TRIGONOMETRY

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PREFACE

There is a dearth of good text books on Trigonometry for Degree classes. This book has been written to meet the requirements of students of Degree classes of most of the Indian Universities. For such students the treatment is adequately rigorous. Exercises for students have been graded and selected from question papers of different Universities.

The author will be grateful for any suggestions for improvement.

Author

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CHAPTER I

Inverse Trigonometrical Functions

1·1. The notation $\sin^{-1}x$, (read sine inverse x), represents an angle whose sine is x. Thus if $\sin\theta=x$, then $\theta=\sin^{-1}x$. Similarly we may define $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, and $\cot^{-1}x$. These functions are called inverse trigonometric (or circular) functions, and, it should be noted, represent angles.

1·11. Evidently these functions are multiple-valued; for infinite values of θ satisfy the equation $\sin\theta=x$, and if α is the smallest positive value of θ, then the general value is given by

$$\sin^{-1}x=\theta=n\pi+(-1)^n\alpha.$$

However the convention is that $\sin^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$ and $\operatorname{cosec}^{-1}x$ represent angles lying between $-\pi/2$ and $+\pi/2$; and $\cos^{-1}x$ and $\sec^{-1}x$ are angles lying between 0 and π.

Thus,

$$\sin^{-1}\frac{1}{2}=\frac{\pi}{6},\ \tan^{-1}(-1)=-\frac{\pi}{4},\quad \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}.$$

1·2. Relations between inverse functions.

If $\quad \sin\theta=x$, $\operatorname{cosec}\theta=1/x$

Hence $\quad \theta=\sin^{-1}x=\operatorname{cosec}^{-1}1/x$

Similarly $\tan^{-1}x=\cot^{-1}1/x$ and $\cos^{-1}x=\sec^{-1}1/x$.

Also $\quad \sin(\sin^{-1}x)=\sin\theta=x$, similarly $\cos(\cos^{-1}x)=x$ and so on.

Again, if $\sin\theta=x$, $\cos\theta=\sqrt{1-x^2}$, $\tan\theta=\dfrac{x}{\sqrt{1-x^2}}$

$$\therefore\quad \theta=\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

1·3. Some Formulae.

(a) Let $\sin\theta = x$, then $x = \cos(\pi/2 - \theta)$

Hence $\theta = \sin^{-1}x$, and $\pi/2 - \theta = \cos^{-1}x$

and $\sin^{-1}x + \cos^{-1}x = \pi/2$

Similarly $\tan^{-1}x + \cot^{-1}x = \pi/2$ and $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$.

(b) Let $\tan^{-1}x = \theta$, $\tan^{-1}y = \phi$,

So that $\tan\theta = x$, $\tan\phi = y$.

We have, $\tan(\theta+\phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \dfrac{x+y}{1-xy}$

Hence $\theta + \phi = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$*.

Similarly $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$

Putting $x = y$, in $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$,

we have $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$

This formula can easily be extended to

*It should be noted that this formula is true when $xy < 1$. If $xy > 1$ it is not tenable. As an illustration, let $x = \sqrt{3}$, $y = 1$.

Hence $\tan^{-1}x + \tan^{-1}y = \dfrac{\pi}{3} + \dfrac{\pi}{4} = \dfrac{7\pi}{12}$

But $\tan^{-1}\dfrac{x+y}{1-xy} = \tan^{-1}\dfrac{1+\sqrt{3}}{1-\sqrt{3}} = \dfrac{-5\pi}{12}$,

since $\tan^{-1}\dfrac{x+y}{1-xy}$ represents, conventionally, an angle in the range $(-\pi/2, \pi/2)$.

If, however, $\tan^{-1}\dfrac{x+y}{1-xy}$, represents an angle in the range $(0, \pi)$ the formula holds true.

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx}.$$

whence, by putting $x=y=z$, we have

$$3\tan^{-1}x=\tan^{-1}\frac{3x-x^3}{1-3x^2}.$$

(c) We know $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

Putting $\sin\theta=x$, we have

$$3\theta=3\sin^{-1}x=\sin^{-1}(3x-4x^3)$$

Similarly, $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$.

