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METHODS IN NUMERICAL ANALYSIS

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PREFACE

The interest in numerical analysis has increased tremendously in recent years, primarily because of the success of the large-scale calculating machines. Problems that used to take hours are now solved in seconds. The ability to obtain numerical answers in a short time has led engineers into lengthy analytical studies, which in turn have placed additional emphasis on numerical methods. Thus, at the annual meeting of the American Society for Engineering Education held in 1954, it was stated that "Every engineer who has mastered the Calculus should also have a course in Numerical Analysis." Practically every university is now teaching a course in numerical analysis, and there is a need for an elementary textbook. It is to meet this need that this book was written.

Since it is a textbook for the practical man, it does not seem appropriate to fill it with mathematical sophistication, but rather to present the methods in a simple manner. The mathematician may peruse this book profitably, however, to obtain a knowledge of today's methods and references to deeper investigations.

Numerical analysis may be divided into two main categories: (1) the analysis of tabulated data, and (2) the numerical method of finding the solutions to equations. Both are treated in this book. Emphasis is placed on the methods which are easily adapted to automatic desk calculators since most engineers and scientists will have these available and because these methods may also be used on the large-scale calculators.

The author is a firm believer in systematic procedures, and many illustrative examples, calculating forms, and schematics are displayed. Many of the numerical methods derived in the book are easily performed with the aid of tables. Consequently, the necessary tables form a part of this book. Exercises with some answers are also given to enable the student to master this subject matter.

An attempt has been made to present not only the classical procedures but also to exhibit the most recent methods which have been developed because of the new calculating machines. It is believed that a knowledge of the material in this book will permit the user to solve the numerical analysis problem that may occur in his investigations.

Many people have contributed to this book, both directly and indirectly. Contribution of subject matter has been acknowledged throughout the text and in the bibliography. The author gratefully acknowledges his indebtedness to his many friends who aided in the preparation of the manuscript and its subsequent publication. He is indebted to Professor Sir Ronald A. Fisher, Cambridge, to Dr. Frank Yates, Rothamsted, and to Messrs. Oliver and Boyd Limited, Edinburgh, for permission to reprint Table No. XVII from their book *Statistical Tables for Biological, Agricultural, and Medical Research.*

KAJ L. NIELSEN

CONTENTS

MATHEMATICAL SYMBOLS

$=$, is equal to;
$\neq$, is not equal to;
$\doteq$, is approximately equal to;
$<$, is less than;
$>$, is greater than;
$\leq$, is less than or equal to;
$\geq$, is greater than or equal to;
$\ll$, is much less than;
$\gg$, is much greater than;
Δ, difference
ln, natural logarithm;

$\dot{x}$, derivative of x with repsect to t;
$\sqrt{n}$, square root of n;
$n \to \infty$, n approaches infinity;
$\angle ABC$, angle with vertex at B;
ΔABC, triangle ABC;
. . . . , and so on;
Δx, increment of x;
P_1, P sub 1;
Σ, sum of;
Π, product of;
log, common logarithm.

GREEK ALPHABET

LETTERS		NAMES	LETTERS		NAMES	LETTERS		NAMES
Α	α	Alpha	Ι	ι	Iota	Ρ	ρ	Rho
Β	β	Beta	Κ	κ	Kappa	Σ	σ	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	Τ	τ	Tau
Δ	δ	Delta	Μ	μ	Mu	Υ	υ	Upsilon
Ε	ϵ	Epsilon	Ν	ν	Nu	Φ	φ	Phi
Ζ	ζ	Zeta	Ξ	ξ	Xi	Χ	χ	Chi
Η	η	Eta	Ο	o	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ω	Omega

METHODS IN NUMERICAL ANALYSIS

CHAPTER I. FUNDAMENTALS

1. Introduction. In general, the application of mathematics eventually requires results in numerical form. The numerical answers desired may result from evaluations of formulas, solutions of equations, or inferences drawn from tabulated data. Numerical analysis is that branch of mathematics which is concerned with obtaining these numerical answers. Frequently, other branches of mathematics will develop theories that yield solutions to problems in "closed form"; that is, formulas which represent exact solutions. The work of the numerical analyst is then reduced to a simple evaluation of these formulas for variations in the parameters and variables. On the other hand, frequently a "closed form" solution cannot be obtained, and in such a case an approximate method is the alternative. The solution takes the form of a formula which approximates the exact solution, and the difference between the two is reduced to a minimum. It is the work of the numerical analyst to develop approximate formulas in such form that they may be evaluated, to evaluate them, and to state the degree of approximation.

A course in methods of numerical analysis is one which teaches how to perform this work. In so doing the evaluation of mathematical expressions is reduced to the fundamental operations of arithmetic. These operations are then carried out by calculating machines of which there are many varieties. The methods of this book are those which are best adapted to electrical desk calculating machines. As is often the case, however, many are equally well adapted to large-scale digital computers, and references will be made to them whenever appropriate.

Since this book will be concerned with numbers, it seems appropriate to begin with a discussion of them. Certain mathematical concepts are also basic to this study, and they will be reviewed for the reader's benefit.

2. The Accuracy of Numbers. There are two kinds of numbers: those which are absolutely exact and those which denote values to a certain degree of accuracy. Examples of absolutely exact numbers are the integers, 1, 2, 3, $\cdots$, rational fractions, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$, the quantities $\sqrt{3}$, π, e, etc., written in this manner. An approximate number is one which expresses a value that is only accurate to the number of digits recorded. Thus, although $\sqrt{3}$ is exact as written, it cannot be expressed exactly by a finite number of digits; we could write for it the number 1.732 which now is an approximate number approximating the value of the square root of 3. We could also have written 1.73205 which is a better approximation. The digits used to express a number are called *significant figures* if they have a meaning in the number. Thus all of the digits in 1.73205 are significant figures; however, in the number 0.00572 only 5, 7, and 2 are significant figures, the 0 having been used to place the decimal point. If a zero, 0, is used at the end of a number, additional information is required to determine whether or not it is significant. Thus $525,000 may be exact or may be an expression of money to the nearest thousand. If exactness is desired, such numbers are usually written in the powers-of-ten notation which is 5.25×10^5 or 5.25000×10^5; the significant figures being written in the left factor.

Frequently arithmetical operations will yield numbers which have no termination, and it is necessary to cut them to a useable number of figures. This cutting-off process is called *rounding-off*. To carry out such a process certain arbitrary rules must be established. The following ones are in practice by most numerical analysis and are recommended. To round off a number to n significant figures, throw away all digits to the right of the nth place, and if this discarded number is

a) greater than half a unit in the nth place, increase the digit in that place by 1;

b) less than half a unit in the nth place, leave the digit in that place unaltered;

c) exactly half a unit in the nth place;
  i) increase an odd digit in the nth place by 1;
  ii) leave an even digit in the nth place unaltered.

The number is then said to be *correct to n significant figures.*

Example 1.1. The following numbers are rounded off to five significant figures.

31.35764 to 31.358
10.19313 to 10.193
14.32250 to 14.322
14.32150 to 14.322

The rule for the case of "exactly half" is quite arbitrary but has been found to be the wisest in most cases. In the case where a set of numbers has to be added, it may be practical to deviate from the rule by increasing in *half* the cases and leaving unchanged in the *other half* of the cases.

In performing a sequence of arithmetical operations on numbers, certain facts must be kept in mind. Of primary importance is the fact that, in general, the results are no more accurate than the original data used. Thus, if the original data are given to three significant figures, the result is accurate to only three significant figures in most cases. However, this does *not* imply that all computations should be rounded off to the significant figures of the data at *each step* of the computation. Quite to the contrary, it is advisable to retain more figures during the computation. A good rule to adopt is the following one.

Rule. *During the computation retain one more figure than that given in the data and round off after the last operation has been performed.*

Another practice, which has become quite widespread when using electric desk calculators, is to set the machine for a fixed number of decimal points employing at least one more figure than that given in the data and then to use all the figures to that many decimal points. Frequently, when this practice is followed no attention is paid to rounding off.

Example 1.2 Compute the value of N

$$N = \frac{27.13 \times 3.157}{11.32} + (5.921)(.3214)$$
$$= 9.469.$$

The machine is set to five decimal places. All computation is carried out employing figures to five decimals and the final answer is rounded off to four significant figures.

This practice is especially prominent if tables of values are employed during the computation. The machine is then set to the same decimal accuracy as the tables which are used.

The two methods described above will, in general, prevent the accumulation of errors during a lengthy computation. However, no rule is infallible. One should always hesitate to throw away numbers during a computation; on the other hand, it is foolhardy to waste a lot of time employing unnecessary numbers.

One of the most troublesome operations in numerical calculations is the one of subtraction when the numbers are nearly equal. Thus, if we calculated the difference .2341 − .2337 we obtain .0004, an answer which has only one significant figure. If this result is used again in a computation, some strange things can happen. In such cases it is desirable to rearrange the computation or change the formula so that the difference may be obtained in a new manner. The classical example is the value of $1 - \cos x$ for small values of x; in this case use the equivalent expression $2 \sin^2 \frac{1}{2} x$ or expand $\cos x$ in a series.

3. Some Basic Concepts. A review of certain basic mathematical concepts will be presented here.

A. Formulas and Functions. Numerical analysis frequently reduces to the evaluation of formulas and functions for variations in the parameters and over a range of values for the independent variable. We shall be concerned with two types of functions: one, those which are specified by formulas; and, two, those which are specified by tables of values. In either case we shall usually assume that the functions are continuous, possess as many derivatives as required, and if singularities exist, they are defined by appropriate additional information.

The evaluation of a function expressed by a formula is easily accomplished by simply substituting the given values of the variables. It must be remembered, however, that formulas can be rearranged. Thus, for example,

$$y = x^2 + 2x + 3 \tag{3.1}$$
$$y - 3 = x(x + 2) \tag{3.2}$$
$$x = -1 \pm \sqrt{y - 2} \tag{3.3}$$

are all equivalent formulas and in algebra express the same relationship. In numerical analysis a formula is more often used to define a process. Formula (3.1) defines the process for finding y for given values of x, formula (3.2) shows a form for computation, whereas formula (3.3) states the process for finding values of x if y is specified.

The evaluation of a function specified at discrete values of the variable may at first seem to be an easy matter since its values are known at the given points, at least to the accuracy of the table. However, to obtain the value of the function at a point not listed may prove difficult and depends greatly on the behavior of the function. The analysis of tabulated data forms an integral part of numerical analysis and is usually divided into two parts: the finding of a value not listed, and the investigation of the behavior of the function. Both topics will be considered in subsequent chapters. However, before leaving the subject matter it is well to mention that plotting or graphing of functions specified by empirical data is a great aid in determining the properties of such functions.

B. Polynomials. The polynomial will be an overworked mathematical expression throughout this book. The function

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n, \tag{3.4}$$

where a_i, $(i = 0, \cdots, n)$, are constants, is a *polynomial* of degree n in the unknown x. In a course in algebra, the value of a polynomial for a given value of x was found by synthetic division. For example, the value of $f(x)$ at $x = 2$ in the polynomial

$$f(x) = 3x^4 - 5x^3 + 2x^2 - 3x + 6 \tag{3.5}$$

was found to be 16 by the schematic

$$\begin{array}{rrrrrr} 3 & -5 & 2 & -3 & 6 & \underline{2} \\ & 6 & 2 & 8 & 10 & \\ \hline 3 & 1 & 4 & 5 & 16 = f(2) & . \end{array}$$

Another method of doing synthetic division readily adapted to calculators is the one which has been termed the *nesting* process. The polynomial (3.4) may be rewritten in the following nested form

$$\text{(3.6)} \quad \begin{aligned} f(x) &= a_n + x\{a_{n-1} + x[a_{n-2} + \cdots + x(a_1 + a_0x)]\} \\ &= \{[(a_0x + a_1)x + a_2]x + a_3\}x + \cdots + a_n. \end{aligned}$$

The value of the polynomial for a given value of x is now found by starting with $a_0x + a_1$ multiplying it by x adding a_2 to the result, etc. This method is applicable to any finite series. In the example given by the polynomial (3.5) we have

$$f(x) = \{[(3x - 5)x + 2]x - 3\}x + 6$$

and at $x = 2$

$$f(2) = \{[(3 \cdot 2 - 5)2 + 2]2 - 3\}2 + 6 = 16.$$

All polynomials of degree n can be factored into n factors;

$$\text{(3.7)} \qquad f(x) = (x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

where x_i, $(i = 0, \cdots, n - 1)$, are the roots of the equation $f(x) = 0$.

The particular polynomial

$$\text{(3.8)} \qquad P^n(x) = x(x - 1)(x - 2) \cdots [x - (n - 1)]$$

is called a *factorial polynomial.* If the product on the right is multiplied out the polynomial may be written in the form

$$\text{(3.9)} \quad P^n(x) = S_{0,n}x^n + S_{1,n}x^{n-1} + S_{2,n}x^{n-2} + \cdots + S_{n-1,n}x$$

where $S_{i,n}$ are the *Stirling Numbers of the first kind.* The second subscript, n, denotes the degree of the polynomial; thus a fifth degree factorial polynomial

$$\begin{aligned} P^5(x) &= x(x - 1)(x - 2)(x - 3)(x - 4) \\ &= S_{0,5}x^5 + S_{1,5}x^4 + S_{2,5}x^3 + S_{3,5}x^2 + S_{4,5}x. \end{aligned}$$

The Stirling Numbers have the most useful property

$$S_{i,n+1} = S_{i,n} - nS_{i-1,n} \tag{3.10}$$

which enables one to easily compute a table of these numbers. Such a table is given as Table II in the list of tables at the back of this book.

Any polynomial may be expressed as the sum of factorial polynomials. For example,

$$\begin{aligned} P(x) &= 2x^4 - 3x^3 + 14x^2 + 22x + 10 \\ &= 2P^4(x) + 9P^3(x) + 10P^2(x) + 35P^1(x) + 10P^0(x). \end{aligned}$$

This sum is obtained by first subtracting $2P^4(x)$ from the original polynomial, leaving a polynomial of degree 3 with leading term $9x^3$. Subtract now $9P^3(x)$ from this difference, etc. Schematically, we have

	x^4	x^3	x^2	x	x^0
$P(x)$	2	-3	14	22	10
$2P^4(x) =$	2	-12	22	-12	
Diff. =		9	-8	34	10
$9P^3(x) =$		9	-27	18	
Diff. =			19	16	10
$10P^2(x) =$			19	-19	
Diff. =				35	10
$35P^1(x) =$				35	
Diff. =					10
$10P^0(x) =$					10

A special factorial polynomial, namely that one which has a term, $x - i$, missing is frequently employed in numerical analysis and is given a special notation. Thus we define

$$\begin{aligned} P_i^n(x) &= (x - i)^{-1}P^n(x) \\ &= x(x-1) \cdots (x - i + 1)(x - i - 1) \cdots (x - n + 1). \end{aligned} \tag{3.11}$$

Factorial polynomials have many useful properties, a few of which will be listed for reference. The reader may like to verify each of these.

Properties.

1. $P^n(i) = 0, \qquad (i = 0, 1, \cdots, n-1).$
2. $P_i^n(j) = 0, \qquad j \neq i.$
3. $P_i^n(i) = (-1)^{n-i-1}i!(n-i-1)!.$
4. $P^n(j) = \dfrac{j!}{(j-n)!}, \qquad (j = n, n+1, \cdots).$
5. $P^n(x+1) - P^n(x) = nP^{n-1}(x).$
6. $\displaystyle\sum_{i=0}^{r} P^n(x+i) = \frac{1}{n+1}[P^{n+1}(x+r+1) - P^{n+1}(x)].$

C. Binomial Coefficients. One of the most elementary series is that one obtained from applying the binomial theorem to the expression $(1+x)$,

$$(3.12) \qquad (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots.$$

The coefficients of this series are called the *binomial coefficients* and occur frequently in numerical analysis. The most commonly employed notation is the following one:

$$(3.13) \qquad \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

where k is any integer. It is easily seen that the numerator on the right-hand side is a factorial polynomial in n of degree k, so that we can write

$$(3.14) \qquad \binom{n}{k} = \frac{P^k(n)}{k!}.$$

The binomial coefficients possess many interesting properties of which only a few will be listed.

Properties. If n is an integer

1. $\binom{n}{k} = \begin{matrix} 1, & \text{if } k = n; \\ 0, & \text{if } k > n. \end{matrix}$
2. $\binom{n}{k} = \binom{n}{n-k}.$

3. $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

As is well known (Pascal's triangle) a table of values for the binomial coefficients can easily be tabulated, and by Property 2 above, only values to the mid-point need be recorded. (*See* Table I in the back of the book).

D. Series. Certain portions of numerical analysis are based on fundamentals obtained from series expansions. It is therefore appropriate to review briefly the concept of series expansion, in particular the representation of functions by power series. The second power series that a student of mathematics usually encounters is the *Maclaurin Series*.

$$(3.15)\quad f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots$$

where

$f^{(n)}(0)$ is the value of the nth derivative of $f(x)$ at $x = 0$.

Another well-known expression is the *Taylor Series*

$$(3.16)\quad f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots.$$

These series may each be written as a finite series plus a remainder by the use of the *Extended Theorem of Mean Value*. Thus we have

$$(3.17)\quad f(x) = f(a) + f'(a)(x-a) + \cdots + f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!} + R$$

where

$$(3.18)\qquad R = f^{(n)}(x_1)\frac{(x-a)^n}{n!}; \qquad (a < x_1 < x).$$

Another form of the same series is the one which is expanded in powers of h, the increment of x; i.e., $x = a + h = x_0 + h$;

(3.19) $$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + \cdots + f^{(n)}(x_0)\frac{h^n}{n!} + \cdots.$$

If we are concerned with a function of two variables, we have the *Taylor Series* about the point (a,b),

(3.20) $$\begin{aligned} f(x,y) = f(a,b) &+ f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &+ \frac{1}{2}[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) \\ &\quad + f_{yy}(a,b)(y-b)^2] \\ &+ \cdots + R_n, \end{aligned}$$

where

$$f_x = \frac{\partial f}{\partial x}, \qquad f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \text{ etc.}$$

A function of several variables may be expanded in a series by Taylor's formula in the form

(3.21) $$\begin{aligned} &f(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_n + \Delta x_n) \\ &= f(x_1, x_2, \cdots, x_n) + \sum_{i=1}^{n} \Delta x_i \frac{\partial f}{\partial x_i} \\ &+ \frac{1}{2}\left[\sum_{i=1}^{n} \Delta x_i^2 \frac{\partial^2 f}{\partial x_i^2} + 2\sum_{i=1}^{n} \Delta x_i \Delta x_{i+1} \frac{\partial^2 f}{\partial x_i \partial x_{i+1}}\right] + \cdots. \end{aligned}$$

For a thorough discussion of series the reader is referred to any standard textbook on advanced calculus.

E. Successive Approximations and Iteration. In this book we shall make a distinction between two very similar processes which are frequently used in numerical analysis; one is called *the method of successive approximations*, and the other *the method of iteration.* Each is applied to the solution for a particular value of a variable which is involved in a formula or functional notation. Let us consider, for example, that it is desired to find a root of the equation

(3.22) $$f(x) = 0.$$

The **method of successive approximations** is concerned with the finding of a sequence of numbers, $x_0, x_1, x_2, \cdots$ which converge to a limit b such that equation (3.22) is satisfied by $x = b$. The process is based upon the development of a recursion formula for x_{i+1} in terms of x_i so that x_{i+1} may be calculated after x_i is known.

The **method of iteration** also calculates a sequence of values $x_0, x_1, x_2, \cdots$ which converge to the desired value. The process, however, is based upon an explicit solution for the variable, x, in terms of the parameters and powers of x directly from the functions, $f(x)$.

Example I.3. Let us find a root of the equation

$$f(x) = x^2 + \sin^2 x - 3 = 0.$$

From elementary calculus it may be recalled that a good recursion formula for x_{i+1} is given by Newton's formula for approximation

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{3.23}$$

If from this formula we now develop a sequence of numbers $x_0, x_1, x_2, \cdots$ which converge to the desired solution, we are employing the *method of successive approximations.*

If, on the other hand, we were to solve the equation explicitly for x,

$$x = \pm\sqrt{3 - \sin^2 x}$$

and determine a sequence of numbers $x_0, x_1, x_2, \cdots$ by substituting the known values into the right-hand side of this equation and determining the new x_i thereby, we would be employing the *method of iteration.*

F. Undetermined Coefficients. A technique which is frequently employed by mathematicians is one which may be called the *method of undetermined coefficients.* In the application of this method the mathematician writes down a desired form of a mathematical expression with arbitrary coefficients. These arbitrary coefficients are at this stage undetermined. A sufficient number of conditions are now imposed upon the expression so that these undetermined coefficients may be found, and the mathematical expression becomes a specific formula.

A good example of this method was encountered in calculus with the introduction of the Maclaurin Series where it is stated that a function of x may be written in the form

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots .$$

Then the question was raised as to what form the coefficients a_i $(i = 0, 1, \cdots)$, must be in order that the function may be represented by a power series. The coefficients were then determined by letting $x = 0$ in the function and its successive derivatives to obtain the expression for the Maclaurin Series (3.15).

The method of undetermined coefficients will be widely employed in this book and many examples will become evident.

G. Derivatives of $\binom{x}{k}$. If we insert a variable x into the binomial coefficient expression, we may differentiate it with respect to x. There are two ways in which this derivative may be expressed. The first is obtained by differentiating formula (3.14) with $n = x$ to obtain

(3.24)
$$\frac{d}{dx}\binom{x}{k} = \frac{1}{k!}\frac{d}{dx}P^k(x)$$
$$= \frac{1}{k!}[kS_{0,k}x^{k-1} + (k-1)S_{1,k}x^{k-2} + (k-2)S_{2,k}x^{k-3} + \cdots + S_{k-1,k}]$$

where $S_{i,k}$ are the Stirling numbers of the first kind. The second is obtained by differentiating the binomial coefficient expression as the product of k variables,

(3.25)
$$\frac{d}{dx}\binom{x}{k} = \frac{d}{dx}[x(x-1)(x-2)\cdots(x-k+1)]\frac{1}{k!}$$
$$= \frac{1}{k!}[(x-1)(x-2)\cdots(x-k+1) + x(x-2)(x-3)\cdots(x-k+1) + x(x-1)(x-3)\cdots(x-k+1) + \cdots + x(x-1)(x-2)\cdots(x-k+2)].$$

It is to be noted that the bracketed expression is a sum of factorial polynomials with one factor, $x - i$, $(i = 0, 1, \cdots, k-1)$, missing from each. If the missing factors be inserted and then divided out, we could write

$$(3.26)\quad \frac{d}{dx}\binom{x}{k} = \binom{x}{k}\left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{x-k+1}\right]$$

$$= \sum_{i=0}^{k-1} \frac{P^k(x)}{k!(x-i)}.$$

A useful recursion formula for varying k may be developed from formula (3.25). Let us introduce the symbol

$$(3.27)\qquad D_k(x) = \frac{d}{dx}\binom{x}{k}$$

and consider $k = i$, $(i = 1, 2, 3, \cdots)$, then

$$D_1(x) = 1;$$

$$D_2(x) = \frac{1}{2}[(x-1) + x] = \frac{1}{2}[(x-1)D_1(x) + xb_1(x)];$$

$$D_3(x) = \frac{1}{3!}[(x-1)(x-2) + x(x-2) + x(x-1)]$$

$$= \frac{1}{3}\left[\frac{1}{2}\{(x-1) + x\}(x-2) + \frac{1}{2}x(x-1)\right]$$

$$= \frac{1}{3}[(x-2)D_2(x) + (x-1)b_2(x)];$$

$$D_4(x) = \frac{1}{4!}[(x-1)(x-2)(x-3) + x(x-2)(x-3) + x(x-1)(x-3) + x(x-1)(x-2)]$$

$$= \frac{1}{4}[(x-3)D_3(x) + (x-2)b_3(x)].$$

In general we have

$$(3.28)\quad D_{i+1}(x) = \frac{1}{i+1}[(x-i)D_i(x) + (x-i+1)b_i(x)],\qquad (i = 1, 2, 3, \cdots),$$

where

$$(3.29)\quad b_i(x) = \frac{1}{i}(x - i + 2)b_{i-1}(x), \qquad (i = 2, 3, 4, \cdots),$$
$$= \frac{1}{i}\binom{x}{i-1}$$

and

$$D_1(x) = b_1(x) = 1.$$

Formula (3.28) permits rapid evaluation of the first derivatives for changing k since it is the sum of products of two factors followed by a division.

It is especially interesting to obtain the values of the derivative at $x = i$, $(i = 0, \cdots, k)$. From formula (3.25) it is easily seen that all the expressions in the bracket are zero except the one in which $x - i$ does not occur. Thus we have for $i < k$

$$(3.30)\quad D_k(i) = \frac{1}{k!}[x(x-1)\cdots(x-i+1)(x-i-1)\cdots(x-k+1)]_{x=i}$$
$$= \frac{1}{k!}[i(i-1)\cdots(2)(1)(-1)(-2)\cdots(i-k+1)]$$
$$= (-1)^{k-i-1}\left[\frac{i!(k-i-1)!}{k!}\right].$$

In particular

$$D_k(0) = \frac{1}{k!}[(x-1)(x-2)\cdots(x-k+1)]_{x=0}$$
$$= (-1)^{k-1}\left[\frac{(k-1)!}{k!}\right] = (-1)^{k-1}\left[\frac{1}{k}\right],$$

and from formula (3.26)

$$(3.31)\quad D_k(k) = \frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \cdots \frac{1}{1} = \sum_{j=1}^{k}\left(\frac{1}{j}\right).$$

By direct substitution into formula (3.30), we can arrive at the recursion formula

$$(3.32)\qquad D_k(i) = -\frac{i}{k-i}[D_k(i-1)], \qquad (i < k).$$

The second derivative is obtained by differentiating the first derivative. From formula (3.24) we obtain

$$(3.33)\quad D_k^2(x) = \frac{1}{k!}[k(k-1)S_{0,k}x^{k-2} + (k-1)(k-2)S_{1,k}x^{k-3} + \cdots + S_{k-2,k}].$$

From formula (3.28) we obtain

$$(3.34)\quad D_{i+1}^2(x) = \frac{1}{i+1}[D_i(x) + (x-i)D_i^2(x) + b_i(x) + (x-i+1)b_i'(x)].$$

Now from formula (3.29)

$$b_i'(x) = \frac{1}{i}\frac{d}{dx}\binom{x}{i-1} = \frac{1}{i}D_{i-1}(x)$$

and by formula (3.28)

$$\frac{1}{i}(x-i+1)D_{i-1}(x) + b_i(x) = \frac{1}{i}[(x-i+1)D_{i-1}(x) + (x-i+2)b_{i-1}(x)] = D_i(x).$$

Thus formula (3.34) becomes

$$(3.35)\quad D_{i+1}^2(x) = \frac{1}{i+1}[2D_i(x) + (x-i)D_i^2(x)], \qquad (i = 1, 2, 3, \cdots),$$

with

$$D_1^2(x) = 0 \qquad \text{and} \qquad D_2^2(x) = 1.$$

For higher derivatives it is easily seen that

$$(3.36)\quad D_{i+1}^{(n)}(x) = \frac{1}{i+1}[nD_i^{(n-1)}(x) + (x-i)D_i^{(n)}(x)].$$

H. Orthogonal Polynomials.* An important property of polynomials which finds wide application in numerical analysis is known as the orthogonality property. In general two functions, $g_m(x)$ and $g_n(x)$, are said to be **orthogonal** if

$$(3.37)\qquad \int_a^b g_m(x)g_n(x)dx = 0$$

* For a thorough discussion of orthogonal polynomials see R. V. Churchill, *Fourier Series and Boundary Value Problems* (New York: The McGraw-Hill Book Company, 1941).

and a set of functions $g_n(x)$, $(n = 1, 2, 3, \cdots)$, is orthogonal in the interval (a,b) if the condition (3.37) is true when $m \neq n$ for all functions of the set.

If in particular we consider the set of polynomials

$$(3.38) \quad P_m(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m, \qquad (m = 0, 1, 2, \cdots),$$

they are said to be **orthogonal** in the interval $[-1,1]$ if

$$(3.39) \qquad \int_{-1}^{1} P_m(x)P_n(x)dx = 0, \qquad \text{if } (m \neq n).$$

A special set of such polynomials in which

$$(3.40) \qquad \begin{cases} a_0 = 1 \\ a_n = \dfrac{(2n-1)(2n-3)\cdots(3)(1)}{n!} \end{cases}$$

are known as the *Legendre polynomials*. They may be generated by *Rodrigues' formula*

$$(3.41) \qquad P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m.$$

This set of polynomials have the further property that when $m = n$

$$(3.42) \quad \int_{-1}^{1} [P_m(x)]^2 dx = \frac{2}{2m+1}, \qquad (m = 0, 1, 2, \cdots).$$

The zeros of the *Legendre polynomials* or the roots of $P_m(x) = 0$ are all real and distinct.

4. Inherent Errors. In numerical analysis there are three kinds of errors: (1) errors due to the fact that the given data is approximate; (2) errors due to the fact that the formulas and procedures employed are approximate; and (3) errors made by the computer. Very little can be done about the first of these. Clearly, attempts can be made to obtain better data, correct obvious errors in the data, and a few things of that nature. The third is, of course, a mistake and is corrected by redoing the problem; diligence on the part of the operator will minimize this error. In general the second error can be evaluated, and it is desirable that it be made as small as possible.

Errors which are due to the approximate procedures are called *inherent errors*, and most formulas and procedures should have some expression for these inherent errors. Errors can be expressed in three ways:

Absolute Error, the numerical difference between the true value and the approximate value;

$$E_a = x - x_1 = \Delta x$$

where x is the true value, and x_1 is the calculated or measured approximate value.

Relative Error, the absolute error divided by the true value

$$E_r = \frac{E_a}{x} = \frac{\Delta x}{x}.$$

Percentage Error, the relative error multiplied by 100, i.e., $100E_r$ is the percentage error.

The error committed by a formula or a functional representative can be determined by a *general error formula*. Let

$$N = f(x_1, x_2, \cdots, x_n) \tag{4.1}$$

denote any function of the independent variables x_i, $(i = 1, \cdots, n)$, each of which is subject to error, say Δx_i, $(i = 1, \cdots, n)$, respectively. There will then be an error ΔN in the function so that we have

$$N + \Delta N = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_n + \Delta x_n). \tag{4.2}$$

The right-hand side of (4.2) may be expanded in a Taylor Series for several variables [see (3.21)] to give us

$$\begin{aligned} f(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_n + \Delta x_n) &= f(x_1, x_2, \cdots, x_n) \\ &+ \left[\Delta x_1 \frac{\partial f}{\partial x_1} + \Delta x_2 \frac{\partial f}{\partial x_2} + \cdots + \Delta x_n \frac{\partial f}{\partial x_n}\right] \\ &+ \frac{1}{2}\left[\Delta x_1^2 \frac{\partial^2 f}{\partial x_1} + \cdots + \Delta x_n^2 \frac{\partial^2 f}{\partial x_n} + 2\Delta x_1 \Delta x_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + \cdots \right. \\ &\qquad \left. + 2\Delta x_{n-1}\Delta x_n \frac{\partial^2 f}{\partial x_{n-1}\partial x_n}\right] \\ &+ \cdots. \end{aligned}$$

Usually the errors in the variables should be relatively small, that is, $\frac{\Delta x_i}{x_i} \ll 1$. It is, therefore, permissible to ignore the squares, cross-products, and higher powers of Δx_i, $(i = 1, \cdots, n)$, and write to a *first-order approximation*

$$(4.3) \qquad N + \Delta N = f(x_1, \cdots, x_n) + \sum_{i=1}^{n} \Delta x_i \frac{\partial f}{\partial x_i}.$$

If we now subtract (4.1) from (4.3), we obtain

$$(4.4) \qquad \Delta N = \frac{\partial f}{\partial x_1} \Delta x_1 + \frac{\partial f}{\partial x_2} \Delta x_2 + \cdots + \frac{\partial f}{\partial x_n} \Delta x_n.$$

Formula (4.4) is a first-order approximation for the error of a function. It may be noted that it is the same expression as that for the total differential of the function N. A formula for the relative error follows directly.

$$(4.5) \qquad E_r = \frac{\Delta N}{N} = \frac{\partial N}{\partial x_1} \frac{\Delta x_1}{N} + \frac{\partial N}{\partial x_2} \frac{\Delta x_2}{N} + \cdots + \frac{\partial N}{\partial x_n} \frac{\Delta x_n}{N}.$$

The error formula may be applied to all functions and serves as an evaluation of all processes which can be expressed in formula form.

Example 1.4. Consider the formula

$$N = \frac{3x^2y}{z^3}.$$

To evaluate the error in N due to errors in x, y, z, we first determine the partial derivatives

$$\frac{\partial N}{\partial x} = \frac{6xy}{z^3},$$
$$\frac{\partial N}{\partial y} = \frac{3x^2}{z^3},$$
$$\frac{\partial N}{\partial z} = -\frac{9x^2y}{z^4}.$$

Then

$$\Delta N = 6xyz^{-3}\Delta x + 3x^2z^{-3}\Delta y - 9x^2yz^{-4}\Delta z.$$

Into this formula are now substituted the values of the errors Δx, Δy, Δz, and the partial derivatives are evaluated at the desired point, (x_1,y_1,z_1). In general the errors, Δx, Δy, Δz, may be either positive or negative, and if the sign is not specified it is possible to compute only the maximum error in N, and in that case we sum the absolute values of the terms on the right-hand side. Thus

$$\Delta N_{max} = |6xyz^{-3}\Delta x| + |3x^2y^{-3}\Delta y| + |9x^2yz^{-4}\Delta z|.$$

It is important to note that this is the maximum error and thus furnishes an upper bound on the error. To continue the example, suppose $\Delta x = \Delta y = \Delta z = .005$, then the maximum error at the point (1,1,1) is

$$\begin{aligned}\Delta N_{max} &= 6(.005) + 3(.005) + 9(.005)\\ &= 18(.005) = .090.\end{aligned}$$

The value of the function at (1,1,1) is $N = 3$ and the relative maximum error is

$$E_{r_{max}} = \left(\frac{\Delta N}{N}\right)_{max} = \frac{1}{3}\,(.090) = .030.$$

It should again be emphasized that this is the maximum error and that the actual error can only be found if the algebraic signs of the individual errors are known.

As another example let us apply the error formula to the simple operation of addition.

$$\begin{aligned}N &= x + y;\\ \Delta N &= \Delta x + \Delta y.\end{aligned}$$

The inclusion of higher-order terms is demanded only in rare situations. Thus, in the Example 1.4, the second-order term is

$$\frac{1}{2}\left[\frac{\partial^2 N}{\partial x^2}\,\Delta x^2 + \frac{\partial^2 N}{\partial y^2}\,\Delta y^2 + \frac{\partial^2 N}{\partial z^2}\,\Delta z^2\right] + \frac{\partial^2 N}{\partial x\partial y}\,\Delta x\Delta y + \frac{\partial^2 N}{\partial y\partial z}\,\Delta y\Delta z + \frac{\partial^2 N}{\partial x\partial z}\,\Delta x\Delta z.$$

The partial derivatives evaluated at (1,1,1) become

$$\frac{\partial^2 N}{\partial x^2} = 6yz^{-3} = 6; \qquad \frac{\partial^2 N}{\partial x \partial y} = 6z^{-3} = 6;$$
$$\frac{\partial^2 N}{\partial y^2} = 0; \qquad \frac{\partial^2 N}{\partial y \partial z} = -9x^2z^{-4} = -9;$$
$$\frac{\partial^2 N}{\partial z^2} = 36x^2yz^{-5} = 36; \qquad \frac{\partial^2 N}{\partial x \partial z} = -18xyz^{-4} = -18.$$

The squares and cross-products of the errors for

$$\Delta x = \Delta y = \Delta z = .005$$

become

$$\Delta x^2 = \Delta y^2 = \Delta z^2 = \Delta x \Delta y = \Delta x \Delta z = \Delta y \Delta z = .000025.$$

The maximum error at (1,1,1) including second order terms is

$$\Delta N_{max} = .090 + [3 + 0 + 18 + 6 + 9 + 18].000025$$
$$= .09135.$$

The errors committed by series expansions can be evaluated by the remainder after n terms. In a normal course in calculus, formulas for the remainders in series expansion were given in terms of the nth derivative at an intermediate point. We shall also list the remainder in integral form which is obtained by repeated integration by parts of

$$\frac{1}{(n-1)!}\int_0^x f^{(n)}(x-t)t^{n-1}dt$$

where $f(x-t)$ is a continuous function of t in the interval from 0 to x. Remainder terms for some well-known series are listed.

I. Maclaurin Series (3.15)

(4.6) $$R_n(x) = \frac{x^n}{n!}f^{(n)}(x_1), \qquad 0 < x_1 < x.$$

(4.7) $$R_n(x) = \frac{1}{(n-1)!}\int_0^x f^{(n)}(x-t)t^{n-1}dt.$$

II. Taylor Series (3.16)

$$R_n(x) = \frac{(x-a)^n}{n!} f^{(n)}(x_1), \qquad (a < x_1 < x), \tag{4.8}$$

$$R_n(x) = \frac{1}{(n-1)!}\int_0^{x-a} f^{(n)}(x-t)t^{n-1}dt. \tag{4.9}$$

III. Taylor Series (3.19)

$$R_n(x) = \frac{h^n}{n!} f^{(n)}(x_1), \qquad (0 < x_1 < x), \tag{4.10}$$

$$R_n(x) = \frac{1}{(n-1)!}\int_0^{h} f^{(n)}(x+h-t)t^{n-1}dt. \tag{4.11}$$

IV. Taylor Series (3.20)

The remainder term now becomes quite complex, and the reader is referred to a book on advanced calculus.

$$R_n = \frac{t^n}{n!}\left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)^n f(a + \theta\alpha t, b + \theta\beta t) \tag{4.12}$$

with $\qquad \alpha t = x - a, \beta t = y - b$, and $0 < \theta < 1$.

The use of remainder terms is common for functions which assume series representation and will be used throughout this book.

The remainder terms above are all given in terms of the nth derivative which may or may not be obtainable. If it is not obtainable, it is replaced by its equivalent in terms of differences which will be explained in Chapter II.

5. Calculating Machines. Since numerical analysis is concerned with calculations, it depends heavily upon calculating machines for the accomplishment of its work. The great amount of labor involved in employing some of the techniques which will be derived in this book would be prohibitive if it were not for the modern automatic calculator. In fact many of the techniques known to mathematicians for years remained dormant awaiting the invention of these machines. The tremendous growth in the computing machinery field in recent years has reactivated interest in numerical analysis and has permitted refinements in techniques dictated by experience and practice. Many of the refinements are presented in this book, but it must be kept in mind that refinements are made every day.

A complete survey of calculating machines will not be attempted here. In fact the operation of calculators will not be taught since each type of machine has excellent handbooks on operational methods.

This book is written on the assumption that the student will have available an automatic desk calculator and will know how to operate the particular one he has. One comment may be in order. If a student writes with his *right* hand, this author recommends very strongly that he learn to operate the calculator with his *left* hand and vice versa.

There are many brands of desk calculators now on the market. Each machine has certain specialized features but, in general, they all perform the arithmetical operations in comparable fashion. In the ordinary calculator the operator copies the answers from the dials to a computing paper, although a desk calculator has been joined to a typewriter so that the answers could be automatically recorded. The speed of operation depends greatly upon the operator, and in the next section we shall discuss the benefits to be derived from programming and systematic arrays.

There are many large-scale digital calculators in operation. These grew from the specialized computers designed for military purposes to commercial ones now on the market. It would be impossible to enumerate the features of each. Fundamentally, the machines receive input data and coded instructions from punched cards or tapes. The operations are performed electronically employing tube, mechanical, or magnetic drum storage. The results may be put on punched cards or tapes or printed by a printer. Each problem is programmed for the machine, and special experience with the particular machine is necessary.

6. Programming. The term *programming* has become popular with the innovation of the large-scale machines; however, its widest connotation simply implies the preparation of a problem for a machine and should also apply to desk calculators. The solution of any mathematical problem should be accomplished in a systematic manner. Each problem must be carefully arranged for the machine, recording of intermediate results should be reduced to a minimum, periodic checks should be inserted to spot mistakes, and the results should be neatly presented.

On most desk calculators, whenever possible the problem should be reduced to a *sum of products of two numbers followed by a division if required.* That is, the formulas should be rearranged into this form:

$$N = \frac{(a)(b) + (c)(d) - (e)(f)}{r}.$$

in so far as is possible.

Furthermore, common factors or constants should be isolated in order that they may be locked into the calculator at the appropriate time. Thus,

$$N = kx$$

would be accomplished by locking k into the keyboard or multiplier and varying x.

In performing lengthy calculations it is recommended that columnar paper be used with the appropriate formulas listed at the top of the page and each column specifically labelled. If it is necessary to scale the problem, such scaling should be clearly indicated at each column.

Many examples will be presented throughout this book. It is hoped that any reader will develop a systematic arrangement of his work, for it has been said that one of the great by-products of mathematics is a development of a logical and systematic approach to the solution of problems, and this applies to the branch of numerical analysis as well as to other branches.

Example I.5. Find y for $x = 1.261$, $.327$, and 2.541 in the formula

$$y = \frac{3.2x^2 - 1.53x + x\sqrt{2x + 5}}{x + 2.12}.$$

Solution. The formula is rewritten in the form $y = \dfrac{N}{D}$ where

$$N = x\,(\sqrt{2x + 5} - 1.53 + 3.2x)$$
$$D = x + 2.12$$

and only D need be recorded as an intermediate value. The values of x are rearranged in ascending order. The calculating sheet would be arranged as follows:

$y = \frac{N}{D}$ $N = x(\sqrt{2x + 5} - 1.53 + 3.2x)$ $D = x + 2.12$				No. 1 12/1/53 KLN
x	D	y		
.327 1.261 2.541	2.447 3.381 4.661	.2531 1.9573 5.3297		

All calculations should be labelled, paged, dated, and initialed by the calculator.

7. Mathematical Tables. One of the greatest aids to computational procedures is a mathematical table. The reader is already familiar with trigonometric, logarithmic, exponential, and other common tables. Whenever possible mathematicians have listed values of functions and placed them in tables for others to use. The use of large-scale calculators has permitted the calculations of many tables, and it is indeed difficult to know of all existing ones. One of the best source materials is the magazine *Mathematical Tables and Other Aids to Computation*, which is published quarterly.*

A *table* is a listing of values of functions or dependent variables for given values of the independent variable usually called the *argument*. All values of the argument cannot be listed and so tables are given at discrete points, usually at equally spaced intervals of the argument. Once the table is listed it may be used to find the value of the function or it may be used to find the value of the argument. If it is desired to find the value of either for a given value which is not identical with the given table value, it becomes necessary to approximate between the given values. This process is called interpolation and forms a large part of this book.

* Published by National Research Council, Washington, D.C.

Of the many tables which are now published, there is one type which is of great importance to the numerical analyst, and that is the type which lists coefficients for numerical processes. This is a table of numbers which are used as multipliers in a numerical method; they are, of course, associated with the process. Those which are most useful are listed in the back of this book.

One word of advice is appropriate at this time and pertains to the finding of an argument from a tabulated function, for example, the finding of an angle from trigonometric functions. Some tables will lead to greater errors than others, and when there is a choice of tables to use, the error should be investigated. Thus it is better to obtain an angle from the tangent function than from the sine function.

8. Exercise I.

1. Round off the following numbers correctly to five significant figures:

$$38.46235,\ 2.37425,\ .00237135,\ .700029.$$

2. Find the sum of

$$12.3172,\ 11.283,\ 1.3496,\ 2.4875.$$

3. Evaluate the formula

$$v = \sqrt{\frac{2ghd}{fx}}$$

if $g = 32.2$, $h = 92$, $d = \frac{3}{8}$, $f = .025$, and $x = 1250$.

4. Find the value of $f(x)$ in

$$f(x) = 7x^4 - 5x^3 + x^2 - 8x + 2$$

for $x = 2, 3, 2.5, 1.3298$.

5. Express the polynomial

$$P(x) = x^5 + 2x^4 - 7x^3 + 8x^2 - x + 4$$

as the sum of factorial polynomials.

6. Prove

$$P_i^{(n)}(i) = (-1)^{n-i-1}i!(n - i - 1)!$$

7. Prove

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

if n is an integer.

8. Expand the following functions in a Maclaurin Series:

$$\sin x,\ e^x,\ \cos x,\ \tan x.$$

9. If f is in error by $\Delta f = .005$, what is the error in z in Problem 3?

10. Evaluate the remainder after 5 terms in the series expansion for $\cos x$ at $x = .5$ radians.

11. Obtain $D_k(x)$ at $x = 0, 1, 2, 3, 4, 5$.

12. Prove $D_k(i) = -\dfrac{1}{k-i}[D_k(i-1)]$.

CHAPTER II. FINITE DIFFERENCES

9. Introduction. The calculus of finite differences is used to a large extent throughout numerical calculations. It forms the basis for many processes and is used in the derivation of many formulas. It is not the intention here to give a complete discussion on the calculus of finite differences but rather to extract only that which is necessary for numerical methods.

In numerical methods we are concerned with functions of a continuous variable. Such a function may be defined by a formula, and thus its value can be calculated for a given value of the independent variable. On the other hand, a function may be specified by a table of values and not explicitly defined by a formula. In this case, where the function is given at discrete tabular values of the independent variable, it is not formally defined at any other value. However, we shall be interested in the function at all values throughout an interval, and under certain assumptions of continuity we can usually obtain the value of the function at any value of the independent variable to any desired degree of accuracy.

10. First Difference. Let the function $y = f(x)$ be given at discrete values of x, so that we have a table of values

x	x_0	x_1	x_2	x_3	x_4	$\cdots$	x_n
y	y_0	y_1	y_2	y_3	y_4	$\cdots$	y_n

If we now subtract from each value of y the preceding value of y, thus

$$(y_1 - y_0),\ (y_2 - y_1),\ \cdots\ (y_n - y_{n-1}),$$

the result is called the **first difference** of the function y. We shall denote these differences by Δy_i in which the subscript of Δy will be the same as the *second member* of the difference. Thus

$$
(10.1) \qquad \begin{cases} y_1 - y_0 = \Delta y_0 \\ y_2 - y_1 = \Delta y_1 \\ \cdots\cdots\cdots\cdots \\ y_n - y_{n-1} = \Delta y_{n-1}. \end{cases}
$$

To illustrate let us consider the following example.

Example II.1.

x	y	Δy
1.0	1.36491	
1.1	1.59432	.22941 = Δy_0
1.2	1.82419	.22987 = Δy_1
1.3	2.14926	.32507 = Δy_2
1.4	2.37812	.22886 = Δy_3
1.5	2.63212	.25400 = Δy_4
1.6	2.94160	.30948 = Δy_5
1.7	3.34217	.40057 = Δy_6
1.8	3.68947	.34730 = Δy_7
1.9	4.01340	.32393 = Δy_8

It is easily seen that

$$\Delta y_0 = 1.59432 - 1.36491 = .22941,$$

etc. The differences are usually written on a line between the two numbers for which they represent the differences. To compute the differences on a calculating machine in the case of an increasing function, start with the bottom number and work up the table: this permits the locking of a number in the keyboard and holding it there for two operations.

11. Higher Differences. If we now consider the differences of the first differences, we will obtain a set of quantities called **second differences** which may be denoted by a superscript on Δ. Thus

$$\text{(11.1)} \qquad \begin{cases} \Delta^2 y_0 = \Delta y_1 - \Delta y_0 \\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1 \\ \cdots\cdots\cdots\cdots \\ \Delta^2 y_i = \Delta y_{i+1} - \Delta y_i. \end{cases}$$

The process could be continued to define **third differences** as the difference of second differences.

$$\text{(11.2)} \qquad \Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i.$$

In general we define the nth **differences** by the formula

$$\text{(11.3)} \qquad \Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i.$$

Although each higher difference is defined in terms of the next lower difference, it is possible by continuous substitution to obtain the higher differences in terms of the values of the function. Thus

$$\text{(11.4)} \qquad \begin{cases} \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0 \\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1 = y_3 - y_2 - (y_2 - y_1) = y_3 - 2y_2 + y_1 \end{cases}$$

and

$$\text{(11.5)} \qquad \begin{aligned} \Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0) \\ &= y_3 - 3y_2 + 3y_1 - y_0. \end{aligned}$$

In general we have

$$\text{(11.6)} \qquad \begin{aligned} \Delta^n y_0 &= y_n - \binom{n}{1} y_{n-1} + \binom{n}{2} y_{n-2} - \cdots + (-1)^n y_0 \\ &= \sum_{i=0}^{n} (-1) \binom{n}{i} y_{n-i}. \end{aligned}$$

where $\binom{n}{i}$ is the binomial coefficient notation.

The procedure may be reversed to express a value of y in terms of the initial value of y and differences. From formula (10.1) we have

$$\text{(11.7)} \qquad y_1 = y_0 + \Delta y_0$$

and from (11.5) we obtain upon substitution of known values

(11.8)
$$\begin{aligned} y_3 &= y_0 - 3y_1 + 3y_2 + \Delta^3 y_0 \\ &= y_0 + 3\Delta y_1 + \Delta^3 y_0 \\ &= y_0 + 3(\Delta^2 y_0 + \Delta y_0) + \Delta^3 y_0 \\ &= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0. \end{aligned}$$

In general we may write

(11.9)
$$\begin{aligned} y_k &= y_0 + \binom{k}{1}\Delta y_0 + \binom{k}{2}\Delta^2 y_0 + \cdots + \Delta^k y_0 \\ &= \sum_{i=0}^{k} \binom{k}{i} \Delta^i y_0. \end{aligned}$$

12. Difference Tables. Differences are usually arranged in tabular form, and one of the most commonly employed schemes is the **diagonal difference table** shown in Table 2.1.

Table 2.1. Diagonal Difference Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$	$\Delta^8 y$
x_0	y_0								
		Δy_0							
x_1	y_1		$\Delta^2 y_0$						
		Δy_1		$\Delta^3 y_0$					
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$				
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$			
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$		
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$		$\Delta^7 y_0$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		$\Delta^6 y_1$		$\Delta^8 y_0$
		Δy_4		$\Delta^3 y_3$		$\Delta^5 y_2$		$\Delta^7 y_1$	
x_5	y_5		$\Delta^2 y_4$		$\Delta^4 y_3$		$\Delta^6 y_2$		
		Δy_5		$\Delta^3 y_4$		$\Delta^5 y_3$			
x_6	y_6		$\Delta^2 y_5$		$\Delta^4 y_4$				
		Δy_6		$\Delta^3 y_5$					
x_7	y_7		$\Delta^2 y_6$						
		Δy_7							
x_8	y_8								

As an example of the diagonal difference table let us complete Example II.1 to six differences.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	1.36491						
		22941					
1.1	1.59432		46				
		22987		9474			
1.2	1.82419		9520		9667		
		32507		19141		−16673	
1.3	2.14926		−9621		−7006		14578
		22886		12135		−2095	
1.4	2.37812		2514		−9101		11723
		25400		3034		9628	
1.5	2.63212		5548		527		720
		30948		3561		10348	
1.6	2.94160		9109		10875		−32669
		40057		14436		−22321	
1.7	3.34217		−5327		−11446		
		34730		2990			
1.8	3.68947		−2337				
		32393					
1.9	4.01340						

In the construction of this table the decimal points have been omitted in the writing of the differences. This is common practice to save time and space; it is understood that all numbers are carried to the same accuracy as the given data, and thus the decimal point and the insignificant zeros can easily be restored. For example,

$$\Delta^4 y_3 = .00527$$

although it is listed in the table as merely 527.

A special form of a diagonal difference table is one which is called a **central difference table** and is identical to the diagonal difference table except that the values of function are ordered around a central value, y_0. The values of the independent variable are numbered $\cdots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \cdots$ and the corresponding values of the function $\cdots, y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3, \cdots$. The form of a central difference table is shown in Table 2.2.

Table 2.2. Central Difference Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$	$\Delta^8 y$
x_{-4}	y_{-4}								
		Δy_{-4}							
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$						
		Δy_{-3}		$\Delta^3 y_{-4}$					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$				
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		$\Delta^7 y_{-4}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$		$\Delta^8 y_{-4}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$		$\Delta^8 y_{-3}$
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$		$\Delta^7 y_{-2}$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		$\Delta^6 y_{-1}$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$			
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$				
		Δy_3		$\Delta^3 y_2$					
x_4	y_4		$\Delta^2 y_3$						
		Δy_4							
x_5	y_5								

In our definition of a central difference table we have preserved the original notation for differences as defined in Section 11 in order not to cause unnecessary confusion. The central difference table is one which focuses attention upon those around a horizontal line through the central value.

	Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
	Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	

It is seen that for this line there are entries only for the even-order differences. For certain kinds of numerical analyses it is desirable to have an entry on this line for the odd differences as well, and this is accomplished by taking the arithmetic mean of the two adjacent differences. Thus, in our notation we can define

$$(12.1)\quad m_{2i-1} = \frac{1}{2}\,(\Delta^{2i-1}y_{-i} + \Delta^{2i-1}y_{-i+1}), \qquad (i = 1, 2, 3, \cdots),$$

so that

$$i = 1: \qquad m_1 = \frac{1}{2}\,(\Delta y_{-1} + \Delta y_0)$$

$$i = 2: \qquad m_3 = \frac{1}{2}\,(\Delta^3 y_{-2} + \Delta^3 y_{-1}), \text{ etc.}$$

The horizontal line through the central value would then have the entries:

y_0	m_1	$\Delta^2 y_{-1}$	m_3	$\Delta^4 y_{-2}$	m_5	$\Delta^6 y_{-3}$	$\cdots$

For some purposes it may be desirable to focus attention on a line between the values y_0 and y_1, which we could call the $y_{\frac{1}{2}}$ line.

y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$	
	Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$
y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$	

Here are recorded only the odd-order differences. Following the same procedure as above, we may now define the arithmetic mean of the adjacent even-order differences by the formula

$$(12.2)\quad m_{2i} = \frac{1}{2}\,[\Delta^{2i}y_{-i} + \Delta^{2i}y_{-i+1}], \qquad (i = 0, 1, 2, \cdots),$$

so that

$$i = 0: \qquad m_0 = \frac{1}{2}\,[y_0 + y_1],$$

$$i = 1: \qquad m_2 = \frac{1}{2}\,[\Delta^2 y_{-1} + \Delta^2 y_0], \text{ etc.}$$

The horizontal line through these values would then have the entries

m_0	Δy_0	m_2	$\Delta^3 y_{-1}$	m_4	$\Delta^5 y_{-2}$	m_6	$\Delta^7 y_{-3}$	. . .

Combining all of the above into one central difference table we can write

Table 2.3. Central Difference Table with Arithmetic Means.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$	$\Delta^8 y$	$\Delta^9 y$
x_{-4}	y_{-4}									
		Δy_{-4}								
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$							
		Δy_{-3}		$\Delta^3 y_{-4}$						
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$					
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$				
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$			
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		$\Delta^7 y_{-4}$		
x_0	y_0	m_1	$\Delta^2 y_{-1}$	m_3	$\Delta^4 y_{-2}$	m_5	$\Delta^6 y_{-3}$	m_7	$\Delta^8 y_{-4}$	
	m_0	Δy_0	m_2	$\Delta_4 y_{-1}$	m_4	$\Delta^5 y_{-2}$	m_6	$\Delta^7 y_{-3}$	m_8	$\Delta^9 y_{-4}$
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$		$\Delta^8 y_{-3}$	
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$		$\Delta^7 y_{-2}$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		$\Delta^6 y_{-1}$			
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$				
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$					
		Δy_3		$\Delta^3 y_2$						
x_4	y_4		$\Delta^2 y_3$							
		Δy_4								
x_5	y_5									

Example II.2. To illustrate the central difference table the differences have been calculated in the following table with $x_0 = 1.5$. The calculation of the values for m_r are simply the arithmetic means of the vertically adjacent differences and are calculated after the differences have been evaluated. In this example all differences above the fifth are zero.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	−3.00000					
		137372				
1.1	−1.62628		40840			
		178212		10620		
1.2	.15584		51460		2160	
		229672		12780		240
1.3	2.45256		64240		2400	
		293912		15180		240
1.4	5.39168		79420		2640	
		373332		17820		240
1.5	9.12500	421952	97240	19260	2880	240
	11.47786	470572	107590	20700	3000	240
1.6	13.83072		117940		3120	
		588512		23820		240
1.7	19.71584		141760		3360	
		730272		27180		240
1.8	27.01856		168940		3600	
		899212		30780		
1.9	36.01068		199720			
		1098932				
2.0	47.00000					

By direct reference to the definitions and some algebraic manipulation, it is possible to solve for any difference in terms of other differences. Of particular importance are formulas which express the differences of y_0 in terms of the entries on the horizontal line through y_0 and also the one through $y_{\frac{1}{2}}$. We shall derive some of these expressions. Consider first those in terms of quantities on the line through y_0. From the definitions (11.1) and (12.1) we have

$$\begin{aligned} \Delta y_0 &= \Delta^2 y_{-1} + \Delta y_{-1} \\ &= \Delta^2 y_{-1} + 2m_1 - \Delta y_0 \end{aligned}$$

and solving for Δy_0 we obtain

$$\Delta y_0 = m_1 + \frac{1}{2}\Delta^2 y_{-1}. \tag{12.3}$$

Before proceeding further we note that in general from the definitions we have

$$\begin{cases} \Delta^{2i}y_{-i} = \Delta^{2i-1}y_{1-i} - \Delta^{2i-1}y_{-i} \\ 2m_{2i-1} = \Delta^{2i-1}y_{1-i} + \Delta^{2i-1}y_{-i} \end{cases}$$

from which we get by an addition and division by 2

$$\textbf{(12.4)} \qquad \Delta^{2i-1}y_{1-i} = m_{2i-1} + \frac{1}{2}\Delta^{2i}y_{-i}.$$

To obtain an expression for $\Delta^2 y_0$ we write from the definition and (12.4) with $i = 2$

$$\textbf{(12.5)} \qquad \begin{aligned} \Delta^2 y_0 &= \Delta^2 y_{-1} + \Delta^3 y_{-1} \\ &= \Delta^2 y_{-1} + m_3 + \frac{1}{2}\Delta^4 y_{-2}. \end{aligned}$$

For $\Delta^3 y_0$ we have

$$\begin{aligned} \Delta^3 y_0 &= \Delta^3 y_{-1} + \Delta^4 y_{-1} \\ &= m_3 + \frac{1}{2}\Delta^4 y_{-2} + \Delta^4 y_{-1} \end{aligned}$$

in which we wish to replace $\Delta^4 y_{-1}$ since it is not on the line through y_0. From the definition and (12.4) with $i = 3$ we have

$$\begin{aligned} \Delta^4 y_{-1} &= \Delta^4 y_{-2} + \Delta^5 y_{-2} \\ &= \Delta^4 y_{-2} + m_5 + \frac{1}{2}\Delta^6 y_{-3} \end{aligned}$$

so that

$$\textbf{(12.6)} \qquad \Delta^3 y_0 = m_3 + \frac{3}{2}\Delta^4 y_{-2} + m_5 + \frac{1}{2}\Delta^6 y_{-3}.$$

The procedure is now quite clear so that for $\Delta^4 y_0$ we have

$$\textbf{(12.7)} \quad \begin{aligned} \Delta^4 y_0 &= \Delta^4 y_{-1} + \Delta^5 y_{-1} \\ &= \Delta^4 y_{-2} + \Delta^5 y_{-2} + \Delta^5 y_{-2} + \Delta^6 y_{-2} \\ &= \Delta^4 y_{-2} + 2\Delta^5 y_{-2} + \Delta^6 y_{-3} + \Delta^7 y_{-3} \\ &= \Delta^4 y_{-2} + 2\left(m_5 + \frac{1}{2}\Delta^6 y_{-3}\right) + \Delta^6 y_{-3} + m_7 + \frac{1}{2}\Delta^8 y_{-4} \\ &= \Delta^4 y_{-2} + 2m_5 + 2\Delta^6 y_{-3} + m_7 + \frac{1}{2}\Delta^8 y_{-4}. \end{aligned}$$

Similarly

$$\Delta^5 y_0 = m_5 + \frac{5}{2}\Delta^6 y_{-3} + 3m_7 + \frac{5}{2}\Delta^8 y_{-4} + m_9 + \frac{1}{2}\Delta^{10} y_{-5}. \tag{12.8}$$

The process may be continued as far as desired.

To obtain y_0 and the corresponding differences in terms of quantities on the line through $y_{\frac{1}{2}}$ we proceed in the same manner as above. First we note that

$$\begin{aligned} y_0 &= y_1 - \Delta y_0 \\ &= 2m_0 - y_0 - \Delta y_0 \end{aligned}$$

and

$$y_0 = m_0 - \frac{1}{2}\Delta y_0. \tag{12.9}$$

Corresponding to formula (12.4) we combine

$$\begin{cases} \Delta^{2i+1} y_{-i} = \Delta^{2i} y_{1-i} - \Delta^{2i} y_{-i} \\ 2m_{2i} = \Delta^{2i} y_{1-i} + \Delta^{2i} y_{-i} \end{cases}$$

to give

$$\Delta^{2i} y_{1-i} = m_{2i} + \frac{1}{2}\Delta^{2i+1} y_{-i}. \tag{12.10}$$

Then with $i = 1$ in (12.10) we have

$$\Delta^2 y_0 = m_2 + \frac{1}{2}\Delta^3 y_{-1}. \tag{12.11}$$

For $\Delta^3 y_0$ we have

$$\begin{aligned} \Delta^3 y_0 &= \Delta^3 y_{-1} + \Delta^4 y_{-1} \\ &= \Delta^3 y_{-1} + m_4 + \frac{1}{2}\Delta^5 y_{-2}. \end{aligned} \tag{12.12}$$

Similarly we obtain

$$\begin{aligned} \Delta^4 y_0 &= \Delta^4 y_{-1} + \Delta^5 y_{-1} \\ &= \Delta^4 y_{-1} + \Delta^5 y_{-2} + \Delta^6 y_{-2} \\ &= m_4 + \frac{1}{2}\Delta^5 y_{-2} + \Delta^5 y_{-2} + m_6 + \frac{1}{2}\Delta^7 y_{-3} \\ &= m_4 + \frac{3}{2}\Delta^5 y_{-2} + m_6 + \frac{1}{2}\Delta^7 y_{-3}. \end{aligned} \tag{12.13}$$

Again the process may be continued as far as desired.

13. Errors in Tabulated Values. One of the simple applications of differences is that of checking for errors in tabulated values of function. Let us suppose that there is an error, ϵ, in the value of y_0; the growth of this error in the difference table is shown in Table 2.4.

Table 2.4. The Growth of an Error.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
y_{-3}				
	Δy_{-3}			
y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4} + \epsilon$
	Δy_{-2}		$\Delta^3 y_{-3} + \epsilon$	
y_{-1}		$\Delta^2 y_{-2} + \epsilon$		$\Delta^4 y_{-3} - 4\epsilon$
	$\Delta y_{-1} + \epsilon$		$\Delta^3 y_{-2} - 3\epsilon$	
$y_0 + \epsilon$		$\Delta^2 y_{-1} - 2\epsilon$		$\Delta^4 y_{-2} + 6\epsilon$
	$\Delta y_0 - \epsilon$		$\Delta^3 y_{-1} + 3\epsilon$	
y_1		$\Delta^2 y_0 + \epsilon$		$\Delta^4 y_{-1} - 4\epsilon$
	Δy_1		$\Delta^3 y_0 - \epsilon$	
y_2		$\Delta^2 y_1$		$\Delta^4 y_0 + \epsilon$
	Δy_2			
y_3				

It is noted that the effect of the error grows in the difference table and spreads fanwise through the higher differences. The following characteristics can also be observed:

1. The coefficients of the ϵ's are the binomial coefficients with alternating signs.

2. The algebraic sum of the errors in any difference column is zero.

3. The maximum error in the even differences is in the same horizontal line as the tabular value which is in error.

These characteristics enable us to locate and correct a random error and may also be used to smooth tabular values obtained from experimental measurements.

Example II.3. In this example the values of x and y are given and the differences calculated.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	1.0000				
		5191			
1.1	1.5191		354		
		5545		−24	
1.2	2.0736		330		24
		5875		0	
1.3	2.6611		330		24
		6205		24	
1.4	3.2816		354		51
		6559		75	
1.5	3.9375		429		−84
		6988		−9	
1.6	4.6363		420		186
		7408		177	
1.7	5.3771		597		−84
		8005		93	
1.8	6.1776		690		51
		8695		144	
1.9	7.0471		834		
		9529			
2.0	8.0000				

It is noted that the fourth differences are oscillating for the larger values of x. The largest numerical 4th difference of 186 is at $x = 1.6$. If the table should be regular this suggests an error in the y value for $x = 1.6$. The dashed fan lines are now drawn, and we note from Table 2.4 that

$$\begin{aligned} \Delta^4 y_{-4} + \epsilon &= 51, \\ \Delta^4 y_{-3} - 4\epsilon &= -84, \\ \Delta^4 y_{-2} + 6\epsilon &= 186, \\ \Delta^4 y_{-1} + 4\epsilon &= -84, \\ \Delta^4 y_0 + \epsilon &= 51. \end{aligned}$$

Assuming now that we want all 4th differences to be alike, we could eliminate $\Delta^4 y$ between any two of the compatible equations and solve for ϵ; thus subtracting the second equation from the first

$$5\epsilon = 135; \qquad \epsilon = 27.$$

Then

$$y(1.6) + \epsilon = 4.6363$$

or

$$y(1.6) + 27 = 4.6363$$

and

$$y(1.6) = 4.6336.$$

If this correction is made, all of the fourth differences become 24 thus defining a regular function for y. It may be noted that the error was one of transposing numbers: i.e., writing 63 instead of 36. This is a common error in numerical analysis and has an additional characteristic. A transposition of two adjacent digits which differ by m will produce an error of $9m$. In the example we noted that $\epsilon = 27 = (9)(3)$ so that $m = 3$ and that the difference between 6 and 3, the transposed numbers, is 3.

Note. A word of caution is necessary. If the values of the function, y, are rounded off to a given number of significant digits, they are in error by not more than $\frac{1}{2}$ in the last significant figure. This error, however, grows in the difference table and it may appear that a correction should be made on the tabular value of y when in reality it should not. A fairly good rule to follow is: *Do not correct a tabular value if the correction is only* **one** *unit in the last significant digit.*

It is also advantageous to have some criterion for the fluctuations in the differences which may be tolerated. We shall state one given by Comrie.* Fluctuations in the nth order differences less than the following limits may be accepted:

n	1	2	3	4	5	6	8	10
limits	± 1	± 2	± 3	± 6	± 12	± 22	± 80	± 300

14. Differences of a Polynomial. The computation of the successive differences of a polynomial will derive a very important property. Let us consider a polynomial of the nth degree:

$$(14.1) \qquad y = f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

* L. J. Comrie, Chambers's 6-Figure Mathematical Tables (New York: Van Nostrand, 1949), Vol. 2, p. xxxi.

and let the increment on x be $h = \Delta x$ and the difference between two consecutive values of y be Δy so that we have

$$(14.2)\quad y + \Delta y = a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n$$

and upon taking the difference (14.2) − (14.1), we obtain

$$\begin{aligned}(14.3)\quad \Delta y &= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \cdots \\ &\quad + a_{n-1}[(x+h) - x] + [a_n - a_n] \\ &= a_0\left[x^n + nhx^{n-1} + \binom{n}{2}h^2x^{n-2} + \cdots + h^n - x^n\right] \\ &\quad + a\left[x^{n-1} + (n-1)hx^{n-2} + \binom{n-1}{2}h^2x^{n-3} + \cdots + h^{n-1} - x^{n-1}\right] \\ &\quad + \cdots \\ &\quad + a_{n-1}[x + h - x]\end{aligned}$$

where $\binom{n}{k}$ are the binomial coefficients. Completing the subtraction and collecting the coefficients of the powers of x, we obtain

$$\begin{aligned}(14.4)\quad \Delta y &= a_0nhx^{n-1} + \left[a_0h^2\binom{n}{2} + a_1h\binom{n-1}{1}\right]x^{n-2} \\ &\quad + \left[a_0h^3\binom{n}{3} + a_1h^2\binom{n-1}{2} + a_2h\binom{n-2}{1}\right]x^{n-3} \\ &\quad + \cdots \\ &\quad + [a_0h^n + a_1h^{n-1} + \ldots + a_{n-1}h].\end{aligned}$$

Since we shall assume that h is a constant, the bracketed expressions are all constants so that we may write

$$(14.5)\quad \Delta y = a_0nhx^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \cdots + b_{n-2}x + b_{n-1}.$$

This is the first difference of the polynomial of the nth degree and is a polynomial of degree $n - 1$.

If we now difference this polynomial in the same manner, we have

$$\text{(14.6)} \quad \Delta y + \Delta(\Delta y) = a_0nh(x+h)^{n-1} + b_1(x+h)^{n-2} + b_2(x+h)^{n-3} + \cdots + b_{n-1}$$

and subtracting (14.5) from (14.6) yields

$$\begin{aligned}\text{(14.7)} \quad \Delta^2 y &= a_0nh[(x+h)^{n-1} - x^{n-1}] + b_1[(x+h)^{n-2} - x^{n-2}] + \cdots + [b_{n-1} - b_{n-1}] \\ &= a_0nh\left[x^{n-1} + (n-1)hx^{n-2} + \binom{n-1}{2}h^2x^{n-3} + \cdots - x^{n-1}\right] \\ &\quad + b_1\left[x^{n-2} + (n-2)hx^{n-3} + \binom{n-2}{2}h^2x^{n-4} + \cdots - x^{n-2}\right] \\ &\quad + \cdots \\ &= a_0n(n-1)h^2x^{n-2} + c_1x^{n-3} + c_2x^{n-4} + \cdots + c_{n-2}.\end{aligned}$$

Thus the second difference is a polynomial of degree $n-2$. The process may be continued until we find for the nth difference

$$\text{(14.8)} \quad \Delta^n y = a_0[n(n-1)(n-2)\cdots 1]h^nx^{n-n} = a_0h^nn!$$

This is a polynomial of degree $n - n = 0$ or a constant, and all higher differences will be zero.

For this derivation we specified that h be a constant or, in other words, that we choose values of the independent variable x at equally spaced intervals. We may now state the proposition:

For equally spaced intervals of the independent variable, the **n***th differences of a polynomial of the* **n***th degree are constant.* Conversely: *If the* **n***th differences of a function tabulated at equally spaced intervals are constant, the function is a polynomial of degree* **n**.

This proposition permits us to analyze a tabulated function. Thus if the nth differences are constants (or nearly so since rounding off errors may prevent them from being exactly equal), we draw the conclusion that the function may be represented by a polynomial of the nth degree. Let us refer back to Example II.2. It is noted that the $\Delta^5 y_i = 240$ are all equal, and we conclude that the tabulated function may be represented by a polynomial of the 5th degree. In fact it is

$$y = 2x^5 - 3x^4 + 5x^3 - x^2 + x - 7.$$

The coefficients may be determined by the method of undetermined coefficients discussed in 3.F.

15. Tabulation of Polynomials. We have already discussed a couple of methods of calculating the values of polynomials (*see* 3.B). The proposition discussed in the last section furnishes still another method for building up a table of values for polynomials if we choose equally spaced intervals for the independent variable. Since we know that the nth differences are constant, we may construct the lower-order differences for the polynomial from the formula

$$\Delta^{n-1} y_{i+1} = \Delta^n y_i + \Delta^{n-1} y_i \tag{15.1}$$

which is obtained from formula (11.3) by solving for $\Delta^{n-1} y_{i+1}$ once we have the value of the nth difference. Thus it is necessary to compute n values and the initial n differences. Then the lower-order differences are computed for the next line by formula (15.1) and yield the next value of the polynomial. It is advisable to check the values periodically. Let us consider an example.

Example II.4. Let us build up the table of values for the polynomial

$$y = x^3 - 3x^2 + 5x + 2$$

for integral values of x in the interval $0 \leq x \leq 10$.

Solution. We calculate the first four values, the initial differences, and the value for $x = 10$ for a check. We then build up the difference table line by line starting with the third difference which is a constant for this 3rd degree polynomial.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	2			
		3		
1	5		0	
		3		6
2	8		6	
		9		6
3	17		12	
		21		6
4	38		18	
		39		6
5	77		24	
		63		6
6	140		30	
		93		6
7	233		36	
		129		6
8	362		42	
		171		6
9	533		48	
		219		
10	752 check			

Although it is only necessary to compute the first four values for a 3rd degree polynomial, it is recommended that one more value and the differences be computed as a check on the start. This is indicated by the dotted line in the table of values. The calculation adapts itself readily to a calculating machine as it is simply a sequence of additions. Thus for the values at

$$\begin{array}{r} x = 5: \\ \hline 6 \\ +12 \\ \hline 18 \\ +21 \\ \hline 39 \\ +38 \\ \hline 77. \end{array} \qquad \begin{array}{r} x = 7: \\ \hline 6 \\ +24 \\ \hline 30 \\ +63 \\ \hline 93 \\ +140 \\ \hline 233. \end{array}$$

In the case where it is necessary to round off the values of the function, an extra value should always be calculated since the nth differences are not always exactly constant to the number of significant figures carried. Furthermore, the differences and initial values should be carried to one more significant number than that desired in the values for the function. This is illustrated in the next example.

Example II.5. List a table of values of the function

$$y = 1.3x^3 - 2.6x^2 + 4.9x - 1$$

in the interval $0 \le x \le .15$ with $\Delta x = .01$ and correct to five decimals.

Solution.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	−1.000000			
		48741		
.01	−.951259		−512	
		48229		8
.02	−.903030		−504	
		47725		7
.03	−.855305		−497	
		47228		8
.04	−.808077		−489	
		46739		8
.05	−.761338		−481	
		46258		8
.06	−.71508		−473	
		45785		8
.07	−.66930		−465	
		45320		8
.08	−.62398		−457	
		44863		8
.09	−.57912		−449	
		44414		8
.10	−.53471		−441	
		43973		8
.11	−.49074		−433	
		43540		8
.12	−.44720		−425	
		43115		8
.13	−.40408		−417	
		42698		8
.14	−.361385		−409	
		42289		
.15	−.31911 check			
	−.31910			

Six initial values were computed and carried to one more significant figure than desired. From the initial third differences the value 8 was decided upon. (The three values could be averaged and rounded off to six decimals which in the example would yield 8.) The function values were then calculated by building up the differences. The check value at $x = 1.5$ shows the value from the differences to be off by one unit in the fifth place. This is to be expected and should it be desired to continue the table, the calculated value at $x = 1.5$ should be used and another periodic check established.

16. Differences of a Function of Two Variables. The differences for a function of two variables are considerably more complex than those for a function of a single variable. Let us consider the function $z = f(x,y)$ to be any function of the two independent variables x and y. The value of the function for given values of the independent variables, say x_i and y_j, may be denoted by $z_{ij} = f(x_i, y_j)$. Following the usual notation of the first subscript denoting the row and the second subscript denoting the column, we may now form a table of values of the function according to this scheme.

Table 2.5. Function of Two Variables.

	y_0	y_1	y_2	y_3	. . .	. . .	. . .	y_m
x_0	z_{00}	z_{01}	z_{02}	z_{03}	. . .	. . .	. . .	z_{0m}
x_1	z_{10}	z_{11}	z_{12}	z_{13}	. . .	. . .	. . .	z_{1m}
x_2	z_{20}	z_{21}	z_{22}	z_{23}	. . .	. . .	. . .	z_{2m}
x_3	z_{30}	z_{31}	z_{32}	z_{33}	. . .	. . .	. . .	z_{3m}
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
x_n	z_{n0}	z_{n1}	z_{n2}	z_{n3}	. . .	. . .	. . .	z_{nm}

In calculating the first differences we can do so either along a row or in a column, and, of course, the distinction must be made. This is very similar to partial differentiation. We may hold y constant and difference with respect to x or vice versa. We shall use a subscript on the difference symbol Δ to denote the variable with respect to which we are differencing. The definition for the **first**

difference of a function of two variables is given by

$$(16.1)\qquad \begin{cases} \Delta_x z_{ij} = z_{i+1,j} - z_{ij} \\ \Delta_y z_{ij} = z_{i,j+1} - z_{ij}. \end{cases}$$

For example

$$(16.2)\qquad \begin{cases} \Delta_x z_{00} = z_{10} - z_{00} \\ \Delta_x z_{01} = z_{11} - z_{01} \\ \Delta_x z_{10} = z_{20} - z_{10} \\ \Delta_x z_{22} = z_{32} - z_{22} \end{cases}$$

and

$$(16.3)\qquad \begin{cases} \Delta_y z_{00} = z_{01} - z_{00} \\ \Delta_y z_{01} = z_{02} - z_{01} \\ \Delta_y z_{10} = z_{11} - z_{10} \\ \Delta_y z_{22} = z_{23} - z_{22}. \end{cases}$$

If now we wish to difference with respect to x and y both, we are automatically dealing with *second differences*. Thus

$$(16.4)\qquad \begin{aligned} \Delta_{xy} z_{ij} &= \Delta_x z_{i,j+1} - \Delta_x z_{ij} \\ &= z_{i+1,j+1} - z_{i,j+1} - (z_{i+1,j} - z_{ij}) \\ &= (z_{i+1,j+1} - z_{i+1,j}) - (z_{i,j+1} - z_{ij}) \\ &= \Delta_y z_{i+1,j} - \Delta_y z_{ij}. \end{aligned}$$

The definition of **second differences** is completed with the following two formulas

$$(16.5)\qquad \begin{cases} \Delta_{x^2} z_{ij} = \Delta_x z_{i+1,j} - \Delta_x z_{ij} \\ \qquad\quad = z_{i+2,j} - 2z_{i+1,j} + z_{ij} \\ \Delta_{y^2} z_{ij} = \Delta_y z_{i,j+1} - \Delta_y z_{ij} \\ \qquad\quad = z_{i,j+2} - 2z_{i,j+1} + z_{ij} \end{cases}$$

in which the symbol $\Delta_{x^2} z_{ij}$ denotes the *second difference of z_{ij} with respect to x* while y is being held constant and the equivalent definition with respect to y.

In general we shall let the symbol $\Delta_{x^m y^n} z_{ij}$ denote the $m + n$ *difference of z_{ij} taking m differences with respect to x and n differences with respect to y.*

To specifically define third differences we would need four formulas, one for each of $\Delta_{x^3} z_{ij}$, $\Delta_{y^3} z_{ij}$, $\Delta_{x^2 y} z_{ij}$, and $\Delta_{xy^2} z_{ij}$. They are

$$
(16.6)\qquad
\begin{cases}
\Delta_{x^3} z_{ij} = \Delta_{x^2} z_{i+1,j} - \Delta_{x^2} z_{ij} \\
\qquad\quad = z_{i+3,j} - 3z_{i+2,j} + 3z_{i+1,j} - z_{ij} \\
\Delta_{y^3} z_{ij} = \Delta_{y^2} z_{i,j+1} - \Delta_{y^2} z_{ij} \\
\qquad\quad = z_{i,j+3} - 3z_{i,j+2} + 3z_{i,j+1} - z_{ij} \\
\Delta_{x^2y} z_{ij} = \Delta_{x^2} z_{i,j+1} - \Delta_{x^2} z_{ij} \\
\qquad\quad = \Delta_y z_{i+2,j} - 2\Delta_y z_{i+1,j} + \Delta_y z_{ij} \\
\Delta_{xy^2} z_{ij} = \Delta_{y^2} z_{i+1,j} - \Delta_{y^2} z_{ij} \\
\qquad\quad = \Delta_x z_{i,j+2} - 2\Delta_x z_{i,j+1} + \Delta_x z_{ij}
\end{cases}
$$

in which the last two formulas may be reduced to the functional values by (16.5) or (16.4).

Certain characteristics in the formulas are beginning to become apparent. It appears that the coefficients of the right-hand side are the binomial coefficients with alternating signs. It is immaterial with respect to which variable we difference first. Keeping these characteristics in mind, we may now formulate the general formula for **higher-order differences of a function of two variables:**

$$
(16.7)\quad \Delta_{x^m y^n} z_{ij} = \Delta_{x^m} z_{i,j+n} - \binom{n}{1} \Delta_{x^m} z_{i,j+n-1} + \binom{n}{2} \Delta_{x^m} z_{i,j+n-2} + \cdots + (-1)^n \Delta_{x^m} z_{ij}
$$
$$
= \Delta_{y^n} z_{i+m,j} - \binom{m}{1} \Delta_{y^n} z_{i+n-1,j} + \binom{m}{2} \Delta_{y^n} z_{i+n-2,j} + \cdots + (-1)^m \Delta_{y^n} z_{ij}
$$

and the right-hand side may be further expanded by a repetition of this general formula; that is,

$$
(16.8)\quad \Delta_{x^m} z_{ij} = z_{i+m,j} - \binom{m}{1} z_{i+m-1,j} + \binom{m}{2} z_{i+m-2,j} + \cdots + \cdots + (-1)^m z_{ij}.
$$

A difference table may be constructed for the first differences, but difference tables for higher differences are too complex to be practical. To construct a difference table for first differences, arrange the functional values as shown in Table 2.5 leaving a blank space on each side of the entry both horizontally and vertically. The appropriate differences may now be entered in these blank spaces. It is further recommended that the differences be entered in a different color to distinguish them from the functional values. In other words, the following two tables are superimposed upon each other.

Table 2.6. Function of Two Variables.

	y_0	y_1	y_2	y_3
x_0	z_{00}	z_{01}	z_{02}	z_{03}
x_1	z_{10}	z_{11}	z_{12}	z_{13}
x_2	z_{20}	z_{21}	z_{22}	z_{23}
x_3	z_{30}	z_{31}	z_{32}	z_{33}
x_4	z_{40}	z_{41}	z_{42}	z_{43}

Table 2.7. First Difference Table For Two Variables.

	y_0		y_1		y_2		y_3
x_0		$\Delta_y z_{00}$		$\Delta_y z_{01}$		$\Delta_y z_{02}$	
	$\Delta_x z_{00}$		$\Delta_x z_{01}$		$\Delta_x z_{02}$		$\Delta_x z_{03}$
x_1		$\Delta_y z_{10}$		$\Delta_y z_{11}$		$\Delta_y z_{12}$	
	$\Delta_x z_{10}$		$\Delta_x z_{11}$		$\Delta_x z_{12}$		$\Delta_x z_{13}$
x_2		$\Delta_y z_{20}$		$\Delta_y z_{21}$		$\Delta_y z_{22}$	
	$\Delta_x z_{20}$		$\Delta_x z_{21}$		$\Delta_x z_{22}$		$\Delta_x z_{23}$
x_3		$\Delta_y z_{30}$		$\Delta_y z_{31}$		$\Delta_y z_{32}$	
	$\Delta_x z_{30}$		$\Delta_x z_{31}$		$\Delta_x z_{32}$		$\Delta_x z_{33}$
x_4		$\Delta_y z_{40}$		$\Delta_y z_{41}$		$\Delta_y z_{42}$	

Example II.6. Form a table of first differences for the function $z = f(x,y)$ as given in the table of values:

Solution.

$x \backslash y$	0		1		2		3
0	1	−5	−4	−5	−9	−5	−14
	4		4		4		4
1	5	−5	0	−5	−5	−5	−10
	6		6		6		6
2	11	−5	6	−5	1	−5	−4
	8		8		8		8
3	19	−5	14	−5	9	−5	4
	10		10		10		10
4	29	−5	24	−5	19	−5	14

It is noted that $\Delta_y z_{ij} = -5$ for all i, j which means that the function is linear in y. It is also noted that the first differences with respect to x differ by 2; i.e., all second differences are constant which means that the function is quadratic in x. The function used is

$$z(x,y) = x^2 + 3x - 5y + 1.$$

Although it is possible to define difference formulas for functions of more than two variables, the formulas are too unwieldy to be practical. In studying functions of more than two variables it is recommended that the variance study be made on families of functions of two variables using the other variables as parameters. Thus if we had a function

$$R = f(x,y,v)$$

we could choose v as a parameter and for chosen values of v, say $v_1, v_2, \cdots$, study the functions

$$R_1 = f(x,y,v_1)$$
$$R_2 = f(x,y,v_2)$$
etc.,

each of which is a function of two variables.

17. Divided Differences. If a difference is divided by the interval length of the independent variable which it spans, it is called a *divided difference*. This general definition is better understood in terms of formulas. Let us again consider the function $y = f(x)$ to be tabulated for $x_i, (i = 0, \cdots, n)$. The **first-order divided difference** is then defined by

$$[x_i x_j] = \frac{y_i - y_j}{x_i - x_j} = \frac{y_j - y_i}{x_j - x_i}. \tag{17.1}$$

If i and j are consecutive integers, the numerator is Δy_i, the *first difference*. Formula (17.1) also shows that the order inside the bracket is immaterial since clearly $[x_i x_j] = [x_j x_i]$; however, it must be remembered that the differences in the numerator and denominator must be in the same direction. Although i and j are usually consecutive integers, the definition is quite general and first-order divided differences can be calculated for any integral values of i and j.

In defining the first-order divided difference, the bracket notation $[x_i x_j]$ was used. There is no generally accepted notation for divided differences, and although the bracket notation is quite popular, it becomes somewhat cumbersome for higher-order divided differences. We shall therefore modify the notation somewhat and adopt the notation

(17.2) $\Delta^n[x_i x_k] \equiv$ the nth order divided differences between x_i and x_k.

