \bar{x}^j\, \partial \bar{x}^\alpha}\dfrac{\partial \bar{x}^i}{\partial x^\mu}\right). \end{cases}$$

Now

$$(11\cdot 7)\qquad \frac{\partial \bar{x}^i}{\partial x^\mu}\frac{\partial x^\mu}{\partial \bar{x}^\alpha} = \delta^i_\alpha,$$

where

$$\delta^i_\alpha = \begin{cases} 1 & \text{if} \quad i = \alpha, \\ 0 & \text{if} \quad i \neq \alpha. \end{cases}$$

On differentiating 11·7 with respect to x^λ, we obtain

$$(11\cdot 8)\qquad \frac{\partial^2 \bar{x}^i}{\partial x^\lambda\, \partial x^\mu}\frac{\partial x^\mu}{\partial \bar{x}^\alpha} + \frac{\partial \bar{x}^i}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial \bar{x}^j\, \partial \bar{x}^\alpha}\frac{\partial \bar{x}^j}{\partial x^\lambda} = 0$$

and hence 11·6 reduces to

$$(11\cdot 9)\qquad \frac{\partial \bar{\xi}^i}{\partial \bar{x}^\alpha} + \bar{\Gamma}^i_{j\alpha}\bar{\xi}^j = \left(\frac{\partial \xi^\lambda}{\partial x^\mu} + \Gamma^\lambda_{s\mu}\xi^s\right)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^i}{\partial x^\lambda}.$$

But 11·9 states that the functions

$$\frac{\partial \xi^\lambda}{\partial x^\mu} + \Gamma^\lambda_{s\mu}\xi^s$$

are the components of a mixed tensor field of rank two. This completes the proof of the result stated at the beginning of this chapter.

By a slight variation of the above method of proof, it can be established that *the functions*

$$(11\cdot 10)\qquad \frac{\partial \xi_i}{\partial x^\alpha} - \Gamma^\sigma_{i\alpha}\xi_\sigma$$

are the components of a covariant tensor field of rank two whenever ξ_i *are the components of a covariant vector field, called the covariant derivative of* ξ_i. As before, $\Gamma^\sigma_{i\alpha}$ are the Euclidean Christoffel symbols. We shall use the notation $\xi_{i,\alpha}$ for the covariant derivative of ξ_i.

Tensor Fields of Rank $r = p + q$, Contravariant of Rank p and Covariant of Rank q.

It is convenient at this point to give the definition of a general tensor field. As in the case of tensor fields of rank two, the definition will be clear if we give the law of transformation of its components under a transformation of coordinates.

$$(11\cdot 11)\quad \bar{T}^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = T^{\gamma_1\cdots\gamma_p}_{\delta_1\cdots\delta_q}(x^1, x^2, x^3)\frac{\partial x^{\delta_1}}{\partial \bar{x}^{\beta_1}}\cdots\frac{\partial x^{\delta_q}}{\partial \bar{x}^{\beta_q}} \times \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{\gamma_1}}\cdots\frac{\partial \bar{x}^{\alpha_p}}{\partial x^{\gamma_p}}.$$

We are now in a position to consider some problems that arise in taking successive covariant derivatives of tensor fields. However, we must first say a word or two about the formula for the covariant derivative of a tensor field. *If* $T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}$ *are the components of a tensor field of rank* $p+q$, *contravariant of rank* p *and covariant of rank* q, *then the functions* $T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q,\gamma}$ *defined by*

$$(11\cdot 12)\quad \begin{cases} T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q,\gamma} = \dfrac{\partial T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}}{\partial x^\gamma} + \Gamma^{\alpha_1}_{\sigma\gamma}T^{\sigma\alpha_2\cdots\alpha_p}_{\beta_1\cdots\beta_q} \\ \qquad + \cdots + \Gamma^{\alpha_p}_{\sigma\gamma}T^{\alpha_1\cdots\alpha_{p-1}\sigma}_{\beta_1\cdots\beta_q} - \Gamma^{\sigma}_{\beta_1\gamma}T^{\alpha_1\cdots\alpha_p}_{\sigma\beta_2\cdots\beta_q} - \cdots - \Gamma^{\sigma}_{\beta_q\gamma}T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_{q-1}\sigma} \end{cases}$$

are the components of a tensor field of rank $p+q+1$, *contravariant of rank* p *and covariant of rank* $q+1$.

$T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q,\gamma}$ will be called the *covariant derivative of* $T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}$. The above result [1] stating that *the covariant* derivative of a tensor field is indeed a tensor field can be proved by a long but quite straightforward calculation analogous to the method of proof given for the covariant derivative of vector fields.

Since the covariant derivative of a tensor field is a tensor field, we can consider the covariant derivative of the latter tensor field, called the second covariant derivative of the original tensor field. In symbols, if $T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}$ is the original tensor field, we can consider its second successive covariant derivative

$$T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q,\gamma,\delta}.$$

The fundamental question arises whether covariant differentiation is a commutative operation, i.e., whether

$$(11\cdot 13)\quad T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q,\gamma,\delta} = T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q,\delta,\gamma}.$$

The answer is in the affirmative.[2] In fact, since all the Euclidean Christoffel symbols are zero in cartesian coordinates, the partial derivatives of all orders of the Christoffel symbols are also zero in cartesian coordinates. Hence if ${}^*T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}(y^1, y^2, y^3)$ are the components of $T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}$ in the *cartesian coordinates* y^i, we find with the obvious repeated use of formula 11·12 that

$${}^*T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q,\gamma,\delta}(y^1, y^2, y^3) = \frac{\partial^2\, {}^*T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}}{\partial y^\gamma\,\partial y^\delta}.$$

In other words, we have the result that *successive covariant derivatives of a tensor field reduce to partial derivatives of the tensor field whenever the tensor field and the operations are evaluated in cartesian coordinates.*

Properties of Tensor Fields.

Perhaps the most important property of tensor fields is the following:

If the components of a tensor field vanish identically (*or at one point, or at a set of points*) *in one coordinate system, they vanish likewise in all coordinate systems.* This result follows immediately on inspecting the law of transformation 11·11 of the components of a tensor field.

If, then, one can demonstrate that a tensor equation

$$T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = 0$$

holds good in one coordinate system, it will necessarily hold good, without further calculation, in all coordinate systems. For example, consider the covariant derivative $g_{ij,k}$ of the Euclidean metric tensor. Now, since

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{\sigma j}\Gamma^\sigma_{ik} - g_{i\sigma}\Gamma^\sigma_{jk},$$

and since the g_{ij} are constants when evaluated in cartesian coordinates, we have $g_{ij,k} = 0$ in *cartesian coordinates*, and hence the tensor equation $g_{ij,k} = 0$ holds in *all coordinates* throughout space.

Exercises

1. Prove that the covariant derivatives of g^{ij} and of δ^i_j are zero. See equation 10·26 for the definition of the tensor g^{ij}. The mixed tensor $\delta^i_j = 1$ if $i = j$ and $\delta^i_j = 0$ if $i \neq j$.

2. If T^α_β is any tensor field, then show that T^α_α is a scalar field. Similarly, if $T^\alpha_{\beta\gamma}$ is a mixed tensor field of rank three, show that $T^\alpha_{\alpha\gamma}$ is a covariant vector field.[3]

3. If the tensor field T^α_β is defined by $T^\alpha_\beta = \lambda^{\alpha\sigma}\mu_{\sigma\beta}$ in terms of the two tensor fields $\lambda^{\alpha\beta}$ and $\mu_{\alpha\beta}$, prove that the following formula (reminiscent of the differentiation of a product in the ordinary differential calculus) holds:

$$T^\alpha_{\beta,\gamma} = \lambda^{\alpha\sigma}_{,\gamma}\,\mu_{\sigma\beta} + \lambda^{\alpha\sigma}\,\mu_{\sigma\beta,\gamma}.$$

CHAPTER 12

LAPLACE EQUATION, WAVE EQUATION, AND POISSON EQUATION IN CURVILINEAR COORDINATES

Some Further Concepts and Remarks on the Tensor Calculus.

We remarked in Chapter 10 that, if $s(x^1, x^2, x^3)$ is a scalar field, then $\dfrac{\partial s}{\partial x^i}$ is a covariant vector field. So, to complete the picture of covariant differentiation, we can call $\dfrac{\partial s}{\partial x^i}$ *the covariant derivative of the scalar field* $s(x^1, x^2, x^3)$.

For some discussions it is convenient to extend the notion of a tensor field. By a *relative tensor field of weight* w, we shall mean an object with components whose transformation law differs from the transformation law of a tensor field by the appearance of the functional determinant (Jacobian) to the wth power as a factor on the right side of the equations. If $w = 0$, we have the previous notion of a tensor field. For example:

$$\bar{s}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \begin{vmatrix} \dfrac{\partial x^1}{\partial \bar{x}^1}, & \dfrac{\partial x^1}{\partial \bar{x}^2}, & \dfrac{\partial x^1}{\partial \bar{x}^3} \\ \dfrac{\partial x^2}{\partial \bar{x}^1}, & \dfrac{\partial x^2}{\partial \bar{x}^2}, & \dfrac{\partial x^2}{\partial \bar{x}^3} \\ \dfrac{\partial x^3}{\partial \bar{x}^1}, & \dfrac{\partial x^3}{\partial \bar{x}^2}, & \dfrac{\partial x^3}{\partial \bar{x}^3} \end{vmatrix}^w s(x^1, x^2, x^3)$$

and

$$\bar{\xi}^i(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \begin{vmatrix} \dfrac{\partial x^1}{\partial \bar{x}^1}, & \dfrac{\partial x^1}{\partial \bar{x}^2}, & \dfrac{\partial x^1}{\partial \bar{x}^3} \\ \dfrac{\partial x^2}{\partial \bar{x}^1}, & \dfrac{\partial x^2}{\partial \bar{x}^2}, & \dfrac{\partial x^2}{\partial \bar{x}^3} \\ \dfrac{\partial x^3}{\partial \bar{x}^1}, & \dfrac{\partial x^3}{\partial \bar{x}^2}, & \dfrac{\partial x^3}{\partial \bar{x}^3} \end{vmatrix}^w \xi^\sigma(x^1, x^2, x^3)\frac{\partial \bar{x}^i}{\partial x^\sigma}$$

are the transformation laws for a relative scalar field of weight w and a relative contravariant vector field of weight w respectively. A relative scalar field of weight *one* is called a *scalar density*, a terminology suggested by the physical example of the density of a solid or fluid. In fact, the mass m is related to the density function $\rho(x^1, x^2, x^3)$ of the solid or fluid by

$$m = \int\int\int \rho(x^1, x^2, x^3)\, dx^1\, dx^2\, dx^3,$$

where the triple integral is extended over the whole extent of the solid or fluid.

Another important example of a *scalar density is given by* $\sqrt{g}$, *where* g *is the determinant of the Euclidean metric tensor* $g_{\alpha\beta}$. In fact,

$$(12\cdot 1) \qquad \bar{g}_{\alpha\beta} = g_{ab}\frac{\partial x^a}{\partial \bar{x}^\alpha}\frac{\partial x^b}{\partial \bar{x}^\beta}.$$

Let $\bar{g} = \left| \bar{g}_{\alpha\beta} \right|$, the determinant of the $\bar{g}_{\alpha\beta}$. By a double use of the formula for the product of two determinants when applied to **12·1**, one can prove that

$$(12\cdot 2) \qquad \bar{g} = \left| \frac{\partial x^a}{\partial \bar{x}^\alpha} \right|^2 g,$$

where $\left| \frac{\partial x^a}{\partial \bar{x}^\alpha} \right|$ stands for the functional determinant of the partial derivatives $\frac{\partial x^a}{\partial \bar{x}^\alpha}$.

On taking square roots in **12·2** we obviously get

$$(12\cdot 3) \qquad \sqrt{\bar{g}} = \left| \frac{\partial x^a}{\partial \bar{x}^\alpha} \right| \sqrt{g},$$

which states that $\sqrt{g}$ is a scalar density.

The $\sqrt{g}$ enters in an essential manner in the formula for the volume enclosed by a closed surface. In fact, the formula for *the volume in general curvilinear coordinates* x^i *is given by the triple integral*

$$(12\cdot 4) \qquad V = \int\int\int \sqrt{g}\, dx^1\, dx^2\, dx^3.$$

This form for the volume can be calculated readily by the following steps. If y^i are rectangular coordinates, then

$$V = \int\int\int dy^1\, dy^2\, dy^3.$$

Hence

$$(12\cdot 5) \qquad V = \int\int\int J\, dx^1\, dx^2\, dx^3,$$

where J stands for the functional determinant

$$\left| \frac{\partial y^i}{\partial x^\alpha} \right|$$

of the transformation of coordinates from the curvilinear coordinates x^i to the rectangular coordinates y^i. Now

$$g_{\alpha\beta}(x^1, x^2, x^3) = \sum_{i=1}^{3} \frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^i}{\partial x^\beta},$$

and hence the determinant

$$g = |\ g_{\alpha\beta}\ |$$

is precisely equal to J^2 on using the rule for the multiplication of two determinants. In other words

$$\sqrt{g} = J, \tag{12·6}$$

from which the formula 12·4 for the volume becomes clear.

Since the formula for volume in general coordinates x^i is given by 12·4 it follows that the mass m of a medium in general coordinates x^i has the form

$$m = \int\int\int \rho_0(x^1, x^2, x^3)\sqrt{g}\, dx^1\, dx^2\, dx^3,$$

where $\rho_0(x^1, x^2, x^3)$ is an absolute scalar field that defines the (physical) density of the medium at each particle (x^1, x^2, x^3) of the medium. Clearly $\rho(x^1, x^2, x^3) = \rho_0(x^1, x^2, x^3)\sqrt{g}$ is a scalar density and, since $\sqrt{g} = 1$ in rectangular coordinates, has the same components as the density $\rho_0(x^1, x^2, x^3)$ of the medium in rectangular cartesian coordinates.

Other concepts and properties of tensors will be discussed later in the book whenever they are needed.

Laplace's Equation.

Let $g^{\alpha\beta}$ be the contravariant tensor field of rank two defined in Chapter 10, i.e.,

$$g^{\alpha\beta} = \frac{\text{Cofactor of } g_{\beta\alpha} \text{ in } g}{g}. \tag{12·7}$$

If $\psi(x^1, x^2, x^3)$ is a scalar field, then the second covariant derivative $\psi_{,\alpha,\beta}$ is a covariant tensor field of rank two. Now we can show that $g^{\alpha\beta}\psi_{,\alpha,\beta}$ *is a scalar field.* In fact,

$$\bar{g}^{\alpha\beta} = g^{\lambda\mu}\frac{\partial \bar{x}^\alpha}{\partial x^\lambda}\frac{\partial \bar{x}^\beta}{\partial x^\mu} \tag{12·8}$$

$$\bar{\psi}_{,\alpha,\beta} = \psi_{,\sigma,\tau}\frac{\partial x^\sigma}{\partial \bar{x}^\alpha}\frac{\partial x^\tau}{\partial \bar{x}^\beta}. \tag{12·9}$$

On multiplying corresponding sides of 12·8 and 12·9 and summing on α and β, we obtain

$$\bar{g}^{\alpha\beta}\bar{\psi}_{,\alpha,\beta} = g^{\lambda\mu}\psi_{,\sigma,\tau}\frac{\partial \bar{x}^\alpha}{\partial x^\lambda}\frac{\partial \bar{x}^\beta}{\partial x^\mu}\frac{\partial x^\sigma}{\partial \bar{x}^\alpha}\frac{\partial x^\tau}{\partial \bar{x}^\beta}$$

and hence the desired result

$$\bar{g}^{\alpha\beta}\bar{\psi}_{,\alpha,\beta} = g^{\sigma\tau}\psi_{,\sigma,\tau}$$

on using the obvious relations

$$\frac{\partial \bar{x}^\alpha}{\partial x^\lambda}\frac{\partial x^\sigma}{\partial \bar{x}^\alpha} = \delta^\sigma_\lambda. \tag{12·10}$$

Let

$$(12\cdot 11) \qquad F(x^1, x^2, x^3) = g^{\alpha\beta}(x^1, x^2, x^3)\psi_{,\alpha,\beta}(x^1, x^2, x^3)$$

for the arbitrarily chosen scalar field $\psi(x^1, x^2, x^3)$. We have just shown that $F(x^1, x^2, x^3)$ is also a scalar field. In rectangular cartesian coordinates, the Euclidean metric tensor has components $\delta_{\alpha\beta}$ equal to 0 for $\alpha \neq \beta$ and equal to 1 for $\alpha = \beta$. Furthermore, we saw in Chapter 11 that in cartesian coordinates, and hence in rectangular cartesian coordinates, y^i,

$$^*\psi_{,\alpha,\beta}(y^1, y^2, y^3) = \frac{\partial^{2*}\psi(y^1, y^2, y^3)}{\partial y^\alpha\,\partial y^\beta}.$$

The component of the scalar field $F(x^1, x^2, x^3)$ in rectangular coordinates y^i is then

$$\delta^{\alpha\beta}\frac{\partial^{2*}\psi(y^1, y^2, y^3)}{\partial y^\alpha\,\partial y^\beta}$$

or

$$(12\cdot 12) \qquad \frac{\partial^{2*}\psi(y^1, y^2, y^3)}{(\partial y^1)^2} + \frac{\partial^{2*}\psi(y^1, y^2, y^3)}{(\partial y^2)^2} + \frac{\partial^{2*}\psi(y^1, y^2, y^3)}{(\partial y^3)^2},$$

the Laplacean of the function $^*\psi(y^1, y^2, y^3)$.

Hence the form of *Laplace's equation in curvilinear coordinates* x^i *with the scalar field* $\psi(x^1, x^2, x^3)$ *as unknown is given by*

$$(12\cdot 13) \qquad g^{\alpha\beta}(x^1, x^2, x^3)\psi_{,\alpha,\beta}(x^1, x^2, x^3) = 0,$$

where $g^{\alpha\beta}(x^1, x^2, x^3)$ *is the contravariant tensor field 12.7 defined in terms of the Euclidean metric tensor* $g_{\alpha\beta}$, *and where* $\psi_{,\alpha,\beta}(x^1, x^2, x^3)$ *is the second covariant derivative of the scalar field* $\psi(x^1, x^2, x^3)$.

If we write 12.13 explicitly in terms of the Euclidean Christoffel symbols $\Gamma^i_{jk}(x^1, x^2, x^3)$, we evidently have

$$(12\cdot 14) \quad g^{\alpha\beta}(x^1, x^2, x^3)\left(\frac{\partial^2\psi(x^1, x^2, x^3)}{\partial x^\alpha\,\partial x^\beta} - \Gamma^\sigma_{\alpha\beta}(x^1, x^2, x^3)\frac{\partial\psi(x^1, x^2, x^3)}{\partial x^\sigma}\right) = 0$$

as the form of Laplace's equation † *in curvilinear coordinates* x^i.

It is worth while at this point to give an example of Laplace's equation in curvilinear coordinates and at the same time review several concepts and formulas that were studied in previous chapters.

† There is another form of Laplace's equation in curvilinear coordinates x^i which is sometimes more useful in numerical calculations than 12·14. It is given by $\dfrac{1}{\sqrt{g}}\dfrac{\partial\left(\sqrt{g}\,g^{\alpha\beta}\dfrac{\partial\psi}{\partial x^\beta}\right)}{\partial x^\alpha} = 0$. For a proof see note 3 to Chapter 13. Similar remarks could be made for the wave equation and Poisson's equation since the Laplace differential expression occurs in them.

Let y^1, y^2, y^3 be rectangular cartesian coordinates and x^1, x^2, x^3 polar spherical coordinates defined by the coordinate transformation

$$(12\cdot15)\qquad \begin{cases} y^1 = x^1 \sin x^2 \cos x^3, \\ y^2 = x^1 \sin x^2 \sin x^3, \\ y^3 = x^1 \cos x^2. \end{cases}$$

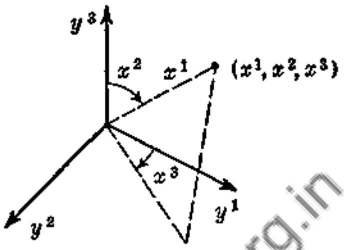


FIG. 12·1.

Clearly the inverse coordinate transformation is given by

$$x^1 = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$$

$$x^2 = \cos^{-1}\left(\frac{y^3}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}\right)$$

$$x^3 = \tan^{-1}\left(\frac{y^2}{y^1}\right).$$

From the definition of the Euclidean metric tensor $g_{\alpha\beta}$ we find

$$(12\cdot16)\qquad g_{11} = 1,\ g_{22} = (x^1)^2,\ g_{33} = (x^1)^2 (\sin x^2)^2, \text{ and all other } g_{ij} = 0,$$

so that the line element ds is given by

$$(12\cdot17)\qquad ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1)^2(\sin x^2)^2(dx^3)^2.$$

Again, from the definition of the tensor $g^{\alpha\beta}$, we find

$$(12\cdot18)\qquad g^{11} = 1,\ g^{22} = \frac{1}{(x^1)^2},\ g^{33} = \frac{1}{(x^1)^2(\sin x^2)^2}, \text{ and all other } g^{ij} = 0.$$

The Euclidean Christoffel symbols can now be computed in spherical polar coordinates; use either formula 10·27 or 10·31. They are as follows:

$$(12\cdot19)\qquad \begin{cases} \Gamma^1_{22} = -x^1,\ \Gamma^1_{33} = -x^1(\sin x^2)^2, \\ \Gamma^2_{12} = \Gamma^2_{21} = \dfrac{1}{x^1},\ \Gamma^2_{33} = -\sin x^2 \cos x^2, \\ \Gamma^3_{13} = \Gamma^3_{31} = \dfrac{1}{x^1},\ \Gamma^3_{23} = \Gamma^3_{32} = \cot x^2, \\ \text{and all other } \Gamma^i_{jk} = 0. \end{cases}$$

On using 12·18 and 12·19 in 12·14 (Γ^1_{22}, Γ^1_{33}, Γ^2_{33} are the only nonzero Christoffel symbols that are actually used), we find that *Laplace's equation in spherical polar coordinates* x^1, x^2, x^3 *is*

$$(12\cdot20)\qquad \begin{cases} \dfrac{\partial^2\psi}{(\partial x^1)^2} + \dfrac{1}{(x^1)^2}\dfrac{\partial^2\psi}{(\partial x^2)^2} + \dfrac{1}{(x^1)^2(\sin x^2)^2}\dfrac{\partial^2\psi}{(\partial x^3)^2} \\ \qquad + \dfrac{2}{x^1}\dfrac{\partial\psi}{\partial x^1} + \dfrac{\cot x^2}{(x^1)^2}\dfrac{\partial\psi}{\partial x^2} = 0 \end{cases}$$

whenever the unknown function is a scalar field $\psi(x^1, x^2, x^3)$.

Laplace's Equation for Vector Fields.

We now turn our attention to the related problem of considering *vector fields whose individual components satisfy Laplace's equation in rectangular coordinates.* The question arises whether *each component* of the vector field will satisfy Laplace's equation 12·14 in curvilinear coordinates x^1, x^2, x^3. The answer is in the negative, as a little reflection will now show. To be specific, let the unknown be a contravariant vector field $\xi^i(x^1, x^2, x^3)$ with components $^*\xi^i(y^1, y^2, y^3)$ in rectangular cartesian coordinates. By hypothesis

$$\frac{\partial^2 {}^*\xi^i}{(\partial y^1)^2} + \frac{\partial^2 {}^*\xi^i}{(\partial y^2)^2} + \frac{\partial^2 {}^*\xi^i}{(\partial y^3)^2} = 0. \tag{12·21}$$

By practically the same type of argument used in deriving equations 12·13, we find that the contravariant vector field $\xi^i(x^1, x^2, x^3)$ in curvilinear coordinates x^1, x^2, x^3 will satisfy the system of three differential equations

$$g^{\alpha\beta}\xi^i_{,\alpha,\beta} = 0, \tag{12·22}$$

where $\xi^i_{,\alpha,\beta}$ is the second covariant derivative of $\xi^i(x^1, x^2, x^3)$. If we expand 12·22 explicitly in terms of the Euclidean Christoffel symbols $\Gamma^i_{\alpha\beta}(x^1, x^2, x^3)$ we find

$$g^{\alpha\beta}\left[\frac{\partial^2\xi^i}{\partial x^\alpha\,\partial x^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial\xi^i}{\partial x^\sigma} + \Gamma^i_{\sigma\alpha}\frac{\partial\xi^\sigma}{\partial x^\beta} + \Gamma^i_{\sigma\beta}\frac{\partial\xi^\sigma}{\partial x^\alpha} + \left(\frac{\partial\Gamma^i_{\sigma\alpha}}{\partial x^\beta} + \Gamma^i_{\tau\beta}\Gamma^\tau_{\sigma\alpha} - \Gamma^i_{\sigma\tau}\Gamma^\tau_{\alpha\beta}\right)\xi^\sigma\right] = 0, \tag{12·23}$$

a system of three differential equations in which *all three unknowns* ξ^1, ξ^2, ξ^3 occur in *each* differential equation.