Examples

1. Show that $\sin^{-1}a+\sin^{-1}b=\sin^{-1}(a\sqrt{1-b^2}+b\sqrt{1-a^2})$

Let $\sin^{-1}a=x$, $\sin^{-1}b=y$; hence $a=\sin x$, $b=\sin y$

$\therefore\ \sin(x+y)=\sin x\cos y+\cos x\sin y=a\sqrt{1-b^2}+b\sqrt{1-a^2}$

Hence $x+y=\sin^{-1}a+\sin^{-1}b=\sin^{-1}(a\sqrt{1-b^2}+b\sqrt{1-a^2})$

2. Prove that

$$2\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\}=\cos^{-1}\frac{b+a\cos x}{a+b\cos x}.$$

Let $2\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\}=\theta$,

or $$\tan\frac{\theta}{2}=\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}$$

Now $$\cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{1-\frac{a-b}{a+b}\tan^2\frac{x}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}$$

$$=\frac{(a+b)\cos^2 x/2-(a-b)\sin^2 x/2}{(a+b)\cos^2 x/2+(a-b)\sin^2 x/2}$$

$$=\frac{b+a\cos x}{a+b\cos x}$$

$\therefore \quad \theta=2\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\}=\cos^{-1}\frac{b+a\cos x}{a+b\cos x}.$

3. Solve $3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}$. (Agra '43)

$$3\tan^{-1}\frac{1}{2+\sqrt{3}}=3\tan^{-1}(2-\sqrt{3})$$

$$=\tan^{-1}\frac{3(2-\sqrt{3})-(2-\sqrt{3})^3}{1-3(2-\sqrt{3})^2},$$

by using the formula $3\tan^{-1}x=\tan^{-1}\frac{3x-x^3}{1-3x^2}$.

Hence $3\tan^{-1}\frac{1}{2+\sqrt{3}}=\tan^{-1}\frac{(2-\sqrt{3})(3-7+4\sqrt{3})}{1-3(7-4\sqrt{3})}$

$$=\tan^{-1}\frac{(\sqrt{3}-1)(2-\sqrt{3})}{3\sqrt{3}-5}=\tan^{-1}\frac{3\sqrt{3}-5}{3\sqrt{3}-5}=\tan^{-1}1$$

The equation becomes

$$\tan^{-1}1/x=\tan^{-1}1-\tan^{-1}\tfrac{1}{3}=\tan^{-1}\frac{1-\frac{1}{3}}{1+\frac{1}{3}}=\tan^{-1}\tfrac{1}{2}$$

$\therefore \quad x=2.$

4. If $\cos^{-1}x/a+\cos^{-1}y/b=\alpha$, prove that

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$ (Cal. '43)

Let $\cos^{-1}x/a=\theta$, $\cos^{-1}y/b=\phi$, so that $\cos\theta=x/a$, $\cos\phi=y/b$, and $\theta+\phi=\alpha$.

Hence $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}$

$$=\cos^2\theta-2\cos\theta\cos\phi\cos(\theta+\phi)+\cos^2\phi$$

$$=\cos^2\theta-2\cos\theta\cos\phi(\cos\theta\cos\phi-\sin\theta\sin\phi)+\cos^2\phi$$

$$=\cos^2\theta(1-\cos^2\phi)+\cos^2\phi(1-\cos^2\theta)+2\cos\theta\cos\phi\sin\theta\sin\phi$$

$$=\cos^2\theta\sin^2\phi+\sin^2\theta\cos^2\phi+2\cos\theta\cos\phi\sin\theta\sin\phi$$

$$=\sin^2(\theta+\phi)=\sin^2\alpha.$$

Exercises I

1. Show that

(i) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

(ii) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$

(iii) $2\tan^{-1}x=\cos^{-1}\dfrac{1-x^2}{1+x^2}=\sin^{-1}\dfrac{2x}{1+x^2}$

(iv) $\cot^{-1}x+\cot^{-1}y=\cot^{-1}\dfrac{xy-1}{x+y}$.