In this notation **i** *and* **k** *will embrace a sufficient number of consecutive integers to permit the taking of* **n** *differences.* They themselves can only be consecutive integers for first-order divided differences. If it is desired to have divided differences for values of the independent variable which cannot be ordered by consecutive integral notation, all values of the x_i will be written out; that is, for example,

(17.3) $\Delta^2[x_1 x_3 x_5] \equiv$ second order divided difference between x_1 and x_5 with x_3 as the intermediate value.

Let us now define the higher-order divided differences. The difference of the first-order divided difference divided by the interval of the independent variable which they span is called the **second-order divided difference.**

(17.4) $$\Delta^2[x_1 x_3] = \frac{\Delta[x_1 x_2] - \Delta[x_2 x_3]}{x_1 - x_3} = \frac{\Delta[x_2 x_3] - \Delta[x_1 x_2]}{x_3 - x_1}.$$

The first of the two formulas is preferred since it gives consecutive sequence to the subscripts.

We may now proceed in a similar manner to formulate the **third-order divided difference**

(17.5) $$\Delta^3[x_0 x_3] = \frac{\Delta^2[x_0 x_2] - \Delta^2[x_1 x_3]}{x_0 - x_3}$$

and the **fourth-order divided difference**

(17.6) $$\Delta^4[x_0 x_4] = \frac{\Delta^3[x_0 x_3] - \Delta^3[x_1 x_4]}{x_0 - x_4}.$$

In general we have the ***n*th-order divided difference**

(17.7) $$\Delta^n[x_0 x_n] = \frac{\Delta^{n-1}[x_0 x_{n-1}] - \Delta^{n-1}[x_1 x_n]}{x_0 - x_n}.$$

A divided difference table can be constructed using the same principle as for ordinary difference tables. The general form is given in Table 2.8. Each difference is obtained by differencing the numbers

Table 2.8. Divided Difference Table.

x	y	$\Delta[x_ix_j]$	$\Delta^2[x_ix_k]$	$\Delta^3[x_ix_k]$	$\Delta^4[x_ix_k]$
x_0	y_0				
		$\Delta[x_0x_1]$			
x_1	y_1		$\Delta^2[x_0x_2]$		
		$\Delta[x_1x_2]$		$\Delta^3[x_0x_3]$	
x_2	y_2		$\Delta^2[x_1x_3]$		$\Delta^4[x_0x_4]$
		$\Delta[x_2x_3]$		$\Delta^3[x_1x_4]$	
x_3	y_3		$\Delta^2[x_2x_4]$		$\Delta^4[x_1x_5]$
		$\Delta[x_3x_4]$		$\Delta^3[x_2x_5]$	
x_4	y_4		$\Delta^2[x_3x_5]$		
		$\Delta[x_4x_5]$			
x_5	y_5				

in the adjacent left-hand column and dividing by the interval length which it spans. The interval length is indicated in the notation or can be found by "fanning" back from the value being computed. Thus, in computing $\Delta^3[x_1x_4]$ the divisor is $x_1 - x_4$ which is indicated by the subscript on the x's in the bracket. The divisor can also be found by following the slant lines which form a fan as shown in Table 2.8.

If the values of the independent variable are given in equally spaced intervals, the divisor for the divided differences is constant for each order. This is easily seen, for in this case

$$x_0 - x_1 = x_1 - x_2 = x_2 - x_3 = \cdots = h$$

for the first order,

$$x_0 - x_2 = x_1 - x_3 = x_2 - x_4 = \cdots = 2h$$

for the second order, and in general

$$x_0 - x_k = x_1 - x_{k+1} = x_2 - x_{k+2} = \cdots = kh$$

for the kth order.

Example II.7. Find the divided differences up to the fifth order for the table of (x,y) as given.

Solution.

x	y	$\Delta[x_ix_j]$	$\Delta^2[x_ix_k]$	$\Delta^3[x_ix_k]$	$\Delta^4[x_ix_k]$	$\Delta^5[x_ix_k]$
−4	−4320					
		2040				
−2	−240		−392			
		80		52		
1	0		−28		−7	
		−60		−11		1
3	−120		−105		3	
		−480		13		1
5	−1080		−40		14	
		−600		125		1
6	−1680		710		23	
		2240		286		
9	5040		2140			
		10800				
10	15840					

The higher-order divided differences which are defined in terms of the next lower-order divided differences may be written in terms of the ordinates, y_i, by continuous substitution. If we keep in mind the symmetric property that we may interchange the order of differencing in numerator and denominator simultaneously, we may write

$$\Delta[x_ix_{i+1}] = \frac{y_i - y_{i+1}}{x_i - x_{i+1}} = \frac{y_i}{x_i - x_{i+1}} + \frac{y_{i+1}}{x_{i+1} - x_i} \tag{17.8}$$

and

$$\begin{aligned} \Delta^2[x_0x_2] &= \frac{\Delta[x_0x_1] - \Delta[x_1x_2]}{x_0 - x_2} \\ &= \frac{1}{x_0 - x_2}\left[\frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} - \frac{y_1}{x_1 - x_2} - \frac{y_2}{x_2 - x_1}\right] \\ &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} \\ &\quad + \frac{1}{x_0 - x_2}\left[\frac{(x_0 - x_2)y_1}{(x_1 - x_0)(x_1 - x_2)}\right] + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} \\ &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} \\ &\quad + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}. \end{aligned} \tag{17.9}$$

Similarly, the third-order divided difference can be reduced to

$$(17.10)\quad \Delta^3[x_0x_3] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + \frac{y_3}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}.$$

In general we have

$$(17.11)\quad \Delta^n[x_0x_n] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} + \cdots + \frac{y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}.$$

It is noted that the denominator of y_i is a product polynomial evaluated at x_i; that is,

$$(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)$$

evaluated at x_i.

18. Difference Operators. The calculus of finite differences employs certain operators which are very useful. We shall define a few of them here. The symbol Δ which we have used throughout this chapter may be thought of as an operator in that when it prefixes the functional notation $y = f(x)$, it defines an operation of taking the difference between two values of y; it may then be called *the difference operator.* The symbol Δ is also used to denote an increment on the independent variable which is in agreement with its use as an operator, since $x_2 = x_1 + \Delta x$. In the following we shall consider $y = f(x)$.

I. The Operator, E. This operator is defined by

$$(18.1)\qquad Ey = f(x + \Delta x)$$

where $y = f(x)$ and Δx is the usual increment on the independent variable. This operator is often referred to as the *shift operator* since it results in another value of the function. If the functional values, y_i, $(i = 0, \cdots, n)$, are given for known values of x_i, $(i = 0, \cdots, n)$, then

$$Ey_i = y_{i+1}. \tag{18.2}$$

The inverse operator E^{-1} is defined by

$$E^{-1}y = f(x - \Delta x) \tag{18.3}$$

so that

$$E^{-1}y_i = y_{i-1}. \tag{18.4}$$

Since we already defined

$$\begin{aligned} \Delta y_i &= y_{i+1} - y_i \\ &= Ey_i - y_i, \end{aligned} \tag{18.5}$$

we see that

$$Ey_i = \Delta y_i + y_i.$$

In other words the operators are related by the expression

$$E = \Delta + 1. \tag{18.6}$$

II. The Central Difference Operator, δ. This operator is defined by

$$\delta y = f\left(x + \frac{1}{2}\Delta x\right) - f\left(x - \frac{1}{2}\Delta x\right). \tag{18.7}$$

If we employ the operator E we can write

$$\delta y = E^{\frac{1}{2}}y - E^{-\frac{1}{2}}y,$$

so that the relationship between the operators is

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}. \tag{18.8}$$

III. The Averaging Operator, μ. This operator is defined by

$$\begin{aligned} \mu y &= \frac{1}{2}\left[f\left(x + \frac{1}{2}\Delta x\right) + f\left(x - \frac{1}{2}\Delta x\right)\right] \\ &= \frac{1}{2}\left(E^{\frac{1}{2}}y + E^{-\frac{1}{2}}y\right) \end{aligned} \tag{18.9}$$

and the relationship between the operators is

$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right). \tag{18.10}$$

A complete discussion of difference operators and their properties is beyond the scope of this book, and we shall conclude this section with the interesting property that

(18.11) $$\mu\delta = \frac{1}{2}[E - E^{-1}]$$

which can be obtained by combining formulas (18.8) and (18.10). From this we have

(18.12) $$\begin{aligned}\mu\delta y &= \frac{1}{2}[f(x + \Delta x) - f(x - \Delta x)] \\ &= \frac{1}{2}[f(x + \Delta x) - f(x) + f(x) - f(x - \Delta x)] \\ &= \frac{1}{2}[\Delta y_i + \Delta y_{i-1}]\end{aligned}$$

which is the arithmetic mean of two consecutive first differences. In Section 12 we denoted these by m_1, thus

(18.13) $$\mu\delta y = m_1.$$

Also, we can obtain

(18.14) $$\mu\delta^3 y = m_3$$

and other similar properties.

19. Differences and Derivatives. The relationship between differences and derivatives can be understood by a consideration of their respective definitions. We recall the definition of a derivative

(19.1) $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

where $y = f(x)$. The numerator of

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

is the first difference of y and the fraction is known as a *difference quotient*. If the function is tabulated at discrete values of x, the above is in accord with the definitions of Section 10.

The second derivative is defined as the derivative of the first derivative, and the difference quotient would be the difference of first differences. This is true of all higher-order derivatives, and thus we can state in general

$$\frac{d^n y}{dx^n} = \lim_{\Delta x \to 0} \frac{\Delta^n y}{(\Delta x)^n}. \tag{19.2}$$

By the *Theorem of Mean Value* we have

$$f(x + \Delta x) - f(x) = (\Delta x) f'(x + \theta \Delta x) \tag{19.3}$$

where $0 < \theta < 1$. In terms of differences this states

$$\Delta y = (\Delta x) f'(x + \theta \Delta x), \qquad (0 < \theta < 1), \tag{19.4}$$

which can be generalized to the nth difference

$$\Delta^n y = (\Delta x)^n f^{(n)}(x + n\theta \Delta x), \qquad (0 < \theta < 1). \tag{19.5}$$

20. Exercise II.

1. Calculate the difference table for

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
y	1.0000	1.5191	2.0736	2.6611	3.2816	3.9375	4.6336	5.3771	6.1776

2. Find and correct the error by means of differences

(a)

x	y
1.0	0
1.1	−.54549
1.2	−.96768
1.3	−1.25307
1.4	−1.39776
1.5	−1.40625
1.6	−1.29024
1.7	−1.06734
1.8	−.76032
1.9	−.39501
2.0	0

(b)

x	y
0	2
1	5
2	8
3	17
4	38
5	75
6	140
7	233
8	362
9	533
10	752

3. By the method of differences, tabulate the values of the polynomial

(a) $y = 2x^3 - 3x^2 + x - 5$ in $0 \leq x \leq 1$, $\Delta x = .1$;
(b) $y = 5x^3 + 2.07x^2 - .3$ in $0 \leq x \leq .1$; $\Delta x = .01$.

4. Compute the values of $z = 2x^2 - 3y^2 + xy$ for $x = 0, 1, 2, 3, 4$; and $y = 0, 1, 2, 3$; and construct a table of first differences.
5. Construct a table of divided differences for the values

x	-1	-2	0	3	5	8
y	-1320	-240	0	-120	-1080	0

6. Show that $\mu\delta^3 y = m_3$.
7. Prove that

$$\Delta[x_1x_2] = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} \div \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}.$$

8. Prove that $\Delta^2[x_0x_2]$ is the quotient of the two determinants

$$\begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} \text{ divided by } \begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix}.$$

9. Find the divided differences up to the fourth order of $y = \cos x$ for $0° \leq x \leq 30°$ with $\Delta x = 5°$.
10. Find the difference table for Λ_0 in Table VIII in the back of the book, for $0 \leq p \leq .1$; $\Delta p = .01$.

CHAPTER III. INTERPOLATION

21. Introduction. Interpolation is a fundamental operation in mathematics. The reader was probably first introduced to the process in a course in trigonometry when it became necessary to find values of the trigonometric functions for the values of the angles other than those given in the table or perhaps when logarithms were first studied. In a lighter vein, interpolation has been said to be the art of reading between the lines of tabulated values of a function. We may now make a distinction between interpolation and extrapolation. The latter is the art of reading before the first line or after the last line of a tabulated function. More specifically, we may define **interpolation** as the *process* of finding the values of a function for any value of the independent variable *within an interval* for which some values are given and **extrapolation** as the *process* of finding the values *outside of this interval.*

The process of interpolation becomes very important in advanced mathematics when dealing with functions which either are not known at every value of the independent variable within an interval, or the expression of which is so complicated that the evaluation of the function is prohibitive. It is then that the function is replaced by a simple function which assumes the known values of the given function and from which the other values may be computed to the desired degree of accuracy. This is the broader sense of interpolation.

In precise mathematical language we are concerned with a function, $y = f(x)$, whose values, $y_0, y_1, \cdots, y_n$, are known for the values $x_0, x_1, \cdots, x_n$ of the independent variable. Interpolation now seeks to replace $f(x)$ by a simpler function, $I(x)$, which has the same value as $f(x)$ for $x_0, x_1, \cdots, x_n$ and from which other values can easily be calculated. The function $I(x)$ is said to be an *inter-*

polating formula or *interpolating function*. In many engineering applications this function is called a *smoothing function*. We shall, however, use the term smoothing in a slightly different sense; namely, a function which does not take on the tabulated values exactly but instead smoothens out these given values into a smooth curve.

A desired characteristic of interpolating functions is that they be simple. Consequently, the most frequently employed forms are the polynomial and the finite trigonometric series. In these cases we refer to the process as *polynomial interpolation* or *trigonometric interpolation*. The latter is used if the given values indicate that the function is periodic. The interpolating function can, of course, be arbitrarily chosen and can take any form; thus it could be exponential, logarithmic, etc. One such form which is frequently used is that of a rational fraction. However, it should always be as simple as possible.

The use of the polynomial and trigonometric series is based on *Weierstrass' Theorems*.

I. Every function, $f(x)$, which is continuous in an interval (a,b) can be represented there, to any degree of accuracy, by a polynomial $P(x)$, i.e.,

$$|f(x) - P(x)| < \epsilon$$

for all $a < x < b$ and where ϵ is any preassigned positive quantity.

II. Every continuous function, $f(x)$, of period 2π can be represented by a finite trigonometric series:

$$T(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx$$

such that

$$|f(x) - T(x)| < \epsilon$$

for $a < x < b$ and $\epsilon > 0$.

22. Linear Interpolation. The simplest of all interpolation is the case in which the interpolating polynomial is linear. In a course in trigonometry, this was accomplished by multiplying the tabular difference, Δy, by the increase in the independent variable, $x - x_0$, divided by the tabular difference in x, Δx, and adding to the value of y corresponding to x_0; thus

$$y = y_0 + \frac{(x - x_0)\Delta y}{\Delta x}. \tag{22.1}$$

One of the most convenient methods of writing the linear interpolating function for machine calculations is given by

$$I(x) = \frac{1}{x_1 - x_0}\begin{vmatrix} y_0 & x_0 - x \\ y_1 & x_1 - x \end{vmatrix}. \tag{22.2}$$

If we expand the determinant, we obtain

$$\begin{aligned} I(x) &= \frac{1}{x_1 - x_0}[y_0(x_1 - x) - y_1(x_0 - x)] \\ &= \frac{1}{x_1 - x_0}[(y_1 - y_0)x + y_0x_1 - y_1x_0] \end{aligned} \tag{22.3}$$

which is linear in x and which reduces to formula (22.1) if we let $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$.

The right-hand side of formula (22.2) is easily evaluated since it is the difference of two products, $y_0(x_1 - x) - y_1(x_0 - x)$, followed by a division by Δx and is thus accomplished on a calculator without recording any intermediate values.

Example III.1. Find y at $x = .3421$ given

i	x_i	y_i	$x_i - x$
0	.3412	.1946	−9
1	.3432	.1273	11

Solution. The given data are augmented by the column $x_i - x$, the values for which are calculated. The difference $x_1 - x_0 = (20)$ is calculated and recorded if it is not a simple subtraction which can be done mentally. Since we can remove common factors from $x_i - x$ and $x_1 - x_0$ which cancel each other in the division, the decimal point and insignificant zeros are ignored. Now

$$y(.3421) = \frac{1}{20}[(.1946)(11) - (.1273)(-9)] = \boxed{.1643}.$$

This method is especially handy when it is desired to obtain one trigonometric function from another and linear interpolation is sufficiently accurate.

Example III.2. Find $\sin A$ if $\cos A = .74061$ and given the first three columns

i	$\cos A_i$	$\sin A_i$	$\cos A_i - \cos A$
0	.74120	.67129	59
1	.74022	.67237	-39

Solution. Compute the fourth column given above. Then

$$\sin A = -\frac{1}{98}[(.67129)(-39) - (.67237)(59)] = .67191.$$

23. Classical Polynomial Formulas. There are many polynomial interpolating functions based on finite differences which have become classical in numerical analysis. We shall now derive and display these formulas. Although they may be derived in a number of ways, we shall use the more classical approach. It is recommended that Chapter II be thoroughly reviewed before proceeding.

In the following we consider $y = f(x)$ to be a function which takes the values $y_0, y_1, \cdots, y_n$ for equidistant values of the independent variable $x_0, x_1, \cdots, x_n$. Let $I(x)$ be a polynomial of the nth degree and write it in the form

$$\textbf{(23.1)}\quad \begin{aligned} I(x) = a_0 &+ a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &+ a_3(x - x_0)(x - x_1)(x - x_2) + \cdots \\ &+ a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}). \end{aligned}$$

It is now desired to determine the coefficients a_i, $(i = 0, \cdots, n)$, such that

$$I(x_0) = y_0,\ I(x_1) = y_1,\ \cdots,\ I(x_i) = y_i,\ \cdots,\ I(x_n) = y_n.$$

Since the values x_i, $(i = 0, \cdots, n)$, are chosen at equidistant values, we have

$$x_i - x_0 = ih, \qquad (i = 1, \cdots, n), \tag{23.2}$$

where h is the interval length, Δx.

Now let x assume the values x_i, $(i = 0, \cdots, n)$, in formula (23.1) and set $I(x_i) = y_i$; then we have

$$\tag{23.3} \left\{ \begin{aligned} I(x_0) &= y_0 = a_0, \\ I(x_1) &= y_1 = a_0 + a_1(x_1 - x_0) = a_0 + a_1 h, \\ I(x_2) &= y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\ &\qquad = a_0 + a_1(2h) + a_2(2h)(h), \\ &\cdots\cdots\cdots\cdots \\ (Ix_i) &= y_i = a_0 + a_1(ih) + a_2(ih)[(i-1)h] + \cdots \\ &\qquad\qquad + a_i(i)!h^i, \\ &\cdots\cdots\cdots\cdots \\ I(x_n) &= y_n = a_0 + a_1(nh) + a_2(n)(n-1)h^2 + \cdots \\ &\qquad\qquad + a_n(n)!h^n. \end{aligned} \right.$$

These equations may be solved for a_i, $(i = 0, \cdots, n)$. By continuous substitution and recalling the definitions of finite differences we have

$$\tag{23.4} \left\{ \begin{aligned} a_0 &= y_0, \\ a_1 &= \frac{y_1 - a_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}, \\ a_2 &= \frac{1}{2h^2}[y_2 - a_0 - 2ha_1] = \frac{1}{2h^2}[y_2 - 2y_1 + y_0] \\ &= \frac{\Delta^2 y_0}{2h^2}, \\ a_3 &= \frac{1}{3!h^3}[y_3 - 3y_2 + 3y_1 - y_0] = \frac{\Delta^3 y_0}{3!h^3}, \\ &\cdots\cdots \\ a_i &= \frac{\Delta^i y_0}{i!h^i}, \\ &\cdots\cdots \\ a_n &= \frac{\Delta^n y_0}{n!h^n}. \end{aligned} \right.$$

If we put these values for the coefficients into formula (23.1), we have

$$(23.5)\quad I(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \cdots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

Let us now make a transformation on the variable by letting

$$(23.6)\qquad x = x_0 + hu \qquad \text{or} \qquad u = \frac{[x - x_0]}{h}$$

and note that

$$(23.7)\qquad \frac{x - x_i}{h} = \frac{x - [x_0 + ih]}{h} = \frac{x - x_0}{h} - i = u - i$$

for $(i = 1, \cdots, n - 1)$. It is also seen that each term of formula (23.5) contains an h in the denominator for each parenthetical expression $(x - x_i)$ so that by (23.7) we have

$$(23.8)\quad I_{N_1}(u) = y_0 + u\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \binom{u}{3}\Delta^3 y_0 + \cdots + \binom{u}{i}\Delta^i y_0 + \cdots + \binom{u}{n}\Delta^n y_0$$

where $\binom{u}{i}$ is the binomial coefficient symbol. This is **Newton's Forward Interpolation Formula.** It is written in terms of u and higher-order finite differences. It is called the forward interpolation formula since it utilizes y_0 and higher-order differences of y_0. Consequently, it is used when it is desired to find values of y at the *beginning of a table.*

We could also write an interpolating function in a form similar to that of (23.1) but focusing our attention on the end point (x_n, y_n) rather than the initial point. Thus

$$(23.9)\quad I(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots + a_n(x - x_n)(x - x_{n-1})\cdots(x - x_1).$$

The formulas for the coefficients $a_i, (i = 0, \cdots, n)$, can be obtained by substituting in the values $x_n, x_{n-1}, \cdots, x_1$ for x and letting $I(x_i) = y_i$, $(i = 1, \cdots, n)$. We will find that

(23.10) $$a_i = \frac{\Delta^i y_{n-i}}{i!h^i}, \qquad (i = 0, \cdots, n).$$

After substituting these values into (23.9) we make a transformation on the variable x by letting

$$u = \frac{x - x_n}{h} \qquad \text{or} \qquad x = x_n + hu$$

so that

(23.11) $$\frac{x - x_i}{h} = \frac{x - [x_n - ih]}{h} = u + i.$$

Formula (23.9) then takes the form

(23.12) $$I_{N_2}(u) = y_n + u\Delta y_{n-1} + \binom{u}{2}^* \Delta^2 y_{n-2} + \binom{u}{3}^* \Delta^3 y_{n-3} + \cdots + \binom{u}{n}^* \Delta^n y_0$$

where

(23.13) $$\binom{u}{i}^* = \frac{u(u+1)(u+2)\cdots(u+i-1)}{i!}, \qquad (i = 1, \cdots, n).$$

This is **Newton's Backward Interpolation Formula.** It utilizes the end value of the function, y_n, and the higher-order differences on the upward diagonal line from y_n. The formula is used when it is desired to find values of the function near the *end of a table.*

Newton's two formulas adapt themselves to the Diagonal Difference Table exhibited in Table 2.1. The initial value of the function, y_0, can, of course, be chosen any place in the table, and thus they could also be adapted to the Central Difference Table shown in Table 2.2. However, in such cases it is better to utilize values of y on both sides of y_0, and we shall now derive some interpolating formulas which employ values on a horizontal line of the central difference table. In Section 12 we developed the relationships between $\Delta^i y_0$, and the differences on the line through y_0. [See formulas (12.3) through (12.8).] If we substitute these expressions into Newton's formula (23.8) we obtain

$$(23.14)\quad I(u) = y_0 + u\left[m_1 + \frac{1}{2}\Delta^2 y_{-1}\right] + \binom{u}{2}\left[\Delta^2 y_{-1} + m_3 + \frac{1}{2}\Delta^4 y_{-2}\right] + \binom{u}{3}\left[m_3 + \frac{3}{2}\Delta^4 y_{-2} + m_5 + \frac{1}{2}\Delta^6 y_{-3}\right] + \binom{u}{4}\left[\Delta^4 y_{-2} + 2m_5 + 2\Delta^6 y_{-3} + m_7 + \frac{1}{2}\Delta^8 y_{-4}\right] + \binom{u}{5}\left[m_5 + \frac{5}{2}\Delta^6 y_{-3} + 3m_7 + \frac{5}{2}\Delta^8 y_{-4} + m_9 + \frac{1}{2}\Delta^{10} y_{-5}\right] + \cdots .$$

Collecting the coefficients of the differences the formula becomes

$$(23.15)\quad I(u) = y_0 + um_1 + \left[\frac{1}{2}u + \binom{u}{2}\right]\Delta^2 y_{-1} + \left[\binom{u}{2} + \binom{u}{3}\right]m_3 + \left[\frac{1}{2}\binom{u}{2} + \frac{3}{2}\binom{u}{3} + \binom{u}{4}\right]\Delta^4 y_{-2} + \left[\binom{u}{3} + 2\binom{u}{4} + \binom{u}{5}\right]m_5 + \left[\frac{1}{2}\binom{u}{3} + 2\binom{u}{4} + \frac{5}{2}\binom{u}{5} + \binom{u}{6}\right]\Delta^6 y_{-3} + \cdots .$$

These coefficients can now be simplified.

$$\frac{1}{2}u + \binom{u}{2} = \frac{1}{2}[u + u(u-1)] = \frac{1}{2}u^2;$$

$$\binom{u}{2} + \binom{u}{3} = \frac{1}{6}[3u(u-1) + u(u-1)(u-2)] = \frac{1}{6}[u(u-1)(u+1)] = \frac{1}{3!}[u(u^2-1)] = \binom{u+1}{3};$$

$$\frac{1}{2}\binom{u}{2} + \frac{3}{2}\binom{u}{3} + \binom{u}{4} = \frac{1}{4!}[12u(u-1) + 6u(u-1)(u-2) + u(u-1)(u-2)(u-3)]$$
$$= \frac{1}{4!}[u^2(u^2-1)];$$
$$\binom{u}{3} + 2\binom{u}{4} + \binom{u}{5} = \frac{1}{5!}[u(u^2-1)(u^2-2^2)] = \binom{u+2}{5}; \text{ etc.}$$

Let us designate the coefficients of m_{2i-1} by S_{2i-1} and those of $\Delta^{2i}y_{-i}$ by S_{2i} for $(i = 0, \cdots, n)$. Then, in general, we have

$$(23.16)\quad \begin{cases} S_0 = 1; \\ S_1 = u; \\ S_{2i-1} = \dfrac{1}{(2i-1)!}\{u(u^2-1^2)(u^2-2^2)\cdots[u^2-(i-1)^2]\} \\ \qquad = \dbinom{u+i-1}{2i-1}; \\ S_{2i} = \dfrac{1}{(2i)!}\{u^2(u^2-1^2)(u^2-2^2)\cdots[u^2-(i-1)^2]\} \\ \qquad = \dfrac{u}{2i}S_{2i-1}. \end{cases}$$

Formula (23.15) then becomes

$$(23.17)\quad I_S(u) = y_0 + S_1m_1 + S_2\Delta^2y_{-1} + S_3m_3 + S_4\Delta^4y_{-2} + S_5m_5 + \cdots + S_{2n-1}m_{2n-1} + S_{2n}\Delta^{2n}y_{-n}$$

where the coefficients S_k are defined by formulas (23.16). This is **Stirling's Interpolation Formula.** It is used with a central differences table in which all the entries on the line through y_0 have been made.

If the values for $\Delta^i y_0$ in terms of the quantities on the line through y_0 [see formulas (12.9) through (12.13)] are substituted into Newton's formula (23.8), the result would be

$$(23.18)\quad I(u) = m_0 - \frac{1}{2}\Delta y_0 + u\Delta y_0 + \binom{u}{2}\left[m_2 + \frac{1}{2}\Delta^3 y_{-1}\right]$$
$$+ \binom{u}{3}\left[\Delta^3 y_{-1} + m_4 + \frac{1}{2}\Delta^5 y_{-2}\right]$$
$$+ \binom{u}{4}\left[m_4 + \frac{3}{2}\Delta^5 y_{-2} + m_6 + \frac{1}{2}\Delta^7 y_{-3}\right]$$
$$+ \cdots .$$

We can again collect the coefficients of the differences, and letting B_{2i} denote the coefficient of m_{2i} and B_{2i+1} denote the coefficient of $\Delta^{2i+1}y_{-i}$ with i assuming the values from 0 to n we find

$$B_0 = 1;$$
$$B_1 = u - \frac{1}{2};$$
$$B_2 = \binom{u}{2};$$
$$B_3 = \frac{1}{2}\binom{u}{2} + \binom{u}{3};$$
$$B_4 = \binom{u}{3} + \binom{u}{4};$$
$$B_5 = \frac{1}{2}\binom{u}{3} + \frac{3}{2}\binom{u}{4} + \binom{u}{5};\ \text{etc.}$$

which upon simplifying become

$$(23.19)\quad \begin{cases} B_0 = 1; \\ B_1 = u - \dfrac{1}{2}; \\ B_{2i} = \dfrac{1}{(2i)!}[u(u-1)(u+1)(u-2)(u+2)\cdots(u-i)(u+i-1)]; \\ B_{2i+1} = \dfrac{1}{(2i+1)!}\left[\left(u - \dfrac{1}{2}\right)(u)(u-1)(u+1)\cdots(u-i)(u+i-1)\right]; \\ \qquad = \dfrac{1}{2i+1}\left(u - \dfrac{1}{2}\right)B_{2i} = \dfrac{1}{2i+1}B_1 B_{2i} \end{cases}$$

and we can write

$$(23.20) \quad I_B(u) = m_0 + B_1\Delta y_0 + B_2 m_2 + B_3\Delta^3 y_{-1} + B_4 m_4 + \cdots + B_{2n}m_{2n} + B_{2n+1}\Delta^{2n+1}y_{-n}$$

where the coefficients B_k are defined by formula (23.19). This is **Bessel's Interpolation Formula.** It is used with a central difference table in which all of the entries on the line through $y_{\frac{1}{2}}$ have been made. The formula can be changed slightly by combining the first two terms. Thus

$$\begin{aligned} m_0 + B_1\Delta y_0 &= \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right)(y_1 - y_0) \\ &= \frac{1}{2}y_0 + \frac{1}{2}y_1 + uy_1 - \frac{1}{2}y_1 - uy_0 + \frac{1}{2}y_0 \\ &= y_0 + u(y_1 - y_0) \\ &= y_0 + u\Delta y_0 \end{aligned}$$

and

$$(23.21) \quad I_B(u) = y_0 + u\Delta y_0 + \sum_{i=1}^{n} B_{2i}m_{2i} + \sum_{i=1}^{n} B_{2i+1}\Delta^{2i+1}y_{-i}.$$

Two other changes are frequently made on this formula. The first is to let $u = \frac{1}{2}$ which results in a formula for *interpolating to halves.* The second is to let $u = v + \frac{1}{2}$ which adds symmetry to the coefficients B_k:

$$(23.22) \quad \begin{cases} B_{2i} = \dfrac{1}{(2i)!}\left\{\left[v^2 - \left(\dfrac{1}{2}\right)^2\right]\left[v^2 - \left(\dfrac{3}{2}\right)^2\right]\cdots\left[v^2 - \left(\dfrac{2i-1}{2}\right)^2\right]\right\} \\ B_{2i+1} = \dfrac{v}{2i+1}B_{2i}. \end{cases}$$

Let us combine two consecutive terms of Bessel's formula (23.20), say

$$B_2 m_2 + B_3\Delta^3 y_{-1} = \frac{1}{2}B_2(\Delta^2 y_{-1} + \Delta^2 y_0) + B_3\Delta^3 y_{-1}.$$

We recall that

$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$

so that upon combining

$$B_2 m_2 + B_3 \Delta^3 y_{-1} = \left(\frac{1}{2} B_2 + B_3\right) \Delta^2 y_0 + \left(\frac{1}{2} B_2 - B_3\right) \Delta^2 y_{-1}.$$

By formula (23.19) we have

$$B_3 = \frac{1}{3}\left(u - \frac{1}{2}\right) B_2$$

so that finally

(23.23) $$B_2 m_2 + B_3 \Delta^3 y_{-1} = \frac{1}{3}(u + 1) B_2 \Delta^2 y_0 + \frac{1}{3}(2 - u) B_2 \Delta^2 y_{-1}.$$

We can thus eliminate all the odd-order differences from Bessel's formula and reduce it to a polynomial in the even-order differences by combining two consecutive terms. The combination of the first two terms yields

$$\begin{aligned} m_0 + B_1 \Delta y_0 &= \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right)(y_1 - y_0) \\ &= \left[\frac{1}{2} - \left(u - \frac{1}{2}\right)\right] y_0 + \frac{1}{2} + u - \frac{1}{2}\, y_1 \\ &= (1 - u) y_0 + u y_1. \end{aligned}$$

In general

$$\begin{aligned} B_{2i} m_{2i} + B_{2i+1} \Delta^{2i+1} y_{-i} &= \frac{1}{2} B_{2i} (\Delta^{2i} y_{-i} + \Delta^{2i} y_{-i+1}) \\ &\qquad + B_{2i+1}(\Delta^{2i} y_{-i+1} - \Delta^{2i} y_{-i}) \\ &= \left(\frac{1}{2} B_{2i} - B_{2i+1}\right) \Delta^{2i} y_{-i} + \left(\frac{1}{2} B_{2i} + B_{2i+1}\right) \Delta^{2i} y_{-i+1} \end{aligned}$$

since

$$B_{2i+1} = \frac{1}{2i + 1}\left(u - \frac{1}{2}\right) B_{2i}$$

we have

$$\begin{cases} \dfrac{1}{2} B_{2i} - B_{2i+1} = \dfrac{i + 1 - u}{2i + 1} B_{2i} \\ \dfrac{1}{2} B_{2i} + B_{2i+1} = \dfrac{u + i}{2i + 1} B_{2i} \end{cases}$$

and

$$(23.24)\quad B_{2i}m_{2i} + B_{2i+1}\Delta^{2i+1}y_{-i} = \frac{i+1-u}{2i+1} B_{2i}\Delta^{2i}y_{-i} + \frac{u+i}{2i+1} B_{2i}\Delta^{2i}y_{-i+1}.$$

By thus changing Bessel's formula we have an interpolating formula in terms of the even-order differences at the beginning and end of an interval; i.e., the even-order differences on the lines through y_0 and y_1 in the central difference table. This formula may be written in the form

$$(23.25)\quad I_E(u) = E_{00}y_0 + E_{10}\Delta^2y_{-1} + E_{20}\Delta^4y_{-2} + E_{30}\Delta^6y_{-3} + \cdots + E_{01}y_1 + E_{11}\Delta^2y_0 + E_{21}\Delta^4y_{-1} + E_{31}\Delta^6y_{-2} + \cdots$$

and is known as **Everett's Central Difference Formula.** The coefficients can be simplified by the introduction of another letter; let $t = 1 - u$. We then have

$$(23.26)\quad \begin{cases} E_{00} = 1 - u = t; \\ E_{10} = \frac{1}{3}(2-u)B_2 = \frac{1}{6}(2-u)u(u-1) \\ \quad = \frac{1}{6}(1+t)(1-t)(-t) \\ \quad = \frac{1}{3!}t(t^2-1^2) = \binom{t+1}{3} \\ E_{20} = \frac{3-u}{5}B_4 \\ \quad = \frac{3-u}{5}\left[\frac{1}{4!}(u)(u-1)(u+1)(u-2)\right] \\ \quad = \frac{1}{5!}[(2+t)(1-t)(-t)(2-t)(-t-1)] \\ \quad = \frac{1}{5!}[t(t-1)(t+1)(t-2)(t+2)] \\ \quad = \frac{1}{5!}[t(t^2-1^2)(t^2-2^2)] = \binom{t+2}{5}; \end{cases}$$

and in general for the coefficients at the beginning of the interval, we have from formulas (23.24) and (23.19)

$$
\begin{aligned}
(23.27)\qquad E_{i0} &= \frac{i+1-u}{2i+1}\,B_{2i}\\
&= \frac{i+1-u}{2i+1}\left(\frac{1}{(2i)!}\right)\\
&\qquad [u(u-1)(u+1)\cdots(u-i)(u+i-1)]\\
&= \frac{1}{(2i+1)!}\,[t(t^2-1^2)(t^2-2^2)\cdots(t^2-i^2)]\\
&= \binom{t+i}{2i+1}.
\end{aligned}
$$

For the coefficients at the end of the interval we have

$$
(23.28)\quad \begin{cases}
E_{01} = u;\\
E_{11} = \frac{1}{3}(u+1)B_2 = \frac{1}{6}(u+1)u(u-1)\\
\qquad = \frac{1}{3!}u(u^2-1^2) = \binom{u+1}{3};\\
E_{21} = \frac{1}{5}(u+2)B_4 = \frac{1}{5!}(u+2)u(u-1)(u+1)(u-2)\\
\qquad = \frac{1}{5!}u(u-1^2)(u-2^2) = \binom{u+2}{5};
\end{cases}
$$

and in general from formulas (23.24) and (23.19)

$$
\begin{aligned}
(23.29)\qquad E_{i1} &= \frac{1}{(2i+1)!}\,[u(u^2-1^2)(u^2-2^2)\cdots(u^2-i^2)]\\
&= \binom{u+i}{2i+1}.
\end{aligned}
$$

It is noted that the form of the function $E_{i1}(u)$ is identical to that of $E_{i0}(t)$.

24. Use of Classical Polynomial Formulas. Each of the classical polynomial formulas may be used in interpolating for values of a function other than those which are tabulated, provided a sufficient number of differences are available to obtain the desired degree of accuracy. There exist, however, special cases for which each is especially efficient, and we shall concentrate on their application to these special cases.

Newton's two formulas [(23.8) and (23.12)] adapt themselves to the cases in which it is desired to find the values of a function near the *beginning* or the *end* of a table. They utilize the diagonal difference table. The interpolating polynomials $I_{N_1}(u)$ and $I_{N_2}(u)$ are evaluated by summing the products of the coefficients and the differences which have been "blocked off" as shown in Table 3.1.

Table 3.1. Use of Newton's Formulas.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0	N_{11}				
		Δy_0	N_{21}			
x_1	y_1		$\Delta^2 y_0$	N_{31}		
				$\Delta^3 y_0$	N_{41}	
x_2	y_2				$\Delta^4 y_0$	N_{51}
						$\Delta^5 y_0$
. . .	. . .	. . .	. . .	. . .	. . .	. . .
						$\Delta^5 y_{n-5}$
x_{n-2}	y_{n-2}				$\Delta^4 y_{n-4}$	N_{52}
				$\Delta^3 y_{n-3}$	N_{42}	
x_{n-1}	y_{n-1}		$\Delta^2 y_{n-2}$	N_{32}		
		Δy_{n-1}	N_{22}			
x_n	y_n	N_{12}				

The coefficients N_{i1} and N_{i2} may be evaluated and entered directly above and below the difference into which each is multiplied as shown in Table 3.1.

Example III.3. Find the values of $f(.53)$ and $f(1.18)$ from the table of values given below.

GIVEN		SOLUTION				
x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
.5	.34375	.3				
		53241	−.105			
.6	.87616		6840	.0595		
		60081		2790	−.040162	
.7	1.47697		9630		600	.029720
		69711		3390		120
.8	2.17408		13020		720	
		82731		4110		120
.9	3.00139		17130		840	
		99861		4950		120
1.0	4.00000		22080		960	−.025536
		121941		5910	−.0336	
1.1	5.21941		27990	−.048		
		149931	−.08			
1.2	6.71872	−.2				

a) For $f(.53)$:

$$N_{11} = u = \frac{x - x_0}{h} = \frac{.53 - .5}{.1} = .3;$$

$$N_{21} = \binom{u}{2} = \frac{1}{2}(.3)(-.7) = -.105;$$

$$N_{31} = \binom{u}{3} = \frac{1}{3} N_{21}(u - 2) = .0595;$$

$$N_{41} = \binom{u}{4} = \frac{1}{4} N_{31}(u - 3) = -.040162;$$

$$N_{51} = \binom{u}{5} = \frac{1}{5} N_{41}(u - 4) = .029720;$$

$$f(.53) = \sum_{i=0}^{5} N_{i1}\Delta^i y_0 = \boxed{.49775}.$$

Note that the differences are numbers to five decimals although the significant digits only are recorded.

b) For $f(1.18)$

$$N_{12} = u = \frac{x - x_n}{h} = \frac{1.18 - 1.20}{.1} = -.2;$$

$$N_{22} = \frac{1}{2}\,u(u + 1) = \frac{1}{2}\,(-.2)(.8) = -.08;$$

$$N_{32} = \frac{1}{3}\,N_{22}(u + 2) = \frac{1}{3}\,(-.08)(1.8) = -.048;$$

$$N_{42} = \frac{1}{4}\,N_{32}(u + 3) = -.0336;$$

$$N_{52} = \frac{1}{5}\,N_{42}(u + 4) = -.025536;$$

$$f(1.18) = \boxed{6.39328}.$$

The coefficients N_{i1} and N_{i2} may be calculated for given values of u once and for all and tabulated. This has been done for N_{i1} with $0 \leq u \leq 2.0$ at intervals of .01 and $1 \leq i \leq 6$. It can be shown that

$$N_{i1}(u) = -N_{i2}(-u), \qquad (i = 1,3,5),$$
$$N_{i1}(u) = N_{i2}(-u), \qquad (i = 2,4,6),$$

so that for $-2 \leq u \leq 0$ for N_{i2} the values for N_{i1} and N_{i2} can be placed in the same table by changing the headings of the columns. These values have been tabulated in Table III, p. 328.

Example III.4. Using Table III find $f(.57)$ and $f(1.11)$ from the data of Example III.3.

Solution.

a) For $f(.57)$; $u = \dfrac{.57 - .5}{.1} = .70.$

From Table III for $u = .70$ we have the coefficients

$$.70,\ -.10500,\ .04550,\ -.026162,\ .017267.$$

Thus $f(.57) = \boxed{.71039}$.

b) For $f(1.11)$: $u = \dfrac{1.11 - 1.20}{.1} = -.9.$

From Table III for $u = -.9$ we have the coefficients

$$-.90, \; -.045, \; -.0165, \; -.008662, \; -.005371.$$

Thus $f(1.11) = \boxed{5.35568}$.

Stirling's and *Bessel's* formulas [(23.17) and (23.20)] employ the central difference table with *Stirling's* formula utilizing the entries on the line through y_0 and *Bessel's* formula utilizing the entries on the line through $y_{\frac{1}{2}}$. For this reason these two formulas should be used whenever it is possible to obtain a sufficient number of higher-order central differences to obtain the desired degree of accuracy. They cannot be used at the beginning nor at the end of a table of values. The coefficients are computed by formulas (23.16) and (23.19).

Example III.5. Find the value of $f(.82)$ from the data of Example III.3.

Solution. Choose $x_0 = .8$. Then $u = \dfrac{x - x_0}{h} = .2$. For Stirling's formula we compute the entries on the line through x_0; and the coefficients

x	y	m_1	$\Delta^2 y$	m_3	$\Delta^4 y$	m_5
.8	2.17408	.76221	.13020	.03750	.00720	.00120
S_i	1	.2	.02	−.032	−.0016	.006336

$$S_1 = u = .2;$$

$$S_2 = \frac{1}{2} u^2 = .02;$$

$$S_3 = \frac{1}{3!}(u)(u^2 - 1) = \frac{1}{6}(.2)(-.96) = -.032;$$

$$S_4 = \frac{1}{4!}(u^2)(u^2 - 1^2) = \frac{u}{4} S_3 = -.0016;$$

$$S_5 = \frac{1}{5!} u(u^2 - 1^2)(u^2 - 2^2) = \frac{1}{20} S_3(u^2 - 4) = .006336;$$

$f(.82)$ = sum of the vertical products of the entries

$= \boxed{2.32792}$.

For Bessel's formula we compute the entries on the line through $x_{\frac{1}{2}}$ and the coefficients

x	m_0	Δy	m_2	$\Delta^3 y$	m_4	$\Delta^5 y$
.85	2.587735	.827310	.150750	.041100	.007800	.001200
B_i	1	$-.3$	$-.08$	.008	.0144	$-.000864$

$$B_1 = u - \frac{1}{2} = .2 - .5 = -.3;$$

$$B_2 = \frac{1}{2}\, u(u-1) = .5(.2)(-.8) = -.08;$$

$$B_3 = \frac{1}{3}\, B_1 B_2 = \frac{1}{3}\,(-.3)(-.08) = .008;$$

$$B_4 = \frac{1}{12}\,(u+1)(u-2)B_2 = \frac{1}{12}\,(1.2)(-1.8)(-.08) = .0144;$$

$$B_5 = \frac{1}{5}\, B_1 B_4 = \frac{1}{5}\,(-.3)(.0144) = -.000864;$$

$$f(.82) = \text{sum of the vertical products of the entries} = \boxed{2.32792}.$$

The coefficients S_k and B_k may be computed and tabulated as was done for the coefficients N_{kj}. They are given in Tables IV and VI, p. 332 and p. 336, respectively.

Everett's formula (23.25) uses only even-order differences at the beginning and at the end of an interval. It is therefore used with tabular values for which these differences have been computed and recorded. Since only the even-order differences are involved, the recording of the values is cut in half, and the trend is to publish tables with the even-order differences recorded.

Example III.6. Work Example III.5 using Everett's formula.

Solution. The beginning of the interval is at $x = .8$ and the end at $x = .9$. The values and even differences are

x	y	$\Delta^2 y$	$\Delta^4 y$
.8	2.17408	13020	720
.9	3.00139	17130	840

The coefficients are now found from $u = .2$ and $t = .8$:

$$E_{00} = t = .8;$$
$$E_{10} = \binom{t+1}{3} = \frac{1}{6}(.8)(.64 - 1) = -.048;$$
$$E_{20} = \binom{t+2}{5} = \frac{1}{20} E_{10}(t^2 - 4) = .008064;$$
$$E_{01} = u = .2;$$
$$E_{11} = \binom{u+1}{3} = \frac{1}{6}(.2)(.04 - 1) = -.032;$$
$$E_{21} = \binom{u+2}{5} = \frac{1}{20} E_{11}(.04 - 4) = .006336.$$

Combining these into one table we have

E_{i0}	.8	−.048	.008064
.8	2.17408	.13020	.00720
.9	3.00139	.17130	.00840
E_{i1}	.2	−.032	.006336

The value of the function is now found by summing the vertical products of this table.

$$f(.82) = (2.17408)(.8) + (.13020)(-.048) + (.00720)(.008064) + (3.00139)(.2) + (.17130)(-.032) + (.00840)(.006336) = \boxed{2.32792}.$$

The coefficients E_{ij} may be computed for given values of u and tabulated. Values for $0 \leq u \leq 1$ are given in Table V, p. 334, at increments of $\Delta u = .01$. Thus the values computed above can be read directly from this table.

When the coefficients are obtained from the tables, they are not written down on the work sheet since they can be entered directly into the calculating machine. It is to be noted that with the use of tables of coefficients all of the formulas of this section are sums of products of two numbers and are therefore easily evaluated on desk calculators. Let us consider a summary example.

Example III.7. Find the value of $f(1.63)$ in the table of values given below using each of the classical polynomial interpolating formulas discussed above.

Solution. Calculate the table of differences.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	
1.3	24.27685						
		10.29755					
1.4	34.57440		3.22180				
		13.51935		.70110			
1.5	48.09375		3.92290		.09480		
		17.44225		.79590		.00600	
1.6	65.53600	19.80165	4.71880	.84630	.10080	.00600	Stirling
	76.616525	22.16105	5.16715	.89670	.10380	.00600	Bessel
1.7	87.69705		5.61550		.10680		
		27.77655		1.00350		.00600	
1.8	115.47360		6.61900		.11280		
		34.39555		1.11630		.00600	
1.9	149.86915		7.73530		.11880		
		42.13085		1.23510			
2.0	192.00000		8.97040				
		51.10125					
2.1	243.10125						

Let $x_0 = 1.6$; then $u = \dfrac{(x - x_0)}{h} = .3$ and $t = .7$. The coefficients are now obtained from the tables and

$$\begin{aligned} f_{N_1}(1.63) &= 71.65004; \\ f_S(1.63) &= 71.65004; \\ f_B(1.63) &= 71.65004; \\ f_E(1.63) &= 71.65004. \end{aligned}$$

The reader should perform the arithmetic and check the calculations.

25. Inherent Errors. In the preceding sections we have derived some polynomial formulas which approximate a given function within an interval. The polynomials were made to coincide with the given function at the points (x_i, y_i), $(i = 0, \cdots, n)$, and by

Weierstrass' Theorem we can make the approximation as close as we desire by increasing the number of points indefinitely. This would, of course, change the polynomials to infinite series, and for this reason they are frequently called *interpolation series*. In application we employ only a finite number of terms and thus, as in the case of all such series, there is a remainder term. This remainder term is a good estimation of the inherent error committed by the interpolating polynomial due to the use of only a finite number of terms.

The remainder terms for the classical interpolating polynomials may be derived in the same manner as those for the well-known series as taught in a study of the calculus. The general procedure may be summarized as follows. Let $f(x)$ denote the given function and $I(x)$ a polynomial interpolation formula of degree n, in the interval (x_0,x_n). Then form the arbitrary function

$$F(z) = f(z) - I(z) - [f(x) - I(x)]\frac{(z - x_0)(z - x_1) \cdots (z - x_n)}{(x - x_0)(x - x_1) \cdots (x - x_n)}.$$

This function satisfies the conditions of Rolle's Theorem to the extent that the $(n + 1)$th derivative of it vanishes at some point $z = \xi$ in the interval (x_0,x_n); thus

$$F^{(n+1)}(z) = f^{(n+1)}(z) - [f(x) - I(x)]\frac{(n + 1)!}{(x - x_0)(x - x_1) \cdots (x - x_n)}$$
$$= 0 \text{ at } z = \xi$$

from which we solve for

$$f(x) - I(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$
$$= R_n = \text{error}.$$

This expresses the remainder in terms of the $(n + 1)$th derivative which may not be available in tabulated function; consequently, it is replaced by its equivalent in terms of differences.

$$f^{(n+1)}(\xi) = \frac{\Delta^{n+1}f(x_0)}{(\Delta x)^{n+1}}$$

[*see* formula (19.5)]. The remainder can then be written in the form

(25.1) $$R_n = \frac{\Delta^{n+1}y_0}{\Delta^{n+1}x}\frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n + 1)!}.$$

For Newton's *forward formula* we have

$$x - x_i = h(u - i), \qquad (i = 0, \cdots, n),$$

so that

$$R_n = \frac{\Delta^{n+1}y_0}{(n+1)!}u(u-1)\cdots(u-n)$$

$$= \Delta^{n+1}y_0\binom{u}{n+1}.$$

The choice of the form of

$$\frac{(z-x_0)(z-x_1)\cdots(z-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)}$$

in the arbitrary function $F(z)$ is arbitrary and can be changed to suit the need as desired. By proper choice of this fraction and observing the maximum derivative which we can make to vanish, we are led to the remainder terms for each of our functions. In summary form we may list them:

1. For Newton's formula (23.8)

(25.2) $$R_n = \Delta^{n+1}y_0\binom{u}{n+1}.$$

2. For Newton's formula (23.12)

(25.3) $$R_n = \Delta^{n+1}y_{-1}\binom{u+n}{n+1}.$$

3. For Stirling's formula (23.17)

(25.4) $$R_n = m_{2n+1}\binom{u+n}{2n+1}.$$

4. For Bessel's formula (23.20)

(25.5) $$R_n = m_{2n+2}\binom{u+n}{2n+2}.$$

5. For Everett's formula (23.25)

(25.6) $$R_n = \Delta^{2n+2}y_{-n-1}\binom{t+n+1}{2n+3} + \Delta^{2n+2}y_{-n}\binom{u+n+1}{2n+3}.$$

To evaluate any remainder term it is necessary to be able to evaluate the appropriate higher-order difference.

Example III.8. Find the error committed by Stirling's formula if we stop with fourth differences in Example III.5.

Solution.

$$R_n = m_5 \binom{u+2}{5} = (.00120)(.006336)$$

$$= \boxed{.00000076}.$$

26. Exercise III.

1. Find the values of $f(153)$, $f(177)$, $f(181)$, $f(189)$ by linear interpolation in the table

x	150	160	170	180	190
$f(x)$	17609	20412	23045	25527	27875

2. Find the values of $f(153)$ and $f(189)$ by use of the appropriate Newton's formula in the table of Problem 1.
3. The following tabulated functions are given:

(a)

x	y
0	1.00000
.1	.95135
.2	.91817
.3	.89747
.4	.88726
.5	.88623
.6	.89352
.7	.90864
.8	.93138
.9	.96177
1.0	1.00000

(b)

x	y
0	2.000000
.2	1.042637
.4	.132454
.6	−.893670
.8	−2.260301
1.0	−4.000000
1.2	−5.460275
1.4	−4.506394
1.6	3.644442
1.8	28.022067
2.0	84.000000

Find y at x = .02, .93, .55, .72, and .13 in (a).
Find y at x = .1, 1.9, .82, 1.15, and 1.32 in (b).

4. Find the inherent errors wherever possible in Problem 3.
5. By the use of a five-place table of tangents and cotangents find tan A if cot A = 1.0902 without finding the angle A. Check by finding the angle A first and then tan A.

27. Interpolation Using Divided Differences. The classical polynomial interpolating formulas derived thus far have all been limited to the case in which the intervals of the independent variable were equally spaced. To overcome this handicap we shall now consider an interpolating polynomial which employs divided differences (see Section 17) and for which the spacing on the independent variable may be arbitrary. As was done in the past let us again consider $y = f(x)$ whose $n + 1$ values, y_i, $(i = 0, \cdots, n)$, are known at x_i, $(i = 0, \cdots, n)$. The problem of interpolation is to obtain a value of y for a given value of x which is not one of those tabulated. We again seek a formula which is a polynomial to accomplish this. Let us form the divided difference in the interval x to x_0 which from the definition (17.1) is

$$\Delta[xx_0] = \frac{y - y_0}{x - x_0}, \tag{27.1}$$

and upon solving for y we obtain

$$y = y_0 + \Delta[xx_0](x - x_0). \tag{27.2}$$

Now

$$\Delta^2[xx_1] = \frac{\Delta[xx_0] - \Delta[x_0x_1]}{x - x_1}$$

from which we get

$$\Delta[xx_0] = \Delta[x_0x_1] + \Delta^2[xx_1](x - x_1). \tag{27.3}$$

Elimination of $\Delta[xx_0]$ between (27.2) and (27.3) yields

$$y = y_0 + \Delta[x_0x_1](x - x_0) + \Delta^2[xx_1](x - x_0)(x - x_1). \tag{27.4}$$

We can now define an expression for $\Delta^3[xx_2]$ by

$$\Delta^3[xx_2] = \frac{\Delta^2[xx_1] - \Delta^2[x_0x_2]}{x - x_2}$$

and solve this expression for $\Delta^2[xx_1]$ to substitute into (27.4). The result would be

$$\begin{aligned} y = y_0 + \Delta[x_0x_1](x - x_0) &+ \Delta^2[x_0x_2](x - x_0)(x - x_1) \\ &+ \Delta^3[xx_2](x - x_0)(x - x_1)(x - x_2). \end{aligned} \tag{27.5}$$

A continuation of this procedure leads to a general formula

$$\begin{aligned} y(x) = y_0 &+ \Delta[x_0x_1](x - x_0) + \Delta^2[x_0x_2](x - x_0)(x - x_1) \\ &+ \Delta^3[x_0x_3](x - x_0)(x - x_1)(x - x_2) + \cdots \\ &+ \Delta^n[x_0x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}) \\ &+ \Delta^{n+1}[xx_n](x - x_0)(x - x_1) \cdots (x - x_n). \end{aligned} \tag{27.6}$$

If we now let

$$(27.7)\quad I(x) = y_0 + \Delta[x_0x_1](x - x_0) + \Delta^2[x_0x_2](x - x_0)(x - x_1) + \Delta^3[x_0x_3](x - x_0)(x - x_1)(x - x_2) + \cdots + \Delta^n[x_0x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

and

$$(27.8)\quad R_n(x) = \Delta^{n+1}[xx_n](x - x_0)(x - x_1)\cdots(x - x_n)$$

formula (27.6) becomes

$$y(x) = I(x) + R_n(x).$$

Formula (27.7) for $I(x)$ exhibits a polynomial of degree not more than n, and if we let $x = x_i$, $(i = 0, \cdots, n)$, we obtain $R_n(x_i) = 0$ and

$$y(x_i) = I(x_i) = y_i, \qquad (i = 0, \cdots, n),$$

so that $I(x)$ satisfies the criteria for an interpolating polynomial and $R_n(x)$ becomes the remainder term or the inherent error. Thus $I(x)$ as given by (27.7) is the interpolating polynomial in terms of divided differences.

Example III.9. Find $y(.3)$ from the table of values (x_i, y_i) given below.

Solution. Form the table of divided differences.

x	y	$\Delta[x_0x_i]$	$\Delta^2[x_ix_k]$	$\Delta^3[x_ix_k]$	$\Delta^4[x_ix_k]$
0	−6.0000	.3			
		4.648	.03		
.2	−5.0704		−1.120	−.003	
		4.200		2.400	.0015
.4	−4.2304		.800		1.000
		4.680		3.400	
.8	−2.3584		3.520		
		6.792			
1.0	−1.0000				

Formula (27.7) is given in terms of the divided differences on the initial diagonal which have been "blocked off." Now compute the product

$$\prod_{i=0}^{k} (x - x_i) = (x - x_0)(x - x_1) \cdots (x - x_k), \qquad (k = 0, 1, 2, 3),$$

and enter them directly above the difference by which it is multiplied in formula (27.7). Notice that the entries are easy to compute since each succeeding one is obtained by multiplying the previous one by $(x - x_i)$. Now calculate the sum of the products of the number in the blocks by the number directly above it.

$$y(.3) = -6.0000 + (4.648)(.3) + (-1.120)(.03) + (2.400)(-.003) + (1.000)(.0015)$$
$$= \boxed{-4.6449}.$$

If it is desired to obtain more than one value of the function, it is convenient to list the divided differences in a column and then to list the values of $\Pi(x - x_i)$ in adjacent columns.

Example III.10. Find y for the following values of x: .3, .35, .5, .6, .7 in Example III.9.

Solution. Calculate the table

k	$\Delta^k[x_0 x_i]$	$\prod_{i=0}^{k-1}(x - x_i)$				
		$x = .3$	$x = .35$	$x = .5$	$x = .6$	$x = .7$
0	−6.000	1	1	1	1	1
1	4.648	.3	.35	.5	.6	.7
2	−1.120	.03	.0525	.15	.24	.35
3	2.400	−.003	−.002625	.015	.048	.105
4	1.000	.0015	.001181	−.0045	−.0096	−.0105
$y(x) =$		−4.6449	−4.4371	−3.8125	−3.3744	−2.8969

The values of $y(x)$ are found by calculating the sum of the products of the number in the second column times the number in the same row in the appropriate succeeding column.

The interpolating polynomial (27.7) employs the divided differences on the top diagonal line in the divided difference table. It is, of course, possible to use some other sequence of divided differences; in fact, for $n + 1$ points there are 2^n different formulas. These formulas may be obtained by following a set of rules which are known as **Sheppard's Rules.** For a given set of data calculate the divided difference table; then

1. Start with any tabulated value of y_i.

2. The next term will be $\Delta[x_ix_j](x - x_i)$, where $j = i + 1$ or $j = i - 1$. Thus in the table we can move either upward or downward.

3. Move to successive divided differences by steps, each of which may be either upward or downward.

4. Each divided difference is multiplied by the product of factors of the form $(x - x_k)$ such that the product contains all the values x_k that were involved in the preceding difference.

It is thus possible to follow any "zigzag" path through the table of differences. If we introduce the notation

$$x - x_k = X_k, \tag{27.9}$$

the formulas are more easily written down and at the same time we can keep track of the coefficients of the difference. As an illustration let us follow the path shown in the following table.

x	y	$\Delta[x_ix_j]$	$\Delta^2[x_ix_k]$	$\Delta^3[x_ix_k]$	$\Delta^4[x_ix_k]$	$\Delta^5[x_ix_k]$
x_0	y_0					
		$\Delta[x_0x_1]$				
x_1	y_1		$\Delta^2[x_0x_2]$			
		$\Delta[x_1x_2]$		$\Delta^3[x_0x_3]$		
x_2	y_2		$\Delta^2[x_1x_3]$		$\Delta^4[x_0x_4]$	
		$\Delta[x_2x_3]$		$\Delta^3[x_1x_4]$		$\Delta^5[x_0x_5]$
x_3	y_3		$\Delta^2[x_2x_4]$		$\Delta^4[x_1x_5]$	
		$\Delta[x_3x_4]$		$\Delta^3[x_2x_5]$		
x_4	y_4		$\Delta^2[x_3x_5]$			
		$\Delta[x_4x_5]$				
x_5	y_5					

Along this path we involve the subscripts on x_i in this order:

$$2, 1, 3, 4, 0, 5.$$

Consequently, the factor of the coefficients will enter in this order:

$$1, X_2, X_1, X_3, X_4, X_0$$

and the formula is

$$(27.10) \quad I(x) = y_2 + X_2\Delta[x_1x_2] + X_2X_1\Delta^2[x_1x_3] + X_2X_1X_3\Delta^3[x_1x_4] + X_2X_1X_3X_4\Delta^4[x_0x_4] + X_2X_1X_3X_4X_0\Delta^5[x_0x_5].$$

28. Aitken's Repeated Process. Formula (22.2) presents a method for linear interpolation which is convenient for machine calculations. As written, the formula interpolates in the interval from x_0 to x_1. It could also be written in a form to interpolate in the interval from x_0 to x_2; thus

$$(28.1) \qquad I(x) = \frac{1}{x_2 - x_0}\begin{vmatrix} y_0 & x_0 - x \\ y_2 & x_2 - x \end{vmatrix}.$$

Some distinction in the notation $I(x)$ must be made in the expressions (22.2) and (28.1). To accomplish this in a logical manner let us order the values for the independent variable x in the sequence $x_0, x_1, x_2, \cdots, x_n$. Thus if any entries in the original tabulation are omitted, those used are reordered so that we always have a sequence of consecutive integers in the subscripts. Furthermore, we shall always consider the beginning of the interval for interpolation to be at x_0. Then we may write

$$y_{11}(x) = I(x) \text{ in the interval } x_0 \text{ to } x_1,$$
$$y_{21}(x) = I(x) \text{ in the interval } x_0 \text{ to } x_2,$$

or in general

$$(28.2) \qquad y_{i1}(x) = I(x) = \frac{1}{x_i - x_0}\begin{vmatrix} y_0 & x_0 - x \\ y_i & x_i - x \end{vmatrix}$$

denotes linear interpolation using the values (x_0, y_0) *and* (x_i, y_i). The function $y_{i1}(x)$ is a linear function of x. At $x = x_i$ we have

(28.3) $$y_{i1}(x_i) = \frac{1}{x_i - x_0}[y_0(0) - y_i(x_0 - x_i)] = y_i$$

and in particular at $x = x_0$, we have

$$y_{i1}(x_0) = \frac{1}{x_i - x_0}[y_0(x_i - x_0) - 0] = y_0.$$

By changing the interval we can thus build up a set of values $y_{i1}(x)$, $(i = 1, 2, \cdots, n)$. If linear interpolation is exact, these values would all be alike. On the other hand if the function $y = f(x)$ is not linear, these values would all differ by some amount. Let us list these values together with the differences $x_i - x$ in the following manner:

$$\begin{array}{lll} x_0 & & \\ x_1 & y_{11}(x) & x_1 - x \\ x_2 & y_{21}(x) & x_2 - x \\ x_3 & y_{31}(x) & x_3 - x \\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ x_i & y_{i1}(x) & x_i - x. \end{array}$$

We could then apply the linear interpolation formula to these entries. Thus

(28.4) $$I(x) = \frac{1}{x_2 - x_1}\begin{vmatrix} y_{11}(x) & x_1 - x \\ y_{21}(x) & x_2 - x \end{vmatrix} = y_{22}(x).$$

If the determinant is expanded we would obtain

$$y_{11}(x)(x_2 - x) - y_{21}(x)(x_1 - x)$$

which is a second degree polynomial in x since $y_i(x)$ is linear in x. Furthermore, at $x = x_0$, $x = x_1$, and $x = x_2$ we have by formula (28.3)

$$y_{22}(x_0) = \frac{1}{x_2 - x_1}\begin{vmatrix} y_0 & x_1 - x_0 \\ y_0 & x_2 - x_0 \end{vmatrix} = y_0\left[\frac{x_2 - x_0 - x_1 + x_0}{x_2 - x_1}\right] = y_0;$$

$$y_{22}(x_1) = \frac{1}{x_2 - x_1}\begin{vmatrix} y_1 & x_1 - x_1 \\ y_1 & x_2 - x_1 \end{vmatrix} = y_1\left[\frac{x_2 - x_1}{x_2 - x_1}\right] = y_1;$$

$$y_{22}(x_2) = \frac{1}{x_2 - x_1}\begin{vmatrix} y_2 & x_1 - x_2 \\ y_2 & x_2 - x_2 \end{vmatrix} = y_2\left[-\frac{x_1 - x_2}{x_2 - x_1}\right] = y_2.$$

Thus $y_{22}(x)$ satisfies all the criteria for a second degree interpolating polynomial. The process may be applied to the interval from x_1 to x_i so that in general we may write

$$\text{(28.5)} \qquad y_{i2}(x) = \frac{1}{x_i - x_1}\begin{vmatrix} y_{11}(x) & x_1 - x \\ y_{i1}(x) & x_i - x \end{vmatrix}.$$

We may now compute a set of values, $y_{i2}(x)$, $(i = 2, \cdots, n)$, and form the table

x_2	$y_{22}(x)$	$x_2 - x$
x_3	$y_{32}(x)$	$x_3 - x$
. . .	. . .	. . .
x_i	$y_{i2}(x)$	$x_i - x.$

Again applying the linear interpolation formula we could obtain

$$\text{(28.6)} \qquad y_{i3}(x) = \frac{1}{x_i - x_2}\begin{vmatrix} y_{22}(x) & x_2 - x \\ y_{i2}(x) & x_i - x \end{vmatrix}$$

which yields a *third degree interpolating polynomial.* The process can be repeated until all of the entries of the original table of values have been consumed. It is easily seen that the general formula is given by

$$\text{(28.7)} \qquad y_{ik}(x) = \frac{1}{x_i - x_{k-1}}\begin{vmatrix} y_{k-1,k-1}(x) & x_{k-1} - x \\ y_{i,k-1}(x) & x_i - x \end{vmatrix}$$

in which k denotes the number of times linear interpolation has been applied and also the degree of the polynomial. The subscript i assumes the values $k, k + 1, \cdots, n$. The process is known as Aitken's method or, more appropriately, as *Aitken's Repeated Process.* It is a very useful process since the calculations are easily performed on a calculating machine, and furthermore, it provides its own criterion of when the process has been carried far enough. The work should be arranged as shown below.

x_0	y_0				$x_0 - x$
x_1	y_1	$y_{11}(x)$			$x_1 - x$
x_2	y_2	$y_{21}(x)$	$y_{22}(x)$		$x_2 - x$
x_3	y_3	$y_{31}(x)$	$y_{32}(x)$	$y_{33}(x)$	$x_3 - x$
x_4	y_4	$y_{41}(x)$	$y_{42}(x)$	$y_{43}(x)$	$x_4 - x$

Computational Form for Aitken's Process.

In accordance with our usual procedure the calculations should be carried to one more decimal place to prevent rounding errors.

Example III.11. Find $f(.53)$ in Example III.3 by Aitken's Method.

Solution. Compute the following table of values:

x_i	y_i	$y_{i1}(x)$	$y_{i2}(x)$	$y_{i3}(x)$	$y_{i4}(x)$	$y_{i5}(x)$	$x_i - x$
.5	.34375						−3
.6	.87616	.503473					7
.7	1.47697	.513733	.496291				17
.8	2.17408	.526783	.495314	.497952			27
.9	3.00139	.543073	.494233	.498040	.497714		37
1.0	4.00000	.563125	.493034	.498137	.497702	.497758	47
1.1	5.21941	.587533	.491705	.498240	.497693	.497753	57

Thus to five figures we write $f(.53) = \boxed{.49775}$. In performing the calculation a common factor is removed from $x_i - x$ and the divisor $x_i - x_{k-1}$ which eliminates the decimal point in these terms. Furthermore, any digits which become identical in the numbers in a column need not be used in the calculation of the next column. Thus the digits "49" in column $y_{i2}(x)$ need not be used in calculating column $y_{i3}(x)$; the results obtained from such calculation are then "attached to" .49 or simply recorded with .49 understood; the latter, however, is not recommended.

Since at the start of the problem it is not known how many $y_{ik}(x)$ need be computed, the placing of the column $x_i - x$ on a work sheet becomes a problem. Consequently, it is often preferred to place this column immediately after the column "x." This may be done without changing the method of calculation *provided* $x - x_i$ is recorded in place of $x_i - x$, since it is well known from the theory of determinants that an interchange of two adjacent columns in the determinant changes the sign of the determinant.

Example III.12. Find $f(.5)$ if the values of the function $y = f(x)$ are those given in the table below.

Solution.

x_i	$x - x_i$	y_i	$y_{i1}(x)$	$y_{i2}(x)$	$y_{i3}(x)$	$y_{i4}(x)$
0	5	−6.0000				
.2	3	−5.0704	−3.676			
.4	1	−4.2304	−3.788	−3.844		
.8	−3	−2.3584	−3.724	−3.700	−3.808	
1.0	−5	−1.0000	−3.500	−3.610	−3.805	−3.8125
1.4	−9	3.6656	−2.548	−3.394	−3.799	−3.8125

Thus $f(.5) = \boxed{-3.8125}$.

29. Inverse Interpolation. This subject matter deals with the *process* of finding the value of the *argument* corresponding to a given value of the function which is between two tabulated values. The reader was probably introduced to this process in a course in trigonometry at the time it became necessary to find an angle from its trigonometric function.

An easy method of doing inverse interpolation is *Aitken's Repeated Process* applied to the data after interchanging the roles of the dependent and independent variables.

Example III.13. From a table of cos A in radians find A if cos A = .671178.

Solution.

$y = \cos A$	$y_i - y_0$	$y_i - y_1$	$y - y_i$	A_i	$A_{i1}(y)$	$A_{i2}(y)$
.674876			−3698	.83		
.667463	−7413		3715	.84	834988	
.659983	−14893	−7480	11195	.85	66	834999
.652437	−22439	−15026	18741	.86	44	99

Thus $A = \boxed{.834999}$.

The denominators, $y_i - y_k$, by which the cross product must be divided, are fairly large numbers in inverse interpolation. The usual scheme for Aitken's process is therefore augmented by columns of these divisors, and they are usually placed at the beginning of the scheme as shown in Example III.13.

This method represents the inverse function $x = g(y)$ by means of an interpolating polynomial and therefore can be used only when this is sufficiently accurate.

A second method is one of *iteration* using the classical polynomial formulas. Let us consider Newton's formula (23.8) which can be written in the form

$$y = y_0 + u\Delta y_0 + \sum_{k=2}^{n} \binom{u}{k} \Delta^k y_0 \tag{29.1}$$

and upon solving for u we obtain

$$u = \frac{y - y_0}{\Delta y_0} - \frac{1}{\Delta y_0} \sum_{k=2}^{n} \binom{u}{k} \Delta^k y_0. \tag{29.2}$$

This is an iteration formula which expresses u explicitly in terms of a function of u. We may obtain a first approximation by letting

$$u_1 = \frac{y - y_0}{\Delta y_0}, \tag{29.3}$$

and introducing

$$g(u) = \frac{1}{\Delta y_0} \sum_{k=2}^{n} \binom{u}{k} \Delta^k y_0$$

we can obtain a second approximation to u by substituting u_1 into $g(u)$ and obtaining

$$u_2 = \frac{y - y_0}{\Delta y_0} - g(u_1).$$

A third approximation is obtained by substituting u_2 into $g(u)$ in formula (29.2). The process may be repeated until there is no change in u_i.

Since all of the classical polynomial interpolating formulas may be solved for u, we can obtain an iteration formula from each of them. In the case of Stirling's formula (23.17) we have

$$y = y_0 + um_1 + \sum_{i=1}^{n} S_{2i}\Delta^{2i}y_{-i} + \sum_{i=2}^{n} S_{2i-1}m_{2i-1}$$

or

$$u = \frac{1}{m_1}[(y - y_0) - g_1(u) - g_2(u)]. \tag{29.4}$$

Example III.14. What is the value of x for which $y = -3.7777$ in the data given below?

Solution. Calculate the central difference table for the data.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	−6.0000				
		9296			
.2	−5.0704		−896		
		8400		1056	
.4	−4.2304	8480	160	1248	384
		8560		1440	
.6	−3.3744		1600		384
		10160		1824	
.8	−2.3584		3424		
		13584			
1.0	−1.0000				

Let $x_0 = .4$ and calculate m_1 and m_3 as shown in the table. We shall stop with the fourth difference so that formula (29.4) reduces to

$$u = \frac{1}{m_1}(y - y_0 - S_2\Delta^2 y_{-1} - S_3 m_3 - S_4\Delta^4 y_{-2})$$

with [*see* formula (23.16)]

$$S_2 = \frac{1}{2}u^2, \qquad S_3 = \frac{1}{6}u(u^2 - 1), \qquad S_4 = \frac{1}{4}uS_3.$$

The calculations in the iteration may be arranged in the following schematic:

	$y - y_0$	$\Delta^2 y_{-1}$	m_3	$\Delta^4 y_{-2}$	m_1
u_1 u_2		$S_2(u_1)$ $S_2(u_2)$	$S_3(u_1)$ $S_3(u_2)$	$S_4(u_1)$ $S_4(u_2)$	
			etc.		

The problem is then completed by

	.4527	.0160	.1248	.0384	.8480
.533844		.142195	−.063617	−.008490	
.540902		.146287	−.063774	−.008623	
.540860		.146264	−.063774	−.008623	
.540860					

from which we get $u = .540860$ and

$$x = x_0 + hu = .4 + (.2)(.540860) = \boxed{.508172}.$$

30. Multiple Interpolation. In the previous discussion we have applied interpolation to a function of a single variable. In the cases where we have functions of two or more variables, the interpolating formulas become very complex, and the recommended procedure is to hold all variables constant except one, interpolate with respect to that variable for a set of values of the remaining ones, and repeat the procedure. It is better understood by referring to some examples.

Multiple interpolation is the process of finding a value of a function of more than one variable for given values of the independent variables which fall between those tabulated.

Example III.15. Given the table of values for $t = f(E,A,r)$.

$E = 20°$

A \ r	4	8	12	16	20
0	.47	1.00	1.61	2.33	3.22
20	.47	1.00	1.61	2.33	3.20
40	.47	1.01	1.62	2.32	3.16
60	.48	1.02	1.62	2.30	3.08
80	.43	1.02	1.60	2.26	2.98
			$E = 0$		
0	.46	.99	1.60	2.31	3.18
20	.46	.99	1.60	2.31	3.17
40	.47	1.00	1.61	2.31	3.13
60	.48	1.01	1.61	2.28	3.05
80	.43	1.02	1.60	2.24	2.95

Find t at $E = 10°$, $A = 50°$, $r = 9.5$.

Solution. Since there are only two entries with respect to E, we are limited to linear interpolation with respect to this variable. The variation with respect to A is small so that linear interpolation is sufficient here. Let us hold E and r "constant" and interpolate linearly with respect to A to obtain

$A = 50°$

E \ r	4	8	12	16	20
20°	.475	1.015	1.62	2.31	3.12
0°	.475	1.005	1.61	2.295	3.09

Now interpolating linearly with respect to E, we have

$A = 50°, E = 10°$

r	4	8	12	16	20
t	.475	1.010	1.615	2.3025	3.105

Let us apply Aitken's process to these values:

r_i	$r - r_i$	t_i	t_{i1}	t_{i2}	t_{i3}
4	5.5	.475			
8	1.5	1.010	1.210625		
12	−2.5	1.615	1.258750	1.228671	
16	−6.5	2.3025	1.312604	1.229746	1.227999
20	−10.5	3.105	1.379062	1.231679	1.227731

Thus $t(10°, 50°, 9.5) = \boxed{1.228}$.

For additional discussion of multiple interpolation see Chapter V.

31. Trigonometric Interpolation. If the data for the given function, $y = f(x)$, are such as to indicate that the function may be periodic, then the interpolating formula should take this into account and trigonometric interpolation should be used. The classical formula for interpolating periodic functions is known as *Hermite's formula*.

$$(31.1) \qquad I_n(x) = H_0(x)y_0 + H_1(x)y_1 + \cdots + H_n(x)y_n$$

where

$$(31.2) \qquad H_i(x) = \frac{\prod_{k=0}^{n} \sin (x - x_k)_{k \neq i}}{\prod_{k=0}^{n} \sin (x_i - x_k)_{k \neq i}}, \qquad (i = 0, \cdots, n),$$

i.e.,

$$\prod_{k=0}^{n} \sin (x - x_k)_{k \neq i} = \sin (x - x_0) \cdots$$
$$\sin (x - x_{i-1}) \sin (x - x_{i+1}) \cdots \sin (x - x_n).$$

The function $I_H(x)$ has the period 2π since

$$\sin (x + 2\pi - x_k) = \sin (2\pi + x - x_k) = \sin (x - x_k).$$

Furthermore, at $x = x_0$ we have

$$H_0(x) = 1 \qquad \text{and} \qquad H_i(x) = 0 \qquad \text{for} \qquad i = 1, \cdots, n$$

so that $I_H(x_0) = y_0$. In general at $x = x_i$ we have $I_H(x_i) = y_i$ so that formula (31.1) passes through the $n + 1$ points (x_i,y_i) which is the criterion for an interpolating function.

To apply formula (31.1) compute the square array

(31.3)

$$\begin{vmatrix} \sin (x - x_0) & \sin (x_0 - x_1) & \sin (x_0 - x_2) & \cdots & \sin (x_0 - x_n) \\ \sin (x_1 - x_0) & \sin (x - x_1) & \sin (x_1 - x_2) & \cdots & \sin (x_1 - x_n) \\ \sin (x_2 - x_0) & \sin (x_2 - x_1) & \sin (x - x_2) & \cdots & \sin (x_2 - x_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sin (x_n - x_0) & \sin (x_n - x_1) & \sin (x_n - x_2) & \cdots & \sin (x - x_n) \end{vmatrix}$$

The product of the elements of each row are the denominators of formula (31.2) times $\sin (x - x_i)$. The product of the elements of the main diagonal is the numerator of formula (31.2) times $\sin (x - x_i)$. Thus

$$(31.4) \qquad H_i(x) = \frac{\Pi[\text{Elements of the main diagonal of (31.3)}]}{\Pi[\text{Elements of the } i\text{th row of (31.3)}]} = \frac{N(x)}{D_i}.$$

Now formula (31.1) can be written in the form

$$(31.5) \qquad I_H(x) = \frac{y_0}{D_0} N(x) + \frac{y_1}{D_1} N(x) + \cdots + \frac{y_n}{D_n} N(x)$$
$$= [y_0 D_0^{-1} + y_1 D_1^{-1} + \cdots y_n D_n^{-1}] N(x).$$

Let us illustrate the procedure of calculation.

Example III.16. Given the following table of values of x and y

x	.2	.3	.4	.5	.6
y	.9801	.9553	.9211	.8776	.8253

Assuming the function to be periodic, find the value of y at $x = .37$.

Solution. Calculate the square array (31.3) with

$x - x_0 = .17$; $\quad x_1 - x_0 = .1 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$;
$x - x_1 = .07$; $\quad x_2 - x_0 = .2 = x_3 - x_1 = x_4 - x_2$;
$x - x_2 = -.03$; $\quad x_3 - x_0 = .3 = x_4 - x_1$;
$x - x_3 = -.13$; $\quad x_4 - x_0 = .4$.
$x - x_4 = -.23$;

					$D_i \times 10^3$	$y_i D_i^{-1} \times 10^{-3}$
.16918	−.09983	−.19867	−.29552	−.38942	.386137	2.538218
.09983	.06994	−.09983	−.19867	−.29552	−.040923	−23.343840
.19867	.09983	−.03000	−.09983	−.19867	−.011801	−78.052707
.29552	.19867	.09983	−.12963	−.09983	.075848	11.570509
.38942	.29552	.19867	.09983	−.22798	−.520350	−1.586047
					$N(x) = -.010491 \times 10^{-3}$	-88.873867×10^{3}

$$y(.37) = N(x) \sum_{i=0}^{4} y_i D_i^{-1} = \boxed{.9324}.$$

There are many ways in which the trigonometric interpolating formula may be written. Thus **Gauss's** formula is identical to formula (31.1) with the angles $(x - x_i)$ and $(x_i - x_k)$ replaced by half of the angles. Other variations can, of course, be made; however, it is believed that they offer little or no advantage over formula (31.1) for trigonometric interpolation.

32. Lagrange's Formula. There is one more classical interpolating polynomial which is known as *Lagrange's Formula*. This formula leads to so many applications that an entire chapter has been devoted to it. The complete discussion is given in Chapter V.

33. Exercise IV.

1. Given the tables of values

(a)

x	y
.2	−5.0704
.6	−3.3744
.8	−2.3584
1.0	−1.0000
1.4	3.6656
1.6	7.5296
2.0	20.0000

(b)

x	y
1.5	48.09375
1.6	65.53600
1.7	87.69705
1.8	115.47360
1.9	149.86915
2.0	192.00000
2.1	243.10125

Find y at $x = .3, .7, .95$ in table (a) by the method of divided differences.
Find y at $x = 1.63, 1.91$ in table (b) by Aitken's process.

2. Find x for which $y = 100.00000$ in table (b).
3. Find x for which $y = 1$ in table (a) using Aitken's method.
4. The function $y = f(x)$ given in the following table is periodic.

x	0	.1	.2	.3	.4	.5	.6	.7	.8
y	.141	.158	.176	.194	.213	.231	.249	.268	.287

Find y if $x = .15, .52$.

CHAPTER IV. DIFFERENTIATION AND INTEGRATION

34. Introduction. In this chapter we shall be concerned with numerical differentiation and integration. We shall derive formulas for the evaluation of the derivative of a function even though that function may be defined only by a table of values at discrete points. We shall also derive formulas to find the value of a definite integral of such functions. For the present we shall limit our discussion to functions whose values are given for equidistant values of the independent variable. The more general case will be considered in Chapter V in which this special case is also given a different treatment.

A table of values does not completely define a function and much less determine its differentiability. In fact the function may not be differentiable anywhere. Furthermore, differentiation exaggerates any rounding errors or other irregularities. Thus, it may be concluded that numerical differentiation is a process which should be viewed with much concern. On the other hand, if the data for the tabulated values are obtained from a physical system, it is reasonable to assume that the rate of change should be well defined and a differentiation may point to errors in the functional values.

Whereas differentiation exaggerates errors in a table of functional values, the process of integration to the contrary tends to smooth the errors. This is understandable since integration is a process of summation of many values, and errors in individual entries will be "averaged out."

In either process we approximate the function by an interpolation formula and derive from it a method for performing the process desired.

35. First Derivatives of Classical Interpolation Formulas. To obtain the values of the first derivative we first derive a formula by differentiating any of the classical interpolation formulas. We recall from differential calculus that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Now $uh = x - x_0$ so that

$$\frac{du}{dx} = \frac{1}{h}$$

and

$$\frac{dy}{dx} = \frac{1}{h}\frac{dy}{du}. \tag{35.1}$$

Let us consider Newton's formula (23.8)

$$y = y_0 + u\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \cdots \binom{u}{i}\Delta^i y_0 + \cdots \binom{u}{n}\Delta^n y_0$$

and differentiate it with respect to x;

$$\frac{dy}{dx} = \frac{1}{h}[\Delta y_0 + D_2(u)\Delta^2 y_0 + \cdots + D_k(u)\Delta^k y_0 + \cdots + D_n(u)\Delta^n y_0] \tag{35.2}$$

where $D_k(u)$ is defined by formula (3.27).

The coefficients of the differences may be evaluated by any one of formulas (3.24), (3.25) or (3.26) with $x = u$. (*See* Section 3.G.) However, for computational purposes, it is more convenient to use (3.28) and arrange the calculations in the following schematic:

u	b	D
$u - 1$	$b_1(u)$	$D_1(u)$
$u - 2$	$b_2(u)$	$D_2(u)$
$u - 3$	$b_3(u)$	$D_3(u)$
$\vdots$	$\vdots$	$\vdots$
$u - i$	$b_i(u)$	$D_i(u)$
		$D_{i+1}(u)$

Example IV.1. Find the value of the derivative of $y = f(x)$ at $x = 1.03$ in Example II.2.

Solution. From the data we have $h = .1$ and at $x = 1.03$, $u = .3$. We now evaluate the derivatives $D_i(u)$ according to formula (3.28) and the schematic above. At the same time we list the differences $\Delta^i y_0$ from Example II.2. The value of the derivative is then the sum of the products of the entries in the last two columns divided by h. The work is all arranged in the following calculation sheet:

$$b_{i+1} = \frac{1}{i+1} b_i(u - i + 1);$$

$$D_{i+1} = \frac{1}{i+1} [(u - i)D_i + (u - i + 1)b_i];$$

$$y' = \frac{1}{h} \Sigma D_i \Delta^i y_0.$$

i	$u - i$	b_i	D_i	$\Delta^i y_0$
0	.3			
1	−.7	1	1	1.37372
2	−1.7	.150000	−.200000	.40840
3	−2.7	−.035000	.078333	.10620
4	−3.7	.014875	−.038000	.02160
5			.020087	.00240

$$y' = \boxed{12.99586}.$$

One of the most important applications of numerical differentiation is to find the value of the derivative at one of the tabulated points. This condition specializes the formula for the derivative considerably. Thus at $x = x_i$ we have

$$u = \frac{x_i - x_0}{h} = \frac{ih}{h} = i.$$

The derivatives which form the coefficients in formula (35.2) can now be evaluated by formula (3.32) for $i < k$ and (3.31) for $i = k$. In the case where $k > i$ we must use formula (3.28). Formula (35.2) is derived from Newton's forward interpolation formula and thus is to be used at the *beginning* of a table of values. Consequently, we may tabulate the values of $D_k(i)$ for small values of i and k. For calculation purposes it is desirable to reduce formula (35.2) to a sum of products of two numbers followed by a division. The values of $D_k(i)$ are reduced to a set of fractions with a lowest common denominator. In the table we list the numerator and a column containing the lowest common denominator (L.C.D.).