Wave Equation.

The propagation of various disturbances in theory of elasticity, hydrodynamics, theory of sound, and electrodynamics is governed by the partial differential equation known as the wave equation. In rectangular cartesian coordinates y^i, the wave equation is

$$\frac{\partial^2 {}^*u(y^1, y^2, y^3, t)}{\partial t^2} = \frac{\partial^2 {}^*u(y^1, y^2, y^3, t)}{(\partial y^1)^2} + \frac{\partial^2 {}^*u(y^1, y^2, y^3, t)}{(\partial y^2)^2} + \frac{\partial^2 {}^*u(y^1, y^2, y^3, t)}{(\partial y^3)^2} \tag{12·24}$$

and hence in curvilinear coordinates x^i

$$\frac{\partial^2 u(x^1, x^2, x^3, t)}{\partial t^2} = g^{\alpha\beta}u_{,\alpha,\beta}, \tag{12·25}$$

where $u_{,\alpha,\beta}$ is the second covariant derivative of the scalar field $u(x^1, x^2, x^3, t)$. If we write 12·25 explicitly in terms of the Euclidean Christoffel symbols we obtain †

$$(12 \cdot 26) \qquad \frac{\partial^2 u(x^1, x^2, x^3, t)}{\partial t^2} = g^{\alpha\beta}\left(\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial u}{\partial x^\sigma}\right).$$

It is to be observed that the right-hand side of 12·25, or equivalently of 12·26, is the Laplacean. If the x^i are spherical polar coordinates, Laplace's equation has the form 12·20. Hence, immediately, we see that the wave equation has the following form in spherical polar coordinates

$$(12 \cdot 27) \qquad \frac{\partial^2 u(x^1, x^2, x^3, t)}{\partial t^2} = \frac{\partial^2 u}{(\partial x^1)^2} + \frac{1}{(x^1)^2}\frac{\partial^2 u}{(\partial x^2)^2} + \frac{1}{(x^1)^2(\sin x^2)^2}\frac{\partial^2 u}{(\partial x^3)^2} + \frac{2}{x^1}\frac{\partial u}{\partial x^1} + \frac{\cot x^2}{(x^1)^2}\frac{\partial u}{\partial x^2}.$$

By exactly the same calculations as we used in obtaining Laplace's equation for contravariant vector fields in curvilinear coordinates, we find that the *wave equation in curvilinear coordinates x^i takes the following form whenever the unknown is a contravariant vector field $\xi^i(x^1, x^2, x^3, t)$ that depends parametrically on the time t:*

$$(12 \cdot 28) \qquad \frac{\partial^2 \xi^i(x^1, x^2, x^3, t)}{\partial t^2} = g^{\alpha\beta}\xi^i_{,\alpha,\beta},$$

or in expanded form

$$(12 \cdot 29) \qquad \frac{\partial^2 \xi^i(x^1, x^2, x^3, t)}{\partial t^2} = g^{\alpha\beta}\left[\frac{\partial^2 \xi^i}{\partial x^\alpha \partial x^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial \xi^i}{\partial x^\sigma} + \Gamma^i_{\sigma\alpha}\frac{\partial \xi^\sigma}{\partial x^\beta} + \Gamma^i_{\sigma\beta}\frac{\partial \xi^\sigma}{\partial x^\alpha} + \left(\frac{\partial \Gamma^i_{\sigma\alpha}}{\partial x^\beta} + \Gamma^i_{\tau\beta}\Gamma^\tau_{\sigma\alpha} - \Gamma^i_{\sigma\tau}\Gamma^\tau_{\alpha\beta}\right)\xi^\sigma\right]$$

on using the corresponding result 12·23 for Laplace's equation. Note that 12·29 is a system of three differential equations for the three unknowns ξ^1, ξ^2, and ξ^3 and *not* just *one* differential equation satisfied by the three functions ξ^1, ξ^2, and ξ^3.

Poisson's Equation.

As a final exercise in this chapter we consider Poisson's differential equation. In rectangular cartesian coordinates y^i, Poisson's equation is

$$(12 \cdot 30) \qquad \frac{\partial^2 {}^*\psi(y^1, y^2, y^3)}{(\partial y^1)^2} + \frac{\partial^2 {}^*\psi(y^1, y^2, y^3)}{(\partial y^2)^2} + \frac{\partial^2 {}^*\psi(y^1, y^2, y^3)}{(\partial y^3)^2} = -4\pi {}^*\sigma(y^1, y^2, y^3).$$

† For another form of the wave equation, see Exercise 1 at the end of the chapter.

On the right-hand side of 12·30, $*\sigma(y^1, y^2, y^3)$ is the component in rectangular coordinates y^i of a scalar field

$$\sigma(x^1, x^2, x^3) = \frac{\rho(x^1, x^2, x^3)}{\sqrt{g}}, \tag{12·31}$$

where $\rho(x^1, x^2, x^3)$ is a scalar density.† Since $\sqrt{g}$ is also a scalar density, $\sigma(x^1, x^2, x^3)$ is obviously a scalar field. From the definition of g as the determinant of the $g_{\alpha\beta}$, we see that in rectangular coordinates

$$*g(y^1, y^2, y^3) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1. \tag{12·32}$$

Hence

$$\sqrt{*g} = 1$$

and consequently

$$*\sigma(y^1, y^2, y^3) = *\rho(y^1, y^2, y^3). \tag{12·33}$$

In other words, the scalar field $\sigma(x^1, x^2, x^3)$ and the scalar density $\rho(x^1, x^2, x^3)$ have equal components in rectangular cartesian coordinates.

On making use of our calculations for Laplace's equation, we can derive corresponding results for Poisson's differential equation. For example, *Poisson's differential equation in curvilinear coordinates* x^i will be

$$g^{\alpha\beta}\left(\frac{\partial^2\psi(x^1, x^2, x^3)}{\partial x^\alpha\, \partial x^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial\psi}{\partial x^\sigma}\right) = -4\pi\sigma(x^1, x^2, x^3), \tag{12·34}$$

whenever the unknown is a scalar field $\psi(x^1, x^2, x^3)$. As before, the $\Gamma^\sigma_{\alpha\beta}$ are the Euclidean Christoffel symbols in the x^i coordinates.

Exercises

1. Show that the wave equation in curvilinear coordinates x^i with scalar $u(x^1, x^2, x^3, t)$ as unknown can be written as

$$\frac{\partial^2 u(x^1, x^2, x^3, t)}{\partial t^2} = \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\, g^{\alpha\beta}\frac{\partial u}{\partial x^\beta}\right)}{\partial x^\alpha}.$$

2. Show that Poisson's equation in curvilinear coordinates x^i with scalar $\psi(x^1, x^2, x^3)$ as unknown can be written as

$$\frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\, g^{\alpha\beta}\frac{\partial\psi}{\partial x^\beta}\right)}{\partial x^\alpha} = -4\pi\sigma(x^1, x^2, x^3).$$

† In most physical problems $\rho(x^1, x^2, x^3) = \rho_0(x^1, x^2, x^3)\sqrt{g}$, where $\rho_0(x^1, x^2, x^3)$ is an absolute scalar field and represents the physical density of a medium.

3. Obtain Laplace's equation 12·20 in spherical polar coordinates from the following form of Laplace's equation: $\dfrac{1}{\sqrt{g}}\dfrac{\partial\left(\sqrt{g}\,g^{\alpha\beta}\dfrac{\partial\psi}{\partial x^{\beta}}\right)}{\partial x^{\alpha}} = 0.$

4. Obtain the wave equation and Poisson equation in spherical polar coordinates on using the form of these equations given in exercises 1 and 2 respectively.

CHAPTER 13

SOME ELEMENTARY APPLICATIONS OF THE TENSOR CALCULUS TO HYDRODYNAMICS

Navier-Stokes Differential Equations for the Motion of a Viscous Fluid.

As an interesting application of the covariant derivative of a tensor field in hydrodynamics, we shall write the famous *Navier-Stokes differential equations in curvilinear coordinates.* Let y^1, y^2, and y^3 be rectangular cartesian coordinates, and let

$$
(13\cdot 1)\quad \begin{cases}
u^i = u^i(y^1,\ y^2,\ y^3,\ t), \text{ the contravariant velocity components of a viscous fluid.} \\
t = \text{time.} \\
p = p(y^1, y^2, y^3, t), \text{ pressure.} \\
\rho = \rho(y^1, y^2, y^3, t), \text{ density.} \\
\mu = \text{coefficient of viscosity, a constant.} \\
\nu = \dfrac{\mu}{\rho}, \text{ kinematic viscosity.} \\
X^i = X^i(y^1, y^2, y^3, t), \text{ contravariant vector components of body force per unit mass.}
\end{cases}
$$

Then the motion of a viscous fluid is governed by the four Navier-Stokes differential equations.[1]

$$
(13\cdot 2)\quad \begin{aligned}
\frac{\partial u^i}{\partial t} &= \nu\left(\frac{\partial^2 u^i}{(\partial y^1)^2} + \frac{\partial^2 u^i}{(\partial y^2)^2} + \frac{\partial^2 u^i}{(\partial y^3)^2}\right) - u^\alpha \frac{\partial u^i}{\partial y^\alpha} + \frac{\nu}{3}\frac{\partial}{\partial y^i}\left(\frac{\partial u^\beta}{\partial y^\beta}\right) \\
&\qquad - \frac{1}{\rho}\frac{\partial p}{\partial y^i} + X^i, \\
\frac{\partial \rho}{\partial t} &+ \frac{\partial(\rho u^\alpha)}{\partial y^\alpha} = 0.
\end{aligned}
$$

The last differential equation is the *equation of continuity,* which expresses the requirement that the *mass* of any portion of the liquid is conserved. For a *non-viscous fluid,* $\mu = 0$ and hence $\nu = 0$, the Navier-Stokes equations *reduce to the Eulerian hydrodynamical equations.*[2]

The expression within the parenthesis in 13.2 is the Laplacean of u^i. If we then make use of the results of the previous chapter on the form of the Laplacean in curvilinear coordinates, the Navier-Stokes differential equations take the following form in curvilinear coordinates x^i

$$(13\cdot3)\qquad \begin{aligned} \frac{\partial u^i}{\partial t} &= \nu g^{\alpha\beta} u^i_{,\alpha,\beta} - u^\alpha u^i_{,\alpha} + \frac{\nu}{3} g^{i\alpha} \frac{\partial}{\partial x^\alpha}(u^\beta_{,\beta}) - \frac{1}{\rho} g^{i\alpha} \frac{\partial p}{\partial x^\alpha} + X^i, \\ \frac{\partial \rho}{\partial t} &+ (\rho u^\alpha)_{,\alpha} = 0, \end{aligned}$$

where, as before, commas denote covariant differentiation based on the Euclidean Christoffel symbols $\Gamma^i_{\alpha\beta}(x^1, x^2, x^3)$. If we expand the covariant derivatives, we can write the *Navier-Stokes differential equations in curvilinear coordinates* x^i as

$$(13\cdot4)\qquad \left\{ \begin{aligned} \frac{\partial u^i}{\partial t} = \nu g^{\alpha\beta} \Bigg[& \frac{\partial^2 u^i}{\partial x^\alpha\, \partial x^\beta} + \Gamma^i_{\sigma\beta} \frac{\partial u^\sigma}{\partial x^\alpha} + \Gamma^i_{\sigma\alpha} \frac{\partial u^\sigma}{\partial x^\beta} - \Gamma^\sigma_{\alpha\beta} \frac{\partial u^i}{\partial x^\sigma} \\ & + \left(\frac{\partial \Gamma^i_{\sigma\alpha}}{\partial x^\beta} + \Gamma^i_{\tau\beta}\Gamma^\tau_{\sigma\alpha} - \Gamma^i_{\sigma\tau}\Gamma^\tau_{\alpha\beta} \right) u^\sigma \Bigg] - u^\alpha \left(\frac{\partial u^i}{\partial x^\alpha} + \Gamma^i_{\sigma\alpha} u^\sigma \right) \\ & + \frac{\nu}{3} g^{i\alpha} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial u^\beta}{\partial x^\beta} + \Gamma^\beta_{\sigma\beta} u^\sigma \right) - \frac{1}{\rho} g^{i\alpha} \frac{\partial p}{\partial x^\alpha} + X^i, \\ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u^\alpha)}{\partial x^\alpha} & + \Gamma^\alpha_{\sigma\alpha} \rho u^\sigma = 0. \end{aligned} \right.$$

It is of interest in itself as well as in the above expansions to have equivalent expressions for the "divergence" $u^\alpha_{,\alpha}$ in curvilinear coordinates. We have used the evident formula

$$(13\cdot5)\qquad u^\alpha_{,\alpha} = \frac{\partial u^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\sigma\alpha} u^\sigma,$$

but *it can also be proved that*[3]

$$(13\cdot6)\qquad u^\alpha_{,\alpha} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}\, u^\alpha)}{\partial x^\alpha}$$

where g is the determinant of the Euclidean metric tensor $g_{\alpha\beta}$. Formula 13·6 is often more useful in the calculation of the divergence than formula 13·5. With the aid of formula 13.6 we see that *the equation of continuity for a moving fluid takes the following form in curvilinear coordinates x^i:*

$$(13\cdot7)\qquad \frac{\partial \rho(x^1, x^2, x^3, t)}{\partial t} + \frac{\partial(\rho u^\alpha)}{\partial x^\alpha} + \tfrac{1}{2}\rho u^\alpha \frac{\partial \log g}{\partial x^\alpha} = 0.$$

Examples

1. We saw in the last chapter that, if the x^i are spherical polar coordinates, $g_{11} = 1$, $g_{22} = (x^1)^2$, $g_{33} = (x^1)^2 (\sin x^2)^2$, and all *other*

$g_{ij} = 0$. Hence

$$g = \begin{vmatrix} 1, & 0, & 0 \\ 0, & (x^1)^2, & 0 \\ 0, & 0, & (x^1)^2(\sin x^2)^2 \end{vmatrix} = (x^1)^4(\sin x^2)^2$$

and

$$\frac{\partial \log g}{\partial x^1} = \frac{4}{x^1}, \qquad \frac{\partial \log g}{\partial x^2} = 2 \cot x^2, \qquad \frac{\partial \log g}{\partial x^3} = 0.$$

The *equation of continuity in spherical polar coordinates* x^i *then becomes*

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u^\alpha)}{\partial x^\alpha} + \rho\left(\frac{2u^1}{x^1} + u^2 \cot x^2\right) = 0.$$

2. As another exercise, we may take the x^i to be cylindrical polar coordinates so that

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (dx^3)^2.$$

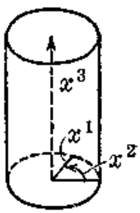


Fig. 13·1.

Evidently

$$g_{11} = 1, \quad g_{22} = (x^1)^2, \quad g_{33} = 1, \quad \text{and all } other\ g_{ij} = 0.$$

Hence the determinant $g = (x^1)^2$ and we find readily the *equation of continuity in cylindrical polar coordinates* x^i

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u^\alpha)}{\partial x^\alpha} + \rho \frac{u^1}{x^1} = 0.$$

Incidentally $g^{11} = 1$, $g^{22} = \dfrac{1}{(x^1)^2}$, $g^{33} = 1$, and all *other* $g^{ij} = 0$, so that the Euclidean Christoffel symbols can easily be computed and found to be $\Gamma^1_{22} = -x^1$, $\Gamma^2_{12} = \Gamma^2_{21} = \dfrac{1}{x^1}$, and all *other* $\Gamma^i_{jk} = 0$. These calculations for the Christoffel symbols are very much simpler than the corresponding ones in Chapter 12 for spherical polar coordinates.

Multiple-Point Tensor Fields. †

The tensor fields that have been studied so far in this book have components that are functions of the coordinates of only one variable point in space. It is possible, however, to consider generalized tensor fields, called multiple-point tensor fields,† whose components depend on the coordinates of several points in space. Perhaps the simplest example of a two-point scalar field is the *distance between two points.*

† The first systematic research on multiple-point tensor fields was initiated by the writer many years ago. See A. D. Michal, *Transactions of American Mathematical Society*, vol. 29 (1927), pp. 612–646.

Let $d(x_1, x_2)$ be the distance between two points having general coordinates (x_1^1, x_1^2, x_1^3) and (x_2^1, x_2^2, x_2^3) respectively. Each point may not necessarily be referred to the same coordinate system. For example, the x_1^i may

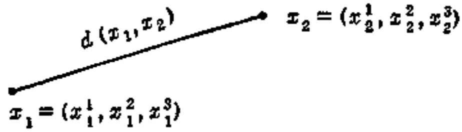


FIG. 13·2.

be spherical coordinates while the x_2^i may be cylindrical coordinates. Under transformation of coordinates

(13·8) $$\bar{x}_1^i = f_1^i(x_1^1, x_1^2, x_1^3)$$

and

(13·9) $$\bar{x}_2^i = f_2^i(x_2^1, x_2^2, x_2^3),$$

the components of distance transform by the rule

(13·10) $$\bar{d}(\bar{x}_1, \bar{x}_2) = d(x_1, x_2).$$

Of course, if both points $y_1 = (y_1^1, y_1^2, y_1^3)$ and $y_2 = (y_2^1, y_2^2, y_2^3)$ are referred to the same rectangular cartesian coordinate system, then the distance is given by the well-known formula

(13·11) $$d(y_1, y_2) = \sqrt{\sum_{i=1}^{3} (y_2^i - y_1^i)^2}.$$

Another simple example of a two-point tensor field is obtained as follows. Define

(13·12) $$s(x_1, x_2) = \tfrac{1}{2}[d(x_1, x_2)]^2,$$

where $d(x_1, x_2)$ is the above two-point distance scalar. Obviously $s(x_1, x_2)$ is also a two-point scalar field. Consider the partial derivatives of $s(x_1, x_2)$ with respect to the coordinates of the second point

(13·13) $$\frac{\partial s(x_1, x_2)}{\partial x_2^i}.$$

Now under transformations of coordinates 13·8 and 13·9 of the two points, we know that

(13·14) $$\bar{s}(\bar{x}_1, \bar{x}_2) = s(x_1, x_2).$$

Differentiating 13·14 we obtain

(13·15) $$\frac{\partial \bar{s}(\bar{x}_1, \bar{x}_2)}{\partial \bar{x}_2^i} = \frac{\partial s(x_1, x_2)}{\partial x_2^\alpha} \frac{\partial x_2^\alpha}{\partial \bar{x}_2^i}.$$

This shows that the partial derivatives 13·13 are the components of a two-point tensor field: a covariant vector field with respect to the second point and a scalar field with respect to the first point. *If both* *points are referred to the same rectangular cartesian coordinate system,* then from 13·11 we see that

$$\frac{\partial s(y_1, y_2)}{\partial y_2^i} = y_2^i - y_1^i. \tag{13·16}$$

If we define

$$s^i(x_1, x_2) = g^{i\alpha}(x_2)\frac{\partial s(x_1, x_2)}{\partial x_2^\alpha}, \tag{13·17}$$

then obviously $s^i(x_1, x_2)$ is a two-point tensor field: a contravariant vector field with respect to the second point and a scalar field with respect to the first point. If both points are referred to the same rectangular coordinate system, then both two-point tensor fields $\dfrac{\partial s(x_1, x_2)}{\partial x_2^i}$ and $s^i(x_1, x_2)$ have the same components $y_2^i - y_1^i$ in that rectangular coordinate system.

We shall have occasion to consider a two-point tensor field of rank two, a contravariant vector field with respect to each of the two points. As in the other examples, the two points need not be referred to the same coordinate system. The transformation law for the components of such a two-point tensor field are

$$\bar{t}^{ij}(\bar{x}_1, \bar{x}_2) = t^{\alpha\beta}(x_1, x_2)\frac{\partial \bar{x}_1^i}{\partial x_1^\alpha}\frac{\partial \bar{x}_2^j}{\partial x_2^\beta}. \tag{13·18}$$

A Two-Point Correlation Tensor Field in Turbulence.

Let $\xi^i(t, x^1, x^2, x^3)$ be the contravariant velocity field of a fluid in motion. We shall denote the mean value of a function $f(t)$ of the time t over the time interval (t_0, t_1) by $M[f(t)]$. For example

$$M[\xi^i(t, x^1, x^2, x^3)] = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \xi^i(t, x^1, x^2, x^3)\, dt. \tag{13·19}$$

Clearly $M[\xi^i(t, x^1, x^2, x^3)]$ is a contravariant vector field. Define a set of functions $C^{ij}(x_1, x_2)$ of two points, $x_1 = (x_1^1, x_1^2, x_1^3)$ and $x_2 = (x_2^1, x_2^2, x_2^3)$, by

(13·20)

$$C^{ij}(x_1, x_2) = \frac{M[\xi^i(t, x_1)\xi^j(t, x_2)]}{\{g_{\lambda\mu}(x_1)M[\xi^\lambda(t, x_1)\xi^\mu(t, x_1)]\}^{\frac{1}{2}}\{g_{pq}(x_2)M[\xi^p[(t, x_2)\xi^q(t, x_2)]\}^{\frac{1}{2}}}$$

where $g_{\alpha\beta}(x^1, x^2, x^3)$ is the Euclidean metric tensor in the general coordinates x^i. It is evident that $C^{ij}(x_1, x_2)$ is a *two-point tensor field of rank two, a contravariant vector field with respect to each of the two points;* we shall call it the *(two-point) correlation tensor field.* If both points are referred to the same rectangular cartesian coordinate system, then the correlation tensor field has components

$$(13\cdot21)\qquad \frac{M[u_1^i, u_2^j]}{\left\{\sum_{\alpha=1}^{3} M[(u_1^\alpha)^2]\right\}^{\frac{1}{2}} \left\{\sum_{\beta=1}^{3} M[(u_2^\beta)^2]\right\}^{\frac{1}{2}}}$$

in terms of the notations

$$u_1^i = \xi^i(t, y_1), \qquad u_2^i = \xi^i(t, y_2).$$

If we now assume that we are dealing with the special case of *isotropic turbulence*, the correlation tensor field 13·21 in rectangular coordinates simplifies still further and has components

$$\frac{1}{3}\frac{M[u_1^i u_2^j]}{M[(u)^2]}$$

on using the isotropic turbulence conditions that $M[(u_1^\alpha)^2]$ is independent of position and the index α, and equal, say, to $M[(u)^2]$.

Except for the numerical factor $\frac{1}{3}$, the above in rectangular coordinates is the correlation tensor used by Kármán in isotropic turbulence. See his paper entitled "The Fundamentals of the Statistical Theory of Turbulence," *Journal of the Aeronautical Sciences*, vol. 4 (1937), pp. 131–138.

CHAPTER 14

APPLICATIONS OF THE TENSOR CALCULUS TO ELASTICITY THEORY

Finite Deformation Theory of Elastic Media.[1]

One of the most natural and fruitful fields of application of the tensor calculus is to the deformation of media, elastic or otherwise. In the next three chapters we shall consider the fundamentals of the deformation of elastic media. We need not and shall not make the usual approximations of the classical ("infinitesimal") theory in the general development of our subject.

Consider a three-dimensional medium (a collection of point particles) in three-dimensional physical Euclidean space. We shall consider a deformation of the medium from the initial (unstrained) to its final (strained) position and obtain the strain tensor field under less stringent restrictions than those imposed in Chapter 10.

Let $({}^1a, {}^2a, {}^3a)$ be the curvilinear coordinates of a representative particle in an elastic medium, and let (x^1, x^2, x^3) be the curvilinear coordinates of the representative particle after deformation. The deformation, a one-one point transformation $A \leftrightarrow X$, will be assumed

FIG. 14·1.

given by differentiable functions

$$x^i = f^i({}^1a, {}^2a, {}^3a). \tag{14·1}$$

It is convenient, and of some importance for the mathematical foundations, to assume that the *representative unstrained particle A is represented in a coordinate system not necessarily the same as the one in which the corresponding strained particle X is represented.* For example, ${}^1a$, ${}^2a$, ${}^3a$ may be cylindrical coordinates while x^1, x^2, x^3 are spherical polar coordinates.

We shall adopt the following notational conventions with respect to one-point and two-point tensor fields. Tensor indices with respect to transformation of *coordinates of strained particles* will be written to the *right*, and tensor indices with respect to transformation of *coordinates of unstrained particles* will be written to the *left*. For example, under a simultaneous transformation of the coordinates ${}^ia$ of point A in the unstrained medium and of the coordinates x^i of point X in the strained medium,

$$(14\cdot2) \qquad {}^ra_{,s} = \frac{\partial\, {}^ra}{\partial x^s}$$

is a two-point tensor field of rank two, contravariant vector field with respect to point A and covariant vector field with respect to point X. In other words, under *transformations of coordinates*

$$(14\cdot3) \qquad \begin{cases} {}^i\bar{a} = {}^i\phi({}^1a, {}^2a, {}^3a) \\ \bar{x}^i = \psi^i(x^1, x^2, x^3) \end{cases}$$

in the unstrained and strained medium respectively, the two-point components ${}^ra_{,s}$ undergo the transformation

$$(14\cdot4) \qquad {}^r\bar{a}_{,s} = {}^\alpha a_{,\beta} \frac{\partial x^\beta}{\partial \bar{x}^s} \frac{\partial\, {}^r\bar{a}}{\partial\, {}^\alpha a}$$

Similarly

$$(14\cdot5) \qquad {}_{s,}x^r = \frac{\partial x^r}{\partial\, {}^sa}$$

is a two-point tensor field of rank two, covariant vector field with respect to point A and contravariant vector field with respect to point X. The relationships

$$(14\cdot6) \qquad ({}^ra_{,\sigma})({}_{s,}x^\sigma) = {}^r_s\delta, \; ({}_{\sigma,}x^r)({}^\sigma a_{,s}) = \delta^r_s$$

are clear, where

$$ {}^r_s\delta = \delta^r_s \begin{cases} = 0 & \text{if} \quad r \neq s, \\ = 1 & \text{if} \quad r = s. \end{cases}$$

Let the initial and final squared elements of arc length in curvilinear coordinates be given respectively by

$$(14\cdot7) \qquad \begin{cases} ds_0^2 = {}_{\alpha\beta}c(a)(d^\alpha a)(d^\beta a), \\ ds^2 = g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta. \end{cases}$$

Clearly, the initial and final squared elements in terms of the final coordinates x^i and initial coordinates ${}^ia$ respectively are

$$(14\cdot8) \qquad \begin{cases} ds_0^2 = h_{\sigma\tau} dx^\sigma\, dx^\tau, \\ ds^2 = {}_{pq}k(d^pa)(d^qa), \end{cases}$$

where

$$(14\cdot9) \qquad \begin{cases} h_{\sigma\tau} = {}_{\alpha\beta}c({}^\alpha a_{,\sigma})({}^\beta a_{,\tau}), \\ {}_{pq}k = g_{\alpha\beta}({}_{p,}x^\alpha)({}_{q,}x^\beta). \end{cases}$$

We are now in a position to write down the change produced by the deformation in the squared arc element. In fact, in terms of the coordinates x^i, we have

$$ds^2 - ds_0^2 = 2\epsilon_{\alpha\beta}(x)\,dx^\alpha\,dx^\beta, \tag{14·10}$$

where

$$\epsilon_{\alpha\beta}(x) = \tfrac{1}{2}(g_{\alpha\beta}(x) - h_{\alpha\beta}(x)) \tag{14·11}$$

Similarly, in terms of the coordinates ${}^ia$, we find

$$ds^2 - ds_0^2 = 2\,{}_{\alpha\beta}\eta(a)(d^\alpha a)(d^\beta a), \tag{14·12}$$

where

$${}_{\alpha\beta}\eta(a) = \tfrac{1}{2}({}_{\alpha\beta}k(a) - {}_{\alpha\beta}c(a)). \tag{14·13}$$

Clearly $\epsilon_{\alpha\beta}(x)$ *is a covariant tensor field of rank two in the "strained" coordinates* x^i while ${}_{\alpha\beta}\eta(a)$ is a *covariant tensor field of rank two in the "unstrained" coordinates* ${}^ia$. We shall call $\epsilon_{\alpha\beta}(x)$ the *Eulerian strain tensor* and ${}_{\alpha\beta}\eta(a)$ the *Lagrangean strain tensor. The Eulerian strain tensor will often be referred to as the strain tensor.* We have chosen this terminology in analogy with the two viewpoints in hydrodynamics represented respectively by the Eulerian and the Lagrangean differential equations of motion. As an immediate consequence of formula 14·10 we find the following fundamental result: *A necessary and sufficient condition that the elastic deformation of the medium be a rigid motion (i.e., a degenerate deformation that merely displaces the medium in space with a preservation of distances between particles) is that the Eulerian strain tensor components be zero.* Equivalently from 14·12 we have: *a necessary and sufficient condition for a rigid motion is the vanishing of all the Lagrangean strain components.* These results justify the use of the word "strain" in connection with the tensor fields $\epsilon_{\alpha\beta}(x)$ and ${}_{\alpha\beta}\eta(a)$.

Strain Tensors in Rectangular Coordinates.

If the *same rectangular cartesian coordinate system* is used for the description of both the initial and final positions of the elastic body, the Eulerian strain tensor reduces to

$$\epsilon_{\alpha\beta}(x^1, x^2, x^3) = \frac{1}{2}\left(\delta_{\alpha\beta} - \sum_{\lambda=1}^{3} \frac{\partial^\lambda a}{\partial x^\alpha}\frac{\partial^\lambda a}{\partial x^\beta}\right). \tag{14·14}$$

In terms of the usual notation (a, b, c) and (x, y, z) for the rectangular coordinates in the same coordinate system of the representative initial and final particles respectively, we have

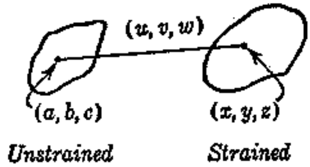


FIG. 14·2.

$$
(14\cdot 15)\quad
\begin{cases}
\epsilon_{xx} = \dfrac{1}{2}\left[1 - \left(\dfrac{\partial a}{\partial x}\right)^2 - \left(\dfrac{\partial b}{\partial x}\right)^2 - \left(\dfrac{\partial c}{\partial x}\right)^2\right], \\[2ex]
\epsilon_{xy} = \dfrac{1}{2}\left[-\dfrac{\partial a}{\partial x}\dfrac{\partial a}{\partial y} - \dfrac{\partial b}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial c}{\partial x}\dfrac{\partial c}{\partial y}\right] = \epsilon_{yx}, \\[2ex]
\epsilon_{xz} = \dfrac{1}{2}\left[-\dfrac{\partial a}{\partial x}\dfrac{\partial a}{\partial z} - \dfrac{\partial b}{\partial x}\dfrac{\partial b}{\partial z} - \dfrac{\partial c}{\partial x}\dfrac{\partial c}{\partial z}\right] = \epsilon_{zx}, \\[2ex]
\epsilon_{yy} = \dfrac{1}{2}\left[1 - \left(\dfrac{\partial a}{\partial y}\right)^2 - \left(\dfrac{\partial b}{\partial y}\right)^2 - \left(\dfrac{\partial c}{\partial y}\right)^2\right], \\[2ex]
\epsilon_{yz} = \dfrac{1}{2}\left[-\dfrac{\partial a}{\partial y}\dfrac{\partial a}{\partial z} - \dfrac{\partial b}{\partial y}\dfrac{\partial b}{\partial z} - \dfrac{\partial c}{\partial y}\dfrac{\partial c}{\partial z}\right] = \epsilon_{zy}, \\[2ex]
\epsilon_{zz} = \dfrac{1}{2}\left[1 - \left(\dfrac{\partial a}{\partial z}\right)^2 - \left(\dfrac{\partial b}{\partial z}\right)^2 - \left(\dfrac{\partial c}{\partial z}\right)^2\right].
\end{cases}
$$

Similarly the Lagrangean strain tensor reduces to

$$
(14\cdot 16)\qquad {}_{\alpha\beta}\eta = \frac{1}{2}\left(\sum_{\lambda=1}^{3} \frac{\partial x^\lambda}{\partial^\alpha a}\frac{\partial x^\lambda}{\partial^\beta a} - {}_{\alpha\beta}\delta\right).
$$

On denoting the "displacement vector" (really a *two-point* tensor field that was discussed in the previous chapter) $(x - a,\ y - b,\ z - c)$ by (u, v, w) we can rewrite 14·15 in the form

$$
(14\cdot 17)\quad
\begin{cases}
\epsilon_{xx} = \dfrac{\partial u}{\partial x} - \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial x}\right)^2\right], \\[2ex]
\epsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) - \dfrac{1}{2}\left[\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\right] = \epsilon_{yx}, \\[2ex]
\epsilon_{xz} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) - \dfrac{1}{2}\left[\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial z}\right] = \epsilon_{zx}, \\[2ex]
\epsilon_{yy} = \dfrac{\partial v}{\partial y} - \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right], \\[2ex]
\epsilon_{yz} = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) - \dfrac{1}{2}\left[\dfrac{\partial u}{\partial y}\dfrac{\partial u}{\partial z} + \dfrac{\partial v}{\partial y}\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z}\right] = \epsilon_{zy}, \\[2ex]
\epsilon_{zz} = \dfrac{\partial w}{\partial z} - \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2 + \left(\dfrac{\partial w}{\partial z}\right)^2\right].
\end{cases}
$$

In the classical theory (the usual approximate theory) of elastic deformations, the squares and products of the partial derivatives of u, v, and w are considered negligible. Hence, to the degree of approximation considered in the classical theory, 14·17 *yields the following well-known formulas for the strain tensor:*

$$(14\cdot18)\qquad \begin{cases} \epsilon_{xx} = \dfrac{\partial u}{\partial x}, & \epsilon_{xy} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) = \epsilon_{yx}, \\ \epsilon_{xz} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) = \epsilon_{zx}, & \epsilon_{yy} = \dfrac{\partial v}{\partial y}, \\ \epsilon_{yz} = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) = \epsilon_{zy}, & \epsilon_{zz} = \dfrac{\partial w}{\partial z}. \end{cases}$$

Change in Volume under Elastic Deformation.

Let us now return to our deformation theory without making the approximations of the classical theory. One of the first fundamental questions that arises is the manner in which volumes behave under a deformation of the medium.

The element of volume in the unstrained medium is, by 12.4,

$$(14\cdot19)\qquad dV_0 = \sqrt{c}\, d^1a\, d^2a\, d^3a,$$

where $c = |\ {}_{\alpha\beta}c\ |$, the determinant of the ${}_{\alpha\beta}c$. Similarly,

Fig. 14·3.

$$(14\cdot20)\qquad dV = \sqrt{g}\, dx^1\, dx^2\, dx^3,$$

where $g = |\ g_{\alpha\beta}\ |$, the determinant of the $g_{\alpha\beta}$.

In terms of the "strained variables" x^i, we have

$$(14\cdot21)\qquad dV_0 = \sqrt{c}\ |\ {}^{\alpha}a_{,i}\ |\ dx^1\, dx^2\, dx^3,$$

where $|\ {}^{\alpha}a_{,i}\ |$ is the determinant of the $\dfrac{\partial^{\alpha}a}{\partial x^i}$.

Now

$$ds_0^2 = {}_{\lambda\mu}c(a)\, d^{\lambda}a\, d^{\mu}a = h_{\alpha\beta}(x)\, dx^{\alpha}\, dx^{\beta},$$

where $h_{\alpha\beta}$ is given by formula 14·9.

Evidently the determinant h of the $h_{\alpha\beta}$ is given by

$$h = c\ |\ {}^{\alpha}a_{,i}\ |^2.$$

From this

$$(14\cdot22)\qquad \sqrt{h} = \sqrt{c}\ |\ {}^{\alpha}a_{,i}\ |$$

and

$$(14\cdot23)\qquad dV_0 = \sqrt{h}\, dx^1\, dx^2\, dx^3.$$

Formulas 14·20 and 14·23 imply that

$$(14\cdot24)\qquad \frac{dV_0}{dV} = \sqrt{\frac{h}{g}}.$$

Define

(14·25) $$h^\alpha_\beta = g^{\alpha\sigma} h_{\sigma\beta}$$

and thus obtain

(14·26) $$h_{\alpha\beta} = g_{\alpha\sigma} h^\sigma_\beta$$

by an application of the properties of $g_{\alpha\beta}$ and $g^{\alpha\beta}$; see formulas 10·26. From 14·26 we compute

(14·27) $$h = g \mid h^\alpha_\beta \mid,$$

where $\mid h^\alpha_\beta \mid$ is the determinant of the h^α_β. On using 14·24, we have immediately

(14·28) $$\frac{dV_0}{dV} = \sqrt{\mid h^\alpha_\beta \mid}.$$

To express this ratio in terms of the strain tensor $\epsilon_{\alpha\beta}$, we first recall the definition 14·11 and obtain

$$h_{\alpha\beta}(x) = g_{\alpha\beta}(x) - 2\epsilon_{\alpha\beta}(x).$$

On raising the indices with the aid of the $g^{\alpha\beta}$, we evidently have

(14·29) $$h^\alpha_\beta = \delta^\alpha_\beta - 2\epsilon^\alpha_\beta,$$

where the mixed tensor field ϵ^α_β of rank two is defined by

(14·30) $$\epsilon^\alpha_\beta(x) = g^{\alpha\sigma}(x)\epsilon_{\sigma\beta}(x).$$

On using 14·29 in 14·28, we arrive at the important result that the ratio of the *element of volume of a set of particles in the unstrained medium to the element of volume of the corresponding particles in the strained position is given in terms of the strain tensor* $\epsilon_{\alpha\beta}(x)$ *by means of the following formula*

(14·31) $$\frac{dV_0}{dV} = \sqrt{\mid \delta^\alpha_\beta - 2\epsilon^\alpha_\beta(x) \mid}.$$

Notice that, with rigid motion of the medium, the strain tensor $\epsilon_{\alpha\beta} = 0$ and hence $\epsilon^\alpha_\beta = 0$. Hence, from 14·31 and $\epsilon^\alpha_\beta = 0$, we see that a *rigid motion preserves volumes.*

We have developed the fundamentals of elastic deformation and strain tensor in three-dimensional space. It is clear, however, that everything we have said can be taken over for the corresponding elastic deformations in the plane. In all the formulas, there will be two variables x^1, x^2, etc., the indices will have the range 1 to 2, and the consequent summations will go from 1 to 2. For example, formulas 14·14, 14·15, 14·17, and 14·18 for the Eulerian strain tensor will be respectively as follows in elasticity theory in the plane.

$$\epsilon_{\alpha\beta}(x^1, x^2) = \frac{1}{2}\left(\delta_{\alpha\beta} - \sum_{\lambda=1}^{2} \frac{\partial^\lambda a}{\partial x^\alpha}\frac{\partial^\lambda a}{\partial x^\beta}\right). \tag{14·32}$$

$$\left\{\begin{aligned} \epsilon_{xx} &= \frac{1}{2}\left[1 - \left(\frac{\partial a}{\partial x}\right)^2 - \left(\frac{\partial b}{\partial x}\right)^2\right], \\ \epsilon_{xy} &= \frac{1}{2}\left[-\frac{\partial a}{\partial x}\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial b}{\partial y}\right] = \epsilon_{yx}, \\ \epsilon_{yy} &= \frac{1}{2}\left[1 - \left(\frac{\partial a}{\partial y}\right)^2 - \left(\frac{\partial b}{\partial y}\right)^2\right]; \end{aligned}\right. \tag{14·33}$$

$$\left\{\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right], \\ \epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{1}{2}\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right] = \epsilon_{yx}, \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right], \end{aligned}\right. \tag{14·34}$$

where $u = x - a$ and $v = y - b$.

$$\left\{\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x}, \\ \epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \epsilon_{yx}, \\ \epsilon_{yy} &= \frac{\partial v}{\partial y}. \end{aligned}\right. \tag{14·35}$$

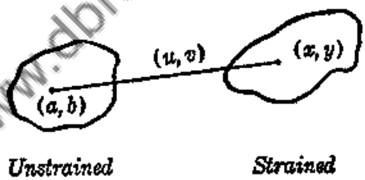


FIG. 14·4.

Exercises

1. Find the components of the Eulerian strain tensor $\epsilon_{\alpha\beta}(x^1, x^2, x^3)$ when the deformation of the elastic body is a stretching whose equations are

$$x^i = A\,{}^ia \qquad (A \text{ is a constant } \textit{greater} \text{ than unity})$$

where both x^i and ${}^ia$ are rectangular coordinates referred to the same rectangular coordinate system. Discuss the change in volume elements. Work out the corresponding problem in plane elasticity.

2. Work out exercise 1 for a contraction so that the constant A is *less* than unity.

CHAPTER 15

HOMOGENEOUS AND ISOTROPIC STRAINS, STRAIN INVARIANTS, AND VARIATION OF STRAIN TENSOR

Strain Invariants.

If we expand the three-rowed determinant $| h^\alpha_\beta |$, we find

(15·1) $$| \delta^\alpha_\beta - 2\epsilon^\alpha_\beta | = 1 - 2I_1 + 4I_2 - 8I_3,$$

where

$I_1 = \epsilon^\alpha_\alpha$, I_2 = sum of the principal two-rowed minors in the determinant

(15·2) $$\Delta = | \epsilon^\alpha_\beta |, \quad \text{and} \quad I_3 = \Delta.$$

A function $f(g_{11}, g_{12}, \cdots, g_{33}, \epsilon_{11}, \epsilon_{12}, \cdots, \epsilon_{33})$ *of the Euclidean metric tensor* $g_{\alpha\beta}$ and the strain tensor $\epsilon_{\alpha\beta}$ *will be called a strain invariant*[1] *if* (*a*) *it is a scalar field;* (*b*) *under all transformations of coordinates* x^i *to* $\bar{x}^i$

(15·3) $$f(\bar{g}_{11}, \bar{g}_{12}, \cdots, \bar{g}_{33}, \bar{\epsilon}_{11}, \bar{\epsilon}_{12}, \cdots, \bar{\epsilon}_{33}) = f(g_{11}, g_{12}, \cdots, g_{33}, \epsilon_{11}, \epsilon_{12}, \cdots, \epsilon_{33}),$$

where the function f, on the left, is the same function of the $\bar{g}_{\alpha\beta}$ *and* $\bar{\epsilon}_{\alpha\beta}$ *as it is, on the right, of the* $g_{\alpha\beta}$ *and* $\epsilon_{\alpha\beta}$.

We shall now prove that *the three functions* I_1, I_2, *and* I_3 *occurring in the expansion of the determinant* 15·1 *are strain invariants.* From the law of transformation of the mixed tensor field ϵ^α_β we readily get

$$\bar{\epsilon}^\alpha_\alpha(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \epsilon^\alpha_\alpha(x^1, x^2, x^3),$$

from which follows that I_1 is a strain invariant on recalling that $\epsilon^\alpha_\beta = g^{\alpha\sigma}\epsilon_{\sigma\beta}$. To prove that I_3 is a strain invariant, we have by hypothesis

$$\bar{\epsilon}^\alpha_\beta(\bar{x}) = \epsilon^\lambda_\mu(x) \frac{\partial x^\mu}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\alpha}{\partial x^\lambda}.$$

On taking the determinant of corresponding sides, we obtain

$$\left| \bar{\epsilon}^\alpha_\beta \right| = \left| \epsilon^\lambda_\mu \right| \cdot \left| \frac{\partial x^r}{\partial \bar{x}^s} \right| \cdot \left| \frac{\partial \bar{x}^p}{\partial x^q} \right|.$$

But the product of the functional determinants is equal to unity. Hence $| \bar{\epsilon}^\alpha_\beta | = | \epsilon^\lambda_\mu |$, and I_3 is a strain invariant. To prove that I_2 is a strain invariant, we first observe that

$$\delta^\alpha_\beta - 2\epsilon^\alpha_\beta$$

is a mixed tensor field of rank two. Hence, by the argument just completed for $\left| \epsilon^\alpha_\beta \right|$, we see that *the determinant* $\left| \delta^\alpha_\beta - 2\epsilon^\alpha_\beta \right|$ *is itself a strain invariant.* But, from the expansion 15·1, we see that

$$(15 \cdot 4) \qquad I_2 = \tfrac{1}{4}\left[\left| \delta^\alpha_\beta - 2\epsilon^\alpha_\beta \right| - 1 + 2I_1 + 8I_3\right].$$

Formula 15·4 expresses I_2 as a linear combination of strain invariants with numerical multipliers. Hence obviously I_2 is itself a strain invariant.

On using the results 14·31 together with what we have just proved, we obtain the result

$$(15 \cdot 5) \qquad \frac{dV_0}{dV} = \sqrt{1 - 2I_1 + 4I_2 - 8I_3},$$

which gives the ratio of the element of the volume of a set of particles in the unstrained medium to the element of volume of the corresponding particles in the strained medium in terms of the three strain invariants I_1, I_2, *and* I_3.

Homogeneous and Isotropic Strains.

Let us now discuss the mathematical description of a homogeneous strain.

Definition of Homogeneous Strain. A strain is homogeneous if the corresponding strain tensor $\epsilon_{\alpha\beta}$ has a zero covariant derivative, i.e., $\epsilon_{\alpha\beta,\,\gamma} = 0$.

In rectangular coordinates, the condition reduces to $\dfrac{\partial \epsilon_{\alpha\beta}}{\partial x^\gamma} = 0$ since all the Euclidean Christoffel symbols are identically zero in rectangular coordinates. In other words, the *strain tensor components* $\epsilon_{\alpha\beta}$ *in rectangular coordinates are constants for a homogeneous strain.*

It readily follows from the definition of the strain invariants I_1, I_2, and I_3 that for a *homogeneous* strain

$$\frac{\partial I_i}{\partial x^j} = 0$$

in rectangular coordinates and hence in all coordinates. (Keep in mind that I_1, I_2, and I_3 are three scalars and *not* the three components of a covariant vector.) Hence, for a homogeneous strain, $\dfrac{dV_0}{dV}$ is a numerical constant for all coordinates. We thus arrive at the important result that *for a homogeneous strain*

$$(15 \cdot 6) \qquad \frac{V_0}{V} = \sqrt{1 - 2I_1 + 4I_2 - 8I_3} = \text{a constant}.$$

This constant is the same for all "unstrained" volumes V_0 *and their cor-*

responding "strained" volumes V, and is independent of the coordinate system.

For the special case of a *homogeneous strain* which is *also isotropic at each point*, we shall have

$$\epsilon^{\alpha}_{\beta} = \epsilon\delta^{\alpha}_{\beta} \tag{15·7}$$

or what amounts to the same thing

$$\epsilon_{\alpha\beta} = \epsilon g_{\alpha\beta}. \qquad (g_{\alpha\beta}, \text{ the Euclidean metric tensor.}) \tag{15·8}$$

Since the strain is homogeneous, we have $\epsilon_{\alpha\beta,\,\gamma} = 0$. We also have (see the end of Chapter 11)

$$g_{\alpha\beta,\,\gamma} = 0.$$

Hence ϵ in 15·7 and 15·8 *is a constant* scalar field, a numerical constant that is independent of position and the coordinate system.

An example of an isotropic homogeneous strain is found in an isotropic medium subjected to uniform hydrostatic pressure, i.e., an isotropic medium subjected to the same pressure in all directions.

Since 15·6 holds and

$$1 - 2I_1 + 4I_2 - 8I_3 = \left| \delta^{\alpha}_{\beta} - 2\epsilon^{\alpha}_{\beta} \right|,$$

we see by an evident calculation that for an *isotropic homogeneous strain*

$$\frac{V_0}{V} = (1 - 2\epsilon)^{3/2}. \tag{15·9}$$

The constant scalar ϵ for an isotropic homogeneous strain is given by the formula

$$\epsilon = \frac{1}{2}\left[1 - \left(\frac{V_0}{V}\right)^{2/3}\right] \tag{15·10}$$

in terms of any one volume before and after deformation.