Table 4.1. Values of $D_k(i) = \dfrac{\text{entry}}{\text{L.C.D.}}$

i \ k	1	2	3	4	5	6	L.C.D.
0	60	−30	20	−15	12	−10	60
1	300	150	−50	25	−15	3	300
2	600	900	200	−50	20	−3	600
3	600	1500	1100	150	−30	3	600
4	300	1050	1300	625	60	−3	300

By use of this table we may write the formulas for the value of the derivative at x_i using up to sixth differences. For example

$$(35.3)\quad y'(x_2) = \frac{1}{600h}\,[600\Delta y_0 + 900\Delta^2 y_0 + 200\Delta^3 y_0 - 50\Delta^4 y_0 + 20\Delta^5 y_0 - 3\Delta^6 y_0].$$

Example IV.2. Find the value of the derivative of $y = f(x)$ at $x = 1.1$ in Example II.2.

Solution. From Table 4.1 and the differences from Example II.2 we write

$D_k(1)$	300	150	−50	25	−15
$\Delta^k y_0$	1.37372	.10840	.10620	.02160	.00240

Since $h = .1$, we have

$$y'(1.1) = \frac{1}{30}\sum_{k=1}^{5} D_k(1)\Delta^k y_0 = \boxed{15.619}.$$

The reader may like to verify that in the same set of data we have $y'(1.0) = 12.$; $y'(1.2) = 20.2$; $y'(1.3) = 25.917$; and $y'(1.4) = 33.088$.

If we differentiate the other classical polynomial interpolation formulas, we obtain a set of formulas from which we can evaluate the derivatives of tabulated functions. Each such formula has its own individual usefulness and is used under the same conditions for which the original interpolation function was devised. In the derivation of the formulas particular attention is paid to the forms assumed by the derivatives of the coefficients of the differences. We shall now list the general formulas and some special cases which are frequently used. The reader should have no difficulty in verifying the particular forms displayed.

From *Newton's Backward Interpolation Formula* (23.12) we have

$$(35.4)\quad y'(x) = \frac{1}{h}[D_1^*(u)\Delta y_{n-1} + D_2^*(u)\Delta^2 y_{n-2} + D_3^*(u)\Delta^3 y_{n-3} + \cdots + D_k^*(u)\Delta^k y_{n-k}]$$

where

$$(35.5)\quad \begin{cases} D_1^*(u) = 1 \\ D_2^*(u) = \frac{1}{2}(2u+1) \\ D_3^*(u) = \frac{1}{3}[(u+2)D_2^*(u) + (u+1)b_2^*(u)] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ D_k^*(u) = \frac{1}{k}[(u+k-1)D_{k-1}^*(u) + (u+k-2)b_{k-1}^*(u)] \end{cases}$$

and

$$b_i^*(u) = \frac{1}{i}[u + i - 2]b_{i-1}^*(u), \qquad (b_1^* = 1;\ i = 2, 3, \cdots),$$

as a special case at $x = x_n$ we have $u = 0$ and (35.4) reduces to

$$(35.6) \quad y'(x_n) = \frac{1}{h}\left[\Delta y_{n-1} + \frac{1}{2}\Delta^2 y_{n-2} + \frac{1}{3}\Delta^3 y_{n-3} + \frac{1}{4}\Delta^4 y_{n-4} + \cdots + \frac{1}{k}\Delta^k y_{n-k}\right].$$

Example IV.3. Find the value of $y'(2)$ in Example II.2.

Solution. From Example II.2 we obtain the values of $\Delta^i y_{n-i}$ and

$$y'(2) = \left[10.98932 + \frac{1}{2}(1.99720) + \frac{1}{3}(.30780) + \frac{1}{4}(.03600) + \frac{1}{5}(.00240)\right]10$$

$$= \frac{1}{6}[60(10.98932) + 30(1.99720) + 20(.30780) + 15(.03600) + 12(.00240)]$$

$$= 121.00.$$

From *Stirling's* formula (23.17) and *Bessel's* formula (23.20) we obtain

$$(35.7) \quad y'(x) = \frac{1}{h}[S_1'm_1 + S_2'\Delta^2 y_{-1} + S_3'm_3 + S_4'\Delta^4 y_{-2} + \cdots + S_{2n-1}'m_{2n-1} + S_{2n}'\Delta^{2n}y_{-n}]$$

and

$$(35.8) \quad y'(x) = \frac{1}{h}[B_1'\Delta y_0 + B_2'm_2 + B_3'\Delta^3 y_{-1} + B_4'm_4 + \cdots + B_{2n}'m_{2n} + B_{2n+1}'\Delta^{2n+1}y_{-n}]$$

where S'_k and B'_k are given by

(35.9)

k	S'_k	B'_k
1	1	1
2	u	$\frac{1}{2}(2u-1)$
3	$\frac{1}{3}\left[uS'_2 + S_2 - \frac{1}{2}S'_1\right]$	$\frac{1}{3}[B_1B'_2 + B'_1B_2]$
4	$\frac{1}{4}[uS'_3 + S_3]$	$\frac{1}{4}[(u-2)S'_3 + S_3]$
5	$\frac{1}{5}[uS'_4 + S_4 - S'_3]$	$\frac{1}{5}[B'_1B_4 + B_1B'_4]$
6	$\frac{1}{6}[uS'_5 + S_5]$	$\frac{1}{6}[(u-3)S'_5 + S_5]$
7	$\frac{1}{7}\left[uS'_6 + S_6 - \frac{3}{2}S'_5\right]$	$\frac{1}{7}[B'_1B_6 + B_1B'_6]$
. . .	. . .	
$2i-1$	$\frac{1}{2i-1}\left[uS'_{2i-2} + S_{2i-2} - \frac{1}{2}(i-1)S'_{2i-3}\right]$	
$2i$	$\frac{1}{2i}[uS'_{2i-1} + S_{2i-1}]$	$\frac{1}{2i}[(u-i)S'_{2i-1} + S_{2i-1}]$
$2i+1$		$\frac{1}{2i+1}[B'_1B_{2i} + B_1B'_{2i}]$

Formulas (35.9) for the coefficients of the differences in the formulas for the derivative are given in the form of recursion formulas; that is, the succeeding ones are given in terms of the preceding ones. They can, of course, be written out in terms of u by continuous substitution. The above method is very advantageous for calculation purposes; however, those in formulas (35.9) necessitate the evaluation of S_k and B_k as well as the derivatives of these quantities. Furthermore, B'_k is given in terms of S_k and S'_k which is undesirable. Nevertheless, since S_k and B_k may be evaluated from recursion formulas, the calculations still proceed quite rapidly.

A special application of formula (35.7) is to find the derivative at $x = x_0$ in which case $u = 0$ and the formula reduces to

$$\textbf{(35.10)} \quad y'(x_0) = \frac{1}{h}\left[m_1 - \frac{1}{6}m_3 + \frac{1}{30}m_5 - \frac{1}{140}m_7 + \frac{1}{630}m_9 + \cdots\right].$$

A special application of formula (35.8) is to find the derivative at $x = x_{\frac{1}{2}}$ in which case $u = \frac{1}{2}$ and the formula reduces to

$$(35.11)\qquad y'(x_{\frac{1}{2}}) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{24}\Delta^3 y_{-1} + \frac{3}{640}\Delta^5 y_{-2} + \cdots\right].$$

Example IV.4. Given the table of values (x_i, y_i) listed below. Find the values of the first derivative at $x = 1.80, 1.85, 1.90, 2.80$.

Solution. First calculate the augmented central difference table, then for

$x = 1.80$	use formula (35.10);
$x = 1.85$	use formula (35.7);
$x = 1.90$	use formula (35.11);
$x = 2.80$	use formula (35.6).

The work is arranged on a calculating sheet as follows:

$$h = .2$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	−2.00000					
		75584				
			11200			
1.2	−1.24416					
				−31680		
		86784				
					3840	
			−20480			
1.4	−.37632					
						7680
				−27840		
		66304				
					11520	
			−48320			
1.6	.28672					
						7680
				−16320		
		17984				
				−6720	19200	7680
1.8	.46656	−14336	−64640			
						7680
				2880		
		−46656				
					26880	
			−61760			
2.0	.00000					
						7680
				29760		
		−108416				
					34560	
			−32000			
2.2	−1.08416					
						7680
				64320		
		−140416				
					42240	
			32320			
2.4	−2.48832					
				106560		
		−108096				
			138880			
2.6	−3.56928					
		30784				
2.8	−3.26144					

At $x = 1.80$: $\quad x_0 = 1.80, \quad u = 0.$

$$y'(1.80) = \frac{1}{6}[30m_1 - 5m_3 + m_5] = \boxed{-.64800}.$$

At $x = 1.85$: $\quad x_0 = 1.80, \quad u = .25.$

k	1	2	3	4	5
S_k		.03125	−.039063	−.002144	[illegible]
S'_k	1	.25	−.135417	−.018229	[illegible]

$$y'(1.85) = \boxed{-1.48691}.$$

The values for S_k are obtained from Table IV in back of the book.

$At\ x = 1.90$: $\quad x_0 = 1.80, \quad u = \frac{1}{2}.$

$$y'(1.90) = \frac{1}{384}[1920\Delta y_0 - 80\Delta^3 y_{-1} + 9\Delta^5 y_{-2}] = \boxed{-2.33700}.$$

$At\ x = 2.80$: $\quad x_n = 2.8; \quad u = 0.$

$$y'(2.80) = \frac{1}{12}[60\Delta y_{n-1} + 30\Delta^2 y_{n-2} + 20\Delta^3 y_{n-3} + 15\Delta^4 y_{n-4} + 12\Delta^5 y_{n-5}]$$

$$= \boxed{7.39200}.$$

Before differentiating Everett's formula (23.25) let us note that

$$\frac{dy}{dx} = \frac{1}{h}\frac{dy}{du} = \frac{1}{h}\frac{dy}{dt}\frac{dt}{du} = \frac{1}{h}\frac{dy}{dt}(-1) = -\frac{1}{h}\frac{dy}{dt}.$$

The differentiation thus yields

(35.12)
$$y'(x) = \frac{1}{h}[-E'_{00}(t)y_0 - E'_{10}(t)\Delta^2 y_{-1} - E'_{20}(t)\Delta^4 y_{-2} - \cdots + E'_{01}(u)y_1 + E'_{11}(u)\Delta^2 y_0 + E'_{21}(u)\Delta^4 y_{-1} + \cdots].$$

Now the form of $E_{i0}(t)$ and $E_{i1}(u)$ are identical for the same $i > 1$ so that the forms of their derivatives are identical. We shall list the first four

(35.13)
$$\begin{cases} E'_{01}(u) = 1 \\ E'_{11}(u) = \frac{1}{6}[u(u-1) + (u-1)(u+1) + (u+1)u] \\ E'_{21}(u) = \frac{1}{20}[(u^2-4)E'_{11}(u) + 2uE_{11}(u)] \\ E'_{31}(u) = \frac{1}{42}[(u^2-9)E'_{21}(u) + 2uE_{21}(u)] \end{cases}$$

and $E'_{i0}(t)$ are identical with u replaced by t. There are two special forms:

I. $u = 0$ and $t = 1$, then

$$(35.14)\quad y'(x) = \frac{1}{h}\left[-y_0 - \frac{1}{3}\Delta^2 y_{-1} + \frac{1}{20}\Delta^4 y_{-2} - \frac{1}{105}\Delta^6 y_{-3} + \cdots + y_1 - \frac{1}{6}\Delta^2 y_0 + \frac{1}{30}\Delta^4 y_{-1} - \frac{1}{140}\Delta^6 y_{-2} + \cdots\right].$$

II. $u = 1$ and $t = 0$, then

$$(35.15)\quad y'(x) = \frac{1}{h}\left[-y_0 + \frac{1}{6}\Delta^2 y_{-1} - \frac{1}{30}\Delta^4 y_{-2} + \frac{1}{140}\Delta^6 y_{-3} - \cdots + y_1 + \frac{1}{3}\Delta^2 y_0 - \frac{1}{20}\Delta^4 y_{-1} + \frac{1}{105}\Delta^6 y_{-2} - \cdots\right].$$

Example IV.5. Find the value of the first derivative at $x = 1.80$, 1.90, and 2.00 for the data of Example IV.4 using Everett's formula.

Solution. We have $h = .2$.

a) At $x = 1.80$ we use $x_0 = 1.80$, thus $u = 0$ and $t = 1$ and we apply formula (35.14):

$$y'(1.80) = \frac{1}{12}[-60(.46656) - 20(-.64640) + 3(.19200) + 60(.00000) - 10(-.61760) + 2(.26880)]$$
$$= \boxed{-.64800}.$$

b) At $x = 1.90$ we use $x_0 = 1.80$, thus $u = t = \frac{1}{2}$ and we apply formula (35.12):

i	$-E'_{i0}(t)$	$\Delta^{2i}y_{-i}$	$E'_{i1}(u)$	$\Delta^{2i}y_{i-1}$	$E_{i0}(t) = E_{i1}(u)$
0	−1	.46656	1	0	.5
1	.041667	−.64640	−.041667	−.61760	−.0625
2	−.004688	.19200	.004688	.26880	

$$y'(1.90) = \boxed{-2.33700}.$$

The values of $E_{i0}(t)$ and $E_{i1}(u)$ may be obtained from Table V, p. 334.

c) At $x = 2.00$. If we take $x_0 = 1.8$, then $u = 1$ and $t = 0$ so that we may apply formula (35.15).

$$y'(x) = \frac{1}{12}[-60(.46656) + 10(-.64640) - 2(.19200) + 60(.00000) + 20(-.61760) - 3(.26880)]$$

$$= \boxed{-4.00000}.$$

The first derivative of Lagrange's formula will be discussed in Chapter V.

36. Higher Derivatives of Classical Interpolation Formulas. The extension to higher derivatives is straightforward. All of the formulas derived in the last section could be differentiated again to yield the second derivative; these could be differentiated again to yield the third derivative; etc. We shall not display all formulas resulting from such operation but shall instead concentrate on a few special cases.

Let us consider the formulas resulting from differentiation of *Newton's* forward formula. The second derivative is obtained by differentiating formula (35.2) and resulting in

$$\textbf{(36.1)} \quad y''(x) = \frac{1}{h^2}[D_2^2(u)\Delta^2 y_0 + D_3^2(u)\Delta^3 y_0 + \cdots + D_k^2(u)\Delta^k y_0 + \cdots]$$

where from formula (3.35) we have

$$\textbf{(36.2)} \quad D_{i+1}^2(u) = \frac{1}{i+1}[2D_i(u) + (u-i)D_i^2(u)], \qquad (i = 2,3, \cdots)$$

and in particular

$$D_2^2(u) = 1.$$

We can easily generalize to the kth derivative

$$\textbf{(36.3)} \qquad y^{(k)}(x) = \frac{1}{h^k}[D_k^k(u)\Delta^k y_0 + D_{k+1}^k(u)\Delta^{k+1} y_0 + \cdots]$$

and

$$(36.4) \qquad D_{i+1}^{(k)}(u) = \frac{1}{i+1}\left[kD_i^{(k-1)}(u) + (u-i)D_i^{(k)}(u)\right].$$

These formulas are used at the beginning of a table of values. It is therefore especially interesting to reduce the formulas at $x = x_0$ and $x = x_1$ or, in other words, at $u = 0$ and $u = 1$. The values of $D_i^{(k)}(0)$ may be put in tabular form and reduced to a lowest common denominator for an appropriate number of differences.

Table 4.2. Values of $D_i^k(0) = \dfrac{\text{entry}}{\text{L.C.D.}}$.

$k \backslash i$	1	2	3	4	5	6	L.C.D.
1	60	−30	20	−15	12	−10	60
2		180	−180	165	−150	137	180
3			8	−12	14	−15	8
4				6	−12	17	6
5					6	−15	6
6						1	1

Table 4.3. Values of $D_i^k(1) = \dfrac{\text{entry}}{\text{L.C.D.}}$.

$k \backslash i$	1	2	3	4	5	6	L.C.D.
1	300	150	−50	25	−15	3	300
2		180	0	−15	15	−13	180
3			8	−4	2	−1	8
4				6	−6	5	6
5					2	−3	2
6						1	1

From these tables we can write, for example,

$$y''(x_0) = \frac{1}{180h^2}\left[180\Delta^2 y_0 - 180\Delta^3 y_0 + 165\Delta^4 y_0 - 150\Delta^5 y_0 + 137\Delta^6 y_0\right]$$

or

$$y'''(x_1) = \frac{1}{8h^3}[8\Delta^3 y_0 - 4\Delta^4 y_0 + 2\Delta^5 y_0 - \Delta^6 y_0].$$

Example IV.6. Find $y''(1)$ from the data of Example IV.4.

Solution.

$$y''(1) = \frac{1}{7.2}[180(.11200) - 180(-.31680) + 165(.03840) - 130(.07680)]$$

$$= \boxed{10}.$$

The values of the higher derivatives at $x = x_n$ may be obtained by repeated differentiation of *Newton's* backward formula, and a table of coefficients similar to Table 4.2 may be calculated. This is left as an exercise for the reader.

Another set of formulas of special interest are those which may be used to evaluate the higher derivatives of a function at $x = x_0$, where x_0 is the central value in a central difference table. To derive such a set we obtain the successive derivatives of *Stirling's* formula and evaluate the coefficients at $x = x_0$; i.e., $u = 0$.

From formula (35.7) we obtain by a differentiation

(36.5) $$y''(x) = \frac{1}{h^2}[S_2''\Delta^2 y_{-1} + S_3'' m_3 + S_4''\Delta^4 y_{-2} + S_5'' m_5 + \cdots]$$

and

(36.6) $$y'''(x) = \frac{1}{h^3}[S_3''' m_3 + S_4'''\Delta^4 y_{-2} + S_5''' m_5 + S_6'''\Delta^6 y_{-3} + \cdots]$$

where

$S_2'' = 1$	
$S_3'' = u$	$S_3''' = 1$
$S_4'' = \frac{1}{4}[uS_3'' + 2S_3']$	$S_4''' = u$
$S_5'' = \frac{1}{5}[uS_4'' + 2S_4' - S_3'']$	$S_5''' = \frac{1}{5}[uS_4''' + 3S_4'' - S_3''']$
$S_6'' = \frac{1}{6}[uS_5'' + 2S_5']$	$S_6''' = \frac{1}{6}[uS_5''' + 3S_5'']$
etc.	etc.

The process may, of course, be continued for higher derivatives. The evaluation of $S_i^{(k)}(u)$ at $u = 0$ is an easy matter, and again we may put the results in tabular form.

Table 4.4. Values of $S_i^k(0) = \dfrac{\text{entry}}{\text{L.C.D.}}$.

k \ i	1	2	3	4	5	6	L.C.D.
1	30	0	−5	0	1	0	30
2		180	0	−15	0	2	180
3			4	0	−1	0	4
4				6	0	−1	6
5					1	0	1
6						1	1

Example IV.7. Find $y''(2)$ in the data of Example IV.3.

Solution.

$$y''(2) = \frac{1}{7.2}[180(-.61760) - 15(.26880)] = -16.$$

The higher derivatives of *Lagrange's* formula will be discussed in Chapter V.

37. Maximum and Minimum Values of Tabulated Functions. The maximum and minimum values of a function are obtained by setting its first derivative equal to zero and solving for the variable. The process may be applied to a tabulated function. We may equate any one of the general formulas for the first derivative derived in Section 35 equal to zero and solve for u. Then the value of x may be found from $x = x_0 + hu$. Let us consider *Stirling's* formula (35.7) and for simplicity terminate it after the sixth difference. We have

$$\textbf{(37.1)} \quad y'(u) = m_1 + S_2'\Delta^2 y_{-1} + S_3' m_3 + S_4'\Delta^4 y_{-2} + S_5' m_5 + S_6'\Delta^6 y_{-3}$$

where

$$
\text{(37.2)} \quad \begin{cases} S'_2 = u \\ S'_3 = \dfrac{1}{2}u^2 - \dfrac{1}{6} \\ S'_4 = \dfrac{1}{6}u^3 - \dfrac{1}{12}u \\ S'_5 = \dfrac{1}{24}u^4 - \dfrac{1}{8}u^2 + \dfrac{1}{30} \\ S'_6 = \dfrac{1}{120}u^5 - \dfrac{1}{36}u^3 + \dfrac{1}{90}u. \end{cases}
$$

By substituting the values (37.2) into formula (37.1) and writing the result as a polynomial in u, we obtain

$$
\text{(37.3)} \qquad y'(u) = a_5u^5 + a_4u^4 + a_3u^3 + a_2u^2 + a_1u + a_0
$$

where

$$
\text{(37.4)} \quad \begin{cases} a_5 = \dfrac{1}{120}\Delta^6 y_{-3} \\ a_4 = \dfrac{1}{24}m_5 \\ a_3 = \dfrac{1}{36}(6\Delta^4 y_{-2} - \Delta^6 y_{-3}) \\ a_2 = \dfrac{1}{8}(4m_3 - m_5) \\ a_1 = \dfrac{1}{180}(180\Delta^2 y_{-1} - 15\Delta^4 y_{-2} + 2\Delta^6 y_{-3}) \\ a_0 = \dfrac{1}{30}(30m_1 - 5m_3 + m_5). \end{cases}
$$

In solving the equation obtained by setting formula (37.3) equal to zero, we may be faced with the problem of finding the roots of an algebraic equation whose coefficients are approximate numbers. Methods for accomplishing this are discussed in Chapter VI.

Example IV.8. Find the maximum and minimum values of the function

x	0	1	2	3	4	5
y	0	.25	0	2.25	16.00	56.25

Solution. Calculate a central difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		.25			
			−.50		
1	.25			3	
		−.25			
			2.50	6	6
2	0	1.00			
				9	
		2.25			6
			11.50		
3	2.25			15	
		13.75			
			26.50		
4	16.00				
		40.25			
5	56.25				

Let $x_0 = 2$, then since $\Delta^5 y = \Delta^6 y = 0$

$$\begin{aligned} y'(u) &= a_3u^3 + a_2u^2 + a_1u + a_0 \\ &= \frac{1}{6}(6)u^3 + \frac{1}{2}(6)u^2 + \frac{1}{12}(24)u + \frac{1}{30}(0) \\ &= u^3 + 3u^2 + 2u \end{aligned}$$

The solutions of $y'(u) = 0$ are $u = 0, -1, -2$. Then since $h = 1$ there are maxima and minima at

$$x = x_0 + u = 2 + u_i = \boxed{2, 1, 0}.$$

The values of the function can now be read out of the table.

The determination as to whether there is a maximum or a minimum at the resulting value of x can easily be determined by inspecting the table of values, and it is not necessary to be concerned with the second derivative. Thus, in the above example it is easily seen that the values at $x = 0$ and $x = 2$ are minima values.

38. Exercise V.

1. Given the data of problem 3(a) in Exercise III. Find the value of the first derivative at $x = 0, .15, .50, .52, .55, 1.0$.
2. Given the data of problem 3(b) in Exercise III. Find the value of the third derivative at $x = .2$ and $x = 1$.

3. Express the first derivative of Newton's forward formula terminated after the sixth differences as a polynomial in u. $\left[\text{Hint. Write each } \binom{u}{i}\right.$ as a polynomial before differentiating.]
4. Find the maxima and minima values of

x	-2	-1	0	1	2	3	[illegible]
y	2	$-.25$	0	$.25$	2	15.75	56

39. Numerical Integration—Introduction. Numerical integration is the *process* of calculating the value of a definite integral from the tabulated values of the integrand. The student of calculus was introduced to this subject at the time he studied the *trapezoidal* rule and *Simpson's* rule. The process is frequently called *mechanical quadrature* if it is applied to the integration of a function of one variable.

The solution to the problem of numerical integration may be obtained by replacing the integrand by an interpolation function and integrating this function between the desired limits. Thus, if we integrate the interpolation formulas of the last chapter, we obtain *quadrature formulas* for the integration of a function whose values are given in tabular form. In carrying out such calculation we must be cognizant of the fact that we are replacing the given integrand by an interpolating function, usually a polynomial, and then integrating this function. The accuracy of the result depends upon the ability of the interpolating function to represent the integrand over the interval of integration. It is frequently desirable to investigate this fact before embarking upon extensive numerical integration.

It is possible to derive a large number of quadrature formulas but quite impractical to exhibit them all. We shall concentrate our efforts on those which are readily adapted to calculating machines and which have proven to be sufficiently accurate for most applications. The particular formulas obtained by integrating Lagrange's interpolation function will be discussed in Chapter V.

40. Quadrature Formula for Equidistant Values. Let us first consider a function whose values are given at equally spaced intervals of the independent variable. In this case we have

(40.1) $$x = x_0 + hu$$

so that

(40.2) $$dx = hdu$$

and

(40.3) $$\int_{x_1}^{x_2} y(x)dx = h \int_{u_1}^{u_2} y(u)du.$$

Let us consider *Newton's* forward formula (23.8)

$$y(x) = y_0 + u\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \binom{u}{3}\Delta^3 y_0 + \cdots + \binom{u}{i}\Delta^i y_0 + \cdots$$

and integrate it between the limits x_0 and x_n which makes the limits on u from 0 to n.

(40.4) $$\int_{x_0}^{x_n} y(x)dx = h \int_0^n \left(y_0 + u\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \cdots \right) du$$
$$= h[a_0 y_0 + a_1\Delta y_0 + a_2\Delta^2 y_0 + \cdots + a_k\Delta^k y_0 + \cdots]$$

where

$$a_0 = \int_0^n du = u \Big|_0^n = n$$
$$a_1 = \int_0^n udu = \frac{1}{2}u^2 \Big|_0^n = \frac{1}{2}n^2$$
$$a_2 = \int_0^n \binom{u}{2} du = \frac{1}{6}u^3 - \frac{1}{4}u^2 \Big|_0^n = \frac{1}{6}n^3 - \frac{1}{4}n^2$$
$$a_k = \int_0^n \binom{u}{k} du = \frac{1}{k!}\int_0^n \prod_{j=0}^{k-1} (u - j)du.$$

By the use of Stirling's numbers we may write a_k as polynomials in n. They will be of degree $k + 1$ and terminate with n^2. A lowest common denominator for the coefficients may be found and factored out. We may thus obtain the following table of the coefficients of the various powers of n:

Table 4.5. Coefficients of n^i in a_k.

k	L.C.D.	n^{10}	n^9	n^8	n^7	n^6	n^5	n^4	n^3	n^2
1	2									1
2	12								2	−3
3	24							1	−4	4
4	720						1	−45	110	−90
5	1440					2	−24	105	−200	144
6	60480				12	−210	1428	−4725	7672	−5040
7	120960			3	72	700	−3528	9744	−14112	8640
8	3628800		10	−315	4140	−29400	121842	−295470	392040	−226800
9	7257600	2	−80	1365	−12960	74830	−269136	590620	−730560	403200

If we specify n, the above polynomials may be evaluated and the a become constants; their values may then be tabulated for given values of n. There exist better formulas, however, and such tabulation is not worth while.

41. Special Rules. It is well known from the study of integral calculus that

$$\int_0^6 y\,dx = \int_0^2 y\,dx + \int_2^4 y\,dx + \int_4^6 y\,dx. \tag{41.1}$$

Let us now divide the set of points (x_i,y_i) into groups of three (x_0,x_1,x_2); (x_2,x_3,x_4); (x_4,x_5,x_6); etc. For each such group there are no differences beyond the second, so that in formula (40.4) $n = 2$, and it becomes

$$\begin{aligned} \int_{x_0}^{x_2} y\,dx &= h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{6} - \frac{4}{4}\right)\Delta^2 y_0\right] \\ &= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right] \\ &= \frac{1}{3}h[y_0 + 4y_1 + y_2]. \end{aligned} \tag{41.2}$$

For the interval from x_2 to x_4 we obtain

$$\begin{aligned} \int_{x_2}^{x_4} y\,dx &= h\left[2y_2 + 2\Delta y_2 + \frac{1}{3}\Delta^2 y_2\right] \\ &= \frac{1}{3}h[y_2 + 4y_3 + y_4]. \end{aligned} \tag{41.3}$$

We may continue this process and then add them all together. Thus, if n is *even* we obtain

$$(41.4)\quad \int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n]$$

$$= \frac{h}{3}\sum_{i=0}^{n} c_i y_i$$

where

i	0	1	2	3	4	5	. . .	$n-2$	$n-1$	n
c_i	1	4	2	4	2	4	. . .	2	4	1

This is the well-known **Simpson's Rule** and is the easiest of the quadrature formulas to apply. In its application there must exist an *odd number of points*, and it is quite accurate for small values of h.

The data may be divided into groups of 7 points $(x_0, \ldots, x_6)$ for which $n = 6$, and by formula (40.4) we have

$$(41.5)\quad \int_{x_0}^{x_6} y\,dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right].$$

If we replace $\frac{41}{140}h\Delta^6 y_0$ by $\frac{3}{10}h\Delta^6 y_0$ we are committing an error of $\frac{h}{140}\Delta^6 y_0$ which may be tolerated provided h and $\Delta^6 y_0$ are sufficiently small. Making this substitution and replacing all the differences by their values in terms of the given values of y reduces formula (41.5) to

$$(41.6)\quad \int_{x_0}^{x_6} y\,dx = .3h[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6].$$

Apply now the same procedure to the points for x_6 and x_{12} to obtain

$$(41.7) \quad \int_{x_6}^{x_{12}} y\,dx = .3h[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}].$$

Continuing the process and adding together the results we obtain

$$(41.8) \quad \int_{x_0}^{x_n} y\,dx = .3h[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + \cdots + 2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

$$= .3h \sum_{i=0}^{n} c_i y_i$$

where

i	0	1	2	3	4	5	6	7	8	9	10	11	. . .	n
c_i	1	5	1	6	1	5	2	5	1	6	1	5		1

The coefficients may best be remembered in groups of six as follows:

First Group	1, 5, 1, 6, 1, 5
All Interior Groups	2, 5, 1, 6, 1, 5
Last Group	2, 5, 1, 6, 1, 5, 1

This is known as **Weddle's Rule.** It requires at least seven consecutive values of the function and uses these values in multiples of *six* if more than the first seven are used. In general it is more accurate than Simpson's Rule.

In applying these rules it is recommended that the multipliers c_i be recorded adjacent to the values of the function that they multiply.

Example IV.9. Find the value of $\int_0^6 y\,dx$ for the function tabulated below.

Solution. We shall find the value by using both of the above rules. In this case we have $h = 1$.

x	y	$c_i(S)$	$c_i(W)$
0	0	1	1
1	−45	4	5
2	−496	2	1
3	−2541	4	6
4	−8184	2	1
5	−19525	4	5
6	−37320	1	1

$$I_S = \frac{1}{3}\sum c_i(S)y_i = \boxed{-47708.0}$$

$$I_W = .3\sum c_i(W)y_i = \boxed{-47723.8}$$

This is a tabulation of the function

$$y(x) = x^6 - 9x^5 - 9x^4 - 9x^3 - 9x^2 - 10x$$

and the integral from $x = 0$ to $x = 6$ correct to one decimal place is $\boxed{-47733.9}$. Thus Simpson's Rule gives a value which is in error by .054% and Weddle's Rule gives a value which is in error by .011%.

42. Integration Formulas Based on Central Differences. In this section we shall integrate *Stirling's* and *Bessel's* interpolation formulas. First let us integrate *Stirling's* formula (23.17) from $x_0 - h$ to $x_0 + h$; i.e., $u = -1$ to $u = 1$. We have

$$(42.1)\quad \int_{-1}^{1} y\,du$$
$$= \int_{-1}^{1} (y_0 + S_1 m_1 + S_2\Delta^2 y_{-1} + S_3 m_3 + S_4\Delta^4 y_{-2} + \cdots)du$$
$$= s_0 y_0 + s_1 m_1 + s_2\Delta^2 y_{-1} + s_3 m_3 + s_4\Delta^4 y_{-2} + \cdots$$

where

$$\begin{cases} s_0 = u \Big|_{-1}^{1} = 2 \\ s_1 = \frac{1}{2} u^2 \Big|_{-1}^{1} = 0 \\ s_2 = \frac{1}{6} u^3 \Big|_{-1}^{1} = \frac{1}{3} \\ s_3 = \frac{1}{24} (u^4 - 2u^2) \Big|_{-1}^{1} = 0 \\ s_4 = \frac{1}{24} \frac{u^5}{5} - \frac{u^3}{3} \Big|_{-1}^{1} = -\frac{1}{90} \\ \text{etc.} \end{cases}$$

We note that the coefficients of m_i contain only even powers of u and, consequently, are zero when integrated from $u = -1$ and $u = 1$. There remains then only the even differences, and we write

$$(42.2) \quad \int_{-1}^{1} y du = 2y_0 + \frac{1}{6} \Delta^2 y_1 - \frac{1}{90} \Delta^4 y_{-2} + \frac{1}{756} \Delta^6 y_{-3} - \cdots .$$

Since we can order x_0 at any interior point of a central difference table, the value of the integral from $u = 0$ to $u = 2$ is identical in form to formula (42.2) with the subscripts on the y's advanced by one unit. Thus

$$(42.3) \quad \int_{0}^{2} y du = 2y_1 + \frac{1}{3} \Delta^2 y_0 - \frac{1}{90} \Delta^4 y_{-1} + \frac{1}{756} \Delta^6 y_{-2} - \cdots .$$

If we continue by finding the integrals from 2 to 4, then 4 to 6, etc. and then add them all together, we obtain for an even value of n

$$(42.4) \quad \int_{x_0}^{x_n} y dx = h[2(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + \frac{1}{3} (\Delta^2 y_0 + \Delta^2 y_2 + \Delta^2 y_4 + \cdots + \Delta^2 y_{n-2}) - \frac{1}{90} (\Delta^4 y_{-1} + \Delta^4 y_1 + \Delta^4 y_3 + \cdots + \Delta^4 y_{n-3}) + \frac{1}{756} (\Delta^6 y_{-2} + \Delta^6 y_0 + \Delta^6 y_2 + \cdots + \Delta^6 y_{n-4}) + \cdots].$$

For machine calculation the formula should be reduced to a least common denominator form before doing the calculation.

Let us now integrate *Bessel's* formula (23.20) from x_0 to x_1; i.e., $u = 0$ to 1. The coefficients of the resulting formula are

$$\begin{cases} b_0 = u \Big|_0^1 = 1 \\ b_1 = \dfrac{1}{2}(u^2 - u)\Big|_0^1 = 0 \\ b_2 = \dfrac{1}{12}(2u^3 - 3u^2)\Big|_0^1 = -\dfrac{1}{12} \\ b_3 = \dfrac{1}{24}(u^4 - 2u^3 + u^2)\Big|_0^1 = 0 \\ b_4 = \dfrac{11}{720} \\ b_5 = 0 \\ b_6 = -\dfrac{191}{60480} \\ \qquad \text{etc.} \end{cases}$$

so that we obtain

$$\text{(42.5)} \quad \int_{x_0}^{x_1} y\,dx = h\left[m_0 - \frac{1}{12}m_2 + \frac{11}{720}m_4 - \frac{191}{60480}m_6 + \cdots\right].$$

If we now proceed as we have in the past and integrate from x_1 to x_2, then x_2 to x_3, etc., we can obtain in each case a formula similar to (42.5) involving the arithmetic mean of the even differences on the lines through $x_{\frac{3}{2}}$, $x_{\frac{5}{2}}$, etc. In order to employ the notation using m_k it is necessary to use a double subscript on m. Let the first subscript denote the line on which the arithmetic means of the even differences are calculated, so that between x_{j-1} and x_j we write $m_{j,2i}$, $(j = 1, \cdots, n)$.

Then, in general, we have

$$\text{(42.6)} \quad \int_{x_{j-1}}^{x_j} y\,dx = h\left[m_{j0} - \frac{1}{12}m_{j2} + \frac{11}{720}m_{j4} - \frac{191}{60480}m_{j6} + \cdots\right].$$

If we now add all of these together, we obtain

$$(42.7)\quad \int_{x_0}^{x_n} y\,dx = h\left[\sum_{j=1}^{n} m_{j0} - \frac{1}{12}\sum_{j=1}^{n} m_{j2} + \frac{11}{720}\sum_{j=1}^{n} m_{j4} - \cdots\right].$$

The evaluation of this formula requires a lengthy calculation which may be somewhat simplified. From the definition we have

$$\sum_{j=1}^{n} m_{j0} = \frac{1}{2}[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \cdots + (y_{n-1} + y_n)]$$

$$= \frac{1}{2}y_0 + y_1 + y_2 + y_3 + \cdots + \frac{1}{2}y_n$$

$$\sum_{j=1}^{n} m_{j2} = \frac{1}{2}[(\Delta^2 y_{-1} + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^2 y_1) + \cdots + (\Delta^2 y_{n-2} + \Delta^2 y_{n-1})]$$

$$= \frac{1}{2}[\Delta^2 y_{-1} + 2\Delta^2 y_0 + 2\Delta^2 y_1 + 2\Delta^2 y_2 + 2\Delta^2 y_3 + \cdots + 2\Delta^2 y_{n-2} + \Delta^2 y_{n-1}]$$

$$= \frac{1}{2}[\Delta y_0 - \Delta y_{-1} + 2\Delta y_1 - 2\Delta y_0 + 2\Delta y_2 - 2\Delta y_1 + 2\Delta y_3 - 2\Delta y_2 + \cdots \Delta y_n - \Delta y_{n-1}]$$

$$= \frac{1}{2}[-\Delta y_{-1} - \Delta y_0 + \Delta y_{n-1} + \Delta y_n]$$

$$= \frac{1}{2}[\Delta y_n + \Delta y_{n-1} - (\Delta y_{-1} + \Delta y_0)]$$

$$= [m_{n1} - m_{01}].$$

The cancellation of the center terms when the even-order differences are reduced to the next lower odd-order differences holds for the rest of the terms of formula (42.7) which then reduces to

$$(42.8)\quad \int_0^{x_n} y\,dx = h\left[\left(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \cdots + \frac{1}{2}y_n\right) - \frac{1}{12}(m_{n1} - m_{01}) + \frac{11}{720}(m_{n-1,3} - m_{03}) - \frac{191}{60480}(m_{n-2,5} - m_{0,5}) + \cdots\right].$$

In this case n may be either even or odd. To better understand formula (42.8) let us indicate clearly in a central difference table which elements are used; they have been blocked off in Table 4.6.

Table 4.6. Elements for Formula (42.8).

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$	$\Delta^8 y$
x_{-4}	y_{-4}								
		Δy_{-4}							
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$						
		Δy_{-3}		$\Delta^3 y_{-4}$					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$				
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		$\Delta^7 y_{-4}$	
x_0	$\boxed{y_0}$	$\boxed{m_{01}}$	$\Delta^2 y_{-1}$	$\boxed{m_{03}}$	$\Delta^4 y_{-2}$	$\boxed{m_{05}}$	$\Delta^6 y_{-3}$	$\boxed{m_{07}}$	$\Delta^8 y_{-4}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$	
x_1	$\boxed{y_1}$	m_{11}	$\Delta^2 y_0$	m_{13}	$\Delta^4 y_{-1}$	m_{15}	$\Delta^6 y_{-2}$	$\boxed{m_{17}}$	$\Delta^8 y_{-3}$
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$		$\Delta^7 y_{-2}$	
x_2	$\boxed{y_2}$	m_{21}	$\Delta^2 y_1$	m_{23}	$\Delta^4 y_0$	$\boxed{m_{25}}$	$\Delta^6 y_{-1}$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$			
x_3	$\boxed{y_3}$	m_{31}	$\Delta^2 y_2$	$\boxed{m_{33}}$	$\Delta^4 y_1$				
		Δy_3		$\Delta^3 y_2$					
x_4	$\boxed{y_4}$	$\boxed{m_{41}}$	$\Delta^2 y_3$						
		Δy_4							
x_5	y_5								

Example IV.10. Find the value of the integral from $x = 3$ to $x = 6$ of the function

x	0	1	2	3	4	5	6	7
y	0	−45	−496	−2541	−8184	−19525	−37320	−58821

Solution. Calculate the difference table and let $x_0 = 3$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	0						
		−45					
1	−45		−406				
		−451		−1188			
2	−496		−1594		−816		
		−2045		−2004		720	
3	−2541	−3844	−3598	−2052	−96	1080	720
		−5643		−2100		1440	
4	−8184		−5698		1344	1800	720
		−11341		−756		2160	
5	−19525		−6454	996	3504		
		−17795		2748			
6	−37320	−19648	−3706				
		−21501					
7	−58821						

Applying formula (42.8) with $h = 1$ we have

$$\int_3^6 y\,dx = -47639.5 - \frac{1}{12}(-15804) + \frac{11}{720}(3048) - \frac{191}{60480}(720)$$
$$= -47639.5 + 1317 + 46.566667 - 2.273809$$
$$= \boxed{-46278.2}.$$

To perform the calculations on a calculating machine write the formula over the least common denominator; thus

$$\int_3^6 y\,dx = \frac{1}{60480}[30240(-2541) + 60480(-8184) + 60480(-19525) + 30240(-37320) - 5040(-19648) + 5040(-3844) + 924(996) - 924(-2052) - 191(1800) + 191(1080)]$$
$$= \boxed{-46278.2}.$$

Although formulas (42.4) and (42.8) have a classical interest, they are seldom used in practice. For more applicable formulas to find the value of an integral for tabulated functions, see Chapter V.

43. Gauss's Formula for Numerical Integration. A very accurate quadrature formula for finding the value of the definite integral

$$I = \int_a^b f(x)dx$$

where $f(x)$ is a known function but whose integral is either not easily evaluated or cannot be conveniently expressed in closed form was derived by Gauss and is based on Legendre polynomials. [*See* section 3.H.] The principle is to obtain the best subdivision of the interval (a,b), the value of the function at these points, and the coefficients to multiply the functional values to yield the value of the definite integral.

The first step is to transform the interval (a,b) to the interval $(-1, 1)$* which is accomplished by letting

(43.1) $$x = \frac{1}{2}(b - a)v + \frac{1}{2}(a + b).$$

Then at $x = a$, we have

$$v = \frac{2a - a - b}{b - a} = -1$$

and at $x = b$, we have

$$v = \frac{2b - a - b}{b - a} = 1.$$

The new form of $f(x)$ is

(43.2) $$f(x) = f\left[\frac{1}{2}(b - a)v + \frac{1}{2}(a + b)\right] = \varphi(v)$$

and

$$dx = \frac{1}{2}(b - a)dv$$

so that

(43.3) $$\int_a^b f(x)dx = \frac{b - a}{2}\int_{-1}^1 \varphi(v)dv.$$

Now it is desired to have a formula

(43.4) $$\int_{-1}^1 \varphi(v)dv = g_1\varphi(v_1) + g_2\varphi(v_2) + g_3\varphi(v_3) + \cdots + g_n\varphi(v_n)$$

* Some authors have transformed to the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ or (0, 1), and care must be exercised when using tables of values for the coefficients.

which is a good evaluation of the integral when $\varphi(v)$ is any polynomial of as high a degree as possible and v_i, $(i = 1, \cdots, n)$, are the points of subdivision of the interval $(-1, 1)$. This is **Gauss's Mechanical Quadrature Formula.** It is clear that we need to determine both the g_i, $(i = 1, \cdots, n)$, and the v_i, $(i, \cdots, n)$; thus $2n$ relations are necessary and the highest degree of the polynomials $\varphi(v)$ will probably be $2n - 1$. Let us therefore write

$$(43.5) \quad \varphi(v) = a_0 + a_1 v + a_2 v^2 + a_3 v^3 + \cdots + a_{2n-1} v^{2n-1}$$

from which is obtained

$$(43.6) \quad \int_{-1}^{1} \varphi(v)dv = \left[a_0 v + \frac{1}{2} a_1 v^2 + \frac{1}{3} a_2 v^3 + \cdots + \frac{1}{2n} a_{2n-1} v^{2n} \right]_{-1}^{1}$$

$$= 2a_0 + \frac{2}{3} a_2 + \frac{2}{5} a_4 + \cdots.$$

Substitution of v_i, $(i = 1, \cdots, n)$, into the polynomial (43.5) yields the n values

$$(43.7) \quad \varphi(v_i) = a_0 + a_1 v_i + a_2 v_i^2 + a_3 v_i^3 + \cdots + a_{2n-1} v_i^{2n-1}.$$

Formula (43.4) then becomes

$$(43.8) \quad \int_{-1}^{1} \varphi(v)dv = g_1(a_0 + a_1 v_1 + a_2 v_1^2 + a_3 v_1^3 + \cdots + a_{2n-1} v_1^{2n-1})$$
$$+ g_2(a_0 + a_1 v_2 + a_2 v_2^2 + a_3 v_2^3 + \cdots + a_{2n-1} v_2^{2n-1})$$
$$+ g_3(a_0 + a_1 v_3 + a_2 v_3^2 + a_3 v_3^3 + \cdots + a_{2n-1} v_3^{2n-1})$$
$$+ \cdots\cdots\cdots\cdots\cdots$$
$$+ g_n(a_0 + a_1 v_n + a_2 v_n^2 + a_3 v_n^3 + \cdots + a_{2n-1} v_n^{2n-1})$$
$$= a_0(g_1 + g_2 + g_3 + \cdots + g_n)$$
$$+ a_1(g_1 v_1 + g_2 v_2 + g_3 v_3 + \cdots + g_n v_n)$$
$$+ a_2(g_1 v_1^2 + g_2 v_2^2 + g_3 v_3^2 + \cdots + g_n v_n^2)$$
$$+ \cdots\cdots\cdots\cdots\cdots$$
$$+ a_{2n-1}(g_1 v_1^{2n-1} + g_2 v_2^{2n-1} + \cdots + g_n v_n^{2n-1}).$$

Since it is desired that formulas (43.6) and (43.8) be identical for *all* values of a_i, the coefficients of a_i must be equal. Thus we obtain the $2n$ equations

$$(43.9)\qquad \left\{\begin{aligned} g_1 + g_2 + g_3 + \cdots + g_n &= 2 \\ g_1v_1 + g_2v_2 + g_3v_3 + \cdots + g_nv_n &= 0 \\ g_1v_1^2 + g_2v_2^2 + g_3v_3^2 + \cdots + g_nv_n^2 &= \frac{2}{3} \\ g_1v_1^3 + g_2v_2^3 + g_3v_3^3 + \cdots + g_nv_n^3 &= 0 \\ \cdots \qquad \cdots \qquad & \cdots \quad \cdots \\ g_1v_1^{2n-1} + g_2v_2^{2n-1} + \cdots + g_nv_n^{2n-1} &= 0. \end{aligned}\right.$$

The solution of this system of nonlinear equations would be quite difficult. However, it can be reduced to a system of linear equations in g_i if we choose the values for v_i to be the zeros of the Legendre polynomials which are known to be all real and distinct. These zeros have been calculated and tabulated and the corresponding values of g_i have also been evaluated. These are given in Table VII, p. 337, to ten significant digits. Since the v_i are negatively symmetrical; i.e., $v_i = -v_{-i}$, and $g_i = g_{-i}$, only half plus one need be recorded.

To illustrate the use of *Gauss's* formula let us consider an example.

Example IV.11. Calculate the value of

$$I = \int^{8} \frac{dx}{x}.$$

Solution. Let

$$\begin{aligned} x &= \frac{1}{2}(b - a)v + \frac{1}{2}(b + a) \\ &= \frac{1}{2}(8 - 2)v + \frac{1}{2}(8 + 2) \\ &= 3v + 5; \end{aligned}$$

then

$$f(x) = \frac{1}{x} = \frac{1}{3v + 5} = \varphi(v).$$

Choose $n = 7$ and evaluate $\varphi(v_i)$ obtaining the v_i from Table VII arranging the work as follows:

i	v_i	$\varphi(v_i)$	g_i
−3	−.9491079123	.4645380344	.1294849662
3	.9491079123	.1274319798	.1294849662
−2	−.7415311856	.3603075875	.2797053915
2	.7415311856	.1384160911	.2797053915
−1	−.4058451514	.2643778911	.3818300505
1	.4058451514	.1608354318	.3818300505
0	0	.2000000000	.4179591837

The value of the integral is then

$$I = \frac{b-a}{2}\sum g_i\varphi(v_i)$$
$$= 3[.4620981736]$$
$$= \boxed{1.386294521}.$$

The true value of the integral is

$$I = \int_2^8 \frac{dx}{x} = \ln\frac{8}{2} = \ln 4 = 1.386294361.$$

Thus the error is

$$E_G = -.000000160.$$

In practice the values of v_i and g_i are not recorded since they may be taken directly out of the table. Furthermore, since $g_i = g_{-i}$ we have

$$g_{-i}\varphi(v_{-i}) + g_i\varphi(v_i) = g_i[\varphi(v_{-i}) + \varphi(v_i)]$$

and therefore the sum

$$\varphi(v_{-i}) + \varphi(v_i)$$

could be computed and recorded thus cutting the recording nearly in half.

Lobatto and *Tchebycheff* have both made variations on Gauss's formula, Lobatto to include the end values and Tchebycheff to make the coefficients of the g's all equal. Such formulas offer advantages only in special cases.

44. Numerical Double Integration. The process of calculating the value of a definite double integral of a function of two variables is called **numerical double integration** and also **mechanical cubature.** A formula for this process may be derived by first obtaining an interpolation function in terms of the differences of a function of two variables. However, this is unnecessary since mechanical cubature may be performed by a double application of a quadrature formula.

Let $z = f(x,y)$ be a function of two independent variables x and y and let its values be given at equally spaced intervals x_i, $(i = 0, \cdots, n)$, of length h and y_i, $(i = 0, \cdots, m)$, of length k. Then, since $dx = hdu$ and $dy = kdv$, we have

$$(44.1)\quad I = \int_{x_0}^{x_n} \int_{y_0}^{y_m} Z(x,y)dydx = hk \int_0^n \int_0^m z(u,v)dvdu.$$

If the values of $Z_{ij}(x_i,y_i)$ are exhibited in a rectangular array (*see* Table 2.5), then the value of I in formula (44.1) may be found by applying the following rule:

Rule. *The value of the double integral may be found by applying to each horizontal row (or to each vertical column) any quadrature formula employing equidistant ordinates. Then, to the results thus obtained for the rows (or columns), again apply a similar formula.*

The value of the double integral can thus be found by repeated application of *Simpson's Rule, Weddle's Rule,* or any other quadrature formula we wish.

Example IV.12. Find the value of the following integral by numerical integration:

$$I = \int_1^{2.2} \int_1^{2.2} \frac{dxdy}{xy}.$$

Solution. Let the values of the integrand be given as shown in the following table which has $h = .3$ and $k = .2$:

$x \backslash y$	1.0	1.2	1.4	1.6	1.8	2.0	2.2	A_x	$c_i(s)$
1.0	1.00000	.83333	.71429	.62500	.55556	.50000	.45455	.788463	1
1.3	.76923	.64103	.54945	.48077	.42735	.38462	.34965	.606513	4
1.6	.62500	.52083	.44643	.39063	.34722	.31250	.28409	.492790	2
1.9	.52632	.43860	.37594	.32895	.29240	.26316	.23923	.414983	4
2.2	.45455	.37879	.32468	.28409	.25253	.22727	.20661	.358393	1
$c_i(W)$	1	5	1	6	1	5	1		

$I = .62184$

Apply Weddle's Rule to each row to obtain

$$A_x = (.06)\Sigma[c_i(\text{W})][\text{Entry}]$$

which have been entered in a column for each x. Then apply Simpson's Rule to the column of A_x to obtain

$$I = .1\Sigma[c_i(s)]A_x = .62184.$$

The true value of the integral is

$$I = (\ln 2.2)(\ln 2.2) = .62167.$$

Thus the error is $-.00017$.

45. Accuracy of Numerical Integration. The formulas which were derived in this chapter for finding the value of a definite integral are in general approximate, and thus there is an inherent error associated with each. It is desirable to be able to evaluate this error, and it is possible to derive expressions for the errors.

Consider first *Simpson's Rule* and let $f(x)$ be a well-behaved function so that it is continuous in the interval under consideration and has as many continuous derivatives as will be required. Then we may write

$$F(x) = \int_a^x f(x)dx, \qquad F'(x) = f(x), \qquad F''(x) = f'(x), \text{ etc.}$$

The definite integral from $x_0 - h$ to $x_0 + h$ is

$$(45.1) \qquad I = \int_{x_0-h}^{x_0+h} f(x)dx = F(x_0 + h) - F(x_0 - h).$$

Simpson's Rule yields

$$(45.2) \qquad I_s = \frac{h}{3}[f(x_0 - h) + 4f(x_0) + f(x_0 + h)]$$

and the difference between these two is the inherent error

$$(45.3) \quad E_s = I - I_s = F(x_0 + h) - F(x_0 - h) - \frac{1}{3}h[f(x_0 - h) + 4f(x_0) + f(x_0 + h)].$$

This expression may now be transformed by expanding each term of the right-hand side in a Taylor's series. That is

$$(45.4)\quad \begin{cases} F(x_0 + h) = F(x_0) + hf(x_0) + \dfrac{1}{2} h^2 f'(x_0) + \dfrac{1}{3!} h^3 f''(x_0) + \cdots \\ F(x_0 - h) = F(x_0) - hf(x_0) + \dfrac{1}{2} h^2 f'(x_0) - \dfrac{1}{3!} h^3 f''(x_0) + \cdots \\ f(x_0 + h) = f(x_0) + hf'(x_0) + \dfrac{1}{2} h^2 f''(x_0) + \dfrac{1}{3!} h^3 f'''(x_0) + \cdots \\ f(x_0 - h) = f(x_0) - hf'(x_0) + \dfrac{1}{2} h^2 f''(x_0) - \dfrac{1}{3!} h^3 f'''(x_0) + \cdots \end{cases}$$

and substituting into (45.3) we see that the terms up to the fourth derivative subtract out

$$F(x_0) - F(x_0) = 0$$

$$hf(x_0) + hf(x_0) - \frac{1}{3} h[f(x_0) + 4f(x_0) + f(x_0)] = 0$$

$$\frac{1}{2} h^2 f'(x_0) - \frac{1}{2} h^2 f'(x_0) - \frac{1}{3} h[hf'(x_0) - hf'(x_0)] = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left[\frac{1}{5!} f^{iv}(x_0) + \frac{1}{5!} f^{iv}(x_0)\right] h^5 - \frac{1}{3} h \left[\frac{1}{4!} h^4 f^{iv}(x_0) + \frac{1}{4!} h^4 f^{iv}(x_0)\right] = \left(\frac{2}{5!} - \frac{1}{3}\frac{2}{4!}\right) h^5 f^{iv}(x_0).$$

Thus

$$(45.5)\qquad E_s = -\frac{1}{90} h^5 f^{iv}(x_0) + \cdots$$

is an expression for the error over the interval from $x_0 - h$ to $x_0 + h$. If we change the interval to be from $x_1 - h$ to $x_1 + h$ we see that $f^{iv}(x_0)$ is replaced by $f^{iv}(x_1)$ in formula (45.5). The entire interval from a to b can now be subdivided into subintervals x_0 to

x_2, x_2 to x_4, etc., so that the total error is

$$(45.6) \qquad E_s = -\frac{1}{90} h^5[f^{iv}(x_1) + f^{iv}(x_3) + f^{iv}(x_5) + \cdots + f^{iv}(x_{n-1})] - \cdots.$$

If we ignore all the terms above the fourth derivative and let $f^{iv}(x_k)$ be the largest value of any of the $f^{iv}(x_i)$, we may replace each by $f^{iv}(x_k)$ and add the $\frac{1}{2}n$ quantities to yield

$$(45.7) \qquad E_s \leq -\frac{nh^5}{180} f^{iv}(x_k) = -\frac{b-a}{180} h^4 f^{iv}(x_k)$$

since

$$b - a = nh.$$

This gives us an expression for the error in Simpson's Rule.

Since the fourth derivative is zero if $f(x)$ is a polynomial of degree no larger than the third, it is seen that Simpson's Rule gives the exact value of the integral for these polynomials.

The above error formula is expressed in terms of the fourth derivative which may not always be conveniently evaluated. It is therefore desirable to replace it by the equivalent differences and then replace those by their values in terms of the given ordinates. The first part can be done by using formula (19.5) to write

$$f^{iv}(x_i) = \frac{\Delta^4 y_{i-2}}{h^4}$$

so that we have

$$(45.8) \quad E_s = -\frac{h}{90} [\Delta^4 y_{-1} + \Delta^4 y_1 + \Delta^4 y_3 + \cdots + \Delta^4 y_{n-3}].$$

To carry the transformation further we use formula (11.6) to write the differences in terms of the given ordinates

$$\begin{cases} \Delta^4 y_{-1} = y_3 - 4y_2 + 6y_1 - 4y_0 + y_{-1} \\ \Delta^4 y_1 = y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta^4 y_{n-3} = y_{n+1} - 4y_n + 6y_{n-1} - 4y_{n-2} + y_{n-3} \end{cases}$$

and upon substituting these into formula (45.8) we have, after collecting terms,

$$(45.9)\quad E_s = -\frac{h}{90}[y_{-1} + y_{n+1} - 4(y_0 + y_n) + 7(y_1 + y_{n-1}) \\ -8(y_2 + y_4 + \cdots + y_{n-2}) + 8(y_3 + y_5 + \cdots + y_{n-3})]$$

for $n \geq 6$.

For n less than six some of the y_i will not exist. Since n must be even for Simpson's Rule we have the following two special forms: $n = 2$:

$$(45.10)\qquad E_s = -\frac{h}{90}[y_{-1} + y_3 - 4(y_0 + y_2) + 6y_1]$$

$n = 4$:

$$(45.11)\quad E_s = -\frac{h}{90}[y_{-1} + y_5 - 4(y_0 + y_4) + 7(y_1 + y_3) - 8y_2].$$

It is to be noted that these formulas utilize two values, y_{-1} and y_{n+1}, which are outside the interval of integration. In order to obtain these values it may be necessary to extrapolate, which can be done by using Newton's forward and backward interpolation formulas.

Example IV.13. Find the approximate error committed by Simpson's Rule in Example IV.9.

Solution. In this case $h = 1$, $n = 6$, and since $y = f(x)$ is a known polynomial, we may evaluate y_{-1} and y_7. They are $y_{-1} = 11$ and $y_7 = -58821$.

$$E_s = -\frac{1}{90}[11 - 58821 - 4(0 - 37320) + 7(-45 - 19525) \\ - 8(-496 - 8184) + 8(-2541)]$$

$$= \boxed{-28.8}.$$

This compares favorably with the true error which is $\boxed{-25.9}$.

The principal part of the error in *Weddle's Rule* stems from the fact that we omitted the quantity

$$(45.12)\qquad E_W = -\frac{h}{140}\Delta^6 y_0 = -\frac{h^7}{140}f^{vi}(x).$$

We may therefore use this for an approximate expression of the error. In so doing we are neglecting only differences of order higher than the sixth, and thus Weddle's Rule gives an exact result for polynomials of degree five or less.

By again employing formula (11.6) we may write

$$(45.13)\quad E_W = -\frac{h}{140}[y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0].$$

Example IV.14. Find the approximate error committed by Weddle's Rule in Example IV.9.

Solution.

$$E_W = -\frac{1}{90}[-37320 - 6(-19525) + 15(-8184) - 20(-2541) + 15(-496) - 6(-45) + 0]$$

$$= \boxed{-8}.$$

The true error is $-47733.9 - (-47728.8) = -5.1$.

As can be seen from the last two examples, the true error is usually numerically less than or equal to the computed error, which adds assurance to the original calculations.

The inherent error in *Gauss's Quadrature Formula* is dependent upon the theory of Legendre polynomials, and the derivation of an error formula is beyond the scope of this book. It is, however, possible to state a rule for approximating the error. First of all, it is dependent upon being able to expand $\varphi(v)$ in a power series:

$$(45.14)\quad \varphi(v) = c_0 + c_1u + c_2u^2 + \cdots + c_nu^n + \cdots.$$

If this is possible we approximate the error by

$$(45.15)\quad E_G = \frac{b-a}{(2n+1)2^{2n}}\left(\frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}\right)^2 \times\left[c_{2n} + \frac{c_{2n+2}}{8}\left(\frac{(n+1)(n+2)}{2n+3} + \frac{n(n-1)}{2n-1}\right)\right].$$

The evaluation of this formula is quite cumbersome and in practice is used only when absolutely necessary. The accuracy of Gauss's formula can, of course, be increased by increasing the number of points.

46. Exercise VI.

1. Given the following data

x	y	x	y
0	0	2.2	166.769856
.2	.949696	2.4	215.660544
.4	2.375424	2.6	268.026304
.6	4.703424	2.8	319.393536
.8	8.605696	3.0	363.000000
1.0	15.000000	3.2	389.287936
1.2	25.003776	3.4	385.351104
1.4	39.841984	3.6	334.335744
1.6	60.708864	3.8	214.795456
1.8	88.583616	4.0	0
2.0	124.000000		

Using Simpson's Rule find

(a) $\int_0^{1.2} y\,dx$; (b) $\int_0^{3.6} y\,dx$; (c) $\int_0^{4.0} y\,dx$; (d) $\int_1^{3} y\,dx$.

Using Weddle's Rule find

(e) $\int_0^{1.2} y\,dx$; (f) $\int_0^{3.6} y\,dx$; (g) $\int_1^{3.4} y\,dx$.

2. Find a sufficient number of differences for the data of problem 1 to evaluate the integral from $x = 2$ to $x = 3$ by formula (42.8).

3. Find $\int_0^1 x\,dx$ using $n = 7$ in Gauss's quadrature formula.

4. Find $\int_1^2 \frac{dx}{x\sqrt{x+1}}$ using $n = 5$ in Gauss's quadrature formula.

5. Find an expression involving the first five terms for the integral of Newton's backward interpolation formula between the limits $x = x_{n-1}$ and $x = x_n$.

CHAPTER V. LAGRANGIAN FORMULAS

47. Introduction. The derivation and discussion of formulas based on the idea of passing an nth degree polynomial through $(n + 1)$ given points has been deferred to this chapter in order that this concept may be presented as a unit. The basic formula is credited to Lagrange and is usually referred to as Lagrange's interpolation formula since its primary use was for interpolation problems. This formula, however, may be used as the basis for many other numerical analysis problems, and in the special cases tables of coefficients may be calculated which reduce the problem to simply one of obtaining the sums of products. The formulas are not limited to equally spaced intervals but may be applied to this special case.

Since we are fitting the data or replacing a function by a polynomial, the formulas should only be used whenever this is possible, and frequent checks should be made to discover any irregularities. Nevertheless, a wide variety of problems may be handled by these formulas.

48. The Fundamental Formula.

Let there be given the values of the ordinates $y_0, y_1, \cdots, y_n$ of the function $y = f(x)$ at the $(n + 1)$ points $x_0, x_1, \cdots, x_n$. The polynomial of the nth degree through these points may be written in the form

(48.1)

$$L(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} y_1$$
$$\cdots\cdots\cdots\cdots$$
$$+ \frac{(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} y_i$$
$$+ \cdots\cdots\cdots\cdots$$
$$+ \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} y_n.$$

If we let

(48.2) $$P(x) = (x - x_0)(x - x_1) \cdots (x - x_n) = \prod_{j=0}^{n} (x - x_j)$$

and

(48.3) $$P_i(x) = (x - x_i)^{-1}P(x) = (x - x_i)^{-1} \prod_{j=0}^{n} (x - x_j)$$

then formula (48.1) becomes

(48.4) $$L(x) = \frac{P_0(x)}{P_0(x_0)} y_0 + \frac{P_1(x)}{P_1(x_1)} y_1 + \cdots + \frac{P_n(x)}{P_n(x_n)} y_n.$$

It is easily seen that

$$P_r(x_i) = 0 \qquad \text{if} \qquad i \neq r$$

so that

(48.5) $$L(x_r) = \frac{P_r(x_r)}{P_r(x_r)} y_r = y_r, \qquad (r = 0, \cdots, n),$$

and the polynomial given by (48.1) is one which passes through the $(n + 1)$ points (x_r,y_r), $(r = 0, \cdots, n)$. In general, $P_i(x)$ is a polynomial of degree n; i.e.,

(48.6) $$P_i(x) = a_{i,n}x^n + a_{i,n-1}x^{n-1} + \cdots + a_{i,0} \text{ with } a_{in} = 1$$

and

(48.7) $$P_i(x_i) = a_{i,n}x_i^n + a_{i,n-1}x_i^{n-1} + \cdots + a_{i,0} = k_i.$$

It is desired that $L(x)$ approximate the function $f(x)$; i.e.,

(48.8) $$f(x) = L(x) + R(x)$$

where the remainder $R(x)$ is such that

(48.9) $$R(x_i) = 0 \quad \text{for} \quad (i = 0, 1, \cdots, n).$$

Furthermore, if we let

$$R(x) = P(x)Q(x)$$

and consider any function

(48.10) $$\varphi(z) = f(z) - L(z) - P(z)Q(x)$$

such that

$$\varphi(x_i) = 0 \quad \text{and} \quad \varphi(x) = 0$$

if $x \neq x_i$ for all $i = 0, 1, \cdots, n$, then $\varphi(z)$ vanishes at $n + 2$ points. By repeated application of Rolle's theorem $\varphi^{n+1}(\xi)$ vanishes where $x_0 \leq \xi \leq x_n$. However,

$$L^{n+1}(z) = 0 \quad \text{and} \quad P^{n+1}(z) = (n + 1)!$$

so that upon differentiating equation (48.10) we have

$$0 = f^{n+1}(\xi) - (n + 1)!Q(x)$$

or

$$Q(x) = \frac{1}{(n + 1)!} f^{n+1}(\xi)$$

and at $z = x$ equation (48.10) yields

(48.11) $$f(x) = L(x) + \frac{P(x)f^{n+1}(\xi)}{(n + 1)!}$$

Thus we may write

(48.12) $$f(x) = \sum_{i=0}^{n} \frac{P_i(x)}{P_i(x_i)} y_i + \frac{P(x)}{(n + 1)!} f^{n+1}(\xi)$$

where ξ lies in the interval $[x_0 \cdots x_n]$.

The last term of formula (48.12) is in a sense a remainder term and is a measure of the accuracy of the fit of the polynomial $L(x)$.

Formula (48.4) has the very useful property that *it is invariant under a linear transformation.* This can be easily shown. Let us make the transformation

$$x = hu + a, \qquad x_i = hu_i + a,$$

then

$$\begin{aligned}
P_i(x) &= (x - x_i)^{-1}P(x) \\
&= (x - x_i)^{-1}(x - x_0)(x - x_1) \cdots (x - x_n) \\
&= (hu + a - hu_i - a)^{-1}(hu + a - hu_0 - a) \\
&\qquad (hu + a - hu_1 - a) \cdots (hu + a - hu_n - a) \\
&= h^{-1}(u - u_i)^{-1}h^n(u - u_0)(u - u_1) \cdots (u - u_n) \\
&= h^{n-1}(u - u_i)^{-1}P(u) \\
&= h^{n-1}P_i(u); \\
P_i(x_i) &= (x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots \\
&\qquad (x_i - x_n) \\
&= (hu_i + a - hu_0 - a) \cdots (hu_i + a - hu_{i-1} - a) \\
&\qquad (hu_i + a - hu_{i+1} - a) \cdots (hu_i + a - hu_n - a) \\
&= h^{n-1}(u_i - u_0)(u_i - u) \cdots (u_i - u_{i-1})(u_i - u_{i+1}) \cdots \\
&\qquad (u_i - u_n) \\
&= h^{n-1}P_i(u_i).
\end{aligned}$$

Therefore

$$L(x) = \sum_{i=0}^{n} \frac{P_i(x)}{P_i(x_i)}\, y_i = \sum_{i=0}^{n} \frac{P_i(u)}{P_i(u_i)}\, y_i = L(u).$$

49. Equally Spaced Intervals. If the values of x_i, $(i = 0, \cdots, n)$, are given at equally spaced intervals, we have

(49.1) $$x_i = x_0 + ih$$

where

$$h = \Delta x = x_1 - x_0 = x_r - x_{r-1}, \qquad (r = 1, \cdots, n),$$

and also let

(49.2) $$u = \frac{x - x_0}{h} \qquad \text{or} \qquad x = x_0 + uh.$$

The polynomial $P_i(x)$ now takes a special form:

$$(49.3) \quad P_i(x) = (x - x_i)^{-1} \prod_{j=0}^{n} (x - x_j)$$

$$= (x_0 + uh - x_0 - ih)^{-1} \prod_{j=0}^{n} (x_0 + uh - x_0 - jh)$$

$$= h^{-1}(u - i)^{-1} \prod_{j=0}^{n} h(u - j)$$

$$= h^n(u - i)^{-1} \prod_{j=0}^{n} (u - j).$$

Now

$$(49.4) \quad \prod_{j=0}^{n} (u - j) = u(u - 1) \cdots (u - n)$$

$$= S_0^{n+1}u^{n+1} + S_1^{n+1}u^n + \cdots + S_n^{n+1}u$$

where

$S_i^{n+1}, (i = 0, \cdots, n)$, are Stirling's Numbers of the first kind.

For equally spaced intervals we have

$$(49.5) \quad P_i(x_i) = \prod_{j=0}^{n} (x_i - x_j)_{j \neq i} = h^n \prod_{j=0}^{n} (i - j)_{j \neq i}$$

$$= h^n i! n - i! (-1)^{n-i}$$

so that

$$(49.6) \quad \frac{P_i(x)}{P_i(x_i)} = \frac{h^n \prod_{j=0}^{n} (u - j)_{j \neq i}}{h^n i! (n - i)!} (-1)^{n-i}$$

$$= (-1)^{n-i}[i!(n - i)!]^{-1} \prod_{j=0}^{n} (u - j)_{j \neq i}$$

and

$$(49.7) \quad L(x) = L_0(u)y_0 + L(u)y_1 + \cdots + L_n(u)y_n$$

where $L_i(u)$ is given by (49.6).

Since $L_i(u)$ are functions of n and u it is possible to compute tables of these coefficients, and thus the value of $L(x)$ for a specified x can be computed by a sum of products. In order to cut down the range of values on u for such a table, a shift to a "center" value is made which changes the notation somewhat. Let us number the $n + 1$ points $x_0, x_1, \cdots, x_n$ in the following manner.

Case 1. Even number points; $n + 1 = 2r$.

In this case we could number the points

$$x_{-r}, x_{-r+1}, \cdots, x_{-2}, x_{-1}, x_1, x_2, \cdots, x_r$$

and we are faced with a choice of picking the "center" value to be either x_{-1} or x_1. The general practice is to pick x_{-1} to be the "center" point and call it x_0. The numbering then becomes

$$x_{-r+1}, x_{-r+2}, \cdots, x_{-1}, x_0, x_1, x_2, \cdots, x_r.$$

For example, if we are given 6 points, they are numbered

$$x_{-2}, x_{-1}, x_0, x_1, x_2, x_3.$$

We now have

$$x_i = x_0 + ih, \qquad (i = -r + 1, \cdots, -1, 1, \cdots, r),$$

and

$$x = x_0 + ph$$

so that

$$(49.8) \quad P_i(x) = (x - x_i)^{-1} \prod_{j=0}^{n} (x - x_j)$$

$$= h^{-1}(p - i)^{-1} \prod_{j=-r+1}^{r} (x_0 + ph - x_0 - jh)$$

$$= h^{2r-1}(p - i)^{-1} \prod_{j=-r+1}^{r} (p - j).$$

The product

$$\prod_{j=-r+1}^{r} (p - j) = (p + r - 1)(p + r - 2) \cdots (p + 1)p(p - 1) \cdots (p - r).$$

Furthermore

$$(49.9)\quad P_i(x_i) = \prod_{j=0}^{n} (x_i - x_j)_{j \neq i} = h^{2r-1} \prod_{j=r+1}^{r} (x_0 + ih - x_0 - jh)$$

$$= h^{2r-1} \prod_{j=-r+1}^{r} (i - j)_{i \neq j}$$

$$= h^{2r-1}[(i + r - 1)(i + r - 2) \cdots (i + 1)(i) \cdots (1)(-1)(-2) \cdots (r - i)]$$

$$= h^{2r-1}[(i + r - 1)!(r - i)!](-1)^{r-i}$$

and the coefficients of y_i become

$$(49.10)\quad A_i(p) = (-1)^{r-i}[(i + r - 1)!(r - i)!]^{-1} \prod_{j=-r+1}^{r} (p - j)_{j \neq i}$$

In the tabulation of these coefficients the following property is useful:

$$(49.11)\quad A_i(p) = A_{1-i}(1 - p).$$

Case 2. Odd number of points; $n + 1 = 2r + 1$.

In this case the points are numbered

$$x_{-r}, x_{-r+1}, \cdots, x_{-1}, x_0, x_1, \cdots, x_r$$

and

$$(49.12)\quad A_i(p) = (-1)^{r-i}[(i + r)!(r - i)!]^{-1} \prod_{j=-r}^{r} (p - j)_{j \neq i}$$

with the property

$$(49.13)\quad A_i(p) = A_{-i}(-p).$$

Let us consider as an example the case of 5 points, and let $p = .7$. It is required to find $A_{-2}, A_{-1}, A_0, A_1, A_2$. Substituting into formula (49.12) we obtain

$$A_{-2}(.7) = (-1)^{2+2}[(-2 + 2)!(2 + 2)!]^{-1}(.7 + 1)(.7 + 0)(.7 - 1)(.7 - 2)$$

$$= (-1)^4[(4)!]^{-1}[(1.7)(.7)(-.3)(-1.3)]$$

$$= \frac{1}{24}(.4641)$$

$$= .0193375.$$

$$A_{-1}(.7) = (-1)^{2+1}[(-1+2)!(2+1)!]^{-1}[(.7+2)(.7+0)(.7-1)(.7-2)]$$
$$= -(3!)^{-1}[(2.7)(.7)(-.3)(-1.3)]$$
$$= -.12285.$$
$$A_0(.7) = (-1)^2[(2!)(2!)]^{-1}[(.7+2)(.7+1)(.7-1)(.7-2)]$$
$$= .447525.$$
$$A_1(.7) = (-1)^1[(3!)(1!)]^{-1}(.7+2)(.7+1)(.7+0)(.7-2)$$
$$= .69615.$$
$$A_2(.7) = (-1)^0[4!(0!)]^{-1}(.7+2)(.7+1)(.7+0)(.7-1)$$
$$= -.0401625.$$

Extensive tables of these coefficients have been published by the National Bureau of Standards. A limited table for five points is given in Table VIII, p. 338.

50. Interpolation. One of the prime applications of Lagrange's formula is in interpolation. The problem is to find a value of a function at a point x which falls between tabulated values at points $x_0, x_1, \cdots$.

A. Equally Spaced Intervals. The simplest case is the one in which the tabulated values are given at equally spaced intervals; then formula (49.7) gives the desired results. Furthermore, the coefficients may be tabulated, and the problem is reduced to simply one of finding the sum of products of two numbers and is easily performed as one operation on a calculating machine. For this purpose the coefficients are tabulated about a "mid-point" and we have

$$f(x) = A_{-r}y_{-r} + A_{-r+1}y_{-r+1} + \cdots + A_0y_0 + A_1y_1 + \cdots + A_ry_r. \tag{50.1}$$

The procedure is explained by an example.

Example V.1. Find $f(1.77)$ by a 5-point Lagrangian formula from the table of values:

x_i	1.5	1.6	1.7	1.8	1.9	2.0
y_i	48.09375	65.53600	87.69705	115.47360	149.86915	192.00000
$A_i(.7)$	.0193375	−.1228500	.4475250	.6961500	−.0401625	
$A_i(-.3)$		−.0261625	.2541500	.8895250	−.1368500	.0193375

Solution. Let $x_0 = 1.7$, then $p = \frac{x - x_0}{h} = \frac{1.77 - 1.7}{.1} = .7$. The Lagrangian coefficients $A_i(.7)$ are obtained from Table VIII, p. 339, and listed above.

$$f(1.77) = \sum_{i=-2}^{2} A_i(.7)y_i = \boxed{106.49336}.$$

If we let $x_0 = 1.8$, $p = \frac{1.77 - 1.8}{.1} = -.3$ we obtain

$$f(1.77) = \sum_{i=-2}^{2} A_i(-.3)y_i = \boxed{106.49348}.$$

B. Unequally Spaced Intervals. The case of unequally spaced intervals makes the method more difficult to handle but increases its importance since fewer methods are now available. It now becomes necessary to return to formula (48.4). A schematic may be devised which greatly aids the computation. First, a linear transformation is made on the given x_i, $(i = 0, \cdots, n)$, in order to obtain small integers in so far as is possible.

Form now the square array

$$(50.2)\quad \begin{cases} x - x_0 & x_0 - x_1 & x_0 - x_2 & \cdots & x_0 - x_n \\ x_1 - x_0 & x - x_1 & x_1 - x_2 & \cdots & x_1 - x_n \\ x_2 - x_0 & x_2 - x_1 & x - x_2 & \cdots & x_2 - x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n - x_0 & x_n - x_1 & x_n - x_2 & \cdots & x - x_n \end{cases}$$

We note that the product of the principal diagonal is

$$(50.3)\quad P(x) = (x - x_0)(x - x_1) \cdots (x - x_n).$$

The products of the elements in each row yield

$$(50.4)\quad R_i = (x_i - x_0) \cdots (x - x_i) \cdots (x_i - x_n) = (x - x_i)p_i(x_i).$$

Thus

$$(50.5)\quad \frac{P_i(x)}{P_i(x_i)} = \frac{(x - x_i)^{-1}P(x)}{(x - x_i)^{-1}R_i} = \frac{P(x)}{R_i}$$

and

$$(50.6)\quad y = L(x) = \frac{P(x)}{R_0}y_0 + \frac{P(x)}{R_1}y_1 + \frac{P(x)}{R_2}y_2 + \cdots \frac{P(x)}{R_n}y_n = P(x)\left[\frac{y_0}{R_0} + \frac{y_1}{R_1} + \cdots \frac{y_n}{R_n}\right].$$

The square array (50.2) may now be augmented by the two columns R_i and $\frac{y_i}{R_i}$ (for speedy operation R_i need not be recorded). The sum of the last column multiplied by the product of the terms of the main diagonal yields the desired result.

Example V.2. Given the values

x	0	3	9	12	15	21	27
y	150	108	0	−54	−100	−144	−84

Find y at $x = 18$.

Solution. We first make the transformation $x = 3s$, then

s_i	0	1	3	4	5	7	9

and $s = 6$. The array of $s - s_i$ and $s_i - s_j$ becomes after being augmented by R_i and $\frac{y_i}{R_i}$

$s_i - s_0$	$s_i - s_1$	$s_i - s_2$	$s_i - s_3$	$s_i - s_4$	$s_i - s_5$	$s_i - s_6$	R_i	y_i/R_i
6	−1	−3	−4	−5	−7	−9	22680	.006613757
1	5	−2	−3	−4	−6	−8	−5760	−.01875
3	2	3	−1	−2	−4	−6	864	0
4	3	1	2	−1	−3	−5	−360	.150
5	4	2	1	1	−2	−4	320	−.3125
7	6	4	3	2	−1	−2	2016	−.071428571
9	8	6	5	4	2	−3	−51840	.001620370
								Sum = −.244444444

The product of terms in principal diagonal, $P(s) = 540$.

$$y = (540)(-.244444444) = -132.$$

C. Inverse Interpolation. Lagrange's formula adapts itself very nicely to inverse interpolation since formula (2) is simply a relation between two variables, either of which may be considered the independent variable. Thus, we can write x as a function of y:

$$(50.7) \quad L(y) = \frac{P_0(y)}{P_0(y_0)}\,x_0 + \frac{P_1(y)}{P_1(y_1)}\,x_1 + \cdots + \frac{P_n(y)}{P_n(y_n)}\,x_n.$$

Normally, the values of y_i will be unequally spaced and the method just described must in general be employed. Furthermore, it will be more difficult to find a linear transformation which will reduce the size of the numbers. Let us consider an example.

Example V.3. From a five-place table of natural sines we have

x	30	31	33	34	36
y	.50000	.51504	.54464	.55919	.58779

Find x when $y = .52992$.

Solution. Let $y = .1s$, then

s_i	5.0000	5.1504	5.4464	5.5919	5.8779	5.2992

and the computational form is

$s_i - s_0$	$s_i - s_1$	$s_i - s_2$	$s_i - s_3$	$s_i - s_4$	R_i	x_i/R_i
.2992	−.1504	−.4464	−.5919	−.8779	.010438234	2874.049
.1504	.1488	−.2960	−.4415	−.7275	−.002127679	−14569.867
.4464	.2960	−.1472	−.1455	−.4315	−.001221146	−27023.796
.5919	.4415	.1455	−.2927	−.2860	.003182957	10681.891
.8779	.7275	.4315	.2860	−.5787	−.045611921	−789.267
						Sum = −28826.990

$$P(x) = -.001110065;\ x = 31.9998.$$

To five-place accuracy $\boxed{x = 32}$.

D. Multiple Interpolation. Interpolation in tables of multiple arguments is a laborious procedure and is usually accomplished by repeated application of interpolation on a single variable. Repeated application of Lagrange's formula can best be illustrated by example. We shall limit our attention to equally spaced intervals; for unequally spaced intervals repeated application of the technique of Section 50.B must be employed.

Consider $y = f(x,r)$ and suppose the table of values is arranged as follows:

$x \backslash r$	r_0	r_1	r_2	r_3	r_4	r_5
x_0	y_{00}	y_{01}	y_{02}	y_{03}	y_{04}	y_{05}
x_1	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}
x_2	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}	y_{25}
x_3	y_{30}	y_{31}	y_{32}	y_{33}	y_{34}	y_{35}
x_4	y_{40}	y_{41}	y_{42}	y_{43}	y_{44}	y_{45}
x_5	y_{50}	y_{51}	y_{52}	y_{53}	y_{54}	y_{55}

Let it be desired to find a value of $y(x,r)$ where $x_2 < x < x_3$ and $r_1 < r < r_2$ by a three-point Lagrangian formula.

We obtain $p_x = \dfrac{x - x_2}{\Delta x}$ and $p_r = \dfrac{r - r_1}{\Delta r}$ and utilize the points x_1, x_2, x_3 and r_0, r_1, r_2, denoting the Lagrangian coefficient by A^x_{-1}, A^x_0, A^x_1 and A^r_{-1}, A^r_0, A^r_1. If we interpolate first with respect to x at each r_i $(i = 0,1,2)$ we have

$$(50.8)\qquad \begin{cases} y_{x0} = A^x_{-1}y_{10} + A^x_0y_{20} + A^x_1y_{30} \\ y_{x1} = A^x_{-1}y_{11} + A^x_0y_{21} + A^x_1y_{31} \\ y_{x2} = A^x_{-1}y_{12} + A^x_0y_{22} + A^x_1y_{32}. \end{cases}$$

Then interpolating with respect to r yields

$$\begin{aligned} (50.9)\qquad y(x,r) &= A^r_{-1}y_{x0} + A^r_0y_{x1} + A^r y_{x2} \\ &= A^r_{-1}[A^x_{-1}y_{10} + A^x_0y_{20} + A^x_1y_{30}] \\ &\quad + A^r_0[A^x_{-1}y_{11} + A^x_0y_{21} + A^x_1y_{31}] \\ &\quad + A^r_1[A^x_{-1}y_{12} + A^x_0y_{22} + A^x_1y_{32}] \\ &= A^r_{-1}A^x_1y_{10} + A^r_{-1}A^x_0y_{20} + A^r_{-1}A^x_1y_{30} \\ &\quad + A^r_0A^x_{-1}y_{11} + A^r_0A^x_0y_{21} + A^r_0A^x_1y_{31} \\ &\quad + A^r_1A^x_{-1}y_{12} + A^r_1A^x_0y_{22} + A^r_1A^x_1y_{32}. \end{aligned}$$

It is seen that nothing is gained by attempting to use the last expression since this would require the recording of the 9 products $A_i^r A_j^x$, $(i,j = -1,0,1)$, while using the first expression only 4 recordings are necessary. The second method would have application only in the special case where p_x and p_r remain constant for a large number of interpolations. The most expeditious method, therefore, is to compute the three y_{xi}, $(i = 0,1,2)$, and then

$$y(x,r) = A_{-1}^r y_{x0} + A_0^r y_{x1} + A_1^r y_{x2}. \tag{50.10}$$

Example V.4. Given a function of two variables, $y = f(r,\beta)$, in tabulated form:

r \ β	80°	81°	82°	83°	84°
5	.17519	.18085	.18721	.19447	.20288
6	.21023	.21702	.22466	.23336	.24346
7	.24526	.25319	.26210	.27226	.28403
8	.28030	.28936	.29954	.31115	.32461
9	.31534	.32553	.33699	.35004	.36519
10	.35038	.36170	.37443	.38894	.40576

Find $f(7.3, 81°25)$.