In the *usual approximate theory* (usual theory of elasticity) higher powers of ϵ than the first are neglected; see Chapter 14. Since

$$(1 - 2\epsilon)^{3/2} = 1 - 3\epsilon, \text{ approximately,}$$

we have

$$\frac{V_0}{V} = 1 - 3\epsilon, \text{ approximately,} \tag{15·11}$$

and

$$\epsilon = \frac{1}{3}\frac{V - V_0}{V} = \frac{1}{3}\frac{\Delta V}{V} = \frac{1}{3}\frac{\Delta V}{V_0}, \text{ approximately.} \tag{15·12}$$

A Fundamental Theorem on Homogeneous Strains.

We shall now outline the proof of the following theorem. *A necessary and sufficient condition that a strain be homogeneous is that, in terms of* *unstrained cartesian coordinates* ${}^{\alpha}z$ *and strained cartesian coordinates* y^i, *the deformation is linear, i.e.,*

(15·13) $$ {}^{\alpha}z = {}^{\alpha}A_i\, y^i + {}^{\alpha}A. $$

Outline of Proof.

From the definition of the strain tensor $\epsilon_{\alpha\beta}$, we have

$$ h_{pq} = g_{pq} - 2\epsilon_{pq}. $$

Since $g_{pq,r} = 0$, and since under our "necessity hypothesis" $\epsilon_{pq,r} = 0$, we obtain the vanishing of the covariant derivative of $h_{pq}\ (x)$. But by definition

$$ h_{pq}\ (x) = ({}_{\alpha\beta}c)({}^{\alpha}a_{,p})({}^{\beta}a_{,q}) $$

so that the x^i are the independent variables and the ${}^{\alpha}a$ are the dependent variables. Expanding the covariant derivative in $h_{pq,r}\ (x) = 0$, and rearranging, we find

(15·14) $$ {}_{\alpha\beta}c\ {}^{\alpha}a_{,pr}\ {}^{\beta}a_{,q} = -\frac{\partial_{\alpha\beta}c}{\partial x^r}\ {}^{\alpha}a_{,p}\ {}^{\beta}a_{,q} - {}_{\alpha\beta}c\ {}^{\alpha}a_{,p}\ {}^{\beta}a_{,qr}, $$

where

(15·15) $$ {}^{\alpha}a_{,pr} = \frac{\partial\, {}^{\alpha}a_{,p}}{\partial x^r} - \Gamma^{\sigma}_{pr}\ {}^{\alpha}a_{,\sigma}. $$

Since

(15·16) $$ {}^{\alpha}a_{,pr} = {}^{\alpha}a_{,rp}, $$

the right side of 15·14 must be symmetric in p and r. Hence

(15·17) $$ {}_{\alpha\beta}c\ {}^{\alpha}a_{,pr}\ {}^{\beta}a_{,q} = -\frac{\partial_{\alpha\beta}c}{\partial x^p}\ {}^{\alpha}a_{,r}\ {}^{\beta}a_{,q} - {}_{\alpha\beta}c\ {}^{\alpha}a_{,r}\ {}^{\beta}a_{,qp}. $$

On equating the corresponding sides of 15·14 and 15·17, rearranging, and interchanging p and q and β and α, we obtain

(15·18) $$ {}_{\alpha\beta}c\ {}^{\alpha}a_{,pr}\ {}^{\beta}a_{,q} = -\frac{\partial_{\alpha\beta}c}{\partial x^r}\ {}^{\alpha}a_{,p}\ {}^{\beta}a_{,q} + \frac{\partial_{\alpha\beta}c}{\partial x^q}\ {}^{\alpha}a_{,r}\ {}^{\beta}a_{,p} + {}_{\alpha\beta}c\ {}^{\alpha}a_{,r}\ {}^{\beta}a_{,qp}. $$

Adding corresponding sides of 15·17 and 15·18 there results

(15·19) $$ 2\,{}_{\alpha\beta}c\ {}^{\alpha}a_{,pr}\ {}^{\beta}a_{,q} = -\frac{\partial_{\alpha\beta}c}{\partial x^p}\ {}^{\alpha}a_{,r}\ {}^{\beta}a_{,q} - \frac{\partial_{\alpha\beta}c}{\partial x^r}\ {}^{\alpha}a_{,p}\ {}^{\beta}a_{,q} + \frac{\partial_{\alpha\beta}c}{\partial x^q}\ {}^{\alpha}a_{,r}\ {}^{\beta}a_{,p}. $$

Recalling that the ${}_{\alpha\beta}c$ are functions of the unstrained coordinates ${}^{i}a$, we find

(15·20) $$ {}_{\alpha\beta}c\ {}^{\alpha}a_{,pr}\ {}^{\beta}a_{,q} = -\frac{1}{2}\left[\frac{\partial_{\alpha\beta}c}{\partial^{\gamma}a} + \frac{\partial_{\gamma\beta}c}{\partial^{\alpha}a} - \frac{\partial_{\alpha\gamma}c}{\partial^{\beta}a}\right]{}^{\alpha}a_{,r}\ {}^{\beta}a_{,q}\ {}^{\gamma}a_{,p}. $$

Multiplying corresponding sides by ${}_{\pi,}x^q$ and summing on q, we obtain

(15·21) $$_{\alpha\pi}c\ {}^{\alpha}a_{,pr} = -\frac{1}{2}\left[\frac{\partial_{\alpha\pi}c}{\partial^{\gamma}a} + \frac{\partial_{\gamma\pi}c}{\partial^{\alpha}a} - \frac{\partial_{\alpha\gamma}c}{\partial^{\pi}a}\right]{}^{\alpha}a_{,r}\ {}^{\gamma}a_{,p}.$$

Finally, if we multiply corresponding sides of 15·21 by $c^{\pi\sigma}$, we arrive readily at the interesting result

(15·22) $${}^{\sigma}a_{,pr} = -{}_{\gamma\alpha}^{\sigma}\Gamma(a)\ {}^{\gamma}a_{,p}\ {}^{\alpha}a_{,r},$$

where ${}_{\gamma\alpha}^{\sigma}\Gamma(a)$ *are the Euclidean Christoffel symbols based on the Euclidean metric tensor* ${}_{\alpha\beta}c(a)$ *in the unstrained coordinates* ${}^{i}a$.

Now from the definition 15·15 of ${}^{\alpha}a_{,pr}$ and from the vanishing of the Euclidean Christoffel symbols $\Gamma^{i}_{jk}(x)$ and ${}^{i}_{jk}\Gamma(a)$ when evaluated in "strained" cartesian coordinates y^i and "unstrained" cartesian coordinates ${}^{i}z$ respectively, we see that 15·22 reduces to

(15·23) $$\frac{\partial^2\, {}^{\sigma}z}{\partial y^p\, \partial y^r} = 0.$$

This implies that the deformation is linear, i.e., of type 15·13.

To prove the converse part of the fundamental theorem on homogeneous strains, we have by hypothesis that the deformation, or strain, is given by a linear transformation 15·13 in cartesian coordinates ${}^{i}z$ and y^i. Now

$$h_{pq}(x) = {}_{\alpha\beta}c(a)({}^{\alpha}a_{,p})({}^{\beta}a_{,q})$$

and ${}_{\alpha\beta}c$ are constants ${}_{\alpha\beta}d$ in cartesian coordinates. From 15·13 we have

$$\frac{\partial^{\alpha}z}{\partial y^p} = {}^{\alpha}A_p$$

and hence the components ${}^{*}h_{pq}(y)$ in cartesian coordinates y^i are given by

$${}^{*}h_{pq}(y) = {}_{\alpha\beta}d\ {}^{\alpha}A_p\ {}^{\beta}A_q,$$

a set of constants. Hence

$$\frac{\partial\, {}^{*}h_{pq}(y)}{\partial y^r} = 0.$$

But this condition implies that the covariant derivative

$$h_{pq,r}\ (x) = 0$$

and hence the covariant derivative $\epsilon_{pq,r}\ (x) = 0$. In other words, the strain is homogeneous, and the proof of the theorem is complete.

Variation of the Strain Tensor.

In preparation for the subject matter of the next chapter, we need to consider deformations that depend on an accessory parameter t, which in dynamical problems can be taken as the time t. So let the coordinates x^i of a representative particle in the strained medium depend on the coordinates ${}^ia$ of the corresponding particle in the unstrained medium and on the accessory parameter t. Let

(15·24) $$\mathrm{D}x^i = \frac{\partial x^i({}^1a, {}^2a, {}^3a, t)}{\partial t}\, dt$$

be the partial differential of x^i in t. If $f^{\cdots}_{\cdots}(x)$ is any tensor field in the strained medium, *define* $\delta f^{\cdots}_{\cdots}(x)$ by

(15·25) $$\delta f^{\cdots}_{\cdots}(x) = f^{\cdots}_{\cdots,i}\, \mathrm{D}x^i,$$

where $f^{\cdots}_{\cdots,i}$ is the covariant derivative of $f^{\cdots}_{\cdots}$. Clearly, if the x^i are cartesian, $\delta f^{\cdots}_{\cdots} = \mathrm{D}f^{\cdots}_{\cdots}$. Moreover, $\delta f(x) = \mathrm{D}f(x)$ for a scalar $f(x)$ in *general* coordinates x^i. To have a well-rounded notation, define $\delta x^r = \mathrm{D}x^r$ and *refer to* δx^r *as the virtual displacement vector*. If $x^i({}^1a, {}^2a, {}^3a, t)$ have continuous second derivatives, then from the commutativity of second derivatives

$$\mathrm{D}({}_{s,}x^k) = \frac{\partial(\delta x^k)}{\partial^s a} = \frac{\partial(\delta x^k)}{\partial x^\alpha}\cdot({}_{s,}x^\alpha).$$

Hence

$$\mathrm{D}(dx^k) = \frac{\partial}{\partial x^\alpha}(\delta x^k)\cdot dx^\alpha,$$

which implies the tensor equation

(15·26) $$\delta(dx^k) = (\delta x^k)_{,\alpha}\, dx^\alpha.$$

Obviously $\delta g_{rs} = 0$, since $g_{rs,t} = 0$. Hence the above tensor equation may be written

(15·27) $$\delta(dx_k) = (\delta x_k)_{,\alpha}\, dx^\alpha.$$

Since $\delta d({}^ra) = 0$, an evident calculation using 15·26 shows that

(15·28) $$\delta({}^ra_{,\beta}) = -{}^ra_{,\alpha}(\delta x^\alpha)_{,\beta}.$$

Recalling that

$$h_{pq}(x) = {}_{\alpha\beta}c(a)({}^\alpha a_{,p})({}^\beta a_{,q})$$

and applying formula 15·28 we find

$$\delta h_{pq} = -{}_{\alpha\beta}c[{}^\alpha a_{,r}(\delta x^r)_{,p}{}^\beta a_{,q} + {}^\alpha a_{,p}\,{}^\beta a_{,r}(\delta x^r)_{,q}],$$

which can be put in the convenient form

(15·29) $$\delta h_{pq} = -h^r_q(\delta x_r)_{,p} - h^r_p(\delta x_r)_{,q}.$$

From this, and from the definition of the strain tensor $\epsilon_{\alpha\beta}$ and the related formula

$$h^p_q = \delta^p_q - 2\epsilon^p_q,$$

we arrive at the *fundamental formula for the variation of the strain tensor*.

$$(15\cdot 30)\quad \delta\epsilon_{pq}(x) = \tfrac{1}{2}[(\delta x_q)_{,p} + (\delta x_p)_{,q}] - [\epsilon_q^r(\delta x_r)_{,p} + \epsilon_p^r(\delta x_r)_{,q}].$$

This formula becomes

$$(15\cdot 31)\qquad \delta\epsilon_{pq}(x) = \tfrac{1}{2}[(\delta x_q)_{,p} + (\delta x_p)_{,q}]$$

within the *approximations of the usual approximate theory of elasticity*.

Returning to our finite deformation theory, we define a *rigid virtual displacement* by the condition $\delta(ds^2) = 0$. On using formulas 15·26 and 15·27 in an evident calculation, we find

$$(15\cdot 32)\qquad \delta(ds^2) = [(\delta x_\alpha)_{,\beta} + (\delta x_\beta)_{,\alpha}]\, dx^\alpha\, dx^\beta$$

for any virtual displacement, rigid or not. Hence the virtual displacement vector must satisfy *Killing's differential equations* for a *rigid* virtual displacement

$$(15\cdot 33)\qquad (\delta x_\alpha)_{,\beta} + (\delta x_\beta)_{,\alpha} = 0.$$

For the sake of completeness, we shall write down the formula (without giving the derivation) for the variation of the *Lagrangean* strain tensor ${}_{pq}\eta$ under an arbitrary virtual displacement.

$$\delta_{pq}\eta = \tfrac{1}{2}[(\delta x_\alpha)_{,\beta} + (\delta x_\beta)_{,\alpha}]\; {}_{p,}x^\alpha\; {}_{q,}x^\beta.$$

Exercise

Calculate the three fundamental strain invariants for a homogeneous isotropic strain. *Hint*: since they are constants, calculate them in rectangular cartesian coordinates.

CHAPTER 16

STRESS TENSOR, ELASTIC POTENTIAL, AND STRESS-STRAIN RELATIONS

Stress Tensor.

Let S be the bounding surface of a portion of the elastic medium in its strained position. The surface element of S may be described by means of the covariant vector dS_r,

(16·1) $dS_1 = \sqrt{g}\, d(x^2, x^3),\ dS_2 = \sqrt{g}\, d(x^3, x^1),\ dS_3 = \sqrt{g}\, d(x^1, x^2),$

where

$$(16\cdot2)\qquad d(x^p, x^q) = \begin{vmatrix} \dfrac{\partial x^p}{\partial u}, & \dfrac{\partial x^p}{\partial v} \\ \dfrac{\partial x^q}{\partial u}, & \dfrac{\partial x^q}{\partial v} \end{vmatrix} du\, dv$$

and u, v are the surface parameters so that the parametric equations of the surface S are given by $x^i = f^i(u, v)$. In rectangular coordinates and in the usual notations x, y, z, the components of the covariant vector dS_r are given by

$$dS_x = d(y, z),\ dS_y = d(z, x),\ dS_z = d(x, y).$$

Exercise

Prove that dS_r is a covariant vector under transformations of the coordinates x^i. *Hint:* use the fact that $\sqrt{g}$ is a scalar density.

Let dS be the magnitude of the surface element, i.e.,

$$(dS)^2 = g^{\alpha\beta}\, dS_\alpha\, dS_\beta.$$

Before we introduce the notion of a stress tensor we must define a stress vector. A *stress vector is a surface force that acts on the surface of a volume.* An example of a surface force is the tension acting on any horizontal section of a steel rod suspended vertically. If one thinks of the rod as cut by a horizontal plane into two parts, then the action of the weight of the lower part of the rod is transmitted to the upper part across the surface of the cut. A hydrostatic pressure on the surface of a submerged solid body provides another example of a surface force.

There are other kinds of forces called body, volume, or *mass forces,* i.e., forces that act throughout the volume. As a typical example of a mass force one can take the force of gravity, $\rho g\,\Delta V$, acting on the mass contained in the volume ΔV of the medium whose density is ρ, and where g is the gravitational acceleration.

A stress tensor $T^{r\alpha}$ is defined implicitly by the relation

$$(16\cdot 3) \qquad F^r\,dS = T^{r\alpha}\,dS_\alpha,$$

where F^r is the stress vector acting on the surface element dS_r.

Let us now consider a virtual displacement of the strained medium corresponding to the accessory parameter t. The *virtual work* of the stresses across the boundary S is

$$(16\cdot 4) \qquad \int_S\!\!\int F^\beta\,\delta x_\beta\,dS = \int_S\!\!\int T^{\beta\alpha}\,\delta x_\beta\,dS_\alpha = \int\!\!\int_V\!\!\int (T^{\beta\alpha}\,\delta x_\beta)_{,\alpha}\,dV,$$

a volume integral extended over the volume V bounded by S and obtained by Green's theorem or generalized Stokes' theorem in curvilinear coordinates.

If there are *mass forces* (M^r per unit mass) acting on the medium, the virtual work of these mass forces is

$$\int\!\!\int_V\!\!\int \rho M^\beta \delta x_\beta\,dV,$$

where ρ is the *mass density*. Hence, the virtual work of *all* the forces acting on any portion of the medium is

$$(16\cdot 5) \qquad \int\!\!\int_V\!\!\int [(T^{\beta\alpha}_{,\alpha} + \rho M^\beta)\delta x_\beta + T^{\beta\alpha}(\delta x_\beta)_{,\alpha}]\,dV.$$

We shall now adopt the

Physical Assumption of Equilibrium: *The virtual work of all the forces acting on any portion of the medium is zero for any* rigid *virtual displacement.*

In particular, the translations, characterized by $(\delta x_p)_{,q} = 0$, are rigid virtual displacements, and so we must have the condition

$$(16\cdot 6) \qquad \int\!\!\int_V\!\!\int (T^{\beta\alpha}_{,\alpha} + \rho M^\beta)\,\delta x_\beta\,dV = 0.$$

Since δx_β is arbitrary at any chosen point and V is arbitrary, we have the following *differential equations for equilibrium:*

$$(16\cdot 7) \qquad T^{\beta\alpha}_{,\alpha} + \rho M^\beta = 0.$$

Consequently the virtual work of all the forces (mass as well as surface) acting upon any portion of the medium in any virtual displacement is given (on using 16·5 and 16·7) by

$$(16\cdot 8) \qquad \text{Total virtual work} = \int\!\!\int_V\!\!\int T^{\beta\alpha}(\delta x_\beta)_{,\alpha}\,dV.$$

Since this must vanish for *any rigid* virtual displacement, i.e., for

$$(\delta x_p)_{,q} + (\delta x_q)_{,p} = 0,$$

the stress tensor must be symmetric:

$$T^{\alpha\beta} = T^{\beta\alpha}. \tag{16·9}$$

Hence 16.8 can be written

$$\text{Total virtual work} = \tfrac{1}{2}\iiint_V T^{\alpha\beta}[(\delta x_\alpha)_{,\beta} + (\delta x_\beta)_{,\alpha}]\, dV. \tag{16·10}$$

Within the approximations of the usual approximate elasticity theory, the total virtual work may be written

$$\iiint_V T^{\alpha\beta}\,\delta\epsilon_{\alpha\beta}\, dV \tag{16·11}$$

since formula 15·31 holds for the approximate theory. But note that this is not a legitimate result for the finite deformation theory; formula 16·10 is the legitimate result for that theory.

Elastic Potential.

We shall now turn our attention to the elastic potential and its relation to the stress tensor. Let ρ be the density of the volume element dV in the strained medium. The element of mass dm is given then by $dm = \rho\, dV$. The *principle of conservation of mass* in a virtual displacement is expressed by

$$\delta(dm) = \delta(\rho\, dV) = 0.$$

Let T be the temperature of the element of mass dm, σ the *entropy density* (per unit mass) so that the entropy of the mass dm is $\sigma\, dm = \rho\sigma\, dV$, and $u\, dm$ the internal energy of the mass dm. Then the fundamental *energy-conservation law of thermodynamics* says that

$$T\delta(\sigma\, dm) = \delta(u\, dm) \tag{16·12}$$

(virtual work of all forces acting on dm). Let

$$\phi = u - T\sigma, \tag{16.13}$$

the free energy density or *elastic potential.*

From the principle of conservation of mass, we have, on integrating over any portion of the strained medium and making use of equations 16·8, 16·12, and 16·13,

$$\iiint_V (\delta\phi)\rho\, dV = \iiint_V T^{\alpha\beta}(\delta x_\alpha)_{,\beta}\, dV - \iiint_V (\delta T)\rho\sigma\, dV. \tag{16·14}$$

Since V is arbitrary, this yields

$$\rho\,\delta\phi = T^{\alpha\beta}(\delta x_\alpha)_{,\beta} - \rho\sigma\,\delta T. \tag{16·15}$$

We shall now work under the following

HYPOTHESIS ON ELASTIC POTENTIAL ϕ: *ϕ is a function of ${}^r a_{,s}$, the Euclidean metric tensor $g_{ij}(x)$ in the strained medium, the Euclidean metric tensor ${}_{\alpha\beta}c(a)$ in the unstrained medium, and the temperature T.*

We shall restrict ourselves to isothermal variations, so that T is a constant parameter in ϕ. From 16·15 and the symmetry of the stress tensor $T^{\alpha\beta}$, we see that $\delta\phi = 0$ for any *isothermal rigid virtual displacement*. Now, for any virtual displacement, $\delta_{\alpha\beta}c = 0$ and $\delta g_{rs} = 0$. Hence from Killing's differential equation 15·33 we have

(16·16) $$\frac{\partial\phi}{\partial({}^{\alpha}a_{,\beta})}\,\delta({}^{\alpha}a_{,\beta}) = 0$$

whenever

(16·17) $$(\delta x_{\alpha})_{,\beta} + (\delta x_{\beta})_{,\alpha} = 0.$$

Let

$${}^{r}a^{\alpha}(x) = g^{\alpha\beta}(x)\,{}^{r}a_{,\beta}$$

and use the formula

$$\delta({}^{r}a_{,\beta}) = -{}^{r}a_{,\alpha}\,(\delta x^{\alpha})_{,\beta} \qquad \text{(see formula 15·28)}$$

and 16·17 in 16·16 to obtain

$$\left[\frac{\partial\phi}{\partial({}^{\alpha}a_{,\beta})}\,{}^{\alpha}a^{\gamma} - \frac{\partial\phi}{\partial({}^{\alpha}a_{,\gamma})}\,{}^{\alpha}a^{\beta}\right](\delta x_{\gamma})_{,\beta} = 0.$$

Hence ϕ must satisfy the following system of partial differential equations

(16·18) $$\frac{\partial\phi}{\partial({}^{\alpha}a_{,\beta})}\,{}^{\alpha}a^{\gamma} = \frac{\partial\phi}{\partial({}^{\alpha}a_{,\gamma})}\,{}^{\alpha}a^{\beta}.$$

This is a complete system of three linear first-order partial differential equations in the nine variables ${}^{\alpha}a_{,\beta}$. There are nine conditions in 16·18 but three are identities and only three of the remaining six are independent. From the theory of such systems of differential equations[1] we know that the *general solution of 16·18 is a function of six functionally independent solutions*. It will take us too far afield to give the theory of such differential equations; we are content here with this mere statement of the result concerning the most general solution ϕ.

There are some particularly interesting solutions of equations 16·18. To consider them it is convenient to define an *isotropic medium*.

Definition of Isotropic Medium. *A medium whose elastic potential is a strain invariant that may depend parametrically on the temperature T will be called an isotropic medium.*

Now it can be shown, but in this brief volume we have not the time to give the details of proof, that the elastic potential for an isotropic medium satisfies the differential equations 16·18. It can also be shown by a long mathematical argument that *any strain invariant is a function of the three fundamental strain invariants I_1, I_2, and I_3 of Chapter 15.* The following important result is immediate. *A necessary and sufficient* *condition that a medium be isotropic is that its elastic potential* $\phi = \phi(I_1, I_2, I_3, T)$ *where* I_1, I_2, *and* I_3 *are the fundamental strain invariants.*

In the usual approximate theory of elasticity, the elastic potential ϕ for crystalline media (another name for non-isotropic media) is taken as a quadratic function of the strain tensor components. It is tacitly assumed in the usual approximate theory that a special privileged reference frame, determined by the axes of the crystal, has been chosen. The coefficients of the quadratic form are accordingly not scalars but components of a tensor of rank four which depends on the orientation of the crystalline axes.

Stress-Strain Relations for an Isotropic Medium.

Consider the elastic potential ϕ for an isotropic medium as a function of the strain tensor ϵ_{rs}. Since ϵ_{rs} is symmetric, we have $\epsilon_{rs} = \frac{1}{2}(\epsilon_{rs} + \epsilon_{sr})$. In ϕ, we shall write $\frac{1}{2}(\epsilon_{rs} + \epsilon_{sr})$ wherever ϵ_{rs} occurs, and thus we see that

$$(16\cdot19) \qquad \frac{\partial\phi}{\partial\epsilon_{rs}} = \frac{\partial\phi}{\partial\epsilon_{sr}},$$

with the understanding that in $\frac{\partial\phi}{\partial\epsilon_{rs}}$, say, all the other ϵ's (including ϵ_{sr} for that s, r) are held constant, so that in this differentiation no attention is paid to the symmetry relations $\epsilon_{sr} = \epsilon_{rs}$.

We saw in Chapter 15 (see 15·29) that under a virtual displacement the variation of h_{pq} and hence of the strain tensor ϵ_{pq} was given by

$$(16\cdot20) \qquad \delta\epsilon_{pq} = -\tfrac{1}{2}\delta h_{pq} = \tfrac{1}{2}[h_q^\tau(\delta x_\tau)_{,p} + h_p^\tau(\delta x_\tau)_{,q}],$$

since $h_{pq} = g_{pq} - 2\epsilon_{pq}$. But $\delta g_{pq} = 0$; hence

$$\delta\phi = \frac{\partial\phi}{\partial\epsilon_{\alpha\beta}}\delta\epsilon_{\alpha\beta} = \frac{1}{2}\frac{\partial\phi}{\partial\epsilon_{\alpha\beta}}[h_\beta^\tau(\delta x_\tau)_{,\alpha} + h_\alpha^\tau(\delta x_\tau)_{,\beta}],$$

or

$$(16\cdot21) \qquad \delta\phi = \frac{\partial\phi}{\partial\epsilon_{\alpha\beta}}h_\beta^\tau(\delta x_\tau)_{,\alpha} \qquad (\alpha \text{ and } \beta \text{ are summation indices †})$$

on using conditions 16·19. Now, for an isothermal virtual displacement, formula 16·15 reduces to $\rho\,\delta\phi = T^{\alpha\beta}(\delta x_\alpha)_{,\beta}$, and so for an isotropic medium

$$(16\cdot22) \qquad \rho\frac{\partial\phi}{\partial\epsilon_{\alpha\beta}}h_\beta^\tau(\delta x_\tau)_{,\alpha} = T^{\alpha\beta}(\delta x_\alpha)_{,\beta}.$$

From the arbitrariness of the virtual displacement and the fact that

† Throughout the remaining part of this chapter, a mere repetition of an index in a term will denote summation over that index.

$h^p_q = \delta^p_q - 2\epsilon^p_q$ we obtain the *stress-strain relations for an isotropic medium*

$$T^{\alpha\beta} = \rho\left(\frac{\partial\phi}{\partial\epsilon_{\beta\alpha}} - 2\epsilon^\alpha_\sigma \frac{\partial\phi}{\partial\epsilon_{\beta\sigma}}\right). \tag{16·23}$$

We shall put this stress-strain relation in another form. Now by definition $\epsilon^\lambda_\mu = g^{\lambda\nu}\epsilon_{\nu\mu}$, and hence

$$\frac{\partial\epsilon^\lambda_\mu}{\partial\epsilon_{ij}} = g^{\lambda i}\,\delta_{\mu j}.$$

Obviously

$$\frac{\partial\phi}{\partial\epsilon^q_{ij}} = \frac{\partial\phi}{\partial\epsilon^\lambda_\mu}\frac{\partial\epsilon^\lambda_\mu}{\partial\epsilon_{ij}},$$

and hence

$$\frac{\partial\phi}{\partial\epsilon_{ij}} = g^{\lambda i}\frac{\partial\phi}{\partial\epsilon^\lambda_j}. \tag{16·24}$$

Define the mixed stress tensor T^α_β by

$$T^\alpha_\beta = g_{\beta\gamma}T^{\alpha\gamma} \tag{16·25}$$

($= g_{\beta\gamma}T^{\gamma\alpha}$ from the symmetry of the stress tensor). Then with the aid of 16·24 in 16·23 one can show readily that the *following stress-strain relations hold for an isotropic medium*

$$T^\alpha_\beta = \rho\left(\frac{\partial\phi}{\partial\epsilon^\beta_\alpha} - 2\epsilon^\alpha_\sigma\frac{\partial\phi}{\partial\epsilon^\beta_\sigma}\right). \tag{16·26}$$

From the principle of conservation of mass $\rho\,dV = \rho_0\,dV_0$, and from the fundamental result 15·5, we see that

$$\rho = \rho_0\sqrt{1 - 2I_1 + 4I_2 - 8I_3} \tag{16·27}$$

in terms of the strain invariants I_1, I_2, and I_3 for media, whether isotropic or not. Since I_1, I_2, and I_3 are respectively first degree, second degree, and third degree in the strain tensor components $\epsilon_{\alpha\beta}$, we see that to a first approximation $\rho = \rho_0$; i.e., volumes are also preserved to a first approximation. *Hence to the same degree of approximation, the stress-strain relations 16·26 for an isotropic medium reduce to Hooke's law of the usual approximate theory*

$$T^\alpha_\beta = \frac{\partial\phi^1}{\partial\epsilon^\beta_\alpha}, \tag{16·28}$$

where $\phi^1 = \rho\phi$.

CHAPTER 17

TENSOR CALCULUS IN RIEMANNIAN SPACES AND THE FUNDAMENTALS OF CLASSICAL MECHANICS

Multidimensional Euclidean Spaces.

In the last two chapters of this book we shall attempt to give some indications of a more general tensor calculus and some of its applications. Although our discussion will of necessity be brief, this fact will not keep us from going to the heart of our subject. Our study of Euclidean tensor analysis can be used advantageously to accomplish this.

First of all the subject matter of Chapters 9, 10, and 11 can obviously be extended to n-dimensional Euclidean spaces, where n is any positive integer. There will be n variables wherever there were three before, and indices will have the range 1, 2, $\cdots$ to n with the consequent summations going from 1 to n. For example, the squared element of arc in rectangular coordinates $y^1, y^2, \cdots, y^n$ is

$$(17\cdot1) \qquad ds^2 = \sum_{i=1}^{n} (dy^i)^2,$$

while in general coordinates $x^1, x^2, \cdots, x^n$

$$(17\cdot2) \qquad ds^2 = g_{\alpha\beta}(x^1, x^2, \cdots, x^n)\, dx^\alpha\, dx^\beta,$$

where

$$(17\cdot3) \qquad g_{\alpha\beta}(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^i}{\partial x^\beta},$$

the n-dimensional Euclidean metric tensor (see 9·15). The n-dimensional Euclidean Christoffel symbols are

$$(17\cdot4) \qquad \Gamma^i_{\alpha\beta}(x^1, x^2, \cdots, x^n) = \frac{1}{2}g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right),$$

where the $g_{\alpha\beta}$ are defined in 17·3 while

$$(17\cdot5) \qquad g^{\alpha\beta}(x^1, x^2, \cdots, x^n) = \frac{\text{Cofactor of } g_{\beta\alpha} \text{ in } g}{g}$$

in terms of the n-rowed determinant

$$(17\cdot 6)\qquad g = \begin{vmatrix} g_{11}, & g_{12}, & \cdots, & g_{1n} \\ g_{21}, & g_{22}, & \cdots, & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1}, & g_{n2}, & \cdots, & g_{nn} \end{vmatrix}.$$

Riemannian Geometry.[1]

An n-dimensional Riemannian space is an n-dimensional manifold with coordinates such that length of curves is determined by means of a symmetric covariant tensor field of rank two $g_{\alpha\beta}(x^1, x^2, \cdots, x^n)$ in such a fashion that the squared element of arc

$$(17\cdot 7)\qquad ds^2 = g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta$$

is positive definite, i.e., $g_{\alpha\beta}\, dx^\alpha\, dx^\beta \geq 0$ and is equal to zero if and only if all the dx^α are zero. The length of a curve $x^i = f^i(t)$ given in terms of a parameter t is then by *definition*

$$(17\cdot 8)\qquad s = \int_{t_0}^{t_1} \sqrt{g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt}}\, dt.$$

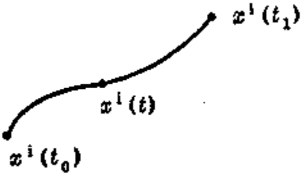


FIG. 17·1.

It can be proved by rather long algebraic manipulations that from the positive definiteness of $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ follows the positive value of the determinant of the $g_{\alpha\beta}$, i.e.,

$$(17\cdot 9)\qquad g = \begin{vmatrix} g_{11}, & g_{12}, & \cdots, & g_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1}, & g_{n2}, & \cdots, & g_{nn} \end{vmatrix} > 0.$$

The theory of a Riemannian space is a Riemannian geometry.

Exactly as in an n-dimensional Euclidean space, we can derive the contravariant tensor field of rank two $g^{\alpha\beta}(x^1, x^2, \cdots, x^n)$ and thus have at our disposal the *Christoffel symbols of our Riemannian geometry*

$$(17\cdot 10)\qquad \Gamma^i_{\alpha\beta}(x^1, x^2, \cdots, x^n) = \frac{1}{2} g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right).$$

Notice that the *Riemannian Christoffel symbols* depend on the fundamental Riemannian metric tensor $g_{\alpha\beta}$ and its first partial derivatives.

Unlike the Euclidean Christoffel symbols, it is impossible in general to find a coordinate system in which all the Riemannian Christoffel symbols are zero *everywhere* in the Riemannian space. This is due to the fact that it is in general impossible to find a coordinate system in which the fundamental metric tensor $g_{\alpha\beta}$ has constant components throughout space. It is to be recalled that in a Euclidean space there do exist just such coordinate systems, i.e., cartesian coordinate systems and rectangular coordinate systems in particular.

We can, however, prove that there exists a coordinate system with any point of the space as the origin, i.e., the $(0, 0, \cdots, 0)$ point, such that all the Christoffel symbols vanish at the origin when they are evaluated in this coordinate system. Such a coordinate system is called a *geodesic coordinate system.* We shall prove that the coordinates y^i defined implicitly by the transformation of coordinates

$$(17\cdot 11) \qquad x^i = q^i + y^i - \tfrac{1}{2}[\Gamma^i_{\alpha\beta}(x^1, x^2, \cdots, x^n)]_{x^r=q^r} y^\alpha y^\beta$$

are *geodesic coordinates.*

A direct calculation from 17·11 yields the needed formulas

$$(17\cdot 12) \qquad \begin{cases} \left(\dfrac{\partial x^i}{\partial y^\alpha}\right)_0 = \delta^i_\alpha = \left(\dfrac{\partial y^i}{\partial x^\alpha}\right)_{x^r=q^r}, \\ \left(\dfrac{\partial^2 x^i}{\partial y^\alpha\, \partial y^\beta}\right)_0 = -[\Gamma^i_{\alpha\beta}(x^1, \cdots, x^n)]_{x^r=q^r}, \end{cases}$$

where the 0 means evaluation at the origin of the y^i coordinates. Now, under a transformation of coordinates, the Christoffel symbols of a Riemannian space transform † by the rule 10·29 for Euclidean Christoffel symbols. Let $^*\Gamma^i_{\alpha\beta}(y^1, y^2, \cdots, y^n)$ be the (Riemannian) Christoffel symbols in the y^i coordinates. Then

$$(17\cdot 13) \quad {}^*\Gamma^i_{\alpha\beta}(y^1, y^2, \cdots, y^n) = \Gamma^\lambda_{\mu\nu}(x^1, x^2, \cdots, x^n) \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial y^i}{\partial x^\lambda} + \frac{\partial^2 x^\lambda}{\partial y^\alpha\, \partial y^\beta} \frac{\partial y^i}{\partial x^\lambda}.$$

Now we see from the transformation of coordinates 17·11 that $x^i = q^i$ when $y^i = 0$. In other words, the origin of the y^i coordinates has coordinates $x^i = q^i$ in the x^i coordinates.

If we evaluate both sides of 17·13 at the origin of the y^i coordinates and if we use formulas 17·12 in the calculations, we find

$$(17\cdot 14) \qquad [{}^*\Gamma^i_{\alpha\beta}(y^1, y^2, \cdots, y^n)]_0 = 0.$$

In other words, the y^i's are geodesic coordinates.

† With the difference that the number of variables now is n and the indices have the range 1 to n. The proof is practically a repetition of that given in note 4 to Chapter 10.

Curved Surfaces as Examples of Riemannian Spaces.

Obviously any Euclidean space is a very special Riemannian space. A simple example of a Riemannian space which is not Euclidean is furnished by a curved surface in ordinary three-dimensional Euclidean space. This can be seen as follows. Let y^i be rectangular coordinates in the three-dimensional Euclidean space, and let the equations of a curved surface be

$$y^i = f^i(x^1, x^2) \tag{17·15}$$

in terms of two parameters x^1 and x^2. Then the squared element of arc *for points on the surface* 17·15 is

$$ds^2 = \sum_{i=1}^{3} (dy^i)^2 = g_{\alpha\beta}(x^1, x^2)\, dx^\alpha\, dx^\beta \tag{17·16}$$

(α and β have the range 1 to 2 and corresponding summations go from 1 to 2), where

$$g_{\alpha\beta}(x^1, x^2) = \sum_{i=1}^{3} \frac{\partial f^i(x^1, x^2)}{\partial x^\alpha} \frac{\partial f^i(x^1, x^2)}{\partial x^\beta}. \tag{17·17}$$

So a surface in three-dimensional Euclidean space is a two-dimensional Riemannian space.

Exercise

The surface of a sphere is a two-dimensional Riemannian space. Find its fundamental metric tensor and its Christoffel symbols. The *surface* of a sphere of fixed radius r is given by

$$y^1 = r \sin x^1 \cos x^2$$
$$y^2 = r \sin x^1 \sin x^2$$
$$y^3 = r \cos x^1$$

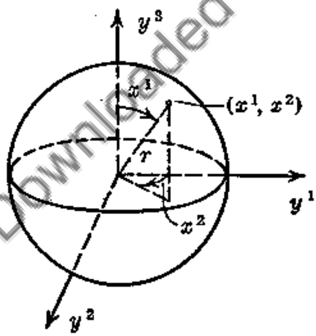


Fig. 17·2.

Therefore *the fundamental metric tensor* is given by $g_{11} = r^2$, $g_{12} = g_{21} = 0$, $g_{22} = r^2(\sin x^1)^2$. Hence

$$g^{11} = \frac{1}{r^2}, \quad g^{12} = g^{21} = 0, \quad g^{22} = \frac{1}{r^2(\sin x^1)^2}.$$

The Christoffel symbols are then

$$\Gamma^1_{22} = -\sin x^1 \cos x^1, \; \Gamma^2_{12} = \Gamma^2_{21} = \cot x^1,$$

and all the other Christoffel symbols of the surface of the sphere are zero.

The Riemann-Christoffel Curvature Tensor.

It was seen in Chapter 11 that covariant differentiation is a commutative operation in three-dimensional Euclidean space, and by exactly the same type of reasoning this is also true in an n-dimensional Euclidean space. To establish this result explicit use was made of cartesian coordinates. Since such coordinates are in general not available in a Riemannian space, we cannot use that type of proof. *In fact, covariant differentiation in a Riemannian space is not in general commutative.* We shall find a formula (see **17·19** below) that makes clear the non-commutativity of covariant differentiation in Riemannian spaces.

In obtaining Laplace's equation in curvilinear coordinates for contravariant vector fields in a Euclidean space, we had to calculate the second covariant derivative of a contravariant vector field. (See the bracket term in 12·23.) The calculation for Riemannian spaces is practically the same, so that we shall write down the second covariant derivative of $\xi^i(x^1, x^2, \cdots, x^n)$ based on the (Riemannian) Christoffel symbols $\Gamma^i_{\alpha\beta}(x^1, x^2, \cdots, x^n)$ without giving any more details (again see bracket term in **12·23**). The result is

$$(17\cdot18) \quad \xi^i_{,\alpha,\beta} = \frac{\partial^2 \xi^i}{\partial x^\alpha \, \partial x^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial \xi^i}{\partial x^\sigma} + \Gamma^i_{\sigma\alpha}\frac{\partial \xi^\sigma}{\partial x^\beta} + \Gamma^i_{\sigma\beta}\frac{\partial \xi^\sigma}{\partial x^\alpha} + \left(\frac{\partial \Gamma^i_{\sigma\alpha}}{\partial x^\beta} + \Gamma^i_{\tau\beta}\Gamma^\tau_{\sigma\alpha} - \Gamma^i_{\sigma\tau}\Gamma^\tau_{\alpha\beta}\right)\xi^\sigma.$$

From the commutativity of the partial derivatives and the symmetry of the Christoffel symbols $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$, we find

$$(17\cdot19) \qquad \xi^i_{,\alpha,\beta} - \xi^i_{,\beta,\alpha} = B^i_{\sigma\alpha\beta}\xi^\sigma,$$

where

$$(17\cdot20) \qquad B^i_{\sigma\alpha\beta} = \frac{\partial \Gamma^i_{\sigma\alpha}}{\partial x^\beta} - \frac{\partial \Gamma^i_{\sigma\beta}}{\partial x^\alpha} + \Gamma^\tau_{\sigma\alpha}\Gamma^i_{\tau\beta} - \Gamma^\tau_{\sigma\beta}\Gamma^i_{\tau\alpha}.$$

To justify the notation $B^i_{\sigma\alpha\beta}$ and prove that they are the components of a tensor field of rank four, contravariant of rank one and covariant of rank three, we first note that the left sides of 17·19 are the components of a tensor field of rank three, contravariant of rank one and covariant of rank two. Hence $B^i_{\sigma\alpha\beta}\xi^\sigma$ is a tensor field of the same type for all contravariant vector fields ξ^σ; i.e.,

$$\overline{B}^i_{\sigma\alpha\beta}(\bar{x})\bar{\xi}^\sigma(\bar{x}) = B^\lambda_{\mu\nu\rho}(x)\xi^\mu(x)\frac{\partial x^\nu}{\partial \bar{x}^\alpha}\frac{\partial x^\rho}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^i}{\partial x^\lambda}.$$

But, writing $\xi^\mu(x)$ in terms of $\bar{\xi}^\sigma(\bar{x})$, we evidently have

$$\overline{B}^i_{\sigma\alpha\beta}(\bar{x})\bar{\xi}^\sigma(\bar{x}) = B^\lambda_{\mu\nu\rho}(x)\bar{\xi}^\sigma(\bar{x})\frac{\partial x^\mu}{\partial \bar{x}^\sigma}\frac{\partial x^\nu}{\partial \bar{x}^\alpha}\frac{\partial x^\rho}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^i}{\partial x^\lambda}.$$

But $\bar{\xi}^{\sigma}(\bar{x})$ are arbitrary, and hence, equating corresponding coefficients, we obtain the tensor law of transformation for $B^{\lambda}_{\mu\nu\rho}(x)$. This tensor field is the famous *Riemann-Christoffel curvature tensor;* it is not a zero tensor in a general Riemannian space. *Hence* $\xi^i_{,\alpha,\beta} \neq \xi^i_{,\beta,\alpha}$ *in a Riemannian space with non-vanishing Riemann-Christoffel curvature tensor.* But obviously the Riemann-Christoffel curvature tensor is zero in Euclidean spaces. Hence $\xi^i_{,\alpha,\beta} = \xi^i_{,\beta,\alpha}$ in Euclidean spaces; this checks a result found earlier, in 11.13.

Geodesics.

A straight line is the shortest distance between two points in Euclidean spaces. There are curves in Riemannian spaces that play a role analogous to the straight lines of Euclidean spaces. Such curves are called *geodesics.* In fact, if a Riemannian space is a Euclidean space, then its geodesics are straight lines. To find the differential equations satisfied by the geodesics of a Riemannian space, we have to get the Euler-Lagrange differential equations for the calculus of variations problem

$$(17\cdot21) \qquad \int_{t_0}^{t_1} \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt = \text{minimum.}$$

It can be shown that the Euler-Lagrange equations for this calculus of variations problem are

$$(17\cdot22) \qquad \frac{d^2x^i(s)}{ds^2} + \Gamma^i_{\alpha\beta}(x)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0,$$

where s is the arc length and $\Gamma^i_{\alpha\beta}(x)$ are the Christoffel symbols of the Riemannian space. In other words, if the coordinates of points on a geodesic are considered as functions $x^i(s)$ of the arc length parameter s, then the n functions $x^i(s)$ satisfy the system 17·22 of n differential equations of the second order.

If the Riemannian space is Euclidean and we choose rectangular cartesian coordinates y^i, equations 17·22 reduce to

$$\frac{d^2y^i(s)}{ds^2} = 0,$$

and hence

$$y^i = \alpha^i s + \beta^i \qquad (\alpha^i \text{ and } \beta^i \text{ are constants}),$$

the parametric equations of straight lines in terms of arc length s.

Equations of Motion of a Dynamical System with n Degrees of Freedom.

In classical mechanics, it is postulated that the motion of a conservative dynamical system of n degrees of freedom with no moving constraints is governed by *Lagrange's equations of motion*[2]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0. \tag{17·23}$$

If the kinetic energy T, in terms of the generalized coordinates $q^1, q^2, \cdots, q^n$ and the generalized velocities $\dot{q}^i(t) = \dfrac{dq^i(t)}{dt}$, is

$$T = \tfrac{1}{2} g_{ij}(q^1, \cdots, q^n)\dot{q}^i\dot{q}^j \qquad (g_{ij} = g_{ji})$$

and if the potential energy is $V(q^1, q^2, \cdots, q^n)$, then the kinetic potential or Lagrangean L is given by $L = T - V$. Now the kinetic energy is positive definite in the velocities $\dot{q}^i$; i.e., $T \geq 0$ and $T = 0$ if and only if $\dot{q}^i = 0$. It can be proved by algebraic reasoning that the determinant g of the g_{ij} is positive so that we can form g^{ij} in terms of the g_{ij} — exactly as in Riemannian geometry. By direct calculation we find

$$\frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \left(\frac{\partial g_{ij}}{\partial q^k}\right)\dot{q}^j\dot{q}^k + g_{ij}\ddot{q}^j \qquad \left(\ddot{q}^j = \frac{d^2q^j}{dt^2}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j}\right)\dot{q}^j\dot{q}^k + g_{ij}\ddot{q}^j,$$

$$\frac{\partial L}{\partial q^i} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial q^i}\right)\dot{q}^j\dot{q}^k - \frac{\partial V}{\partial q^i}.$$

Hence Lagrange's equations of motion 17·23 can be written in the form

$$g_{ij}\ddot{q}^j + \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\right)\dot{q}^j\dot{q}^k = -\frac{\partial V}{\partial q^i}. \tag{17·24}$$

Multiplying corresponding sides of 17·24 by $g^{\alpha i}$ and summing on i, we obtain the *following form for Lagrange's equation of motion.*[3]

$$\ddot{q}^\alpha + \Gamma^\alpha_{jk}(q^1, \cdots, q^n)\dot{q}^j\dot{q}^k = -g^{\alpha i}\frac{\partial V}{\partial q^i}, \tag{17·25}$$

where $\Gamma^\alpha_{jk}(q^1, \cdots, q^n)$ *are the Christoffel symbols based on the* g_{ij} $(q^1, \cdots, q^n)$ *of the kinetic energy of the dynamical system.*

Exercise

A symmetrical gyroscope with a point O fixed on the axis is acted upon by gravity. Let I, I, and J be the principal moments of inertia. Then the kinetic energy is given by

$$T = \frac{1}{2}I\left(\frac{dq^1}{dt}\right)^2 + \frac{1}{2}I \sin^2 q^1\left(\frac{dq^2}{dt}\right)^2 + \frac{1}{2}J\left(\frac{dq^3}{dt} + \cos q^1 \frac{dq^2}{dt}\right)^2$$

and the potential energy by

$V = Mgh \cos q^1$. The coordinates q^1, q^2, and q^3 are the Eulerian angles, M is the mass of the gyroscope, and h is the distance of the center of gravity from O. Find the Lagrangean equations of motion of the symmetrical gyroscope. Compute also the element of arc length of the three-dimensional Riemannian space associated with the symmetrical gyroscope.

CHAPTER 18

APPLICATIONS OF THE TENSOR CALCULUS TO BOUNDARY–LAYER THEORY

Incompressible and Compressible Fluids.