Solution. First hold β constant and interpolate with respect to r using the 5-point formula with $r_0 = 7$, $u = .3$, and $A_i(.3)$ from Table VIII, p. 338. We obtain

β	80°	81°	82°	83°	84°
$f(7.3,\beta)$	.255771	.264041	.273332	.283928	.296203

Now, using the 5-point formula with $\beta_0 = 82$, $u = -.75$ and $A_i(-.75)$, from Table VIII, p. 339, we get

$$f(7.3, 81°25) = \boxed{.26626}.$$

51. Differentiation. Lagrange's formula may be used to find the derivative of a function $y = f(x)$ which is known only at discrete values x_i, $(i = 0, \cdots, n)$. From formula (8) we have

$$f(x) = L(x) + R(x)$$

so that

$$\frac{dy}{dx} = f'(x) = L'(x) + R'(x). \tag{51.1}$$

Since by formula (48.11)

$$R(x) = \frac{P(x)f^{(n+1)}(\xi)}{(n+1)!}$$

$$R'(x) = \frac{P'(x)f^{(n+1)}(\xi)}{(n+1)!} + \frac{P(x)f^{(n+2)}(\xi)}{(n+1)!}. \tag{51.2}$$

The second term of this expression is difficult to find even if it is known that $f^{(n+2)}(x)$ exists. However, if it is evaluated at a given point, x_i, then $P(x_i) = 0$ and only the first term remains. In practice the remainder term is only used to check the accuracy.

The derivative of $L'(x)$ is given by

$$L'(x) = L_0'(x)y_0 + L_1'(x)y_1 + \cdots + L_n'(x)y_n \tag{51.3}$$

and

$$\begin{aligned} L_i'(x) &= \frac{1}{P_i(x_i)} \frac{d}{dx} P_i(x) \\ &= \frac{1}{P_i(x_i)} [(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots \\ &\qquad (x - x_n)] \left[\frac{1}{x - x_0} + \cdots + \frac{1}{x - x_{i-1}} \right. \\ &\qquad \left. + \frac{1}{x - x_{i+1}} + \cdots + \frac{1}{x - x_n} \right] \end{aligned} \tag{51.4}$$

or

$$\begin{aligned} L_i'(x) &= \frac{1}{P_i(x_i)} \left[\prod_{j=0}^{n} (x - x_j)_{j \neq i} \right] \left[\sum_{j=0}^{n} \frac{1}{x - x_j} \right]_{j \neq i} \\ &= \frac{P_i(x)}{P_i(x_i)} \left[\sum_{j=0}^{n} \frac{1}{x - x_j} \right]_{j \neq i}. \end{aligned} \tag{51.5}$$

Formula (51.3) gives an expression for finding the derivative of a function at a general value of x. The computation of the coefficients of y_i, $(i = 0, \cdots, n)$; namely, $L_i'(x)$, as given by formula (51.5) is somewhat difficult to manage. It can best be handled by forming the square array (50.2) and computing $\frac{P_i(x)}{P_i(x_i)}$ as was done by formula (50.5).

A. The Derivative at x_k for Unequally Spaced Intervals. Of special interest is the derivative evaluated at one of the given points, x_k, $(i = 0, \cdots, n)$. Formula (51.5) takes on two forms:

a) $k \neq i$. In this case the product $\prod_{j=0}^{n} (x - x_j)_{j \neq i} = 0$ except when it is multiplied by $(x_k - x_k)^{-1}$. Thus

(51.6)

$$L_i'(x) = \frac{1}{P_i(x_i)} \left[\prod_{j=0}^{n} (x_k - x_j)_{\substack{j \neq i \\ j \neq k}} \right]$$
$$= \frac{1}{P_i(x_i)} [(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_{i-1}) \cdots (x_k - x_n)]$$
$$= \frac{1}{P_i(x_i)} \frac{[(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)]}{x_k - x_i}$$
$$= \frac{1}{P_i(x_i)} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} (x_k - x_i)^{-1}.$$

If we let

(51.7) $$D_{ik} = (x_k - x_i) P_i(x_i), \qquad (i \neq k),$$

we have

(51.8) $$L_i'(x) = \frac{1}{D_{ik}} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k}.$$

b) $k = i$. In this case we have

(51.9) $$\Pi(x_i - x_j)_{j \neq i} = P_i(x_i)$$

and

(51.10) $$L_i'(x) = \frac{P_i(x_i)}{P_i(x_i)} \sum_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x_i - x_j}$$
$$= \sum_{j=0, \neq i}^{n} (x_i - x_j)^{-1}.$$

Thus the complete expression for $L'(x_k)$ becomes

$$\begin{aligned}
(51.11)\quad L'(x_k) &= L_0'(x_k)y_0 + \cdots + L_k'(x_k)y_k + \cdots + L_n'(x_k)y_n \\
&= \frac{y_0}{D_{0k}} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} + \frac{y_1}{D_{1k}} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} \\
&\qquad + \cdots + y_k \sum_{j=0}^{n} (x_k - x_j)^{-1}_{j \neq k} + \cdots \\
&\qquad + \frac{y_n}{D_{nk}} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} \\
&= \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} \left[\frac{y_0}{D_{0k}} + \frac{y_1}{D_{1k}} + \cdots + \frac{y_n}{D_{nk}} \right] \\
&\qquad + y_k \sum_{j=0}^{n} (x_k - x_j)^{-1}_{j \neq k}
\end{aligned}$$

or

$$\begin{aligned}
(51.12)\quad L'(x) &= \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} \left[\sum_{i=0}^{n} y_i D_{ik}^{-1} \right]_{i \neq k} \\
&\qquad + y_k \sum_{j=0}^{n} (x_k - x_j)^{-1}_{j \neq k}.
\end{aligned}$$

This formula for the derivative of $y = f(x)$ at $x = x_k$ can be found by the use of a schematic similar to the one given in 50.B. Form the square array

$$\begin{vmatrix}
 & x_0 - x_1 & x_0 - x_2 & \cdots & x_0 - x_k & \cdots & x_0 - x_n \\
x_1 - x_0 & & x_1 - x_2 & \cdots & x_1 - x_k & \cdots & x_1 - x_n \\
x_2 - x_0 & x_2 - x_1 & & \cdots & x_2 - x_k & \cdots & x_2 - x_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_k - x_0 & x_k - x_1 & x_k - x_2 & \cdots & & \cdots & x_k - x_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_n - x_0 & x_n - x_1 & x_n - x_2 & \cdots & x_n - x_k & \cdots &
\end{vmatrix}$$

Notice that no element appears on the main diagonal. Now the product of the elements of each row is $P_i(x_i)$. If this be again multiplied by $(x_i - x_k)$ we have $-D_{ik}$. Thus we can augment this array by two columns, D_{ik} and $y_i D_{ik}^{-1}$; the first being obtained by multiplying the product of each row by the element in the kth column (i.e., this element twice in the product) and changing sign. There will be no entry in the $(i = k)$ row. The product $\prod_{j=0}^{n} (x_k - x_j)_{j \neq k}$ $= \prod_{j=0}^{n} [-(x_j - x_k)]_{j \neq k}$ is obtained by multiplying together the negative of the elements in the kth column.

The elements of the second augmented column are now summed and multiplied by this product to yield the first term of the formula (51.12). The second term is obtained by summing the negative reciprocals of the terms in the kth column as can be easily seen from its expression. The final value of the derivative is then given by formula (51.12). Let us illustrate the procedure with an example.

Example V.5. Let us consider the example V.2 given in 50.B, and let it be desired to obtain the derivative at $s_i = 5$; i.e., $x = 12$.

Solution.

$s_i - s_0$	$s_i - s_1$	$s_i - s_2$	$s_i - s_3$	$s_i - s_4$	$s_i - s_5$	$s_i - s_6$	D_{ii}	$y_i D_{ii}^{-1}$
	−1	−3	−4	−5	−7	−9	18900	.007936507
1		−2	−3	−4	−6	−8	−4608	−.023437500
3	2		−1	−2	−4	−6	576	0
4	3	1		−1	−3	−5	−180	.300000000
5	4	2	1		−2	−4		
7	6	4	3	2		−2	4032	−.035714285
9	8	6	5	4	2		−69120	.001215278
			$\pi(s_4 - s_i) = 320.$					.250

$$\sum (s_4 - s_i)^{-1} = \frac{1}{5} + \frac{1}{4} + \frac{1}{2} + \frac{1}{1} - \frac{1}{2} - \frac{1}{4} = \frac{1}{5} + 1 = \frac{6}{5} = 1.2000$$

$$L'(x) = (320)(.25) + (1.2)(-100) = \boxed{-40}.$$

B. The Derivative at x_k for Equally Spaced Intervals. If the values of the independent variable are given at equally spaced intervals, we have from equation (49.7)

(51.13) $$L(x) = L_0(u)y_0 + L_1(u)y_1 + \cdots + L_n(u)y_n$$

where

(51.14) $$L_i(u) = (-1)^{n-i}[i!(n-i)!]^{-1} \prod_{j=0}^{n} (u-j)_{j\neq i}.$$

Since $x = uh + x_0$ we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

and

(51.15) $$y' \doteq L'(x) = \frac{1}{h}\sum_{i=0}^{n} L_i'(u)y_i$$

and

(51.16) $$L_i'(u) = (-1)^{n-i}[i!(n-i)!]^{-1} \frac{d}{du}\prod_{j=0}^{n} (u-j)_{j\neq i}.$$

Furthermore [see formula (51.5)]

(51.17) $$\frac{d}{du}\prod_{j=0}^{n} (u-j)_{j\neq i} = \left[\prod_{j=0}^{n} (u-j)_{j\neq i}\right]\left[\sum_{j=0}^{n} (u-j)^{-1}_{j\neq i}\right].$$

When $x = x_k$ we have

(51.18) $$u = \frac{x - x_0}{h} = \frac{x_k - x_0}{h} = k$$

so that at this point formula (51.17) takes on two special forms:

a) $k \neq i$. In this case we have

(51.19)

$$\begin{aligned}\frac{d}{du}\prod_{j=0}^{n} (u-j)_{j\neq i} &= [u(u-1)\cdots(u-k+1)(u-k-1)\cdots \\ &\qquad (u-i+1)(u-i-1)\cdots(u-n)] \\ &= k(k-1)\cdots 1(-1)\cdots \\ &\qquad (k-i+1)(k-i-1)\cdots(k-n) \\ &= (k-i)^{-1}k!(n-k)!(-1)^{n-k}\end{aligned}$$

and

$$\text{(51.20)}\quad L_i'(u) = (-1)^{n-i}[i!(n-i)!]^{-1}(k-i)^{-1}k!(n-k)!(-1)^{n-k}$$
$$= (-1)^{i+k}\,\frac{k!(n-k)!}{i!(n-i)!(k-i)}.$$

Note: $\quad (-1)^{2n-i-k} = (-1)^{2n}(-1)^{-(i+k)} = (-1)^{i+k}.$

b) $k = i$. For this case we have

(51.21)

$$\frac{d}{du}\prod_{j=0}^{n}(u-j)_{j\neq i} = [i(i-1)\cdots(2)(1)(-1)(-2)\cdots(i-n)]$$
$$\left[\frac{1}{i}+\frac{1}{i-1}+\cdots\frac{1}{1}-\frac{1}{1}+\cdots+\frac{1}{i-n}\right]$$
$$= i!(n-i)!(-1)^{n-i}\sum_{j=0}^{n}\frac{1}{i-j}\Bigg|_{j\neq i}$$

so that

$$\text{(51.22)}\quad L_i'(u) = (-1)^{n-i}\frac{i!(n-i)!(-1)^{n-i}}{i!(n-i)!}\left[\sum_{j=0}^{n}\frac{1}{i-j}\right]_{j\neq i}$$
$$= \frac{1}{i}+\frac{1}{i-1}+\cdots+\frac{1}{2}+\frac{1}{1}-\frac{1}{1}-\frac{1}{2}-\cdots-\frac{1}{n-i}$$
$$= \sum_{j=1}^{i}\frac{1}{j}-\sum_{j=1}^{n-i}\frac{1}{j}$$
$$= -\sum_{j=i+1}^{n-i}\frac{1}{j}\quad\text{if}\quad 2i\le n-1,$$
$$= \sum_{j=n-i+1}^{i}\frac{1}{j}\quad\text{if}\quad 2i> n-1.$$

Thus the coefficients of y_i, $(i = 0, \cdots, n)$, are all functions of n, k, and i, and may be computed once and for all, and tabulated. Furthermore, the table of coefficients is "negatively symmetric" about the "mid-point" so that it is necessary to tabulate only half of them. Let the coefficients be denoted by A'_{ki}, then for equally spaced intervals

$$(51.23) \quad y'(x_k) = L'(x_k) = \frac{1}{h}[A'_{k0}y_0 + A'_{k1}y_1 + \cdots + A'_{kn}y_n],$$

and A'_{ki} are given by formulas (51.20) and (51.22).

Now for $k > m$ where $m = \frac{1}{2}n + 1$ if n is even and $m = \frac{1}{2}(n + 1)$ if n is odd, we have

$$A'_{ki} = -A'_{n-k,n-i}.$$

By formula (51.20) we have for $k \neq i$

$$A'_{ki} = (-1)^{i+k}\frac{k!(n-k)!}{i!(n-i)!(k-i)}$$

and

$$A'_{n-k,n-i} = (-1)^{(n-k)+(n-i)}\frac{(n-k)![n(nk)]!}{(n-i)![n-(n-i)]![n-k-(n-i)]}$$

$$= (-1)^{k+i}\frac{(n-k)!(k!)}{(n-i)!(i)!(i-k)}$$

$$= (-1)^{k+i-1}\frac{(n-k)!k!}{(n-i)!(i)!(k-i)},$$

thus

$$A'_{ki} = -A'_{n-k,n-i}.$$

For $k = i$ we have by formula (51.22)

$$A'_{kk} = \sum_{j=n-k+1}^{k}\frac{1}{j} \quad \text{and} \quad A'_{n-k,n-k} = -\sum_{j=n-k+1}^{n-n+k}\frac{1}{j}$$

so that

$$A'_{kk} = -A'_{n-k,n-k}.$$

The tabulation of A'_{ki} can be made in two ways. The first is to obtain the lowest common denominator for the A'_{ki} for each n and tabulate the coefficients of y_i as integers with a final division to obtain the derivative. Thus

$$y'(x_k) = \frac{1}{hD_n} \sum_{i=0}^{n} a'_{ki} y_i$$

where

$$a'_{ki} = A'_{ki} D_n$$

and D_n is the lowest common denominator of A'_{ki}. Table IX, p. 342, gives the values of D_n and a'_{ki} for 10 points. The second method is to publish A'_{ki} in decimal form (*see* Table X, p. 344).

Example V.6.

Find $\frac{dy}{dx}$ at $x = 1.7$ in the data given in Example V.1.

Solution. Here $u = \frac{1.7 - 1.5}{.1} = 2 = k$; $n = 5$; $h = .1$.

From Tables IX, p. 342 ($D_n = 60$) and X, p. 344, we have

y_i	48.09375	65.53600	87.69705	115.47360	149.86915	192.00000
a'_{2i}	3	−30	−20	60	−15	2
A'_{2i}	.050000	−.500000	−.333333	1.000000	−.250000	.033333

$$y'_2 = 10 \frac{\left[\sum_{i=0}^{5} a'_{2i} y_i\right]}{60} = \boxed{248.10650}.$$

$$y'_2 = 10 \left[\sum_{i=0}^{5} A'_{2i} y_i\right] = \boxed{248.10652}.$$

The formulas derived in this section may be used as the basis for obtaining methods of solving differential equations. This will be discussed in Chapter VII.

52. Higher Derivatives. The higher derivatives are obtained by repeated differentiation of equation (51.3). Thus

(52.1) $$L''(x) = L''_0(x)y_0 + L''_1(x)y_1 + \cdots + L''_n(x)y_n$$

where

(52.2) $$L_i''(x) = \frac{1}{P_i(x_i)} \frac{d^2}{dx^2} [P_i(x)].$$

Although a notation and schematic can be devised for the general case it is very unwieldy. Consequently, we shall develop only the special case of equally spaced intervals. From equation (51.15) we have

(52.3) $$y'(x) = \frac{1}{h} \sum_{i=0}^{n} L_i'(u) y_i$$

so that

(52.4) $$y''(x) = \frac{1}{h^2} \sum_{i=0}^{n} L_i''(u) y_i.$$

Now

$$L_i(u) = (-1)^{n-i} \frac{n!}{i!(n-i)!} (u - i) \binom{u}{n}$$

where $\binom{u}{n}$ is the binomial coefficient notation.

Let

(52.5) $$b_i = \frac{(-1)^{n-i}}{i!(n-i)!}$$

then $$L_i'(u) = b_i \frac{d}{du} \left[\prod_{j=0}^{n} (u - j)_{j \neq i} \right],$$

and

(52.6) $$L_i''(u) = b_i \frac{d^2}{du^2} \left[\prod_{j=0}^{n} (u - j)_{j \neq i} \right].$$

These derivatives may now be evaluated at $u = 0, 1, 2, 3, \cdots, n$ to yield the values of the derivatives at $x = x_0, x_1, \cdots, x_n$. Thus we can write

(52.7) $$y''(x_k) = h''(x_k) = \frac{1}{h^2} [A_{k0}'' y_0 + A_{k1}'' y_1 + A_{k2}'' y_2 + \cdots + A_{kn}'' y_n]$$

where

$$A_{ki}'' = L_i''(k)$$

which may be computed once and for all and have been tabulated in Table XI, p. 346.

The continuation to higher derivatives is immediate. The third derivative is given by

$$(52.8) \quad y'''(x_k) = L'''(x_k) = \frac{1}{h^3}[A'''_{k0}y_0 + A'''_{k1}y_1 + \cdots + A'''_{kn}y_n]$$

and the values for A'''_{ki} have been tabulated in Table XII, p. 347.

Example V.7. Find the first, second, and third derivatives at $x = 1$ of the function whose values are

x	-2	-1	0	1	2	3	4
y	104	17	0	-1	8	69	272

Solution. Using Tables IX, XI, XII, at the back of the book, with $n = 6$, $h = 1$, $x_3 = 1$, $u = 3$ we write

$$y'(1) = \frac{1}{60}[-1(104) + 9(17) - 45(0) + 0(-1) + 45(8) - 9(69) + 1(272)]$$

$$= \boxed{1}.$$

$$y''(1) = \frac{1}{180}[2(104) - 27(17) + 270(0) - 490(-1) + 270(8) - 27(69) + 2(272)]$$

$$= \boxed{6}.$$

$$y'''(1) = \frac{1}{24}[3(104) - 24(17) + 39(0) + 0(-1) - 39(8) + 24(69) - 3(272)]$$

$$= \boxed{18}.$$

53. Integration. As in the case of differentiation Lagrange's formula may also be used to find the integral of a function $f(x)$ which is known only at discrete values x_i, $(i = 0, \cdots, n)$. The integral of $L(x)$ will be a good approximation of the integral of $f(x)$ so that we may write

$$(53.1) \quad \int_a^b f(x)dx = a_0y_0 + a_1y_1 + \cdots + a_ny_n + R$$

where

$$(53.2)\qquad a_i = \frac{1}{P_i(x_i)} \int_a^b P_i(x)dx, \qquad (i = 0, \cdots, n),$$

and

$$(53.3)\qquad R = \frac{1}{(m+1)!} \int_a^b f^{(m+1)}(\xi)P(x)dx.$$

The problem of finding the integral of $f(x)$ then reduces to the evaluation of the coefficients a_i and the sum of products. The coefficients a_i depend upon the $n + 1$ points, x_i, and the limits of integration (a,b). If we write $P_i(x)$ as a polynomial in descending powers of x

$$(53.4)\qquad P_i(x) = x^n + a_{i,n-1}x^{n-1} + \cdots + a_{i,1}x + a_{i,0}$$

the integral may be written in the form

$$(53.5)\qquad \int P_i(x)dx = b_{i,n+1}x^{n+1} + b_{i,n}x^n + \cdots + b_{i,1}x + b_{i,0}$$

where

$$(53.6)\qquad \begin{cases} b_{i,n+1} = \dfrac{1}{n+1}, & (i = 0, \cdots, n), \\ b_{i,j} = \dfrac{1}{j}(a_{i,j-1}), & (i = 0, \cdots, n; j = 1, \cdots, n), \\ b_{i0} = \text{constants of integration.} \end{cases}$$

The coefficients b_{ij} may be replaced in a tabular form and augmented by a row of the constants $P_i(x_i)$. It is not necessary to write b_{i0} since they will disappear when the limits of integration are inserted. Thus

x^{n+1}	x^n	x^{n-1}	$\cdots$	x^2	x	$P_i(x_i)$
$b_{0,n+1}$	$b_{0,n}$	$b_{0,n-1}$		$b_{0,2}$	$b_{0,1}$	$P_0(x_0)$
$\cdots$	$\cdots$	$\cdots$		$\cdots$	$\cdots$	$\cdots$
$b_{n,n+1}$	$b_{n,n}$	$b_{n,n-1}$		$b_{n,2}$	$b_{n,1}$	$P_n(x_n)$

There remains now to consider the range of integration. If this range changes so that it is desired to have many integrals, the values for α_i may be tabulated in the following manner:

i	$x_0 - x_1$	$x_0 - x_2$	$\cdots$	$x_1 - x_5$	$\cdots$	$x_3 - x_9$	$\cdots$
0	α_0	α_0	$\cdots$	α_0	$\cdots$	α_0	$\cdots$
1	α_1	α_1	$\cdots$	α_1	$\cdots$	α_1	$\cdots$
2	α_2	α_2	$\cdots$	α_2	$\cdots$	α_2	$\cdots$
.	.	.		.		.	
.	.	.	$\cdots$	.	$\cdots$	.	$\cdots$
.	.	.		.		.	
n	α_n	α_n	$\cdots$	α_n	$\cdots$	α_n	$\cdots$

For the case of equally spaced intervals

$$(53.7)\quad P_i(x) = P_i(u) = u(u-1)\cdots(u-i+1)(u-i-1)\cdots(u-n)$$

$$= \sum_{j=0}^{n} a_{i,j}u^j$$

and since $dx = h\,du$

$$(53.8)\qquad \int P_i(x)dx = h\int P_i(u)du.$$

The values of a_{ij} may be found easily by use of Stirling's numbers of the first kind. Consider for example the case of 4 points:

$$\begin{cases} P_0(u) = (u-1)(u-2)(u-3) = u^3 - 6u^2 + 11u - 6, \\ P_1(u) = u(u-2)(u-3) = u^3 - 5u^2 + 6u, \\ P_2(u) = u(u-1)(u-3) = u^3 - 4u^2 + 3u, \\ P_3(u) = u(u-1)(u-2) = u^3 - 3u^2 + 2u. \end{cases}$$

If we note that

$$(53.9)\qquad P_i(u) = \frac{P(u)}{u-i}$$

the polynomial $P(u)$ has for its coefficients the Stirling numbers of the first kind and from which we remove the factor $(u - i)$. This can be done by a synthetic division and easily put into a schematic.

The removal of $(u - 0)$ does not change the numbers so that the first polynomial has the Stirling numbers of the first kind as coefficients.

	u^3	u^2	u	u^0	DIVISION
a_{0j}	1	-6	11	-6	0
		1	-5		
a_{1j}	1	-5	6		1
		2	-8		
a_{2j}	1	-4	3		2
		$+3$	-9		
a_{3j}	1	-3	2		3

The coefficients b_{ij} are now easily written down

	u^4	u^3	u^2	u	$P_i(u_i)$	L.C.D.
b_{0j}	$\frac{1}{4}$	$-\frac{6}{3}$	$\frac{11}{2}$	$-\frac{6}{1}$	-6	12
b_{1j}	$\frac{1}{4}$	$-\frac{5}{3}$	$\frac{6}{2}$		2	12
b_{2j}	$\frac{1}{4}$	$-\frac{4}{3}$	$\frac{3}{2}$		-2	12
b_{3j}	$\frac{1}{4}$	$-\frac{3}{3}$	$\frac{2}{2}$		6	12

$P_i(u_i)$ are easily found by the formula

$$P_i(u_i) = i!(n - i)!(-1)^{n-i} \tag{53.10}$$

and realizing that $P_0(u_0) = a_{00}$ and then they alternate in sign. A column for the lowest common denominator (L.C.D.) is attached in case it is desired to factor out the denominator; it is to be recalled that $P_i(u_i)$ is a denominator.

Let us now consider a specific integral, say

$$\int_0^2 f(x)dx$$

for 4 points. We have

$$\begin{aligned}
\alpha_0 &= \frac{h}{P_0(u_0)} \int^2 P(u) \\
&= \frac{h}{P_0(u_0)} [b_{04}u^4 + b_{03}u^3 + b_{02}u^2 + b_{01}u]_0^2 \\
&= \frac{h}{-6} \left[\frac{1}{4}(2)^4 - \frac{6}{3}(2)^3 + \frac{11}{2}(2)^2 - 6(2)\right] \\
&= -\frac{h}{6}\left[\frac{1}{12}\right] [(3)(2)^4 - 24(2)^3 + 66(2)^2 - 72(2)] \\
&= -\frac{h}{72}[48 - 192 + 264 - 144] \\
&= -\frac{h}{72}(-24) \\
&= \frac{2}{6}h.
\end{aligned}$$

The schematic eases the computation

	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	2	L.C.D.	$P_i(u_i)$	$\alpha_i h^{-1}$
b_{0j}	3	−24	66	−72	12	−6	$\frac{1}{3}$
b_{1j}	3	−20	36		12	2	$\frac{4}{3}$
b_{2j}	3	−16	18		12	−2	$+\frac{1}{3}$
b_{3j}	3	−12	12		12	6	0

$$\int_0^2 f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2 + 0y_3].$$

For equally spaced intervals the values of α_i can be tabulated for the range of integration 0–1, 0–2, etc. The value of an integral between other integral limits can be found by recalling that

$$\int_2^4 f(x)dx = \int_0^4 f(x)dx - \int_0^2 f(x)dx.$$

Values of α_i are given in Table XIII, p. 348, and their use can be illustrated by an example.*

Example V.8. Find the $\int_{1.5}^{1.8} ydx$ given the table

x_i	1.5	1.6	1.7	1.8	1.9	2.0
y_i	48.09375	65.53600	87.69705	115.47360	149.86915	192

Solution.

$$\int_{1.5}^{1.8} f(x)dx = h\int_0^3 f(u)du$$

$$= h\left[\sum_{i=0}^{5} \alpha_i(0\text{–}3)y_i\right].$$

From Table XIII, p. 349, for six points in the interval 0–3 we have

α_i	.31875	1.36875	.71250	.71250	−.13125	.01875

and since $h = .1$

$$\int_{.5}^{1.8} ydx = .1(233.721045) = 23.372104.$$

54. Accuracy of Lagrangian Formulas. All of the formulas derived in this chapter are based on the assumption that the given data can be fitted by a polynomial in the region under discussion, and this initial assumption should frequently be checked. The inherent error in the Lagrangian formulas is exhibited in the remainder term of equation (48.12)

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}P(x)$$

* Values of α_i may also be computed around a "central value." See *Journal of Mathematics and Physics* (1945), Vol. 24, No. 1, pp. 1–21.

where ξ is an arbitrary point in the interval $(0,n)$. This remainder term may be carried throughout all of the derivations of the formulas and is a measure of their accuracy. Thus, in formula (51.23) for the derivative there should be added the remainder

$$\frac{h^n}{c}\, y^{(n+1)}.$$

For the integrals we have the expression (53.3).

The accuracy should always be tested whenever possible.

55. Lagrangian Multiplier. Although the Lagrangian multiplier is not directly associated with the Lagrangian formulas discussed thus far, it still seems appropriate to include it here because it is concerned with the finding of an extremum of a function which is constrained by some additional condition. In the principle of least squares which is so often used in numerical analysis, we are concerned with the problem of minimizing the sum of the squares of the residuals of the analytical expression and the values found by substitution of the known points. If now an additional constraint is imposed upon the parameters in the analytical expression, the problem can be solved by use of the Lagrangian multiplier.

Let us consider a function of two variables, $f(x,y)$, and let it be desired to find the maximum and minimum values of this function if, furthermore, x and y are related by $g(x,y) = 0$.

The necessary condition for an extremum is that the total differential vanishes. Thus, for the function $f(x,y)$

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial g}\, dy = 0. \tag{55.1}$$

The total differential of the constrained relation is

$$\frac{\partial g}{\partial x}\, dx + \frac{\partial g}{\partial g}\, dy = 0. \tag{55.2}$$

If now equation (55.2) is multiplied by an undetermined multiplier, λ, and added to equation (55.1), we have

$$\left(\frac{\partial f}{\partial x} + \lambda\,\frac{\partial g}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda\,\frac{\partial g}{\partial y}\right) dy = 0. \tag{55.3}$$

Equation (55.3) would be satisfied if λ were determined so that

$$(55.4)\qquad \begin{cases} \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial g}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial g}{\partial y} = 0 \\ g(x,y) = 0. \end{cases}$$

The multiplier λ is called the **Lagrangian multiplier.**

Example V.9. Find the dimensions of a rectangular box, without a top, having maximum volume and whose surface is 66 square inches.

Solution. Let the dimensions of the box be x, y, and z. The volume function to be maximized is

$$V(x,y,z) = xyz$$

subject to the constraint

$$xy + 2xz + 2yz = 66.$$

The system (55.4) for this problem is

$$\begin{cases} yz + \lambda(y + 2z) = 0 \\ xz + \lambda(x + 2z) = 0 \\ xy + \lambda(2x + 2y) = 0 \\ xy + 2xz + 2yz = 66. \end{cases}$$

Multiplying the first equation by x, the second by y, the third by z, and adding, we obtain

$$3xyz + 2\lambda(xy + 2xz + 2yz) = 0.$$

Substituting the last equation into this result we obtain

$$3xyz + 132\lambda = 0 \qquad \text{or} \qquad \lambda = -\frac{xyz}{44}.$$

The first three equations of the system now take the form

$$\begin{cases} \dfrac{yz}{44}(44 - xy - 2xz) = 0 \\ \dfrac{xz}{44}(44 - xy - 2yz) = 0 \\ \dfrac{xy}{44}(44 - 2xz - 2yz) = 0. \end{cases}$$

The solutions are $x = y = \sqrt{22}$, $z = \dfrac{11}{\sqrt{22}}$.

Another example will be discussed in Chapter VIII.

56. Exercise VII.

In the following exercises employ the Lagrangian Formulas.

1. In the data of Exercise III.3(a) find y at $x = .55$ and $x = .72$.
2. In the data of Exercise III.3(b) find y at $x = .82$, 1.15, and 1.32.
3. Solve Example III.12 using Lagrange's formula.
4. Apply Lagrange's formula to the last step of Example III.15.
5. Solve problem 1(a) at $x = .7$ and 1(b) at $x = 1.91$ of Exercise IV.
6. Solve problem 1 of Exercise V by the methods of this chapter.
7. Solve problem 2 of Exercise V by the methods of this chapter.
8. Solve problem 1 of Exercise VI by the methods of this chapter.
9. Divide the interval (0,1) into ten equal parts and find $\int_0^1 x\,dx$ by the method of this chapter. Compare answer with that found in problem 3 of Exercise VI.
10. Find the dimensions of a closed cylinder having a maximum volume and a total surface of 20π square inches.

CHAPTER VI. ORDINARY EQUATIONS AND SYSTEMS

57. Introduction. The solution of algebraic equations is thoroughly discussed in a course in algebra. There, however, one is primarily concerned with the existence of solutions and expressing the solution in closed form which may be very difficult to evaluate. One numerical method for finding the roots of an algebraic equation is Horner's method which is discussed in most textbooks on algebra. Very little, if anything, is said about transcendental equations.

It is the purpose of this chapter to present methods for finding the roots of any ordinary equation with numerical coefficients. Systems of such equations will also be discussed.

58. Some Fundamental Concepts. Before proceeding to the detailed methods it will be useful to review some of the fundamental concepts from algebra. The existence of roots of an algebraic equation is based on a number of theorems all of which are quite important. Since they are adequately developed in a number of books, it is only necessary to list them here. Let us consider a polynomial $y = f(x)$ of the nth degree. If this is set equal to zero, $f(x) = 0$, we have an equation and the **solutions** of this equation are called the **roots** of the equation or the **zeros** of the function $f(x)$. If n is a positive integer the equation $f(x) = 0$ is said to be a **rational integral equation.** We now state some important theorems.

Remainder Theorem. If a polynomial $f(x)$ is divided by $(x - r)$ until a remainder independent of x is obtained, this remainder is equal to $f(r)$.

Factor Theorem. If r is a zero of the polynomial $f(x)$, then $(x - r)$ is a factor of $f(x)$.

Fundamental Theorem of Algebra. Every rational integral equation, $f(x) = 0$, of the nth degree has n and only n roots.

Rational Roots. If a rational number, $\frac{b}{c}$, a fraction in its lowest form, is a root of the rational integral equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

with integral coefficients, then b is a factor of a_n and c is a factor of a_0.

Complex Roots. Complex roots occur in conjugate pairs, i.e., $r_1 = a + bi$ and $r_2 = a - bi$. The same is true for a quadratic surd.

Reducing the Degree of the Equation. Once a root of an equation has been found, it may be "divided out" leaving a quotient of degree one less than the original equation. Further investigation may then be applied to this quotient.

Descartes' Rule of Signs. The number of positive roots of an equation $f(x) = 0$ with real coefficients cannot exceed the number of variations in sign in the polynomial $f(x)$, and the number of negative roots cannot exceed the number of variations in sign in $f(-x)$.

Location Principle. If $f(x)$ is continuous from $x = a$ to $x = b$ and if $f(a)$ and $f(b)$

i) have opposite signs, then $f(x) = 0$ has an odd number of roots between a and b.

ii) have like signs, then $f(x) = 0$ either has no roots or an even number of roots between a and b.

Rules of Transformation

I. To form an equation each of whose roots is k times the corresponding roots of the given equation, multiply the coefficients a_0 by 1, a_1 by k, a_2 by k^2, $\cdots$ a_i by k^i, $\cdots$, a_n by k^n.

II. To form an equation each of whose roots is equal to the *negative* of the corresponding roots of the given equation, change the signs of the *odd-degree* terms.

III. To form an equation each of whose roots is less by h than a corresponding root of a given equation, divide $f(x)$ and each successive resulting quotient by $(x - h)$ until n divisions have been performed. The remainders obtained in each division form the coefficients of the new equation in ascending order.

Roots' and Coefficients' Relationship. In any rational integral equation of the nth degree

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

$\frac{a_1}{a_0} = -$ (the sum of the roots)

$\frac{a_2}{a_0} =$ the sum of the products of the roots taken 2 at a time

$\frac{a_3}{a_0} = -$ (the sum of the products of the roots taken 3 at a time)

. . . .

$\frac{a_n}{a_0} = (-1)^n$ (the product of all the roots).

Special Transformations

EQUATION	TRANSFORMED TO	BY
$x^3 + bx^2 + cx + d = 0$	$y^3 + py + q = 0$	$x = y - \frac{1}{3}b$
$x^4 + bx^3 + cx^2 + dx + e = 0$	$y^4 + qy^2 + ry + s = 0$	$x = y - \frac{1}{4}b$

Splitting the Equation. The roots of an equation may be found by obtaining the abscissas of the points of intersection of two functions into which the original function has been decomposed. For example, the solution of ($x - \cos x = 0$) is the value of x for which $y_1 = x$ intersects $y_2 = \cos x$.

Graphing. Many clues to the solutions of equations may be obtained from the plots of the functions involved, and the value of a graph cannot be overemphasized.

59. One Equation in One Unknown. Most numerical methods for the solution of ordinary equations are based on the method of successive approximations. The problem is then one of finding a recurrence relation which permits the calculation of a sequence of numbers $x_0, x_1, x_2, \ldots$ which converge to the desired root, r. For the equation

$$y = f(x) = 0 \tag{59.1}$$

the common recurrence relation is

$$x_{i+1} = x_i - \frac{f(x_i)}{g(x_i)} \tag{59.2}$$

in which $g(x_i)$ may be the slope of an appropriate line. The correct value for $g(x_i)$ would be the slope of the line from (x_i,y_i) to $(r,0)$ for then

$$g(x_i) = \frac{y_i - 0}{x_i - r}$$

and

$$x_{i+1} = x_i - \frac{y_i}{(y_i)/(x_i - r)} = \frac{[y_i x_i - y_i(x_i - r)]}{y_i} = r.$$

The most classical form of $g(x_i)$ is the value of the derivative of $f(x)$ at x_i, $f'(x_i)$. Formula (59.2) then becomes

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{59.3}$$

and the method is known as the **Newton-Raphson method** or more popularly as simply Newton's method.

Other common forms of $g(x_i)$ are

a) slope of the chord joining two points already calculated

$$g(x_i) = \frac{y_k - y_i}{x_k - x_i}; \tag{59.4}$$

b) constant slope between two points (x_1, y_1) and (x_2, y_2) for which $f(x_1)$ and $f(x_2)$ have unlike signs

$$g(x_i) = \frac{y_1 - y_2}{x_1 - x_2}; \tag{59.5}$$

c) Constant value of the derivative at a chosen point, x_0,

$$g(x_i) = f'(x_0);$$

d) An arithmetic mean between (b) and (c)

$$g(x_i) = \frac{1}{2}\left[f'(x_0) + \frac{y_1 - y_2}{x_1 - x_2}\right]. \tag{59.6}$$

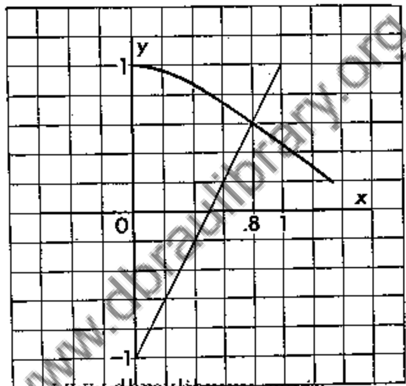
The most popular method is the Newton-Raphson method provided it is not too difficult to calculate the value of the derivative.

Example VI.1. Find the real root of

$$2x - \cos x - 1 = 0.$$

Solution. Find the first approximation by locating graphically the point of intersection of

$$\begin{cases} y_1 = 2x - 1 \\ y_2 = \cos x. \end{cases}$$



Now

$$f(x) = 2x - \cos x - 1$$
$$f'(x) = 2 + \sin x.$$

Then

i	x_i	$f(x_i)$	$f'(x_i)$
0	.8	−.096702	2.717356
1	.835588	.000443	2.741699
2	.835427	.000067	2.741583
3	.835403		

Example VI.2. Find the root between 2 and 3 of the equation

$$x^4 - x^3 - 2x^2 - 6x - 4 = 0.$$

Solution. We write the function and its derivative in their nested form.

$$\begin{cases} f(x) = \{[(x-1)x - 2]x - 6\}x - 4. \\ f'(x) = [(4x - 3)x - 4]x - 6. \end{cases}$$

Since the desired root is between 2 and 3 let us arbitrarily choose $x_0 = 2.5$.

i	x_i	$f(x_i)$	$f'(x_i)$
0	2.5	−8.0625	27.75
1	2.790540	2.591414	46.397506
2	2.734688	.111655	42.431352
3	2.732057	.000259	42.249140
4	2.732051		

Note: In the computation the values of $f'(x_i)$ are computed before those of $f(x_i)$.

Example VI.3. Find the root between 0 and 1 of the equation

$$x^{1.84} - 5.2211x + 2.0123 = 0.$$

Solution. In this case let us take a constant value of $g(x_i) = \frac{y_0 - y_1}{x_0 - x_1}$ choosing for the two points the values (0, 2.0123) and (1, −2.2088). Thus

$$g(x) = \frac{2.0123 + 2.2088}{0 - 1} = -4.2211.$$

Since the absolute values of y_0 and y_1 are nearly equal, we choose $x_0 = 0.5$.

i	x_i	$\log x_i$	$f(x_i)$	$g(x)$
0	.5	−.30103	−.31892	−4.2211
1	.424447	−.37217	.002860	
2	.425124	−.371486	−.000075	
3	.425107			

Thus $r =$ .42511. Check: $f(r) = -.00002$.
Greater improvement can only be achieved by using more extensive logarithmic tables.

It is noted that formula (59.2) fails when $g(x_i) = 0$. For the above expressions of $g(x_i)$ this will occur at or near the root of $f(x) = 0$ when the slope of the tangent line is nearly horizontal. The procedure then is to solve for the root of the derivative equation $f'(x) = 0$ and denote it by $x = a$. Then we have the following three possibilities:

(1) If $f(a) = 0$, then a is a double root.

(2) If $f(a) \neq 0$ and $f(a)$ and $f''(a)$ have like signs, then there is no root of $f(x) = 0$ in the neighborhood of $x = a$.

(3) If $f(a) \neq 0$ and $f(a)$ and $f''(a)$ have unlike signs, then there should be two roots of $f(x) = 0$ in the neighborhood of a, one less than a and one greater than a.

To find the two roots in case (3) above, consider the Taylor series expansion of $f(x)$ about $x = a$; i.e.,

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \cdots.$$

If we neglect all terms higher than the second and remember that $(x) = f'(a) = 0$, we have

$$f(a) + \frac{1}{2}f''(a)(x - a)^2 = 0$$

or

$$x^2 - 2ax + a^2 + 2\frac{f(a)}{f''(a)} = 0$$

and upon solving this quadratic for x we obtain a first approximation to the two roots

(59.7) $$x = a \pm \sqrt{-2f(a)/f''(a)}.$$

The two values may be improved upon by successive approximations using any of the above formulas for $g(x_i)$.

Example VI.4. Find the small positive roots of

$$f(x) = .054x^3 + 89.973x^2 - 60x + 10 = 0.$$

Solution. Tabulate the values of the function at small values of x.

x	0	.5	1
$f(x)$	10	2.5	40.027

From these values it appears that there is a possibility of an even number of roots in the neighborhood of $x < .5$. It is therefore desirable to solve first the derivative equation $f'(x) = 0$ for its roots. We have

$$f'(x) = .162x^2 + 179.946x - 60 = 0.$$

Since this is a quadratic equation we may solve it by using the quadratic formula. Had it been a higher degree equation we could use the Newton-Raphson method. We find

$$x = \frac{-179.946 \pm \sqrt{(179.946)^2 - 4(.162)(60)}}{2(.162)}$$
$$= \frac{-179.946 \pm 180.054}{.324}$$
$$x(>0) = \frac{.108}{.324} = \frac{1}{3} = .333333.$$

The second derivative is

$$f''(x) = .324x + 179.946$$

and at $x = \frac{1}{3}$ we have

$$f\left(\frac{1}{3}\right) = -.001000 \quad \text{and} \quad f''(x) = 180.054.$$

Since these have unlike signs there should be two roots in the neighborhood of $x = \frac{1}{3}$. Let us obtain the first approximations by formula (59.7).

$$x = .333333 \pm \sqrt{\frac{.002}{180.054}}$$
$$= .333333 \pm .003317$$
$$x_1 = .330016 \quad \text{and} \quad x_2 = .336650.$$

We can improve upon these roots by Newton's method.

	x_1			x_2		
i	x_i	$f(x_i)$	$f'(x_i)$	x_i	$f(x_i)$	$f'(x_i)$
0	.330016	−.000009	−.597297	.336650	−.000009	.597181
1	.330001	−.00000017	−.599998	.336665	−.00000056	.599882
2	.3300007			.3366659		

Thus we have

$$x_1 = \boxed{.330001} \qquad \text{and} \qquad x_2 = \boxed{.336666}$$

to six decimals.

60. The Iteration Method. Another method of solving an equation in one unknown is the method of iteration. This method is applicable when the equation $f(x) = 0$ can be solved for the unknown x in terms of a function of x, i.e., it is possible to write

(60.1) $$x = F(x).$$

The procedure of **iteration** is to guess a first approximation of the root, say x_0, substitute this value into $F(x)$, and obtain a second approximation,

(60.2) $$x_1 = F(x_0).$$

Now repeat the process by substituting x_1 into $F(x)$ to get

$$x_2 = F(x_1).$$

By repeated continuation of the process, we can calculate a sequence of numbers $\{x_i\}$ which should converge to the desired root, r. The method is especially suited to the finding of the roots of an equation in which the function is given in the form of a power series such as the interpolating polynomials.

Example VI.5. Find the root of the equation

$$3x - \sqrt{1 + \sin x} = 0.$$

Solution. To get a first approximation we plot roughly the graph of the two functions

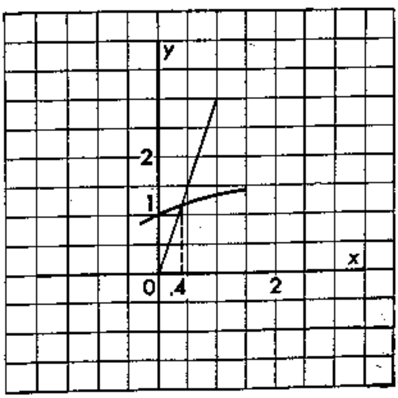
$$y_1 = 3x$$

and

$$y_2 = \sqrt{1 + \sin x}$$

and roughly determine the abscissa of the point of intersection to be $x = .4$. Next we solve $f(x)$ for x

$$x = \frac{1}{3}\sqrt{1 + \sin x}$$



and set up the iteration calculation

x_0	x_1	x_2	x_3	x_4	x_5	x_6
.4	.3929	.391985	.391865	.391849	.391847	.391847

Thus $\boxed{x = .391847}$ to six decimals accuracy.

The success of the iteration process depends on the convergence of the sequence $\{x_i\}$ to a number r which is the desired root. It is, of course, possible that the method will not converge. However, the condition for convergence can be established by a repeated application of the theorem of mean value.* We shall assume the development and simply state the condition which is that

$$|F'(x)| < 1$$

in the neighborhood of the desired root. Furthermore, the smaller the value of $F(x')$, the more rapid the convergence. A check on the above example shows that

$$F'(x) = \frac{1}{6}(1 + \sin x)^{-\frac{1}{2}}(\cos x)$$

and at $x = 0.4$, $F'(.4) = .130$.

61. Systems of Linear Equations. The first discussion of systems of linear equations occurs in an elementary course in algebra, where the reader is introduced to the method of elimination and the solution of the system by determinants. These are basic methods which should be employed whenever possible. We shall assume the reader to be familiar with the theory of determinants and briefly review the method of solving equations employing them.† Let us consider a simple system of three linear equations in three unknowns

$$\text{(61.1)} \qquad \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

* See J. B. Scarborough, *Numerical Mathematical Analysis* (2nd ed.; Baltimore: Johns Hopkins University Press, 1950), p. 201.

† See, for example, P. S. Dwyer, *Linear Computations* (New York: John Wiley & Sons, Inc., 1951), Chapter 9.

The determinant made up of the coefficients of the unknowns is called the **determinant of the system** and we shall denote it by $\boldsymbol{D}$.

(61.2)
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The values of the unknowns are then found by

(61.3)
$$x = \frac{1}{D}\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}; \quad y = \frac{1}{D}\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix};$$
$$z = \frac{1}{D}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix};$$

and there are three distinct values providing $\boldsymbol{D \neq 0}$. We recall that a determinant is said to be of **rank** r if r is the *order* of the highest-order nonvanishing minor. A **matrix** is a rectangular array of mn quantities arranged in m rows and n columns. It is not one quantity like a determinant but is an array of quantities. The *rank of a matrix* is the highest rank of the determinants of the highest order that can be formed from the matrix by striking out rows or columns. A matrix may be square, i.e., m may be equal to n. Thus we shall say that D is the matrix of the coefficients. If to this matrix we attach the column of constant terms, we may call the resulting matrix the **augmented matrix.** We may now state the conditions for solution of equations (61.1).

a) If the matrix of the coefficients is of rank 3 (in general n), i.e., $D \neq 0$, the equations have a unique solution.

b) If the rank of the augmented matrix is greater than the rank of the matrix of coefficients, the equations have *no* solution.

c) If for n equations in n unknowns the ranks of the augmented matrix and of the matrix of coefficients are both equal to $n - r$, then there are an infinite number of solutions expressible in terms of r arbitrary parameters.

The solution of the simple system of two equations in two unknowns is readily adapted to a calculating machine by the following schematic. Consider

(61.4)
$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2. \end{cases}$$

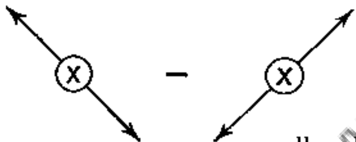
Arrange the numbers in the following array:

a_1	b_1	c_1
a_2	b_2	c_2
D	x	y

where

$$(61.5)\quad \begin{cases} D = a_1b_2 - b_1a_2 \\ x = -\dfrac{1}{D}(b_1c_2 - c_1b_2) \qquad \text{Note the } (-) \text{ sign.} \\ y = \dfrac{1}{D}(a_1c_2 - c_1a_2) \end{cases}$$

which is accomplished on the machine by the cross multiplication



followed by a division if required.

Example VI.6. Solve for x and y in

$$\begin{cases} 3.1416x + 1.3456y = 7.2184 \\ 1.3111x - 2.1221y = 3.1234. \end{cases}$$

Solution. The computation is arranged on the calculating sheet as follows. The student should repeat the calculations to check the answers.

$$D = a_1b_2 - b_1a_2$$
$$x = -\frac{b_1c_2 - c_1b_2}{D}$$
$$y = \frac{a_1c_2 - c_1a_2}{D}$$

a	b	c
3.1416 1.3111	1.3456 −2.1221	7.2184 3.1234
−8.431006	2.3154	−.0413

Thus $x =$ 2.3154 and $y =$ −.0413.

For systems involving more than two equations the evaluation of the determinants is fairly difficult on calculating machines, and in such cases we rely on the method of elimination which has been developed into a schematic. We shall now describe this method in detail.

62. Crout's Method. One of the best methods for solving systems of linear equations on desk calculating machines was developed by P. D. Crout in 1941.* The method is essentially based on elimination with the work arranged in a schematic. A check is provided so that the work may be checked at regular points, and this is highly recommended. The method is best explained by an example. To keep it simple, let there be given three equations in three unknowns, x, y, and z.

(62.1)
$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

Form the matrix of the system plus a check column

(62.2)
$$\begin{Vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \end{Vmatrix}$$

where
$$e_i = a_i + b_i + c_i + d_i, \qquad (i = 1,2,3).$$

A derived matrix

(62.3)
$$\begin{Vmatrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 \\ A_3 & B_3 & C_3 & D_3 & E_3 \end{Vmatrix}$$

is now computed in the following manner:

Step 1. First Column

(62.4)
$$A_i = a_i, \qquad (i = 1,2,3).$$

* Prescott D. Crout, "A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients," *Transactions of the American Institute of Electrical Engineers* (1941), Vol. 60, p. 1235.

Step 2. First Row

$$B_1 = \frac{b_1}{a_1};\ C_1 = \frac{c_1}{a_1};\ D_1 = \frac{d_1}{a_1};\ E_1 = \frac{e_1}{a_1}. \tag{62.5}$$

Step 3. Remaining Second Column

$$\begin{cases} B_2 = b_2 - B_1 A_2, \\ B_3 = b_3 - B_1 A_3. \end{cases} \tag{62.6}$$

Step 4. Remaining Second Row

$$\begin{cases} C_2 = \dfrac{c_2 - C_1 A_2}{B_2}, \\ D_2 = \dfrac{d_2 - D_1 A_2}{B_2}, \\ E_2 = \dfrac{e_2 - E_1 A_2}{B_2}. \end{cases} \tag{62.7}$$

Step 5. Remaining Third Column

$$C_3 = c_3 - C_2 B_3 - C_1 A_3. \tag{62.8}$$

Step 6. Remaining Third Row

$$\begin{cases} D_3 = \dfrac{d_3 - D_2 B_3 - D_1 A_3}{C_3}, \\ E_3 = \dfrac{e_3 - E_2 B_3 - E_1 A_3}{C_3}. \end{cases} \tag{62.9}$$

The solutions are then given by

$$\begin{cases} z = D_3, \\ y = D_2 - C_2 z, \\ x = D_1 - C_1 z - B_1 y. \end{cases} \tag{62.10}$$

The check column (E_i) serves as a continuous check on the calculations in that the E_i are equal to *one* plus the sum of the elements in the same row to the right of the principal diagonal; thus

$$
(62.11)\qquad \begin{cases} E_1 = 1 + B_1 + C_1 + D_1, \\ E_2 = 1 + C_2 + D_2, \\ E_3 = 1 + D_3, \end{cases}
$$

and the check should be made after completing each row.

The calculations adapt themselves readily to desk machine operations and only the elements of the derived matrix need be recorded, i.e., no intermediate quantities have to be written down. The arrangement is illustrated in the following example.

Example VI.7. Solve for x, y, and z in the following system of equations:

$$
\begin{cases} x + y + z = 1, \\ 3x + y - 3z = 5, \\ x - 2y - 5z = 10. \end{cases}
$$

Solution.

	x	y	z	k	CHECK
GIVEN MATRIX	1	1	1	1	4
	3	1	−3	5	6
	1	−2	−5	10	4
DERIVED MATRIX	1	1	1	1	4
	3	−2	3	−1	3
	1	−3	3	2	3
SOLUTIONS	6	−7	2		

A check can also be made on the computations of the solutions by using the check column. Thus

$$
(62.12)\qquad \begin{cases} z' = E_3 \\ y' = E_2 - C_2 z' \\ x' = E_1 - C_1 z' - B_1 y' \end{cases}
$$

and

$$
(62.13)\qquad \begin{cases} z' = 1 + z \\ y' = 1 + y \\ x' = 1 + x. \end{cases}
$$

This is illustrated in the following example.

Example VI.8.

x	y	z	w	k	CHECK
2.462	1.349	−2.390	−3.400	0.903	−1.076
−1.000	2.000	3.000	2.000	1.340	7.340
0.983	0.220	−1.600	−0.930	3.000	1.673
1.310	−3.000	2.100	1.200	2.560	4.170
2.462	0.5479	−0.9708	−1.3810	0.3668	−0.4370
−1.000	2.5479	0.7964	0.2429	0.6699	2.7092
0.983	−0.3186	−0.3920	−1.2881	−7.2779	−7.5659
1.310	−3.7178	6.3326	12.0692	4.1973	5.1973
3.721	1.141	−1.871	4.197		SOLUTIONS
4.721	2.141	−.871	5.197		CHECK

A. The Case of Symmetrical Coefficients. If the elements of the given matrix are symmetrical about the principal diagonal, the work of computing the auxiliary matrix is cut almost in half. This can be seen by noting that the elements of the auxiliary matrix below the principal diagonal will yield, if divided by its diagonal element, the symmetrically opposite element above this diagonal. Let us follow this through. If the system given by (63.1) were symmetrical, we would have

$$b_1 = a_2, \qquad c_1 = a_3, \qquad c_2 = b_3,$$

so that the given matrix would be

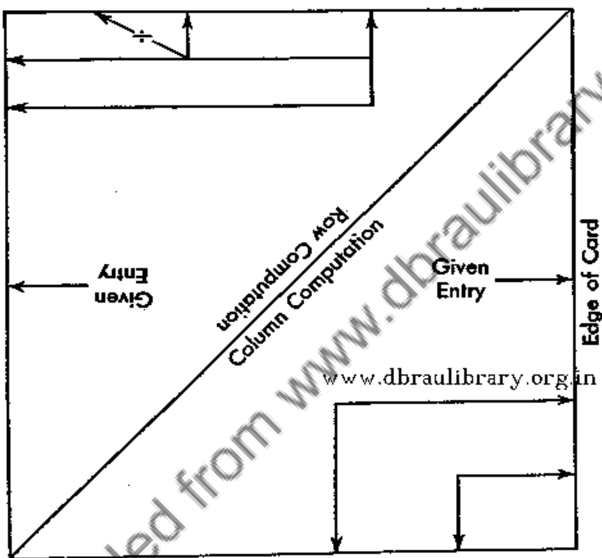
$$\text{(62.14)} \qquad \left\| \begin{matrix} a_1 & a_2 & a_3 & d_1 \\ a_2 & b_2 & b_3 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix} \right\|$$

and in the derived matrix we would have

$$\text{(62.15)} \qquad \begin{cases} B_1 = \dfrac{b_1}{a_1} = \dfrac{A_2}{A_1}, \\ C_1 = \dfrac{c_1}{a_1} = \dfrac{A_3}{A_1}, \\ C_2 = \dfrac{c_2 - C_1 A_2}{B_2} = b_3 - \dfrac{B_1 A_3}{B_2} = \dfrac{B_3}{B_2}. \end{cases}$$

During the computation the symmetrically corresponding elements of the derived matrix could therefore be computed as one operation with an additional division.

Various stencils have been devised to guide the computer. One such stencil which the author has found convenient to prepare is a rectangular guide card divided by a diagonal so that half is used for calculating the columns and half for the rows by simply rotating the card through 180° each time a change is made.



B. The General Case. Crout's method is easily generalized. Let (a_{ij}) be the elements of the given matrix and (A_{ij}), the elements of the derived matrix, then

$$A_{ii} = a_{ii} - \sum_{k=1}^{i-1} A_{ik}A_{ki},$$

(62.16) $$A_{ij} = a_{ij} - \sum_{k=1}^{j-1} A_{ik}A_{kj}, \qquad (\text{if } i > j),$$

$$A_{ij} = \left[a_{ij} - \sum_{k=1}^{i-1} A_{ik}A_{kj}\right]\frac{1}{A_{ii}}, \qquad (\text{if } i < j),$$

and if we have n variables, x_i, the solution matrix, can be written as a row matrix,

$$(62.17) \quad x_{1i} = A_{i,n+1} - \sum_{k=i+1}^{n} A_{ik}x_{ik}, \qquad (i = 1, \cdots, n),$$

where it is understood that any sum whose lower limit exceeds its upper limit as assumed to be zero.

C. Accuracy Improvements in the Solutions. The values obtained for the unknowns are usually not exact, and the substitutions of them back in the original equations may not make the equations balance exactly. Improvements on the solutions may be obtained by calculating the differences and, treating them as the column of constants, solving for corrections to the first solutions of the unknowns. Since all of the derived matrices except the column of constants remain the same, it is only necessary to annex an additional column to obtain these corrections. For example, suppose that the solutions to Example VI.8 had been

$$(62.18) \qquad \begin{cases} x = 3.72 \\ y = 1.13 \\ z = -1.88 \\ w = 4.20. \end{cases}$$

If these were then substituted into the original equations, we would have the values

Substituted	Given	Differences
0.896	0.903	.007
1.300	1.340	.040
3.007	3.000	−.007
2.575	2.560	−.015

Employing the difference column as the column of constants in the given matrix, we obtain the corresponding column in the derived matrix

$$\begin{array}{r} .002843 \\ .016815 \\ .011318 \\ -.002296 \end{array}$$

and solving in the usual manner we obtain

$$\Delta w = -.0023; \qquad \Delta z = .0084; \qquad \Delta y = .01068; \qquad \Delta x = .0020.$$

Thus

$$x = 3.722; \qquad y = 1.141; \qquad z = -1.872; \qquad w = 4.198$$

which compares favorably with the solutions obtained earlier. This correction procedure may be continued as many times as desired.

D. Evaluation of Determinants. Crout's method may be used to find the value of a determinant. Refer back to the system of three equations in three unknowns and consider the product of the elements of the main diagonal of the derived matrix, $A_1B_2C_3$. We have

$$\begin{aligned}
A_1 &= a_1 \\
B_2 &= b_2 - B_1A_2 = b_2 - \frac{b_1}{a_1}a_2 = \frac{1}{a_1}(a_1b_2 - b_1a_2) \\
C_3 &= c_3 - C_2B_3 - c_1A_3 \\
&= c_3 - \left(\frac{c_2 - C_1A_2}{B_2}\right)(b_3 - B_1A_3) - \frac{c_1}{a_1}(a_3) \\
&= c_3 - \left(\frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}\right)\left(\frac{a_1b_3 - b_1a_3}{a_1}\right) - \frac{c_1a_3}{a_1} \\
&= \frac{1}{a_1}\left[a_1c_3 - c_1a_3 - \frac{(a_1c_2 - c_1a_2)(a_1b_3 - b_1a_3)}{a_1b_2 - b_1a_2}\right].
\end{aligned}$$

The product becomes after some simplification

$$\begin{aligned}
(62.19) \quad A_1B_2C_3 &= \frac{1}{a_1}[(a_1c_3 - c_1a_3)(a_1b_2 - b_1a_2) \\
&\qquad - (a_1c_2 - c_1a_2)(a_1b_3 - b_1a_3)] \\
&= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 \\
&\qquad - a_1b_3c_2
\end{aligned}$$

which is the value of the determinant of the coefficients. This can be easily proved for the general case so that we may state: *The value of a determinant is equal to the product of the elements of the main diagonal of the derived matrix.*

Example VI.9. Find the value of the determinant

$$D = \begin{vmatrix} 1 & 3 & 2 \\ -1 & 4 & 5 \\ 2 & 1 & 6 \end{vmatrix}.$$

Solution. The derived matrix is

$$\begin{Vmatrix} 1 & 3 & 2 \\ -1 & 7 & 1 \\ 2 & -5 & 7 \end{Vmatrix}.$$

The value of the determinant is

$$D = (1)(7)(7) = \boxed{49}.$$

It is advantageous to obtain the value of the determinant when solving a system of linear equations. It may be conveniently entered in our computational scheme at the end of the diagonal line through the elements of the main diagonal of the derived matrix.

E. Cofactors and Matrix Inversion. It is frequently desirable to evaluate the cofactors of the elements of a determinant, and this may be done by following Crout's method. At the same time we can obtain the conjugate of the inverse matrix of the given set of equations. Consider again the set of equations in three unknowns (62.1) and recall that the cofactor of any element a_{ij} of a determinant is the minor of that element multiplied by $(-1)^{i+j}$. In the determinant of (62.1) the cofactors of a_1, b_1, and c_1 are

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \qquad (-)\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \qquad \text{and} \qquad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

respectively. Consider now the system

$$\text{(62.20)} \qquad \begin{cases} a_1x_1 + b_1y_1 + c_1z_1 = 1 \\ a_2x_1 + b_2y_1 + c_2z_1 = 0 \\ a_3x_1 + b_3y_1 + c_3z_1 = 0. \end{cases}$$

The solution for x is given by

$$x_1 = \frac{1}{D}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \frac{1}{D}\,(\text{cofactor of } a_1).$$

Thus

$$\text{cofactor of } a_1 = Dx_1.$$

Similarly,

$$\begin{aligned}\text{cofactor of } b_1 &= Dy_1,\\ \text{cofactor of } c_1 &= Dz_1.\end{aligned}$$

If we replaced the constants of the system by 0, 1, 0 we would obtain a new set of equations in x, y, z for which the left-hand side is identical to that of (62.1), and if we let the solution of this system be x_2, y_2, z_2 we have

$$\begin{aligned}\text{cofactor of } a_2 &= Dx_2,\\ \text{cofactor of } b_2 &= Dy_2,\\ \text{cofactor of } c_2 &= Dz_2.\end{aligned}$$

The final step is to form the system with constant terms 0, 0, 1 and arrive at solutions x_3, y_3, z_3 from which

$$\begin{aligned}\text{cofactor of } a_3 &= Dx_3,\\ \text{cofactor of } b_3 &= Dy_3,\\ \text{cofactor of } c_3 &= Dz_3.\end{aligned}$$

The entire process may be combined into one operation. First augment the matrix of the coefficients by the unit matrix (I)

$$(62.21)\qquad \begin{Vmatrix} a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{Vmatrix}.$$

Calculate the derived matrix

$$(62.22)\qquad \begin{Vmatrix} A_1 & B_1 & C_1 & D_1 & 0 & 0 \\ A_2 & B_2 & C_2 & D_2 & E_2 & 0 \\ A_3 & B_3 & C_3 & D_3 & E_3 & F_3 \end{Vmatrix}.$$

Now solve for x_i, y_i, z_i, $(i = 1,2,3)$, using the constants D_i, E_i, F_i, $(i = 1,2,3)$, in turn. We obtain

$$\begin{Vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{Vmatrix}. \tag{62.23}$$

The elements of this matrix are the cofactors of the elements of the determinant of the system divided by this determinant, D. They are also the elements of the *conjugate* of the *inverse* matrix of the original system. Thus, to find the cofactors simply multiply each element of (62.23) by D.

Example VI.10. Solve the following system of equations. Evaluate the determinant of the system and obtain the value of the cofactors of each element of the determinant of the coefficients.

Solution.

x	y	z	k	CHECK	(I)		
1.36	2.54	−1.62	1.97	4.25	1	0	0
−1.82	3.65	1.81	2.42	6.06	0	1	0
2.38	−1.42	3.24	1.11	5.31	0	0	1
1.36	1.868	−1.191	1.449	3.125	.735	0	0
−1.82	7.050	−.051	.717	1.666	.190	.142	0
2.38	−5.866	5.775	.323	1.324	−.110	.144	.173

$D = 55.37$

.464	.733	.323	SOLUTION
1.463	1.734	1.324	CHECK

CONJUGATE INVERSE			COFACTORS		
.260	.184	−.110	14.40	10.19	−6.09
−.107	.149	.144	−5.92	8.25	7.97
.189	.009	.173	10.46	.498	9.58

F. Systems Having Complex Coefficients. If the coefficients of the system are complex numbers, the solutions will be complex numbers. The method of solution is the same except that some scheme must be devised for recording the real and imaginary part at each step. Since the product of two complex numbers

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \tag{62.24}$$

is the sum of products of real numbers, it is conceivable that a scheme for desk machines is feasible. In Crout's method the elements of the derived matrix which lie on or below the principal diagonal can be obtained by two machine operations, one for the real part and one for the imaginary part. The elements which lie to the right of the principal diagonal give more trouble because they require a division after the sum of products has been obtained. The division by a complex number may be carried out by a multiplication.

$$\frac{1}{A} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \tag{62.25}$$

where

$$A = a + bi.$$

The method may therefore be carried out by writing the complex number in two-element form in the matrix and augmenting the derived matrix. Let us consider the system

$$\begin{array}{cccc} x & y & z & k \\ \end{array}$$
$$\left\| \begin{array}{cccc} a_{11} + b_{11}i & a_{12} + b_{12}i & a_{13} + b_{13}i & a_{14} + b_{14}i \\ a_{21} + b_{21}i & a_{22} + b_{22}i & a_{23} + b_{23}i & a_{24} + b_{24}i \\ a_{31} + b_{31}i & a_{32} + b_{32}i & a_{33} + b_{33}i & a_{34} + b_{34}i \end{array} \right\|. \tag{62.26}$$

When numbers are involved this is conveniently written on two lines for each element instead of two columns. (*See* example below.)

The derived matrix will be

$$\left\| \begin{array}{cccc} A_{11} + B_{11}i & A_{12} + B_{12}i & A_{13} + B_{13}i & A_{14} + B_{14}i \\ A_{21} + B_{21}i & A_{22} + B_{22}i & A_{23} + B_{23}i & A_{24} + B_{24}i \\ A_{31} + B_{31}i & A_{32} + B_{32}i & A_{33} + B_{33}i & A_{34} + B_{34}i \end{array} \right\| \tag{62.27}$$

where the elements are obtained in the regular manner. However, to accomplish this it is necessary to record additional data. Following Crout, we supplement the derived matrix both on the left and right. In the left supplementary matrix, we form the sum of the squares of the real numbers and the reciprocals of each element of the principal diagonal. Thus

(62.28)
$$\left\|\begin{array}{ll} A_{ii}^2 + B_{ii}^2 & \dfrac{1}{A_{ii} + B_{ii}}\, i \\ \hline A_{11}^2 + B_{11}^2 & \dfrac{A_{11}}{LS_{11}} - \dfrac{B_{11}}{LS_{11}}\, i \\ A_{22}^2 + B_{22}^2 & \dfrac{A_{22}}{LS_{21}} - \dfrac{B_{22}}{LS_{21}}\, i \\ A_{33}^2 + B_{33}^2 & \dfrac{A_{33}}{LS_{31}} - \dfrac{B_{33}}{LS_{31}}\, i \end{array}\right\|$$

where LS_{ij} are the elements of the left suppiementary matrix.

In the right supplementary matrix is recorded the sums of the products which form the numerators of the elements to the right of the principal diagonal before each is divided by the corresponding diagonal elements. Thus in the example

(62.29)
$$\left\|\begin{array}{ll} RS_{23} & RS_{24} \\ & RS_{34} \end{array}\right\|$$

we have

(62.30)
$$\begin{cases} RS_{23} = (a_{23} + b_{23}i) - (A_{21} + B_{21}i)(A_{13} + B_{13}i) \\ \qquad = (a_{23} - A_{21}A_{13} + B_{21}B_{13}) + (b_{23} - B_{21}A_{13} - A_{21}B_{13})i \\ RS_{24} = (a_{24} + b_{24}i) - (A_{21} + B_{21}i)(A_{14} + B_{14}i) \\ \qquad = (a_{24} - A_{21}A_{14} + B_{21}B_{14}) + (b_{24} - B_{21}A_{14} - A_{21}B_{14})i \\ RS_{34} = (a_{34} + b_{34}i) - (A_{31} + B_{31}i)(A_{14} + B_{14}i) \\ \qquad\qquad + (A_{32} + B_{32}i)(A_{24} + B_{34}i) \\ \qquad = a_{34} - A_{31}A_{14} + B_{31}B_{14} - A_{32}A_{24} + B_{32}B_{34} \\ \qquad\qquad + (b_{34} - B_{31}A_{14} - A_{31}B_{14} - B_{32}A_{24} - A_{32}B_{34})i. \end{cases}$$

Example VI.11*

		x_1	x_2	x_3	x_4	k	CHECK				
		8.342 7.130	−9.012 1.132	3.345 −1.248	−4.518 −3.362	65.65 −1.810	63.807 1.842				
		9.123 1.071 3.789 −1.242	4.567 5.432 −2.421 7.321	−2.222 7.444 7.342 −2.181	8.041 −2.111 3.467 −7.182	−87.30 6.500 −34.25 −35.45	−67.791 18.336 −22.073 −38.734				
		−4.142 −3.181	8.042 −2.131	3.732 −7.801	−2.111 −2.932	−42.63 14.50	−37.109 −1.545				
$A^2 + B^2$	$\frac{1}{A + Bi}$	DERIVED MATRIX						*RS*			
120.43	.069268 −.059204	8.342 7.130	−.55722 .61196	.15781 −.28448	−.5120 .03460	4.4403 −4.0121	4.5288 −3.6500				
106.41	.096851 −.004190	9.123 1.071	10.306 .44587	−.34279 .97257	1.2269 −.23533	−2.6339 4.2675	−10.7497 5.0047	−3.9664 9.8703	12.749 −1.8783	−132.106 38.347	−113.016 46.7846
121.90	.089603 −.013213	3.789 −1.242	−1.0697 4.3102	10.9227 1.6107	.32911 −1.28346	−3.10756 4.51272	−1.77845 3.22921		5.6621 −13.4889	−41.2120 44.2863	−24.6270 32.4075
115.84	.050184 −.078192	−4.142 −3.181	3.78735 −1.3688	5.2576 −12.6300	5.81342 9.05788	−7.1094 −3.4450	−6.1094 −3.4450			−10.12549 −84.42388	−4.3122 −75.3656
		−.5507 .4159	−5.2309 2.0754	3.6537 −3.4781	−7.1094 −3.4450	SOLUTION					

* The second number in each group is the imaginary part. Thus, $x_1 = -.5507 + .4159i$

63. Inherent Errors in Systems of Linear Equations. The computational methods for solving systems of linear equations can be checked for their accuracy, and errors due to computation can be evaluated and corrected. Systems involving approximate numbers may have errors due to the nature of the equations, and these inherent errors may be very serious. However, it is possible to find expressions which permit us to study such errors. Following the procedure of Section 4 (Chapter I) we shall obtain expressions for the errors in the variables due to errors in the coefficients and constant terms. Consider the simple system of two equations in two unknowns

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2. \end{cases} \tag{63.1}$$

Let the known errors in a_i, b_i, c_i be Δa_i, Δb_i, Δc_i, $(i = 1,2)$. Then by taking the differentials of the equations of the given system, we can write to a *first-order* approximation

$$\begin{cases} a_1\Delta x + b_1\Delta y = \Delta c_1 - x\Delta a_1 - y\Delta b_1 \\ a_2\Delta x + b_2\Delta y = \Delta c_2 - x\Delta a_2 - y\Delta b_2 \end{cases} \tag{63.2}$$

which gives us a system of linear equations for the inherent errors in the solution, Δx and Δy. Since the right-hand side of this system is known as we know the errors of the coefficients and we are able to solve to obtain the values of x and y, this system may now be solved for Δx and Δy. In practice, of course, the actual errors of a_i, b_i, and c_i are not known precisely. What we do know is that they do not exceed a known magnitude. Thus we proceed not to find the actual inherent error but the bound for the largest values of the errors, i.e., the right-hand members of equations (63.2) do not exceed a known constant. This constant will be the sum of the absolute values of these terms; thus

$$\begin{cases} k_1 = |\Delta c_1| + |x\Delta a_1| + |y\Delta b_1| \\ k_2 = |\Delta c_2| + |x\Delta a_2| + |y\Delta b_2| \end{cases} \tag{63.3}$$

and we solve for the upper bounds of Δx and Δy by any known method in terms of k_i. If the numbers in the original system are all given to the same degree of accuracy, then the errors on each of them are equal and $k_1 = k_2$.

For example, if a_i, b_i, c_i are given to two decimal places, then

$$|\Delta a_i| \leq .005, \qquad |\Delta b_i| \leq .005, \qquad \text{and} \qquad |\Delta c_i| \leq .005.$$

Let the upper bound of these errors be denoted by ϵ. Then the upper bound of the right-hand side of equations (63.2) is given by

$$k_1 = k_2 = (1 + |x| + |y|)\epsilon. \tag{63.4}$$

The solution for Δx and Δy in equation (63.2) can now be easily accomplished. We note first that the determinant of the system is exactly the same as that of the original equations so that in applying Crout's method we need only attach a column of constant terms. If all the k_i are equal, we shall replace them by 1 and multiply the results by k. The solutions obtained before multiplying by k is in a sense a measure of sensitivity. Furthermore, since we are seeking the upper bound and the errors may be either positive or negative, we shall always assume the worst case and *always add* regardless of sign. Let us illustrate with an example.

Example VI.12. Find the inherent errors in Example VI.10.

Solution. Augment the given matrix by a column whose elements are all unity. Calculate the corresponding elements in the derived matrix always adding regardless of sign. Now using these elements calculate the one-row solution matrix again always adding. Thus

			DERIVED MATRIX
			.735 .332 .813
2.400	.373	.813	SOLUTION MATRIX

Now $\epsilon = .005$ and

$$k = (1 + |x| + |y| + |z|) = 2.520(.005) = .0126.$$

Multiplying the solutions by k we obtain

$$\Delta x = .030, \qquad \Delta y = .005, \qquad \text{and} \qquad \Delta z = .010.$$

The above example illustrates the effect of inherent errors. The errors computed are, of course, the upper bounds, and the actual errors should be less than these. Nevertheless, if the errors all combine to approach the upper bound, we see that the solutions are not as accurate as the given data.

As to whether or not it is realistic to check the solutions by an upper bound on the errors is often questioned. As a compromise it seems reasonable to compute k as above using the sum of the absolute values, but in the computation of the sensitivity factors *take into account the algebraic signs* and solve in the usual manner. For the above example we have

x	y	z	DERIVED MATRIX
			.7353 .3317 .2071
.3425 .0043	.3423 .0043	.2071 .0026	SENSITIVITY Δ

In general no technique will improve on the inherent errors, but some measures of safety may be taken. The greatest source of inherent errors arises from the loss of leading significant figures by subtraction. Thus in Example VI.10 we see that C_2 (in the derived matrix) is $-.051$ which does not have as many significant figures as the given data.* This must be guarded against and can be done by carrying the calculations to more figures than the data. In other words, *do not round off until the very end.*

The most troublesome thing in solving systems of linear equations occurs when the determinant of the system, D, is near zero, not exactly zero, mind you, but close to it to the degree of accuracy of the approximate numbers being used. The solutions are then very sensitive, and some absurd results may occur. Usable answers may still be attainable, however. In using Crout's method there will be

* For this reason Example VI.10 is actually done incorrectly, the mistake being intentional.

an indication of this situation as one of the elements along the main diagonal of the derived matrix, A_{ii}, will become small. It is best that this not occur at an early stage of the computation, but if it does, it can sometimes be corrected by recording the equations. In fact, in applying Crout's method particular attention should be paid to the elements of the main diagonal of the derived matrix, A_{ii}. They should be allowed to become neither extremely large nor small. If it happens, rearrange the equations. Should D be near zero, the ideal situation is to have the smallest element of the A_{ii} occur at A_{nn}, i.e., the last one. This will prevent the numbers from getting "out of hand." We shall illustrate with an example.

Example VI.13

x	y	z	k	CHECK
.91143	.90274	.89318	.62433	3.33168
.36518	.36161	.34622	.24561	1.31862
.24375	.22100	.65521	.35000	1.46996
.91143	.990466	.979977	.685000	3.655443
.36518	−.000088	132.363646	51.571590	184.939485
.24375	−.020426	3.720000	.396292	1.396320
			−.000250	
1.171288	−.883064	.396292	SOLUTIONS	
2.170723	.117479	1.396320	CHECK	

We note that the value of $D = -.000250$ is small and that the sensitivities will be large. Equally important, we see that $A_{22} = -.000088$ is nearly zero, and as a result other numbers become large and the computational checks fall off. Let us reorder the equations and at the same time compute a set of reasonable inherent errors. Note that the reordering changes the algebraic sign of D but not its absolute value. This is consistent with the theory of determinants.

y	z	x	k	CHECK	I.B.
.22100	.65521	.24375	.35000	1.46996	1
.36161	.34622	.36518	.24561	1.31862	1
.90274	.89318	.91143	.62433	3.33168	1
.221000	2.964751	1.102941	1.583710	6.651403	4.524887
.361610	−.725864	.046364	.450601	1.496966	.876533
.902740	−1.783219	−.001562	1.170359	2.169865	974.229315
			.000250		
−.882170	.396338	1.170359	SOLUTIONS		
.118304	1.396362	2.169865	CHECK		
				$k =$ .000017	
−894.382	−44.293	974.229	SENSITIVITY		
.0152	.00075	.0166	Δ		

64. Other Methods for Systems of Linear Equations. Extensive use of large-scale calculating machines has greatly increased the number of methods of solving systems of linear equations. Many of the new methods are special adaptations of classical methods to particular calculating machines, depending primarily on the machines' storage capacity and the effect of round-off errors. An entire book might be written on these methods. In fact some of the recent numerical analysis books are largely devoted to this particular subject. It is not the intention of this book to specialize to such an extent.

In general there is a particular method which is most easily adapted to a specific calculator. However, the present state of calculating machines would make it inadvisable to state that one method is the best possible method.

In 1953 the IBM Applied Science Division made a survey of the methods of inverting matrices and solving systems of simultaneous linear algebraic equations. In this survey they listed the following sixteen methods:

The Adjoint Method
Gaussian Elimination
Crout's Method
Triangular Square Root Method
Methods of Sub-Matrices or Partitioning
Error Squaring Method
Shur's Method
Relaxation Methods
Richardson's Method
The Method of Simultaneous Displacements
The Method of Successive Displacements
Block Iteration
The Method of Steepest Descent
The N Step or Conjugate Gradient Method
Monte Carlo Technique
Improvements in Iterative Methods.

These methods may be classified as *direct*, *iterative*, or *statistical* methods. The direct methods are those which obtain the solutions after a fixed number of steps, such as Crout's method. The major handicap to these methods is the round-off errors. The iterative methods get approximations and then improve upon them; more accurately, they are methods of successive approximations. Usually they require more computations and the major handicap is the speed of convergence; they do tend to eliminate the round-off errors. There are also various aids to improve the convergence. The statistical method is the Monte Carlo method which is iterative in nature; the results should be considered to be an estimate rather than a true approximation to the solutions.

The literature on this subject is very extensive and the reader is referred to the bibliography in the back of the book.

65. Exercise VIII.

1. Find the smallest positive root of

$$x^4 - x^3 - 1 = 0.$$

2. Find the positive roots of

$$49x^4 + 84x^3 + 22x^2 - 12x + 1 = 0.$$

3. Find a real root of

$$2x - \cos x - 3 = 0.$$

4. Find a root of

$$x \log_{10} x = -0.15.$$

5. Find the smallest positive root of

$$1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} = 0.$$

6. Investigate the existence of a small positive real root of

$$x^4 - 1.47x^3 + .056x^2 - .944x + 1.528 = 0.$$

7. Find the positive roots of

$$x^3 + 9.531x^2 - 4.6355x + .5449 = 0.$$

8. Solve the following system of linear equations:

$$\begin{aligned} .534x - .329y &= .720 \\ .128x + .832y &= .314. \end{aligned}$$

9. Solve the following system of linear equations:

$$\begin{aligned} 1.932x + 3.864y + 2.898z + 7.728w &= 1.462 \\ .349x + .700y + 2.439z + 2.000w &= 3.832 \\ 1.682x + 2.264y + 1.463z + .468w &= 4.291 \\ 1.244x + 1.866y + 2.488z + 3.732w &= 2.943. \end{aligned}$$

10. Solve the following system of linear equations:

$$\begin{cases} (2.3 + .5i)x + (1.9 - .2i)y + (.75 + .32i)z = 1.2 + .3i \\ (1.9 - .2i)x + (1.45 + .25i)y + (1.12 - .18i)z = 2.3 + .4i \\ (.75 + .32i)x + (1.12 - .18i)y + (2.38 - .62i)z = 1.25 - .2i. \end{cases}$$

11. Find the real root of $x^3 - 17 = 0$ by Newton's method.
12. Find the real root of $x^5 - 17 = 0$ by Newton's method.

66. Systems of Nonlinear Equations. The solutions of systems of simultaneous nonlinear equations is accomplished either by successive approximations or by the method of iteration. In either case the first approximations are usually obtained from a rough graph. We shall consider the method of successive approximations first; the best-known such method is the *Newton-Raphson* Method.

A. The Newton-Raphson Method. Let us consider first the simple case of two equations in two unknowns,

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \tag{66.1}$$

and let (x_0,y_0) be the initial approximate values. The method seeks to obtain a correction, Δx and Δy, on x_0 and y_0 so that the corrected values will be

$$\begin{cases} x = x_0 + \Delta x \\ y = y_0 + \Delta y \end{cases} \tag{66.2}$$

for which

$$\begin{cases} f(x_0 + \Delta x,\ y_0 + \Delta y) = 0 \\ g(x_0 + \Delta x,\ y_0 + \Delta y) = 0. \end{cases} \tag{66.3}$$

We may expand equations (66.3) by Taylor's theorem for a function of two variables [see formula (3.20), Chapter I] to obtain

$$\begin{cases} f(x + \Delta x,\ y + \Delta y) = f(x_0,y_0) + f_x(x_0,y_0)\Delta x \\ \qquad\qquad + f_y(x_0,y_0)\Delta y + \cdots \\ \qquad = 0 \\ g(x + \Delta x,\ y + \Delta y) = g(x_0,y_0) + g_x(x_0,y_0)\Delta x \\ \qquad\qquad + g_y(x_0,y_0)\Delta y + \cdots \\ \qquad = 0 \end{cases} \tag{66.4}$$

where f_x, f_y, g_x, g_y are the usual notations for the partial derivatives. If we ignore all terms of order higher than the first, we are left with a system of two linear equations in the two unknowns Δx and Δy

$$\begin{cases} f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y = -f(x_0,y_0) \\ g_x(x_0,y_0)\Delta x + g_y(x_0,y_0)\Delta y = -g(x_0,y_0) \end{cases} \tag{66.5}$$

from which we solve to obtain the corrections. The process may now be repeated by using

$$\begin{cases} x_1 = x_0 + \Delta x \\ y_1 = y_0 + \Delta y \end{cases} \tag{66.6}$$

and evaluating the functions and their partial derivatives at (x_1,y_1). The repetition is carried to the desired degree of accuracy.

The computational form is fairly simple.

$f(x,y) =$ ________	$f_x(x,y) =$ ________	$f_y(x,y) =$ ________
$g(x,y) =$ ________	$g_x(x,y) =$ ________	$g_y(x,y) =$ ________

f g	f_x g_x	f_y g_y
D	$(-)\ \Delta x$ x	Δy y

The entries are obtained by a cross multiplication as was done in Section 61.

Example VI.14. Find the root in the neighborhood of (3.8, −1.8) satisfying the system

$$\begin{cases} x^2 + y - 11 = 0 \\ y^2 + x - 7 = 0. \end{cases}$$

Solution.