The constancy of volume of all parts of a fluid in motion sometimes plays an important role in the theory of fluid flows. A fluid in motion with this property is called an *incompressible fluid*, whereas a fluid in motion without this property is called a *compressible fluid*. If u^i are the contravariant components of velocity of the fluid in motion in general coordinates x^i, then

$$\frac{dx^i}{dt} = u^i \tag{18·1}$$

are the differential equations whose integration gives the paths of the fluid particles in the coordinates x^i.

It can be proved by a direct calculation that a necessary and sufficient condition that the volume

$$\int\int\int \sqrt{g}\, dx^1\, dx^2\, dx^3 \tag{18·2}$$

of arbitrary portions of the moving fluid be preserved [1] is that the divergence of the velocity field u^i be zero, i.e.,

$$u^\alpha_{,\alpha} = 0. \tag{18·3}$$

In 18·2, g is the determinant of the Euclidean metric tensor g_{ij} ($ds^2 = g_{ij}\, dx^i\, dx^j$), and the comma in $u^\alpha_{,\alpha}$ stands for covariant differentiation based on the Euclidean Christoffel symbols Γ^i_{jk}. In other words, *a necessary and sufficient condition for an incompressible fluid is that the velocity vector field u^i satisfy the partial differential equation 18·3.* A glance at formula 13·6 shows that the condition of incompressibility is equivalent to

$$\frac{\partial(\sqrt{g}u^\alpha)}{\partial x^\alpha} = 0. \tag{18·4}$$

If we recall the Navier-Stokes equations 13·3 for the motion of a viscous fluid, incompressible or compressible, we know that the equation of continuity

$$\frac{\partial \rho}{\partial t} + (\rho u^\alpha)_{,\alpha} = 0 \tag{18·5}$$

in general coordinates x^i merely states the constancy[1] of the mass m

$$m = \int\int\int \rho(x^1, x^2, x^3, t)\sqrt{g}\, dx^1\, dx^2\, dx^3 \tag{18·6}$$

of any portion of the moving fluid. An evident consequence of the conditions 18·3 and 18·5 is that the density $\rho(x^1, x^2, x^3, t)$ (an absolute scalar) satisfies the condition

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x^\alpha} u^\alpha = 0, \tag{18·7}$$

which states that

$$\frac{d\rho}{dt} = 0 \tag{18·8}$$

along any chosen path of fluid particles. This means that *18·7 can be taken as the defining condition for an incompressible fluid* in view of the continuity equation 18.5. The Navier-Stokes equations for an incompressible viscous fluid reduce then to the following system of four differential equations in general coordinates x^i:

$$\begin{cases} \dfrac{\partial u^i}{\partial t} = \nu g^{\alpha\beta} u^i_{,\alpha,\beta} - u^\alpha u^i_{,\alpha} - \dfrac{1}{\rho} g^{i\alpha} \dfrac{\partial p}{\partial x^\alpha} + X^i \\ u,^\alpha_\alpha = 0. \end{cases} \tag{18·9}$$

For an incompressible fluid, the density $\rho(x^1, x^2, x^3, t)$ is given subject to condition 18·7. Then the four differential equations 18·9 will have as unknowns the three velocity components u^1, u^2, u^3 and the pressure $p(x^1, x^2, x^3, t)$ of the fluid.

The situation is different for *compressible fluids*. The Navier-Stokes equations 13.3 are four in number with five unknown functions u^1, u^2, u^3, ρ, and p. To make the problem determinate a fifth condition must be imposed. This is usually furnished by the "equation of state," which in the isothermal case is of the form

$$p = f(\rho). \tag{18·10}$$

Boundary-Layer Equations for the Steady Motion of a Homogeneous Incompressible Fluid.†

We shall now restrict ourselves to the steady motion of a fluid without any external forces, so that $X^i = 0$ and all the quantities u^i, ρ, p are independent of the time t. If in addition we assume that the fluid is *homogeneous*, i.e., ρ is a constant, and incompressible, the four unknowns

† The remaining part of this chapter is an exposition of some unpublished researches of Dr. C. C. Lin. These results were presented by Dr. Lin in my seminar on applied mathematics.

u^i, p must, by a reference to 18·9, satisfy the four differential equations

$$(18\cdot11)\qquad \begin{cases} u^j u^i_{,j} = \nu g^{jk} u^i_{,j,k} - g^{ij}\dfrac{\partial \pi}{\partial x^j}, \\ u^j_{,j} = 0, \end{cases}$$

where π is the pressure p divided by the constant density ρ of the fluid, and the constant ν is the kinematical viscosity. Since the covariant derivative of the Euclidean metric tensor g_{ij} is zero, it follows from 18·11 that the covariant vector components $u_i = g_{is}u^s$ of the velocity field and the function π will satisfy the system of differential equations

$$(18\cdot12)\qquad \begin{cases} u^j u_{i,j} = \nu g^{jk} u_{i,j,k} - \dfrac{\partial \pi}{\partial x^i}, \\ u_{,i}^{\;\;i} = 0. \end{cases}$$

For the treatment of "boundary-layer" problems connected with an arbitrary surface, it is convenient to take a system of *space coordinates* in which x^1, x^2 are surface coordinates and x^3 is a coordinate measured along the normals to the surface. Thus $x^3 = 0$ will be the equation of the given surface. If we allow Latin indices to run over the range (1, 2, 3) and Greek indices over the range (1, 2), we have the following fundamental metrics:

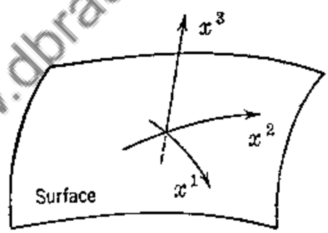


FIG. 18·1.

$$(18\cdot13)\qquad ds^2 = g_{ij}(x^1, x^2, x^3)\,dx^i\,dx^j \quad \text{in 3-space,}$$

and

$$(18\cdot14)\qquad ds^2 = g_{\alpha\beta}(x^1, x^2, C)\,dx^\alpha\,dx^\beta$$

over the surface

$$x^3 = C, \quad \text{a constant.}$$

From the manner in which the coordinate x^3 was chosen, it follows that in 3-space ‡

$$(18\cdot15)\qquad ds^2 = g_{ij}\,dx^i\,dx^j = g_{\alpha\beta}(x^1, x^2, x^3)\,dx^\alpha\,dx^\beta + (dx^3)^2.$$

‡ In Riemannian geometry, this is sometimes called the "geodesic form" of the line element. Such forms of the line element were used recently by Dr. W. Z. Chien in connection with his researches on the intrinsic theory of plates and shells (see references). Dr. Chien presented some of his work in my seminar on applied mathematics and made some exceedingly helpful calculations in connection with interesting geometric ideas arising in his and Dr. C. C. Lin's work.

In other words, the Euclidean metric tensor $g_{ij}(x^1, x^2, x^3)$ is such that

(18·16) $$g_{33} = 1, \quad g_{3\alpha} = g_{\alpha 3} = 0.$$

We shall *henceforth consider transformations of surface coordinates* x^1, x^2 *alone* so that the coordinate x^3 may be regarded as a *scalar parameter* under a transformation of surface coordinates. To emphasize this fact we shall *use the notation* $x^0 = x_0 = x^3$. Thus, in the new notation, 18·16 can be written

(18·17) $$g_{00} = 1, \quad g_{0\alpha} = g_{\alpha 0} = 0.$$

We saw in the previous chapter that a *surface* can be considered as a *two-dimensional Riemannian space.* There is thus at our disposal the Riemannian tensor calculus of the previous chapter for immediate use in connection with the surface x^0 = constant. We shall use a *semicolon to denote surface covariant differentiation* in contradistinction to the comma for covariant differentiation in the enveloping three-dimensional Euclidean space.

To express all covariant differentiations with respect to space coordinates in terms of covariant differentiations with respect to surface coordinates x^1, x^2 and partial differentiations with respect to x^0, consider first the Euclidean Christoffel symbols Γ^i_{jk} in the coordinates x^1, x^2, x^3. If i, j, k are all in the range 1, 2 no reduction is possible unless a special surface coordinate system is chosen; if one of the three indices is zero, we have

(18·18) $$\Gamma^0_{\alpha\beta} = -\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^0}, \quad \Gamma^\alpha_{0\beta} = \Gamma^\alpha_{\beta 0} = -\frac{1}{2} g^{\alpha\gamma}\frac{\partial g_{\beta\gamma}}{\partial x^0}.$$

These are evidently tensor fields with respect to transformation of surface coordinates, and they shall be denoted by $\Gamma_{\alpha\beta}$ and Γ^α_β respectively. The other Christoffel symbols Γ^i_{jk}, in which two or all of the i, j, k are zero, vanish identically.

With the help of these relations, it can be easily verified that

(18·19) $$\begin{cases} u_{0,0} = \dfrac{\partial u_0}{\partial x^0}, \quad u_{0,\alpha} = \dfrac{\partial u_0}{\partial x^\alpha} + \Gamma^\beta_\alpha u_\beta, \\ u_{\alpha,0} = \dfrac{\partial u_\alpha}{\partial x^0} + \Gamma^\beta_\alpha u_\beta, \quad u_{\alpha,\beta} = u_{\alpha;\beta} + \Gamma_{\alpha\beta} u_0, \end{cases}$$

(18·20) $$u^j_{,j} = \frac{\partial u_0}{\partial x^0} + u^\beta_{;\beta} - \Gamma^\beta_\beta u_0,$$

(18·21) $$\begin{cases} g^{jk} u_{0,j,k} = \dfrac{\partial^2 u_0}{\partial x^{02}} - \Gamma^\alpha_\alpha \dfrac{\partial u_0}{\partial x^0} + \Phi_0, \\ g^{jk} u_{\alpha,j,k} = \dfrac{\partial^2 u_\alpha}{\partial x^{02}} + 2\Gamma^\beta_\alpha \dfrac{\partial u_\beta}{\partial x^0} - \Gamma^\beta_\beta \dfrac{\partial u_\alpha}{\partial x^0} + \Phi_\alpha, \end{cases}$$

where

$$(18\cdot 22)\begin{cases}\Phi_0 = g^{\alpha\beta}u_{0;\alpha;\beta} + 2\Gamma^{\alpha\beta}u_{\alpha;\beta} - \Gamma^{\alpha}_{\beta}\Gamma^{\beta}_{\alpha}u_0 + \Gamma^{\alpha\beta}_{;\beta}u_\alpha,\\ \Phi_\alpha = 3\Gamma^{\gamma}_{\alpha}\Gamma^{\beta}_{\gamma}u_\beta + \Gamma^{\beta}_{\alpha}u_\beta + g^{\beta\gamma}u_{\alpha;\beta;\gamma} - 2\Gamma^{\beta}_{\alpha}\dfrac{\partial u_0}{\partial x^\beta} - \Gamma^{\beta}_{\alpha;\beta}u_0\\ \qquad\qquad - (\Gamma^{\gamma}_{\alpha}\Gamma^{\beta}_{\gamma} + \Gamma^{\gamma}_{\gamma}\Gamma^{\beta}_{\alpha})u_\beta,\end{cases}$$

do not involve differentiation of u_i with respect to x^0.

Let us now consider the analytical nature of the system 18·12 of four partial differential equations in the four unknowns π, u_0, u_α. By using 18·19, 18·20, and 18·21, we can put this system in the normal form with respect to x^0 by solving for $\dfrac{\partial\pi}{\partial x^0}$, $\dfrac{\partial^2 u_\alpha}{\partial x^{02}}$ from the equations of motion and for $\dfrac{\partial u_0}{\partial x^0}$ from the equation of continuity. Thus

$$(18\cdot 23)\begin{cases}\dfrac{\partial\pi}{\partial x^0} = -u^0\dfrac{\partial u_0}{\partial x^0} - u^\alpha\left(\dfrac{\partial u_0}{\partial x^\alpha} + \Gamma^{\beta}_{\alpha}u_\beta\right) + \nu\left(\dfrac{\partial^2 u_0}{\partial x^{02}} - \Gamma^{\alpha}_{\alpha}\dfrac{\partial u_0}{\partial x^0} + \Phi_0\right),\\ \dfrac{\partial^2 u_\alpha}{\partial x^{02}} = \dfrac{1}{\nu}u^0\left(\dfrac{\partial u_\alpha}{\partial x^0} + \Gamma^{\beta}_{\alpha}u_\beta\right) + \dfrac{1}{\nu}u^\beta\left(u_{\alpha;\beta} + \Gamma_{\alpha\beta}u_0\right) + \dfrac{1}{\nu}\dfrac{\partial\pi}{\partial x^\alpha}\\ \qquad\qquad - \left(2\Gamma^{\beta}_{\alpha}\dfrac{\partial u_\beta}{\partial x^0} - \Gamma^{\beta}_{\beta}\dfrac{\partial u_\alpha}{\partial x^0} + \Phi_\alpha\right),\\ \dfrac{\partial u_0}{\partial x^0} = -u^{\beta}_{;\beta} + \Gamma^{\beta}_{\beta}u_0,\end{cases}$$

where $\dfrac{\partial u_0}{\partial x^0}$ and $\dfrac{\partial^2 u_0}{\partial x^{02}}$ in the first equation may be expressed in terms of u_0 and $\dfrac{\partial u_\alpha}{\partial x^0}$ by using the last equation. Thus, the highest derivatives of all the variables with respect to x^0 have the coefficient unity in these equations. Hence, if π, u_0, u_α, $\dfrac{\partial u_\alpha}{\partial x^0}$ are given as functions of x^1 and x^2 on the surface $x^0 = 0$, the solution of the problem is uniquely determined.

This normal form 18·23 of the system of differential equations, however is not analytic in the small parameter ν, the *important case in aeronautics*, in the neighborhood of $\nu = 0$, and is consequently inconvenient for the application of the method of successive approximations. We therefore make the transformation of variables

$$(18\cdot 24)\qquad \zeta = \frac{x^0}{\sqrt{\nu}},\quad w = \frac{u_0}{\sqrt{\nu}}$$

to bring it into the desired form. We then have

$$(18\cdot25)\quad \begin{cases} \dfrac{\partial\pi}{\partial\zeta} = -\nu^{\frac{1}{2}}\Gamma_{\alpha\beta}u^{\alpha}u^{\beta} - \nu\left(w\dfrac{\partial w}{\partial\zeta} + u^{\alpha}\dfrac{\partial w}{\partial x^{\alpha}}\right) + \nu^{\frac{1}{2}}\left(\dfrac{\partial^2 w}{\partial\zeta^2} - \Gamma^{\alpha}_{\alpha}\dfrac{\partial w}{\partial\zeta} + \Phi_0\right) \\ \dfrac{\partial^2 u_{\alpha}}{\partial\zeta^2} = w\dfrac{\partial u_{\alpha}}{\partial\zeta} + u^{\beta}u_{\alpha;\beta} + \dfrac{\partial\pi}{\partial x^{\alpha}} + \nu^{\frac{1}{2}}\left(2w\Gamma^{\beta}_{\alpha}u_{\beta} - 2\Gamma^{\beta}_{\alpha}\dfrac{\partial u_{\beta}}{\partial\zeta}\right. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\left. + \Gamma^{\beta}_{\beta}\dfrac{\partial u_{\alpha}}{\partial\zeta}\right) - \nu\Phi_{\alpha} \\ \dfrac{\partial w}{\partial\zeta} = -u^{\beta}_{;\beta} + \nu^{\frac{1}{2}}\Gamma^{\beta}_{\beta}w. \end{cases}$$

Let us note that Φ_0 and Φ_{α} are linear in the small parameter $\nu^{\frac{1}{2}}$ through the term in u_0 (cf. 18·22), and also depend on $\nu^{\frac{1}{2}}$ through the geometrical quantities, which are functions of $x^0 = \nu^{\frac{1}{2}}\zeta$. Indeed, it can be shown[2] that the *surface metric tensor $g_{\alpha\beta}$ is a quadratic function of x^0, while all other geometrical quantities may be expanded as power series of x^0* convergent for $|x^0| < R_m$, R_m being the minimum magnitude of the principal radii of curvature over the surface under consideration. Hence, the right-hand sides of the equations 18·25 are Taylor series in $\nu^{\frac{1}{2}}$, and the solution of 18·25 may be carried out by expanding each of the dependent variables as a power series of $\nu^{\frac{1}{2}}$, convergent for all finite values of $\nu^{\frac{1}{2}}$ for which $|\nu^{\frac{1}{2}}\zeta| < R_m$.

If we try to solve 18·23 by the same type of expansion, either the series are asymptotic, or they may terminate; but in general we cannot find a solution satisfying all the required boundary conditions. In fact, the initial approximation is easily verified to satisfy the non-viscous equation ($\nu = 0$). The boundary conditions at infinity and the condition $u_0 = 0$ at $x^0 = 0$ are then sufficient to determine this approximation completely. Indeed, the boundary conditions at infinity are usually such that the resultant solution is potential. Then the initial approximation is an *exact* solution of the complete equations 18·23. However, the boundary conditions $u_0 = 0$ at $x^0 = 0$ cannot be satisfied in general. The effect of viscosity can never be brought into evidence. *This shows that the more elaborate treatment described above is absolutely necessary.* The non-viscous solution (usually potential), however, serves as a guide for making the exact solutions satisfy the boundary conditions at infinity. This point will be discussed in more detail below.

Let us now proceed with the solution of 18.25 by writing

$$(18\cdot26)\qquad \begin{cases} \pi = \pi^{(0)} + \sqrt{\nu}\pi^{(1)} + \nu\pi^{(2)} + \cdots + \cdots, \\ u_{\alpha} = u^{(0)}_{\alpha} + \sqrt{\nu}u^{(1)}_{\alpha} + \nu u^{(2)}_{\alpha} + \cdots + \cdots, \\ w = w^{(0)} + \sqrt{\nu}w^{(1)} + \nu w^{(2)} + \cdots + \cdots, \end{cases}$$

Corresponding developments for the geometrical quantities $g_{\alpha\beta}$, $\Gamma_{\alpha\beta}$, $\cdots$ must also be used. The initial approximation gives

$$(18\cdot27)\qquad \begin{cases} u^{\beta}u_{\alpha;\beta} + w\dfrac{\partial u_{\alpha}}{\partial \zeta} = -\dfrac{\partial \pi}{\partial x^{\alpha}} + \dfrac{\partial^2 u_{\alpha}}{\partial \zeta^2}, \\ 0 = \dfrac{\partial \pi}{\partial \zeta}, \\ u^{\beta}_{;\beta} + \dfrac{\partial w}{\partial \zeta} = 0, \end{cases}$$

where the superscripts of the initial approximation are dropped. In these equations, ζ is a scalar, and the metric tensor is $a_{\alpha\beta}(x^1, x^2)$, being $g_{\alpha\beta}(x^1, x^2, x^0)$ evaluated at $x^0 = 0$. The conditions over the surface $\zeta = 0$ are $u_{\alpha} = 0$ and $w = 0$. The condition at infinite ζ is set according to the following considerations. For a large but finite value of ζ, the value of x^0 is still small. Hence, the solution may be expected to pass into the non-viscous solution close to the surface if ζ is large. Thus, for the initial approximation, we may lay down the conditions

$$(18\cdot28)\qquad u_{\alpha} = \bar{u}_{\alpha},\ \pi = \bar{\pi} \quad \text{for } \zeta \to \infty,$$

where $\bar{u}_{\alpha}$ and $\bar{\pi}$ are functions of x^1 and x^2, being the values of u_{α} and π of the non-viscous solution at $x^0 = 0$. The initial approximation is then completely determined.

If u_{α} and π differ from $\bar{u}_{\alpha}$ and $\bar{\pi}$ by quantities of the order of ν for $\zeta = h$, then an approximate solution of 18·12 is usually taken to be given (*a*) by the non-viscous solution for $\zeta > h$, and (*b*) by the solution of 18·27 for $\zeta < h$. *The quantity $h\sqrt{\nu}$ is known as the "thickness of the boundary layer"* and is arbitrary to a certain extent. For example, we may define h to be given by (say) three times $\bar{h}$ of the equation

$$(18\cdot29)\qquad \int_0^{\infty} (\bar{u}_{\alpha} - u_{\alpha})\, d\zeta = \bar{u}_{\alpha}\bar{h},$$

which is in general different according to whether $\alpha = 1$ or 2. This initial approximation is usually known as the *boundary-layer theory* of Prandtl. Incidentally, we note that π is a function of x^1 and x^2 alone, by the second equation of 18·27. Hence, by 18·28, $\pi = \bar{\pi}(x^1, x^2)$, which is known from the non-viscous solution. The first and third equations of 18·27 then serve as three equations for the velocities u_{α} and w.

The higher approximations in 18·26 satisfy certain differential equations obtained together with the derivation of 18·27. The boundary conditions at $\zeta = 0$ are $u^{n}_{\alpha} = 0$, for any approximation. The boundary conditions for the nth approximation at infinity will be specified by using the nth approximation of the asymptotic solution, which will in turn be determined from certain boundary conditions related to the $(n - 1)$st approximation of the convergent solution. Since we are never concerned with higher approximations in practice, we shall not go into further details.

NOTES ON PART I

Chapter 1

1. In the modern quantum mechanics of theoretical physics, matrices with an infinite number of elements as well as with a finite number of elements are used very widely. The elements of these matrices are often complex numbers. The reader is referred to the bibliographical entries under Born, Jordan, and Dirac.

2. For applications of matrices to the social sciences the reader is referred to the references given in a recent paper by Hotelling.

3. We shall deal for the most part with matrices whose elements are real or complex numbers. It is possible, however, to deal with *matrices whose elements are themselves matrices*. We shall have occasion to use a few such matrices in connection with our discussion of aircraft flutter in Chapter 7.

Chapter 2

1. For the properties of determinants, linear equations, and related questions on the algebra of matrices, see Bocher's *Introduction to Higher Algebra*.

2. Cramer's rule for the solution of linear algebraic equations is given in most books on algebra. In our notations it can be stated in the following manner. *If the determinant* $a = | a_j^i |$ *of the n equations*

$$a_j^i x^j = b^i$$

in the n unknowns $x^1, x^2, \cdots, x^n$ *is not zero, then the equations have a unique solution given by*

$$x^i = \frac{A^i}{a},$$

where A^i *is the n-rowed determinant obtained from a by replacing the elements* $a_i^1, a_i^2, \cdots, a_i^n$ *of the ith column by the corresponding elements* $b^1, b^2, \cdots, b^n$.

3. The rule for the multiplication of two determinants takes the following form in our notations. *If* $a = | a_j^i |$ *and* $b = | b_j^i |$ *are two n-rowed determinants, then the numerical product* $c = ab$ *is itself an n-rowed determinant with elements* c_j^i *given by the formula*

$$c_j^i = a_\alpha^i b_j^\alpha.$$

4. It can be shown that s_r, the trace of the matrix A^r, is also equal to the sum of the rth powers of the n characteristic roots of the matrix A.

5. The recurrence formula 2·6 for the coefficients $a_1, \cdots, a_n$ of the characteristic function of a matrix can be derived from Newton's formulas; see Bocher's *Introduction to Higher Algebra*, pp. 243–244, for a derivation of Newton's formulas.

Chapter 3

1. To prove that the matric exponential e^A is convergent for all square matrices A, let $A = \| a^i_j \|$ be an n-rowed square matrix, and let $V(A)$ be the greatest of the numerical values of the n^2 numbers a^i_j — the greatest of the moduli of the a^i_j if the a^i_j are complex numbers. Then each element in the matrix A^i will not exceed $n^{i-1}V^i$ in numerical value. Hence each of the n^2 infinite series in e^A will be dominated by series

$$1 + V + \frac{nV^2}{2!} + \frac{n^2V^3}{3!} + \cdots + \cdots = \frac{1}{n}(e^{nV} - 1) + 1.$$

Hence all the n^2 numerical series in e^A converge. This means that e^A is convergent for all square matrices A.

In the terminology of modern functional analysis and topological spaces, the $V(A)$ is called the *norm* of the matrix A, and the class of n-rowed matrices with the operations of addition and multiplication of matrices, multiplication by numbers, and convergence of matrices defined by means of the norm $V(A)$ — in other words, $V(A)$ plays an analogous role to the absolute value or modulus of a number in the convergence of numbers — is called a *normed linear ring.* Other equivalent definitions of the norm of a matrix are possible. For example, $V(A)$ can be taken as

$$\sqrt{\sum_{i,j=1}^{n}(a^i_j)^2}$$

if the elements a^i_j of the matrix A are real numbers. Whatever suitable definition of a norm is adopted, the norm $V(A)$ of a matrix will have the following properties:

(1) $V(A) \geq 0$ and $= 0$ if and only if A is the zero matrix.
(2) $V(A + B) \leq V(A) + V(B)$ (triangular inequality).
(3) $V(AB) \leq V(A)V(B)$.