$f(x,y) = x^2 + y - 11$	$f_x = 2x$	$f_y = 1$
$g(x,y) = y^2 + x - 7$	$g_x = 1$	$g_y = 2y$

	$x = 3.8$	$y = -1.8$
1.64 .04	7.6 1	1 −3.6
−28.36 SOLUTION	−.2096 3.5904	−.0471 −1.8471
.043872 .002178	7.180800 1	1 −3.694200
−27.527311 SOLUTION	−.005966 3.584434	−.001025 −1.848125
.000042 0	7.168868 1	1 −3.696250
−27.497928 SOLUTION	−.000006 3.584428	−.000002 −1.848127

The extension of this method to more than two equations is straightforward. As the number of equations increases, it becomes more cumbersome to solve the resulting system of linear equations for the corrections on the variables. Some time can be saved by noting that the values of the partial derivatives do not change much. Thus, we can leave the determinant of the coefficients unchanged and simply solve for changing constant terms. This is especially easily done by Crout's method. Let us illustrate with an example of three equations in three unknowns:

$$f(x,y,z) = 0; \qquad g(x,y,z) = 0; \qquad h(x,y,z) = 0.$$

The correction equations are

$$\text{(66.7)} \qquad \begin{cases} f_x\Delta x + f_y\Delta y + f_z\Delta z = -f \\ g_x\Delta x + g_y\Delta y + g_z\Delta z = -g \\ h_x\Delta x + h_y\Delta y + h_z\Delta z = -h \end{cases}$$

where the partial derivatives and the functions are evaluated at the approximate point (x_i,y_i,z_i).

Example VI.15. Find a solution to the system

$$\begin{cases} x^2 + y^2 + z^2 - 1 = 0 & = f(x,y,z) \\ 2x^2 + y^2 - 4z = 0 & = g(x,y,z) \\ 3x^2 - 4y + z^2 = 0 & = h(x,y,z). \end{cases}$$

Solution. The partial derivatives are

$$\begin{aligned} f_x &= 2x & f_y &= 2y & f_z &= 2z \\ g_x &= 4x & g_y &= 2y & g_z &= -4 \\ h_x &= 6x & h_y &= -4 & h_z &= 2z. \end{aligned}$$

Assume $x_0 = y_0 = z_0 = .5$.

Δx	Δy	Δz	k			
.5	.5	.5				
1	1	1	.25			
2	1	−4	1.25			
3	−4	1	1			
1	1	1	.25			
2	−1	6	−.75			
3	−7	40	−.125			
.375	0	−.125	Δ			
.875	.5	.375	$u + \Delta$			
			k_2	k_3	k_4	k_5
1.75	1	.75	−.156250	−.007744	−.000770	−.000078
3.50	1	−4.00	−.281250	−.015425	−.001538	−.000156
5.25	−4	.75	−.437500	−.023170	−.002311	−.000249
1.75	.571429	.428572	−.089286	−.004425	−.000440	−.000045
3.50	−1	5.5	−.031251	−.000062	−.000002	−.0000015
5.25	−7	37.0	−.005067	−.000010	0	−.0000006
−.084896	−.00382	−.005067	Δ			
.790104	.496618	.369933	$u + \Delta$			
−.004417	−.000007	−.000010	Δ			
.785687	.496611	.369923	$u + \Delta$			
−.000439	−.000002	0	Δ			
.785248	.496609	.369923	$u + \Delta$			
−.000046	.0000018	−.0000006	Δ			
.785202	.496611	.369922	$u + \Delta$			
x	y	z				

In the above example the initial values are merely guessed. Some writers prefer always to guess $x_0 = y_0 = z_0 = 0$. However, this author has found it more advantageous to give some value other than zero. After the second correction has been made it is assumed that the determinant of the coefficients will not change much, and even though slightly incorrect it is used for the calculations of the subsequent corrections. These corrections are easily found by Crout's method by simply computing a new column of constant terms for each "go around" and then solving for Δx, Δy, and Δz.

Should the convergence be slow or the solutions "jump around," then a new determinant of the coefficients should be calculated.

B. The Exceptional Cases. The above method fails if the determinant, D, vanishes at or near a solution to the original system. However, this gives additional information about the problem, as the vanishing of D indicates

a) multiple solutions, $f(x,y) = 0$ and $g(x,y) = 0$ are tangent to each other or

b) two or more solutions close together or

c) no solution in this neighborhood.

Let us again consider the simple system of two equations in two unknowns, so that for the special case we have

$$D(x_i,y_i) = f_x(x_i,y_i)g_y(x_i,y_i) - f_y(x_i,y_i)g_x(x_i,y_i) = 0. \tag{66.8}$$

Now the locus of points satisfying $D(x,y) = 0$ is the curve on which the loci of $f(x,y) = k_1$ and $g(x,y) = k_2$ have either common tangents or singular points. The procedure is to solve either of the two systems

$$\begin{cases} D(x,y) = 0 \\ f(x,y) = 0 \end{cases} \quad \text{or} \quad \begin{cases} D(x,y) = 0 \\ g(x,y) = 0 \end{cases} \tag{66.9}$$

which can be done provided one of the determinants

$$\begin{vmatrix} D_x & D_y \\ f_x & f_y \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} D_x & D_y \\ g_x & g_y \end{vmatrix}$$

does not vanish in the neighborhood of the point we are considering. Suppose that we solve the first of these two systems and find the solution to be $x = a$ and $y = b$. We now calculate $g(a,b)$ and if this is zero, the curves are tangent at (a,b), and this point is said to be a double solution. If $g(a,b) \neq 0$ then there is either no solution or two solutions close together in the neighborhood of (a,b). Usually the case can be decided by graphing the two functions $f(x,y) = 0$ and $g(x,y) = 0$ and determining from the graph if the two functions intersect. Let us assume that they do. Then we may expand the two functions in Taylor Series about the point (a,b) and take into account second-order terms. We obtain

$$(66.10)\quad \begin{cases} f(a,b) + f_x\Delta x + f_y\Delta y + \frac{1}{2}f_{xx}\Delta x^2 + f_{xy}\Delta x\Delta y + \frac{1}{2}f_{yy}\Delta y^2 = 0 \\ g(a,b) + g_x\Delta x + g_y\Delta y + \frac{1}{2}g_{xx}\Delta x^2 + g_{xy}\Delta x\Delta y + \frac{1}{2}g_{yy}\Delta y^2 = 0 \end{cases}$$

where

$$\Delta x = x - a, \qquad \Delta y = y - b$$

and all the partial derivatives are evaluated at (a,b). We have then a system of simultaneous quadratic equations which we shall simplify somewhat. First, we have $f(a,b) = 0$, and since

$$(66.11)\qquad D = f_xg_y - f_yg_x = 0$$

we have

$$(66.12)\qquad f_xg_y = f_yg_x \quad \text{or} \quad \frac{g_y}{f_y} = \frac{g_x}{f_x} = k$$

so that

$$(66.13)\qquad g_x = kf_x \quad \text{and} \quad g_y = kf_y.$$

Thus if we multiply the first equation of (66.10) by k and subtract from the second, the linear terms are eliminated and we obtain

$$(66.14)\quad g + \frac{1}{2}(g_{xx} - kf_{xx})\Delta x^2 + (g_{xy} - kf_{xy})\Delta x\Delta y + \frac{1}{2}(g_{yy} - kf_{yy})\Delta y^2 = g + A\Delta x^2 + B\Delta x\Delta y + C\Delta y^2 = 0.$$

We shall now divide the first equation of (66.10) by f_x or f_y, whichever is the larger. Let us suppose it is f_y, to obtain

$$\frac{f_x}{f_y}\Delta x + \Delta y + \frac{1}{2f_y}[f_{xx}\Delta x^2 + 2f_{xy}\Delta x\Delta y + f_{yy}\Delta y^2] = 0.$$

Divide this equation once more, this time by Δx and solve for $\frac{\Delta y}{\Delta x} = m$, to obtain

$$(66.15)\quad \frac{\Delta y}{\Delta x} = m = -\frac{1}{f_y}\left[f_x + \frac{\Delta x}{2}(f_{xx} + 2f_{xy}m + f_{yy}m^2)\right].$$

Consider equation (66.14) and write it as

$$g + \Delta x^2 \left[A + B \frac{\Delta y}{\Delta x} + C \left(\frac{\Delta y}{\Delta x} \right)^2 \right] = 0$$

and solve for Δx

$$\Delta x = \pm \left[\frac{-g}{A + Bm + Cm^2} \right]^{\frac{1}{2}}. \tag{66.16}$$

The system (66.15) and (66.16) may now be solved by the method of iteration. Let $m_0 = -\frac{f_x}{f_y}$. Using this value of m solve for Δx by (66.16) and using m_0 and the new Δx obtain a new m by equation (66.15). Eventually we have values for Δx and m from which we get approximations to the two solutions (x_1, y_1) and (x_2, y_2) by

$$\begin{cases} \Delta y = m\Delta x, \\ x = a \pm \Delta x, \\ y = b \pm \Delta y. \end{cases}$$

For these values of x and y, D will not be zero, and they may be improved by the usual method of successive approximations.

Example VI.16. Investigate the following system for possible solution in the first quadrant.

$$\begin{cases} x^2 + 12y^2 - 1 = 0. \\ 49x^2 + 49y^2 + 84x + 2324y - 681 = 0. \end{cases}$$

Solution. Since the first equation is an ellipse lying inside the rectangle $\left(\pm 1, \pm \frac{1}{2}\right)$, we shall assume small values for x_0, y_0, say (.1, .1).

$$f_x = 2x \qquad f_y = 24y$$
$$g_x = 98x + 84. \qquad g_y = 98y + 2324.$$

The first step of the solution is

	$x = .1$	$y = .1$
−.87 −439.22 241.64	.2 93.8 4.040	2.4 2333.8

Although $D = 241.64$ is not zero for these values of x and y, it is much too small since the correction on x, $\Delta x = 4.04$, places it outside the x interval for the ellipse. Let us therefore consider the system

$$f(x,y) = x^2 - 12y^2 - 1 = 0$$
$$D(x,y) = 28(166x - 77xy - 72y) = 0$$

with $D_x = 166 - 77y$ and $D_y = -(77x + 72)$ after dividing out the coefficient 28.

	$x = .1$	$y = .1$	
−.87 8.63 −395.860	.2 158.3 .123	2.4 −79.7 .352	
1.501377 −3.287292 −1462.984	.223 .446 131.196 −.067135	.452 10.848 −89.171 −.135641	
.225290 −.701051 −1101.607	.155865 .311730 141.640357 −.012347	.316359 7.592616 −84.001605 −.029165	
.010357 −.027766	.143518 −.000598	.287194 −.001339	(x,y) $(\Delta x,\Delta y)$
.000982 −.002598	.142920 −.000056	.285855 −.000126	(x,y) $(\Delta x,\Delta y)$
.000103 −.000204	.142864 −.000006	.285729 −.000013	(x,y) $(\Delta x,\Delta y)$
	.142858	.285716	(x,y)

After the third step the values of the partial derivatives and the determinant are held constant and only new values of $f(x,y)$ and $D(x,y)$ are computed. These are put in the extreme left-hand column, and corrections and new values for x and y follow. The value of $g(x,y)$ is now calculated for the last values of x and y to give

$$g(.142858, .285716) = .004114$$

which is a small number but not quite equal to zero to our degree of accuracy. The values of x and y are now found more accurately, and continuing the above process we obtained

$$\boxed{x = .142857142 \qquad \text{and} \qquad y = .285714285}$$

and

$$g(x,y) = -.0000017.$$

Thus it is assumed that $g(x,y) = 0$ and the above solution is a double root.

C. The Method of Iteration. The second method of solving systems of simultaneous equations is the method of iteration. This may be applied in each case in which it is possible to solve explicitly for each of the variables in terms of functions of the variables involved. Suppose we have three equations in three unknowns

$$\begin{cases} f(x,y,z) = 0, \\ g(x,y,z) = 0, \\ h(x,y,z) = 0. \end{cases} \tag{66.17}$$

If these can be solved to give

$$\begin{cases} x = F_1(x,y,z) \\ y = F_2(x,y,z) \\ z = F_3(x,y,z) \end{cases} \tag{66.18}$$

and the initial approximate values are x_0, y_0, z_0, then we proceed in the following manner:

$$\begin{array}{c}\text{First}\\ \text{Approximation}\end{array} \begin{cases} x_1 = F_1(x_0,y_0,z_0) \\ y_1 = F_2(x_1,y_0,z_0) \\ z_1 = F_3(x_1,y_1,z_0) \end{cases}$$

$$\text{Second Approximation} \begin{cases} x_2 = F_1(x_1,y_1,z_1) \\ y_2 = F_2(x_2,y_1,z_1) \\ z_3 = F_3(x_2,y_2,z_1) \\ \quad \text{etc.} \end{cases}$$

The iteration is continued until the values for x, y, and z converge to the desired degree of accuracy. Note that *each new value of a variable is used as soon as it is found.* Thus x_1 is used in finding y_1, and both x_1 and y_1 are used in finding z_1, etc.

Example VI.17. Find the smallest positive root of

$$\sin x - y - .25 = 0.$$
$$\cos y - x + .25 = 0.$$

Solution. Solve for y in the first and x in the second equation.

$$\begin{cases} y = \sin x - .25. \\ x = \cos y + .25. \end{cases}$$

Let $x_0 = 1.00$, then

i	0	1	2	3	4	5	6	7
x	1.00	1.08	1.0568	1.0634	1.0615	1.0621	1.0617	1.0620
y	.59	.6320	.6208	.6240	.6231	.6234	.6233	.6233

Therefore

$$\boxed{x = 1.0620, \qquad y = .6233}.$$

The most troublesome thing about the method of iteration is the fact that the process may converge very slowly or may not converge at all. It is therefore necessary to have a criterion for convergence, and in the case of systems of equations this is given by

$$\text{(66.19)} \quad \begin{cases} \left|\frac{\partial F_1}{\partial x}\right| + \left|\frac{\partial F_2}{\partial x}\right| + \left|\frac{\partial F_3}{\partial x}\right| < 1 \\ \left|\frac{\partial F_1}{\partial y}\right| + \left|\frac{\partial F_2}{\partial y}\right| + \left|\frac{\partial F_3}{\partial y}\right| < 1 \\ \left|\frac{\partial F_1}{\partial z}\right| + \left|\frac{\partial F_2}{\partial z}\right| + \left|\frac{\partial F_3}{\partial z}\right| < 1 \end{cases}$$

in the neighborhood of (x_0,y_0,z_0). In fact, in order for the method to be practical the above criterion must be satisfied very well, i.e., the sums of the absolute value of the partials must be considerably less than 1.

Example VI.18. Consider again Example VI.16, and investigate the possibility of using the method of iteration.

Solution. Solve for x and y

$$\begin{cases} x = \pm\sqrt{1 - 12y^2} & = F_1(y) \\ y = \dfrac{1}{2324}(681 - 84x - 49y^2 - 49x^2) & = F_2(x,y). \end{cases}$$

Now

$$\frac{\partial F_1}{\partial x} = 0, \qquad \frac{\partial F_2}{\partial x} = \frac{1}{2324}(-84 - 98x)$$

$$\frac{\partial F_1}{\partial y} = \frac{1}{2}(1 - 12y^2)^{-\frac{1}{2}}(-24y)$$

$$\frac{\partial F_2}{\partial y} = \frac{1}{2324}(-98y)$$

and at (.14, .28)

$$|F_{1x}| + |F_{2x}| = .042$$
$$|F_{1y}| + |F_{2y}| = 13.81 + .012 = 13.822.$$

Thus the method of iteration would not converge.

It is sometimes possible to speed the convergence by rearranging the functional expressions $F_i(x,y,z)$. However, slowly converging processes are frustrating to the computer, and the criterion should be checked whenever possible.

67. Complex Roots of Algebraic Equations. The finding of complex roots of an algebraic equation is more difficult than that of finding the real roots. Consequently, it is advisable to find all the real roots first and reduce the degree of the equation by dividing out the real roots. In so doing as many significant numbers as possible should be carried since round-off errors become quite significant in this procedure. If it is possible to reduce the equation to a quadratic, the complex roots can then be found by the quadratic formula. When there are more than one pair of conjugate complex roots, we shall employ a numerical procedure which divides out a quadratic factor from the original equation. Many writers have discussed this method in recent years. It is usually credited to S. N. Lin with the notable improvement by B. Friedman and Y. L. Luke. (*See* the bibliography.) We shall develop the simple process.

Since the complex roots occur in conjugate pairs, $a + bi$ and $a - bi$, there exists a real quadratic factor,

$$(x - a - bi)(x - a + bi) = x^2 + px + q, \tag{67.1}$$

of the original function $f(x)$. The solution thus depends upon the values of p and q and if these can be determined, we have

$$\begin{aligned} a &= -\frac{1}{2}p \\ b &= \sqrt{q - a^2} \end{aligned} \tag{67.2}$$

It is therefore desired to obtain a procedure for the determination of p and q. Suppose we divide the original function

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n \tag{67.3}$$

by the quadratic

$$x^2 + px + q$$

to obtain the quotient

$$Q(x) = b_0x^{n-2} + b_1x^{n-3} + \cdots + b_{n-2} + b_{n-1}x^{-1} + b_nx^{-2} + R_3 \tag{67.4}$$

where

$$\begin{cases} b_0 = a_0 \\ b_1 = a_1 - b_0p \\ b_2 = a_2 - b_0q - b_1p \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_i = a_i - b_{i-2}q - b_{i-1}p, \qquad (i = 2, \cdots, n). \end{cases} \tag{67.5}$$

Now, if $b_{n-1} = b_n = 0$, there would be no remainder and the quadratic term would be a factor of the original function. Thus it is desired that

$$\begin{cases} b_{n-1} = a_{n-1} - b_{n-3}q - b_{n-2}p = 0 = F(p,q) \\ b_n = a_n - b_{n-2}q - b_{n-1}p = 0 = G(p,q). \end{cases} \tag{67.6}$$

Remembering that the b_i are functions of p and q, it is easily seen that this is a system of two *nonlinear* equations in the two unknowns p and q. The exact expression for $F(p,q)$ and $G(p,q)$ can be obtained by continuous substitution of the b_i, $(i = 0, \cdots, n-2)$, yielding an equation $G(p,q) = 0$ of degree n and $F(p,q) = 0$ of degree $n - 1$. The system may be solved by either of the two methods discussed in Section 66.

The division of the original function by the quadratic factor could have stopped as soon as the linear term was reached, and this remainder could have been set equal to zero. The result would be the following system of equations:

$$\text{(67.7)} \qquad \begin{cases} a_{n-1} - b_{n-3}q - b_{n-2}p = 0 = r_1(p,q) \\ a_n - b_{n-2}q \qquad\qquad\quad = 0 = r_2(p,q). \end{cases}$$

This system is especially adaptable to the method of iteration for solution since

$$\text{(67.8)} \qquad \begin{cases} q = \dfrac{a_n}{b_{n-2}} \\[2ex] p = \dfrac{a_{n-1} - b_{n-3}q}{b_{n-2}} \end{cases}$$

give the explicit solutions for p and q. However, frequently this is slowly converging, and usually the method of successive approximations is faster. On the other hand the method of successive approximations requires the partial derivatives of $F(p,q)$ and $G(p,q)$ which are somewhat laborious to find. Various computational schemes have been devised to reduce the amount of labor. One of these for the general case is to divide the function $Q(x)$ by the quadratic $x^2 + px + q$ to obtain a second quotient

$$\text{(67.9)} \quad Q_2(x) = c_0x^{n-4} + c_1x^{n-5} + \cdots + c_{n-3}x^{-1} + c_{n-2}x^{-2} + c_{n-1}x^{-3}$$

where

$$\text{(67.10)} \qquad \begin{cases} c_0 = b_0 = a_0 \\ c_1 = b_1 - c_0p \\ c_2 = b_2 - c_0q - c_1p \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_i = b_i - c_{i-2}q - c_{i-1}p, \qquad (i = 2, \cdots, n-2), \\ c_{n-1} = b_{n-1} - c_{n-3}q - c_{n-2}p \end{cases}$$

and it can be shown that

$$
\text{(67.11)} \qquad \begin{cases} F_p = -c_{n-2} = G_q \\ F_q = -c_{n-3} \\ G_p = -c_{n-1}. \end{cases}
$$

The computational scheme may be arranged in the following manner where b_i and c_i are computed from formulas (67.5) and (67.10) with $p = p_i$ and $q = q_i$, the values p_i and q_i being the last approximate values in the sequence of successive approximations.

a_0	b_0	c_0
a_1	b_1	c_1
a_2	b_2	c_2
.	.	.
.	.	.
.	.	.
a_{n-4}	b_{n-4}	c_{n-4}
a_{n-3}	b_{n-3}	c_{n-3}
a_{n-2}	b_{n-2}	c_{n-2}
a_{n-1}	b_{n-1}	c_{n-1}
a_n	b_n	
b_{n-1}	c_{n-2}	c_{n-3}
b_n	c_{n-1}	c_{n-2}
D	Δp	$(-)\Delta q$
	p_{i+1}	q_{i+1}

Note that the true values of c_i are entered in the determinant of the system. The fact that they are the negative of the partial derivatives is accounted for in the solution for Δp and Δq as shown by the $(-)$ on Δq.

After a few steps there is little change in the c_i, and it may suffice to take the last values of c_i and D, thus saving some computation. The initial approximations of p and q may be guessed, the simplest being $p = q = 0$. Also a good initial value is $q_0 = \dfrac{a_n}{a_{n-2}}$ and $p_0 = \dfrac{a_{n-1}}{a_{n-2}}$.

Example VI.19. Find the roots of

$$7x^4 + 38x^3 + 61x^2 - 30x + 12 = 0.$$

First Solution.

Method of Successive Approximations

$$b_0 = a_0 = c_0$$
$$b_1 = a_1 - b_0p$$
$$b_2 = a_2 - b_0q - b_1p$$
$$b_3 = a_3 - b_1q - b_2p$$
$$b_4 = a_4 - b_2q - b_3p$$

$$c_1 = b_1 - c_0p$$
$$c_2 = b_2 - c_0q - c_1p$$
$$c_3 = b_3 - c_1q - c_2p$$

b_3	c_2	c_3
b_4	c_1	c_2
D	Δp	$(-)\Delta q$

i	a_i	b_i	c_i		b_i	c_i		b_i	c_i
p,q	0	0			−.4703	−.0346		−.457646	.146514
0	7	7	7		7	7		7	7
1	38	38	38		41.2921	44.5842		41.203522	44.407044
2	61	61	61		80.6619	101.8720		78.831029	98.128137
3	−30	−30	−30		9.3640	58.8170		.039812	38.441508
4	12	12			19.1948			.468370	
	−30	61	38	9.3640	101.8720	44.5842	.039812	98.128137	44.407044
	12	−30	61	19.1948	58.8170	101.8720	.468370	38.441508	98.128137
D,Δ	4861	−.4703	−.0346	7755.5955	.012654	.181114	7922.0575	−.002132	.005608

	b_i		b_i				
p,q	−.459778	.152122	−.4597756	.1521175			
	7		7				
	41.218446		41.2184292				
	78.886481		78.8864055				
	.000036		.0000000			98.128137	44.407044
	−.000353		−.0000028			38.441508	98.128137
Δ	.0000024	−.0000045	0	0	7922.075		

After the third step, c_i and D are held constant, the values being entered at the extreme right of the calculating sheet. New values of b_i and corrections are then calculated. After five complete steps we have to seven decimal places

$$p = -.4597756 \qquad \text{and} \qquad q = .1521175.$$

From these we get

$$a = -\frac{1}{2}p = .2298878$$
$$b = \sqrt{q - a^2} = .3150700$$

and

$$\boxed{x = .2298878 \pm .3150700i}.$$

The quotient $Q(x)$ yields a second quadratic

$$\begin{aligned} Q(x) &= b_0x^2 + b_1x + b_2 \\ &= 7x^2 + 41.2184292x + 78.8864055 \end{aligned}$$

which is solved by the quadratic formula to give

$$\boxed{x = -2.9441735 \pm 1.6128635i}.$$

It should be noted that the computed value for b_0, b_1, and b_2 are used in the quotient which is solved by the quadratic formula and that the solutions obtained from this equation cannot be any more accurate than the degree of accuracy to which b_0, b_1, and b_2 have been obtained. In fact, rounding-off errors usually make the second pair of roots less accurate. The usual practice is to obtain the first pair of roots more accurately than necessary.

Second Solution.

Method of Iteration

$$b_0 = a_0 \qquad q = \frac{a_4}{b_2}$$

$$b_1 = a_1 - b_0 p$$

$$b_2 = a_2 - b_0 q - b_1 p \qquad p = \frac{a_3 - b_1 q}{b_2}$$

a_0	a_1	a_2	a_3	a_4
7	38	61	−30	12

i	q	p	b_0	b_1	b_2
0	0	0	7	38	61
1	.1967	−.6143	7	42.3001	85.6081
2	.140173	−.419695	7	40.937865	77.200206
3	.155439	−.471026	7	41.297182	79.363973
4	.151202	−.456683	7	41.196781	78.755456
5	.152370	−.460630	7	41.224410	78.922610
6	.152047	−.459539	7	41.216773	78.876386
7	.152136	−.459840	7	41.218880	78.889138
8	.152112	−.459757	7	41.218299	78.885617
9	.152118	−.459780	7	41.218460	78.886598
10	.152117	−.459774			

With the solution for p and q the values for x are found in the same manner as in the first method. Note that in finding p_i we use q_i and not q_{i-1}.

The apparent large amount of writing in the first solution can be greatly reduced by the use of stencils. However, since we are here illustrating a method, all the details must necessarily be exhibited.

A combination of the two methods is frequently advantageous, especially in troublesome problems. The combination consists of first applying the method of iteration for 3 or 4 steps and then shifting to the method of successive approximations. If this is done, then in many cases only one calculation of the c_i and D is necessary.

Many calculators prefer to divide through by a_0, thus always making the leading coefficients equal to one. If this is done care must be taken in determining the sufficient number of places to carry.

If there are two pairs of conjugate complex roots close together, the value of D in the method of successive approximations is near zero and the method of iteration converges very, very slowly, if at all. The method of successive approximations may still be used by resorting to the approach employed in the exceptional case discussed in Section 66.B. It is then necessary to obtain the values of the partial derivatives of $D(p,q) = 0$. We have

$$D(p,q) = c_{n-2}^2(p,q) - c_{n-1}(p,q)c_{n-3}(p,q) \tag{67.12}$$

and

$$\begin{cases} D_p(p,q) = 2c_{n-2}\dfrac{\partial c_{n-2}}{\partial p} - c_{n-1}\dfrac{\partial c_{n-3}}{\partial p} - c_{n-3}\dfrac{\partial c_{n-1}}{\partial p} \\ D_q(p,q) = 2c_{n-2}\dfrac{\partial c_{n-2}}{\partial q} - c_{n-1}\dfrac{\partial c_{n-3}}{\partial q} - c_{n-3}\dfrac{\partial c_{n-1}}{\partial q}. \end{cases} \tag{67.13}$$

Now

$$\frac{\partial c_i}{\partial p} = -2d_{i-1} \qquad \text{and} \qquad \frac{\partial c_i}{\partial q} = -2d_{i-2} \tag{67.14}$$

where

$$\begin{cases} d_0 = c_0 \\ d_1 = c_1 - d_0 p \\ d_i = c_i - d_{i-2}q - d_{i-1}p \end{cases} \tag{67.15}$$

so that

$$\begin{cases} D_p(p,q) = 2[d_{n-2}c_{n-3} + d_{n-4}c_{n-1} - 2d_{n-3}c_{n-2}] \\ D_q(p,q) = 2[d_{n-3}c_{n-3} + d_{n-5}c_{n-1} - 2d_{n-4}c_{n-2}]. \end{cases} \tag{67.16}$$

The above procedure becomes quite laborious and its solution still depends on the determinant for the system

$$\begin{cases} F(p,q) = 0 \\ D(p,q) = 0. \end{cases} \tag{67.17}$$

A method of separating the roots will be discussed in the next section.

Example VI.20. Find the roots of

$$x^4 + x^3 + 4.2505x^2 + 2x + 4 = 0.$$

Solution. First obtain an approximate root by the method of iteration.

a_i	1	1	4.2505	2	4
i	q	p	b_0	b_1	b_2
0	.94	.47	1	.53	3.0614
1	1.31	.43	1	.57	2.6954
2	1.48	.43	1	.57	2.5254
3	1.58	.44			

Now, with the values $p = .45$ and $q = 1.60$, change to the method of successive approximations.

p,q		.45	1.6		.475	1.797		.487214	1.895661
i	a_i	b_i	c_i		b_i	c_i		b_i	c_i
0	1	1	1		1	1		1	1
1	1	.55	.10		.525	.050		.512786	.025572
2	4.2505	2.403	.758		2.204125	.383375		2.105002	.196882
3	2								
4	4								
	.03865	.758	.10	.009616	.383375	.050	.002345	.196882	.025572
	.137808	−.46245	.758	.034620	−.262337	.383375	.008487	−.142055	.196882
D,Δ	.620809	.024993	.197052	.160093	.012214	.098661	.042395	.005770	.047271

We notice that D is getting smaller which indicates the roots are close together. In this fourth degree equation this can also be seen by the fact that $p \sim b_1$ and $q \sim b_2$. However, continuing three more times yields

$$p = .495985 \qquad \text{and} \qquad q = 1.968143$$

with the last value of $D = .004106$. These values satisfy $F(p,q) = 0$ and $G(p,q) = 0$ to six decimal places. The solutions are

$$x = -.247992 \pm 1.380812i$$

and

$$x = -.252008 \pm 1.403162i.$$

68. Graeffe's Root-Squaring Method. It was seen in the last section that if the roots of an algebraic equation are close together, there is considerable trouble in finding them to a high degree of accuracy. Some of this trouble may be alleviated by transforming the original equation into one whose roots are powers of those of the original equation and thus are widely separated. The most easily applied procedure is to transform the original equation into one whose roots are the squares of the original and then repeat this process. This method is known as **Graeffe's Root-Squaring Process.**

Consider the equation

$$(68.1) \quad f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

with the n roots $x_1, x_2, \cdots, x_n$. By the Factor Theorem it can be written in the form

$$(68.2) \quad f(x) = a_0(x - x_1)(x - x_2) \cdots (x - x_n) = 0.$$

If we multiply the equation by

$$(68.3) \quad (-1)^n f(-x) = a_0(x + x_1)(x + x_2) \cdots (x + x_n)$$

we obtain

$$(68.4) \quad \begin{cases} F(x^2) = (-1)^n f(-x)f(x) \\ \qquad\quad = a_0(x^2 - x_1^2)(x^2 - x_2^2) \cdots (x^2 - x_n^2). \end{cases}$$

Letting $y = x^2$ and setting the function equal to zero

$$(68.5) \quad F(y) = a_0(y - x_1^2)(y - x_2^2) \cdots (y - x_n^2) = 0$$

we see that the roots of this equation are the square of the roots of the original. The multiplication of the two functions (68.1) and (68.3) can be easily accomplished by a schematic. We notice that by arranging the coefficients of the two functions in the following manner

a_0	a_1	a_2	a_3	a_4	a_5	$\cdots$
a_0	$-a_1$	a_2	$-a_3$	a_4	$-a_5$	$\cdots$
a_0^2	$-a_1^2$	a_2^2	$-a_3^2$	a_4^2	$-a_5^2$	$\cdots$
	$2a_0a_2$	$-2a_1a_3$	$2a_2a_4$	$-2a_3a_5$	$2a_4a_6$	$\cdots$
		$2a_0a_4$	$-2a_1a_5$	$2a_2a_6$	$-2a_3a_7$	$\cdots$
			$2a_0a_6$	$-2a_1a_7$	$2a_2a_8$	$\cdots$
				$2a_0a_8$	$-2a_1a_9$	$\cdots$
					$2a_0a_{10}$	$\cdots$
b_0	b_1	b_2	b_3	b_4	b_5	$\cdots$

we obtain the coefficients of the new function

$$F(x^2) = b_0(x^n)^2 + b_1(x^{n-1})^2 + b_2(x^{n-2})^2 + \cdots + b_{n-1}x^2 + b_n \tag{68.6}$$

with

$$\begin{cases} b_0 = a_0^2 \\ b_1 = -a_1^2 + 2a_0a_2 \\ b_2 = a_2^2 - 2a_1a_3 + 2a_0a_4 \\ \cdots\cdots\cdots \\ b_5 = -a_5^2 + 2a_4a_6 - 2a_3a_7 + 2a_2a_8 - 2a_1a_9 + 2a_0a_{10} \\ \cdots\cdots\cdots \end{cases} \tag{68.7}$$

These coefficients have the following characteristics:

a) the terms alternate in sign in both directions;

b) the first term is the square of the corresponding original coefficient;

c) to this is added, algebraically, twice the product of the coefficients equally removed from the one under consideration until either the first or last one is reached.

Consider, for example, b_3:

a) the first term is $-a_3^2$;

b) the second term is $2a_2a_4$, a_2 and a_4 being adjacent to a_3;

c) the third term is $-2a_1a_5$, a_1 and a_5 each being two terms removed from a_3;

d) the fourth is $2a_0a_6$, a_0 and a_6 each being three terms removed from a_3; this is also the last term since it employs a_0.

The general procedure is to continue the process until the original equation has been broken up into n simple equations, from which the desired roots can easily be found. The stopping point is established as being reached when the *double products in the second row have no effect on the coefficients of the next transformed equation.* Suppose we have carried the process k times. Then our last equation is

$$b_0(x^k)^n + b_1(x^k)^{n-1} + b_2(x^k)^{n-2} + \cdots + b_{n-1}(x^k) + b_n = 0$$

from which we write

$$\textbf{(68.8)} \quad \begin{cases} x_1^k = \dfrac{-b_1}{b_0} & \text{and} \quad x_1 = \left(\dfrac{-b_1}{b_0}\right)^{\frac{1}{k}} \\ x_2^k = \dfrac{-b_2}{b_1} & \text{and} \quad x_2 = \left(\dfrac{-b_2}{b_1}\right)^{\frac{1}{k}} \\ x_i^k = \dfrac{-b_i}{b_{i-1}} & \text{and} \quad x_i = \left(\dfrac{-b_i}{b_{i-1}}\right)^{\frac{1}{k}} \end{cases}$$

$$(i = 1, \cdots, n).$$

If the equation has complex roots it cannot be broken into the linear factors (68.8), and this will become evident by the fact that the double products do not all disappear and that the signs of some of the coefficients fluctuate as the process continues.

Since Graeffe's method does not adapt itself readily to calculating machines, we shall not give it any more detailed treatment as the other methods discussed in this chapter will solve the algebraic equations. If an automatic square root machine is available, it is sometimes convenient to apply Graeffe's method once or twice in order to separate the roots somewhat, and then apply one of the other methods.

Example VI.21. Find the roots of

$$2x^4 - 6.001x^3 + 3.999x^2 - 8.002x + 16 = 0.$$

Solution. A tabulation yields

x	0	1	2	3
$f(x)$	16	7.996	$-.016$	27.28

which indicates two roots in the neighborhood of $x = 2$. Instead of using the exceptional case of the Newton-Raphson method, let us apply Graeffe's method twice to obtain

$$f(x) = 16z^4 - 529.02432z^3 + 4865.02432z^2 - 12304.38912z + 65536$$

$$f'(x) = 64z^3 - 1587.07296z^2 + 9730.04864z - 12304.38912$$

where

$$z = x^4.$$

Then

i	z	$f(z)$	$f'(z)$
0	16	−4195. +	−770.28864
1	11	48980. +	−12125.68224
2	15	143. +	−7445.07552
3	15.019281	1.01+	−7342.016163
4	15.019418		
0	17	−407. +	8874.35232
1	17.045870	11.89323	9394.27144
2	17.044604	−.01047	9379.82802
3	17.044605		

Therefore

$$x_1 = z_1^{\frac{1}{4}} = \boxed{1.968626}$$

$$x_2 = z_2^{\frac{1}{4}} = \boxed{2.031874}.$$

Dividing out these two roots from the original equation by synthetic division, we are left with

$$2x^2 + 2x + 4 = 0$$

which we solve by the quadratic equation to get

$$x = \boxed{-.5 \pm 1.322876i}.$$

69. Exercise IX.

1. Solve the following systems of nonlinear equations:

(a) $\begin{cases} x^3 - 3x^2y + y^2 = 7 \\ x^2 - 4x + y^2 - 4y + 4 = 0; \end{cases}$

(b) $\begin{cases} x^2 - 3 \sin y - 4 = 0 \\ y^2 - 3 \sin x - 4 = 0; \end{cases}$ For positive x

(c) $\begin{cases} 121x^2 - 32y^2 = 121 \\ 7x^2 + 7xy + 7y^2 + 70x - 63y = 34. \end{cases}$ For positive x

2. Find all the roots of the following algebraic equations:

(a) $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0;$

(b) $x^5 - 2x^4 + 3x^2 - 4x + 2 = 0;$

(c) $63x^5 - 134x^4 + 8x^3 + 197x^2 - 268x + 142 = 0.$

CHAPTER VII. DIFFERENTIAL AND DIFFERENCE EQUATIONS

70. Introduction. The numerical solution of differential and difference equations is a subject which could constitute an entire book in itself.* For this introductory book on numerical methods we shall limit ourselves to a discussion of the fundamental methods which will solve practically any such equations or systems thereof. Ordinary differential equations will be considered first with difference equations and their application to partial differential equations following. In the first case the problem is to find the values of a function satisfying the differential equation having numerical coefficients and given initial conditions. These values are found by starting with the initial values and then constructing the function by short steps for (usually) equal intervals of the independent variable. Thus the function is generated over a certain range. The numerical solution of differential equations is troublesome in three ways: (1) getting the solution started, (2) choosing the interval length large enough to reduce the amount of labor but not too large to make the solution inaccurate, and (3) checking the solution for errors. Appropriate remarks will be made as each method is discussed.

71. Euler's Method. The simplest method for the numerical solution of a differential equation is due to Euler. It should, however, be employed with caution since it can be very inaccurate. Consider a first-order differential equation:

$$\frac{dy}{dx} = f(x,y) \tag{71.1}$$

* See W. E. Milne, *Numerical Solution of Differential Equations* (New York: John Wiley & Sons, Inc., 1953).

the solution of which may be

$$y = F(x). \tag{71.2}$$

Let us now assume that the function (71.2) has a smooth curve so that for a short distance we may approximate it by a straight line increment, i.e.,

$$\Delta y_i \doteq \left(\frac{dy}{dx}\right)_i \Delta x \tag{71.3}$$

and

$$y_{i+1} \doteq y_i + \Delta y_i = y_i + \left(\frac{dy}{dx}\right)_i \Delta x \tag{71.4}$$

where

$$\left(\frac{dy}{dx}\right)_i = f(x_i,y_i).$$

Formulas (71.3) and (71.4) permit us to generate the function $y = F(x)$ in a step-by-step procedure. The assumptions made, however, necessitate the taking of very small increments on x, and the error can grow as we proceed. The advantage lies in the fact that the computational procedure is very simple.

Example VII.1. Solve the differential equation

$$\frac{dy}{dx} = x + y$$

if $y_0 = 1$ at $x_0 = 0$.

Solution. The computation is arranged in the following schematic. Each value of y is computed by formula (71.4) and the value of $y' = f(x,y)$ from the given equation. The value of Δx is chosen to be 0.1.

x	y	y'
0	1.0000	1.0000
.1	1.1000	1.2000
.2	1.2200	1.4200
.3	1.3620	1.6620
.4	1.5282	1.9282
.5	1.7210	2.2210
.6	1.9431	2.5431
.7	2.1974	2.8974
.8	2.4871	3.2871
.9	2.8158	3.7158
1.0	3.1874	4.1874

A continuation of the method with the same increment value of x yields

x	1.5	2.0	2.5
y	5.8543	10.4548	18.1689

The exact solution to this simple equation may be found to be

$$y = 2e^x - x - 1,$$

and we may find the values at given values for x.

x	0	.5	1.0	1.5	2.0	2.5
y (EULER)	1.000	1.7210	3.1874	5.8543	10.4548	18.1689
y (EXACT)	1.000	1.7974	3.4366	6.4634	11.7781	20.8650

We notice that the values found by Euler's method are getting farther and farther away from the true values. This is characteristic of Euler's method, and for this reason the method should not be employed for a large range of the independent variable.

To overcome the errors of Euler's method it may be modified to give somewhat better results. The modification consists of finding the *average* value of the derivative over the interval for x by a series of successive approximations. The procedure is as follows:

a) compute $y'_{00} = f(x_0,y_0)$ and $y_{10} = y_0 + y'_{00}\Delta x$;

b) compute $y'_{10} = f(x_1,y_{10})$ and $\frac{1}{2}(y'_{00} + y'_{10})$;

c) compute $y_{11} = y_0 + \frac{1}{2}(y'_{00} + y'_{10})\Delta x$ and $y'_{11} = f(x_1,y_{11})$;

d) compute $\frac{1}{2}(y'_{00} + y'_{11})$ and $y_{12} = y_0 + \frac{1}{2}(y'_{00} + y'_{11})\Delta x$,

etc.

The calculations are continued until agreement is reached to the desired degree of accuracy. The procedure is then repeated for the next interval. Applying it to the above example we may arrange the calculations as follows:

x	y_{i0}	y'_{i0}	y_{i1}	y'_{i1}	y_{i2}	y'_{i2}	y_{i3}
0	1.0000	1.0000					
.1	1.1000	1.2000	1.1100	1.2100	1.1105	1.2105	1.1105
.2	1.2316	1.4316	1.2426	1.4426	1.2432	1.4432	1.2432
.3	1.3875	1.6875	1.3997	1.6997	1.4003	1.7003	1.4004
.4	1.5704	1.9704	1.5839	1.9839	1.5846	1.9846	1.5846
.5	1.7831	2.2831	1.7980	2.2980	1.7987	2.2987	1.7988

It is apparent that accurate results with Euler's method are obtained only after considerable labor. The method is therefore recommended only for "quick and dirty" answers. It is also sometimes used to start a solution.

72. Milne's Method. We shall now consider a method devised by W. E. Milne.* Fundamentally, it employs two quadrature formulas, one for predicting ahead and the second for checking the prediction. There are many ways to derive the prediction formula, but we shall limit our derivation to a simple algebraic development from the derivative formulas given in Chapter V.

Let us write our equation (51.23) for the derivative at x_1, x_2, x_3 and x_4, for five points,

$$(72.1)\quad \begin{cases} y'_1 = \dfrac{1}{12h}[-3y_0 - 10y_1 + 18y_2 - 6y_3 + y_4] \\ y'_2 = \dfrac{1}{12h}[y_0 - 8y_1 + 8y_3 - y_4] \\ y'_3 = \dfrac{1}{12h}[-y_0 + 6y_1 - 18y_2 + 10y_3 + 3y_4] \\ y'_4 = \dfrac{1}{12h}[3y_0 - 16y_1 + 36y_2 - 48y_3 + 25y_4], \end{cases}$$

* "Numerical Integration of Ordinary Differential Equations," *American Mathematical Monthly* (1926), Vol. 33, pp. 455–460.

and solve this system for y_4 in terms of y_0, y_1', y_2', and y_3'. This is easily accomplished by adding the first and third to obtain

$$y_1 - y_3 = y_4 - 3h(y_1' + y_3') - y_0 \tag{72.2}$$

and substituting this value into the second equation obtaining

$$12hy_2' = y_0 - 8[y_4 - 3h(y_1' + y_3') - y_0] - y_4 \tag{72.3}$$

from which we get

$$y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3'). \tag{72.4}$$

Since we can choose the points x_i to suit ourselves, we may write the general formula

$$y_{i+1} = y_{i-3} + \frac{4h}{3}(2y_{i-2}' - y_{i-1}' + 2y_i') \tag{72.5}$$

where $h = \Delta x$. This formula yields a value of y in terms of the value of y four steps previous and the values of the derivative of y at the preceding three steps. With this predictor or extrapolation formula there is used a corrector formula which may be obtained from the last three equations of (72.1) by simply multiplying y_3' by 4 and adding:

$$\begin{aligned}
y_2' &= \frac{1}{12h}[y_0 - 8y_1 + 8y_3 - y_4] \\
4y_3' &= \frac{1}{12h}[-4y_0 + 24y_1 - 72y_2 + 40y_3 + 12y_4] \\
y_4' &= \frac{1}{12h}[3y_0 - 16y_1 + 36y_2 - 48y_3 + 25y_4] \\
\hline
y_2' + 4y_3' + y_4' &= \frac{1}{12h}[-36y_2 + 36y_4]
\end{aligned}$$

or

$$y_4 = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']. \tag{72.6}$$

Formulas (72.5) and (72.6) will keep the solution going once it has been started, but the method is dependent upon knowing four values. These initial four values may be obtained by using Taylor's Series expansion or the modified Euler method. Still a third and quite favorable method results from formulas obtained from Taylor's Series. Let us rewrite formula (3.19) in two forms:

$$
(72.7)\quad \begin{cases} f(x_0 + h) = y_1 = y_0 + y_0'h + \frac{1}{2} y_0''h^2 + \frac{1}{3!} y_0'''h^3 \\ \qquad\qquad + \frac{1}{4!} y_0^{iv} h^4 + \cdots \\ f(x_0 - h) = y_{-1} = y_0 - y_0'h + \frac{1}{2} y_0''h^2 - \frac{1}{3!} y_0'''h^3 \\ \qquad\qquad + \frac{1}{4!} y_0^{iv} h^4 + \cdots . \end{cases}
$$

The function $y' = f(x)$ may also be represented in the neighborhood of $x = x_0$ by Taylor's Series:

$$
\begin{cases} y_1' = y_0' + y_0''h + \frac{1}{2} y_0'''h^2 + \frac{1}{3!} y^{iv}h^3 + \frac{1}{4!} y^{v}h^4 + \cdots \\ y_{-1}' = y_0' - y_0''h + \frac{1}{2} y_0'''h^2 - \frac{1}{3!} y^{iv}h^3 + \frac{1}{4!} y^{v}h^4 + \cdots . \end{cases}
$$

By adding and subtracting the last two equations, we may solve for y_0''' and y_0^{iv} to get

$$
\begin{cases} h^2 y_0''' = y_1' + y_{-1}' - 2y_0' - \frac{1}{12} y_0^{v}h^4 - \cdots \\ \frac{1}{3} h^3 y_0^{iv} = y_1' - y_{-1}' - 2y_0''h - \cdots \end{cases}
$$

and upon substituting into equation (72.7) we have

$$
(72.8)\quad y_1 = y_0 + y_0'h + \frac{1}{2} y_0''h^2 + \frac{1}{6} h(y_1' + y_{-1}' - 2y_0')
$$
$$
+ \frac{1}{8} h[y_1' - y_{-1}' - 2y_0''h] + \text{higher-order terms}
$$
$$
= y_0 + \frac{h}{24} [y_{-1}' + 16y_0' + 7y_1'] + \frac{1}{4} y_0''h^2 + \text{h.o.t.}
$$

Similarly

$$
(72.9)\quad y_{-1} = y_0 - \frac{h}{24} [7y_{-1}' + 16y_0' + y_1'] + \frac{1}{4} y_0''h^2 + \text{h.o.t.}
$$

We may repeat the entire procedure for y_2 to get

$$
(72.10)\quad y_2 = y_0 + \frac{2h}{3} [5y_1' - y_0' - y_{-1}'] - 2h^2 y_0'' + \text{h.o.t.}
$$

We now have a sufficient number of formulas to get the solution started. The procedure is as follows:

1. We are given $y' = f(x,y)$, x_0, y_0.
2. Differentiate y' to obtain $y'' = f'(x,y)$ and choose $h = \Delta x$.
3. Calculate y'_0, y''_0, $2h^2y''_0$, and $\frac{1}{4}h^2y''_0$. These remain constant.
4. Use Euler's formulas for trial values

$$y'_1 = y'_0 + hy''_0$$
$$y'_{-1} = y'_0 - hy''_0.$$

5. Compute trial values of y_1 and y_{-1} by formulas (72.8) and (72.9).
6. Compute new values of y'_1 and y'_{-1} from given differential equation.
7. Recompute y_1 and y_{-1} by formulas (72.8) and (72.9).
8. Continue steps 6 and 7 until values converge to desired degree of accuracy.
9. Compute y_2 by formula (72.10).
10. Compute y'_2 from given differential equation.
11. Check y_2 by formula (72.6), i.e.,

$$y_2 = y_0 + \frac{h}{3}(y'_0 + 4y'_1 + y'_2).$$

12. Recompute y'_2 and y_2.
13. Continue until y_2 converges to desired degree of accuracy.

Let us illustrate the entire Milne method by an example.

Example VII.2. Solve the differential equation

$$y' = \frac{2x - 1}{x^2}y + 1$$

with $x_0 = 1$, $y_0 = 2$.

Solution. By differentiation we find

$$y'' = \frac{1}{x^2}[-y' + 2y + 2x].$$

Choose $\Delta x = .1$, let $A(x) = \frac{2x - 1}{x^2}$, and arrange the work as follows:

1. Getting the Solution Started

x	y	y'	$A(x)$	
1.0	2.0000	3.0000		$y_0'' = 3.0000$ $2h^2y_0'' = .0600$
0.9	1.7150	2.7000	.987654	$\frac{1}{4}h^2y_0'' = .0075$
1.1	2.3150	3.3000	.991736	$16y_0' = 48.0000$
0.9	1.715197	2.693827		$y_0 + \frac{1}{4}y_0''h^2 = 2.0075$
1.1	2.314854	3.295867		
0.9	1.715192	2.694021		
1.1	2.314850	3.295724		
0.9	1.715192	2.694016		
1.1	2.314850	3.295720		
1.2	2.658972	3.585111	.972222	
1.2	2.658933	3.585073		
1.2	2.658932	3.585072		

2. Continuing the Solution

	CORRECTED		PREDICTED			
x	y	y'	y	y'	$A(x)$	D
0.9	1.715192	2.694016				
1.0	2.000000	3.000000				
1.1	2.314850	3.295720				
1.2	2.658932	3.585072				
1.3	3.031728	3.870276	3.031782	3.870327	.946746	−54
1.4	3.432894	4.152657	3.432923	4.152683	.918367	−29
1.5	3.862194	4.433062	3.862208	4.433074	.888889	−14
1.6	4.319459	4.712035	4.319468	4.712043	.859375	−9
1.7	4.804566	4.989952	4.804571	4.989956	.830450	−5
1.8	5.317423	5.267067	5.317426	5.267070	.802469	−3
1.9	5.857959	5.543568	5.857961	5.543569	.775623	−2
2.0	6.426121	5.819591	6.426122	5.819592	.750000	−1

The choice of Δx should always be such that only one correction is necessary at each step. A check on this can be established. If the remainder term is carried in the derivation of the predictor and corrector formulas, it can be shown that their respective errors are approximated by

$$+\frac{28}{90}h^5y^v \qquad \text{and} \qquad -\frac{1}{90}h^5y^v.$$

Thus the error of the predictor is approximately 28 times the error in the corrector and in the opposite direction. We may thus compute the difference, $D = y_c - y_p$, between the two values, and if $\frac{D}{29}$ is not significant we assume y_c to be correct. In the example above it is significant at $x = 1.3$, and a more correct value would be $y = 3.031726$. However, it is seen that the values of D get better, and thus there is no need to make Δx smaller.

The differential equation of this example may be solved by analytical methods, and the solution is

$$y = x^2(1 + e^{\frac{1}{x}-1}).$$

At $x = 2$ we have $y = 6.42612$ from a five-place table of the exponential function.

The predictor formula (72.5) employs four previous points. It is, of course, possible to develop formulas which employ more points by considering derivative formulas which employ more points. The utilization of more points, however, increases the amount of labor in keeping the solution going and in starting it.

73. The Runge-Kutta Method. The second method that we shall consider in detail is known as the *Runge-Kutta method* and is essentially a refinement of what may be called averaging methods. Consider again a first-order differential equation,

$$\frac{dy}{dx} = y' = f(x,y), \tag{73.1}$$

with initial values x_0 and y_0. The increment for advancing the dependent variable is now given by

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{73.2}$$

where

$$(73.3)\qquad \begin{cases} k_1 = hf(x_0,y_0) \\ k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) \\ k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) \\ k_4 = hf(x_0 + h,\ y_0 + k_3). \end{cases}$$

The values at (x_1,y_1) are then given by

$$(73.4)\qquad x_1 = x_0 + h \qquad \text{and} \qquad y_1 = y_0 + \Delta y.$$

The increment on y for the second interval is computed by the same formulas, with (x_0,y_0) replaced by (x_1,y_1). Thus all intervals are computed in the same manner, using for the initial values the values at the beginning of each interval. The method does not need any special formulas to get the solution started, and it adapts itself very nicely to a computational form. In order to exhibit this computational form, let us employ a double subscript notation in which the first subscript denotes the interval in which we are working and the second the entries in that interval. Thus we have

$$x_{11} = x_0,\ y_{11} = y_0 \qquad \text{and} \qquad y'_{11} = f(x_0,y_0) = f(x_{11}, y_{11}).$$

Now we see that

$$k_1 = hf(x_0,y_0) = hy'_{11}$$

and

$$y_0 + \frac{1}{2}k_1 = y_0 + \frac{1}{2}hy'_{11} = y_0 + \frac{\Delta x}{2}y'_{11}.$$

Thus we have

$$x_{12} = x_{11} + \frac{\Delta x}{2}, \qquad y_{12} = y_{11} + \frac{\Delta x}{2}y'_{11} \qquad \text{and} \qquad y'_{12} = f(x_{12},y_{12}).$$

Continuing, we see that

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = \Delta x y'_{12}$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = \Delta x y'_{13}$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = \Delta x y'_{14}$$

so that

$$(73.5) \qquad \Delta y_1 = \frac{\Delta x}{6}[y'_{11} + 2y'_{12} + 2y'_{13} + y'_{14}].$$

The computational form for a first-order differential equation follows.

x	y	$\frac{dy}{dx} = y'$	AUXILIARY COMPUTATIONS
$x_{11} = x_0$	$y_{11} = y_0$	$y'_{11} = f(x_{11}, y_{11})$	Whatever is necessary to compute $f(x,y)$
$x_{12} = x_{11} + \frac{\Delta x}{2}$	$y_{12} = y_{11} + y'_{11}\frac{\Delta x}{2}$	$y'_{12} = f(x_{12}, y_{12})$	
$x_{13} = x_{11} + \frac{\Delta x}{2}$	$y_{13} = y_{11} + y'_{12}\frac{\Delta x}{2}$	$y'_{13} = f(x_{13}, y_{13})$	
$x_{14} = x_{11} + \Delta x$	$y_{14} = y_{11} + y'_{13}\Delta x$	$y'_{14} = f(x_{14}, y_{14})$	
		$\Delta y_1 = \frac{\Delta x}{6}(y'_{11} + 2y'_{12} + 2y'_{13} + y'_{14})$	
$x_{21} = x_{11} + \Delta x$	$y_{21} = y_{11} + \Delta y_1$	$y'_{21} = f(x_{21}, y_{21})$	
$x_{22} = x_{21} + \frac{\Delta x}{2}$	$y_{22} = y_{21} + y'_{21}\frac{\Delta x}{2}$	$y'_{22} = f(x_{22}, y_{22})$	
$x_{23} = x_{21} + \frac{\Delta x}{2}$	$y_{23} = y_{21} + y'_{22}\frac{\Delta x}{2}$	$y'_{23} = f(x_{23}, y_{23})$	
$x_{24} = x_{21} + \Delta x$	$y_{24} = y_{21} + y'_{23}\Delta x$	$y'_{24} = f(x_{24}, y_{24})$	
		$\Delta y_2 = \frac{\Delta x}{6}(y'_{21} + 2y'_{22} + 2y'_{23} + y'_{24})$	
$x_{31} = x_{21} + \Delta x$	$y_{31} = y_{21} + \Delta y_2$	$y'_{31} = f(x_{31}, y_{31})$	

The required values for x, y, and $f(x,y)$ are given in the first line of each box.

Note that at each step within an interval the values at the beginning of the interval are used.

Example VII.3. Let us again solve the differential equation

$$y' = \frac{2x - 1}{x^2}y + 1$$

with $x_0 = 1$, $y_0 = 2$.

Solution. Let $A(x) = \dfrac{2x - 1}{x^2}$. Choose $\Delta x = .2$. Then

$$y' = A(x)y + 1 \qquad \text{and} \qquad \frac{\Delta x}{6} = .033333,$$

and the calculations are arranged as follows:

x	y	y'	$A(x)$
1.0	2.000000	3.000000	1.000000
.1	2.300000	3.280993	.991736
.1	2.328099	3.308860	
.2	2.661772	3.587833	.972222
		.658913	
1.2	2.658913	3.585054	
.3	3.017418	3.856728	.946746
.3	3.044586	3.882450	
.4	3.435403	4.154961	.918367
		.773938	
1.4	3.432851	4.152617	
.5	3.848113	4.420545	.888889
.5	3.874906	4.444361	
.6	4.321723	4.713981	.859375
		.886538	
1.6	4.319389	4.711975	
.7	4.790386	4.978342	.830450
.7	4.817223	5.000463	
.8	5.319482	5.268719	.802469
		.997933	
1.8	5.317322	5.266986	
.9	5.844021	5.532757	.775623
.9	5.870598	5.553371	
2.0	6.427996	5.820997	.750000
		1.108664	
2.0	6.425986		

The error at $x = 2.00$ is .00013 or .002%.

74. Accuracy and Choice of Method. The accuracy of a step-by-step solution of a differential equation is often difficult to determine. The Milne method offers a check by the computation of the quantities, D. The Runge-Kutta method has no such check, and the error cannot be determined although it is near the order of h^5. Improvement on the accuracy of any method can be achieved by taking smaller intervals. However, a decrease of interval size adds to the mount of labor and increases the possible round-off-error. The choice as to which method to use is also difficult. The advantages of the Runge-Kutta method are that no special methods are needed to start the solution, the interval length can be changed at any time, and its computational routine has fewer formulas and is easily mastered. The advantages of the Milne method are that it may be more accurate for some problems, it offers a check at each step, and the computational procedure requires fewer evaluations of the derivative.

As a check on the calculations it is advantageous to plot the resulting functions and periodically check by recomputing portions with out-of-phase intervals. We may also remark again that the most common mistake in the use of the Runge-Kutta method is that the student fails to use the initial value of y at each interval, i.e., in computing y_{i3} the incorrect y_{i2} is used instead of the correct y_{i1}, and similarly for y_{i4}.

It is believed that the methods discussed in this chapter will solve most ordinary differential equations with numerical coefficients and sufficient initial conditions. Special equations often require special treatment, but the addition of more methods would further complicate the choice on simple problems.

The reader should also consider the possibility of combining the two methods. Thus, the Runge-Kutta method could be used to start the solution which after four or more points could be continued by the Milne method. Also the corrector formula (72.6) could be used as a periodic check on the solution obtained by the Runge-Kutta method.

75. Higher-Order Differential Equations. The solution of higher-order differential equations in which we can solve for the highest-order derivative may be obtained by repeated application of the methods already discussed. We first apply the method to obtain the next lower-order derivative and then apply the method again to obtain the next lower-order derivative and so on until we arrive at the function. We shall show the details of both the Milne method and the Runge-Kutta method.

Consider first the second-order differential equation

$$\frac{d^2x}{dt^2} = f(x,\dot{x},t) \tag{75.1}$$

in which t is the independent variable with initial conditions, t_0, x_0, $\dot{x}_0$, and use the popular notation of a dot denoting differentiation with respect to t, i.e.,

$$\frac{dx}{dt} = \dot{x}; \qquad \frac{d^2x}{dt^2} = \ddot{x}; \text{ etc.} \tag{75.2}$$

A. The Milne Method.

1. To start the solution, differentiate $\ddot{x} = f(x,\dot{x},t)$ to obtain $\dddot{x} = \dot{f}(x,\dot{x},t)$ and calculate the constants

$a_1 = \dddot{x}_0$ $b_1 = \ddot{x}_0$	$a_2 = 2h^2a_1$ $b_2 = 2h^2b_1$	$a_3 = .25h^2a_1$ $b_3 = .25h^2b_1$	$a_4 = \dot{x}_0 + a_3$ $b_4 = x_0 + b_3$

Record also $16\ddot{x}_0$ and $16\dot{x}_0$.

2. Obtain trial values of $\ddot{x}_i$ from

$$\begin{cases} \ddot{x}_{-1} = \ddot{x}_0 - ha_1 \\ \ddot{x}_1 = \ddot{x}_0 + ha_1. \end{cases}$$

3. Compute trial values of $\dot{x}_i$, $(i = -1,1)$, by

$$\begin{cases} \dot{x}_{-1} = a_4 - \dfrac{h}{24}[7\ddot{x}_{-1} + 16\ddot{x}_0 + \ddot{x}_1] \\ \dot{x}_1 = a_4 + \dfrac{h}{24}[\ddot{x}_{-1} + 16\ddot{x}_0 + 7\ddot{x}_1]. \end{cases} \tag{75.3}$$

4. Compute trial values of x_i, $(i = -1,1)$, by

$$\begin{cases} x_{-1} = b_4 - \dfrac{h}{24}[7\dot{x}_{-1} + 16\dot{x}_0 + \dot{x}_1] \\ x_1 = b_4 + \dfrac{h}{24}[\dot{x}_{-1} + 16\dot{x}_0 + 7\dot{x}_1]. \end{cases} \tag{75.4}$$

5. Compute new values of $\ddot{x}_1$ and $\ddot{x}_{-1}$ from given equation.
6. Repeat steps 3 to 5 until there is no change.
7. Compute $\dot{x}_2$ and x_2 by

$$\text{(75.5)}\qquad \begin{cases} \dot{x}_2 = \dot{x}_0 + \dfrac{2h}{3}[5\ddot{x}_1 - \ddot{x}_0 - \ddot{x}_1] - a_2 \\ x_2 = x_0 + \dfrac{2h}{3}[5\dot{x}_1 - \dot{x}_0 - \dot{x}_{-1}] - b_2. \end{cases}$$

8. Compute $\ddot{x}_2$ from given equation.
9. Check $\dot{x}_2$ and x_2 by

$$\text{(75.6)}\qquad \begin{cases} \dot{x}_2 = \dot{x}_0 + \dfrac{1}{3}h[\ddot{x}_0 + 4\ddot{x}_1 + \ddot{x}_2] \\ x_2 = x_0 + \dfrac{1}{3}h[\dot{x}_0 + 4\dot{x}_1 + \dot{x}_2]. \end{cases}$$

10. Repeat steps 8 and 9 until convergence.
11. Compute $\dot{x}_3$ by

$$\text{(75.7)}\qquad \dot{x}_3 = \dot{x}_{-1} + \frac{4h}{3}(2\ddot{x}_0 - \ddot{x}_1 + 2\ddot{x}_2).$$

12. Compute x_3 by

$$\text{(75.8)}\qquad x_3 = x_1 + \frac{h}{3}(\dot{x}_1 + 4\dot{x}_2 + \dot{x}_3).$$

13. Compute $\ddot{x}_3$ from given equation.
14. Recompute $\dot{x}_3$ by

$$\text{(75.9)}\qquad \dot{x}_3 = \dot{x}_1 + \frac{h}{3}(\ddot{x}_1 + 4\ddot{x}_2 + \ddot{x}_3).$$

15. Repeat steps 12 to 14 until there is no change.
16. Continue the process by steps 11 to 15.

Example VII.4. Find the values of the function satisfying

$$\ddot{x} = \dot{x} + 6x$$

with $x = 2$, $\dot{x} = 1$ at $t = 0$ in the interval $0 \le t \le .5$.

Solution. Choose $\Delta t = .1$. Differentiate to get

$$\dddot{x} = \ddot{x} + 6\dot{x}.$$

The calculations are then arranged as follows:

t	x	$\dot{x}$	$\ddot{x}$	
0	2.0000	1.0000	13.0000	$a_1 =$ 19.0000
				$a_2 =$ .3800
−.1	1.9618	−.2050	11.1	$a_3 =$ .0475
.1	2.1682	2.3950	14.9	$b_2 =$ .2600
				$b_3 =$ .0325
−.1	1.962222	−.220687	11.5658	$16\ddot{x}_0 =$ 208.0000
.1	2.168587	2.411647	15.4042	$16\dot{x}_0 =$ 16.0000
				$a_4 =$ 1.0475
−.1	1.962211	−.220382	11.552645	$b_4 =$ 2.0325
.1	2.168603	2.412145	15.423169	
−.1	1.962211	−.220391	11.552884	
.1	2.168603	2.412163	15.423763	
−.1	1.962211	−.220391	11.552875	
.1	2.168603	2.412164	15.423781	
.2	2.492080	4.124402	19.076882	
.2	2.492480	4.125734	19.080614	
.2	2.492484	4.125858	19.080762	
.2	2.492484	4.125863	19.080767	

CORRECTED

t	x	$\dot{x}$	$\ddot{x}$
−.1	1.962211	−.220391	11.552875
0	2.000000	1.000000	13.000000
.1	2.168603	2.412164	15.423781
.2	2.492484	4.125863	19.080767
.3	3.008506	6.281470	24.332506
.4	3.769616	9.062213	31.679909
.5	4.849856	12.710176	41.809312

APPROXIMATIONS

t	x	$\dot{x}$	$\ddot{x}$
.3	3.008389	6.277976	24.328310
.3	3.008501	6.281336	24.332342
.4	3.769461	9.057574	31.674400
.4	3.769610	9.062036	31.679696
.5	4.849650	12.704010	41.801910
.5	4.849848	12.709938	41.809026

The calculations for continuing the solution are stopped after three approximations. If these are not sufficiently accurate, the interval length should be shortened.

B. The Runge-Kutta Method. In applying the Runge-Kutta method to higher-order equations, simply repeat the process for each derivative until the function is obtained. The schematic is readily adapted to machine calculations which are again worked from the right to the left.

Second-Order Differential Equation

$$\boxed{\frac{d^2x}{dt^2} = f(x,\dot{x},t)}$$

Initial conditions: x_0, $\dot{x}_0$, t_0

t	x	$\dot{x}$	$\ddot{x}$	AUXILIARY COMPUTATIONS
$t_{11} = t_0$	$x_{11} = x_0$	$\dot{x}_{11} = \dot{x}_0$	$\ddot{x}_{11} = f(x_0,\dot{x}_0,t)$	Whatever is necessary to compute $f(x,\dot{x},t)$
$t_{12} = t_{11} + \frac{\Delta t}{2}$	$x_{12} = x_{11} + \dot{x}_{11}\frac{\Delta t}{2}$	$\dot{x}_{12} = \dot{x}_{11} + \ddot{x}_{11}\frac{\Delta t}{2}$	$\ddot{x}_{12} = f(x_{12},\dot{x}_{12},t_{12})$	
$t_{13} = t_{11} + \frac{\Delta t}{2}$	$x_{13} = x_{11} + \dot{x}_{12}\frac{\Delta t}{2}$	$\dot{x}_{13} = \dot{x}_{11} + \ddot{x}_{12}\frac{\Delta t}{2}$	$\ddot{x}_{13} = f(x_{13},\dot{x}_{13},t_{13})$	
$t_{14} = t_{11} + \Delta t$	$x_{14} = x_{11} + \dot{x}_{13}\Delta t$	$\dot{x}_{14} = \dot{x}_{11} + \ddot{x}_{13}\Delta t$	$\ddot{x}_{14} = f(x_{14},\dot{x}_{14},t_{14})$	
		$\Delta x_1 = \frac{\Delta t}{6}(\dot{x}_{11} + 2\dot{x}_{12} + 2\dot{x}_{13} + \dot{x}_{14})$	$\Delta\dot{x}_1 = \frac{\Delta t}{6}(\ddot{x}_{11} + 2\ddot{x}_{12} + 2\ddot{x}_{13} + \ddot{x}_{14})$	
$t_{21} = t_{11} + \Delta t$	$x_{21} = x_{11} + \Delta x_1$	$\dot{x}_{21} = \dot{x}_{11} + \Delta\dot{x}_1$	$\ddot{x}_{21} = f(x_{21},\dot{x}_{21},t_{21})$	
$t_{22} = t_{21} + \frac{\Delta t}{2}$	$x_{22} = x_{21} + \dot{x}_{21}\frac{\Delta t}{2}$	$\dot{x}_{22} = \dot{x}_{21} + \ddot{x}_{21}\frac{\Delta t}{2}$	$\ddot{x}_{22} = f(x_{22},\dot{x}_{22},t_{22})$	
$t_{23} = t_{21} + \frac{\Delta t}{2}$	$x_{23} = x_{21} + \dot{x}_{22}\frac{\Delta t}{2}$	$\dot{x}_{23} = \dot{x}_{21} + \ddot{x}_{22}\frac{\Delta t}{2}$	$\ddot{x}_{23} = f(x_{23},\dot{x}_{23},t_{23})$	
$t_{24} = t_{21} + \Delta t$	$x_{24} = x_{21} + \dot{x}_{23}\Delta t$	$\dot{x}_{24} = \dot{x}_{21} + \ddot{x}_{23}\Delta t$	$\ddot{x}_{24} = f(x_{24},\dot{x}_{24},t_{24})$	
		$\Delta x_2 = \frac{\Delta t}{6}(\dot{x}_{21} + 2\dot{x}_{22} + 2\dot{x}_{23} + \dot{x}_{24})$	$\Delta\dot{x}_2 = \frac{\Delta t}{6}(\ddot{x}_{21} + 2\ddot{x}_{22} + 2\ddot{x}_{23} + \ddot{x}_{24})$	
$t_{31} = t_{21} + \Delta t$	$x_{31} = x_{21} + \Delta x_2$	$\dot{x}_{31} = \dot{x}_{21} + \Delta\dot{x}_2$	$\ddot{x}_{31} = f(x_{31},\dot{x}_{31},t_{31})$	

Example VII.5. Solve Example VII.4 by the Runge-Kutta method.

Solution.

t	x	$\dot{x}$	$\ddot{x}$
0	2.000000	1.000000	13.000000
.05	2.050000	1.650000	13.950000
.05	2.082500	1.697500	14.192500
.10	2.169750	2.419250	15.437750
		.168571	1.412046
.10	2.168571	2.412046	15.423472
.15	2.289173	3.183220	16.918258
.15	2.327732	3.257959	17.224351
.20	2.494367	4.134481	19.100683
		.323815	1.713490
.20	2.492386	4.125536	19.079852
.25	2.698663	5.079529	21.271507
.25	2.746362	5.189111	21.667283
.30	3.011297	6.292264	24.360046
		.515918	2.155291
.30	3.008304	6.280827	24.330651
.35	3.322345	7.497360	27.431430
.35	3.383172	7.652399	27.951431
.40	3.773544	9.075970	31.717234
		.760939	2.780227
.40	3.769243	9.061054	31.676512
.45	4.222296	10.644880	35.978656
.45	4.301487	10.859987	36.668909
.50	4.855242	12.727945	41.859397
		1.079979	3.647184
.50	4.849222	12.708238	41.803570

The particular differential equation used in these examples may be solved by analytical means to yield the solution

$$x = e^{3t} + e^{-2t}.$$

At $t = 0.5$ we have $x = 4.849568$. Thus the errors committed by our step-by-step methods are

MILNE METHOD	−.000288	.006%
RUNGE-KUTTA METHOD	.000346	.007%

Improvements on the numerical solutions may be obtained by taking smaller interval length. This, however, adds considerably to the amount of labor involved.

76. Systems of Equations. Systems of differential equations may be solved by either of our two methods. The procedure is to apply the method of our choice to each of the equations of our system. As before it must be possible to express the highest-order derivative as a function of the variables and lower-order derivatives. We shall illustrate by an example.

Example VII.6. Find the values of the functions satisfying the system of differential equations

$$\begin{cases} 2\ddot{y} - \dot{x} - 4y - 4t = 0 \\ 4\dot{x} + 2\dot{y} - 3x = 0 \end{cases}$$

with $x = \frac{7}{3}$, $y = 4$, $\dot{y} = -3.5$ at $t = 0$.

Solution. Transform the given equations into the following system:

$$\begin{cases} \ddot{y} = .5\dot{x} + 2y + 2t \\ \dot{x} = .75x - .5\dot{y} \end{cases}$$

and choose $\Delta t = .1$. For the Milne method we need

$$\begin{cases} \dddot{y} = .5\ddot{x} + 2\dot{y} + 2 \\ \ddot{x} = .75\dot{x} - .5\ddot{y}. \end{cases}$$

t	x	$\dot{x}$	y	$\dot{y}$	$\ddot{y}$	CONSTANTS	FOR (x)	FOR (y)
0	2.3333	3.5000	4.0000	−3.5000	9.7500	a_1		−6.1250
						a_2		−.1225
−.1	1.9718	3.7250	4.3998	−4.5056	10.3625	a_3		−.0153
.1	2.6718	3.2750	3.6978	−2.5556	9.1375	b_1	−2.25	
						b_2	−.0450	.1950
−.1	1.9716	3.7316	4.3999	−4.5090	10.4654	b_3	−.0056	.0244
.1	2.6720	3.2816	3.6978	−2.5523	9.2364	$16\ddot{u}_0$		156.0000
						$16\dot{u}_0$	56.0000	−56.0000
−.1	1.9715	3.7332	4.3999	−4.5091	10.4664	a_4		−3.5153
.1	2.6720	3.2802	3.6978	−2.5523	9.2357	b_4	2.3274	4.0244
.2	2.9921	3.0674	3.4882	−1.6467	8.9101			
.2	2.9896	3.0655	3.4881	−1.6466	8.9090			
.2	2.9895	3.0654	3.4881	−1.6466	8.9089			
−.1	1.9715	3.7332	4.3999	−4.5091	10.4664			
0	2.3333	3.5000	4.0000	−3.5000	9.7500			
.1	2.6720	3.2802	3.6978	−2.5523	9.2357			
.2	2.9895	3.0654	3.4881	−1.6466	8.9089			
.3	3.2849	2.8461	3.3677	−.7648	8.7585	PREDICTED		
.3	3.2849	2.8460	3.3677	−.7646	8.7584	CORRECTED		
.4	3.5583	2.6134	3.3350	.1106	8.7767	PREDICTED		
.4	3.5583	2.6134	3.3350	.1107	8.7767	CORRECTED		
.5	3.8069	2.3571	3.3902	.9961	8.9590	PREDICTED		
.5	3.8068	2.3570	3.3902	.9962	8.9589	CORRECTED		

The solution using the Runge-Kutta method is given on page 244. The true solutions are $x(.5) = 3.8071$ and $y(.5) = 3.3902$. The fewer formulas used by the Runge-Kutta method adds to its advantages in the solution of systems of differential equations. Since one of its disadvantages is that there is no check on the accuracy, we may use the correction formulas of Milne's method as a periodic check.

77. Difference Equations. A very general definition of a differential equation is that it is one which involves one or more derivatives of the dependent variable. In a similar manner we may say that a difference equation is one which involves one or more of the differences of the dependent variable. To be a little more precise, let us recall the definition of a derivative of a simple function $f(x)$ which states that it is the limiting value of the quotient

$$\frac{f(x+h) - f(x)}{h}. \tag{77.1}$$

The Runge-Kutta method for Example VII.6:

t	x	$\dot{x}$	y	$\dot{y}$	$\ddot{y}$
0	2.3333	3.5000	4.0000	−3.5000	9.7500
.05	2.5083	3.3875	3.8250	−3.0125	9.4438
.05	2.5027	3.3090	3.8486	−3.0278	9.4926
.10	2.6724	3.2796	3.6972	−2.5507	9.2342
		.3389		−.3022	.9476
.10	2.6722	3.2804	3.6978	−2.5524	9.2358
.15	2.8362	3.1724	3.5702	−2.0906	9.0266
.15	2.8308	3.1736	3.5933	−2.1011	9.0734
.20	2.9896	3.0648	3.4877	−1.6451	8.9078
		[illegible]		−.2097	.9057
.20	2.9895	3.0655	3.4881	−1.6467	8.9090
.25	3.1428	2.9577	3.4058	−1.2012	8.7904
.25	3.1374	2.9566	3.4280	−1.2072	8.8343
.30	3.2852	2.8456	3.3674	−.7633	8.7576
		.2957		−.1204	.8819
.30	3.2852	2.8463	3.3677	−.7648	8.7586
.35	3.4275	2.7341	3.3295	−.3269	8.7260
.35	3.4219	2.7307	3.3514	−.3285	8.7682
.40	3.5583	2.6127	3.3348	.1120	8.7760
		.2731		−.0327	.8754
.40	3.5583	2.6134	3.3350	.1106	8.7767
.45	3.6890	2.4920	3.3405	.5494	8.8270
.45	3.6829	2.4862	3.3625	.5520	8.8681
.50	3.8069	2.3565	3.3902	.9974	8.9586
		.2488		.0552	.8854
.50	3.8071	2.3573	3.3902	.9960	8.9590

The expression (77.1) is known as a **difference quotient,** and it is a good approximation to the derivative especially if h is very small. Thus, if we were to replace all the derivatives in a differential equation by the corresponding difference quotients, we would have a *difference equation.* The subject of difference equations is quite extensive. We shall, however, limit ourselves to those which permit us to obtain numerical solutions of partial differential equations.

We concern ourselves, therefore, with partial difference quotients of the second and higher orders. Let us begin with a function $u(x,y)$ of two variables and let us divide the x,y-plane into a network by two families of parallel lines

$$\begin{cases} x = mh, & (m = 0, 1, 2, \cdots), \\ y = nh, & (n = 0, 1, 2, \cdots), \end{cases} \tag{77.2}$$

as is done in the construction of ordinary graph paper. The points of intersection of these lines are called **lattice points.**

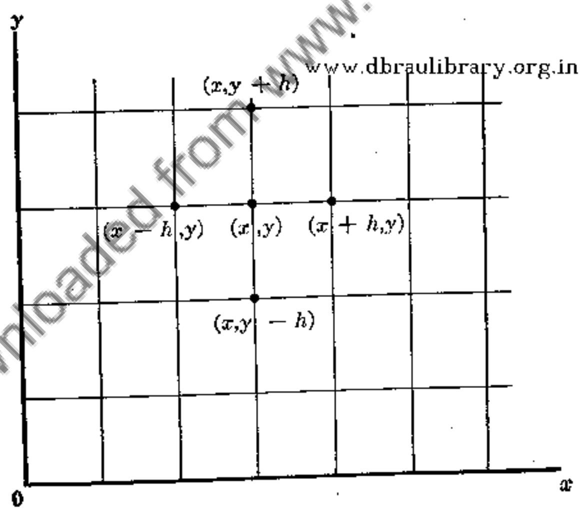


Fig. 7.1

For each of the variables of the function $u(x,y)$, we have a forward and backward difference quotient. Thus with respect to x

$$(77.3)\qquad \begin{cases} u_x = \dfrac{u(x+h,\, y) - u(x,y)}{h} \\[2ex] u_{\bar{x}} = \dfrac{u(x,y) - u(x-h,\, y)}{h} \end{cases}$$

and with respect to y

$$(77.4)\qquad \begin{cases} u_y = \dfrac{u(x,y+h) - u(x,y)}{h} \\[2ex] u_{\bar{y}} = \dfrac{u(x,y) - u(x,y-h)}{h} \end{cases}$$

in which the forward difference quotients are denoted by u_x and u_y, the backward difference quotients by $u_{\bar{x}}$ and $u_{\bar{y}}$.

We may now define a second difference quotient of $u(x,y)$ with respect to x as the difference quotient of the first difference quotients, i.e.,

$$(77.5)\qquad u_{\bar{x}x} = \frac{u_x - u_{\bar{x}}}{h} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$$

and similarly

$$(77.6)\qquad u_{\bar{y}y} = \frac{u_y - u_{\bar{y}}}{h} = \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

It is our purpose to employ these definitions of difference quotients to replace partial derivatives in a partial differential equation and then solve the resulting difference equation. In so doing it should be remembered that the functions occurring in the difference equations are defined at the lattice points only. To get a better description of the function it then becomes necessary to increase the number of lattice points.

78. Partial Differential Equations. In our discussion of the numerical solution of partial differential equations, we shall limit ourselves to those equations which can be replaced by the equivalent difference equation and consider the methods of solving the resulting difference equations. Furthermore, we shall consider only a few methods which, however, are sufficient to solve many problems.*

* For a more detailed discussion of partial differential equations, see W. E. Milne, *Numerical Solutions of Differential Equations* (New York: John Wiley & Sons, Inc., 1953), Part II. Or see F. D. Murnaghan, *Introduction to Applied Mathematics* (New York: John Wiley & Sons, Inc., 1948), Chapters 5–7.

The general procedure is to replace the partial derivatives by the equivalent difference quotients and then to obtain the solution at the lattice points. Let us consider some well-known partial differential equations.

I. Laplace's Equation in Two Dimensions.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{78.1}$$

Transforming to a difference equation by

$$u_{xx} \sim \frac{\partial^2 V}{\partial x^2} \quad \text{and} \quad u_{\bar{y}y} \sim \frac{\partial^2 V}{\partial y^2} \tag{78.2}$$

we have

$$h^{-2}[u(x + h,y) - 2u(x,y) + u(x - h,y)] + h^{-2}[u(x,y + h) - 2u(x,y) + u(x,y - h)] = 0.$$

Solving this equation for $u(x,y)$, we obtain

$$u(x,y) = \frac{1}{4}[u(x + h,y) + u(x,y + h) + u(x - h,y) + u(x,y - h)]. \tag{78.3}$$

We see that in this equation the value of $u(x,y)$ at any interior lattice point is the arithmetic mean of the values of $u(x,y)$ at the four lattice points surrounding it.

II. Poisson's Equation in Two Dimensions.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi\rho(x,y). \tag{78.4}$$

Again using the transformation (78.2) and solving for $u(x,y)$ we obtain

$$u(x,y) = \frac{1}{4}[u(x + h,y) + u(x,y + h) + u(x - h,y) + u(x,y - h)] + \pi h^2\rho(x,y). \tag{78.5}$$

In this case the value of $u(x,y)$ at an interior lattice point depends upon the values of $u(x,y)$ at the four adjacent points, the value of h, and the value of the function $\rho(x,y)$.

III. The Parabolic Equation.

(78.6) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

In dividing the xt-plane into a network let us choose different increments on x and t,

$$\Delta t = k \quad \text{and} \quad \Delta x = h.$$

The transformation to a difference equation yields

(78.7) $$k^{-2}[u(x,t + k) - u(x,t)] = c^2h^{-2}[u(x + h,t) - 2u(x,t) + u(x - h,t)].$$

Let $c^2kh^{-2} = r$. Then we have

(78.8) $$u(x,t + k) = r[u(x + h,t) + u(x - h,t)] + (1 - 2r)u(x,t)$$

from which we can calculate the values of $u(x,t + k)$ from the values at $u(x,t)$, $u(x + h,t)$ and $u(x - h,t)$ once we have chosen k and h so that r is known.

IV. Two-Dimensional Heat-Flow Equation.

(78.9) $$\frac{\partial T}{\partial t} = \alpha^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right).$$

Let $\Delta t = k$, $\Delta y = \Delta x = h$; then the difference equation is

(78.10) $$k^{-1}[T(x,y,t + k) - T(x,y,t)] = \alpha^2h^{-2}[T(x + h,y,t) - 2T(x,y,t) + T(x - h,y,t) + T(x,y + h,t) - 2T(x,y,t) + T(x,y - h,t)].$$

If we choose $4k = h^2\alpha^{-2}$ we obtain

(78.11) $$T(x,y,t + k) = \frac{1}{4}[T(x + h,y,t) + T(x,y + h,t) + T(x - h,y,t) + T(x,y - h,t)],$$

an equation which gives the temperature at any interior lattice point at a time $t + k$ in terms of the temperatures at four adjacent points at time t.

It should now be clear how each partial differential equation is changed to an equivalent difference equation. This difference equation is rearranged to give an expression for a particular value of the function in terms of the values of the function at adjacent points and other constants or known functional values. Frequently, the resulting expression also depends upon the choice of the increments on the independent variables. Since these expressions give the values of the function at interior lattice points, its values on the boundaries must be known. We shall now turn to the problem of finding the values at the interior points, having given the necessary boundary conditions.

79. The Method of Iteration. To illustrate this method let us first consider a simple equation in two variables such as Laplace's equation. In the last section we obtained the equivalent difference equation

$$(79.1)\quad u(x,y) = \frac{1}{4}[u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h)]$$

for a network of small squares of side h. Let the known boundary values of the function $u(x,y)$ be denoted by a_i. We shall first cover the area with a coarse network of squares as shown in Fig. 7.2.

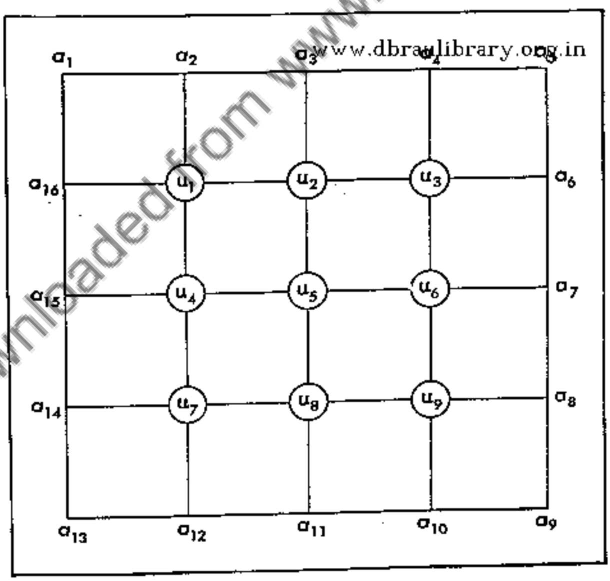


Fig. 7.2

The first approximations to the interior points are now computed in the following order:

(79.2)

$$u_5 = \frac{1}{4}[a_7 + a_3 + a_{15} + a_{11}] \qquad u_2 = \frac{1}{4}[u_3 + a_3 + u_1 + u_5]$$

$$u_1 = \frac{1}{4}[u_5 + a_3 + a_1 + a_{15}] \qquad u_4 = \frac{1}{4}[u_5 + u_1 + a_{15} + u_7]$$

$$u_3 = \frac{1}{4}[a_7 + a_5 + a_3 + u_5] \qquad u_6 = \frac{1}{4}[a_7 + u_3 + u_5 + u_9]$$

$$u_7 = \frac{1}{4}[u_5 + a_{15} + a_{13} + a_{11}] \qquad u_8 = \frac{1}{4}[u_9 + u_5 + u_7 + a_{11}]$$

$$u_9 = \frac{1}{4}[a_7 + u_5 + a_{11} + a_9]$$

We note that the values of u_i, $(i = 5,2,4,6,8)$, are calculated according to the schematic

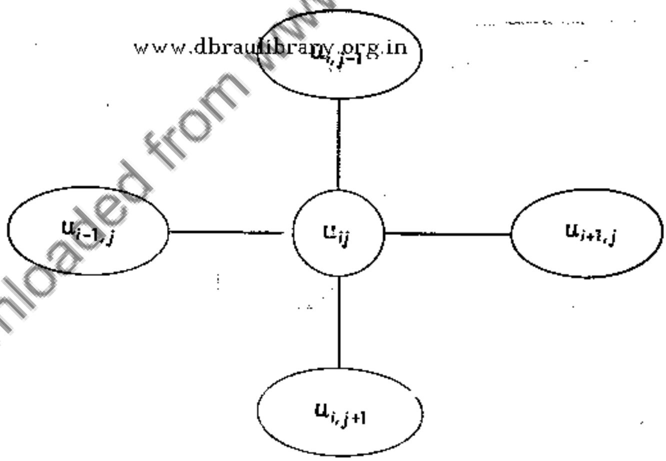


Fig. 7.3

which is the general scheme by equation (79.1).

The values of u_i, $(i = 1,3,7,9)$, are calculated according to the schematic

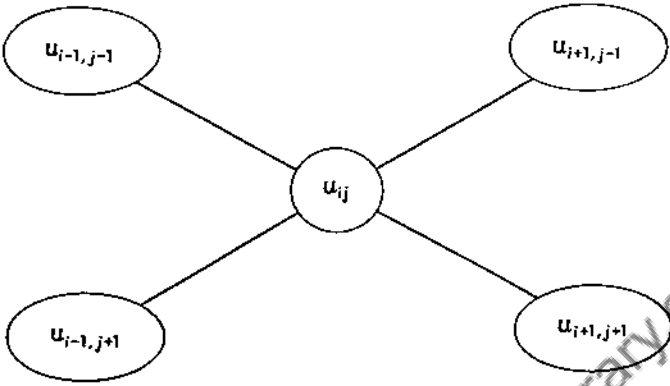


Fig. 7.4

The schematic of Fig. 7.4 is used only to get the first approximation.

The second approximations are now computed in order by the schematic of Fig. 7.3.

(79.3)

$$
\begin{aligned}
{}_2u_1 &= \frac{1}{4}[u_2 + a_2 + a_{16} + u_4] \\
{}_2u_2 &= \frac{1}{4}[u_3 + a_3 + {}_2u_1 + u_5] \\
{}_2u_3 &= \frac{1}{4}[a_5 + a_4 + {}_2u_2 + u_6] \\
{}_2u_4 &= \frac{1}{4}[u_5 + {}_2u_1 + a_{15} + u_7] \\
{}_2u_5 &= \frac{1}{4}[u_6 + {}_2u_2 + {}_2u_4 + u_8]
\end{aligned}
\qquad
\begin{aligned}
{}_2u_6 &= \frac{1}{4}[a_7 + {}_2u_3 + {}_2u_5 + u_9] \\
{}_2u_7 &= \frac{1}{4}[u_8 + {}_2u_4 + a_{14} + a_{12}] \\
{}_2u_8 &= \frac{1}{4}[u_9 + {}_2u_5 + {}_2u_7 + a_{11}] \\
{}_2u_9 &= \frac{1}{4}[a_8 + {}_2u_6 + {}_2u_8 + a_{10}]
\end{aligned}
\tag{79.3}
$$

It is to be noted that the new values of u_i are used as soon as they are available. The successive approximations are calculated by the formulas (79.3) with the new values just previously found.

Example VII.7. Find the function satisfying Laplace's two-dimensional equation with the boundary conditions

$(u = 0, x = 0)$; $\left(u = \frac{1}{2}x^2, y = 0\right)$; $(u = 8 + 2y, x = 4)$;
$(u = x^2, y = 4)$.

Solution. For the first coarse network take $x = y = 0, 1, 2, 3, 4$. The boundary values are calculated at these points and placed on the square array. The calculations are then carried out in the manner outlined above.

$y \backslash x$	0	1.0	2.0	3.0	4
0	0	.5	2.0	4.5	8
1	0	1.625 1.5625 1.5547 1.5640 . . . . . . 1.5670	3.6875 3.6719 3.6982 3.7088 . . . . . . 3.7054	6.625 6.5586 6.5722 6.5698 . . . . . . 6.5670	10
2	0	2.0625 2.0469 2.0576 [illegible] 2.0626	4.50 4.6797 4.6992 4.6981 . . . . . . 4.6876	8.0625 8.0908 8.0704 8.0656 . . . . . . 8.0626	12
3	0	2.125 1.9961 2.0002 1.9983 . . . . . . 1.9956	4.9375 4.9502 4.9274 4.9227 . . . . . . 4.9197	9.125 9.0102 8.9994 8.9971 . . . . . . 8.9956	14
4	0	1.0	4.0	9.0	16

The calculations are continued until there is no change, with the above numbers obtained after ten calculations. The network is now changed to a finer mesh by halving the interval on both x and y. This necessitates calculating the boundary values from the given conditions and then proceeding with the calculations according to formula (79.1) and the schematics of Figs. 7.4 and 7.3. The values from the coarse network are used as the first approximations. After an entry at each interior point has been established, only the schematic of Fig. 7.3 is used. The calculations are arranged as follows:

y \ x	0	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
0		.125	.500	1.125	2.000	3.125	4.500	6.125	8
.5	0	.5168 .5011 .4992 ...... .5060	1.1317 1.1310 1.1299 ...... 1.1414	1.9431 1.9421 1.9432 ...... 1.9581	2.9604 2.9602 2.9629 ...... 2.9782	4.1931 4.1930 4.1944 ...... 4.2081	5.6317 5.6317 5.6316 ...... 5.6414	7.2668 7.2511 7.2510 ...... 7.2560	9
1.0	0	.7478 .7439 .7460 ...... .7577	1.5670 1.5782 1.5805 ...... 1.6017	2.5553 2.5578 2.5615 ...... 2.5879	3.7054 3.7155 3.7184 ...... 3.7468	5.0553 5.0578 5.0626 ...... 5.0879	6.5670 6.5807 6.5827 ...... 6.6017	8.2478 8.2473 8.2480 ...... 8.2577	10
1.5	0	.9074 .9064 .9075 ...... .9233	1.8856 1.8882 1.8924 ...... 1.9198	3.0056 3.0069 3.0104 ...... 3.0452	4.2886 4.2914 4.2967 ...... 4.3333	5.7556 5.7570 5.7614 ...... 5.7952	7.3856 7.3894 7.3946 ...... 7.4198	9.1574 9.1582 9.1606 ...... 9.1733	11
2.0	0	.9962 .9959 .9992 ...... 1.0159	2.0626 2.0748 2.0783 ...... 2.1090	3.2930 3.2964 3.3019 ...... 3.3400	4.6876 4.7009 4.7054 ...... 4.7462	6.2933 6.2967 6.3028 ...... 6.3400	8.0626 8.0762 8.0808 ...... 8.1090	9.9962 9.9998 10.0020 ...... 10.0159	12
2.5	0	1.0146 1.0145 1.0161 ...... 1.0313	2.1223 2.1253 2.1327 ...... 2.1604	3.4164 3.4180 3.4236 ...... 3.4598	4.9225 4.9262 4.9333 ...... 4.9717	6.6664 6.6682 6.6736 ...... 6.7098	8.6223 8.6262 8.6352 ...... 8.6604	10.7646 10.7664 10.7709 ...... 10.7813	13
3.0	0	.9398 .9398 .9380 ...... .9488	1.9956 2.0184 2.0208 ...... 2.0416	3.3276 3.3337 3.3401 ...... 3.3672	4.9197 4.9361 4.9417 ...... 4.9711	6.8276 6.8322 6.8415 ...... 6.8672	8.9956 9.0198 9.0260 ...... 9.0416	11.4398 11.4463 11.4420 ...... 11.4488	14
3.5	0	.7489 .7176 .7167 ...... .7222	1.6808 1.6787 1.6793 ...... 1.6900	2.9788 2.9798 2.9826 ...... 2.9964	4.6568 4.6612 4.6638 ...... 4.6785	6.7288 6.7310 6.7357 ...... 6.7464	9.1808 9.1874 9.1832 ...... 9.1900	11.9989 11.9709 11.9688 ...... 11.9722	15
4.0	0	.25	1.00	2.25	4.00	6.25	9.00	12.25	16

The last set of entries were obtained after 30 iterations.

If the boundary values are obtained from empirical data, the values at the desired points when the mesh is halved are approximated by graphical means or fitting the known points. The values at the interior points cannot be any more accurate than the approximated boundary values.

80. The Method of Relaxation. This is a second method for solving the problem which was discussed in the last section. In the iteration method we employed the four adjacent points and sought to obtain the values such that equation (79.1) was satisfied, i.e.,

$$u_{ij} = \frac{1}{4}(u_{i+1,j} + u_{i,j-1} + u_{i-1,j} + u_{i,j+1}). \tag{80.1}$$

This condition, of course, is not satisfied until the very last set of entries. Thus, at any prior set of entries there is a *residual*, R_{ij}, at each point.

$$R_{ij} = u_{i+1,j} + u_{i,j-1} + u_{i-1,j} + u_{i,j+1} - 4u_{ij} \tag{80.2}$$

has a definite non-zero value. The method of relaxation seeks to make all of these residuals zero by continuously altering the values of the function at the interior points. Therefore it is desirable to develop a process for doing these alterations or, as it has been termed, relaxations.

Let us again make up a lattice network and this time use the double subscript notation

$$\begin{matrix} u_{11} & u_{12} & u_{13} & u_{14} & \cdots & u_{1n} \\ u_{21} & u_{22} & u_{23} & u_{24} & \cdots & u_{2n} \\ u_{31} & u_{32} & u_{33} & u_{34} & \cdots & u_{3n} \\ u_{41} & u_{42} & u_{43} & u_{44} & \cdots & u_{4n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{n1} & u_{n2} & u_{n3} & u_{n4} & \cdots & u_{nn} \end{matrix}$$

Consider as an illustration the point, P_{33}. The residual at this point is

$$R_{33} = u_{34} + u_{23} + u_{32} + u_{43} - 4u_{33}. \tag{80.3}$$

If we alter the value of the function at this point by an amount Δu_{33} the residual will also be altered, and we have

$$R_{33} + \Delta R_{33} = u_{34} + u_{23} + u_{32} + u_{43} - 4(u_{33} + \Delta u_{33}). \tag{80.4}$$

By a simple subtraction we see that

$$\Delta R_{33} = -4\Delta u_{33} \tag{80.5}$$

or that the resulting change in the residual is a negative four times the change in the function at that point. If we now wish to make the residual zero, i.e., make $\Delta R_{33} = -R_{33}$, then we simply change the value of the function by an amount

$$\Delta u_{33} = \frac{1}{4} R_{33} \tag{80.6}$$

and $R_{33} = 0$.

However, a change in u_{33} will affect the residuals at the four adjacent points. Thus

$$R_{23} = u_{24} + u_{13} + u_{22} + u_{33} - 4u_{23} \tag{80.7}$$

and a change in u_{33} gives

$$R_{23} + \Delta R_{23} = u_{24} + u_{13} + u_{22} + (u_{33} + \Delta u_{33}) - 4u_{23} \tag{80.8}$$

from which we see that

$$\Delta R_{23} = \Delta u_{33}. \tag{80.9}$$

The same is true for R_{34}, R_{32}, and R_{43}. Consequently, if the value of a function is changed (relaxed), the residuals of the four adjacent interior points must be changed by the same amount.

The above defines the process for performing the method of relaxation. Having obtained values at all the interior points, we then calculate all the residuals at these points. The method of relaxation then begins by "relaxing" the value of the function with the *largest* residual by the amount $\Delta u_{ij} = \frac{1}{4} R_{ij}$. *This makes* $R_{ij} = 0$ *and necessitates the changing of* $R_{i+1,j}$, $R_{i-1,j}$, $R_{i,j+1}$, $R_{i,j-1}$ *by the amount* Δu_{ij}. The latter is very important, and the failure to carry it out constitutes the most frequent error made by the inexperienced computer. All the corrections resulting from a change in a point must be made before proceeding to another point. After the corrections reducing the largest residual to zero have been made, we proceed to the residual which now has become the largest. The corrections are made until all the R_{ij} are zero or as nearly zero as possible.

Let us illustrate the method by an example.

Example VII.8. Solve the problem of Example VII.7 by the method of relaxation.

Solution. Consider first the coarse network of $x = y = i$, $(i = 0,1,2,3,4)$. The first values of the interior points are computed in exactly the same manner as in Example VII.7. The residuals R_{ij}

are now calculated and recorded in the left column of each block. The largest residual, $R_{22} = .7500$, is at the point P_{22} which has a function value of $u_{22} = 4.5000$. This value is now altered by the amount $\frac{1}{4} R_{22} = \frac{1}{4}(.7500) = .1875$ which is recorded in the right column of each block. This reduces R_{22} to zero and changes R_{23} to .1875, R_{12} to .1875, R_{21} to .1875, and R_{32} to .1875. After these changes have been made, the largest residual is then $R_{31} = R_{33} = -.500$. Suppose we choose to work on R_{33} and change u_{33} by the amount $\frac{1}{4} R_{33} = -.125$. This changes R_{33} to zero, R_{23} to .0625, and R_{32} to .0625. Note that there are no residuals at the boundary points. Next we change the functional value u_{31} to make R_{31} zero, etc. The entire calculations are arranged as follows:

y \ x	0	1			2			3			4
		R	u	Δu	R	u	Δu	R	u	Δu	
0	0		.5			2			4.5		8
1	0	−.25	1.625		0	3.6875		−.25	6.625		10
		0	1.5625	−.0625	.1875			0	6.5625	−.0625	
					.1250						
					.0625		.0156				
		.0156		.0039	.0001	3.7031		.0156		.0039	
		0	1.5664		.0040			0	6.5664		
					.0079		.0020				
		.0020		.0005	−.0001	3.7051		.0020		.0005	
		0	1.5669		.0004			0	6.5669		
		.0002			.0009		.0002	.0002			
					.0001	3.7053					
2	0	0	2.0625		.75	4.5	.1875	0	8.0625		12
		.1875			0	4.6875		.1875			
		.0625			.0156			.0625			
		0			0			0			
		.0039			.0020			.0039			
		0			0			0			
		.0005			.0002			.0005			
		0			0			0			
3	0	−.5	2.125		0	4.9375		−.5	9.125		14
		0	2.000	−.125	.1875			0	9.000	−.125	
					.0625						
		−.0156		−.0039	−.0625		−.0156	−.0156		−.0039	
		0	1.9961		−.0001	4.9219		0	8.9961		
					−.0040						
					−.0079	4.9199	−.0020	−.0020			
		−.0020		−.0005	.0001			0	8.9956	−.0005	
		0	1.9956		−.0004						
					−.0009		−.0002				
		−.0002			−.0001	4.9197		−.0002			
4	0		1			4			9		16

If it is desired to have the functional values at a finer mesh, we could again halve the intervals as we did in Example VII.7 and obtain the values at the finer mesh points by the method of relaxation. We should obtain the same values as were obtained previously. The calculations are left as an exercise for the reader. The schematic for performing the calculations exhibited above may be slightly improved by noting the order in which the points are relaxed. Thus the corrections listed in the right column could be numbered by a colored pencil.

81. Direct Step-by-Step Method. Equations of the parabolic or hyperbolic type may be solved by a direct step-by-step method showing the growth of the function, provided sufficient boundary conditions are given. Let us consider the parabolic equation (78.6). We saw in Section 78 that we could obtain an expression for for $u(x,t + k)$ in terms of $u(x,t)$, $u(x + h,t)$ and $u(x - h,t)$, i.e.,

(81.1) $$u(x,t + k) = r[u(x + h,t) + u(x - h,t)] + (1 - 2r)u(x,t),$$

where $r = c^2kh^{-2}$.

This expression gives the growth of the function as t increases; it is dependent upon the choice of h and k and having sufficient boundary values. Let us consider an example.

Example VII.9. Find the values of the function satisfying

$$\frac{\partial U}{\partial t} = 4\frac{\partial^2 U}{\partial x^2}$$

with the boundary conditions

$$U = 0 \text{ at } x = 0 \text{ and } x = 8$$

$$U = 4x - \frac{1}{2}x \text{ at } t = 0.$$

Solution. The formula (81.1) would simplify to

(81.2) $$u(x,t + k) = \frac{1}{2}[u(x + h,t) + u(x - h,t)]$$

if we could choose h and k such that $r = c^2kh^{-2}$ is $\frac{1}{2}$. Let us therefore choose $h = 1$. Then $r = 4k = \frac{1}{2}$ would yield $k = \frac{1}{8}$ making the increment on $t = \frac{1}{8}$. We may now arrange the work according to the following schematic. The entries in the first row, first and last columns, are calculated from the boundary conditions, and the other entries from formula (81.2).

$t \backslash x$	0	1	2	3	4	5	6	7	8
0	0	3.5000	6.0000	7.5000	8.0000	7.5000	6.0000	3.5000	0
1/8	0	3.0000	5.5000	7.0000	7.5000	7.0000	5.5000	3.0000	0
2/8	0	2.7500	5.0000	6.5000	7.0000	6.5000	5.0000	2.7500	0
3/8	0	2.5000	4.625	6.0000	6.5000	6.0000	4.6250	2.5000	0
4/8	0	2.3125	4.2500	5.5625	6.0000	5.5625	4.2500	2.3125	0
5/8	0	2.1250	3.9375	5.1250	5.5625	5.1250	3.9375	2.1250	0
6/8	0	1.9688	3.6250	4.7500	5.1250	4.7500	3.6250	1.9688	0
7/8	0	1.8125	3.3594	4.3750	4.7500	4.3750	3.3594	1.8125	0
8/8	0	1.6797	3.0938	4.0547	4.3750	4.0547	3.0938	1.6797	0

Formula (81.2) is a simple averaging formula and could yield quite large errors. The formula can be improved if the function $U(x,t)$ has continuous partial derivatives with respect to x of order 6 and with respect to t of order 3. Then it can be shown* that

$$(81.3) \quad U(x,t+k) = \frac{1}{6}[U(x+h,t) + 4U(x,t) + U(x-h,t)] + \text{error function}$$

if h and k are chosen so that $r = \frac{1}{6}$. Improvements on the functional values may also be obtained by choosing a finer mesh, i.e., smaller values for h and k. Remember, however, that the formula for the calculation of the values of the function is dependent upon h and k through r, and thus it may change if h and k are chosen arbitrarily.

82. Remarks on the Solution of Partial Differential Equations. We have described three methods for the numerical solution of partial differential equations. All are based on the idea of replacing the partial derivatives by difference quotients and then solving the resulting difference equation. There is indeed much more to the

* W. E. Milne, *Numerical Solution of Differential Equations* (New York: John Wiley & Sons, Inc., 1953), Chapter 8.

theory than this. In general, however, most of the methods have been developed in order to solve specific problems. The above has been intended only as an introduction.*

83. Exercise X.

1. Solve the following differential equations by both Milne's method and the Runge-Kutta method.

(a) $y' = \dfrac{y^2x^2 + y}{x}$ with $y = -3$ at $x = 1$.

(b) $y' = x^2y^2 - xy$ with $y = 1$ at $x = 0$.

(c) $xy''' - 2y'' - 12x^3 = 0$ with $y'' = y = 0,\ y' = 1$ at $x = 1$.

(d) $\begin{cases}\dot{x} - 2y = 0\\ \dot{y} - z = 0\\ \dot{z} + 2x = 0.\end{cases}$ with $x = 1,\ y = 2,\ z = 3$ at $t = 0$

(e) $\begin{cases}\ddot{x} + 2\dot{y} + 150A(x,y)x = 0\\ \ddot{y} + 150A(x,y)y = 2(\dot{x} - 1)\end{cases}$ where $A(x,y) = 1 - (x^2 + y^2)^{-\frac{1}{2}}$

and at $t = 0$

$x = .63813,\ \dot{x} = -.91781,\ y = -.82945,\ \dot{y} = -1.70032.$

2. Continue the Milne method solution of Example VII.4 to $t = 1.0$ using $\Delta t = .1$.
3. Solve Example VII.4 by the Runge-Kutta method with $\Delta t = .2$, $\Delta t = .4$, and $\Delta t = .8$ and compare the answers at $t = .8$.
4. Solve Example VII.6 by the Runge-Kutta method with $\Delta t = .25$ and compare the answers at $t = .5$.
5. Solve the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi xy$$

with the boundary conditions

$$V = 0 \text{ at } x = 0 \qquad V = x - 2y \text{ at } x = 4$$
$$V = x \text{ at } y = 0 \qquad V = 4 - x - y \text{ at } y = 4.$$

* If the reader finds a need for additional information he is advised to consult the references already given. For information on triangular networks, block relaxation, and the Rayleigh-Ritz method consult: J. B. Scarborough, *Numerical Mathematical Analysis* (2nd ed.; Baltimore: Johns Hopkins University Press, 1950), Chapter 12; or *R. V. Southwell*, *Relaxation Methods in Theoretical Physics* (London: Oxford University Press, 1946); and H. W. Emmons, "The Numerical Solution of Partial Differential Equations," *Quarterly of Applied Mathematics*, October, 1924, Vol. 2, No. 3, pp. 173–195.

CHAPTER VIII. LEAST SQUARES AND THEIR APPLICATION

84. Introduction. The fitting of empirical data by formulas or equations may be accomplished in two distinct manners. One is to have the formula satisfied exactly at the observational points; this was the case of the interpolation formulas discussed in the preceding chapters. The other is to have the approximating function come as close as possible to the values at the given points and still retain a predetermined characteristic which will show the general nature of the data but will not necessarily pass exactly through the given points. This second manner can be a most desirable way to study data which have been obtained from experimental observations and thus contain various errors of measurements.

The most frequently employed method to obtain functional representation of the second kind is that which is known as the **method of Least Squares.** We shall now discuss this method and its application to the fitting of empirical data.

85. The Principle of Least Squares. Let us consider a set of m points (x_j,y_j), $(j = 1, \cdots, m)$, which have been obtained by measurements and which should be related by some function, $y = f(x)$. As a first consideration let us designate the function to be a polynomial of degree $n < m$,

$$(85.1) \qquad y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{i=0}^{n} a_ix^i.$$

We propose to determine this polynomial, i.e., to find the values of the coefficients a_i, $(i = 0, \cdots, n)$, such that the polynomial (85.1) is a "good fit" to the data (x_j, y_j). If we substitute the points into the polynomial we have the m equations

$$(85.2)\quad \begin{cases} R_1 = a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_nx_1^n - y_1 \\ R_2 = a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_nx_2^n - y_2 \\ \cdots \\ R_m = a_0 + a_1x_m + a_2x_m^2 + \cdots + a_nx_m^n - y_m \end{cases}$$

which are not equal to zero since the polynomial does not necessarily pass exactly through the points. The difference between the polynomial value and the observed functional value,

$$\sum_{i=0}^{n} a_ix_j^i - y_j, \qquad (j = 0, \cdots, m),$$

is called the **residual** and is denoted by R_j. Thus, we have the m *residual equations* exhibited in (85.2). The *principle of least squares* states that the best representation of the data is that which makes the *sum of the squares of the residuals a minimum.* We therefore seek to make the function

$$(85.3)\quad f(a_0,a_1, \cdots ,a_n) = R_1^2 + R_2^2 + R_3^2 + \cdots + R_m^2$$

a minimum. The condition which fulfills this requirement is that the partial derivatives be zero.* Thus

$$(85.4)\quad \begin{cases} \dfrac{\partial f}{\partial a_0} = 2\left[R_1\dfrac{\partial R_1}{\partial a_0} + R_2\dfrac{\partial R_2}{\partial a_0} + \cdots + R_m\dfrac{\partial R_m}{\partial a_0}\right] = 0 \\ \dfrac{\partial f}{\partial a_1} = 2\left[R_1\dfrac{\partial R_1}{\partial a_1} + R_2\dfrac{\partial R_2}{\partial a_1} + \cdots + R_m\dfrac{\partial R_m}{\partial a_1}\right] = 0 \\ \cdots\cdots \\ \dfrac{\partial f}{\partial a_n} = 2\left[R_1\dfrac{\partial R_1}{\partial a_n} + R_2\dfrac{\partial R_2}{\partial a_n} + \cdots + R_m\dfrac{\partial R_m}{\partial a_n}\right] = 0. \end{cases}$$

* See P. Franklin, *Methods of Advanced Calculus* (New York: McGraw-Hill Book Company, 1944).

Now

$$
\textbf{(85.5)}\quad
\begin{cases}
\dfrac{\partial R_j}{\partial a_0} = \dfrac{\partial}{\partial a_0}[a_0 + a_1x_j + a_2x_j^2 + \cdots + a_nx_j^n - y_j] = 1 \\
\dfrac{\partial R_j}{\partial a_1} = \dfrac{\partial}{\partial a_1}[a_0 + a_1x_j + a_2x_j^2 + \cdots + a_nx_j^n - y_j] = x_j \\
\dfrac{\partial R_j}{\partial a_2} = \dfrac{\partial}{\partial a_2}[a_0 + a_1x_j + a_2x_j^2 + \cdots + a_nx_j^n - y_j] = x_j^2 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\dfrac{\partial R_j}{\partial a_n} = \dfrac{\partial}{\partial a_n}[a_0 + a_1x_j + a_2x_j^2 + \cdots + a_nx_j^n - y_j] = x_j^n
\end{cases}
$$

for $(j = 1, \cdots, m)$. The system (85.4) thus becomes

$$
\textbf{(85.6)}\quad
\begin{cases}
R_1 + R_2 + R_3 + \cdots + R_m = 0 \\
x_1R_1 + x_2R_2 + x_3R_3 + \cdots + x_mR_m = 0 \\
x_1R_1 + x_2^2R_2 + x_3^2R_3 + \cdots + x_m^2R_m = 0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
x_1^nR_1 + x_2^nR_2 + x_3^nR_n + \cdots + x_m^nR_m = 0.
\end{cases}
$$

If we replace R_j by their values from equations (85.2) and collect the coefficients of the $n + 1$ unknowns a_i, $(i = 0, \cdots, n)$, we have

$$
\textbf{(85.7)}\quad
\begin{cases}
ma_0 + \Sigma x_ja_1 + \Sigma x_j^2a_2 + \cdots + \Sigma x_j^na_n - \Sigma y_j = 0 \\
\Sigma x_ja_0 + \Sigma x_j^2a_1 + \Sigma x_j^3a_2 + \cdots + \Sigma x_j^{n+1}a_n - \Sigma x_jy_j = 0 \\
\Sigma x_j^2a_0 + \Sigma x_j^3a_1 + \Sigma x_j^4a_2 + \cdots + \Sigma x_j^{n+2}a_n - \Sigma x_j^2y_j = 0 \\
\cdot\ \cdot \\
\Sigma x_j^na_0 + \Sigma x_j^{n+1}a_1 + \Sigma x_j^{n+2}a_2 + \cdots + \Sigma_j^{2n}a_n - \Sigma x_j^ny_j = 0
\end{cases}
$$

where all summations are from 1 to m, i.e.,

$$
\Sigma x_j^3 = x_1^3 + x_2^3 + x_3^3 + \cdots + x_m^3
$$
$$
\Sigma x_j^2y_j = x_1^2y_1 + x_2^2y_2 + \cdots + x_m^2y_m.
$$

The equations (85.7) are known as the **normal equations.** All of these summations are known so that the system (85.7) is a system of $n + 1$ linear equations in the $n + 1$ unknowns a_i, $(i = 0, \cdots, n)$, the solution of which will yield the coefficients a_i so that the polynomial

$$
y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n
$$

is now determined. The procedure is to calculate all the sums indicated from the given data, insert these sums into the system (85.7), and solve for a_i, $(i = 0, \cdots, n)$. Various short cuts for accomplishing this will be discussed later.

The principle of least squares is not limited to polynomials. The desired function could take any known form as long as the resulting normal equations can be solved. This, of course, is easiest if the normal equations are linear as was the case above.

86. Application to Polynomial Fitting of Data. We shall now apply the principle of least squares to the fitting of empirical data by polynomials. Consider the m points

x	x_1	x_2	x_3	$\cdots$	x_m
y	y_1	y_2	y_3	$\cdots$	y_m

and let it be desired to fit these points with a third degree polynomial, i.e., $n = 3$ in the last section.

(86.1) $$y = a_0 + a_1x + a_2x^2 + a_3x^3.$$

It is required to obtain the coefficients a_i, $(i = 0, \cdots, 3)$, from the normal equations (85.7). For convenience let us denote the sums by

(86.2) $$\begin{cases} S_0 = m \\ S_1 = \sum_{i=1}^{m} x_i \\ S_2 = \sum_{i=1}^{m} x_i^2 \\ \cdots \\ S_k = \sum_{i=1}^{m} x_i^k \end{cases}$$

and

$$(86.3) \qquad \begin{cases} k_0 = \sum_{j=1}^{m} y_j \\ k_1 = \sum_{j=1}^{m} x_j y_j \\ \cdots \\ k_k = \sum_{j=1}^{m} x_j^k y_j. \end{cases}$$

A schematic for calculating these sums can be arranged as follows:

x^0	x	x^2	x^3	x^4	x^5	x^6	y	xy	x^2y	x^3y
1	x_1	x_1^2	x_1^3	x_1^4	x_1^5	x_1^6	y_1	x_1y_1	$x_1^2y_1$	$x_1^3y_1$
1	x_2	x_2^2	x_2^3	x_2^4	x_2^5	x_2^6	y_2	x_2y_2	$x_2^2y_2$	$x_2^3y_2$
1	x_3	x_3^2	x_3^3	x_3^4	x_3^5	x_3^6	y_3	x_3y_3	$x_3^2y_3$	$x_3^3y_3$
...	...	...	...	...	...	...	...	...	...	...
1	x_m	x_m^2	x_m^3	x_m^4	x_m^5	x_m^6	y_m	x_my_m	$x_m^2y_m$	$x_m^3y_m$
$S_0 = m$	S_1	S_2	S_3	S_4	S_5	S_6	k_0	k_1	k_2	k_3

The normal equations can then be written in this manner:

$$(86.4) \qquad \begin{cases} S_0a_0 + S_1a_1 + S_2a_2 + S_3a_3 = k_0 \\ S_1a_0 + S_2a_1 + S_3a_2 + S_4a_3 = k_1 \\ S_2a_0 + S_3a_1 + S_4a_2 + S_5a_3 = k_2 \\ S_3a_0 + S_4a_1 + S_5a_2 + S_6a_3 = k_3. \end{cases}$$

We note that this system of linear equations is symmetric and its solution is considerably simplified. Let us consider an example.

Example VIII.1. Fit a cubic to the data

x	−4	−2	−1	0	1	3	4	6
y	−35.1	15.1	15.9	8.9	.1	.1	21.1	135

Solution. Let us first obtain the required sums.

x^0	x	x^2	x^3	x^4	x^5	x^6	y	xy	x^2y	x^3y
1	−4	16	−64	256	−1024	4096	−35.1	140.4	−561.6	2246.4
1	−2	4	−8	16	−32	64	15.1	−30.2	60.4	−120.8
1	−1	1	−1	1	−1	1	15.9	−15.9	15.9	−15.9
1	0	0	0	0	0	0	8.9	0	0	0
1	1	1	1	1	1	1	.1	.1	.1	.1
1	3	9	27	81	243	729	.1	.3	.9	2.7
1	4	16	64	256	1024	4096	21.1	84.4	337.6	1350.4
1	6	36	216	1296	7776	46656	135.0	810.0	4860.0	29160.0
8	7	83	235	1907	7987	55643	161.1	989.1	4713.3	32622.9

The normal equations are

$$\begin{cases} 8a_0 + 7a_1 + 83a_2 + 235a_3 = 161.1 \\ 7a_0 + 83a_1 + 235a_2 + 1907a_3 = 989.1 \\ 83a_0 + 235a_1 + 1907a_2 + 7987a_3 = 4713.3 \\ 235a_0 + 1907a_1 + 7987a_2 + 55643a_3 = 32622.9. \end{cases}$$

Solving by Crout's method we obtain the values of a_i,

a_0	a_1	a_2	a_3
9.011039	−8.966140	−1.000093	.999074.

The cubic is given by

$$.999x^3 - 1.000x^2 - 8.966x + 9.011 = 0.$$

A check on the given points shows a good fit.

x	−4	−2	−1	0	1	3	4	6
y (GIVEN)	−35.1	15.1	15.9	8.9	.1	.1	21.1	135
y (CALC.)	−35.1	15.0	16.0	9.0	0	.1	21.1	135
DIFF.	0	.1	−.1	−.1	.1	0	0	0

87. Evenly Spaced Intervals. As we have seen so often in the past great simplification occurs if the data are given at equally spaced values of the argument. In such a case we have

$$x_j = x_1 + (j - 1)h, \qquad (j = 1, \cdots, m), \tag{87.1}$$

where h is the interval length between the equidistant x-points. There are four main advantages which may now be realized.

I. The Values of the Argument May Be Transformed to the Integers. $0, 1, 2, \cdots, m.$

This fact is accomplished by changing the variable to

$$x'_j = \frac{x_j - x_1}{h}. \tag{87.2}$$

It is now easily seen that

$$x'_1 = \frac{x_1 - x_1}{h} = 0$$

$$x'_2 = \frac{x_2 - x_1}{h} = \frac{x_1 + h - x_1}{h} = 1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x'_j = \frac{x_j - x_1}{h} = \frac{x_1 + (j - 1)h - x_1}{h} = j - 1.$$

II. The Data May Be Easily Centralized about the Arithmetic Mean. Centralizing the data about the arithmetic mean is accomplished by the transformation

$$X_j = x_j - \bar{x} \qquad \text{and} \qquad Y_j = y_j - \bar{y} \tag{87.3}$$

where

$$\bar{x} = \frac{1}{m}\sum_{j=1}^{m} x_j \qquad \text{and} \qquad \bar{y} = \frac{1}{m}\sum_{j=1}^{m} y_j. \tag{87.4}$$

For equally spaced intervals we have

(87.5)
$$\bar{x} = \frac{1}{m} \sum_{j=1}^{m} [x_1 + (j - 1)h]$$
$$= \frac{1}{m} \left[mx_1 + h \sum_{j=1}^{m-1} j \right]$$
$$= x_1 + \frac{1}{2}(m - 1)h,$$
since
$$\sum_{k=1}^{m-1} k = \frac{1}{2}(m - 1)m.$$
Thus

(87.6)
$$X_j = x_j - \bar{x} = x_1 + (j - 1)h - x_1 - \frac{1}{2}(m - 1)h$$
$$= \frac{1}{2}(2j - m - 1)h.$$

We now have some very useful properties.

Property A. *If m is odd, $\bar{x} = x_M$ where x_M is the "middle" value of the set x_j.*

Proof. Let $m = 2r - 1$, an odd number; then
$$M = \frac{1}{2}(m + 1) = \frac{1}{2}(2r - 1 + 1) = r$$
and
$$\bar{x} = x_1 + \frac{1}{2}(m - 1)h = x_1 + \frac{1}{2}(2r - 1 - 1)h$$
$$= x_1 + (r - 1)h$$
$$= x_r = x_M.$$

Corollary. In this case $X_M = 0$.

Proof.
$$X_M = x_M - \bar{x} = x_M - x_M = 0.$$

Property B. *The set X_j consists of two identical sets except for algebraic sign.*

Proof. By formula (87.6) we have

$$\begin{cases} X_1 = \frac{1}{2}(2 - m - 1)h = \frac{1}{2}(1 - m)h \\ X_2 = \frac{1}{2}(4 - m - 1)h = \frac{1}{2}(3 - m)h \\ X_3 = \frac{1}{2}(6 - m - 1)h = \frac{1}{2}(5 - m)h \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ X_M = \frac{1}{2}(2M - m - 1)h \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ X_{m-2} = \frac{1}{2}(2m - 4 - m - 1)h = \frac{1}{2}(m - 5)h \\ X_{m-1} = \frac{1}{2}(2m - 2 - m - 1)h = \frac{1}{2}(m - 3)h \\ X_m = \frac{1}{2}(2m - m - 1)h = \frac{1}{2}(m - 1)h \end{cases}$$

from which it is easily seen that

$$(87.7) \qquad \begin{cases} X_1 = -X_m \\ X_2 = -X_{m-1} \\ X_3 = -X_{m-2} \\ \cdots\cdots\cdots \\ X_{M-1} = -X_{M+1} \end{cases}$$

and at the mid-point we have $X_M = 0$ if m is odd and X_M is not a member of the set if m is even.

Property C. *The summation of X_j^i over values for j assumes three forms:*

a) *if* **i** *is an odd integer*

$$(87.8) \qquad \sum_{j=1}^{m} X_j^i = 0;$$

b) *if* **i** *is an even integer and* $\mathbf{m} = \mathbf{2r - 1}$ *is an odd integer*

$$(87.9) \qquad \sum_{j=1}^{m} X_j^i = 2h^i \sum_{k=1}^{r-1} k^i;$$

c) *if* **i** *is an even integer and* **m** = **2r** *is an even integer*

$$\sum_{j=1}^{m} X_j^i = \left(\frac{1}{2}\right)^i 2h^i \sum_{k=1}^{r} (2k-1)^i. \tag{87.10}$$

Proof. The truth of the property may be established directly from Property B. By the equalities (87.7) we see that a summation of X_j would result in a cancellation of all the terms. Since the sign will be retained when the term is raised to an odd power, we see that the cancellation will still hold when summing X_j^i provided i is odd. Thus a) of Property C follows directly.

To prove b), we see that for $m = 2r - 1$, $X_j = (j - r)h$ or the set (X_j) is composed of two sets $[(k - r)h]$ and $[(r - k)h]$ with k running from 1 to r. If these quantities are raised to an even power, they are all positive. Thus

$$\sum_{j=1}^{m} X_j^i = \sum_{j=1}^{r} 2[(j - r)h]^i = 2h \sum_{k=1}^{r-1} k^i.$$

Part c) follows in the same manner. As with $m = 2r$ we have

$X_j = \frac{1}{2}(2j - 2r - 1)h$ which consists of two sets $\left[\frac{1}{2}(n - 2r)h\right]$ and $\left[\frac{1}{2}(2r - n)h\right]$ with n being the odd integers from 1 to $2r$. Thus, with i even, we have

$$\sum_{j=1}^{r} X_j^i = \left(\frac{1}{2}\right)^i \sum_{k=1}^{r} (2k-1)^i.$$

III. The Normal Equations Are Greatly Simplified. To ease the notation let us define

$$S_i = \begin{cases} \displaystyle\sum_{k=1}^{r-1} k^i, \text{ if } m \text{ is odd} \\ \displaystyle\left(\frac{1}{2}\right)^i \sum_{k=1}^{r} (2k-1)^i, \text{ if } m \text{ is even.} \end{cases} \qquad (i = 2, 4, \cdots, 2n) \tag{87.11}$$

Then for equally spaced intervals the normal equations (85.7) may be transformed to the simple system which is represented schematically by

(87.12)

b_0	b_1	b_2	b_3	$\cdots$	b_n	c
m	0	$2S_2h^2$	0	$\cdots$	$2S_nh^n$	ΣY_j
0	$2S_2h^2$	0	$2S_4h^4$	$\cdots$	$2S_{n+1}h^{n+1}$	ΣX_jY_j
$2S_2h^2$	0	$2S_4h^4$	0	$\cdots$	$2S_{n+2}h^{n+2}$	$\Sigma X_j^2Y_j$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$2S_nh^n$	$2S_{n+1}h^{n+1}$	$\cdots$	$\cdots$	$\cdots$	$2S_{2n}h^{2n}$	$\Sigma X_j^nY_j$

where in the last row and last column of b_n, $S_{n+q} = 0$ if $n + q$ is odd, $(q = 0,1, \cdots, n)$.

IV. The Computation of the Constants in the Normal Equations May Be Simplified. The centralization of the data about the arithmetic mean permits us to reduce the calculation of the constant terms in the normal equation considerably. First, we have

(87.13)
$$\sum_{j=1}^{m} Y_j = \sum_{j=1}^{m} (y_j - \bar{y}) = \sum_{j=1}^{m} y_j - \sum_{j=1}^{m} \bar{y} = m\bar{y} - m\bar{y} = 0.$$

Secondly, we have by (87.7)

(87.14)
$$\begin{aligned}\sum_{j=1}^{m} X_jY_j &= X_1Y_1 + X_2Y_2 + X_3Y_3 + \cdots + X_{m-2}Y_{m-2} + X_{m-1}Y_{m-1} + X_mY_m \\ &= X_1Y_1 + X_2Y_2 + X_3Y_3 + \cdots - X_3Y_{m-2} - X_2Y_{m-1} - X_1Y_m \\ &= X_1(Y_1 - Y_m) + X_2(Y_2 - Y_{m-1}) + X_3(Y_3 - Y_{m-2}) + \cdots \\ &= \sum_{j=1}^{r} X_jD(Y_j)\end{aligned}$$

where

(87.15)
$$D(Y_j) = Y_j - Y_{m-j+1} = y_{M+j} - y_{M-j}.$$

In general, if i is *odd*, we have

$$\sum_{j=1}^{m} X_j^i Y_j = \sum_{j=1}^{r} X_j^i D(Y_j). \tag{87.16}$$

Thirdly, we have by (87.7) if i is *even*

$$\sum_{j=1}^{m} X_j^i Y_j = \sum_{j=1}^{r} X_j^i S(Y_j) \tag{87.17}$$

where

$$S(Y_j) = Y_j + Y_{m-j+1} = y_{M+i} + y_{M-i} - 2\bar{y}. \tag{87.18}$$

Let us put all of these advantages from equidistant data together and see how they ease the calculation.

Example VIII.2. Fit a cubic to the data

x	1	2	3	4	5	6	7	8	9	10	11
y	108.0	55.9	.1	−54.1	−100.0	−131.9	−143.9	−129.9	−84.1	0	127.9

Solution. Since h is already equal to 1 we begin by centralizing the data. We have $m = 11$ (an odd number); therefore

$$\bar{x} = x_M = 6, \quad \bar{y} = -32.00, \quad \text{and} \quad 2\bar{y} = -64.00.$$

We now arrange the data in the following columns using the formulas above. To obtain $S(Y)$ and $D(Y)$ we add a column of identifying numbers, I, which has 0 at (x_M, y_M) and then order the terms in each direction. This identifying column is also carried over to the transformed data. Thus, to find the entry at $I = 2$, we have

$$D_2(Y) = y \text{ (at } I = 2 \text{ below 0)} - y \text{ (at } I = 2 \text{ above 0), etc.}$$

At $I = 0$ we have

$$S(Y) = D(Y) = y_M - \bar{y}.$$

As a check $\Sigma S(Y) = 0$.

x	y	I	X	I	$S(Y)$	$D(Y)$
1	108.0	5	0	0	−99.9	−99.9
2	55.9	4	1	1	−179.9	−43.9
3	.1	3	2	2	−120.0	−75.8
4	−54.1	2	3	3	−20.0	−84.2
5	−100.0	1	4	4	119.9	−55.9
6	−131.9	0	5	5	299.9	19.9
7	−143.9	1			0	
8	−129.9	2				
9	−84.1	3				
10	0	4				
11	127.9	5				

The normal equations (87.12) are now easily obtained.

b_0	b_1	b_2	b_3	c
11	0	110	0	0
0	110	0	1958	−572.2
110	0	1958	0	8576.0
0	1958	0	41030	−4013.8

These can be solved directly by elimination to yield

$$\begin{cases} b_0 = -99.95337 \\ b_1 = -22.983804 \\ b_2 = 9.995337 \\ b_3 = .998990 \end{cases}$$

and we have

$$Y = .998990X^3 + 9.995337X^2 - 22.983804X - 99.95337.$$

The transformation back to (x,y) can easily be accomplished by Horner's method division using -6 as the divisor and adding $\bar{y}$ to the constant term.

.998990	9.995337	−22.983804	−99.95337	−6
	4.001397	−46.992186	181.999746	
	−1.992543	−35.036928	−32.00000	
	−7.986483		149.999746	
.998990				

Thus

$$y = .999x^3 - 7.986x^2 - 35.037x + 150.000.$$

88. The Nielsen-Goldstein Method.* It was seen in the last section that for equally spaced values of the argument the normal equations simplify to those given in (87.12). If we further transform these equations by dividing *successive* equations by $\frac{1}{2}h^i$, $(i = 0, \cdots, n)$, and then replacing b_i by $a_i h^{-i}$ we have a system of equations in a_i which may be represented by

$$(88.1)\quad \begin{array}{ccccccc} a_0 & a_1 & a_2 & a_3 & & a_n & c \\ \hline \frac{1}{2}m & 0 & S_2 & 0 & \cdots & S_n & \frac{1}{2}\Sigma Y_j \\ 0 & S_2 & 0 & S_4 & \cdots & S_{n+1} & \frac{1}{2}h^{-1}\Sigma X_j Y_j \\ S_2 & 0 & S_4 & 0 & \cdots & S_{n+2} & \frac{1}{2}h^{-2}\Sigma X_j^2 Y_j \\ 0 & S_4 & 0 & S_6 & \cdots & S_{n+3} & \frac{1}{2}h^{-3}\Sigma X_j^3 Y_j \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_n & S_{n+1} & S_{n+2} & S_{n+3} & \cdots & S_{2n} & \frac{1}{2}h^{-n}\Sigma X_j^n Y_j \end{array}$$

where $S_i = 0$ if i is an odd number.

Before proceeding with the solution of this system of equations it has been found convenient first to divide each equation, except the first, by S_2. Letting $S'_i = S_i S_2^{-1}$ we have the following system:

* K. L. Nielsen and L. Goldstein, "An Algorithm for Least Squares," *Journal of Mathematics and Physics* (July, 1947), Vol. 26, pp. 120–132.

$$
\begin{array}{ccccccc}
a_0 & a_1 & a_2 & a_3 & & a_n & c \\
\hline
\end{array}
$$

$$
(88.2)\qquad
\left\{
\begin{array}{lllllll}
\frac{1}{2}m & 0 & S_2 & 0 & \cdots & S_n & 0 \\
0 & 1 & 0 & S'_4 & \cdots & S'_{n+1} & c_2 \\
1 & 0 & S'_4 & 0 & \cdots & S'_{n+2} & c_3 \\
0 & S'_4 & 0 & S'_6 & \cdots & S'_{n+3} & c_4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
S'_n & S'_{n+1} & S'_{n+2} & S'_{n+3} & \cdots & S'_{2n} & c_{n+1}
\end{array}
\right.
$$

where again $S'_i = S_i = 0$ if i is an odd number, and

$$
(88.3)\qquad c_i = \frac{1}{2} S_2^{-1} h^{-i+1} \sum_{j=1}^{m} X_j^{i-1} Y_j, \qquad (i = 2, \cdots, n+1).
$$

The sums S_i [see equation (87.11)] and S'_i are functions of the number of observations only. In fact they can be calculated by the formulas.*

Case I. $m = 2r + 1.$

$$
(88.4)\qquad \sum_{k=1}^{r} k^i = \frac{1}{i+1} r^{i+1} + \frac{1}{2} r^i + \frac{1}{2}\binom{i}{1} B_1 r^{i-1} - \frac{1}{4}\binom{i}{3} B_2 r^{i-3} + \cdots .
$$

Case II, $m = 2r$.

$$
(88.5)\qquad \left(\frac{1}{2}\right)^i \sum_{k=1}^{r} (2k-1)^i = \left(\frac{1}{2}\right)^i \left[\sum_{k=1}^{2r} k^i - 2^i \sum_{k=1}^{r} k^i\right]
$$

where $B_s (s = 1, 2, \cdots)$ are the Bernoulli numbers.

Let us now solve the system (88.2) by Crout's method. We would first have the derived matrix

* E. P. Adams, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions* (Washington, D.C.: Smithsonian Institution, 1947), p. 27.

(88.6)
$$\begin{Vmatrix} A_{11} & A_{12} & \cdots & A_{1,n+1} & K_1 \\ A_{21} & A_{22} & \cdots & A_{2,n+1} & K_2 \\ A_{31} & A_{32} & \cdots & A_{3,n+1} & K_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n+1,1} & A_{n+1,2} & \cdots & A_{n+1,n+1} & K_{n+1} \end{Vmatrix}$$

where

(88.7)
$$\begin{cases} A_{ii} = S_{2i-2} - \sum_{w=1}^{i-2} A_{iw}A_{wi} \\ A_{ij} = \dfrac{S_{i+j-2} - \sum_{w=1}^{i-2} A_{iw}A_{wj}}{A_{ii}}, \qquad \text{if } (j > i) \\ A_{ij} = A_{ji}A_{jj}, \qquad \text{if } (j < i), \end{cases}$$

$(i = 2, \cdots, n + 1)$,
and

(88.8)
$$K_i = \frac{c_i - \sum_{j=1}^{i-1} K_jA_{ij}}{A_{ii}}, \qquad (i = 1, \cdots, n + 1),$$

with

$$A_{ij} = 0 \text{ if } i + j \text{ is an odd number.}$$

The elements of the solution matrix are given by

(88.9)
$$a_i = K_{i+1} - \sum_{j=i+2}^{n} a_jA_{i+1,j+1}, \qquad (i = 0, \cdots, n).$$

Now, since S_i and S'_i depend only upon the number of observations, so do the A_{ij} which, consequently, can then be computed once and for all and tabulated. It is not necessary to tabulate all of the A_{ij} since there exist certain interrelations. In fact the relationships among the A_{ij} form a nice algebraic study. Table XVI, p. 356, exhibits values of A_{ij} sufficient to compute fourth degree polynomials for $5 \leq m \leqq 100$ observations and sixth degree polynomials for $6 \leq m = 50$ observations. By use of these tables we may simultaneously compute these polynomials up to the sixth degree (or fourth) which give the least square fit to the data. This is accomplished by a simple substitution into a set of formulas.

Let us concentrate on polynomials up to and including the sixth degree. The derived matrix of the normal equations will be

$$(88.10)\qquad \left\|\begin{array}{cccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & c \\ \hline A_{11} & 0 & A_{13} & 0 & A_{15} & 0 & A_{17} & 0 \\ 0 & 1 & 0 & A_{24} & 0 & A_{26} & 0 & K_2 \\ 1 & 0 & A_{33} & 0 & A_{35} & 0 & A_{37} & K_3 \\ 0 & A_{24} & 0 & A_{44} & 0 & A_{46} & 0 & K_4 \\ A_{24} & 0 & A_{53} & 0 & A_{55} & 0 & A_{57} & K_5 \\ 0 & A_{26} & 0 & A_{64} & 0 & A_{66} & 0 & K_6 \\ A_{26} & 0 & A_{73} & 0 & A_{75} & 0 & A_{77} & K_7 \end{array}\right\|$$

Now it can be shown that

$$(88.11)\qquad \begin{cases} A_{15} = A_{13}A_{24}; & A_{53} = A_{35}A_{33}; \qquad A_{73} = A_{37}A_{33}; \\ A_{17} = A_{13}A_{26}; & A_{64} = A_{46}A_{44}; \qquad A_{75} = A_{57}A_{55}. \end{cases}$$

Using these relations we see that the values of K_i are given by

$$(88.12)\qquad \begin{cases} K_1 = 0 \\ K_2 = c_2 \\ K_3 = \dfrac{c_3}{A_{33}} \\ K_4 = \dfrac{c_4 - c_2A_{24}}{A_{44}} = \dfrac{K'_4}{A_{44}} \\ K_5 = \dfrac{c_5 - c_3A_{35}}{A_{55}} = \dfrac{K'_5}{A_{55}} \\ K_6 = \dfrac{c_6 - c_2A_{26} - A_{46}K'_4}{A_{66}} \\ K_7 = \dfrac{c_7 - c_3A_{37} - A_{57}K'_5}{A_{77}} \end{cases}$$

where

$$(88.13)\qquad \begin{cases} K'_4 = c_4 - c_2A_{24} \\ K'_5 = c_5 - c_3A_{35}. \end{cases}$$

The solution for α_i, $(i = 0, \cdots, 6)$, depends upon the degree, n, of the equation desired. Their formulas can be exhibited in the following table:

Table 8.1. Formulas for α_i.

α_i	$n = 6$	$n = 5$	$n = 4$	$n = 3$	$n = 2$	$n = 1$
α_6	K_7					
α_5	K_6	K_6				
α_4	$K_5 - \alpha_6 A_{47}$	K_5	K_5			
α_3	$K_4 - \alpha_5 A_{46}$	$K_4 - \alpha_5 A_{46}$	K_4	K_4		
α_2	$K_3 - \alpha_6 A_{37} - \alpha_4 A_{35}$	$K_3 - \alpha_4 A_{35}$	$K_3 - \alpha_4 A_{35}$	K_3	K_3	
α_1	$K_2 - \alpha_5 A_{26} - \alpha_3 A_{24}$	$K_2 - \alpha_5 A_{26} - \alpha_3 A_{24}$	$K_2 - \alpha_3 A_{24}$	$K_2 - \alpha_3 A_{24}$	K_2	K_2
α_0	$-A_{13}(\alpha_6 A_{26} + \alpha_4 A_{24} + \alpha_2)$	$-A_{13}(\alpha_4 A_{24} + \alpha_2)$	$-A_{13}(\alpha_4 A_{24} + \alpha_2)$	$-\alpha_2 A_{13}$	$-\alpha_2 A_{13}$	0

In Table 8.1 the values of α_i must be obtained from the same column in which they are used. Even so, we can see that many of the values are the same. Thus, using the notation $[\alpha_i]_n$ where n designates the degree of the equation, we have

$$(88.14) \quad \begin{cases} [\alpha_5]_6 = [\alpha_5]_5;\ [\alpha_4]_5 = [\alpha_4]_4;\ [\alpha_3]_4 = [\alpha_3]_3;\ [\alpha_2]_3 = [\alpha_2]_2; \\ [\alpha_1]_2 = [\alpha_1]_1; \\ [\alpha_3]_6 = [\alpha_3]_5;\ [\alpha_2]_5 = [\alpha_2]_4;\ [\alpha_1]_4 = [\alpha_1]_3;\ [\alpha_0]_3 = [\alpha_0]_2; \\ [\alpha_1]_6 = [\alpha_1]_5;\ [\alpha_0]_5 = [\alpha_0]_4. \end{cases}$$

The identities (88.14) save much computation when we find the polynomials of degree one through six simultaneously.

The calculating procedure may now be summarized into the following steps:

1. Locate the mid-point of the data and order the identifying numbers.
2. Calculate $\bar{x}$, $\bar{y}$, X, $S(Y)$, $D(Y)$ and check $\Sigma S(Y) = 0$.
3. Look up S_2 and A_{ij} in Table XVI, p. 356.
4. Calculate

$$(88.15) \quad \begin{cases} p_i = \sum_{j=1}^{M} X_j^{i-1} D_j(Y), & (i = 2,4,6), \\ p_i = \sum_{j=1}^{M} X_j^{i-1} S_j(Y), & (i = 3,5,7). \end{cases}$$

5. Calculate

$$c_i = \frac{p_i h^{-i+1}}{2S_2}.$$

6. Calculate K_i by (88.12).

7. Calculate a_i using the formulas of Table 8.1. A column for each degree of the polynomial desired is needed.

8. The next step is to obtain the b_i, $(i = 0, \cdots, n)$, from the formula

$$b_i = [a_i]_n h^{-i}, \qquad (i = 1, \cdots, n),$$

and

$$b_0 = [a_0]_n + \bar{y}.$$

9. The mean values are now removed from the computation and a_i are obtained by a synthetic division by $-\bar{x}$.

$$\begin{array}{llllll}
b_4 & b_3 & b_2 & b_1 & b_0 & \lfloor -\bar{x} \\
\downarrow & b_{31} & b_{21} & b_{11} & \boxed{b_{01}} = a_0 & \\
\downarrow & b_{32} & b_{22} & \boxed{b_{12}} = a_1 & & \\
\downarrow & b_{33} & \boxed{b_{23}} = a_2 & & & \\
\downarrow & \boxed{b_{34}} = a_3 & & & & \\
\boxed{b_4} = a_4 & & & & &
\end{array}$$

where

$b_{31} = b_3 + b_4(-\bar{x})$ $b_{32} = b_{31} + b_4(-\bar{x})$ $b_{33} = b_{32} + b_4(-\bar{x})$ $b_{34} = b_{33} + b_4(-\bar{x})$	$b_{21} = b_2 + b_{31}(-\bar{x})$ $b_{22} = b_{21} + b_{32}(-\bar{x})$ $b_{23} = b_{22} + b_{33}(-\bar{x})$
$b_{11} = b_1 + b_{21}(-\bar{x})$ $b_{12} = b_{11} + b_{22}(-\bar{x})$	$b_{01} = b_0 + b_{11}(-\bar{x})$

All of the formulas can be placed on one sheet which also gives a pattern for a calculation schematic. We exhibit one for polynomials up to and including the fourth degree as shown on pages 280 and 281.

Example VIII.3. Fit the data (x_i, y_i) by a polynomial.

Solution.

x	y	I	X	I	$S(Y)$	$D(Y)$	CONSTANTS	
0	.5179	8	0	0	−3.0490	−3.0490	$m = 17$	$A_{13} = 24$
.2	.7049	7	.2	1	−5.8520	2.5360	$h = .2$	$A_{33} = 19$
.4	.9625	6	.4	2	−5.1110	5.1390	$\bar{x} = 1.6$	
.6	1.3090	5	.6	3	−3.8690	7.8730	$\bar{y} = 8.2240$	$A_{24} = 43$
.8	1.7650	4	.8	4	−2.1140	10.8040	$2\bar{y} = 16.4480$	$A_{44} = 342$
1.0	2.3530	3	1.0	5	.1690	13.9990	$S_2 = 204$	
1.2	3.0990	2	1.2	6	3.0015	17.5245	$2S_2 = 408$	$A_{35} = 61$
1.4	4.0300	1	1.4	7	6.4079	21.4461	$p_2 = 122.30882$	$A_{55} = 5928$
1.6	5.1750	0	1.6	8	10.4166	25.8288	$\rho_3 = 23.5995$	
1.8	6.5660	1			0		$p_4 = 216.5056$	
2.0	8.2380	2					$\rho_5 = 97.7682$	
2.2	10.2260	3						
2.4	12.5690	4						
2.6	15.3080	5						
2.8	18.4870	6						
3.0	22.1510	7						
3.2	26.3467	8						

i	h^{-i+1}	C_i	K_i	$[\alpha_{i-1}]_4$	$[\alpha_{i-1}]_3$	$[\alpha_{i-1}]_2$	$[\alpha_{i-1}]_1$
1	1	0	0	−3.048999	−3.089736	−3.089736	0
2	5	1.498883	1.498883	1.262598	1.262598	1.498883	1.498883
3	25	2.446048	.128739	.122987	.128739	.128739	
4	125	66.331372	.005495	.005495	.005495		
5	625	149.767474	.942937*	.942937*		*(10^{-4})	

b_4	b_3	b_2	b_1	b_0	$-\bar{x}$	
.058933	.686875	3.074675	6.312990	5.175001	−1.6	
↓	.592582	2.126544	2.910520	.518169	← a_0	
↓	.498289	1.329282	.783669	← a_1	quartic fit	
↓	.403996	.682888	← a_2			
↓	.309703	← a_3				
.058933	← a_4					
	.686875	3.218475	6.312990	5.134264	−1.6	
	↓	2.119475	2.921830	.459336	← a_0	
	↓	1.020475	1.289070	← a_1	cubic fit	
	↓	−.078525	← a_2			
	.686875	← a_3				

Quartic fit:

$$.0589x^4 + .3097x^3 + .6829x^2 + .7837x + .5182 = y.$$

Cubic fit:

$$.6869x^3 - .0785x^2 + 1.2891x + .4593 = y.$$

Polynomial Fitting of Data

$$y = \sum_{i=0}^{n} a_i x^i \qquad (n \le 4)$$

Formulas

x	y	I	X	l	$S(Y)$	$D(Y)$	AUXILIARY	CONSTANTS	
x_1	y_1							m = No. of Observations	A_{12}
x_2	y_2							$h = x_2 - x_1$	A_{32}
x_3	y_3							$\bar{x} = \frac{1}{m} \sum_{i=1}^{m} x_i$	A_{24}
.	.		$x_i - \bar{x}$						A_{44}
.	.								A_{35}
.	.				$y_{M+i} + y_{M-i} - 2\bar{y}$	$y_{M+i} - y_{M-i}$		$\bar{y} = \frac{1}{m} \sum_{i=1}^{m} y_i$	A_{55}
.	.								
x_{M-3}	y_{M-3}	3						S_2 = [Table]	
x_{M-2}	y_{M-2}	2			Check:				
x_{M-1}	y_{M-1}	1							
x_M	y_M	0			$\sum_{i=1}^{m} S_i(Y) = 0$			$p_2 = \sum_{j=1}^{M} X_j D_j(Y)$	
x_{M+1}	y_{M+1}	1							From Table XVI, p. 356
x_{M+2}	y_{M+2}	2							
x_{M+3}	y_{M+3}	3						$p_3 = \sum_{j=1}^{M} X_j^2 S_j(Y)$	
.	.								
.	.							$p_4 = \sum_{j=0}^{M} X_j^3 D_j(Y)$	
.	.								
x_{m-1}	y_{m-1}								
x_m	y_m							$p_5 = \sum_{j=1}^{M} X_j^4 S_j(Y)$	

i	h^{-i+1}	C_i	K_i	$[a_{i-1}]_4$	$[a_{i-1}]_3$	$[a_{i-1}]_2$	$[a_{i-1}]_1$
1 2 3 4 5	1 h^{-1} h^{-2} h^{-3} h^{-4}	0 $p_2h^{-1}/2S_2$ $p_3h^{-2}/2S_2$ $p_4h^{-3}/2S_2$ $p_5h^{-4}/2S_2$	0 C_2 C_3/A_{33} $[C_4 - K_2A_{24}]/A_{44}$ $[C_5 - C_3A_{35}]/A_{55}$	$-A_{13}(a_4A_{24} + a_2)$ $K_2 - a_3A_{24}$ $K_3 - a_4A_{35}$ K_4 K_5	$-a_2A_{13}$ $K_2 - a_3A_{24}$ K_3 K_4	$-a_2A_{13}$ K_2 K_3	0 K_2
b_4		b_3	b_2	b_1	b_0	$-\bar{x}$	
$[a_4]_5h^{-4}$		$[a_3]_4h^{-3}$ $b_{31} = b_3 - \bar{x}b_4$ $b_{32} = b_{31} - \bar{x}b_4$ $b_{33} = b_{32} - \bar{x}b_4$ $b_{34} = b_{33} - \bar{x}b_4$ ← (a_4)	$[a_2]_4h^{-2}$ $b_{21} = b_2 - \bar{x}b_{31}$ $b_{22} = b_{21} - \bar{x}b_{32}$ $b_{23} = b_{22} - \bar{x}b_{33}$ ← (a_3)	$[a_1]_4h^{-1}$ $b_{11} = b_1 - \bar{x}b_{21}$ $b_{12} = b_{11} - \bar{x}b_{22}$ ← (a_2)	$[a_0]_4h + \bar{y}$ $b_{01} = b_0 - \bar{x}b_{11}$ ← (a_1)	← (a_0)	
		$[a_3]_3h^{-3}$	$[a_2]_3h^{-2}$ $b_{21} = b_2 - \bar{x}b_3$ $b_{22} = b_{21} - \bar{x}b_3$ $b_{23} = b_{22} - \bar{x}b_3$ ← (a_3)	$[a_1]_3h^{-1}$ $b_{11} = b_1 - \bar{x}b_{21}$ $b_{12} = b_{11} - \bar{x}b_{22}$ ← (a_2)	$[a_0]_3 + \bar{y}$ $b_{01} = b_0 - \bar{x}b_{11}$ ← (a_1)	← (a_0)	
			$[a_2]_2h^{-2}$	$[a_1]_2h^{-1}$ $b_{11} = b_1 - \bar{x}b_2$ $b_{12} = b_{11} - \bar{x}b_2$ ← (a_2)	$[a_0]_2 + \bar{y}$ $b_{01} = b_0 - \bar{x}b_{11}$ ← (a_1)	← (a_0)	
				$[a_1]_1h^{-1}$	$[a_0]_1 + \bar{y}$ $b_{01} = b_0 - \bar{x}b_1$ ← (a_1) ↰ (a_0)		

Example VIII.4. Fit a sixth degree polynomial to the data given below.

Solution.

x	y	I	X	$S(Y)$	$D(Y)$	CONSTANTS	
−2.0	175.0	8	0	−25.3	−25.3	$m = 17$	$A_{13} = 24$
−1.5	97.2	7	.5	−39.8	103.2	$h = .5$	$A_{33} = 19$
−1.0	50.8	6	1.0	−11.2	189.6	$\bar{x} = 2.0$	$A_{24} = 43$
−0.5	20.0	5	1.5	23.5	244.9	$\bar{y} = 104.8$	$A_{44} = 342$
0	.5	4	2.0	47.9	256.5	$2\bar{y} = 209.6$	$A_{35} = 61$
0.5	−5.9	3	2.5	48.4	218.0	$S_2 = 204$	$A_{55} = 5928$
1.0	4.4	2	3.0	19.2	127.2	$2S_2 = 408$	$A_{26} = 2191$
1.5	33.3	1	3.5	−25.0	−9.8		$A_{46} = 78\ 1/3$
2.0	79.5	0	4.0	−37.7	−178.1		$A_{66} = 98800$
2.5	136.5	1		check			$A_{37} = 3601$
3.0	194.0	2					$A_{57} = 95$
3.5	239.0	3					$A_{77} = 1573200$
4.0	257.0	4					
4.5	238.0	5					
5.0	178.0	6					
5.5	87.4	7					
6.0	−3.1	8					

i	h^{-i+1}	C_i	K_i	K_i'	a_{i-1}	b_{i-1}
1	1	0	0		−25.381860	79.418140
2	2	6.379657	6.379657		52.907237	105.814474
3	4	−2.066912	−.108785		5.678039	22.712156
4	8	−37.193873	−.9108875	−311.519124	−1.400547	−11.204377
5	16	−356.284558	−.03883315	−230.202926	−.18683232	−2.989317
6	32	−9806.890720	.006251134		.006251134	.200036
7	64	−26861.361024	.001557886		.001557886	.099705

b_6	b_5	b_4	b_3	b_2	b_1	b_0	$-\bar{x}$
.099705	.200036	−2.989317	−11.204377	22.712156	105.814474	79.418140	−2
↓	.000627	−2.990571	−5.223236	33.158627	39.497220	.423700	
	−.198783	−2.593006	−.037224	33.233075	−26.968930		
	−.398192	−1.796622	3.556020	26.121035			
	−.597601	−.601419	4.758859				
	−.797011	.992602					
	−.996420						
.099705							

$$y = .0997x^6 - .9964x^5 + .9926x^4 + 4.759x^3 + 26.12x^2 - 26.97x + .4237.$$

89. Use of Orthogonal Polynomials. The polynomial fitting of data seeks to fine the coefficients a_i, $(i = 0, \cdots, n)$, such that

(89.1) $$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

is a good fit. Suppose instead we consider fitting the data with

(89.2) $$y = b_0P_0 + b_1P_1(x) + b_2P_2(x) + \cdots + b_nP_n(x)$$

where

$$P_i(x) = i\text{th degree polynomial in } x.$$

The principle of least squares seeks to minimize the squares of the residuals

(89.3) $$\sum_{j=0}^{m} (b_0P_0 + b_1P_1 + \cdots + b_nP_n - y_j)^2 = R^2(b_i).$$

The normal equations are

(89.4) $$\begin{cases} (\Sigma P_0^2)b_0 + (\Sigma P_0P_1)b_1 + \cdots + (\Sigma P_0P_n)b_n - \Sigma P_0y_j = 0 \\ (\Sigma P_0P_1)b_0 + (\Sigma P_1^2)b_1 + \cdots + (\Sigma P_1P_n)b_n - \Sigma P_1y_j = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\Sigma P_0P_n)b_0 + (\Sigma P_1P_n)b_1 + \cdots + (\Sigma P_n^2)b_n - \Sigma P_ny_j = 0. \end{cases}$$

Now if

(89.5) $$\Sigma P_iP_j = 0 \qquad \text{when} \qquad (i \neq j),$$

then the normal equations reduce to

(89.6) $$\begin{cases} \Sigma P_0^2b_0 - \Sigma P_0y_j = 0 \\ \Sigma P_1^2b_1 - \Sigma P_1y_j = 0 \\ \cdots\cdots \\ \Sigma P_n^2b_n - \Sigma P_ny_j = 0 \end{cases}$$

from which

(89.7) $$b_i = \frac{\Sigma P_iy_j}{\Sigma P_i^2}$$

and we have the coefficients for the equation (89.2).

A set of polynomials which satisfies the condition (89.5) are the orthogonal polynomials.* In particular, we shall choose the following set of polynomials

(89.8) $$\begin{cases} P_0(x) = 1 \\ P_1(x) = x - \bar{x} \\ \cdots\cdots \\ P_{i+1}(x) = P_1P_i - \dfrac{i^2(m^2 - i^2)}{4(4i^2 - 1)} P_{i-1} \end{cases}$$

* See Chapter I, Section 3.H.

where m is the number of observations in the given data (x_i, y_i) and we have transformed x to integral values by (87.2). Let us have a closer look at these polynomials by considering a specific number of observations, say $m = 11$. Then substituting into the formula (89.8) we have

$$(89.9)\qquad \begin{cases} P_0 = 1 \\ P_1 = x - \bar{x} \\ P_2 = P_1^2 - 10P_0 = (x - \bar{x})^2 - 10 \\ P_3 = P_1P_2 - \dfrac{117}{15} P_1 = \dfrac{1}{5}(5P_1P_2 - 39P_1) \\ P_4 = P_1P_3 - \dfrac{36}{5} P_2 = \dfrac{1}{5}(5P_1P_3 - 36P_2) \\ P_5 = P_1P_4 - \dfrac{20}{3} P_3 = \dfrac{1}{3}(3P_1P_4 - 20P_3) \end{cases}$$

and since x_i has the values 1, 2, 3, $\cdots$, 11 we see that

x_i	1	2	3	4	5	6	7	8	9	10	11	C.F.
$P_0(x_i)$	1	1	1	1	1	1	1	1	1	1	1	
$P_1(x_i)$	−5	−4	−3	−2	−1	0	1	2	3	4	5	
$P_2(x_i)$	15	6	−1	−6	−9	−10	−9	−6	−1	6	15	
$P_3(x_i)$	−30	6	22	23	14	0	−14	−23	−22	−6	30	6/5
$P_4(x_i)$	6	−6	−6	−1	4	6	4	−1	−6	−6	6	12
$P_5(x_i)$	−3	6	1	−4	−4	0	4	4	−1	−6	3	40

where C.F. = common factor which has been removed from each value of $P_j(x_i)$. The condition

$$\sum_{i=1}^{m} P_kP_n(x_i) = 0, \qquad k \neq n,$$

can easily be verified for these numbers.

Since the values of $P_i(x_j)$ for integral values of x are dependent only upon the number of observations, these values can be computed once and for all and tabulated. In tabulating them we note the "symmetry" about x_M and see we need only record half of the values if we use $S(Y)$ and $D(Y)$. It is also necessary to have ΣP_i^2

which can be recorded at the bottom of each column. Since it is convenient to use only integral values for the entries, common factors will be factored out so that the entries are really related to the polynomial values by

$$P_i(x_j) = \text{C.F. (Entry)} \tag{89.10}$$

and a transformation to the true value in terms of x is accomplished by replacing P_i' which is obtained from the entry by λP_i where $\lambda = \dfrac{1}{\text{C.F.}}$. Values of λ are also recorded for each polynomial. The tabular values for $m = 11$ would then take this form:

X	P_1'	P_2'	P_3'	P_4'	P_5'
0	0	−10	0	6	0
1	1	−9	−14	4	4
2	2	−6	−23	−1	4
3	3	−1	−22	−6	−1
4	4	6	−6	−6	−6
5	5	15	30	6	3
$\Sigma P_i'^2$	110	858	4290	286	156
λ	1	1	5/6	1/12	1/40

The values of $P_0(x_i) \equiv 1$ and therefore are not recorded. To compute b_i from formula (89.7) we obtain $S(Y)$ and $D(Y)$ and then

$$\text{(89.11)} \quad \begin{cases} b_0 = \dfrac{1}{m} \sum y_j = \bar{y} \\ b_1 = \dfrac{\Sigma P_1'(X)\, D(Y)}{\Sigma P_1'^2(x)} \\ b_2 = \dfrac{\Sigma P_2'(X)\, S(Y)}{\Sigma P_2'^2(x)} \\ b_3 = \dfrac{\Sigma P_3'(X)\, D(Y)}{\Sigma P_3'^2(x)} \\ b_4 = \dfrac{\Sigma P_4'(X)\, S(Y)}{\Sigma P_4'^2(x)} \\ b_5 = \dfrac{\Sigma P_5'(X)\, D(Y)}{\Sigma P_5'^2(x)}. \end{cases}$$

This obtains the polynomial expression

$$(89.12)\quad y = b_0 + b_1P_1'(x) + b_2P_2'(x) + \cdots + b_nP_n'(x).$$

If it is desired to express y as a polynomial in x we make the further transformation

$$P_i'(x_j) = \lambda P_i(x_j)$$

and calculate the form of $P_i(x_j)$ by (89.8). Let us consider an example.

Example VIII.5. Solve Example VIII.2. using the table of orthogonal polynomials.

Solution.

x	y	I	X	I	$S(Y)$	$D(Y)$	b_i
1	108.0	5	0	0	−99.9	−99.9	−32
2	55.9	4	1	1	−179.9	−43.9	−5.201818
3	.1	3	2	2	−120.0	−75.8	9.995337
4	−54.1	2	3	3	−20.0	−84.2	1.198787
5	−100.0	1	4	4	115.9	−55.9	
6	−131.9	0	5	5	299.9	19.9	
7	−143.9	1					
8	−129.9	2					
9	−84.1	3					
10	0	4					
11	127.9	5					

The values of b_i are computed by (89.11) using the table for $m = 11$. This yields

$$y = 1.198787P_3'(x) + 9.995337P_2'(x) - 5.201818P_1'(x) - 32.$$

To transform this to a power series in x we have

$$P_1'(x) = 1P_1(x) = 1(x - \bar{x}) = x - 6$$

$$P_2'(x) = 1P_2(x) = (x - \bar{x})^2 - 10 = x^2 - 12x + 26$$

$$P_3'(x) = \frac{5}{6}P_3(x) = \frac{5}{6}\left(\frac{1}{5}\right)(5P_1P_2 - 39P_1)$$

$$= \frac{1}{6}(5x^3 - 90x^2 + 451x - 546)$$

and

$$y = .99899x^3 - 7.98647x^2 - 35.03704x + 150$$

Values of the modified orthogonal polynomials for $m = 4$ to $m = 33$ are given in Table XVII, p. 362.*

90. Smoothing of Data. The formulas derived in the last sections may be used to obtain smoother data than that given by the observations. This smoothing of data is accomplished by replacing the observed data by calculated data based on fitting the original data by a polynomial. Thus, for equidistant data we calculate the values from the approximating polynomial (89.2)

$$y(x) = \sum_{i=0}^{n} b_i P_i(x) \tag{90.1}$$

where

$$b_i = \frac{\Sigma P_i y_i}{\Sigma P_i^2}. \tag{90.2}$$

At a given value of the argument we may now express the value of the function in terms of the observed values of the function. To clearly illustrate this let us consider 5 equidistant points,

x_0	x_1	x_2	x_3	x_4
y_0	y_1	y_2	y_3	y_4

and fit them with a third degree polynomial,

$$y(x) = b_0P_0(x) + b_1P_1(x) + b_2P_2(x) + b_3P_3(x). \tag{90.3}$$

Now let

$$S_i = \sum_{j=0}^{4} P_i^2(x_j), \qquad (i = 0,1,2,3), \tag{90.4}$$

* An extensive table up to $m = 75$ may be found in Fisher and Yates, *Statistical Tables* (New York: Hafner Publishing Company, Inc., 1949).

so that

$$(90.5)\quad b_i = \frac{1}{S_i}\sum_{j=0}^{4} P_i(x_j)y_j$$

$$= \frac{1}{S_i}[P_i(0)y_0 + P_i(1)y_1 + P_i(2)y_2 + P_i(3)y_3 + P_i(4)y_4]$$

if the x_i have been transformed to the integers. Then

$$\begin{aligned} y(x) = {} & \frac{1}{S_0}[P_0(0)y_0 + P_0(1)y_1 + P_0(2)y_2 + P_0(3)y_3 + P_0(4)y_4]P_0(x) \\ & + \frac{1}{S_1}[P_1(0)y_0 + P_1(1)y_1 + P_1(2)y_2 + P_1(3)y_3 \\ & \qquad + P_1(4)y_4]P_1(x) \\ & + \frac{1}{S_2}[P_2(0)y_0 + P_2(1)y_1 + P_2(2)y_2 + P_2(3)y_3 \\ & \qquad + P_2(4)y_4]P_2(x) \\ & + \frac{1}{S_3}[P_3(0)y_0 + P_3(1)y_1 + P_3(2)y_2 + P_3(3)y_3 \\ & \qquad + P_3(4)y_4]P_3(x) \\ = {} & c_0y_0 + c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 \end{aligned}$$

where

$$(90.6)\quad \begin{cases} c_0(x) = \frac{1}{S_0}P_0(0)P_0(x) + \frac{1}{S_1}P_1(0)P_1(x) \\ \qquad\qquad + \frac{1}{S_2}P_2(0)P_2(x) + \frac{1}{S_3}P_3(0)P_3(x) \\ c_1(x) = \sum_{j=0}^{3}\frac{1}{S_j}P_j(1)P_j(x) \\ \cdots\cdots \\ c_i(x) = \sum_{j=0}^{3}\frac{1}{S_j}P_j(i)P_j(x). \end{cases}$$

Now if we pick a specific point, $x = x_i$, then $P_j(x_i)$ can be evaluated and $y(x_j)$ is the sum of known numbers. For the values $x_i = 0,1,2,3,4$ we have $\bar{x} = 2$ so that by formula (89.8)

(90.7)
$$\begin{cases} P_0(x) = 1 \\ P_1(x) = x - 2 \\ P_2(x) = x^2 - 4x + 2 \\ P_3(x) = \frac{1}{5}(x-2)(5x^2 - 20x + 3) \end{cases}$$

and the values are

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$	S_i
$P_0(x)$	1	1	1	1	1	5
$P_1(x)$	-2	-1	0	1	2	10
$P_2(x)$	2	-1	-2	-1	2	14
$P_3(x)$	$-\frac{6}{5}$	$\frac{12}{5}$	0	$-\frac{12}{5}$	$\frac{6}{5}$	$\frac{72}{5}$

The values of $c_i(x)$ at $x = x_j$ are easily calculated from (90.6) to yield

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
$c_0(x)$	69	2	-3	2	-1
$c_1(x)$	4	27	12	-8	4
$c_2(x)$	-6	12	17	12	-6
$c_3(x)$	4	-8	12	27	4
$c_4(x)$	-1	2	-3	2	69
L.C.D.	70	35	35	35	70

Thus

(90.8)
$$\begin{cases} y'_0 = \frac{1}{70}[69y_0 + 4y_1 - 6y_2 + 4y_3 - y_4] \\ y'_1 = \frac{1}{35}[2y_0 + 27y_1 + 12y_2 - 8y_3 + 2y_4] \\ y'_2 = \frac{1}{35}[-3y_0 + 12y_1 + 17y_2 + 12y_3 - 3y_4] \\ y'_3 = \frac{1}{35}[2y_0 - 8y_1 + 12y_2 + 27y_3 + 2y_4] \\ y'_4 = \frac{1}{70}[-y_0 + 4y_1 - 6y_2 + 4y_3 + 69y_4]. \end{cases}$$

The mid-point formula y_2' is most frequently used when smoothing the data by fitting a cubic to five points since it can apply to any portion of the data for which $h = 1$.

Example VIII.6. Obtain a smoother set of values to (x_i, y_i) given below by a cubic fit to five consecutive points.

Solution. The formulas for y_0', y_1', and y_2' of (90.8) are used to calculate these new values. Then the mid-point formula, y_2', is used until y_{17}' is reached. Then the formula for y_3' and y_4' are used to give y_{18}' and y_{19}'.

x	y	y'
0	0	−.1
1	3	3.3
2	7	6.5
3	9	9.5
4	12	11.6
5	13	13.3
6	14	13.9
7	14	14.0
8	13	12.8
9	11	11.1
10	10	10.4
11	11	10.6
12	10	9.9
13	8	8.3
14	7	6.8
15	6	6.1
16	6	6.3
17	7	6.1
18	5	5.6
19	6	5.9

The above formulas are based upon the idea of fitting the best cubic polynomial through five consecutive points with $h = 1$. It is, of course, easy to derive some other smoothing formula based on different choices of the *degree of the polynomial* and on *the number of points*. The procedure would be identical to that above. The usual procedure is to keep both the degree of the polynomial and the number of points odd. The results would be a set of multiplying coefficients for the y_i entries, the most important being the mid-point formula. These coefficients are given in the following table.

Table 8.2. Smoothing Formulas.
Based on Third Degree Polynomial

i \ NO POINTS	5	7	9	11	13	15	17	19
DENOMINATOR	35	21	231	429	143	1105	323	2261
0	−3	−2	−21	−36	−11	−78	−21	−136
1	12	3	14	9	0	−13	−6	−51
2	17	6	39	44	9	42	7	24
3	12	7	54	69	16	87	18	89
4	−3	6	59	84	21	122	27	144
5		3	54	89	24	147	34	189
6		−2	39	84	25	162	39	224
7			14	69	24	167	42	249
8			−21	44	21	162	43	264
9				9	16	147	42	269
10				−36	9	122	39	264
11					0	87	34	249
12					−11	42	27	224
13						−13	18	189
14						−78	7	144
15							−6	89
16							−21	24
17								−51
18								−136

Example VIII.7. Check the value at $x = 12$ of Example VIII.6 by the 11-point formula for both a third degree and a fifth degree fit.

Solution.

a) Third degree:

$$y'(12) = [-36(14) + 9(13) + 44(11) + 69(10) + 84(11) + 89(10) + 84(8) + 69(7) + 44(6) + 9(6) - 36(7)]/429$$

$$= 8.9.$$

b) Fifth degree (see p. 292):

$$y'(12) = [18(14) - 45(13) - 10(11) + 60(10) + 120(11) + 143(10) + 120(8) + 60(7) - 10(6) - 45(6) + 18(7)]/429$$

$$= 9.5.$$

Table 8.3. Smoothing Formulas.
Based on Fifth Degree Polynomial

i \ NO POINTS	7	9	11	13	15	17	19
DENOMINATOR	231	429	429	2431	46189	4199	7429
0	5	15	18	110	2145	195	340
1	−30	−55	−45	−198	−2860	−195	−255
2	75	30	−10	−135	−2937	−260	−420
3	131	135	60	110	−165	−117	−290
4	75	179	120	390	3755	135	18
5	−30	135	143	600	7500	415	405
6	5	30	120	677	10125	660	790
7		−55	60	600	11063	825	1110
8		15	−10	390	10125	883	1320
9			−45	110	7500	825	1393
10			18	−135	3755	660	1320
11				−198	−165	415	1110
12				110	−2937	135	790
13					−2860	−117	405
14					2145	−260	18
15						−195	−290
16						195	−420
17							−255
18							340

91. Weighted Residuals. For some data it may be desired to give more weight to some observations than others. In that case the best fit is that for which the sum of the weighted squares of the residuals is a minimum, i.e., it is desired to have

$$\Sigma wR^2 = w_1R_1^2 + w_2R_2^2 + \cdots + w_nR_n^2 \tag{91.1}$$

a minimum, where w_i is a set of weights. To minimize the function (91.1) we set the partial derivatives equal to zero as before, thus obtaining a set of weighted normal equations.

Let us consider a simple problem of fitting the data having weights w_j with a quadratic:

$$y = a_0 + a_1x + a_2x^2. \tag{91.2}$$

The weighted residuals are

$$w_jR_j = w_j(a_0 + a_1x_j + a_2x_j^2 - y_j) \tag{91.3}$$

and the sum of the weighted squares of the residuals is

$$\Sigma w_jR_j^2 = \Sigma w_j(a_0 + a_1x_j + a_2x_j^2 - y_j)^2 \tag{91.4}$$

the summation being over j from 0 to m for the m observations. Taking the partial derivative of (91.4) with respect to a_i and setting each equal to zero yields the weighted normal equations

$$\begin{cases} w_1(a_0 + a_1x_1 + a_2x_1^2 - y_1) + \cdots \\ \qquad + w_m(a_0 + a_1x_m + a_2x_m^2 - y_m) = 0 \\ w_1x_1(a_0 + a_1x_1 + a_2x_1^2 - y_1) + \cdots \\ \qquad + w_mx_m(a_0 + a_1x_m + a_2x_m^2 - y_m) = 0 \\ w_1x_1^2(a_0 + a_1x_1 + a_2x_1^2 - y_1) + \cdots \\ \qquad + w_mx_m(a_0 + a_1x_m + a_2x_m^2 - y_m) = 0 \end{cases} \tag{91.5}$$

which may be transformed into

$$\begin{cases} (\Sigma w_j)a_0 + (\Sigma w_jx_j)a_1 + (\Sigma w_jx_j^2)a_2 = \Sigma w_jy_j \\ (\Sigma w_jx_j)a_0 + (\Sigma w_jx_j^2)a_1 + (\Sigma w_jx_j^3)a_2 = \Sigma w_jx_jy_j \\ (\Sigma w_jx_j^2)a_0 + (\Sigma w_jx_j^3)a_1 + (\Sigma w_jx_j^4)a_2 = \Sigma w_jx_j^2y_j \end{cases} \tag{91.6}$$

from which we solve for a_i to obtain the fit (91.2).