From property 3 it follows that $V(A^n) \leq (V(A))^n$, a result that makes obvious the usefulness of the notion of a norm for matrices in the treatment of convergence properties of matrices.

The class of matrices discussed above is only one example of a normed linear ring. The first general theory of normed linear rings was initiated in 1932 by Michal and Martin in a paper entitled "Some Expansions in Vector Space," *Journal de mathématiques pures et appliquées.*

2. The special case of the expansion 3·7 when $F(A)$ is a matric *polynomial* is known as *Sylvester's theorem.*

If the characteristic equation of a matrix A has multiple roots, then the expansion 3·7 is not valid. However, a more general result can be proved. For the case of matrix polynomials $F(A)$ see the Duncan, Frazer, and Collar book. The more general cases of matric power series expansions are treated briefly in a paper by L. Fantappie with the aid of the theory of functionals, since the elements $F^i_j(A)$ in the $F(A)$ are *functionals* of the numerical function $F(\lambda)$. See Volterra's book on functionals (Blackie, 1930).

3. Another equivalent form for the n matrices $G_1, G_2, \cdots, G_n$ is

$$G_i = \frac{M(\lambda_i)}{\left(\dfrac{df(\lambda)}{d\lambda}\right)_{\lambda=\lambda_i}},$$

where the λ_i are the characteristic roots of the matrix A, $f(\lambda)$ is the characteristic determinant of A, and $M(\lambda) = \| M_j^i(\lambda) \|$ is a matrix whose element $M_j^i(\lambda)$ is the cofactor of $\lambda\delta_i^j - a_i^j$ in the characteristic determinant $f(\lambda)$. Notice carefully the position of the indices i and j.

Chapter 4

1. A good approximation to the solution $X(t) = [e^{(t-t_0)A}]X_0$ of $\frac{dX(t)}{dt} = AX(t)$ can be obtained by taking $1 + (t - t_0)A + \frac{(t-t_0)^2}{2!}A^2 + \cdots + \frac{(t-t_0)^n}{n!}A^n$ in the place of the infinite expansion for $e^{(t-t_0)A}$. For many practical purposes $n = 2$ would be large enough to give a good approximate solution. The approximate solution can then be written

$$X(t) = X_0 + (t - t_0)AX_0 + \frac{(t-t_0)^2}{2!}A^2X_0,$$

where X_0 is the column matrix for $t = t_0$ initially given.

2. If the solution of the matric differential equation

$$(1) \qquad \frac{dX(t)}{dt} = AX(t) \qquad (X(t_0) = X_0)$$

has been found, then the solution of

$$(2) \qquad \frac{dX(t)}{dt} = (A + bI)X(t) \qquad (X(t_0) = X_0)$$

can be written down immediately. In fact, from the second property of the matric exponential given in Chapter 2, we see that $e^{A+bI} = e^be^A$, where e^b is the numerical exponential. Hence by formula 4·3 we see that the solution of equation 2 is obtained by a mere multiplication by e^b of the solution of equation 1.

Chapter 5

1. The reader is referred to Whittaker's *Analytical Dynamics* for a treatment of Lagrange's differential equations of motion of particle dynamics. For some engineering applications, the reader is referred to *Mathematical Methods in Engineering* by Kármán and Biot.

2. Consult the references in the above note.

Chapter 6

1. If only the fundamental frequency is wanted but not the corresponding amplitudes, then the application of Rayleigh's principle may be preferable. See *Elementary Matrices* by Frazer, Duncan, and Collar, pp. 310, 299–301.

2. A special case of Sylvester's theorem is what is actually used; see expansion 3·9.

NOTES ON PART II

Chapter 9

1. Although little use has been made of the tensor calculus in plastic deformations, one would suspect that a thoroughgoing application of the tensor calculus to the fundamentals of plastic deformation theory would prove fruitful.

2. For some elementary applications of the tensor calculus to dynamic meteorology, the reader is referred to Ertel's monograph.

3. We shall deal briefly with Riemannian spaces (certain curved spaces) and their applications to classical dynamical systems with a finite number of degrees of freedom (see Chapter 17), and to fluid mechanics (see Chapter 18).

4. A discussion of the fundamentals of coordinates, coordinate systems, and the transformation of coordinates in the various spaces, including Euclidean spaces, is out of the question here. The readers who are interested in modern differential geometry and topology will find ample references in the bibliography under the entries for Veblen, Whitehead, Thomas, and Michal.

Chapter 10

1. Some writers, especially those dealing with physical applications, like to think of the contravariant and covariant components of *one* object called a vector. For example, if ξ^i are the contravariant components of a velocity vector field, then ξ^i and $g_{ij}\xi^j$ can be considered the contravariant vector and covariant vector "representations" respectively of the same physical object called "velocity vector field." This point of view, however, is untenable in spaces without a metric g_{ij}.

2. The importance of the Euclidean Christoffel symbols for Euclidean spaces is, even now, not very well known.

3. Since

$$g_{\alpha\beta}(x^1, x^2, x^3) = \sum_{i=1}^{3} \frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^i}{\partial x^\beta} \qquad (y^1, y^2, y^3 \text{ are rectangular coordinates}),$$

we obtain, from the rule for the multiplication of two determinants, the result that $g = |g_{\alpha\beta}| = J^2$, where $J = \left|\frac{\partial y^i}{\partial x^\alpha}\right|$ is the Jacobian determinant, or the functional determinant, of the transformation of coordinates to rectangular coordinates y^i from general coordinates x^i. Hence $J \neq 0$ since we deal with transformations of coordinates that have inverses. This means that the determinant $g \neq 0$ for all our "admissible" transformations of coordinates.

4. The following steps establish the law of transformation 10·29 of the Euclidean Christoffel symbols; exactly the same method establishes the corresponding law of transformation for the Riemannian Christoffel symbols to be discussed in Chapter 17.

Since $g_{\mu\nu}$ are the components of the Euclidean metric tensor we have under a transformation of coordinates from coordinates x^i to coordinates $\bar{x}^i$

$$(a) \qquad \bar{g}_{\sigma\beta}(\bar{x}^1, \bar{x}^2, \bar{x}^3) = g_{\mu\nu}(x^1, x^2, x^3)\frac{\partial x^\mu}{\partial \bar{x}^\sigma}\frac{\partial x^\nu}{\partial \bar{x}^\beta}.$$

Differentiating corresponding sides of (a) we obtain (considering the $\bar{x}^i$ as independent variables)

$$(b)\qquad \frac{\partial \bar{g}_{\sigma\beta}}{\partial \bar{x}^\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{\partial x^\mu}{\partial \bar{x}^\sigma}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\frac{\partial x^\rho}{\partial \bar{x}^\alpha} + \left[g_{\mu\nu}\frac{\partial^2 x^\mu}{\partial \bar{x}^\sigma\,\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\right] + g_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\sigma}\frac{\partial^2 x^\nu}{\partial \bar{x}^\beta\,\partial \bar{x}^\alpha}.$$

Interchange α and σ in (b) and, noting that

$$\frac{\partial^2 x^\mu}{\partial \bar{x}^\sigma\,\partial \bar{x}^\alpha} = \frac{\partial^2 x^\mu}{\partial \bar{x}^\alpha\,\partial \bar{x}^\sigma},$$

obtain

$$(c)\qquad \frac{\partial \bar{g}_{\alpha\beta}}{\partial \bar{x}^\sigma} = \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\frac{\partial x^\rho}{\partial \bar{x}^\sigma} + \left[g_{\mu\nu}\frac{\partial^2 x^\mu}{\partial \bar{x}^\sigma\,\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\right] + \left[g_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial^2 x^\nu}{\partial \bar{x}^\beta\,\partial \bar{x}^\sigma}\right].$$

Interchange β and σ in (c) and, noting that

$$\frac{\partial^2 x^\nu}{\partial \bar{x}^\beta\,\partial \bar{x}^\sigma} = \frac{\partial^2 x^\nu}{\partial \bar{x}^\sigma\,\partial \bar{x}^\beta},$$

obtain

$$(d)\qquad \frac{\partial \bar{g}_{\alpha\sigma}}{\partial \bar{x}^\beta} = \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\sigma}\frac{\partial x^\rho}{\partial \bar{x}^\beta} + g_{\mu\nu}\frac{\partial^2 x^\mu}{\partial \bar{x}^\beta\,\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\sigma} + \left[g_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial^2 x^\nu}{\partial \bar{x}^\beta\,\partial \bar{x}^\sigma}\right].$$

Add corresponding sides of (b) and (d) and subtract corresponding sides of (c) after interchanging μ and ρ in the first terms of (b) and after interchanging ν and ρ in the first terms of (d). Then take $\frac{1}{2}$ of both sides, obtaining

$$(e)\quad \frac{1}{2}\left(\frac{\partial \bar{g}_{\sigma\beta}}{\partial \bar{x}^\alpha} + \frac{\partial \bar{g}_{\alpha\sigma}}{\partial \bar{x}^\beta} - \frac{\partial \bar{g}_{\alpha\beta}}{\partial \bar{x}^\sigma}\right) = \frac{1}{2}\left(\frac{\partial g_{\rho\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\frac{\partial x^\rho}{\partial \bar{x}^\sigma} + \frac{1}{2}g_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\sigma}\frac{\partial^2 x^\nu}{\partial \bar{x}^\beta\,\partial \bar{x}^\alpha} + \frac{1}{2}g_{\mu\nu}\frac{\partial^2 x^\mu}{\partial \bar{x}^\beta\,\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\sigma},$$

the terms enclosed in brackets in (b), (c) and (d) canceling out in the additions and subtractions. On interchanging μ and ν in the last terms of (e) and on recalling that $g_{\nu\mu} = g_{\mu\nu}$, we get

$$(f)\quad \frac{1}{2}\left(\frac{\partial \bar{g}_{\sigma\beta}}{\partial \bar{x}^\alpha} + \frac{\partial \bar{g}_{\alpha\sigma}}{\partial \bar{x}^\beta} - \frac{\partial \bar{g}_{\alpha\beta}}{\partial \bar{x}^\sigma}\right) = \frac{1}{2}\left(\frac{\partial g_{\rho\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\frac{\partial x^\rho}{\partial \bar{x}^\sigma} + g_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\sigma}\frac{\partial^2 x^\nu}{\partial \bar{x}^\beta\,\partial \bar{x}^\alpha}.$$

Now

$$(g)\qquad \bar{g}^{i\sigma} = g^{\lambda\omega}\frac{\partial \bar{x}^i}{\partial x^\lambda}\frac{\partial \bar{x}^\sigma}{\partial x^\omega}.$$

Multiplying corresponding sides of (f) and (g), summing on σ, and using the identities

$$\frac{\partial x^\rho}{\partial \bar{x}^\sigma}\frac{\partial \bar{x}^\sigma}{\partial x^\omega} = \delta^\rho_\omega, \qquad g^{\lambda\mu}g_{\mu\nu} = \delta^\lambda_\nu,$$

we readily obtain

$$\frac{1}{2}\bar{g}^{i\sigma}\left(\frac{\partial \bar{g}_{\sigma\beta}}{\partial \bar{x}^\alpha} + \frac{\partial \bar{g}_{\alpha\sigma}}{\partial \bar{x}^\beta} - \frac{\partial \bar{g}_{\alpha\beta}}{\partial \bar{x}^\sigma}\right) = \frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\rho\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^i}{\partial x^\lambda} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^\beta\,\partial \bar{x}^\alpha}\frac{\partial \bar{x}^i}{\partial x^\lambda},$$

from which the desired transformation law 10·29 follows immediately on recalling the definition of $\bar{\Gamma}^i_{\alpha\beta}$ and $\Gamma^\lambda_{\mu\nu}$.

Chapter 11

1. Formula 11·12 for the covariant derivative of a tensor field can be established very quickly with the aid of normal coordinate methods of modern differential geometry. See the two Michal and Thomas 1927 papers.

2. In Chapter 17, we shall see that the answer is in general in the negative for the more general Riemannian spaces.

3. The operation of putting a covariant index equal to a contravariant index in a tensor and summing over that common index is called *contraction*. A contraction reduces the rank of a tensor by two: by one contravariant index and by one covariant index.

Chapter 13

1. The Navier-Stokes differential equations of hydrodynamics are discussed in Lamb's *Hydrodynamics*.

2. There are two viewpoints in hydrodynamics: one is the Eulerian point of view in terms of the Eulerian variables; the other is the Lagrangean point of view in terms of the Lagrangean variables. For a non-viscous fluid, the Eulerian hydrodynamical equations are the equations of motion of the fluid from the Eulerian point of view, and the Lagrangean hydrodynamical equations are the equations of motion of the fluid from the Lagrangean point of view. The Eulerian hydrodynamical equations in rectangular coordinates y^i are obtained by putting $\nu = 0$ in the Navier-Stokes equations 13·2.

Eulerian Hydrodynamical Equations.

$$\begin{cases} \dfrac{\partial u^i}{\partial t} = -u^\alpha \dfrac{\partial u^i}{\partial y^\alpha} - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y^i} + X^i \\ \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u^\alpha)}{\partial y^\alpha} = 0. \end{cases}$$

If a^i are the rectangular coordinates of a fluid particle in the initial state of the fluid, and if the $y^i(a^1, a^2, a^3, t)$ are the coordinates of the particle at time t, then the *Lagrangean hydrodynamical equations* are

$$\sum_{i=1}^{3}\left(\frac{\partial^2 y^i}{\partial t^2} - X^i\right)\frac{\partial y^i}{\partial a^j} + \frac{1}{\rho}\frac{\partial p}{\partial a^j} = 0,$$

$$\rho(y^1, y^2, y^3)\begin{vmatrix} \dfrac{\partial y^1}{\partial a^1}, & \dfrac{\partial y^1}{\partial a^2}, & \dfrac{\partial y^1}{\partial a^3} \\ \dfrac{\partial y^2}{\partial a^1}, & \dfrac{\partial y^2}{\partial a^2}, & \dfrac{\partial y^2}{\partial a^3} \\ \dfrac{\partial y^3}{\partial a^1}, & \dfrac{\partial y^3}{\partial a^2}, & \dfrac{\partial y^3}{\partial a^3} \end{vmatrix} = \rho_0(a^1, a^2, a^3).$$

For a treatment of classical hydrodynamics, including treatments of the Eulerian differential equations and the Lagrangean differential equations, the reader is referred to Lamb's *Hydrodynamics* and to Webster's *Dynamics*; cf. the references at the end of Part II. Ertel's monograph on dynamic meteorology has some interesting remarks.

3. To obtain expansion 13·6 for the divergence $u^\alpha_{,\alpha}$ we can proceed as follows. Since g, the determinant of the Euclidean metric tensor g_{ij}, is a relative scalar of weight two, it can be shown by the usual methods of obtaining covariant derivatives of absolute tensors that the covariant derivative $g_{,i}$ is given by

$$g_{,i} = \frac{\partial g}{\partial x^i} - 2g\Gamma^\alpha_{\alpha i}$$

But, in rectangular coordinates, g is unity and the Euclidean Christoffel symbols $\Gamma^i_{\alpha\beta}$ are zero. Hence $g_{,i}$ is zero in rectangular coordinates and consequently $g_{,i} = 0$ in all coordinates. This means that

$$\frac{\partial \log \sqrt{g}}{\partial x^i} = \Gamma^\alpha_{i\alpha}.$$

But the divergence $u^\alpha_{,\alpha}$ of u^α is by definition

$$u^\alpha_{,\alpha} = \frac{\partial u^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{i\alpha} u^i,$$

so that by the above result we find the following *equivalent expression for the divergence* $u^\alpha_{,\alpha}$:

$$u^\alpha_{,\alpha} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}\, u^\alpha)}{\partial x^\alpha}.$$

If in particular $u^i = g^{i\beta} \dfrac{\partial \psi}{\partial x^\beta}$, where $\psi(x^1, x^2, x^3)$ is a scalar field, we see that for this u^i

$$u^\alpha_{,\alpha} = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\, g^{\alpha\beta} \dfrac{\partial \psi}{\partial x^\beta}\right)}{\partial x^\alpha}.$$

But in rectangular cartesian coordinates $g = 1$, $g^{\alpha\beta} = \delta^{\alpha\beta}$, and so this scalar $u^\alpha_{,\alpha}$ reduces to the Laplacean in rectangular cartesian coordinates. Hence *Laplace's equation in general coordinates* x^i *is given by*

$$\frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\, g^{\alpha\beta} \dfrac{\partial \psi}{\partial x^\beta}\right)}{\partial x^\alpha} = 0 \quad \textit{when the unknown } \psi(x^1, x^2, x^3) \textit{ is a scalar field.}$$

Chapter 14

1. The fundamentals of a finite elastic deformation theory are not new. Kirchhoff in 1852 made the first systematic study, and E. and F. Cosserat in 1896 made an extensive investigation of the subject. Ricci and Levi-Civita in 1900 made brief but important contributions to the applications of the tensor calculus to elasticity theory. Léon Brillouin in 1924 simplified and recast Cosserat's treatment with the aid of the tensor calculus, and in 1937 F. D. Murnaghan, among several other authors, made contributions to the tensor theoretic treatment of elasticity theory.

Chapter 15

1. One can consider strain differential invariants of order r, i.e., scalar fields that retain their forms as functions of the metric tensor $g_{\alpha\beta}$, the strain tensor $\epsilon_{\alpha\beta}$,

and the derivatives of $\epsilon_{\alpha\beta}$ up to order r. Since successive covariant differentiation reduces to a corresponding order of partial differentiation in cartesian coordinates, a strain differential invariant can be written exclusively in terms of tensor fields by merely *replacing* the derivatives of $\epsilon_{\lambda\mu}$ by corresponding covariant derivatives. The Michal-Thomas methods (see the 1927 papers of these authors) can be used to carry on some interesting researches on strain differential invariants. Strain differential tensors can also be considered.

Examples of strain differential invariants of order one are

$$H = g^{\alpha\lambda}g^{\beta\mu}g^{\gamma\nu}\epsilon_{\alpha\beta,\gamma}\epsilon_{\lambda\mu,\nu} \qquad \text{and} \qquad g^{\alpha\beta}\frac{\partial I_i}{\partial x^\alpha}\frac{\partial I_j}{\partial x^\beta},$$

where $\epsilon_{\alpha\beta,\gamma}$ is the first covariant derivative of the strain tensor $\epsilon_{\alpha\beta}$, and where I_1, I_2, and I_3 are the three fundamental strain invariants. The question arises whether successive covariant differentiation of I_1, I_2, I_3 and combination with $g_{\alpha\beta}$ would yield all the fundamental strain differential invariants. Clearly there exist no strain *differential* invariants for homogeneous strains. Note that the vanishing of H is the necessary and sufficient condition for a homogeneous strain.

Chapter 16

1. The theory of complete systems of partial differential equations is discussed in Hedrick's translation of Goursat's *Cours d'analyse*, Vol. II, part II.

Chapter 17

1. For an account of Riemannian geometry, see Eisenhart's *Riemannian Geometry*. This reference, of course, treats the (classical) *finite* dimensional Riemannian geometries. Infinite dimensional and dimensionless "Riemannian" geometries were first studied by A. D. Michal (see paper 2 under Michal in the references for Part II). The applications to vibrations of elastic media are now being studied (see papers 4, 5, and 6 under Michal in the references).

2. Several engineering applications of Lagrange's equations of motion are to be found in the Kármán and Biot book.

3. We have seen that the Lagrangean equations of motion for a conservative dynamical system with no constraints and n degrees of freedom were

$$(a) \qquad \ddot{q}^\alpha + \Gamma^\alpha_{jk}\dot{q}^j\dot{q}^k = -g^{\alpha i}\frac{\partial V}{\partial q^i}.$$

This means that the *dynamical trajectories can be considered as curves in an n-dimensional Riemannian space whose element of arc length ds is given by*

$$(b) \qquad ds^2 = g_{ij}(q^1, \cdots, q^n)dq^i\, dq^j,$$

where the g_{ij} are the functions occurring in the kinetic energy $T = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j$. The differential equations of the dynamical curves are the second-order differential equations (a).

These curves are not, in general, geodesics in the Riemannian space with arc lengths ds given by formula (b). The question then arises whether it is possible to define a Riemannian space whose geodesics are some of the curves whose differential equations are Lagrange's equations of motion (a) for the given n-degree dynamical system. We shall show briefly that it is possible to define such a Riemannian space. To do this we must first show that the Lagrangean equations of motion have the energy integral

$$T + V = C$$

i.e., *the sum of the kinetic and potential energies is a constant along any chosen dynamical trajectory.* In fact,

$$\frac{dL}{dt} = \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i + \frac{\partial L}{\partial q^i}\dot{q}^i.$$

Hence on using Lagrange's equations of motion

$$\frac{dL}{dt} = \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\dot{q}^i$$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) = \frac{d}{dt}(2T).$$

We have then immediately $T + V = C$, a constant, since the kinetic potential $L = T - V$.

Consider now the *dynamical curves that correspond to any chosen energy constant* C. We shall show that *these particular dynamical curves are the geodesics of the n-dimensional Riemannian space whose element of arc length ds is given by*

(c) $$ds^2 = 2(C - V)g_{ij}(q^1, \cdots, q^n)\, dq^i\, dq^j,$$

where, as in (a), the *g_{ij} are the coefficients in the kinetic energy T.* For convenience in computation, let us define $A = 2(C - V)$ and $a_{ij} = Ag_{ij}$ so that (c) can be written

(d) $$ds^2 = a_{ij}\, dq^i\, dq^j.$$

By definition

$$a^{ij} = \frac{\text{Cofactor of } a_{ji} \text{ in } a}{a}$$

where a = the determinant of the a_{ij}. Hence, since A^{n-1} factors out in the numerator and A^n in the denominator respectively of a^{ij}, we see that

$$a^{ij} = \frac{g^{ij}}{A}.$$

Let $^*\Gamma^i_{jk}$ be the Christoffel symbols based on the metric tensor a_{ij}. Clearly

(e) $$^*\Gamma^i_{jk} = \frac{1}{2A}g^{i\sigma}\left(\frac{\partial Ag_{\sigma k}}{\partial q^j} + \frac{\partial Ag_{j\sigma}}{\partial q^k} - \frac{\partial Ag_{jk}}{\partial q^\sigma}\right)$$
$$= \Gamma^i_{jk} + \frac{1}{2A}\left(\delta^i_k\frac{\partial A}{\partial q^j} + \delta^i_j\frac{\partial A}{\partial q^k} - g^{i\sigma}\frac{\partial A}{\partial q^\sigma}g_{jk}\right)$$

By definition

$$ds^2 = 2(C - V)g_{ij}\, dq^i\, dq^j$$

and hence along a dynamical trajectory with energy constant C we have

$$\left(\frac{ds}{dt}\right)^2 = [2(C - V)]^2$$

so that

(f) $$\frac{dt}{ds} = A^{-1}$$

along a dynamical trajectory with energy constant C. By elementary calculus we have therefore

$$(g) \qquad \frac{dq^i}{ds} = \frac{dq^i}{dt}A^{-\frac{1}{2}}, \quad \frac{d^2q^i}{ds^2} = \frac{d^2q^i}{dt^2}A^{-2} - \frac{dq^i}{dt}\frac{dA}{dt}A^{-3}.$$

The differential equations for the geodesics of the Riemannian space whose ds is given by (c) is

$$(h) \qquad \frac{d^2q^i}{ds^2} + {}^*\Gamma^i_{jk}\frac{dq^j}{ds}\frac{dq^k}{ds} = 0.$$

On employing results (e), (f), and (g) in (h), we get after obvious simplifications

$$(i) \qquad \frac{d^2q^i}{dt^2} + \Gamma^i_{jk}\frac{dq^j}{dt}\frac{dq^k}{dt} = \frac{1}{2A}g^{i\sigma}\frac{\partial A}{\partial q^\sigma}g_{jk}\frac{dq^j}{dt}\frac{dq^k}{dt}.$$

But

$$g_{jk}\frac{dq^j}{dt}\frac{dq^k}{dt} = A$$

along a dynamical trajectory with energy constant C. Hence equations (i) reduce to

$$(j) \qquad \frac{d^2q^i}{dt^2} + \Gamma^i_{jk}\frac{dq^j}{dt}\frac{dq^k}{dt} = -g^{i\sigma}\frac{\partial V}{\partial q^\sigma}.$$

But these differential equations are another form of the Lagrangean differential equations of motion for our dynamical system. We have thus proved that *the geodesics of the Riemannian space with a ds given by* (c) *are dynamical trajectories of the dynamical system with kinetic energy* $T = \frac{1}{2}g_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt}$ and potential energy V.