The weights, w_j, may be arbitrarily assigned from previous knowledge of the observations or they may be determined by some probable error study. In general, the weights are inversely proportional to the squares of the probable errors. If the data are obtained statistically the weights are usually chosen to be the reciprocal to the square of the standard deviation.

For the logarithmic function it can be shown* that if the weights of y are all equal, then the weights of $\log y$ are given by

$$w_j = \frac{y^2}{M^2} \tag{91.7}$$

where $M = .43429$.

* See J. B. Scarborough, *Numerical Mathematical Analysis* (2nd ed.; Baltimore: Johns Hopkins University Press, 1950), p. 460.

Example VIII.8. Fit the following data, given the values of the weights, with the expression

$$\log y = \log k + n \log x$$

x	172	210	320	400
y	66	80	100	120
w	4356	6400	10000	14400

Solution. We seek to determine $\log k$ and n. The equation may be written as

$$Y = K + nX$$

where

$$Y = \log y, \qquad K = \log k, \qquad \text{and} \qquad X = \log x$$

so that the data is changed to

X	2.23553	2.32222	2.50515	2.60206
Y	1.81954	1.90309	2.00000	2.07918
w	4356	6400	10000	14400

The weighted normal equations [see (91.6)] are easily calculated:

$$\begin{cases} 35156K + 87121.34n = 70045.88 \\ 87121.34K + 216538.92n = 174011.92, \end{cases}$$

the solution of which yields

$$K = 0.33346, \qquad n = .66944.$$

The equation is

$$\boxed{\log y = .33346 + .66944 \log x}.$$

92. Exercise XI.

1. Given the data

x	-2	0	1	3	4
y	18.1	3.9	3.0	12.9	24.1

Obtain the best fitting quadratic function.

2. Given the data

x	0	1	2	3	4	5	6	7	8	9	10
y	4	88	158	211	240	250	240	210	160	100	0

Obtain the best fitting polynomial of degree 4, 3, and 2. Use both the Nielsen-Goldstein method and the method of orthogonal polynomials and compare results.

3. Obtain the best cubic fit to the data

x	y	x	y
0	.127462	3.2	.255362
.2	.133271	3.4	.263251
.4	.139413	3.6	.270285
.6	.145905	3.8	.276221
.8	.152753	4.0	.280822
1.0	.159964	4.2	.283815
1.2	.167528	4.4	.285005
1.4	.175459	4.6	.284214
1.6	.183731	4.8	.281286
1.8	.192319	5.0	.276144
2.0	.201186	5.2	.268764
2.2	.210262	5.4	.259179
2.4	.219482	5.6	.247465
2.6	.228741	5.8	.233738
2.8	.237937	6.0	.218226
3.0	.246848		

4. Prove by direct substitution that

$$\sum_{j=1}^{m} X_j^i = 2h^i \sum_{k=1}^{r-1} k$$

if $m = 2r - 1 = 17$ and $i = 4$.

5. Prove that

$$\sum_{k=1}^{r} (2k-1)^i = \sum_{k=1}^{2r} k^i - 2^i \sum_{k=1}^{r} k^i.$$

6. Show that in the method of orthogonal polynomials, $S(Y)$ may be replaced by $S^*(Y) = y_{M+i} + y_{M-i}$ and $S^*(Y_0) = y_M$, if $m = 11$. Generalize this statement.

7. Given the data

x	y	x	y
0	1	13	15
1	3	14	14
2	4	15	16
3	6	16	15
4	5	17	15
5	7	18	14
6	6	19	13
7	9	20	12
8	10	21	12
9	10	22	10
10	12	23	11
11	11	24	9
12	14	25	7

Smooth the values by formulas (90.8) at the beginning and end and use
a) an 11-point cubic
b) a 9-point fifth degree
wherever possible and compare values.

8. Fit a quadratic to the weighted observations of

x	0	1	2	3	4	5	6
y	1	3	4	6	6	7	8
w	1	1	.75	.5	.5	1	1

CHAPTER IX. PERIODIC AND EXPONENTIAL FUNCTIONS

93. Introduction. In the previous chapters we have concerned outselves with the representation of functions or data in what essentially was power series. For certain functions this may not be desirable. This is especially true for periodic functions or functions which have the characteristics of the exponential function. We shall now concern ourselves with the analysis of such functions.

94. Trigonometric Approximations. A function for which

$$f(x + p) = f(x)$$

is said to be a periodic function with period p. The period can always be changed to 2π by the transformation

(94.1) $$x = \left(\frac{p}{2\pi}\right) x'$$

so that we can consider only functions of period 2π.

A trigonometric series is one of the form

(94.2) $$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_n and b_n are constants.

Any periodic function $f(x)$ can be represented by the trigonometric series (94.2), and if we choose

$$(94.3) \qquad \begin{cases} a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{cases}$$

we have the well-known Fourier Series.*

This series will have additional characteristics if $f(x)$ is either an even function or an odd function. It is recalled that the definition of such functions is:

a) $f(x)$ is said to be an even function if $f(-x) \equiv f(x)$;

b) $f(x)$ is said to be an odd function if $f(-x) \equiv -f(x)$.

For even functions we have $b_n = 0$ and $a'_n = 2a_n$; for odd functions we have $a_n = 0$ and $b'_n = 2b_n$.

The representation of a function in a given interval by a Fourier Series is a trigonometric approximation which is easily accomplished providing a_n by b_n may be determined. Consider, e.g., the following classical example.

Example IX.1. Find the Fourier Series representation of the function

$$\begin{cases} f(x) = -x & \text{for} \quad -\pi < x \leq 0 \\ f(x) = x & \text{for} \quad 0 < x < \pi \end{cases}$$

Solution.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{0} (-x) dx + \frac{1}{\pi} \int_{0}^{\pi} x dx = \frac{1}{2} (\pi + \pi) = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-x) \cos nx dx + \int_{0}^{\pi} x \cos nx dx \right]$$

$$= \frac{2}{\pi n^2} (\cos n\pi - 1)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

* See H. W. Reddick and F. H. Miller, *Advanced Mathematics for Engineers* (2nd ed.; New York: John Wiley & Sons, Inc., 1950), Chapter 5.

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-x) \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right]$$

$$= \frac{1}{n}[\cos n\pi - \cos n\pi] = 0.$$

Thus the series is

$$f(x) = \frac{1}{2}\pi - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right].$$

We shall concern ourselves not with the representation of the functions which have an analytic definition but with those functions which are known only at discrete points.

95. Harmonic Analysis. The problem of finding the coefficients a_n, b_n of the trigonometric series when we are given only the equidistant values of the function within an interval is called *harmonic analysis*. Let us divide the interval from $-\pi$ to π into $2k$ equal parts so that

$$x_j = \frac{j}{k}\pi.$$

We now seek to minimize the residual equation

$$(95.1) \quad \Sigma R^2 = \sum_{j=-k}^{k-1}\left[y_j - \sum_{n=0}^{m}(a_n \cos nx_j + b_n \sin nx_j)\right]^2.$$

Take the partial derivatives with respect to a_n and b_n and set them equal to zero to obtain the normal equations

$$(95.2) \quad \begin{cases} \displaystyle\sum_{n=0}^{m}\left[a_n \sum_{j=-k}^{k-1} \cos nx_j \cos rx_j + b_n \sum_{j=-k}^{k-1} \sin nx_j \cos rx_j\right] = \sum_{j=-k}^{k-1} y_j \cos rx_j \\ \displaystyle\sum_{n=0}^{m}\left[a_n \sum_{j=-k}^{k-1} \cos nx_j \sin rx_j + b_n \sum_{j=-k}^{k-1} \sin nx_j \sin rx_j\right] = \sum_{j=-k}^{k-1} y_j \sin rx_j \end{cases}$$

with $(r = 0, \cdots, m)$.

By the use of trigonometric identities and remembering that we have $2k$ divisions of the interval $(-\pi, \pi)$ with $x_{-s} = -x_s$, we can show that

$$(95.3)\quad \begin{cases} \sum_{j=-k}^{k-1} \sin nx_j \cos rx_j = 0 \\ \sum_{j=-k}^{k-1} \cos nx_j \cos rx_j = \begin{cases} 0, & \text{if } n \neq r \\ k, & \text{if } n = r \neq 0, k \\ 2k, & \text{if } n = r = 0, k \end{cases} \\ \sum_{j=-k}^{k-1} \sin nx_j \sin rx_j = \begin{cases} 0, & \text{if } n \neq r \\ k, & n = r \neq 0, k \\ 0, & \text{if } n = r = 0, k \end{cases} \end{cases}$$

so that the normal equations (95.2) take the form

$$(95.4)\quad \begin{cases} 2ka_0 = \sum_{j=-k}^{k-1} y_j \\ ka_1 = \sum_{j=-k}^{k-1} y_j \cos x_j \\ ka_r = \sum_{j=-k}^{k-1} y_j \cos rx_j \\ kb_r = \sum_{j=-k}^{k-1} y_j \sin rx_j \end{cases}$$

The calculation of these coefficients may be simplified somewhat by first calculating $S(y_j)$ and $D(y_j)$ as was done in the last chapter. Thus let us define

$$(95.5)\quad \begin{cases} F_j = f(x_j) + f(-x_j), & (j = 1, \cdots, k-1), \\ G_j = f(x_j) - f(-x_j), & (j = 1, \cdots, k-1), \\ F_0 = f(0) \quad \text{and} \quad F_k = f(x_k). \end{cases}$$

Then

$$
(95.6)\quad \begin{cases}
a_0 = \dfrac{1}{2k} \displaystyle\sum_{j=0}^{k} F_j \\
a_r = \dfrac{1}{k} \displaystyle\sum_{j=-k}^{k-1} y_j \cos rx_j = \dfrac{1}{k} \displaystyle\sum_{j=0}^{k} F_j \cos rx_j, \\
\qquad\qquad\qquad\qquad (r = 1, \cdots, k-1), \\
a_k = \dfrac{1}{2k} \displaystyle\sum_{j=-k}^{k-1} F_j \cos kx_j \\
b_r = \dfrac{1}{k} \displaystyle\sum_{j=-k}^{k-1} y_j \sin rx_j = \dfrac{1}{k} \displaystyle\sum_{j=1}^{k-1} G_j \sin rx_j.
\end{cases}
$$

The computation may be arranged in a schematic which we will display for $k = 6$, i.e., the interval $(-\pi, \pi)$ divided into twelve parts.

x	F	$\cos x$	$\cos 2x$	$\cos 3x$	$\cos 4x$	$\cos 5x$	$\cos 6x$
0	$F_0 = y_0$	1	1	1	1	1	1
$\frac{\pi}{6}$	$F_1 = y_1 + y_{-1}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	-1
$\frac{2\pi}{6}$	$F_2 = y_2 + y_{-2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{3\pi}{6}$	$F_3 = y_3 + y_{-3}$	0	-1	0	1	0	-1
$\frac{4\pi}{6}$	$F_4 = y_4 + y_{-4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1
$\frac{5\pi}{6}$	$F_5 = y_5 + y_{-5}$	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	-1
$\frac{6\pi}{6}$	$F_6 = y_6$	-1	1	-1	1	-1	1
	a_0	a_1	a_2	a_3	a_4	a_5	a_6

x	G	sin x	sin $2x$	sin $3x$	sin $4x$	sin $5x$	sin $6x$
$\frac{\pi}{6}$	$G_1 = y_1 - y_{-1}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	0
$\frac{2\pi}{6}$	$G_2 = y_2 - y_{-2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{3}$	0	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	0
$\frac{3\pi}{6}$	$G_3 = y_3 - y_{-3}$	1	0	-1	0	1	0
$\frac{4\pi}{6}$	$G_4 = y_4 - y_{-4}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	0	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	0
$\frac{5\pi}{6}$	$G_5 = y_5 - y_{-5}$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	1	$-\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	0
		b_1	b_2	b_3	b_4	b_5	b_6

The coefficients are now obtained according to formula (95.6) by simply summing the products of the elements in F column with the corresponding elements in appropriate cos rx column and similarly for the sin rx. A closer look at the table of values for the trigonometric functions above will indicate some advantageous grouping. Thus

$$
(95.7)\quad
\begin{cases}
a_0 = \dfrac{1}{12}[F_0 + F_1 + \cdots + F_6] \\
a_1 = \dfrac{1}{6}\left[F_0 - F_6 + \dfrac{1}{2}\sqrt{3}\,(F_1 - F_4) + \dfrac{1}{2}(F_2 - F_3)\right] \\
a_2 = \dfrac{1}{6}\left[F_0 - F_3 + F_6 + \dfrac{1}{2}(F_1 - F_2 - F_4 + F_5)\right] \\
a_3 = \dfrac{1}{6}[F_0 - F_2 + F_4 - F_6] \\
a_4 = \dfrac{1}{6}\left[F_0 + F_3 + F_6 - \dfrac{1}{2}(F_1 + F_2 + F_4 + F_5)\right] \\
a_5 = \dfrac{1}{6}\left[[F_0 - F_6 - \dfrac{1}{2}\sqrt{3}\,(F_1 - F_5) + \dfrac{1}{2}(F_2 - F_4)\right] \\
a_6 = \dfrac{1}{12}[F_0 - F_1 + F_2 - F_3 + F_4 - F_5 + F_6]
\end{cases}
$$

and

$$
(95.8)\quad \begin{cases}
b_1 = \dfrac{1}{6}\left[G_3 + \dfrac{1}{2}(G_1 + G_5) + \dfrac{1}{2}\sqrt{3}\,(G_2 + G_4)\right] \\
b_2 = \dfrac{1}{6}\left[\dfrac{1}{2}\sqrt{3}\,(G_1 + G_2 - G_4 - G_5)\right] \\
b_3 = \dfrac{1}{6}[G_1 - G_3 + G_5] \\
b_4 = \dfrac{1}{6}\left[\dfrac{1}{2}\sqrt{3}\,(G_1 - G_2 + G_4 - G_5)\right] \\
b_5 = \dfrac{1}{6}\left[G_3 + \dfrac{1}{2}(G_1 + G_5) - \dfrac{1}{2}\sqrt{3}\,(G_2 + G_4)\right].
\end{cases}
$$

Such grouping is extremely advantageous when the computing equipment is rather limited. However, with an automatic desk calculator it may be just as rapid to place the F and G columns with a stencil of the table of values and simply sum the products of two terms. We shall therefore present a six-place table which may be used.

Table 9.1. Harmonic Analysis—12 Observations

x	cos x	cos $2x$	cos $3x$	cos $4x$	cos $5x$	cos $6x$
0	1	1.0	1	1.0	1	1
30	.866025	0.5	0	−0.5	−.866025	−1
60	.500000	−0.5	−1	−0.5	.500000	1
90	0	−1.0	0	1.0	0	−1
120	−.500000	−0.5	1	−0.5	−.500000	1
150	−.866025	0.5	0	−0.5	.866025	−1
180	−1	1.0	−1	1.0	−1	1
	sin x	sin $2x$	sin $3x$	sin $4x$	sin $5x$	
30	.500000	.866025	1	.866025	.500000	
60	.866025	.866025	0	−.866025	−.866025	
90	1	0	−1	0	1	
120	.866025	−.866025	0	.866025	−.866025	
150	.500000	−.866025	1	−.866025	.500000	

Example IX.2. Obtain the trigonometric approximation to

x^0	−150	−120	−90	−60	−30	0	30	60	90	120	150	180
y	10	16	18	24	38	32	16	5	−7	−13	−14	−5

Solution.

x	F	G		
0	32		$a_0 = 10$	
30	54	−22	$a_1 = 15.872$	$b_1 = 14.928$
60	29	−19	$a_2 = 4.167$	$b_2 = 1.732$
90	11	−25	$a_3 = 1.833$	$b_3 = -3.500$
120	3	−29	$a_4 = -.500$	$b_4 = -1.155$
150	−4	−24	$a_5 = -.038$	$b_5 = -1.072$
180	−5		$a_6 = -.167$	

$$\begin{aligned} y = 10 &+ 15.872 \cos x + 4.167 \cos 2x + 1.833 \cos 3x - .5 \cos 4x \\ &- .038 \cos 5x - .167 \cos 6x \\ &- 14.928 \sin x + 1.732 \sin 2x - 3.5 \sin 3x - 1.155 \sin 4x \\ &- 1.072 \sin 5x. \end{aligned}$$

Check.

x^0	−150	−120	−90	−60	−30	0	30	60	90	120	150	180
y (GIVEN)	10	16	18	24	38	32	16	5	−7	−13	−14	−5
y (CALC.)	10.89	16.25	18.17	23.42	37.45	31.17	15.44	4.42	−6.83	−12.75	−13.11	−4.3

Another convenient division of the interval $(-\pi, \pi)$ is into twenty-four equal subdivisions ($k = 12$). The multipliers of F_i and G_i now become

Table 9.2. Harmonic Analysis—24 Observations.

x	cos x	cos $2x$	cos $3x$	cos $4x$	cos $5x$	cos $6x$	cos $7x$	cos $8x$	cos $9x$	cos $10x$	cos $11x$	cos $12x$
0	1	1	1	1.0	1	1	1	1	1	1	1	1
15	.965926	.866025	.707107	.5	.258819	0	−.258819	−.5	−.707107	−.866025	−.965926	−1
30	.866025	.500000	0	−.5	−.866025	−1	−.866025	−.5	0	.500000	.866025	1
45	.707107	0	−.707107	−1.0	−.707107	0	.707107	1	.707107	0	−.707107	−1
60	.500000	−.500000	−1	−.5	.500000	1	.500000	−.5	−1	−.500000	.500000	1
75	.258819	−.866025	−.707107	.5	.965926	0	−.965926	−.5	.707107	.866025	−.258819	−1
90	0	−1	0	1.0	0	−1	0	1	0	−1	0	1
105	−.258819	−.866025	.707107	.5	−.965926	0	.965926	−.5	−.707107	.866025	.258819	−1
120	−.500000	−.500000	1	−.5	−.500000	1	−.500000	−.5	1	−.500000	−.500000	1
135	−.707107	0	.707107	−1.0	.707107	0	−.707107	1	−.707107	0	.707107	−1
150	−.866025	.500000	0	−.5	.866025	−1	.866025	−.5	0	.500000	−.866025	1
165	−.965926	.866025	−.707107	.5	−.258819	0	.258819	−.5	.707107	−.866025	.965926	−1
180	−1	1	−1	1.0	−1	1	−1	1	−1	1	−1	1

x	sin x	sin $2x$	sin $3x$	sin $4x$	sin $5x$	sin $6x$	sin $7x$	sin $8x$	sin $9x$	sin $10x$	sin $11x$
15	.258819	.500000	.707107	.866025	.965926	1	.965926	.866025	.707107	.500000	.258819
30	.500000	.866025	1	.866025	.500000	0	−.500000	−.866025	−1	−.866025	−.500000
45	.707107	1	.707107	0	−.707107	−1	−.707107	0	.707107	1	.707107
60	.866025	.866025	0	−.866025	−.866025	0	.866025	.866025	0	−.866025	−.866025
75	.965926	.500000	−.707107	−.866025	.258819	1	.258819	−.866025	−.707107	.500000	.965926
90	1	0	−1	0	1	0	−1	0	1	0	−1
105	.965926	−.500000	−.707107	.866025	.258819	−1	.258819	.866025	−.707107	−.500000	.965926
120	.866025	−.866025	0	.866025	−.866025	0	.866025	−.866025	0	.866025	−.866025
135	.707107	−1	.707107	0	−.707107	1	−.707107	0	.707107	−1	.707107
150	.500000	−.866025	1	−.866025	.500000	0	−.500000	.866025	−1	.866025	−.500000
165	.258819	−.500000	.707107	−.866025	.965926	−1	.965926	−.866025	.707107	−.500000	.258819

Example IX.3. Obtain the trigonometric approximation to the data given below.

Solution.

x	y	x	y	F	G	i	a_i	b_i
0	32			32		0	10.167	
−15	37	15	25	62	−12	1	16.581	−14.915
−30	38	30	16	54	−22	2	4.321	1.908
−45	32	45	10	42	−22	3	1.506	−3.518
−60	24	60	5	29	−19	4	−.208	−.938
−75	20	75	0	20	−20	5	.019	−.739
−90	18	90	−7	11	−25	6	−.167	.333
−105	17	105	−11	6	−28	7	−.057	.332
−120	16	120	−13	3	−29	8	−.292	.217
−135	14	135	−15	−1	−29	9	.327	−.018
−150	10	150	−14	−4	−24	10	−.154	.176
−165	5	165	−10	−5	−15	11	.124	.013
		180	−5	−5		12	−.167	

$$
\begin{aligned}
y = 10.167 &+ 16.581 \cos x + 4.321 \cos 2x + 1.506 \cos 3x \\
&- .208 \cos 4x \\
&+ .019 \cos 5x - .167 \cos 6x - .057 \cos 7x \\
&- .292 \cos 8x \\
&+ .327 \cos 9x - .154 \cos 10x + .124 \cos 11x \\
&- .167 \cos 12x \\
&- 14.915 \sin x + 1.908 \sin 2x - 3.518 \sin 3x \\
&- .938 \sin 4x \\
&- .739 \sin 5x + .333 \sin 6x + .332 \sin 7x \\
&+ .217 \sin 8x \\
&- .018 \sin 9x + .176 \sin 10x + .013 \sin 11x
\end{aligned}
$$

Check.

x	−120	−90	−30	0	30	90	120	180
y (GIVEN)	16	18	38	32	16	−7	−13	−5
y (CALC.)	16.00	17.97	38.00	32.00	16.00	−7.00	−13.00	−5.00

96. The Exponential Type Function. Frequently in physical problems a given set of data may be well fitted by a function of the type

$$y = aN^x \tag{96.1}$$

or

$$y = ae^{bx}. \tag{96.2}$$

From the properties of logarithms they may be changed,

$$\log y = \log a + x \log N \tag{96.3}$$

and

$$\begin{aligned} \log y &= \log a + bx \log e \\ \ln y &= \ln a + bx, \end{aligned} \tag{96.4}$$

where $\ln N$ is the natural logarithm of N, (to the base e). By a substitution of variables we can let

$$\begin{cases} \log y = z \\ \log a = c \\ \log N = d \\ x \log e = w \end{cases} \tag{96.5}$$

to obtain

$$z = c + dx \tag{96.6}$$

and

$$z = c + bw, \tag{96.7}$$

both of which are linear expressions to which we may apply the principle of least squares to determine b, c, and d.

Another function which may be treated in the same manner is the function

$$y = ax^n \tag{96.8}$$

which may be transformed to

$$\log y = \log a + n \log x. \tag{96.9}$$

In order to determine if the given data are suitable to be fitted by these functions it is recommended that they be plotted on various graph papers from which we establish the following criterion:

a) If the data appear to lie on a straight line when plotted on *semilogarithmic* graph paper, use the formula

$$y = ae^{bx} \qquad \text{or} \qquad y = aN^x.$$

b) If the data appear to lie on a straight line when plotted on *logarithmic* graph paper, use the formula

$$y = ax^n.$$

Example IX.4. Determine a functional fit to the data

x	1	2	3	4	5	6	7	8	9	10	11
y	1.00	1.15	1.30	1.50	1.75	2.00	2.30	2.65	3.00	3.50	4.00

Solution.

A plot on semilogarithmic paper reveals a straight line tendency. We therefore will fit the data with

$$y = aN^x \qquad \text{or} \qquad z = c + dx$$

where $z = \log y$, $c = \log a$, and $d = \log N$. Using a five-place logarithmic table, we have

x	1	2	3	4	5	6	7	8	9	10	11
z	0	.06070	.11394	.17609	.24304	.30103	.36173	.42325	.47712	.54407	.60206

The Nielsen-Goldstein method yields

$$d = (2S_2)^{-1}\Sigma X_j D(Y) = (110)^{-1}(6.64633) = .060421$$

$$c = \bar{y} - \bar{x}d = .300275 - 6(.060421) = -.062251.$$

Therefore

$$a = \text{antilog}\,(-.062251) = \text{antilog}\,(9.93775 - 10) = .86646$$

$$N = \text{antilog}\,(.06042) = 1.1493$$

and

$$y = .86646(1.1493)^x.$$

Example IX.5. Determine a functional fit to the data

x	1	2	3	4	5	6	7
y	5.0	7.5	9.5	11.3	13.0	14.5	16.0

Solution.

A plot on log paper shows a straight line tendency. Thus we try

$$y = ax^n \qquad \text{or} \qquad z = c + nw$$

where $z = \log y$, $c = \log a$, and $w = \log x$.
The transformed data is

w	0	.30103	.47712	.60206	.69897	.77815	.84510
z	.69897	.87506	.97772	1.05308	1.11394	1.16137	1.20412

The values are no longer given at equidistant values of the independent variable (w) so that we need to obtain the normal equations to apply the principle of least squares. For a linear expression these are

$$\begin{cases} mc + \Sigma w_i n = \Sigma z_i \\ \Sigma w_i c + \Sigma w_i n = \Sigma w_i z_i \end{cases}$$

and in our example, using the schematic of Section 61,

c	n	k
7 3.70243	3.70243 2.489009	7.08426 4.063859
3.715075	.69625	.59704

Now

$$a = \text{antilog } (.69625) = 4.9688$$

so that we have

$$y = 4.9688x^{.59704}.$$

97. The Method of Differential Correction. In the previous sections we have been discussing the fitting of empirical data by some well-known functions which adapt themselves to simple procedures. Thus, we can usually transform the functions to a polynomial and then apply the method of least squares. This is the most desirable procedure since it is advantageous to deal with well-known functions. Occasionally, however, certain empirical data can only be fitted by complicated functions, and it is necessary to have a general method for handling these cases. Such a method is one which we shall call the *method of differential correction.*

Let us suppose that we want a formula which is to relate the two variables x and y and which will have a number of undetermined constants; for simplicity let us choose three, a, b, and c. Symbolically, we write

(97.1) $$y = f(x,a,b,c).$$

This formula is to be a good fit to the data (x_i, y_i), $(i = 1, \ldots, m)$. The residuals are given by

(97.2) $$\begin{cases} R_1 = f(x_1,a,b,c) - y_1 \\ R_2 = f(x_2,a,b,c) - y_2 \\ \cdots\cdots\cdots \\ R_m = f(x_m,a,b,c) - y_m \end{cases}$$

where y_i, $(i = 1, \cdots, m)$, are the given (observed) values from the original data. Let us make a plot of the given data and from this plot determine approximate values to the constants, a, b, c, and call them a_0, b_0, c_0. It is desired to correct these approximate values by some incremental amount α, β, γ such that

(97.3) $$\begin{cases} a = a_0 + \alpha \\ b = b_0 + \beta \\ c = c_0 + \gamma \end{cases}$$

will yield better values and the formula will fit the data better.

If we substitute the values (97.3) into the residuals (97.2) and transpose the y_i, we have

(97.4) $$R_i + y_i = f(x_i, a_0 + \alpha, b_0 + \beta, c_0 + \gamma).$$

We may expand the right-hand side by Taylor's theorem for a function of several variables [see Formula (3.21)] to obtain the set of equations

$$R_i + y_i = f(x_i, a_0, b_0, c_0) + \alpha\left(\frac{\partial f_i}{\partial a}\right)_0 + \beta\left(\frac{\partial f_i}{\partial b}\right)_0 + \gamma\left(\frac{\partial f_i}{\partial c}\right)_0 + \text{higher-order terms in } \alpha, \beta, \gamma, \tag{97.5}$$

where

$$\left(\frac{\partial f_i}{\partial u}\right)_0 \equiv \text{the value of the partial derivative } \frac{\partial f}{\partial u} \text{ at } x = x_i, \; a = a_0,\; b = b_0, \text{ and } c = c_0 \equiv f_{iu}. \tag{97.6}$$

A first approximation is obtained from

$$y' = f(x, a_0, b_0, c_0)$$

so that we have

$$f(x_i, a_0, b_0, c_0) = y'_i \tag{97.7}$$

and these first approximations can be put into (97.5). For simplicity let

$$r_i = y'_i - y_i. \tag{97.8}$$

If we ignore the high-order terms we then have a set of residual equations of the form

$$R_i = \alpha\left(\frac{\partial f_i}{\partial a}\right)_0 + \beta\left(\frac{\partial f_i}{\partial b}\right)_0 + \gamma\left(\frac{\partial f_i}{\partial c}\right)_0 + r_i \tag{97.9}$$

which are *linear* in α, β, γ. Thus we may determine the corrections by the method of least squares. We minimize

$$\Sigma R_i^2 = g(\alpha, \beta, \gamma) \tag{97.10}$$

from which the normal equations are

$$\begin{cases} (\Sigma f_{ia}^2)\alpha + (\Sigma f_{ia}f_{ib})\beta + (\Sigma f_{ia}f_{ic})\gamma + \Sigma f_{ia}r_i = 0 \\ (\Sigma f_{ia}f_{ib})\alpha + (\Sigma f_{ib}^2)\beta + (\Sigma f_{ib}f_{ic})\gamma + \Sigma f_{ib}r_i = 0 \\ (\Sigma f_{ic}f_{ia})\alpha + (\Sigma f_{ic}f_{ib})\beta + (\Sigma f_{ic}^2)\gamma + \Sigma f_{ic}r_i = 0. \end{cases} \tag{97.11}$$

Example IX.6. Find a functional representation for the data

x	1	2	3	4	5	6	7
y	.5	1.6	2.4	3.2	3.9	4.6	5.2

Solution.

A plot of the function shows a logarithmic tendency with something added. A simple expression is

$$y = ax + b \log x.$$

Since $y = .5$ when $x = 1$ and $\log x$ (at 1) $= 0$ we choose $a_0 = .5$ and guess $b_0 = 1$ so that the first approximation is

$$y' = .5x + \log x.$$

Furthermore,

$$\frac{\partial y}{\partial a} = x \qquad \text{and} \qquad \frac{\partial y}{\partial b} = \log x.$$

We then calculate

x	y	y'	r	$\frac{\partial y}{\partial a}$	$\frac{\partial y}{\partial b}$
1	.5	.5	0	1	0
2	1.6	1.30103	−.29897	2	.30103
3	2.4	1.97712	−.42288	3	.47712
4	3.2	2.60206	−.59794	4	.60206
5	3.9	3.19897	−.70103	5	.69897
6	4.6	3.77815	−.82185	6	.77815
7	5.2	4.34510	−.85490	7	.84510

Using (97.11) we calculate the normal equations

$$\begin{cases} 140\alpha + 18.52111\beta = 18.67889 \\ 18.5211\alpha + 2.48901\beta = 2.50376 \end{cases}$$

with $D = 5.42988$, from which we get

$$\alpha = .022 \qquad \beta = .842$$

and

$$a = .5 + .022 = .522$$

$$b = 1 + .842 = 1.842$$

so that our function becomes

$$\boxed{y = .522x + 1.842 \log x}.$$

Although the correction β is large, it turns out that in this example no improvement to our degree of accuracy is made by finding another set of corrections using the above expression as a second approximation. There are times, though, when this may be necessary and it is simple to do if a and b are not involved in the partial derivatives, for then we simply calculate new values for r_i and the constant terms in the normal equations.

The above answer yields a set of residuals for which

$$\Sigma r_i^2 = .0038.$$

98. Summary of Data Fitting. Analysis of data forms a large part of modern scientific work. It is therefore very desirable to establish some kind of criteria for determining the best mathematical expression which will represent a set of experimental data. Unfortunately, there is no clear-cut method for establishing this. It is, however, possible to set forth some logical step to follow, and we recommend this procedure.

a) Plot the data on graph paper, first choosing ordinary rectangular coordinate graph paper. A study of the graph will yield certain clues to the type of mathematical formula which may be used to fit the data.

(i) If it is fairly smooth try a polynomial fit using as simple a polynomial as possible. In such a case it is recommended that the Nielsen-Goldstein method be used and all the polynomials of degree 1 to 4 be computed simultaneously. An indication of the "goodness" of the fit may be seen by comparing the calculated values for $[a_{i-1}]_n$ and $Y_0 = Y_M - \bar{y}$; the closer these agree the better is the fit.

(ii) If the graph shows a tendency to be periodic, investigate the data for a trigonometric formula and possibly a harmonic analysis.

(iii) If the graph shows a logarithmic or exponential tendency, ignore this graph and consider the plots below.

(iv) If the graph is highly irregular, attempt to determine its critical region and consider the possibility of obtaining more data at certain portions or of smoothing the data by some smoothing formulas.

b) Plot the data on logarithmic paper, and if the graph is nearly a straight line consider the formula

$$y = ax^n.$$

c) Plot the data on semilogarithmic paper, and if the graph is nearly a straight line consider the formula

$$y = ae^{bx} \qquad \text{or} \qquad y = aN^x.$$

d) If the above procedure does not yield the desired formulation, choose a formula and obtain the best possible fit to this formula by any of the above methods or the method of differential correction.

There is still another type of fitting which is sometimes used and that is the one of constrained fitting. In this case we desire not only to fit the data but to do so in a particular manner, that is, we put an additional constraint on the formula. This is accomplished by the use of Lagrangian multipliers (see Section 55). To illustrate let us consider the fitting of a set of data by a linear expression,

(98.1) $$y = mx + b,$$

by the principle of least squares and, furthermore, let us constrain the expression so that the slope is minus twice the intercept,

(98.2) $$g(m,b) = m + 2b = 0.$$

The principle of least squares minimizes the sum of the squares of the residuals,

(98.3) $$f(m,b) = (mx_i + b - y_i)^2,$$

with

(98.4) $$\begin{cases} \dfrac{\partial f}{\partial m} = 2m\,\Sigma x_i^2 + 2\Sigma x_i b - 2\Sigma x_i y_i \\ \dfrac{\partial f}{\partial b} = 2m\,\Sigma x_i + 2nb - 2\Sigma y_i \end{cases}$$

for n observations (x_i, y_i), $\quad (i = 1, \cdots, n)$.
Furthermore,

(98.5) $$\frac{\partial g}{\partial m} = 1 \qquad \text{and} \qquad \frac{\partial g}{\partial b} = 2$$

so that the equations (55.4) become

(98.6) $$\begin{cases} (2\Sigma x_i^2)m + (2\Sigma x_i)b - 2\Sigma x_i y_i + \lambda = 0 \\ (\Sigma x_i)m + nb - \Sigma y_i + \lambda = 0 \\ m + 2b = 0 \end{cases}$$

and we have three linear equations in the three unknowns, m, b, and λ. The solutions are

$$\text{(98.7)} \qquad \begin{cases} \lambda = 2k_2 + (4S_2 - 2S_1)b \\ b = \dfrac{k_1 - 2k_2}{n + 4S_2 - 4S_1} \\ m = -2b \end{cases}$$

where

$$S_1 = \Sigma x_i,\ S_2 = \Sigma x_i^2,\ k_1 = \Sigma y_i,\ k_2 = \Sigma x_i y_i.$$

Example IX.7. Fit a straight line to the following data such that the slope is a minus twice the y-intercept.

x	1	2	3	4	5	6	7
y	.4	1.1	1.5	2.2	2.6	3.1	3.8

Solution. We have

$$n = 7,\ S_1 = 28,\ S_2 = 140,\ k_1 = 14.7,\ k_2 = 74.1.$$

Consequently,

$$b = \frac{14.7 - 2(74.1)}{7 + 4(140) - 4(28)} = -.2934$$

$$m = .5868$$

and

$$\boxed{y = .5868x - .2934}.$$

Check.

x	1	2	3	4	5	6	7
y	.2934	.8802	1.467	2.0538	2.6406	3.2274	3.8142

This, of course, is not the best fitting straight line; it is the best fitting straight line for which $m = -2b$.

99. The Autocorrelation Function. A function which has become quite prominent in modern electrical engineering is one known as the autocorrelation function. It is of special interest in the theory of servomechanisms and in general analysis of "noise" data.* We shall not be concerned with the theory at all, which is beyond the scope of this book. We shall only explain how an autocorrelation function may be calculated from a finite set of discrete data. It is usually associated with functions which have time as the independent variable, and the general definition is that the **autocorrelation function of a function $y(t)$ is the time average of $y(t)y(t+\tau)$.** Mathematically this may be expressed by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t)y(t+\tau)dt. \tag{99.1}$$

To obtain an autocorrelation function from experimental data, we have a record of $y(t)$ for a finite interval of time, $0 \le t \le T$. It should be obtained at equidistant values so that we have values y_i at t_i with $t_i = t_1 + (i-1)h$ and $(i = 1, 2, \cdots, N)$ where $N = \frac{T}{h}$. The time interval h should be chosen quite small, at least small enough so that the function $y(t)$ does not vary significantly in the interval, h. The autocorrelation function may then be approximated by

$$R(m) \doteq \frac{1}{N-m}\sum_{i=1}^{N-m} y_i y_{i+m}, \qquad (m > 0). \tag{99.2}$$

For this calculation we must be careful in choosing m, and, according to Phillips, m should not exceed $\frac{1}{5}N$.

For the computation it is advantageous first to centralize the data about the mean value so that we first calculate a new set of data, $Y_i = y_i - \bar{y}$, where

$$\bar{y} = \frac{1}{N}\sum_{1}^{N} y_i \tag{99.3}$$

and consider

$$R'(m) \doteq \frac{1}{N-m}\sum_{i=1}^{N-m} Y_i Y_{i+m}. \tag{99.4}$$

* For a good discussion of its application see H. M. James, N. B. Nichols, and R. S. Phillips, *Theory of Servomechanisms* (New York: McGraw-Hill Book Company, 1947), Chapter 6.

To illustrate the calculations we shall consider a very simple example. In practice a large amount of data is usually obtained, and the calculations are performed on large-scale calculating machines. The following naive example will, however, indicate the procedure for the calculation of an autocorrelation function.

Example IX.8. Obtain the autocorrelation function for the data given below.

Solution. For this data we have $N = 61$, $\bar{y} = .1$; we shall calculate $R'(m)$ for $(m = 0, \cdots, 10)$ by formula (99.4).

t	y	Y	t	y	Y	m	$R(m)$
0	−2.0	−2.1	.31	−1.5	−1.6	0	1.368
.01	−1.7	−1.8	.32	−0.8	−0.9	1	1.236
.02	−1.4	−1.5	.33	−0.5	−0.6	2	1.001
.03	−1.2	−1.2	.34	−0.1	−0.2	3	.709
.04	−0.7	−0.8	.35	−0.1	−0.2	4	.390
.05	−0.3	−0.4	.36	0.4	0.3	5	.082
.06	−0.2	−0.3	.37	1.0	0.9	6	−.178
.07	−0.3	−0.4	.38	2.0	1.9	7	−.385
.08	−0.6	−0.7	.39	2.8	2.7	8	−.545
.09	−0.9	−1.0	.40	2.7	2.6	9	−.679
.10	−1.2	−1.3	.41	2.9	2.8	10	−.784
.11	−0.6	−0.7	.42	2.5	2.4		
.12	−0.2	−0.3	.43	1.0	0.9		
.13	−0.3	−0.4	.44	0	−0.1		
.14	0.2	0.1	.45	−0.4	−0.5		
.15	0.8	0.7	.46	−0.5	−0.6		
.16	0.5	0.4	.47	−0.3	−0.4		
.17	1.4	1.3	.48	0.1	0		
.18	1.3	1.2	.49	0	−0.1		
.19	2.2	2.1	.50	0.2	0.1		
.20	2.0	1.9	.51	0.5	0.4		
.21	1.8	1.7	.52	0	−0.1		
.22	1.5	1.4	.53	−0.5	−0.6		
.23	1.0	0.9	.54	−0.1	−0.2		
.24	0.3	0.2	.55	0.3	0.2		
.25	−0.3	−0.4	.56	0.1	0		
.26	−1.0	−1.1	.57	−0.2	−0.3		
.27	−1.0	−1.1	.58	0	−0.1		
.28	−1.1	−1.2	.59	0.2	0.1		
.29	−1.6	−1.7	.60	0	−0.1		
.30	−2.0	−2.1					

In performing the calculations on a desk calculator it is strongly recommended that a stencil be employed as m gets large.

100. Exercise XII.

1. Obtain the trigonometric approximations to

x^0	-150	-120	-90	-60	-30	0	30	60	90	120	150	180
y	102	161	179	235	360	312	170	53	-53	-110	-135	-120

2. Obtain the trigonometric approximation to

x^0	y	x^0	y
-165	2.0	15	4.6
-150	3.0	30	3.3
-135	3.5	45	1.9
-120	3.0	60	0.6
-105	1.0	75	-1.0
-90	-1.0	90	-1.0
-75	-1.6	105	0.9
-60	-1.1	120	1.8
-45	0.4	135	1.4
-30	2.1	150	0.3
-15	4.2	165	-0.5
0	6.1	180	-0.5

3. Divide the interval $(-\pi, \pi)$ into twelve equal parts and prove

(a) $$\sum_{j=-6}^{5} \sin x_j \cos 2x_j = 0$$

(b) $$\sum_{j=-6}^{5} \cos^2 x_j = 6$$

(c) $$\sum_{j=-6}^{5} \sin^2 2x_j = 6$$

(d) $$\sum_{j=-6}^{5} \sin x_j \sin 2x_j = 0.$$

4. Determine a functional fit to the data

(a)

x	y
1	2.10
2	2.71
3	3.42
4	4.35
5	5.60
6	7.15
7	9.00
8	12.85
9	14.30
10	17.50
11	24.00

(b)

x	y
1	5.00
2	6.51
3	7.63
4	8.48
5	8.93
6	8.76
7	7.72
8	5.61
9	3.87
10	3.00
11	2.65
12	2.41
13	2.52

(c)

x	y
1	1.10
2	1.62
3	2.15
4	2.63
5	3.05
6	3.49
7	3.87
8	4.26
9	4.59
10	4.89
11	5.10

5. Fit the following data by an expression of the form

$$y = ax + b \ln x$$

x	1	2	3	4	5	6	7
y	4.24	8.48	12.29	15.92	19.45	26.38	29.81

6. Fit the following data with a straight line such that the slope is three times the y-intercept.

x	1	2	3	4	5	6	7	8	9	10
y	1.36848	2.39484	3.42120	4.44756	5.47292	6.50028	7.52664	8.55300	9.57936	10.60572

7. Fit the following data with a quadratic ($y = ax^2 + bx + c$) such that it passes through the point $x = 1$, $y = 2$.

x	−3	−2	−1	0	1	2	3
y	17.48	8.36	2.74	.62	2.00	6.88	15.26

8. Find the autocorrelation function for the data

t	y	t	y	t	y
0	1.0	.31	2.4	.61	0.7
.01	1.2	.32	2.7	.62	0.5
.02	1.5	.33	1.0	.63	0.1
.03	1.9	.34	−1.0	.64	0
.04	2.2	.35	−3.0	.65	−0.3
.05	2.8	.36	−2.2	.66	−0.7
.06	2.9	.37	−1.6	.67	−1.0
.07	2.5	.38	−2.5	.68	−1.0
.08	2.2	.39	−2.3	.69	−0.5
.09	1.9	.40	−1.9	.70	0
.10	1.4	.41	−1.4	.71	0.1
.11	0.7	.42	−0.7	.72	0.3
.12	0.3	.43	−0.1	.73	0.6
.13	−0.2	.44	0.5	.74	0.6
.14	−1.0	.45	1.1	.75	0.4
.15	−1.0	.46	1.7	.76	0.2
.16	−1.5	.47	2.2	.77	0
.17	−3.1	.48	1.9	.78	−0.3
.18	−3.1	.49	1.4	.79	−0.6
.19	−3.0	.50	0.9	.80	−0.7
.20	−2.0	.51	0.1	.81	−0.8
.21	−1.0	.52	−1.0	.82	−0.8
.22	0	.53	−1.5	.83	−0.7
.23	0.2	.54	−1.7	.84	−0.6
.24	2.0	.55	−1.4	.85	−0.4
.25	2.6	.56	−1.2	.86	−0.1
.26	2.8	.57	−0.7	.87	0.1
.27	2.6	.58	−0.1	.88	0.3
.28	2.0	.59	0.5	.89	0.5
.29	1.7	.60	0.8	.90	0.5
.30	1.9				

BIBLIOGRAPHY

I. BOOKS

D. N. de G. Allen. *Relaxation Methods*. New York: McGraw-Hill Book Company, 1954.

Harry Bateman. *Higher Transcendental Functions*. New York: McGraw-Hill Book Company, 1953. Vol. 2, Chapter 10.

R. V. Churchill, Fourier Series and Boundary Value Problems. New York: McGraw-Hill Book Company, 1941.

C. W. E. Deming. *The Statistical Adjustment of Data*. New York: John Wiley & Sons, Inc., 1943.

P. S. Dwyer. *Linear Computations*. New York: John Wiley & Sons, Inc., 1951.

R. A. Frazer, W. J. Duncan, A. R. Collar, *Elementary Matrices*. Cambridge, The University Press, 1950.

H. Freeman. *Mathematics for Actuarial Students*. London: Cambridge University Press, 1948.

D. R. Hartree. *Numerical Analysis*. London: Oxford University Press, 1952.

A. S. Householder. *Principles of Numerical Analysis*. New York: McGraw-Hill, Book Company, 1953.

H. M. James, N. B. Nichols, R. S. Phillips. *Theory of Servomechanisms*. New York: McGraw-Hill Book Company, 1947.

W. E. Milne. *Numerical Calculus*. Princeton: Princeton University Press, 1949.

W. E. Milne. *Numerical Solutions of Differential Equations*. New York: John Wiley & Sons, Inc., 1953.

L. M. Milne-Thomson. *The Calculus of Finite Differences*. New York: The Macmillan Company, 1933.

F. D. Murnaghan. *Introduction to Applied Mathematics*. New York: John Wiley & Sons, Inc., 1948.

M. G. Salvadori and M. L. Baron. *Numerical Methods in Engineering*. New York: Prentice-Hall, Inc., 1952.

J. B. Scarborough. *Numerical Mathematical Analysis*, 2nd Ed. Baltimore: Johns Hopkins University Press, 1950.

R. V. Southwell. *Relaxation Methods in Theoretical Physics*. London: Oxford University Press, 1946.

E. T. Whittaker and G. Robinson. *The Calculus of Observations*. London: Blackie and Sons, 1948.

F. A. Willers. *Practical Analysis*. New York: Dover Publications, Inc., 1948.

II. TABLES

E. P. Adams. *Smithsonian Mathematical Formulae*. Washington: Smithsonian Institution, 1947.

R. S. Burington. *Handbook of Mathematical Tables and Formulas*. Sandusky: Handbook Publishers, Inc., 1943.

L. J. Comrie. *Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots, and Reciprocals*. New York: Chemical Publishing Company, Inc., 1944.

R. A. Fisher and F. Yates. *Statistical Tables*. New York: Hafner Publishing Company, Inc., 1949.

C. D. Hodgman. *Mathematical Tables*, 7th Ed. Cleveland: Chemical Rubber Publishing Company, 1941.

National Bureau of Standards

Table of Arc Tan X. Washington, 1953.

Table of Circular and Hyperbolic Sines and Cosines for Radian Arguments. Washington, 1949.

Tables of the Exponential Function. New York, 1939.

Tables of Lagrangian Coefficients for Sexagesimal Interpolation. Washington, 1954.

Tables of Lagrangian Interpolation Coefficients. New York, 1944.

Tables of Natural Logarithms. New York, 1941.

J. Peters, *Seven-Place Values of Trigonometric Functions*. New York: D. Van Nostrand Company, Inc., 1942.

A. J. Thompson. *Table of Coefficients of Everett's Central Difference Interpolation Formula*, 2nd Ed. London: Cambridge University Press, 1943.

III. ARTICLES

A. C. Aitken. "On Interpolation by Iteration of Proportional Parts without the Use of Differences," *Edinburgh Mathematical Society Proceedings*, Ser. 2, Vol. 3 (1932), pp. 56–76.

Gertrude Blanch. "On Modified Divided Differences, I," *Mathematical Tables and Other Aids to Computation*, Vol. 8 (January 1954), pp. 1–11.

P. D. Crout. "A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real and Complex Coefficients," *Transactions of the A.I.E.E.*, Vol. 60 (1941), pp. 1235–41.

H. W. Emmons. "The Numerical Solution of Partial Differential Equations," *Quarterly of Applied Mathematics*, Vol. 2, No. 3 (October 1944), pp. 173–95.

G. E. Forsythe and R. A. Leiber. "Matrix inversion by a Monte Carlo method," *Mathematical Tables and Other Aids to Computation*, Vol. 4, 1950, pp. 127–29.

G. F. Freeman. "On the Iterative Solution of Linear Simultaneous Equations," *Phil. Mag.*, Vol. 34 (1943), pp. 409–16.
B. Friedman. "Note on Approximating Complex Zeros of a Polynomial," *Communications on Pure and Applied Mathematics*, Vol. II, No. 2–3 (1949), pp. 195–208.
A. Geiringer. "On the Solution of Systems of Linear Equations by Certain Iteration Methods" (Ann Arbor, Michigan), *Reissner Anniversary Volume* (1949), pp. 365–93.
H. H. Goldstein and J. von Neumann. "Numerical Inverting of Matrices of High Order," *Bulletin of the American Mathematical Society*, Vol. 53 (1947), pp. 1021–99.
P. G. Guest. "Note on the Fitting of Polynomials to Equally-spaced Observations," *Journal of Mathematics and Physics*, Vol. 32, No. 1 (1953), pp. 68–71.
E. V. Hankam. "Linear Equations and Matrix Inversion," *IBM Technical Newsletter*, No. 3 (1951), pp. 26–34.
A. S. Householder. "Some Numerical Methods for Solving Systems of Linear Equations," *American Mathematical Monthly*, Vol. 57 (1950), pp. 453–59.
W. M. Kincaid. "Solution of Equations by Interpolation," *The Annals of Mathematical Statistics*, Vol. 19 (1948), pp. 207–19.
Y. H. Ku. "A Method for Solving Third and Higher Order Nonlinear Differential Equations," *Journal of Franklin Institute*, Vol. 256 (September 1953).
J. Laderman. "The Square Root Method for Solving Simultaneous Linear Equations," *Mathematical Tables and Other Aids to Computation*, Vol. 3 (1948), pp. 13–16.
C. Lanczos. "Trigonometric Interpolation of Empirical and Analytical Functions," *Journal Mathematics and Physics*, Vol. 17 (1938) pp. 123–98.
A. N. Lowan, N. Davids, and A. Levenson. "Tables of Zeros of the Legendre Polynomials of Order 1–16 and the Weight Coefficients for Gauss' Mechanical Quadrature Formula," *Bulletin of the American Mathematical Society*, Vol. 48, No. 10 (1942), pp. 739–43.
Y. L. Luke and D. Ufford. "On the Roots of Algebraic Equations," *Journal of Mathematics and Physics*, Vol. 30, No. 2 (1951), pp. 94–101.
R. H. Merson. "The Stability of the Runge-Kutta Method of Solution of Linear Differential Equations," *Royal Aircraft Establishment* (Farnborough, England), Tech. Note G.W. 320 (June, 1954).
W. E. Milne. "Numerical Integration of Ordinary Differential Equations," *American Mathematical Monthly*, Vol. 33 (1926) pp. 455–60.
F. R. Moulton. "On the Solutions of Linear Equations Having Small Determinants," *American Mathematical Monthly*, Vol. 20 (1913), pp. 242–49.
E. H. Neville. "Iterative Interpolation," *Indian Mathematical Society (Madras) Journal*, Vol. 20 (1933), pp. 87–120.

K. L. Nielsen and L. Goldstein. "An Algorithm for Least Squares," *Journal of Mathematics and Physics*, Vol. 26, Nol 2 (1947), pp. 120–32.

T. E. Sterne. "The Accuracy of Numerical Solutions of Ordinary Differential Equations," *Mathematical Tables and Other Aids to Computation*, Vol. 7, No. 43 (July, 1953), pp. 159–64.

J. L. Synge. "A Geometric Interpretation of the Relaxation Method," *Quarterly of Applied Mathematics*, Vol. 2 (1944), pp. 87–89.

L. B. Tuckerman. "On the Mathematical Significant Figures in the Solution of Simultaneous Linear Equations," *The Annals of Mathematical Statistics*, Vol. 12 (1941), pp. 307–16.

M. C. K. Tweedie. "A Modification of the Aitken-Neville Linear Iterative Procedures for Polynomial Interpolation," *Mathematical Tables and Other Aids to Computation*, Vol. 8, No. 45 (January, 1954), pp. 13–16.

TABLES

Table I. Binomial Coefficients $\binom{n}{i}$.

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66
13	1	13	78	286	715	1287	1716	1716	1287	715	286
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003
16	1	16	120	560	1820	4368	8008	11440	12870	11440	8008
17	1	17	136	680	2380	6188	12376	19448	24310	24310	19448
18	1	18	153	816	3060	8568	18564	31824	43758	48620	43758
19	1	19	171	969	3876	11628	27132	50388	75582	92378	92378
20	1	20	190	1140	4845	15504	38760	77520	125970	167960	184756

$$\binom{n}{k} = \binom{n}{n-k}$$

Table II. Stirling Numbers of the First Kind.

$$S_i^{(n+1)} = S_i^{(n)} - nS_{i-1}^{(n)}$$

n	$S_0^{(n)}$	$S_1^{(n)}$	$S_2^{(n)}$	$S_3^{(n)}$	$S_4^{(n)}$	$S_5^{(n)}$	$S_6^{(n)}$	$S_7^{(n)}$	$S_8^{(n)}$	$S_9^{(n)}$	$S_{10}^{(n)}$	$S_{11}^{(n)}$
1	1											
2	1	−1										
3	1	−3	2									
4	1	−6	11	−6								
5	1	−10	35	−50	24							
6	1	−15	85	−225	274	−120						
7	1	−21	175	−735	1624	−1764	720					
8	1	−28	322	−1960	6769	−13132	13068	−5040				
9	1	−36	546	−4536	22449	−67284	118124	−109584	40320			
10	1	−45	870	−9450	63273	−269325	723680	−1172700	1026576	−3628800		
11	1	−55	1320	−10320	157773	−902055	3416930	−8409500	12753576	−13894560	36288000	
12	1	−66	1925	−24840	271293	−2637558	13339535	−45995730	10528076	−154183896	189128160	399168000

Table III. Newton's Interpolation Coefficients.

u or $-u$	$N_{21} = N_{22}$	$N_{31} = -N_{32}$	$N_{41} = N_{42}$	$N_{51} = -N_{52}$	$N_{61} = N_{62}$	u or $-u$
.01	−0.0049500	0.0032835	−0.0024544	0.0019586	−0.0016289	.01
.02	−.0098000	.0064680	−.0048187	.0038357	−.0031836	.02
.03	−.0145500	.0095545	−.0070942	.0056328	−.0046658	.03
.04	−.0192000	.0125440	−.0092826	.0073518	−.0060775	.04
.05	−.2375000	.0154375	−.0113852	.0089943	−.0074203	.05
.06	−.0282000	.0182360	−.0134035	.0105619	−.0086960	.06
.07	−.0325500	.0209405	−.0153389	.0120564	−.0099063	.07
.08	−.0368000	.0235520	−.0171930	.0134793	−.0110530	.08
.09	−.0409500	.0260715	−.0189670	.0148322	−.0121377	.09
.10	−.0450000	.0285000	−.0206625	.0161168	−.0131620	.10
.11	−.0489500	.0308385	−.0222808	.0173345	−.0141276	.11
.12	−.0528000	.0330880	−.0238234	.0184869	−.0150360	.12
.13	−.0565500	.0352495	−.0252915	.0195756	−.0158889	.13
.14	−.0602000	.0373240	−.0266867	.0206021	−.0166877	.14
.15	−.0637500	.0393125	−.0280102	.0215678	−.0174340	.15
.16	−.0672000	.0412160	−.0292634	.0224743	−.0181292	.16
.17	−.0705500	.0430355	−.0304476	.0233229	−.0187749	.17
.18	−.0738000	.0447720	−.0315643	.0241151	−.0193725	.18
.19	−.0769500	.0464265	−.0326146	.0248523	−.0199233	.19
.20	−.0800000	.0480000	−.0336000	.0255360	−.0204288	.20
.21	−.0829500	.0494935	−.0345217	.0261675	−.0208904	.21
.22	−.0858000	.0509080	−.0353811	.0267481	−.0213093	.22
.23	−.0885500	.0522445	−.0361793	.0272792	−.0216870	.23
.24	−.0912000	[illegible]	−.0369178	.0277622	−.0220246	.24
.25	−.0937500	.0546875	−.0375977	.0281982	−.0223236	.25
.26	−.0962000	.0557960	−.0382203	.0285888	−.0225851	.26
.27	−.0985500	.0568305	−.0387868	.0289350	−.0228104	.27
.28	−.1008000	.0577920	−.0392986	.0292381	−.0230007	.28
.29	−.1029500	.0586815	−.0397567	.0294995	−.0231571	.29
.30	−.1050000	.0595000	−.0401625	.0297202	−.0232809	.30
.31	−.1069500	.0602485	−.0405171	.0299016	−.0233731	.31
.32	−.1088000	.0609280	−.0408218	.0300448	−.0234350	.32
.33	−.1105500	.0615395	−.0410776	.0301510	−.0234675	.33
.34	−.1122000	.0620840	−.0412859	.0302212	−.0234718	.34
.35	−.1137500	.0625625	−.0414477	.0302568	−.0234490	.35
.36	−.1152000	.0629760	−.0415642	.0302587	−.0234001	.36
.37	−.1165500	.0633255	−.0416365	.0302281	−.0233260	.37
.38	−.1178000	.0636120	−.0416659	.0301661	−.0232279	.38
.39	−.1189500	.0638365	−.0416533	.0300737	−.0231066	.39
.40	−.1200000	.0640000	−.0416000	.0299520	−.0229632	.40
.41	−.1209500	.0641035	−.0415070	.0298020	−.0227986	.41
.42	−.1218000	.0641480	−.0413755	.0296248	−.0226136	.42
.43	−.1225500	.0641345	−.0412064	.0294214	−.0224093	.43
.44	−.1232000	.0640640	−.0410010	.0291927	−.0221864	.44
.45	−.1237500	.0639375	−.0407602	.0289397	−.0219459	.45
.46	−.1242000	.0637560	−.0404851	.0286634	−.0216887	.46
.47	−.1245500	.0635205	−.0401767	.0283648	−.0214154	.47
.48	−.1248000	.0632320	−.0398362	.0280447	−.0211270	.48
.49	−.1249500	.0628915	−.0394644	.0277040	−.0208242	.49
.50	−.1250000	.0625000	−.0390625	.0273438	−.0205078	.50

Table III (*continued*). Newton's Interpolation Coefficients.

u or $-u$	$N_{21} = N_{22}$	$N_{31} = -N_{32}$	$N_{41} = N_{42}$	$N_{51} = -N_{52}$	$N_{61} = N_{62}$	u or $-u$
.51	−0.1249500	0.0620585	−0.0386314	0.0269647	−0.0201786	.51
.52	−.1248000	.0615680	−.0381722	.0265678	−.0198373	.52
.53	−.1245500	.0610295	−.0376857	.0261539	−.0194846	.53
.54	−.1242000	.0604440	−.0371731	.0257238	−.0191213	.54
.55	−.1237500	.0598125	−.0366352	.0252783	−.0187480	.55
.56	−.1232000	.0591360	−.0360730	.0248182	−.0183655	.56
.57	−.1225500	.0584155	−.0354874	.0243444	−.0179743	.57
.58	−.1218000	.0576520	−.0348795	.0238576	−.0175751	.58
.59	−.1209500	.0568465	−.0342500	.0233585	−.0171685	.59
.60	−.1200000	.0560000	−.0336000	.0228480	−.0167552	.60
.61	−.1189500	.0551135	−.0329303	.0223268	−.0163357	.61
.62	−.1178000	.0541880	−.0322419	.0217955	−.0159107	.62
.63	−.1165500	.0532245	−.0315355	.0212549	−.0154807	.63
.64	−.1152000	.0522240	−.0308122	.0207058	−.0150462	.64
.65	−.1137500	.0511875	−.0300727	.0201487	−.0146078	.65
.66	−.1122000	.0501160	−.0293179	.0195843	−.0141660	.66
.67	−.1105500	.0490105	−.0285486	.0190134	−.0137213	.67
.68	−.1088000	.0478720	−.0277658	.0184365	−.0132743	.68
.69	−.1069500	.0467015	−.0269701	.0178542	−.0128253	.69
.70	−.1050000	.0455000	−.0261625	.0172673	−.0123749	.70
.71	−.1029500	.0442685	−.0253437	.0166762	−.0119235	.71
.72	−.1008000	.0430080	−.0245146	.0160816	−.0114715	.72
.73	−.0985500	.0417195	−.0236758	.0154840	−.0110194	.73
.74	−.0962000	.0404040	−.0228283	.0148841	−.0105677	.74
.75	−.0937500	.0390625	−.0219727	.0142822	−.0101166	.75
.76	−.0912000	.0376960	−.0211098	.0136791	−.0096666	.76
.77	−.0885500	.0363055	−.0202403	.0130752	−.0092180	.77
.78	−.0858000	.0348920	−.0193651	.0124711	−.0087713	.78
.79	−.0829500	.0334565	−.0184847	.0118672	−.0083268	.79
.80	−.0800000	.0320000	−.0176000	.0112640	−.0078848	.80
.81	−.0769500	.0305235	−.0167116	.0106620	−.0074456	.81
.82	−.0738000	.0290280	−.0158203	.0100617	−.0070096	.82
.83	−.0705500	.0275145	−.0149266	.0094635	−.0065771	.83
.84	−.0672000	.0259840	−.0140314	.0088678	−.0061484	.84
.85	−.0637500	.0244375	−.0131352	.0082751	−.0057236	.85
.86	−.0602000	.0228760	−.0122387	.0076859	−.0053033	.86
.87	−.0565500	.0213005	−.0113425	.0071004	−.0048875	.87
.88	−.0528000	.0197120	−.0104474	.0065192	−.0044765	.88
.89	−.0489500	.0181115	−.0095538	.0059425	−.0040706	.89
.90	−.0450000	.0165000	−.0086625	.0053708	−.0036700	.90
.91	−.0409500	.0148785	−.0077740	.0048043	−.0032750	.91
.92	−.0368000	.0132480	−.0068890	.0042436	−.0028856	.92
.93	−.0325500	.0116095	−.0060079	.0036889	−.0025023	.93
.94	−.0282000	.0099640	−.0051315	.0031405	−.0021250	.94
.95	−.0237500	.0083125	−.0042602	.0025987	−.0017541	.95
.96	−.0192000	.0066560	−.0033946	.0020639	−.0013897	.96
.97	−.0145500	.0049955	−.0025352	.0015363	−.0010319	.97
.98	−.0098000	.0033320	−.0016827	.0010163	−.0006809	.98
.99	−.0049500	.0016665	−.0008374	.0005041	−.0003369	.99
1.00	0	0	0	0	0	1.00

Table III (*continued*). Newton's Interpolation Coefficients.

u or $-u$	$N_{21} = N_{22}$	$N_{31} = -N_{32}$	$N_{41} = N_{42}$	$N_{51} = -N_{52}$	$N_{61} = N_{62}$	u or $-u$
1.01	0.0050500	−0.0016665	0.0008291	−0.0004958	0.0003297	1.01
1.02	.0102000	−.0033120	.0016394	−.0009771	.0006481	1.02
1.03	.0154500	−.0049955	.0024603	−.0014614	.0009670	1.03
1.04	.0208000	−.0066560	.0032614	−.0019307	.0012743	1.04
1.05	.0262500	−.0083125	.0040523	−.0023909	.0015740	1.05
1.06	.0318000	−.0099640	.0048325	−.0028415	.0018659	1.06
1.07	.0374500	−.0116095	.0056016	−.0032825	.0021500	1.07
1.08	.0432000	−.0132480	.0063590	−.0037137	.0024263	1.08
1.09	.0490500	−.0148785	.0071045	−.0041348	.0026954	1.09
1.10	.0550000	−.0165000	.0078375	−.0045458	.0029548	1.10
1.11	.0610500	−.0181115	.0085577	−.0049464	.0032069	1.11
1.12	.0672000	−.0197120	.0092646	−.0053364	.0034509	1.12
1.13	.0734500	−.0213005	.0099580	−.0057159	.0036867	1.13
1.14	.0798000	−.0228760	.0106373	−.0060846	.0039144	1.14
1.15	.0862500	−.0244375	.0113023	−.0064423	.0041338	1.15
1.16	.0928000	−.0259840	.0119526	−.0067891	.0043450	1.16
1.17	.0994500	−.0275145	.0125879	−.0071247	.0045480	1.17
1.18	.1062000	−.0290280	.0132077	−.0074492	.0047427	1.18
1.19	.1130500	−.0305235	.0138119	−.0077623	.0049290	1.19
1.20	.1200000	−.0320000	.0144000	−.0080640	.0051072	1.20
1.21	.1270500	−.0334565	.0149718	−.0083543	.0052771	1.21
1.22	.1342000	−.0348920	.0155269	−.0086330	.0054388	1.22
1.23	.1414500	−.0363055	.0160652	−.0089001	.0055922	1.23
1.24	.1488000	−.0376960	.0165862	−.0091556	.0057375	1.24
1.25	.1562500	−.0390625	.0170898	−.0093994	.0058746	1.25
1.26	.1638000	−.0404040	.0175757	−.0096315	.0060036	1.26
1.27	.1714500	−.0422910	.0182909	−.0099868	.0062085	1.27
1.28	.1792000	−.0430080	.0184934	−.0100604	.0062375	1.28
1.29	.1870500	−.0442685	.0189248	−.0102572	.0063424	1.29
1.30	.1950000	−.0455000	.0193375	−.0104423	.0064394	1.30
1.31	.2030500	−.0467015	.0197314	−.0106155	.0065285	1.31
1.32	.2112000	−.0478720	.0201062	−.0107769	.0066099	1.32
1.33	.2194500	−.0490105	.0204619	−.0109266	.0066835	1.33
1.34	.2278000	−.0501160	.0207981	−.0110646	.0067494	1.34
1.35	.2362500	−.0511875	.0211148	−.0111909	.0068078	1.35
1.36	.2448000	−.0522240	.0214118	−.0113055	.0068587	1.36
1.37	.2534500	−.0532245	.0216890	−.0114084	.0069021	1.37
1.38	.2622000	−.0541880	.0219461	−.0114998	.0069382	1.38
1.39	.2710500	−.0551135	.0221832	−.0115796	.0069671	1.39
1.40	.2800000	−.0460000	.0224000	−.0116480	.0069888	1.40
1.41	.2890500	−.0568465	.0225965	−.0117050	.0070035	1.41
1.42	.2982000	−.0576520	.0227725	−.0117506	.0070112	1.42
1.43	.3074500	−.0584155	.0229281	−.0117850	.0070121	1.43
1.44	.3168000	−.0591360	.0230630	−.0118083	.0070062	1.44
1.45	.3262500	−.0598125	.0231773	−.0118204	.0069937	1.45
1.46	.3358000	−.0604440	.0232709	−.0118216	.0069748	1.46
1.47	.3454500	−.0610295	.0233438	−.0118120	.0069494	1.47
1.48	.3552000	−.0615680	.0233958	−.0117915	.0069177	1.48
1.49	.3650500	−.0620585	.0234271	−.0117604	.0068798	1.49
1.50	.3750000	−.0625000	.0234375	−.0117188	.0068359	1.50

Table III (*continued*). Newton's Interpolation Coefficients.

u or $-u$	$N_{21} = N_{22}$	$N_{31} = -N_{32}$	$N_{41} = N_{42}$	$N_{51} = -N_{52}$	$N_{61} = N_{62}$	u or $-u$
1.51	0.3850500	−0.0628915	0.0234271	−0.0116667	0.0067861	1.51
1.52	.3952000	−.0632320	.0233958	−.0116043	.0067305	1.52
1.53	.4054500	−.0635205	.0233438	−.0115318	.0066692	1.53
1.54	.4158000	−.0637560	.0232709	−.0114493	.0066024	1.54
1.55	.4262500	−.0639375	.0231773	−.0113569	.0065302	1.55
1.56	.4368000	−.0640640	.0230630	−.0112548	.0064527	1.56
1.57	.4474500	−.0641345	.0229281	−.0111430	.0063701	1.57
1.58	.4582000	−.0641480	.0227725	−.0110219	.0062825	1.58
1.59	.4690500	−.0641035	.0225965	−.0108915	.0061900	1.59
1.60	.4800000	−.0640000	.0224000	−.0107520	.0060928	1.60
1.61	.4910500	−.0638365	.0221832	−.0106036	.0059910	1.61
1.62	.5022000	−.0636120	.0219461	−.0104464	.0058848	1.62
1.63	.5134500	−.0633255	.0216890	−.0102806	.0057743	1.63
1.64	.5248000	−.0629760	.0214118	−.0101064	.0056596	1.64
1.65	.5362500	−.0625625	.0211148	−.0099240	.0055409	1.65
1.66	.5478000	−.0620840	.0207981	−.0097335	.0054183	1.66
1.67	.5594500	−.0615395	.0204619	−.0095352	.0052921	1.67
1.68	.5712000	−.0609280	.0201062	−.0093293	.0051622	1.68
1.69	.5830500	−.0602485	.0197314	−.0091159	.0050289	1.69
1.70	.5950000	−.0595000	.0193375	−.0088953	.0048924	1.70
1.71	.6070500	−.0586815	.0189248	−.0086676	.0047527	1.71
1.72	.6192000	−.0577920	.0184934	−.0084330	.0046100	1.72
1.73	.6314500	−.0568305	.0180437	−.0081918	.0044645	1.73
1.74	.6438000	−.0557960	.0175757	−.0079442	.0043164	1.74
1.75	.6562500	−.0546875	.0170898	−.0076904	.0041606	1.75
1.76	.6688000	−.0535040	.0165862	−.0074306	.0040125	1.76
1.77	.6814500	−.0522445	.0160652	−.0071651	.0038572	1.77
1.78	.6942000	−.0509080	.0155269	−.0068940	.0036998	1.78
1.79	.7070500	−.0494935	.0149718	−.0066175	.0035404	1.79
1.80	.7200000	−.0480000	.0144000	−.0063360	.0033792	1.80
1.81	.7330500	−.0464265	.0138119	−.0060496	.0032164	1.81
1.82	.7462000	−.0447720	.0132077	−.0057586	.0030520	1.82
1.83	.7594500	−.0430355	.0125879	−.0054631	.0028864	1.83
1.84	.7728000	−.0412160	.0119526	−.0051635	.0027195	1.84
1.85	.7862500	−.0393125	.0113023	−.0048600	.0025515	1.85
1.86	.7998000	−.0373240	.0106373	−.0045528	.0023826	1.86
1.87	.8134500	−.0352495	.0099580	−.0042421	.0022130	1.87
1.88	.8272000	−.0330880	.0092646	−.0039282	.0020427	1.88
1.89	.8410500	−.0308385	.0085577	−.0036113	.0018719	1.89
1.90	.8550000	−.0285000	.0078375	−.0032918	.0017007	1.90
1.91	.8690500	−.0260715	.0071045	−.0029697	.0015294	1.91
1.92	.8832000	−.0235520	.0063590	−.0026454	.0013580	1.92
1.93	.8974500	−.0209405	.0056016	−.0023191	.0011866	1.93
1.94	.9118000	−.0182360	.0048325	−.0019910	.0010154	1.94
1.95	.9262500	−.0154375	.0040523	−.0016615	.0008446	1.95
1.96	.9408000	−.0125440	.0032614	−.0013307	.0006742	1.96
1.97	.9554500	−.0095545	.0024603	−.0009989	.0005044	1.97
1.98	.9702000	−.0064680	.0016493	−.0006663	.0003354	1.98
1.99	.9850500	−.0032835	.0008291	−.0003333	.0001672	1.99
2.00	1.0000000	.0000000	.0000000	.0000000	.0000000	2.00

Table IV. Stirling's Interpolation Coefficients.

u	S_2	S_3	S_4	S_6	S_8	u
.01	.0000500	−.0016665	−.0000042	.0003333	.0000006	.01
.02	.0002000	−.0033320	−.0000167	.0006663	.0000022	.02
.03	.0004500	−.0049955	−.0000375	.0009989	.0000050	.03
.04	.0008000	−.0066560	−.0000666	.0013307	.0000089	.04
.05	.0012500	−.0083125	−.0001039	.0016615	.0000138	.05
.06	.0018000	−.0099640	−.0001495	.0019910	.0000199	.06
.07	.0024500	−.0116095	−.0002032	.0023191	.0000271	.07
.08	.0032000	−.0132480	−.0002650	.0026454	.0000353	.08
.09	.0040500	−.0148785	−.0003348	.0029697	.0000445	.09
.10	.0050000	−.0165000	−.0004125	.0032918	.0000549	.10
.11	.0060500	−.0181115	−.0004981	.0036113	.0000662	.11
.12	.0072000	−.0197120	−.0005914	.0039282	.0000786	.12
.13	.0084500	−.0213005	−.0006923	.0042421	.0000919	.13
.14	.0098000	−.0228760	−.0008007	.0045528	.0001062	.14
.15	.0112500	−.0244375	−.0009164	.0048600	.0001215	.15
.16	.0128000	−.0259840	−.0010394	.0051635	.0001377	.16
.17	.0144500	−.0275145	−.0011694	.0054631	.0001548	.17
.18	.0162000	−.0290280	−.0013063	.0057586	.0001728	.18
.19	.0180500	−.0305235	−.0014499	.0060496	.0001916	.19
.20	.0200000	−.0320000	−.0016000	.0063360	.0002112	.20
.21	.0220500	−.0334565	−.0017565	.0066175	.0002316	.21
.22	.0242000	−.0348920	−.0019191	.0068940	.0002528	.22
.23	.0264500	−.0363055	−.0020876	.0071651	.0002747	.23
.24	.0288000	−.0376960	−.0022618	.0074306	.0002972	.24
.25	.0312500	−.0390625	−.0024414	.0076904	.0003204	.25
.26	.0338000	−.0404040	−.0026263	.0079442	.0003443	.26
.27	.0364500	−.0417195	−.0028161	.0081918	.0003686	.27
.28	.0392000	−.0430080	−.0030106	.0084330	.0003935	.28
.29	.0420500	−.0442685	−.0032095	.0086676	.0004189	.29
.30	.0450000	−.0455000	−.0034125	.0088952	.0004448	.30
.31	.0480500	−.0467015	−.0036194	.0091159	.0004710	.31
.32	.0512000	−.0478720	−.0038298	.0093293	.0004976	.32
.33	.0544500	−.0490105	−.0040434	.0095352	.0005244	.33
.34	.0578000	−.0501160	−.0042599	.0097335	.0005516	.34
.35	.0612500	−.0511875	−.0044798	.0099259	.0005790	.35
.36	.0648000	−.0522240	−.0047002	.0101064	.0006064	.36
.37	.0684500	−.0532245	−.0049233	.0102806	.0006340	.37
.38	.0722000	−.0541880	−.0051479	.0104464	.0006616	.38
.39	.0760500	−.0551135	−.0053736	.0106036	.0006892	.39
.40	.0800000	−.0560000	−.0056000	.0107520	.0007168	.40
.41	.0840500	−.0568465	−.0058268	.0108915	.0007443	.41
.42	.0882000	−.0576520	−.0060535	.0110219	.0007715	.42
.43	.0924500	−.0584155	−.0062797	.0111430	.0007986	.43
.44	.0968000	−.0591360	−.0065050	.0112548	.0008253	.44
.45	.1012500	−.0598125	−.0067289	.0113569	.0008518	.45
.46	.1058000	−.0604440	−.0069511	.0114493	.0008778	.46
.47	.1104500	−.0610295	−.0071710	.0115318	.0009033	.47
.48	.1152000	−.0615680	−.0073882	.0116043	.0009283	.48
.49	.1200500	−.0620585	−.0076022	.0116667	.0009528	.49
.50	.1250000	−.0625000	−.0078125	.0117188	.0009766	.50

Table IV (*continued*). Stirling's Interpolation Coefficients.

u	S_2	S_3	S_4	S_5	S_6	u
.51	.1300500	−.0628915	−.0080187	.0117604	.0009996	.51
.52	.1352000	−.0632320	−.0082202	.0117915	.0010219	.52
.53	.1404500	−.0635205	−.0084165	.0118120	.0010434	.53
.54	.1458000	−.0637560	−.0086071	.0118216	.0010639	.54
.55	.1512500	−.0639375	−.0087914	.0118204	.0010835	.55
.56	.1568000	−.0640640	−.0089690	.0118083	.0011021	.56
.57	.1624500	−.0641345	−.0091392	.0117850	.0011196	.57
.58	.1682000	−.0641480	−.0093015	.0117506	.0011359	.58
.59	.1740500	−.0641035	−.0094553	.0117050	.0011510	.59
.60	.1800000	−.0640000	−.0096000	.0116480	.0011648	.60
.61	.1860500	−.0638365	−.0097351	.0115796	.0011773	.61
.62	.1922000	−.0636120	−.0098599	.0114998	.0011883	.62
.63	.1984500	−.0633255	−.0099738	.0114084	.0011979	.63
.64	.2048000	−.0629760	−.0100762	.0113055	.0012059	.64
.65	.2112500	−.0625625	−.0101664	.0111909	.0012123	.65
.66	.2178000	−.0620840	−.0102439	.0110646	.0012171	.66
.67	.2244500	−.0615395	−.0103079	.0109266	.0012201	.67
.68	.2312000	−.0609280	−.0103578	.0107769	.0012214	.68
.69	.2380500	−.0602485	−.0103929	.0106155	.0012208	.69
.70	.2450000	−.0595000	−.0104125	.0104423	.0012183	.70
.71	.2520500	−.0586815	−.0104160	.0102572	.0012138	.71
.72	.2592000	−.0577920	−.0104026	.0100604	.0012073	.72
.73	.2664500	−.0568305	−.0103716	.0098519	.0011986	.73
.74	.2738000	−.0557960	−.0103223	.0096315	.0011879	.74
.75	.2812500	−.0546875	−.0102539	[illegible]	.0011749	.75
.76	.2888000	−.0535040	−.0101658	.0091556	.0011597	.76
.77	.2964500	−.0522445	−.0100571	.0089001	.0011422	.77
.78	.3042000	−.0509080	−.0099271	.0086330	.0011223	.78
.79	.3120500	−.0494935	−.0097750	.0083543	.0011000	.79
.80	.3200000	−.0480000	−.0096000	.0080640	.0010752	.70
.81	.3280500	−.0464265	−.0094014	.0077623	.0010479	.81
.82	.3362000	−.0447720	−.0091783	.0074492	.0010181	.82
.83	.3444500	−.0430355	−.0089299	.0071247	.0009856	.83
.84	.3528000	−.0412160	−.0086554	.0067891	.0009505	.84
.85	.3612500	−.0393125	−.0083539	.0064423	.0009127	.85
.86	.3698000	−.0373240	−.0080247	.0060846	.0008721	.86
.87	.3784500	−.0352495	−.0076668	.0057159	.0008288	.87
.88	.3872000	−.0330880	−.0072794	.0053364	.0007827	.88
.89	.3960500	−.0308385	−.0068616	.0049463	.0007337	.89
.90	.4050000	−.0285000	−.0064125	.0045458	.0006819	.90
.91	.4140500	−.0260715	−.0059313	.0041348	.0006271	.91
.92	.4232000	−.0235520	−.0054170	.0037137	.0005694	.92
.93	.4324500	−.0209405	−.0048687	.0032825	.0005088	.93
.94	.4418000	−.0182360	−.0042855	.0028415	.0004452	.94
.95	.4512500	−.0154375	−.0036664	.0023909	.0003786	.95
.96	.4608000	−.0125440	−.0030106	.0019308	.0003089	.96
.97	.4704500	−.0095545	−.0023170	.0014614	.0002363	.97
.98	.4802000	−.0064680	−.0015847	.0009830	.0001606	.98
.99	.4900500	−.0032835	−.0008127	.0004958	.0000818	.99
1.00	.5000000	.0000000	.0000000	.0000000	.0000000	1.00

Table V. Coefficients E_{ij} for Everett's Formula.

u	E_{1j}	E_{2j}	E_{3j}	u
.01	−.0016665	.0003333	−.0000714	.01
.02	−.0033320	.0006663	−.0001428	.02
.03	−.0049955	.0009989	−.0002140	.03
.04	−.0066560	.0013307	−.0002851	.04
.05	−.0083125	.0016615	−.0003560	.05
.06	−.0099640	.0019910	−.0004265	.06
.07	−.0116095	.0023191	−.0004967	.07
.08	−.0132480	.0026454	−.0005665	.08
.09	−.0148785	.0029697	−.0006358	.09
.10	−.0165000	.0032918	−.0007046	.10
.11	−.0181115	.0036113	−.0007728	.11
.12	−.0197120	.0039282	−.0008404	.12
.13	−.0213005	.0042421	−.0009073	.13
.14	−.0228760	.0045528	−.0009735	.14
.15	−.0244375	.0048600	−.0010388	.15
.16	−.0259840	.0051635	−.0011033	.16
.17	−.0275145	.0054631	−.0011669	.17
.18	−.0290280	.0057586	−.0012295	.18
.19	−.0305235	.0060496	−.0012911	.19
.20	−.0320000	.0063360	−.0013517	.20
.21	−.0334565	.0066175	−.0014111	.21
.22	−.0348920	.0068940	−.0014693	.22
.23	−.0363055	.0071651	−.0015264	.23
.24	−.0376960	.0074306	−.0015821	.24
.25	−.0390625	.0076904	−.0016365	.25
.26	−.0404040	.0079442	−.0016895	.26
.27	−.0417195	.0081918	−.0017412	.27
.28	−.0430080	.0084330	−.0017913	.28
.29	−.0442685	.0086676	−.0018400	.29
.30	−.0455000	.0088952	−.0018871	.30
.31	−.0467015	.0091159	−.0019325	.31
.32	−.0478720	.0093293	−.0019764	.32
.33	−.0490105	.0095352	−.0020185	.33
.34	−.0501160	.0097335	−.0020590	.34
.35	−.0511875	.0099259	−.0020980	.35
.36	−.0522240	.0101064	−.0021345	.36
.37	−.0532245	.0102806	−.0021695	.37
.38	−.0541880	.0104464	−.0022026	.38
.39	−.0551135	.0106036	−.0022338	.39
.40	−.0560000	.0107520	−.0022630	.40
.41	−.0568465	.0108915	−.0022903	.41
.42	−.0576520	.0110219	−.0023155	.42
.43	−.0584155	.0111430	−.0023387	.43
.44	−.0591360	.0112548	−.0023599	.44
.45	−.0598125	.0113569	−.0023789	.45
.46	−.0604440	.0114493	−.0023957	.46
.47	−.0610295	.0115318	−.0024104	.47
.48	−.0615680	.0116043	−.0024230	.48
.49	−.0620585	.0116667	−.0024333	.49
.50	−.0625000	.0117188	−.0024414	.50

Table V (*continued*). Coefficients E_{ij} for Everett's Formula.

u	E_{1j}	E_{2j}	E_{3j}	u
.51	−.0628915	.0117604	−.0024473	.51
.52	−.0632320	.0117915	−.0024508	.52
.53	−.0635205	.0118120	−.0024521	.53
.54	−.0637560	.0118216	−.0024511	.54
.55	−.0639375	.0118204	−.0024478	.55
.56	−.0640640	.0118083	−.0024422	.56
.57	−.0641345	.0117850	−.0024342	.57
.58	−.0641480	.0117506	−.0024239	.58
.59	−.0641035	.0117050	−.0024112	.59
.60	−.0640000	.0116480	−.0023962	.60
.61	−.0638365	.0115796	−.0023788	.61
.62	−.0636120	.0114998	−.0023590	.62
.63	−.0633255	.0114084	−.0023368	.73
.64	−.0629760	.0113055	−.0023124	.64
.65	−.0625625	.0111909	−.0022855	.65
.66	−.0620840	.0110646	−.0022562	.66
.67	−.0615395	.0109266	−.0022246	.67
.68	−.0609280	.0107769	−.0021907	.68
.69	−.0602485	.0106155	−.0021544	.69
.70	−.0595000	.0104423	−.0021158	.70
.71	−.0586815	.0102572	−.0020749	.71
.72	−.0577920	.0100604	−.0020316	.72
.73	−.0568305	.0098519	−.0019861	.73
.74	−.0557960	.0096315	−.0019383	.74
.75	−.0546875	.0093994	[illegible]	.75
.76	−.0535040	.0091556	−.0018360	.76
.77	−.0522445	.0089001	−.0017815	.77
.78	−.0509080	.0086330	−.0017249	.78
.79	−.0494935	.0083543	−.0016661	.79
.80	−.0480000	.0080640	−.0016051	.80
.81	−.0464265	.0077623	−.0015421	.81
.82	−.0447720	.0074492	−.0014770	.82
.83	−.0430355	.0071247	−.0014099	.83
.84	−.0412160	.0067891	−.0013408	.84
.85	−.0393125	.0064423	−.0012697	.85
.86	−.0373240	.0060846	−.0011967	.86
.87	−.0352495	.0057159	−.0011218	.87
.88	−.0330880	.0053364	−.0010451	.88
.89	−.0308385	.0049463	−.0009666	.89
.90	−.0285000	.0045458	−.0008864	.90
.91	−.0260715	.0041348	−.0008045	.91
.92	−.0235520	.0037137	−.0007210	.92
.93	−.0209405	.0032825	−.0006358	.93
.94	−.0182360	.0028415	−.0005491	.94
.95	−.0154375	.0023909	−.0004610	.95
.96	−.0125440	.0019308	−.0003714	.96
.97	−.0095545	.0014614	−.0002804	.97
.98	−.0064680	.0009830	−.0001882	.98
.99	−.0032835	.0004958	−.0000947	.99
1.00	.0000000	.0000000	.0000000	1.00

Table VI. Bessel's Interpolation Coefficients.