By retracing our steps of proof, we can show that *any dynamical trajectory with energy constant C, i.e., any curve that satisfies* (j) *with energy constant C, can be considered a geodesic in the Riemannian space with an element of arc length ds given in* (c).

Illustrative Example of a Shaft Carrying Four Disks.

A shaft is fixed at one end and carries four disks at a distance l apart. If μ is the moment of inertia of each disk and q^1, q^2, q^3, q^4 the respective angular deflections of the four disks, then, if the shaft has a uniform torsional stiffness τ, the kinetic and potential energies are given respectively by

$$T = \frac{\mu}{2}\left[\left(\frac{dq^1}{dt}\right)^2 + \left(\frac{dq^2}{dt}\right)^2 + \left(\frac{dq^3}{dt}\right)^2 + \left(\frac{dq^4}{dt}\right)^2\right]$$

and

$$V = \frac{\tau}{2l}[(q^1)^2 + (q^2 - q^1)^2 + (q^3 - q^2)^2 + (q^4 - q^3)^2].$$

This is a conservative dynamical system of four degrees of freedom with no moving constraints. Hence the dynamical trajectories with total energy constant C can be represented as the *geodesics* of the four-dimensional Riemannian space whose element of arc length ds is given by

$$ds^2 = F(q^1, q^2, q^3, q^4)[(dq^1)^2 + (dq^2)^2 + (dq^3)^2 + (dq^4)^2],$$

where

$$F(q^1, q^2, q^3, q^4) = 2\mu\left\{C - \frac{\tau}{2l}[(q^1)^2 + (q^2 - q^1)^2 + (q^3 - q^2)^2 + (q^4 - q^3)^2]\right\}.$$

Numerous other engineering examples can be given, some simpler and some more sophisticated. For example, if the above shaft carries only *two disks*, the dynamical trajectories with total energy constant C can be represented as the geodesics of the *surface* whose ds is given by

$$ds^2 = 2\mu\left\{C - \frac{\tau}{2l}[(q^1)^2 + (q^2 - q^1)^2]\right\}[(dq^1)^2 + (dq^2)^2].$$

A more sophisticated example is given by the *symmetrical gyroscope;* see the exercise at the end of Chapter 17. Here the dynamical trajectories with total energy constant C can be represented as the geodesics of the three-dimensional Riemannian space whose element of arc length ds is given by

$$ds^2 = 2[C - Mgh\cos q^1][I(dq^1)^2 + (I\sin^2 q^1 + J\cos^2 q^1)(dq^2)^2 + 2J\cos q^1\, dq^2\, dq^3 + J(dq^3)^2].$$

Chapter 18

1. The conditions for an incompressible fluid and the continuity equations for a fluid flow state the invariance of two integrals: the integral for volume and the integral for mass, respectively. In other words, here we have two important examples of *integral invariants*. For the theory of integral invariants and its modern generalizations, the reader is referred to paper 1 under Michal in the references. There the reader will find ample references to the earlier work on integral invariants by H. Poincaré, S. Lie, E. Cartan, and E. Goursat.

2. We saw in Chapter 17 that the Riemann-Christoffel curvature tensor B^i_{jkl} is a zero tensor in any Euclidean space and hence in our three-dimensional Euclidean space. Define the tensor (field) R_{ijkl} by

(a) $$R_{ijkl} = g_{ir}B^r_{jkl}.$$

It is evident that

(b) $$R_{ijkl} = 0$$

holds throughout our three-dimensional Euclidean space. It can be shown that there are only six independent equations in (b). Three of them are included in

(c) $$R_{\alpha o\beta o} = 0;$$

two of them are included in

(d) $$R_{\alpha\beta\gamma o} = 0;$$

and the sixth one is given by

(e) $$R_{1212} = 0.$$

By straightforward calculation it can be shown that

$$\begin{cases} R_{\alpha o\beta o} = \dfrac{1}{2}\dfrac{\partial^2 g_{\alpha\beta}}{\partial x^{02}} - \dfrac{1}{4}g^{\gamma\delta}\dfrac{\partial g_{\alpha\gamma}}{\partial x^0}\dfrac{\partial g_{\beta\delta}}{\partial x^0} \\ R_{\alpha\beta\gamma o} = \Gamma_{\beta\gamma;\alpha} - \Gamma_{\alpha\gamma;\beta} \\ R_{1212} = {}^*R_{1212} + (\Gamma_{11}\Gamma_{22} - \Gamma_{12}\Gamma_{21}), \end{cases}$$

where, it is to be recalled, a semicolon denotes covariant differentiation on the surface $x^0 = C$ and $\Gamma_{\alpha\beta} = -\dfrac{1}{2}\dfrac{\partial g_{\alpha\beta}}{\partial x^0}$. The ${}^*R_{\alpha\beta\gamma\delta}$ stands for the curvature tensor on the surface. Define

$$b_{\alpha\beta}(x^1, x^2) = \left[\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^0}\right]_{x^0=0}.$$

If we evaluate the last two sets of conditions (*f*) on the surface $x^0 = 0$, we now see readily that the vanishing of the curvature tensor R_{ijkl} in the three-dimensional Euclidean space implies the following three sets of conditions:

$$(g)\qquad \frac{\partial^2 g_{\alpha\beta}}{\partial x^{02}} = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\alpha\mu}}{\partial x^0} \frac{\partial g_{\beta\nu}}{\partial x^0}$$

$$(h)\qquad b_{\alpha\beta;\gamma} - b_{\alpha\gamma;\beta} = 0$$

$$(i)\qquad {}^*R_{\rho\alpha\beta\gamma} = b_{\rho\beta} b_{\alpha\gamma} - b_{\rho\gamma} b_{\alpha\beta}.$$

Equations (*h*) and (*i*) are the well-known Codazzi and Gauss equations of the surface $x^0 = 0$. (Cf. McConnell's *Applications of the Absolute Differential Calculus*, p. 204, 1931.)

Conversely it can be shown that equations (*g*) *in 3-space* and equations (*h*) and (*i*) *over the surface* $x^0 = 0$ imply $R_{ijkl} = 0$ *throughout the 3-space*. But we shall not go into this matter any further.

If we differentiate (*g*) with respect to x^0 and if in this result we eliminate the second derivatives $\frac{\partial^2 g_{\alpha\beta}}{\partial x^{02}}$ by means of (*g*), we find

$$(j)\qquad \frac{\partial^3 g_{\alpha\beta}}{\partial x^{03}} = \frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^0} \frac{\partial g_{\alpha\mu}}{\partial x^0} \frac{\partial g_{\beta\nu}}{\partial x^0} + \frac{1}{2} g^{\mu\nu} g^{\lambda\gamma} \frac{\partial g_{\beta\nu}}{\partial x^0} \frac{\partial g_{\alpha\lambda}}{\partial x^0} \frac{\partial g_{\mu\gamma}}{\partial x^0}.$$

On differentiating the well-known identity

$$g_{\lambda\gamma} g^{\gamma\nu} = \delta^\nu_\lambda,$$

we can solve for $\frac{\partial g^{\mu\nu}}{\partial x^0}$ and obtain

$$\frac{\partial g^{\mu\nu}}{\partial x^0} = -g^{\lambda\mu} g^{\gamma\nu} \frac{\partial g_{\lambda\gamma}}{\partial x^0}.$$

If we substitute this expression in (*j*), we evidently obtain

$$\frac{\partial^3 g_{\alpha\beta}}{\partial x^{03}} = 0.$$

Hence *the surface tensor components* $g_{\alpha\beta}(x^1, x^2, x^0)$ *are quadratic* expressions of x^0. Indeed, if we write

$$c_{\alpha\beta} = \left[\frac{1}{2} \frac{\partial^2 g_{\alpha\beta}}{\partial x^{02}}\right]_{x^0=0},$$

then

$$(k)\qquad g_{\alpha\beta} = a_{\alpha\beta} + 2b_{\alpha\beta} x^0 + c_{\alpha\beta}(x^0)^2.$$

A glance at (*g*) shows that

$$c_{\alpha\beta} = a^{\pi\sigma} b_{\alpha\pi} b_{\beta\sigma}.$$

Hence, $a_{\alpha\beta}$, $b_{\alpha\beta}$, and $c_{\alpha\beta}$ *are respectively the tensor coefficients of the first, second, and third fundamental forms of the surface* $x^0 = 0$.

It can be shown that we may write (*i*) in the form

$$(l)\qquad K \epsilon_{\beta\gamma} \epsilon_{\rho\alpha} = b_{\rho\beta} b_{\alpha\gamma} - b_{\rho\gamma} b_{\alpha\beta},$$

where

$$\epsilon_{\alpha\beta} = a^{\frac{1}{2}} \eta_{\alpha\beta}, \quad a = \left| \, a_{\alpha\beta} \, \right|, \quad \eta_{11} = \eta_{22} = 0, \quad \eta_{12} = 1, \quad \eta_{21} = -1,$$

and

$$K = \tfrac{1}{4} \epsilon^{\rho\alpha} \epsilon^{\beta\gamma} {}^*R_{\rho\alpha\beta\gamma} = \tfrac{1}{2}[(a^{\pi\sigma} b_{\pi\sigma})^2 - b^{\pi\sigma} b_{\pi\sigma}],$$

the *total curvature* (*Gaussian curvature*) *of the surface* $x^0 = 0$. If we apply the tensor $a^{\gamma\rho}$ to (*l*), we find

$$c_{\alpha\beta} - 2Hb_{\alpha\beta} + Ka_{\alpha\beta} = 0, \qquad (m)$$

where H is the *mean curvature of the surface* $x^0 = 0$,

$$H = \tfrac{1}{2}a^{\pi\sigma}b_{\pi\sigma}.$$

If R_1 and R_2 are the principal radii of curvature, it is well known in surface theory that

$$H = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad K = \frac{1}{R_1 R_2}.$$

With these relations, we can easily calculate

$$g = \left| \ g_{\alpha\beta} \ \right| = \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}g_{\alpha\gamma}g_{\beta\delta}.$$

For this purpose, we have to calculate the quantities

$$\eta^{\alpha\beta}\eta^{\gamma\delta}a_{\alpha\gamma}a_{\beta\delta}, \quad \eta^{\alpha\beta}\eta^{\gamma\delta}a_{\alpha\gamma}b_{\beta\delta}, \quad \eta^{\alpha\beta}\eta^{\gamma\delta}b_{\alpha\gamma}b_{\beta\delta},$$

if similar quantities involving $c_{\mu\nu}$ have been reduced with the help of (m). Now

$$\eta^{\alpha\beta}\eta^{\gamma\delta}a_{\alpha\gamma} = aa^{\beta\delta}.$$

Hence

$$\eta^{\alpha\beta}\eta^{\gamma\delta}a_{\alpha\gamma}a_{\beta\delta} = 2a, \quad \eta^{\alpha\beta}\eta^{\gamma\delta}a_{\alpha\gamma}b_{\beta\delta} = 2Ha. \qquad (n)$$

Applying $\eta^{\alpha\beta}\eta^{\gamma\delta}$ to (*l*), we obtain

$$\eta^{\alpha\beta}\eta^{\gamma\delta}b_{\alpha\gamma}b_{\beta\delta} = 2K. \qquad (o)$$

If we substitute (n) and (o) in the determinant g, we find

$$g = (1 + 2Hx^0 + K(x^0)^2)^2 = a\left(1 + \frac{x^0}{R_1}\right)^2\left(1 + \frac{x^0}{R_2}\right)^2.$$

Hence

$$g^{\alpha\beta} = a^{-1}\left(1 + \frac{x^0}{R_1}\right)^{-1}\left(1 + \frac{x^0}{R_2}\right)^{-1}\eta^{\alpha\gamma}\eta^{\beta\delta}g_{\gamma\delta}.$$

Thus, if we expand $g^{\alpha\beta}$ as a power series in x^0, the series are convergent if $|x^0| < R_m$, where R_m is the minimum value of the principal radii of curvature of the surface $x^0 = 0$. The same is true of many other geometrical quantities derived from $g_{\alpha\beta}$ and $g^{\alpha\beta}$.

REFERENCES FOR PART I

AITKEN, A. C. 1. "Studies in Practical Mathematics. I. The Evaluation of a Certain Triple Matrix Product," *Proc. Roy. Soc. Edinburgh*, vol. 57 (1937), pp. 172–181.

2. "Studies in Practical Mathematics. II. The Evaluation of the Latent Roots and Latent Vectors of a Matrix," *Proc. Roy. Soc. Edinburgh*, vol. 57 (1937), pp. 269–304.

3. "On Bernoulli's Numerical Solution of Algebraic Equations," *Proc. Roy. Soc. Edinburgh*, vol. 46 (1926), p. 289.

BINGHAM, M. D. "A New Method for Obtaining the Inverse Matrix," *J. Am. Statistical Assoc.*, vol. 36 (1941), pp. 530–534.

BLEAKNEY, H. M. *J. Aeronaut. Sci.*, vol. 9 (1941), pp. 56–63.

BOCHER, *Introduction to Higher Algebra.*

BORN and JORDAN. *Zeitschrift für Physik*, vol. 34 (1925), p. 858.

CARSLAW and JAEGER. *Operational Calculus in Applied Mathematics*, Oxford University Press, 1941.

DEN HARTOG. *Mechanical Vibrations*, McGraw-Hill Book Co., 1934.

DIRAC, P. A. M. *The Principles of Quantum Mechanics*, Oxford University Press, 1930.

DUNCAN, W. J., and A. R. COLLAR. 1. "A Method for the Solution of Oscillation Problems by Matrices," *Philosophical Magazine and Journal of Science*, vol. 17 (1934), pp. 865–909.

2. "Matrices Applied to the Motions of Damped Systems," *Philosophical Magazine and Journal of Science*, vol. 19 (1935), pp. 197–219.

FANTAPPIE, L. "Le calcul des matrices," *Comptes rendus* (Paris), vol. 186 (1928), pp. 619–621.

FRAZER, DUNCAN, and COLLAR. *Elementary Matrices and Some Applications to Dynamics and Differential Equations*, Cambridge University Press, 1938.

FRAZER, R. A., and W. J. DUNCAN. "The Flutter of Aeroplane Wings," *Reports and Memoranda of the (British) Aeronautical Research Committee*, 1155, August, 1928.

HOLZER, H. *Die Berechnung der Drehschwingungen*, Julius Springer, Berlin.

HOTELLING, H. "Some New Methods in Matrix Calculation," *Annals of Mathematical Statistics*, vol. 14 (March, 1943), pp. 1–34.

KÁRMÁN, THEODORE VON. *60th Anniversary Volume*, California Institute of Technology, 1941.

KÁRMÁN and BIOT. *Mathematical Methods in Engineering*, McGraw-Hill Book Co., 1940.

KUSSNER, H. G., and L. SCHWARZ. "The Oscillating Wing with Aerodynamically Balanced Elevator," *N.A.C.A. Technical Memorandum* 991, October, 1941.

LIEBER, PAUL. 1. "Outline of Four Degrees of Freedom Flutter Analysis," *S. M. Douglas Aircraft Corporation Report* 3572, March 20, 1942.

2. "Three Dimensional Four Degrees of Freedom Flutter Analysis," *S. M. Douglas Aircraft Corporation Report* 3587, April 4, 1942.

3. "An Iteration Method for Calculating the Latent Roots of a Flutter Matrix," *S. M. Douglas Aircraft Corporation Report* 3855, Sept. 29, 1942.

4. "A General Three Dimensional Flutter Theory as Applied to Aircraft," *S. M. Douglas Aircraft Corporation Report* 8127.

5. "Matric Methods for Calculating Wing Divergence Speed, Aileron Reversal Speed, and Effective Control," *S. M. Douglas Aircraft Corporation Report* 8142, Oct. 13, 1943.

MacDuffee, C. C. *The Theory of Matrices*, Julius Springer, Berlin, 1933; a bibliographical report.

Michal, A. D., and R. S. Martin. "Some Expansions in Vector Space," *Journal de mathématiques pures et appliquées*, vol. 13 (1934), pp. 69–91.

Oldenburger, Rufus. 1. "Infinite Powers of Matrices and Characteristic Roots," *Duke Mathematical Journal*, vol. 6 (1940), pp. 357–361.

2. "Convergence of Hardy Cross's Balancing Process," *Journal of Applied Mechanics*, 1940.

Pipes, Louis A. 1. "Matrices in Engineering," *Electrical Engineering*, 1937, p. 1177.

2. "Matrix Theory of Oscillatory Networks," *Journal of Applied Physics*, vol. 10 (1939), pp. 849–860.

3. "The Matrix Theory of Four Terminal Networks," *Philosophical Magazine*, 1940, pp. 370–395.

4. "The Transient Behavior of Four-Terminal Networks," *Philosophical Magazine*, 1942, pp. 174–214.

5. "The Matrix Theory of Torsional Oscillations," *Journal of Applied Physics*, vol. 13, No. 7 (1942), pp. 434–444.

Theodorsen, T. "General Theory of Aerodynamic Instability and the Mechanism of Flutter," *N.A.C.A. Report* 496, 1934, pp. 413–433, especially pp. 419–420.

Theodorsen, T., and I. E. Garrick. 1. "Mechanism of Flutter," *N.A.C.A. Report* 685, 1940.

2. "Non-Stationary Flow about a Wing-Aileron-Tab Combination Including Aerodynamic Balance," *Advanced Restricted Report* 334, Langley Memorial Aeronautical Library, March, 1942.

Timoshenko, S. *Vibration Problems in Engineering*, D. Van Nostrand Co., New York, 1928.

Volterra, V. *Theory of Functionals*, Blackie & Son, London, 1930.

Wedderburn, J. H. M. *Lectures on Matrices* (American Mathematical Society Colloquium Publications, 1934); contains a good bibliography on pure matrix theory up to 1934.

Whittaker, E. T. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, 1927.

REFERENCES FOR PART II

Appell, P. *Cours de mécanique rationelle*, Tome V, Gauthier-Villars, 1926.

Brillouin, Léon. 1. Toronto Mathematical Congress, 1924.

2. "Les lois de l'élasticité sous forme tensorielle valable pour des coordonnées quelconques," *Annales de physique*, vol. 3 (1925), pp. 251–298.

3. *Les tenseurs en mécanique et en elasticité*, Masson, Paris, 1938.

Chien, W. Z. "The Intrinsic Theory of Thin Shells and Plates, Part I — General Theory," *Quarterly of Applied Mathematics*, vol. 1 (1944), pp. 297–327.

Cosserat, E. and F. "Sur la théorie de l'élasticité," *Annales de Toulouse*, vol. 10 (1896).

Darboux, G. *Leçons sur la théorie des surfaces*, vol. 2, pp. 452–527, Gauthier-Villars, Paris, 1915.

Eisenhart, L. P. *Riemannian Geometry*, Princeton University Press, 1926.

Ertel, H. *Methoden und Probleme der dynamischen Meteorologie,* Julius Springer, Berlin, 1938; Edwards Brothers, Ann Arbor, Mich., 1943.

Jeffreys, H. *Cartesian Tensors*, Cambridge University Press, 1931.

Kármán, Theodore von. 1. "The Fundamentals of the Statistical Theory of Turbulence," *Journal of the Aeronautical Sciences*, vol. 4 (1937), pp. 131–138.

2. "Turbulence," *Journal of the Royal Aeronautical Society*, December, 1937, Twenty-fifth Wilbur Wright Memorial Lecture.

Kármán, Theodore von, and Leslie Howarth. "On the Statistical Theory of Turbulence," *Proceedings of the Royal Society of London*, vol. 164 (1938), pp. 192–215.

Kasner, E. *Differential-Geometric Aspects of Dynamics*, American Mathematical Society, New York, 1913.

Kirchhoff, G. "Über die Gleichungen des Gleichgewichtes eines elastischen Korpers bei nicht unendlich kleinen Verschiebungen seiner Theile," *Sitz. Math.-Nat. Klasse der kaiserlichen Akad. der Wiss.*, Vol. 9 (1852), p. 762.

Lamb, H. *Hydrodynamics*, Cambridge University Press, 1924.

Lamé, G. *Leçons sur les coordonnés curvilignes et leurs diverses applications*, Paris, 1859.

Levi-Civita, T. *The Absolute Differential Calculus* (*Calculus of Tensors*), Blackie and Son, London, 1927.

Love, A. E. H. *A Treatise on the Mathematical Theory of Elasticity*, Cambridge University Press, 1927).

Michal, A. D. 1. "Functionals of *R*-Dimensional Manifolds Admitting Continuous Groups of Transformations," *Trans. American Mathematical Society*, vol. 29 (1927), pp. 612–646. (This paper includes an introduction to multiple-point tensor fields.)

2. "General Differential Geometries and Related Topics," *Bull. American Mathematical Society*, vol. 45 (1939), pp. 529–563.

3. "Recent General Trends in Mathematics," *Science*, vol. 92 (1940), pp. 563–566.

4. "An Analogue of the Maupertuis-Jacobi 'Least' Action Principle for Dynamical Systems of Infinite Degrees of Freedom," *Bull. American Mathematical Society*, vol. 49 (1943), abstract 288, p. 862.

5. "Physical Models of Some Curved Differential-Geometric Metric Spaces of Infinite Dimensions. I. Wave Motion as a Study in Geodesics," *Bull. American Mathematical Society*, vol. 49 (1943), abstract 289, p. 862.

6. "Physical Models of Some Curved Differential-Geometric Metric Spaces of Infinite Dimensions. II. Vibrations of Elastic Beams and Other Elastic Media as Studies in Geodesics," *Bull. American Mathematical Society*, vol. 50 (1944), abstract 154, p. 345.

Michal, A. D., and T. Y. Thomas. 1. "Differential Invariants of Affinely Connected Manifolds," *Annals of Mathematics*, vol. 28 (1927), pp. 196–236.

2. "Differential Invariants of Relative Quadratic Differential Forms," *Annals of Mathematics*, vol. 28 (1927), pp. 631–688.

McConnell, A. J. *Applications of the Absolute Differential Calculus*, Blackie and Son, London, 1931.

Murnaghan, F. D. "Finite Deformations of an Elastic Solid," *American Journal of Mathematics*, vol. 59 (1937), pp. 235–260.

REUTTER, F. "Eine Anwendung des absoluten Parallelismus auf die Schalentheorie," *Z. angew. Math. Mech.*, vol. 22 (1942), pp. 87–98.

RICCI, G., and T. LEVI-CIVITA. "Methodes de calcul differentiel absolu et leurs applications," *Mathematische Annalen*, vol. 54 (1900), pp. 125–201.

RIEMANN, B. *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, 1854 (*Gesammelte Werke*, 1876, pp. 254–269).

SYNGE, J. L. "On the Geometry of Dynamics," *Phil. Trans. Royal Society London, A*, vol. 226 (1926–1927), pp. 31–106.

SYNGE, J. L., and W. Z. CHIEN. "The Intrinsic Theory of Elastic Shells and Plates," *Theodore von Kármán Anniversary Volume*, 1941, pp. 103–120.

THOMAS, T. Y. *Differential Invariants of Generalized Spaces*, Cambridge University Press, 1934.

VEBLEN, O. "Invariants of Quadratic Differential Forms," *Cambridge Tract* 24, Cambridge University Press, 1927.

VEBLEN, O., and J. H. C. WHITEHEAD. "The Foundations of Differential Geometry," *Cambridge Tract* 29, Cambridge University Press, 1932.

WEBSTER, A. G. *The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies*, G. E. Stechert & Co., New York, 1942 reprint.

WHITTAKER, E. T. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, 1927.

WRIGHT, J. E. "Invariants of Quadratic Differential Forms," *Cambridge Tract* 9, Cambridge University Press, 1908.

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