For $.50 < u \leq .99$ use headings at bottom of page.
Numbers with less than 7 decimal places are exact.

u	B_1	B_2	B_3	B_4	B_5	B_6	u
.01	−.49	−.00495	.0008085	.0008291	.0000813	.0001661	.99
.02	−.48	−.00980	.0015680	.0016493	.0001583	.0003309	.98
.03	−.47	−.01455	.0022795	.0024603	.0002313	.0004944	.97
.04	−.46	−.01920	.0029440	.0032614	.0003000	.0006582	.96
.05	−.45	−.02375	.0035625	.0040523	.0003647	.0008169	.95
.06	−.44	−.02820	.0041360	.0048325	.0004253	.0009756	.94
.07	−.43	−.03255	.0046655	.0056016	.0004817	.0011325	.93
.08	−.42	−.03680	.0051520	.0063590	.0005342	.0012874	.92
.09	−.41	−.04095	.0055965	.0071045	.0005826	.0014403	.91
.10	−.40	−.04500	.0060000	.0078375	.0006270	.0015910	.90
.11	−.39	−.04895	.0063635	.0085577	.0006675	.0017395	.89
.12	−.38	−.05280	.0066880	.0092646	.0007041	.0018855	.88
.13	−.37	−.05655	.0069745	.0099580	.0007369	.0020291	.87
.14	−.36	−.06020	.0072240	.0106373	.0007659	.0021702	.86
.15	−.35	−.06375	.0074375	.0113023	.0007912	.0023085	.85
.16	−.34	−.06720	.0076160	.0119526	.0008128	.0024441	.84
.17	−.33	−.07055	.0077605	.0125879	.0008308	.0025768	.83
.18	−.32	−.07380	.0078720	.0132077	.0008453	.0027065	.82
.19	−.31	−.07695	.0079515	.0138119	.0008563	.0028332	.81
.20	−.30	−.08000	.0080000	.0144000	.0008640	.0029568	.80
.21	−.29	−.08295	.0080185	.0149718	.0008684	.0030772	.79
.22	−.28	−.08580	.0080080	.0155269	.0008695	.0031942	.78
.23	−.27	−.08855	.0079695	.0160652	.0008675	.0033079	.77
.24	−.26	−.09120	.0079040	.0165862	.0008625	.0034181	.76
.25	−.25	−.09375	.0078125	.0170898	.0008545	.0035248	.75
.26	−.24	−.09620	.0076960	.0175757	.0008436	.0036279	.74
.27	−.23	−.09855	.0075555	.0180437	.0008300	.0037273	.73
.28	−.22	−.10080	.0073920	.0184937	.0008137	.0038230	.72
.29	−.21	−.10295	.0072065	.0189248	.0007948	.0039148	.71
.30	−.20	−.10500	.0070000	.0193375	.0007735	.0040029	.70
.31	−.19	−.10695	.0067735	.0197314	.0007498	.0040870	.69
.32	−.18	−.10880	.0065280	.0201062	.0007238	.0041671	.68
.33	−.17	−.11055	.0062645	.0204619	.0006957	.0042432	.67
.34	−.16	−.11220	.0059840	.0207981	.0006655	.0043152	.66
.35	−.15	−.11375	.0056875	.0211148	.0006334	.0043831	.65
.36	−.14	−.11520	.0053760	.0214118	.0005995	.0044468	.64
.37	−.13	−.11655	.0050505	.0216890	.0005639	.0045063	.63
.38	−.12	−.11780	.0047120	.0219461	.0005267	.0045616	.62
.39	−.11	−.11895	.0043615	.0221832	.0004880	.0046126	.61
.40	−.10	−.12000	.0040000	.0224000	.0004480	.0046592	.60
.41	−.09	−.12095	.0036285	.0225965	.0004067	.0047015	.59
.42	−.08	−.12180	.0032480	.0227725	.0003644	.0047394	.58
.43	−.07	−.12255	.0028595	.0229281	.0003210	.0047729	.57
.44	−.06	−.12320	.0024640	.0230630	.0002768	.0048020	.56
.45	−.05	−.12375	.0020625	.0231773	.0002318	.0048267	.55
.46	−.04	−.12420	.0016560	.0232709	.0001862	.0048469	.54
.47	−.03	−.12455	.0012455	.0233438	.0001401	.0048626	.53
.48	−.02	−.12480	.0008320	.0233958	.0000936	.0048738	.52
.49	−.01	−.12495	.0004165	.0234271	.0000469	.0048806	.51
.50	.00	−.12500	.0000000	.0234375	.0000000	.0048828	.50
u	$-B_1$	B_2	$-B_3$	B_4	$-B_5$	B_6	u

Table VII. Gauss's Quadrature Coefficients.

i	v_i	g_i
	$n = 2$	
1	0.57735 02692	1.00000 00000
	$n = 3$	
0	0.00000 00000	0.88888 88889
1	0.77459 66692	0.55555 55556
	$n = 4$	
1	0.33998 10436	0.65214 51549
2	0.86113 63116	0.34785 48451
	$n = 5$	
0	0.00000 00000	0.56888 88889
1	0.53846 93101	0.47862 86705
2	0.90617 98459	0.23692 68851
	$n = 6$	
1	0.23861 91861	0.46791 39346
2	0.66120 93865	0.36076 15730
3	0.93246 95142	0.17132 44924
	$n = 7$	
0	0.00000 00000	0.41795 91837
1	0.40584 51514	0.38183 00505
2	0.74153 11856	0.27970 53915
3	0.94910 79123	0.12948 49662
	$n = 8$	
1	0.18343 46425	0.36268 37834
2	0.52553 24099	0.31370 66459
3	0.79666 64774	0.22238 10345
4	0.96028 98565	0.10122 85363
	$n = 9$	
0	0.00000 00000	0.33023 93550
1	0.32425 34234	0.31234 70770
2	0.61337 14327	0.26061 06964
3	0.83603 11073	0.18064 81607
4	0.96816 02395	0.08127 43884
	$n = 10$	
1	0.14887 43390	0.29552 42247
2	0.43339 53941	0.26926 67193
3	0.67940 95683	0.21908 63625
4	0.86506 33667	0.14945 13492
5	0.97390 65285	0.06667 13443
	$n = 11$	
0	0.00000 00000	0.27292 50868
1	0.26954 31560	0.26280 45445
2	0.51909 61291	0.23319 37646
3	0.73015 20056	0.18629 02109
4	0.88706 25998	0.12558 03695
5	0.97822 86581	0.05566 85671
	$n = 12$	
1	0.12533 34085	0.24914 70458
2	0.36783 14989	0.23349 25365
3	0.58731 79543	0.20316 74267
4	0.76990 26742	0.16007 83285
5	0.90411 72564	0.10693 93260
6	0.98156 06342	0.04717 53364
	$n = 13$	
0	0.00000 00000	0.23255 15532
1	0.23045 83160	0.22628 31803
2	0.44849 27510	0.20781 60475
3	0.64234 93394	0.17814 59808
4	0.80157 80907	0.13887 35102
5	0.91759 83992	0.09212 14998
6	0.98418 30547	0.04048 40048
	$n = 14$	
1	0.10805 49487	0.21526 38535
2	0.31911 23689	0.20519 84637
3	0.51524 86364	0.18553 83975
4	0.68729 29048	0.15720 31672
5	0.82720 13151	0.12151 85707
6	0.92843 48837	0.08015 80872
7	0.98628 38087	0.03511 94603
	$n = 15$	
0	0.00000 00000	0.20257 82419
1	0.20119 40940	0.19843 14853
2	0.39415 13471	0.18616 10000
3	0.57097 21726	0.16626 92058
4	0.72441 77314	0.13957 06779
5	0.84820 65834	0.10715 92205
6	0.93727 33924	0.07036 60475
7	0.98799 25180	0.03075 32420
	$n = 16$	
1	0.09501 25098	0.18945 06105
2	0.28160 35508	0.18260 34150
3	0.45801 67777	0.16915 65194
4	0.61787 62444	0.14959 59888
5	0.75540 44084	0.12462 89713
6	0.86563 12024	0.09515 85117
7	0.94457 50231	0.06225 35239
8	0.98940 09350	0.02715 24594

Table VIII. Lagrangian Interpolation Cofficients. 5-Points

p	A_{-2}	$-A_{-1}$	A_0	A_1	$-A_2$	p
0.00	.0000000	.0000000	1.0000000	.0000000	.0000000	0.00
.01	.0008291	.0065998	.9998750	.0067332	.0008374	.01
.02	.0016493	.0130654	.9995000	.0135986	.0016827	.02
.03	.0024603	.0193956	.9988752	.0205954	.0025352	.03
.04	.0032614	.0255898	.9980006	.0277222	.0033946	.04
0.05	.0040523	.0316469	.9968766	.0349781	.0042602	0.05
.06	.0048325	.0375662	.9955032	.0423618	.0051315	.06
.07	.0056016	.0433468	.9938810	.0498722	.0060079	.07
.08	.0063590	.0489882	.9920102	.0575078	.0068890	.08
.09	.0071045	.0544894	.9898914	.0652676	.0077740	0.09
0.10	.0078375	.0598500	.9875250	.0731500	.0086625	0.10
.11	.0085577	.0650692	.9849116	.0811538	.0095538	.11
.12	.0092646	.0701466	.9820518	.0892774	.0104474	.12
.13	.0099580	.0750814	.9789464	.0975196	.0113425	.13
.14	.0106373	.0798734	.9755960	.1058786	.0122387	.14
0.15	.0113023	.0845219	.9720016	.1143531	.0131352	0.15
.16	.0119526	.0890266	.9681638	.1229414	.0140314	.16
.17	.0125879	.0933870	.9640838	.1316420	.0149266	.17
.18	.0132077	.0976030	.9597624	.1404530	.0158203	.18
.19	.0138119	.1016740	.9552008	.1493730	.0167116	.19
0.20	.0144000	.1056000	.9504000	.1584000	.0176000	0.20
.21	.0149718	.1093806	.9453612	.1675324	.0184847	.21
.22	.0155269	.1130158	.9400856	.1767682	.0193651	.22
.23	.0160652	.1165062	.9345746	.1861058	.0202403	.23
.24	.0165862	.1198490	.9288294	.1955430	.0211098	.24
0.25	.0170898	.1230469	.9228516	.2050781	.0219727	0.25
.26	.0175757	.1260990	.9166424	.2147090	.0228283	.26
.27	.0180437	.1290052	.9102036	.2244338	.0236758	.27
.28	.0184934	.1317658	.9035366	.2342502	.0245146	.28
.29	.0189248	.1343806	.8966432	.2441564	.0253437	.29
0.30	.0193375	.1368500	.8895250	.2541500	.0261625	0.30
.31	.0197314	.1391740	.8821838	.2642290	.0269701	.31
.32	.0201062	.1413530	.8746214	.2743910	.0277658	.32
.33	.0204619	.1433870	.8668398	.2846340	.0285486	.33
.34	.0207981	.1452766	.8588408	2949554	.0293179	.34
0.35	.0211148	.1470219	.8506266	.3053531	.0300727	.35
.36	.0214118	.1486234	.8421990	.3158246	.0308122	.36
.37	.0216890	.1500814	.8335604	.3263676	.0315355	.37
.38	.0219461	.1513966	.8247128	.3369794	.0322419	.38
.39	.0221832	.1525692	.8156586	.3476578	.0329303	.39
0.40	.0224000	.1536000	.8064000	.3584000	.0336000	0.40
.41	.0225965	.1544894	.7969394	.3692036	.0342500	.41
.42	.0227725	.1552382	.7872792	.3800658	.0348795	.42
.43	.0229281	.1558468	.7774220	.3909842	.0354874	.43
.44	.0230630	.1563162	.7673702	.4019558	.0360730	.44
0.45	.0231773	.1566469	.7571266	.4129781	.0366352	.45
.46	.0232709	.1568398	.7466936	.4240482	.0371731	.46
.47	.0233438	.1568956	.7360742	.4351634	.0376857	.47
.48	.0233958	.1568154	.7252710	.4463206	.0381722	.48
.49	.0234271	.1565998	.7142870	.4575172	.0386314	.49
0.50	.0234375	.1562500	.7031250	.4687500	.0390625	0.50
$-p$	A_2	$-A_1$	A_0	A_{-1}	$-A_{-2}$	$-p$

Table VIII (*continued*). Lagrangian Interpolation Coefficients.
5-Points

p	A_{-2}	$-A_{-1}$	A_0	A_1	$-A_2$	p
0.50	.0234375	.1562500	.7031250	.4687500	.0390625	0.50
.51	.0234271	.1557668	.6917880	.4800162	.0394644	.51
.52	.0233958	.1551514	.8602790	.4913126	.0398362	.52
.53	.0233438	.1544046	.6686012	.5026364	.0401767	.53
.54	.0232709	.1535278	.6567576	.5139842	.0404851	.54
0.55	.0231773	.1525219	.6447516	.5253531	.0407602	0.55
.56	.0230630	.1513882	.6325862	.5367398	.0410010	.56
.57	.0229281	.1501278	.6202650	.5481412	.0412064	.57
58	.0227725	.1487422	.6077912	.5595538	.0413755	.58
.59	.0225965	.1472324	.5951684	.5709746	.0415070	.59
0.60	.0224000	.1456000	.5824000	.5824000	.0416000	0.60
.61	.0221832	.1438462	.5694896	.5938268	.0416533	.61
.62	.0219461	.1419726	.5564408	.6052514	.0416659	.62
.63	.0216890	.1399804	.5432574	.6166706	.0416363	.63
.64	.0214118	.1378714	.5299430	.6280806	.0415642	.64
0.65	.0211148	.1356469	.5165016	.6394781	.0414477	0.65
.66	.0207981	.1333086	.5029368	.6508594	.0412859	.66
.67	.0204619	.1308580	.4892528	.6622210	.0410776	.67
.68	.0201062	.1282970	.4754534	.6735590	.0408218	.68
.69	.0197314	.1256270	.4615428	.6848700	.0405171	.69
0.70	.0193375	.1228500	.4475250	.6961500	.0401625	0.70
.71	.0189248	.1199676	.4334042	.7073954	.0397567	.71
.72	.0184934	.1169818	.4191846	.7186022	.0392986	.72
.73	.0180437	.1138942	.4048706	.7297668	.0387868	.73
.74	.0175757	.1107070	.3904664	.7408850	.0382203	.74
0.75	.0170898	.1074219	.3759766	.7519531	.0375977	0.75
.76	.0165862	.1040410	.3614054	.7629670	.0369178	.76
.77	.0160652	.1005662	.3467576	.7739228	.0361793	.77
.78	.0155269	.0969998	.3320376	.7848162	.0353811	.78
.79	.0149718	.0933436	.3172502	.7956434	.0345217	.79
0.80	.0144000	.0896000	.3024000	.8064000	.0336000	0.80
.81	.0138119	.0857710	.2874918	.8170820	.0326146	.81
.82	.0132077	.0818590	.2725304	.8276850	.0315643	.82
.83	.0125879	.0778660	.2575208	.8382050	.0304476	.83
.84	.0119526	.0737946	.2424678	.8486374	.0292634	.84
0.85	.0113023	.0696469	.2273766	.8589781	.0280102	0.85
.86	.0106373	.0654254	.2122520	.8692226	.0266867	.86
.87	.0099580	.0611324	.1970994	.8793666	.0252915	.87
.88	.0092646	.0567706	.1819238	.8894054	.0238234	.88
.89	.0085577	.0523422	.1667306	.8993348	.0222808	.89
0.90	.0078375	.0478500	.1515250	.9091500	.0206625	0.90
.91	.0071045	.0432964	.1363124	.9188466	.0189670	.91
.92	.0063590	.0386842	.1210982	.9284198	.0171930	.92
.93	.0056016	.0340158	.1058880	.9378652	.0153389	.93
.94	.0048325	.0292942	.0906872	.9471778	.0134035	.94
0.95	.0040523	.0245219	.0755016	.9563531	.0113852	0.95
.96	.0032614	.0197018	.0603366	.9653862	.0092826	.96
.97	.0024603	.0148366	.0451982	.9742724	.0070942	.97
.98	.0016493	.0099294	.0300920	.9830066	.0048187	.98
.99	.0008291	.0049828	.0150240	.9915842	.0024544	.99
1.00	.0000000	.0000000	.0000000	1.0000000	.0000000	1.00
$-p$	A_2	$-A_1$	A_0	A_{-1}	$-A_{-2}$	$-p$

Table VIII (*continued*). Lagrangian Interpolation Coefficients.

5-Points

p	A_{-2}	$-A_{-1}$	A_0	A_1	$-A_2$	p
1.00	.0000000	.0000000	.0000000	1.000000	.0000000	1.00
.01	.0008374	.0050162	.0149740	1.0082492	.0025461	.01
.02	.0016827	.0100626	.0298920	1.0163266	.0051853	.02
.03	.0025352	.0151364	.0447478	1.0242274	.0079193	.03
.04	.0033946	.0202342	.0595354	1.0319462	.0107494	.04
1.05	.0042602	.0253531	.0742484	1.0394781	.0136773	1.05
.06	.0051315	.0304898	.0888808	1.0468178	.0167045	.06
.07	.0060079	.0356412	.1034260	1.0539602	.0198326	.07
.08	.0068890	.0408038	.1178778	1.0608998	.0230630	.08
.09	.0077740	.0459746	.1322296	1.0676316	.0263975	.09
1.10	.0086625	.0511500	.1464750	1.0741500	.0298375	1.10
.11	.0095538	.0563268	.1606074	1.0804498	.0333847	.11
.12	.0104474	.0615014	.1746202	1.0865254	.0370406	.12
.13	.0113425	.0666706	.1885066	1.0923716	.0408070	.13
.14	.0122387	.0718306	.2022600	1.0979826	.0446853	.14
1.15	.0131352	.0769781	.2158734	1.1033531	.0486773	1.15
.16	.0140314	.0821094	.2293402	1.1084774	.0527846	.16
.17	.0149266	.0872210	.2426532	1.1133500	.0570089	.17
.18	.0158203	.0923090	.2558056	1.1179650	.0613517	.18
.19	.0167116	.0973700	.2687902	1.1223170	.0658149	.19
1.20	.0176000	.1024000	.2816000	1.1264000	.0704000	1.20
.21	.0184847	.1073954	.2942278	1.1302084	.0751088	.21
.22	.0193651	.1123522	.3066664	1.1337362	.0799429	.22
.23	.0202403	.1172668	.3189084	1.1369778	.0849042	.23
.24	.0211098	.1221350	.3309466	1.1399270	.0899942	.24
1.25	.0219727	.1269531	.3427734	1.1425781	.0952148	1.25
.26	.0228283	.1317170	.3543816	1.1449250	.1005677	.26
.27	.0236758	.1364228	.3657634	1.1469618	.1060547	.27
.28	.0245146	.1410662	.3769114	1.1486822	.1116774	.28
.29	.0253437	.1456434	.3878178	1.1500804	.1174378	.29
1.30	.0261625	.1501500	.3984750	1.1511500	.1233375	1.30
.31	.0269701	.1545820	.4088752	1.1518850	.1293784	.31
.32	.0277658	.1589350	.4190106	1.1522790	.1355622	.32
.33	.0285486	.1632050	.4288732	1.1523260	.1418909	.33
.34	.0293179	.1673874	.4384552	1.1520194	.1483661	.34
1.35	.0300727	.1714781	.4477484	1.1513531	.1549898	1.35
.36	.0308122	.1754726	.4567450	1.1503206	.1617638	.36
.37	.0315355	.1793666	.4654366	1.1489156	.1686900	.37
.38	.0322419	.1831554	.4738152	1.1471314	.1757701	.38
.39	.0329303	.1868348	.4818724	1.1449618	.1830062	.39
1.40	.0336000	.1904000	.4896000	1.1424000	.1904000	1.40
.41	.0342500	.1938466	.4969896	1.1394396	.1979535	.41
.42	.0348795	.1971698	.5040328	1.1360738	.2056685	.42
.43	.0354874	.2003652	.5107210	1.1322962	.2135471	.43
.44	.0360730	.2034278	.5170458	1.1280998	.2215910	.44
1.45	.0366352	.2063531	.5229984	1.1234781	.2298023	1.45
.46	.0371731	.2091362	.5285704	1.1184242	.2381829	.46
.47	.0376857	.2117724	.5337528	1.1129314	.2467348	.47
.48	.0381722	.2142566	.5385370	1.1069926	.2554598	.48
.49	.0386314	.2165842	.5429140	1.1006012	.2643601	.49
1.50	.0390625	.2187500	.5468750	1.0937500	.2734375	1.50
$-p$	$-A_2$	A_1	$-A_0$	A_{-1}	A_{-2}	$-p$

Table VIII (*continued*). **Lagrangian Interpolation Coefficients.**
5-Points

p	A_{-2}	$-A_{-1}$	A_0	A_1	$-A_2$	p
1.50	.0390625	.2187500	.5468750	1.0937500	.2734375	1.50
.51	.0394644	.2207492	.5504110	1.0864322	.2826941	.51
.52	.0398362	.2225766	.5535130	1.0786406	.2921318	.52
.53	.0401767	.2242274	.5561718	1.0703684	.3017528	.53
.54	.0404851	.2256962	.5583784	1.0616082	.3115589	.54
1.55	.0407602	.2269781	.5601234	1.0523531	.3215523	1.55
.56	.0410010	.2280678	.5613978	1.0425958	.3317350	.56
.57	.0412064	.2289602	.5621920	1.0323292	.3421091	.57
.58	.0413755	.2296498	.5624968	1.0215458	.3526765	.58
.59	.0415070	.2301316	.5623026	1.0102386	.3634395	.59
1.60	.0416000	.2304000	.5616000	.9984000	.3744000	1.60
.61	.0416533	.2304498	.5603794	.9860228	.3855602	.61
.62	.0416659	.2302754	.5586312	.9730994	.3969221	.62
.63	.0416365	.2298716	.5563456	.9596226	.4084880	.63
.64	.0415642	.2292326	.5535130	.9455846	.4202598	.64
1.65	.0414477	.2283531	.5501234	.9309781	.4322398	1.65
.66	.0412859	.2272274	.5461672	.9157954	.4444301	.66
.67	.0410776	.2258500	.5416342	.9000290	.4568329	.67
.68	.0408218	.2242150	.5365146	.8836710	.4694502	.68
.69	.0405171	.2223170	.5307982	.8667140	.4822844	.69
1.70	.0401625	.2201500	.5244750	.8491500	.4953375	1.70
.71	.0397567	.2177084	.5175348	.8309714	.5086118	.71
.72	.0392986	.2149862	.5099674	.8121702	.5221094	.72
.73	.0387868	.2119778	.5017624	.7927388	.5358327	.73
.74	.0382203	.2086770	.4929096	.7726690	.5497837	.74
1.75	.0375977	.2050781	.4833984	.7519531	.5639648	1.75
.76	.0369178	.2011750	.4732186	.7305830	.5783782	.76
.77	.0361793	.1969618	.4623594	.7085508	.5930262	.77
.78	.0353811	.1924322	.4508104	.6858482	.6079109	.78
.79	.0345217	.1875804	.4385608	.6624674	.6230348	.79
1.80	.0336000	.1824000	.4256000	.6384000	.6384000	1.80
.81	.0326146	.1768850	.4119172	.6136380	.6540089	.81
.82	.0315643	.1710290	.3975016	.5881730	.6698637	.82
.83	.0304476	.1648260	.3823422	.5619970	.6859669	.83
.84	.0292634	.1582694	.3664282	.5351014	.7023206	.84
1.85	.0280102	.1513531	.3497484	.5074781	.7189273	1.85
.86	.0266867	.1440706	.3322920	.4791186	.7357893	.86
.87	.0252915	.1364156	.3140476	.4500146	.7529090	.87
.88	.0238234	.1283814	.2950042	.4201574	.7702886	.88
.89	.0222808	.1199618	.2751504	.3895388	.7879307	.89
1.90	.0206625	.1111500	.2544750	.3581500	.8058375	1.90
.91	.0189670	.1019396	.2329666	.3259826	.8240115	.91
.92	.0171930	.0923238	.2106138	.2930278	.8424550	.92
.93	.0153389	.0822962	.1874050	.2592772	.8611706	.93
.94	.0134035	.0718498	.1633288	.2247218	.8801605	.94
1.95	.0113852	.0609781	.1383734	.1893531	.8994273	1.95
.96	.0092826	.0496742	.1125274	.1531622	.9189734	.96
.97	.0070942	.0379314	.0857788	.1161404	.9388013	.97
.98	.0048187	.0257426	.0581160	.0782786	.9589133	.98
.99	.0024544	.0131012	.0295270	.0395682	.9793121	.99
2.00	.0000000	.0000000	.0000000	.0000000	1.0000000	2.00
$-p$	$-A_2$	A_1	$-A_0$	A_{-1}	A_{-2}	$-p$

Table IX. Differential Coefficients—Fractional Form.

$$A'_{ki} = \left.\frac{dA_i}{du}\right|_{u=k}$$

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}	A'_{k5}	A'_{k6}	A'_{k7}	A'_{k8}	A'_{k9}	k
					$n = 2$ (3 points) $D_n = 2$						
0	−3	4	−1								2
1	−1	0	1								
					$n = 3$ (4 points) $D_n = 6$						
0	−11	18	−9	2							3
1	−2	−3	6	−1							2
					$n = 4$ (5 points) $D_n = 12$						
0	−25	48	−36	16	−3						4
1	−3	−10	18	−6	1						3
2	1	−8	0	8	−1						
					$n = 5$ (6 points) $D_n = 60$						
0	−137	300	−300	200	−75	12					5
1	−12	−65	120	−60	20	−3					4
2	3	−30	−20	60	−15	2					3

$n = 6$ (7 points) $D_n = 60$

k	$-A_{k,n}$	$-A_{k,n-1}$	$-A_{k,n-2}$	$-A_{k,n-3}$	$-A_{k,n-4}$	$-A_{k,n-5}$	$-A_{k,n-6}$				k
0	−147	360	−450	400	−225	72	−10				6
1	−10	−77	150	−100	50	−15	2				5
2	2	−24	−35	80	−30	8	−1				4
3	−1	9	−45	0	45	−9	1				

$n = 7$ (8 points) $D_n = 420$

k	$-A_{k,n}$	$-A_{k,n-1}$	$-A_{k,n-2}$	$-A_{k,n-3}$	$-A_{k,n-4}$	$-A_{k,n-5}$	$-A_{k,n-6}$	$-A_{k,n-7}$			k
0	−1089	2940	−4410	4900	−3675	1764	−490	60			7
1	−60	−609	1260	−1050	700	−315	84	−10			6
2	10	−140	−329	700	−350	140	−35	4			5
3	−4	42	−252	−105	420	−42	28	−3			4

$n = 8$ (9 points) $D_n = 840$

k	$-A_{k,n}$	$-A_{k,n-1}$	$-A_{k,n-2}$	$-A_{k,n-3}$	$-A_{k,n-4}$	$-A_{k,n-5}$	$-A_{k,n-6}$	$-A_{k,n-7}$	$-A_{k,n-8}$		k
0	−2283	6720	−11760	15680	−14700	9804	−3920	960	−105		8
1	−105	−1338	2940	−2940	2450	−1470	588	−140	15		7
2	15	−240	−798	1680	−1050	560	−210	48	−5		6
3	−5	60	−420	−378	1050	−420	140	−30	3		5
4	3	−32	168	−672	0	672	−168	32	−3		

$n = 9$ (10 points) $D_n = 2520$

k	$-A_{k,n}$	$-A_{k,n-1}$	$-A_{k,n-2}$	$-A_{k,n-3}$	$-A_{k,n-4}$	$-A_{k,n-5}$	$-A_{k,n-6}$	$-A_{k,n-7}$	$-A_{k,n-8}$	$-A_{k,n-9}$	k
0	−7129	22680	−45360	70560	−79380	63504	−35280	12960	−2835	280	9
1	−280	−4329	10080	−11760	11760	−8820	4704	−1680	360	−35	8
2	35	−630	−2754	5880	−4410	2940	−1470	504	−105	10	7
3	−10	135	−1080	−1554	3780	−1890	840	−270	54	−5	6
4	5	−60	360	−1680	−504	2520	−840	240	−45	4	5

Table X. Differentiation Coefficients—Decimal Form.

$$A'_{ki} = \left.\frac{dA_i}{du}\right|_{u=k}$$

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}	A'_{k5}	A'_{k6}	A'_{k7}	A'_{k8}	A'_{k9}
n = 2										
0	−1.5	2.0	−.5							
1	−.5	0	.5							
n = 3										
0	−1.833333	3.000000	−1.500000	.333333						
1	−.333333	−.500000	1.000000	−.166667						
n = 4										
0	−2.083333	4.000000	−3.000000	1.333333	−.250000					
1	−.250000	−.833333	1.500000	−.500000	.083333					
2	.083333	−.666667	0	.666667	−.083333					
n = 5										
0	−2.283333	5.000000	−5.000000	3.333333	−1.250000	.200000				
1	−.200000	−1.083333	2.000000	−1.000000	.333333	−.050000				
2	.050000	−.500000	−.333333	1.000000	−.250000	.033333				

$$A'_{ki} = -A'_{n-k,\,n-i} \qquad k > \tfrac{1}{2}n$$

$$A'_{ki} = \left.\frac{dA_i}{du}\right|_{u=k}$$

$n = 6$

0	−2.450000	6.000000	−7.500000	6.666667	−3.750000	1.200000	−.166667
1	−.166667	−1.283333	2.500000	−1.666667	.833333	−.250000	.033333
2	.033333	−.400000	−.583333	1.333333	−.500000	.133333	−.016667
3	−.016667	.150000	−.750000	0	.750000	−.150000	.016667

$n = 7$

0	−2.592857	7.000000	−10.500000	11.666667	−8.750000	4.200000	−1.166667	.142857
1	−.142857	−1.450000	3.000000	−2.500000	1.666667	−.750000	.200000	−.023810
2	.023810	−.333333	−.783333	1.666667	−.833333	.333333	−.083333	.009524
3	−.009524	.100000	−.600000	−.250000	1.000000	−.100000	.066667	−.007143

$n = 8$

0	−2.717857	8.000000	−14.000000	18.666667	−17.500000	11.200000	−4.666667	1.142857	−.125000
1	−.125000	−1.592857	3.500000	−3.500000	2.116667	−1.750000	.700000	−.166667	.017857
2	.017857	−.285714	−.950000	2.000000	−1.550000	.666667	−.250000	.057143	−.005952
3	−.005952	.071429	−.500000	−.450000	1.250000	−.500000	.166667	−.035714	.003571
4	.003571	−.038095	.200000	−.800000	0	.800000	−.200000	.038095	−.003571

$n = 9$

0	−2.828968	9.000000	−18.000000	28.000000	−31.500000	25.200000	−14.000000	5.142857	−1.125000	.111111
1	−.111111	−1.717857	4.000000	−4.666667	4.666667	−3.500000	1.866667	−.666667	.142857	−.013889
2	.013889	−.250000	−1.092857	2.333333	−1.750000	1.166667	−.583333	.200000	−.041667	.003968
3	−.003968	.053571	−.428571	−.616667	1.500000	−.750000	.333333	−.107143	.021429	−.001984
4	.001984	−.023810	.142857	−.666667	−.200000	1.000000	−.333333	.095238	−.017857	.001587

Table XI. Second Derivative Coefficients.

$$A''_{ki} = \left. \frac{dA'_{ki}}{du} \right|_{u=k}$$

k	A''_{k0}	A''_{k1}	A''_{k2}	A''_{k3}	A''_{k4}	A''_{k5}	A''_{k6}	A''_{k7}
				$n = 2$ (3 Points) $D_n = 1$				
All	1	−2	1					
				$n = 3$ (4 Points) $D_n = 1$				
0	2	−5	4	−1				
1	1	−2	1	0				
2	0	1	−2	1				
3	−1	4	−5	2				
				$n = 4$ (5 Points) $D_n = 12$				
0	35	−104	114	−56	11			
1	11	−20	6	4	−1			
2	−1	16	−30	16	−1			
3	−1	4	6	−20	11			
4	11	−56	114	−104	35			
				$n = 5$ (6 Points) $D_n = 12$				
0	45	−154	214	−156	61	−10		
1	10	−15	−4	14	−6	1		
2	−1	16	−30	16	−1	0		
3	0	−1	16	−30	16	−1		
4	1	−6	14	−4	−15	10		
5	−10	61	−156	214	−154	45		
				$n = 6$ (7 Points) $D_n = 180$				
0	812	−3132	5265	−5080	2970	−972	137	
1	137	−207	−255	470	−285	93	−13	
2	−13	228	−420	200	15	−12	2	
3	2	−27	270	−490	270	−27	2	
4	2	−12	15	200	−420	228	−13	
5	−13	93	−285	470	−255	−207	137	
6	137	−972	2970	−5080	5265	−3132	812	
				$n = 7$ (8 Points) $D_n = 180$				
0	938	−4014	7911	−9490	7380	−3618	1019	−126
1	126	−70	−486	855	−670	324	−90	11
2	−11	214	−378	130	85	−102	16	−2
3	2	−27	270	−490	270	−27	2	0
4	0	2	−27	270	−490	270	−27	2
5	−2	16	−102	85	130	−378	214	−11
6	11	−90	324	−670	855	−486	−70	126
7	−126	1019	−3618	7380	−9490	7911	−4014	938

Table XII. Third Derivative Coefficients.

$$A'''_{ki} = \left.\frac{dA''_{ki}}{du}\right|_{u=k}$$

$n = 3$ (4 Points) $D_n = 1$

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
All	−1	3	−3	1				

$n = 4$ (5 Points) $D_n = 2$

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
0	−5	18	−24	14	−3			
1	−3	10	−12	6	−1			
2	−1	2	0	−2	1			
3	1	−6	12	−10	3			
4	3	−14	24	−18	5			

$n = 5$ (6 Points) $D_n = 4$

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
0	−17	71	−118	98	−41	7		
1	−7	25	−34	22	−7	1		
2	−1	−1	10	−14	7	−1		
3	1	−7	14	−10	1	1		
4	−1	7	−22	34	−25	7		
5	−7	41	−98	118	−71	17		

$n = 6$ (7 Points) $D_n = 24$

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
0	−147	696	−1383	1488	−921	312	−45	
1	−45	168	−249	192	−87	24	−3	
2	−3	−24	105	−144	87	−24	3	
3	3	−24	39	0	−39	24	−3	
4	−3	24	−87	144	−105	24	3	
5	3	−24	87	−192	249	−168	45	
6	45	−312	921	−1488	1383	−696	147	

$n = 7$ (8 Points) $D_n = 120$

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
0	−967	5104	−11787	15560	−12725	6432	−1849	232
1	−232	889	−1392	1205	−680	267	−64	7
2	−7	−176	693	−1000	715	−288	71	−8
3	8	−71	48	245	−440	267	−64	7
4	−7	64	−267	440	−245	−48	71	−8
5	8	−71	288	−715	1000	−693	176	7
6	−7	64	−267	680	−1205	1392	−889	232
7	−232	1849	−6432	12725	−15560	11787	−5104	967

Table XIII. Integration Coefficients (α_i).

$n = 2$ (3 Points)

α_i	0–1	0–2
α_0	.416666667	.333333333
α_1	.666666667	1.333333333
α_2	−.083333333	.333333333

$n = 3$ (4 Points)

α_i	0–1	0–2	0–3
α_0	.375000000	.333333333	.375000000
α_1	.791666667	1.333333333	1.125000000
α_2	−.208333333	.333333333	1.125000000
α_3	.041666667	0	.375000000

$n = 4$ (5 Points)

α_i	0–1	0–2	0–3	0–4
α_0	.348611111	.322222222	.337500000	.311111111
α_1	.897222222	1.377777778	1.275000000	1.422222222
α_2	−.366666667	.266666667	.900000000	.533333333
α_3	.147222222	.044444444	.525000000	1.422222222
α_4	−.026388889	−.011111111	−.037500000	.311111111

$n = 5$ (6 Points)

a_i	0–1	0–2	0–3	0–4	0–5
a_0	.329861111	.311111111	.318750000	.311111111	.329861111
a_1	.990972222	1.433333333	1.368750000	1.422222222	1.302083333
a_2	−.554166667	.155555556	.712500000	.533333333	.868055556
a_3	.334722222	.155555556	.712500000	1.422222222	.868055556
a_4	−.120138889	−.066666667	−.131250000	.311111111	1.302083333
a_5	.018750000	.011111111	.018750000	0	.329861111

$n = 6$ (7 Points)

a_i	0–1	0–2	0–3	0–4	0–5	0–6
a_0	.315591931	.301322751	.305803571	.302645503	.307126323	.292857143
a_1	1.076587301	1.492063492	1.446428571	1.473015873	1.438492063	1.542857143
a_2	−.768204365	.008730159	.518303571	.406349206	.527033730	.192857143
a_3	.620105820	.351322751	.971428571	1.591534392	1.322751323	1.942857143
a_4	−.334176587	−.213492064	−.325446428	.184126984	.961061508	.192857143
a_5	.104365079	.069841270	.096428571	.050793651	.466269841	1.542857143
a_6	−.014269179	−.009788359	−.012946428	−.008465609	−.022734788	.292857143

Table XIII (*continued*). Integration Coefficients (a_i).

$n = 7$ (8 Points)

a_i	0–1	0–2	0–3	0–4	0–5	0–6	0–7
a_0	.304224537	.292857143	.295758928	.294179894	.295758928	.292857143	.304224537
a_1	1.156159060	1.551322751	1.516741071	1.532275132	1.518063822	1.542857143	1.449016203
a_2	−1.006919642	−.169047619	.307366071	.228571428	.288318452	.192857143	.535937500
a_3	1.017964616	.647619047	1.322991071	1.887830687	1.720610119	1.942857143	1.210821759
a_4	−.732035383	−.509788359	−.677008928	−.112169312	.563202711	.192857143	1.210821759
a_5	.343080357	.247619048	.307366071	.228571428	.704985119	1.542857143	.535937500
a_6	−.093840939	−.069047619	−.083258928	−.067724867	−.102306547	.292857143	1.449016203
a_7	.011367394	.008465608	.010044643	.008465608	.011367394	0	.304224537

$n = 8$ (9 Points)

a_i	0–1	0–2	0–3	0–4	0–5	0–6	0–7	0–8
a_0	.294868000	.285511463	.287522321	.286631393	.287319499	.286428571	.288439428	.279082892
a_1	1.231011353	1.610088183	1.582633928	1.592663139	1.585579254	1.594285714	1.575297067	1.661516754
a_2	−1.268902667	−.374726631	.076741071	.017213404	.05201440	.012857143	.093954475	−.261869488
a_3	1.541930665	1.058977072	1.784241071	2.310546737	2.193218143	2.302857143	2.094787808	2.961834215
a_4	−1.386992945	−1.023985890	−1.253571428	−.640564373	−.027557319	−.257142857	.105864197	−1.281128747
a_5	.867046406	.658977072	.768616071	.651287477	1.177593143	1.902857143	1.419903549	2.961834215
a_6	−.355823964	−.274726631	−.313883928	−.279082892	−.338610560	.112857143	1.007033179	−.261869488
a_7	.086219687	.067231040	.075937500	.068853615	.078882826	.051428571	.430505401	1.661516754
a_8	−.009356536	−.007345679	−.008236607	−.007548501	−.008439429	−.006428571	−.015785108	.279082892

$n = 9$ (10 Points)

a_i	0–1	0–2	0–3	0–4	0–5
a_0	.286975446	.279082892	.280546875	.280000000	.280344053
a_1	1.302044339	1.667945326	1.645412946	1.652345679	1.648358272
a_2	−1.553034611	−.606155202	−.174375000	−.221516755	−.199101631
a_3	2.204905202	1.598977072	2.370178571	2.867583774	2.779155643
a_4	−2.381454750	−1.833985890	−2.132477678	−1.476119929	−.906463569
a_5	1.861508212	1.468977072	1.647522321	1.486843033	2.056499393
a_6	−1.018798500	−.814726631	−.899821428	−.836119929	−.924548059
a_7	.370351631	.298659612	.327053571	.307583774	.329998898
a_8	−.080389523	−.065202822	−.071015625	−.067231040	−.071218447
a_9	.007892554	.006428571	.006975446	.006631393	.006975446

a_i	0–6	0–7	0–8	0–9
a_0	.280000000	.280546875	.279082892	.286975446
a_1	1.652142857	1.646330054	1.661516754	1.581127232
a_2	−.218571429	−.190177469	−.261869488	.108482143
a_3	2.842857143	2.757762345	2.961834215	1.943035714
a_4	−1.067142857	−.888597608	−1.281128747	.580379464
a_5	2.712857143	2.414365354	2.961834215	.580379464
a_6	−.427142857	.344058641	−.261869488	1.943035714
a_7	.282857143	.714637345	1.661516754	.108482143
a_8	−.064285714	−.086818094	.279082892	1.581127232
a_9	.006428571	.007892554	0	.286975446

Table XIII (*continued*). Integration Coefficients (a_i).

$n = 10$ (11 Points)

a_i	0–1	0–2	0–3	0–4	0–5
a_0	.280189596	.273403746	.274510450	.274153172	.274342768
a_1	1.369902839	1.724736785	1.705777191	1.710813959	1.708371120
a_2	−1.858397861	−.861716770	−.446014103	−.484624017	−.469159445
a_3	3.019207201	2.280474587	3.094549512	3.569203142	3.499309814
a_4	−3.806483247	−3.026606541	−3.400126826	−2.703953823	−2.166733367
a_5	3.571542408	2.900121853	3.168701298	2.960243706	3.568823152
a_6	−2.443826997	−2.007347282	−2.167470576	−2.063953823	−2.184817858
a_7	1.184653629	.980157126	1.051424512	1.009203142	1.050153068
a_8	−.385752772	−.320764389	−.342654728	−.330338303	−.341276260
a_9	.075751054	.063220031	.067339691	.065099674	.066988295
a_{10}	−.006785850	−.005679146	−.006036424	−.005846828	−.006001285

a_i	0–6	0–7	0–8	0–9	0–10
a_0	.274188311	.274377908	.274020633	.275127333	.268341483
a_1	1.710259740	1.708019723	1.712139383	1.699608360	1.775359414
a_2	−.480097402	−.467780978	−.489671316	−.424682934	−.810435705
a_3	3.540259740	3.498038370	3.569305755	3.364809253	4.549462882
a_4	−2.287597402	−2.184080650	−2.344203944	−1.907724228	−4.351551226
a_5	4.177402597	3.968945006	4.237524450	3.566103896	7.137646304
a_6	−1.647597402	−.951424400	−1.324944684	−.545067978	−4.351551226
a_7	.980259740	1.454913370	2.268988295	1.530255681	4.549462882
a_8	−.325811688	−.364421603	.051281064	1.047962155	−.810435705
a_9	.064545455	.069582222	.050622628	.405456574	1.775359414
a_{10}	−.005811688	−.006168967	−.005062263	−.011848112	.268341483

Table XIV. Powers of Integers, N^i, $(i = 1, \cdots, 6)$.
$(i = 1, \cdots, 6), 1 \leq N \leq 40$

N	N^2	N^3	N^4	N^5	N^6
1	1	1	1	1	1
2	4	8	16	32	64
3	9	27	81	243	729
4	16	64	256	1024	4096
5	25	125	625	3125	15625
6	36	216	1296	7776	46656
7	49	343	2401	16807	117649
8	64	512	4096	32768	262144
9	81	729	6561	59049	531441
10	100	1000	10000	100000	1000000
11	121	1331	14641	161051	1771561
12	144	1728	20736	248832	2985984
13	169	2197	28561	371293	4826809
14	196	2744	38416	537824	7529536
15	225	3375	50625	759375	11390625
16	256	4096	65536	1048576	16777216
17	289	4913	83521	1419857	24137569
18	324	5832	104976	1889568	34012224
19	361	6859	130321	2476099	47045881
20	400	8000	160000	3200000	64000000
21	441	9261	194481	4084101	85766121
22	484	10648	234256	5153632	113379904
23	529	12167	279841	6436343	148035889
24	576	13824	331776	7962624	191102976
25	625	15625	390625	9765625	244140625
26	676	17576	456976	11881376	308915776
27	729	19683	531441	14348907	387420489
28	784	21952	614656	17210368	481890304
29	841	24389	707281	20511149	594823321
30	900	27000	810000	24300000	729000000
31	961	29791	923521	28629151	887503681
32	1024	32768	1048576	33554432	1073741824
33	1089	35937	1185921	39135393	1291467969
34	1156	39304	1336336	45435424	1544804416
35	1225	42875	1500625	52521875	1838265625
36	1296	46656	1679616	60466176	2176782336
37	1369	50653	1874161	69343957	2565726409
38	1444	54872	2085136	79235168	3010936384
39	1521	59319	2313441	90224199	3518743761
40	1600	64000	2560000	102400000	4096000000

Table XV. Sum of Powers of Integers, $\sum_{k=1}^{n} k^m$

$(m = 1, 2, 3, 4);\ 1 \leq n \leq 40$

n	Σk	Σk^2	Σk^3	Σk^4
1	1	1	1	1
2	3	5	9	17
3	6	14	36	98
4	10	30	100	354
5	15	55	225	979
6	21	91	441	2275
7	28	140	784	4676
8	36	204	1296	8772
9	45	285	2025	15333
10	55	385	3025	25333
11	66	506	4356	39974
12	78	650	6084	60710
13	91	819	8281	89271
14	105	1015	11025	127687
15	120	1240	14400	178312
16	136	1496	18496	243848
17	153	1785	23409	327369
18	171	2109	29241	432345
19	190	2470	36100	562666
20	210	2870	44100	722666
21	231	3311	53361	917147
22	253	3795	64009	1151403
23	276	4324	76176	1431244
24	300	4900	90000	1763020
25	325	5525	105625	2153645
26	351	6201	123201	2610621
27	378	6930	142884	3142062
28	406	7714	164836	3756718
29	435	8555	189225	4463999
30	465	9455	216225	5273999
31	496	10416	246016	6197520
32	528	11440	278784	7246096
33	561	12529	314721	8432017
34	595	13685	354025	9768353
35	630	14910	396900	11268978
36	666	16206	443556	12948594
37	703	17575	494209	14822755
38	741	19019	549081	16907891
39	780	20540	608400	19221332
40	820	22140	672400	21781332

Table XVI. *N-G* Polynomial Fitting Constants for $n = 1, 2, 3, 4$.

m	S_2	A_{13}	A_{33}	A_{24}	A_{44}	A_{35}	A_{55}
5	5.00	2.000000	1.400000	3.40	1.440000	4.428571	.822857
6	8.75	2.916667	2.133333	5.05	3.702857	6.785714	4.702041
7	14.00	4.000000	3.000000	7.00	7.714286	9.571429	16.163265
8	21.00	5.250000	4.000000	9.25	14.142857	12.785714	43.102041
9	30.00	6.666667	5.133333	11.80	23.760000	16.428571	98.057143
10	41.25	8.250000	6.400000	14.65	37.440000	20.500000	199.680000
11	55.00	10.000000	7.800000	17.80	56.160000	25.000000	374.400000
12	71.50	11.916667	9.333333	21.25	81.000000	29.928571	658.285714
13	91.00	14.000000	11.000000	25.00	113.142857	35.285714	1099.102041
14	113.75	16.250000	12.800000	29.05	153.874286	41.071428	1758.563265
15	140.00	18.666667	14.733333	33.40	204.582857	47.285714	2714.782041
16	170.00	21.250000	16.800000	38.05	266.760000	53.928571	4064.914285
17	204.00	24.000000	19.000000	43.00	342.000000	61.000000	5928.000000
18	242.25	26.916667	21.333333	48.25	432.000000	68.500000	8448.000000
19	285.00	30.000000	23.800000	53.80	538.560000	76.428571	11797.028571
20	332.50	33.250000	26.400000	59.65	663.582857	84.785714	16178.782040
21	385.00	36.666667	29.133333	65.80	809.074286	93.571429	21832.163265
22	442.75	40.250000	32.000000	72.25	977.142857	102.785714	29035.102041
23	506.00	44.000000	35.000000	79.00	1170.000000	112.428571	38108.571429
24	575.00	47.916667	38.133333	86.05	1389.960000	122.500000	49420.800000
25	650.00	52.000000	41.400000	93.40	1639.440000	133.000000	63391.680000

26	731.25	56.250000	44.800000	101.05	1920.960000	143.928571	80497.371429
27	819.00	60.666667	48.333333	109.00	2237.142857	155.285714	101275.102041
28	913.50	65.250000	52.000000	117.25	2590.714286	167.071429	126328.163265
29	1015.00	70.000000	55.800000	125.80	2984.502857	179.285714	156331.102041
30	1123.75	74.916667	59.733333	134.65	3421.440000	191.928571	192035.108571
31	1240.00	80.000000	63.800000	143.80	3904.560000	205.000000	234273.600000
32	1364.00	85.250000	68.000000	153.25	4437.000000	218.500000	283968.000000
33	1496.00	90.666667	72.333333	163.00	5022.000000	232.428571	342133.714286
34	1636.25	96.250000	76.800000	173.05	5662.902857	246.785714	409886.302041
35	1785.00	102.000000	81.400000	183.40	6363.154286	261.571429	488447.843265
36	1942.50	107.916667	86.133333	194.05	7126.302857	276.785714	579153.502041
37	2109.00	114.000000	91.000000	205.00	7956.000000	292.428571	683458.285714
38	2284.75	120.250000	96.000000	216.25	8856.000000	308.500000	802944.000000
39	2470.00	126.666667	101.133333	227.80	9830.160000	325.000000	939326.400000
40	2665.00	133.250000	106.400000	239.65	10882.440000	341.928571	1094462.537143
41	2870.00	140.000000	111.800000	251.80	12016.902857	359.285714	1270358.302041
42	3085.25	146.916667	117.333333	264.25	13237.714286	377.071429	1469176.163265
43	3311.00	154.000000	123.000000	277.00	14549.142857	395.285714	1693243.102041
44	3547.50	161.250000	128.800000	290.05	15955.560000	413.928571	1945058.742857
45	3795.00	168.666667	134.733333	303.40	17461.440000	433.000000	2227303.680000
46	4053.75	176.250000	140.800000	317.05	19071.360000	452.500000	2542848.000000
47	4324.00	184.000000	147.000000	331.00	20790.000000	472.428571	2894760.000000
48	4606.00	191.916667	153.333333	345.25	22622.142857	492.785714	3286315.102041
49	4900.00	200.000000	159.800000	359.80	24572.674286	513.571429	3721004.963265
50	5206.25	208.250000	166.400000	374.65	26646.582857	534.785714	4202546.782041

Table XVI (*continued*). *N-G* Polynomial Fitting Constants for $n = 1, 2, 3, 4$.

m	S_2	A_{13}	A_{33}	A_{24}	A_{44}	A_{35}	A_{55}
51	5525.00	216.666667	173.133333	389.80	28848.960000	556.428571	4734892.800000
52	5856.50	225.250000	180.000000	405.25	31185.000000	578.500000	5322240.000000
53	6201.00	234.000000	187.000000	421.00	33660.000000	601.000000	5969040.000000
54	6558.75	242.916667	194.133333	437.05	36279.360000	623.928571	6680009.142857
55	6930.00	252.000000	201.400000	453.40	39048.582857	647.285714	7460138.782041
56	7315.00	261.250000	208.800000	470.05	41973.274286	671.071429	8314705.763265
57	7714.00	270.666667	216.333333	487.00	45059.142857	695.285714	9249283.102041
58	8127.25	280.250000	224.000000	504.25	48312.000000	719.928571	10269750.857142
59	8555.00	290.000000	231.800000	521.80	51737.760000	745.000000	11382307.200000
60	8997.50	299.916667	239.733333	539.65	55342.440000	770.500000	12593479.680000
61	9455.00	310.000000	247.800000	557.80	59132.160000	796.428571	13910136.685714
62	9927.75	320.250000	256.000000	576.25	63113.142857	822.785714	15339499.102041
63	10416.00	330.666667	264.333333	595.00	67291.714286	849.571429	16889152.163265
64	10920.00	341.250000	272.800000	614.05	71674.302857	876.785714	18567057.502041
65	11440.00	352.000000	281.400000	633.40	76267.440000	904.428571	20381565.394286
66	11976.25	362.916667	290.133333	653.05	81077.760000	932.500000	22341427.200000
67	12529.00	374.000000	299.000000	673.00	86112.000000	961.000000	24455808.000000
68	13098.50	385.250000	308.000000	693.25	91377.000000	989.928571	26734299.428571
69	13685.00	396.666667	317.133333	713.80	96879.702857	1019.285714	29186932.702041
70	14288.75	408.250000	326.400000	734.65	102627.154286	1049.071429	31824191.843265
71	14910.00	420.000000	335.800000	755.80	108626.502857	1079.285714	34657027.102041
72	15549.00	431.916667	345.333333	777.25	114885.000000	1109.928571	37696868.571429
73	16206.00	444.000000	355.000000	799.00	121410.000000	1141.000000	40955640.000000
74	16881.25	456.250000	364.800000	821.05	128208.960000	1172.500000	44445772.800000
75	17575.00	468.666667	374.733333	843.40	135289.440000	1204.428571	48180220.251429

76	18287.50	481.250000	384.800000	866.05	142659.102857	1236.785714	52172471.902041
77	19019.00	494.000000	395.000000	889.00	150325.714286	1269.571429	56436568.163265
78	19769.75	506.916667	405.333333	912.25	158297.142857	1302.785714	60987115.102041
79	20540.00	520.000000	415.800000	935.80	166581.360000	1336.428571	65839299.428571
80	21330.00	533.250000	426.400000	959.65	175186.440000	1370.500000	71008903.680000
81	22140.00	546.666667	437.133333	983.80	184120.560000	1405.000000	76512321.600000
82	22970.25	560.250000	448.000000	1008.25	193392.000000	1439.928571	82366573.714286
83	23821.00	574.000000	459.000000	1033.00	203009.142857	1475.285714	88589323.102041
84	24692.50	587.916667	470.133333	1058.05	212980.474286	1511.071429	95198891.363265
85	25585.00	602.000000	481.400000	1083.40	223314.582857	1547.285714	102214274.782041
86	26498.75	616.250000	492.800000	1109.05	234020.160000	1583.928571	109655160.685714
87	27434.00	630.666667	504.333333	1135.00	245106.000000	1621.000000	117541944.000000
88	28391.00	645.250000	516.000000	1161.25	256581.000000	1658.500000	125895744.000000
89	29370.00	660.000000	527.800000	1187.80	268454.160000	1696.428571	134738421.257143
90	30371.25	674.916667	539.733333	1214.65	280734.582857	1734.785714	144092594.782041
91	31395.00	690.000000	551.800000	1241.80	293431.474286	1773.571429	153981659.363265
92	32441.50	705.250000	564.000000	1269.25	306554.142857	1812.785714	164429803.102041
93	33511.00	720.666667	576.333333	1297.00	320112.000000	1852.428571	175462025.142857
94	34603.75	736.250000	588.800000	1325.05	334114.560000	1892.500000	187104153.600000
95	35720.00	752.000000	601.400000	1353.40	348571.440000	1933.000000	199382863.680000
96	36860.00	767.916667	614.133333	1382.05	363492.360000	1973.928571	212325696.000000
97	38024.00	784.000000	627.000000	1411.00	378887.142857	2015.285714	225961075.102041
98	39212.25	800.250000	640.000000	1440.25	394765.714286	2057.071429	240318328.163265
99	40425.00	816.666667	653.133333	1469.80	411138.102857	2099.285714	255427703.902041
100	41662.50	833.250000	666.400000	1499.65	428014.440000	2141.928571	271320391.680000

Table XVI (*continued*). *N-G* Polynomial Fitting Constants for $n = 5,6$.

m	A_{26}	A_{46}	A_{66}	A_{37}	A_{57}	A_{77}
6	29.20535711	8.05555556	3.26530625			
7	56.71428571	11.66666667	24.48979545	86.7142857	13.18181818	20.037105
8	99.70535714	15.83333333	106.12244898	155.6875000	18.29545455	187.012988
9	163.00000000	20.55555556	346.66666667	258.1428571	24.09090909	981.818182
10	252.06250000	25.83333333	945.45454545	403.1875000	30.56818182	3808.264462
11	373.000000	31.66666667	2269.090909	601.000000	37.72727273	12138.84297
12	532.562500	38.05555556	4945.454545	862.830357	45.56818182	33615.25746
13	738.142857	45.00000000	9991.836734	1201.000000	54.09090909	83637.96203
14	997.776786	52.50000000	18984.489796	1628.901784	63.29545455	191172.5642
15	1320.142857	60.55555556	34277.551023	2161.000000	73.18181818	407735.0649
16	1714.562500	69.16666667	59280.0000	2812.830354	83.75000000	820800.000
17	2191.000000	78.33333333	98800.0000	3601.000000	95.00000000	1573200.000
18	2760.062500	88.05555556	159466.0000	4543.187500	106.93181818	2890472.727
19	3433.000000	98.33333333	250240.0000	5658.142857	119.54545455	5118545.452
20	4221.705357	109.16666667	383020.4081	6965.687500	132.84090909	8774649.349
21	5138.714286	120.55555556	573369.9442	8486.71429	146.81818182	14614919.20
22	6197.205357	132.50000000	841358.0694	10243.18750	161.47727273	23722767.38
23	7411.000000	145.00000000	1212545.4545	12258.14286	176.81818182	37622826.44
24	8794.562500	158.05555556	1719120.0000	14555.68750	192.84090909	58426036.36
25	10363.000000	171.66666667	2401200.0000	17161.00000	209.54545455	89012316.08

	$\times 10^4$		$\times 10^6$	$\times 10^4$		$\times 10^8$
26	1.213206250	185.8333333	3.308320000	2.010033035	226.9318182	1.332582042
27	1.411814285	200.5555556	4.501115644	2.340100000	245.0000000	1.963178900
28	1.633827678	215.8333333	6.053224491	2.709140178	263.7500000	2.849671837
29	1.881014286	231.6666667	8.053420406	3.120100000	283.1818182	4.080211947
30	2.155206250	248.0555556	10.608000000	3.576033035	303.2954545	5.768378178
			$\times 10^7$			$\times 10^9$
31	2.458300000	265.0000000	1.384344000	4.081000000	324.0909091	0.805920545
32	2.792256250	282.5000000	1.790934545	4.635568750	345.5681818	1.113635662
33	3.159100000	300.5555556	2.298170909	5.245814286	367.7272727	1.523060538
34	3.560920535	319.1666667	2.926650300	5.914318750	390.5681818	2.062981469
35	3.999871429	338.3333333	3.700362448	6.644671429	414.0909091	2.769061437
				$\times 10^5$		
36	4.478170535	358.0555556	4.647121850	0.744056875	438.2954545	3.685200123
37	4.998100000	378.3333333	5.799040000	0.830581429	463.1818182	4.865110688
38	5.562006250	399.1666667	7.193040000	0.924431875	488.7500000	6.374140057
39	6.172300000	420.5555556	8.871416000	1.026010000	515.0000000	8.291361872
40	6.831456250	442.5000000	10.882440000	1.135728304	541.9318182	10.711973792
	$\times 10^5$		$\times 10^8$			$\times 10^{10}$
41	0.754201429	465.0000000	1.328101861	1.254010000	569.5454545	1.375003359
42	0.830657768	488.0555556	1.612940274	1.381275000	597.8409091	1.754157109
43	0.912781429	511.6666667	1.949795037	1.518010000	626.8181818	2.224811637
44	1.000845625	535.8333333	2.346595491	1.664628304	656.4772727	2.806068732
45	1.095130000	560.5555556	2.812252121	1.821610000	686.8181818	3.520428334
46	1.195920625	585.8333333	3.356752000	1.989431875	717.8409091	4.394293525
47	1.303510000	611.6666667	3.991260000	2.168581429	749.5454545	5.458536488
48	1.418197054	638.0555556	4.728227346	2.359556875	781.9318182	6.749131226
49	1.540287142	665.0000000	5.581507444	2.562867143	815.0000000	8.307859153
50	1.670092054	692.5000000	6.566479350	2.779031875	848.7500000	10.183094125

Table XVII. Modified Orthogonal Polynomials.*

m	4			5				6					7				
X	P'_1	P'_2	P'_3	P'_1	P'_2	P'_3	P'_4	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	1	−1	−3	0	−2	0	6	1	−4	−4	2	10	0	−4	0	6	0
2	3	1	1	1	−1	−2	−4	3	−1	−7	−3	−5	1	−3	−1	1	5
3				2	2	1	1	5	5	5	1	1	2	0	−1	−7	−4
4													3	5	1	3	1
$\Sigma P'^2$	20	4	20	10	14	10	70	70	84	180	28	252	28	84	6	154	84
λ	2	1	$\frac{10}{3}$	1	1	$\frac{5}{6}$	$\frac{35}{12}$	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{21}{10}$	1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{20}$

m	8					9					10				
X	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	1	−5	−3	9	15	0	−20	0	18	0	1	−4	−12	18	6
2	3	−3	−7	−3	17	1	−17	−9	9	9	3	−3	−31	3	11
3	5	1	−5	−13	−23	2	−8	−13	−11	4	5	−1	−35	−17	1
4	7	7	7	7	7	3	7	−7	−21	−11	7	2	−14	−22	−14
5						4	28	14	14	4	9	6	42	18	6
$\Sigma P'^2$	168	168	264	616	2184	60	2,772	990	2,002	468	330	132	8,580	2,860	780
λ	2	1	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{7}{10}$	1	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{3}{20}$	2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{5}{12}$	$\frac{1}{10}$

m	11					12					13				
X	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	0	−10	0	6	0	1	−35	−7	28	20	0	−14	0	84	0
2	1	−9	−14	4	4	3	−29	−19	12	44	1	−13	−4	64	20
3	2	−6	−23	−1	4	5	−17	−25	−13	29	2	−10	−7	11	26
4	3	−1	−22	−6	−1	7	1	−21	−33	−21	3	−5	−8	−54	11
5	4	6	−6	−6	−6	9	25	−3	−27	−57	4	2	−6	−96	−18
6	5	15	30	6	3	11	55	33	33	33	5	11	0	−66	−33
											6	22	11	99	22
$\Sigma P'^2$	110	858	4,290	286	156	572	12,012	5,148	8,008	15,912	182	2,002	572	68,068	6,188
λ	1	1	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{40}$	2	3	$\frac{2}{3}$	$\frac{7}{24}$	$\frac{3}{20}$	1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{120}$

* Table XVII is abridged from Table XXIII of Fisher and Yates: *Statistical Tables for Biological, Agricultural, and Medical Research*, published by Oliver and Boyd Limited, Edinburgh, by permission of the authors and publishers.

Table XVII (*continued*). Modified Orthogonal Polynomials.

m	14					15				
X	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	1	−8	−24	108	60	0	−56	0	756	0
2	3	−7	−67	63	145	1	−53	−27	621	675
3	5	−5	−95	−13	139	2	−44	−49	251	1000
4	7	−2	−98	−92	28	3	−29	−61	−249	751
5	9	2	−66	−132	−132	4	−8	−58	−704	−44
6	11	7	11	−77	−187	5	19	−35	−869	−979
7	13	13	143	143	143	6	52	13	−429	−1144
8						7	91	91	1001	1001
$\Sigma P'^2$	910	728	97,240	136,136	235,144	280	37,128	39,780	6,466,460	10,581,480
λ	2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{7}{30}$	1	3	$\frac{5}{6}$	$\frac{35}{12}$	$\frac{21}{20}$

m	16					17				
X	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	1	−21	−63	189	45	0	−24	0	36	0
2	3	−19	−179	129	115	1	−23	−7	31	55
3	5	−15	−265	23	131	2	−20	−13	17	88
4	7	−9	−301	−101	77	3	−15	−17	−3	83
5	9	−1	−267	−201	−33	4	−8	−18	−24	36
6	11	9	−143	−221	−143	5	1	−15	−39	−39
7	13	21	91	−91	−143	6	12	−7	−39	−104
8	15	35	455	273	143	7	25	7	−13	−91
9						8	40	28	52	104
$\Sigma P'^2$	1,360	5,712	1,007,760	470,288	201,552	408	7,752	3,876	16,796	100,776
λ	2	1	$\frac{10}{3}$	$\frac{7}{12}$	$\frac{1}{10}$	1	1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$

Table XVII (*continued*). Modified Orthogonal Polynomials.

m	18					19				
X	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	1	−40	−8	44	220	0	−30	0	396	0
2	3	−37	−23	33	583	1	−29	−44	352	44
3	5	−31	−35	13	733	2	−26	−83	227	74
4	7	−22	−42	−12	588	3	−21	−112	42	79
5	9	−10	−42	−36	156	4	−14	−126	−168	54
6	11	5	−33	−51	−429	5	−5	−120	−354	3
7	13	23	−13	−47	−871	6	6	−89	−453	−58
8	15	44	20	−12	−676	7	19	−28	−388	−98
9	17	68	68	68	884	8	34	68	−68	−68
10						9	51	204	612	102
$\Sigma P'^2$	1,938	23,256	23,256	28,424	6,953,544	570	13,566	213,180	2,288,132	89,148
λ	2	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{3}{10}$	1	1	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{1}{40}$

m	20					21				
X	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	1	−33	−99	1188	396	0	−110	0	594	0
2	3	−31	−287	948	1076	1	−107	−54	540	1404
3	5	−27	−445	503	1441	2	−98	−103	385	2444
4	7	−21	−553	−77	1351	3	−83	−142	150	2819
5	9	−13	−591	−687	771	4	−62	−166	−130	2354
6	11	−3	−539	−1187	−187	5	−35	−170	−406	1063
7	13	9	−377	−1402	−1222	6	−2	−149	−615	−788
8	15	23	−85	−1122	−1802	7	37	−98	−680	−2618
9	17	39	357	−102	−1122	8	82	−12	−510	−3468
10	19	57	969	1938	1938	9	133	114	0	−1938
11						10	190	285	969	3876
$\Sigma P'^2$	2,660	17,556	4,903,140	22,881,320	31,201,800	770	201,894	432,630	5,720,330	121,687,020
λ	2	1	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{7}{20}$	1	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{21}{40}$

Table XVII (*continued*). Modified Orthogonal Polynomials.

m = 22, X	P_1'	P_2'	P_3'	P_4'	P_5'
1	1	−20	−12	702	390
2	3	−19	−35	585	1079
3	5	−17	−55	365	1509
4	7	−14	−70	70	1554
5	9	−10	−78	−258	1158
6	11	−5	−77	−563	363
7	13	1	−65	−775	−663
8	15	8	−40	−810	−1598
9	17	16	0	−570	−1938
10	19	25	57	57	−969
11	21	35	133	1197	2261
$\Sigma P'^2$	3,542	7,084	96,140	8,748,740	40,562,340
λ	2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{7}{12}$	$\frac{7}{30}$

m = 23, X	P_1'	P_2'	P_3'	P_4'	P_5'
1	0	−44	0	858	0
2	1	−43	−13	793	65
3	2	−40	−25	605	116
4	3	−35	−35	315	141
5	4	−28	−42	−42	132
6	5	−19	−45	−417	87
7	6	−8	−43	−747	12
8	7	5	−35	−955	−77
9	8	20	−20	−950	−152
10	9	37	3	−627	−171
11	10	56	35	133	−76
12	11	77	77	1463	209
$\Sigma P'^2$	1,012	35,420	32,890	13,123,110	340,860
λ	1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{1}{60}$

m = 24, X	P_1'	P_2'	P_3'	P_4'	P_5'
1	1	−143	−143	143	715
2	3	−137	−419	123	2005
3	5	−125	−665	85	2893
4	7	−107	−861	33	3171
5	9	−83	−987	−27	2721
6	11	−53	−1023	−87	1551
7	13	−17	−949	−137	−169
8	15	25	−745	−165	−2071
9	17	73	−391	−157	−3553
10	19	127	133	−97	−3743
11	21	187	847	33	−1463
12	23	253	1771	253	4807
$\Sigma P'^2$	4,600	394,680	17,760,600	394,680	177,928,920
λ	2	3	$\frac{10}{3}$	$\frac{1}{12}$	$\frac{3}{10}$

m = 25, X	P_1'	P_2'	P_3'	P_4'	P_5'
1	0	−52	0	858	0
2	1	−51	−77	803	275
3	2	−48	−149	643	500
4	3	−43	−211	393	631
5	4	−36	−258	78	636
6	5	−27	−285	−267	501
7	6	−16	−287	−597	236
8	7	−3	−259	−857	−119
9	8	12	−196	−982	−488
10	9	29	−93	−897	−753
11	10	48	55	−517	−748
12	11	69	253	253	−253
13	12	92	506	1518	1012
$\Sigma P'^2$	1,300	53,820	1,480,050	14,307,150	7,803,900
λ	1	1	$\frac{5}{6}$	$\frac{5}{12}$	$\frac{1}{20}$

Table XVII (*continued*). Modified Orthogonal Polynomials.

m	26					27				
X	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	1	−28	−84	1386	330	0	−182	0	1638	0
2	3	−27	−247	1221	935	1	−179	−18	1548	3960
3	5	−25	−395	905	1381	2	−170	−35	1285	7304
4	7	−22	−518	466	1582	3	−155	−50	870	9479
5	9	−18	−606	−54	1482	4	−134	−62	338	10058
6	11	−13	−649	−599	1067	5	−107	−70	−262	8803
7	13	−7	−637	−1099	377	6	−74	−73	−867	5728
8	15	0	−560	−1470	−482	7	−35	−70	−1400	1162
9	17	8	−408	−1614	−1326	8	10	−60	−1770	−4188
10	19	17	−171	−1419	−1881	9	61	−42	−1872	−9174
11	21	27	161	−759	−1771	10	118	−15	−1587	−12144
12	23	38	598	506	−506	11	181	22	−782	−10879
13	25	50	1150	2530	2530	12	250	70	690	−2530
14						13	325	130	2990	16445
$\Sigma P'^2$	5,850	16,380	7,803,900	40,060,020	48,384,180	1,638	712,530	101,790	56,448,210	2,032,135,560
λ	2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{1}{10}$	1	3	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{21}{40}$

Table XVII (*continued*). Modified Orthogonal Polynomials.

m	28					29				
X	P_1'	P_2'	P_3'	P_4'	P_5'	P_1'	P_2'	P_3'	P_4'	P_5'
1	1	−65	−39	936	1560	0	−70	0	2184	0
2	3	−63	−115	840	4456	1	−69	−104	2080	1768
3	5	−59	−185	655	6701	2	−66	−203	1775	3298
4	7	−53	−245	395	7931	3	−61	−292	1290	4373
5	9	−45	−291	81	7887	4	−54	−366	660	4818
6	11	−35	−319	−259	6457	5	−45	−420	−66	4521
7	13	−23	−325	−590	3718	6	−34	−449	−825	3454
8	15	−9	−305	−870	−22	7	−21	−448	−1540	1694
9	17	7	−255	−1050	−4182	8	−6	−412	−2120	−556
10	19	25	−171	−1074	−7866	9	11	−336	−2460	−2946
11	21	45	−49	−879	−9821	10	30	−215	−2441	−4958
12	23	67	115	−395	−8395	11	51	−44	−1930	−5885
13	25	91	325	455	−1495	12	74	182	−780	−4810
14	27	117	585	1755	13455	13	99	468	1170	−585
15						14	126	819	4095	8190
$\Sigma P'^2$	7,308	95,004	2,103,660	19,634,160	1,354,757,040	2,030	113,274	4,207,320	107,987,880	500,671,080
λ	2	1	$\frac{2}{3}$	$\frac{7}{24}$	$\frac{7}{20}$	1	1	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{7}{40}$

Table XVII (*continued*). **Modified Orthogonal Polynomials.**

m	30					31				
X	P_1'	P_2'	P_3'	P_4'	P_5'	P_1'	P_2'	P_3'	P_4'	P_5'
1	1	−112	−112	12376	1768	0	−80	0	408	0
2	3	−109	−331	11271	5083	1	−79	−119	391	221
3	5	−103	−535	9131	7753	2	−76	−233	341	416
4	7	−94	−714	6096	9408	3	−71	−337	261	561
5	9	−82	−858	2376	9768	4	−64	−426	156	636
6	11	−67	−957	−1749	8679	5	−55	−495	33	627
7	13	−49	−1001	−5929	6149	6	−44	−539	−99	528
8	15	−28	−980	−9744	2384	7	−31	−553	−229	343
9	17	−4	−884	−12704	−2176	8	−16	−532	−344	88
10	19	23	−703	−14249	−6821	9	1	−471	−429	−207
11	21	53	−427	−13749	−10535	10	20	−365	−467	−496
12	23	86	−46	−10504	−11960	11	41	−209	−439	−715
13	25	122	450	−3744	−9360	12	64	2	−324	−780
14	27	161	1071	7371	−585	13	89	273	−99	−585
15	29	203	1827	23751	16965	14	116	609	261	0
16						15	145	1015	783	1131
$\Sigma P'^2$	8,990	302,064	21,360,240	3,671,587,920	2,145,733,200	2,480	158,224	6,724,520	4,034,712	9,536,592
λ	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{35}{12}$	$\frac{3}{10}$	1	1	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{60}$

Table XVII (*continued*). Modified Orthogonal Polynomials.

m	32					33				
X	P'_1	P'_2	P'_3	P'_4	P'_5	P'_1	P'_2	P'_3	P'_4	P'_5
1	1	−85	−51	459	255	0	−272	0	3672	0
2	3	−83	−151	423	737	1	−269	−27	3537	2565
3	5	−79	−245	353	1137	2	−260	−53	3139	4864
4	7	−73	−329	253	1407	3	−245	−77	2499	6649
5	9	−65	−399	129	1509	4	−224	−98	1652	7708
6	11	−55	−451	−11	1419	5	−197	−115	647	7883
7	13	−43	−481	−157	1131	6	−164	−127	−453	7088
8	15	−29	−485	−297	661	7	−125	−133	−1571	5327
9	17	−13	−459	−417	51	8	−80	−132	−2616	2712
10	19	5	−399	−501	−627	9	−29	−123	−3483	−519
11	21	25	−301	−531	−1267	10	28	−105	−4053	−3984
12	23	47	−161	−487	−1725	11	91	−77	−4193	−7139
13	25	71	25	−347	−1815	12	160	−38	−3756	−9260
14	27	97	261	−87	−1305	13	235	13	−2581	−9425
15	29	125	551	319	87	14	316	77	−493	−6496
16	31	155	899	899	2697	15	403	155	2697	899
17						16	496	248	7192	14384
$\Sigma P'^2$	10,912	185,504	5,379,616	5,379,616	54,285,216	2,992	1,947,792	417,384	348,330,136	1,547,128,656
λ	2	1	$\frac{2}{3}$	$\frac{1}{12}$	$\frac{1}{30}$	1	3	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{3}{20}$

Table XVIII. "Even" Angles, $\frac{n}{12}\pi$.

n	RADIAN	DEGREES	SIN	COS	TAN	COT
0	0	0	0	1	0	∞
1	.261799	15	.2588190	.9659258	.2679492	3.7320508
2	.523599	30	.5000000	.8660254	.5773503	1.7320508
3	.785398	45	.7071068	.7071068	1	1
4	1.047198	60	.8660254	.5000000	1.7320508	.5773503
5	1.308997	75	.9659258	.2588190	3.7320508	.2679492
6	1.570796	90	1	0	∞	0
7	1.832596	105	.9659258	−.2588190	−3.7320508	−.2679492
8	2.094395	120	.8660254	−.5000000	−1.7320508	−.5773503
9	2.356194	135	.7071068	−.7071068	−1	−1
10	2.617994	150	.5000000	−.8660254	−.5773503	−1.7320508
11	2.879793	165	.2588190	−.9659258	−.2679492	−3.7320508
12	3.141593	180	0	−1	0	∞
13	3.403392	195	−.2588190	−.9659258	.2679492	3.7320508
14	3.665191	210	−.5000000	−.8660254	.5773503	1.7320508
15	3.926991	225	−.7071068	−.7071068	1	1
16	4.188790	240	−.8660254	−.5000000	1.7320508	.5773503
17	4.450590	255	−.9659258	−.2588190	3.7320508	.2679492
18	4.712389	270	−1	0	∞	0
19	4.974188	285	−.9659258	.2588190	−3.7320508	−.2679492
20	5.235988	300	−.8660254	.5000000	−1.7320508	−.5773503
21	5.497787	315	−.7071068	.7071068	−1	−1
22	5.759587	330	−.5000000	.8660254	−.5773503	−1.7320508
23	6.021386	345	−.2588190	.9659258	−.2679492	−3.7320508
24	6.283185	360	0	1	0	∞

Table XIX. Constants.

TERM	VALUE	RECIPROCAL
$\frac{\pi}{2}$	1.57079 63268	0.63661 97724
π	3.14159 26536	0.31830 98862
2π	6.28318 53072	0.15915 49431
$(\pi/2)^2$	2.46740 11003	0.40528 47346
π^2	9.86960 44011	0.10132 11836
$(2\pi)^2$	39.47841 76044	0.02533 02959
$\sqrt{\pi/2}$	1.25331 41373	0.79788 45608
$\sqrt{\pi}$	1.77245 38509	0.56418 95835
$\sqrt{2\pi}$	2.50662 82746	0.39894 22804
e	2.71828 18285	0.36787 94412
e^2	7.38905 60989	0.13533 52832
$\sqrt{e}$	1.64872 12707	0.60653 06597
$\log_{10} e$	0.43429 44819	2.30258 50930

1 radian = 57.29577 95131 degrees
1 degree = 0.01745 32925 radians

Bernoulli Numbers

$B_1 = \frac{1}{6}$	$B_6 = \frac{691}{2730}$
$B_2 = \frac{1}{30}$	$B_7 = \frac{7}{6}$
$B_3 = \frac{1}{42}$	$B_8 = \frac{3617}{510}$
$B_4 = \frac{1}{30}$	$B_9 = \frac{43867}{798}$
$B_5 = \frac{5}{66}$	$B_{10} = \frac{174611}{330}$

Euler Numbers

$E_1 = 1$	$E_4 = 1385$
$E_2 = 5$	$E_5 = 50521$
$E_3 = 61$	$E_6 = 2702765$

ANSWERS TO THE EXERCISES

Exercise I.

1. 38.462, 2.3742, .0023714, .70003.
2. 30.437.
3. 8.432.
4. 62, 419, 183.5625, 1.2620.
5. $P^{(5)}(x) + 12P^{(4)}(x) + 30P^{(3)}(x) + 16P^{(2)}(x) + 3P^{(1)}(x) + 4P^{(0)}(x)$.
9. .8432.
10. $R_{10}(.5) < 1.29 \times 10^{-10}$.

Exercise II.

I.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	1.0000				
		5191			
1.1	1.5191		354		
		5545		−24	
1.2	2.0736		330		24
		5875		0	
1.3	2.6611		330		24
		6205		24	
1.4	3.2816		354		24
		6559		48	
1.5	3.9375		402		24
		6961		72	
1.6	4.6336		474		24
		7435		96	
1.7	5.3771		570		
		8005			
1.8	6.1776				

2. (a) $y(1.7) = -1.06743$; (b) $y(5) = 77$.

3. (a)

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
y	−5.000	−4.928	−4.904	−4.916	−4.952	−5.000	−5.048	−5.084	−5.096	−5.072	−5.000

(b)

x	0	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
y	−.3000	−.2998	−.2991	−.2980	−.2964	−.2942	−.2915	−.2881	−.2842	−.2796	−.2743

4.

x \ y	0		1		2		3
0	0	−3	−3	−9	−12	−15	−27
	2		3		4		5
1	2	−2	0	−8	−8	−14	−22
	6		7		8		9
2	8	−1	7	−7	0	−13	−13
	10		11		12		13
3	18	0	18	−6	12	−12	0
	14		15		16		17
4	32	1	33	−5	28	−11	17

5.

x	y	$\Delta[x_ix_j]$	$\Delta^2[x_ix_k]$	$\Delta^3[x_ix_k]$	$\Delta^4[x_ix_k]$	$\Delta^5[x_ix_k]$
−4	−4320					
		2040				
−2	−240		−480			
		120		64		
0	0		−32		−8	
		−40		−8		1
−3	−120		−88		4	
		−480		32		
5	−1080		168			
		360				
8	0					

9.

x	$\Delta[x_ix_j]$	$\Delta^2[x_ix_j]$	$\Delta^3[x_ix_j]$	$\Delta^4[x_ix_j]$
0	1.00000			
		−.000762		
			−.0001514	
5	.99619			
				.0000001
		−.002276		
			−.0001500	
10	.98481			
				.0000002
		−.003776		
			−.0001472	
15	.96593			
				.0000003
		−.005248		
			−.0001428	
20	.93969			
				.0000003
		−.006676		
			−.0001380	
25	.90631			
		−.008056		
30	.86603			

10.

p	A_0	ΔA_0	$\Delta^2 A_0$	$\Delta^3 A_0$
0	1.0000000			
		−.0001250		
.01	.9998750		−.0002500	
		−.0003750		.0000002
.02	.9995000		−.0002498	
		−.0006248		0
.03	.9988752		−.0002498	
		−.0008746		.0000004
.04	.9980006		−.0002494	
		−.0011240		0
.05	.9968766		−.0002494	
		−.0013734		.0000006
.06	.9955032		−.0002488	
		−.0016222		.0000002
.07	.9938810		−.0002486	
		−.0018708		.0000006
.08	.9920102		−.0002480	
		−.0021188		.0000004
.09	.9898914		−.0002476	
		−.0023664		
.10	.9875250			

Exercise III.

1. 18449.9, 24782.4, 25761.8, 27640.2.
2. 18469.0, 27645.8.
3. (a) .988845, .972405, .888873, .912580, .939931.
   (b) 1.509643, 50.784809, −2.420785, −5.203276, −5.407975.
5. .91725.

Exercise IV.

1. (a) −4.6449, −2.8969, −1.3831.
   (b) 71.641006, 153.714717.
2. 1.747022.
3. 1.2068.
4. .167, .235.

Exercise V.

1. −.5771, −.3306, .0324.
2. −12.373.

Exercise VI.

1. (a) 8.638566; (c) 8.637235.
2. 242.7165.
3. .499986.

Exercise VII.

1. .888874, .912581.
2. -2.422880, -5.212814, -5.424428.
3. -3.8125.
4. 1.231 (with $p = -.62$); 1.225 (with $p = -.63$).
5. -2.9212 (omit value at $x = 2.0$); 153.71470.
6. $y'(0) = -.57711$, $y'(.5) = .0324$.
7. -12.373.
8. (a) 8.637301.

Exercise VIII.

1. 1.380278.
2. .142857 (double root).
3. 1.5236.
4. .50382; .24698.
5. 1.177115.
6. No real positive root.
7. .212139; .256861.
8. (1.444, .155).
9. ($-.46558$, 1.04132, 2.18812, -1.03563).

Exercise IX.

1. (a) 1.914751, .001817; are there more solutions?
   (c) 1.285714286, 1.571428571.
2. (b) 1, 1.414214, -1.414214; $.5 \pm .8660254i$.

Exercise X.

1. (a) True Solution.

x	1.0	1.1	1.2	1.3	1.4	1.5
y	-3.0	-2.47934	-2.08333	-1.77515	-1.53061	-1.33333

(b) Runge-Kutta Method.

x	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
y	.99534	.98279	.96445	.94224	.91796	.89301	.86858	.84565	.82508	.80769

2. True value at $t = 1.0$ is $x = 20.22087$.
3. True value at $t = 0.8$ is $x = 11.22507$.

Exercise XI.

1. $y = 3.9429 - 3.0422x + 2.0180x^2$.

2. By the Nielsen-Goldstein method

$$y = -.003788x^4 + .034383x^3 - 9.731655x^2 + 97.659134x + 2.580355.$$
$$y = -.041375x^3 - 9.258163x^2 + 96.712139x + 2.853135.$$
$$y = -9.878788x^2 + 99.078789x + 1.363636.$$

3. The Nielsen-Goldstein method yields

$$y = -.003041x^3 + .019176x^2 + .008818x + .132066.$$

Exercise XII.

1.

i	0	1	2	3	4	5	6
a_i	96.17	161.30	24.17	32.5	−16.67	30.65	−7.67
b_i		−139.63	19.63	−32.5	−6.06	−8.87	

4. (a) $y = 1.6691(1.2733)^x$.
5. $y = 3.2x + 1.5 \ln 2x$.
6. $y = 1.02636x + .34212$.
7. $y = 1.75x^2 - .37x + .62$.

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