2. Prove that

(i) $\tan^{-1}\frac{1}{3}+\frac{1}{2}\tan^{-1}\frac{1}{7}=\pi/8$

(ii) $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\pi/4$

(iii) $\sin^{-1}\dfrac{1}{\sqrt{5}}+\cot^{-1}3=\pi/4$

(iv) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\pi/4$ (Agra '48)

(v) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\pi/4$

3. Show that

(i) $\tan(2\tan^{-1}a)=2\tan(\tan^{-1}a+\tan^{-1}a^3)$ (Cal. '44)

(ii) $\frac{1}{2}\tan^{-1}x=\cos^{-1}\left(\dfrac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right)^{\frac{1}{2}}$ (Agra '34)

(iii) $\dfrac{2b}{a}=\tan\left(\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)$ (Cal. '48)

(iv) $\tan^{-1}\dfrac{2a-b}{b\sqrt{3}}+\tan^{-1}\dfrac{2b-a}{a\sqrt{3}}=\dfrac{\pi}{3}$

(v) $\tan^{-1}a=\tan^{-1}\dfrac{a-b}{1+ab}+\tan^{-1}\dfrac{b-c}{1+bc}+\tan^{-1}c$

(vi) $\tan^{-1}(\frac{1}{2}\tan 2\theta)+\tan^{-1}(\cot\theta)+\tan^{-1}(\cot^3\theta)=0$

(vii) $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\pi/2$

(viii) $\tan^{-1}x=2\tan^{-1}[\text{cosec}(\cot^{-1}1/x)-\tan(\cot^{-1}x)]$.

Solve the equations :

4. $\tan^{-1}\dfrac{1}{2x+3}+\tan^{-1}\dfrac{1}{3x+4}=\tan^{-1}\dfrac{2}{x+1}$.

5. $\tan^{-1}x+\tan^{-1}(x+1)=\cot^{-1}\{\frac{1}{4}(1-x)\}$.

6. $\tan^{-1}\frac{1}{4}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}1/x=\pi/4$. (Agra '42)

7. $\tan^{-1}\dfrac{1}{2x+1}+\tan^{-1}\dfrac{1}{4x+1}=\tan^{-1}\dfrac{2}{x^2}$. (Agra '47)

8. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}(3x)$. (Gauhati '53)

9. $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}]=0$. (Agra '58)

10. $\frac{1}{2}\cot^{-1}\left(\dfrac{1-x^2}{2x}\right)+\frac{1}{3}\cot^{-1}\left(\dfrac{1-3x^2}{3x-x^3}\right)$

$=\pi+2\tan^{-1}(1-x-x^2)$. (Cal. '44)

11. $\sin^{-1}\dfrac{2a}{1+a^2}-\cos^{-1}\dfrac{1-b^2}{1+b^2}=\tan^{-1}\dfrac{2x}{1-x^2}$.

12. $\cot^{-1}x+\cot^{-1}(a^2-x+1)=\cot^{-1}(a-1)$.

13. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that
$x^2+y^2+z^2+2xyz=1$. (J & k '53)

14. Show that

(i) $\tan^{-1}\dfrac{a(b-c)}{1+a^2bc}+\tan^{-1}\dfrac{b(c-a)}{1+b^2ca}+\tan^{-1}\dfrac{c(a-b)}{1+c^2ab}=n\pi$

n being an integer or zero. (Andhra '36)

(ii) $\tan(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z)=$
$\cot(\cot^{-1}x+\cot^{-1}y+\cot^{-1}z)$. (Nagpur '35)

15. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, prove that
$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$. (Delhi '46)

16. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, prove $\theta=\pm\frac{1}{2}\sin^{-1}\frac{3}{4}$.

17. If $\phi=\tan^{-1}\dfrac{x\sqrt{3}}{2k-x}$, and $\theta=\tan^{-1}\dfrac{2x-k}{k\sqrt{3}}$, show that one value of $\phi-\theta$ is $\pi/6$.

18. If $\tan^{-1} ax + \frac{1}{2}\sec^{-1} bx = \pi/4$, then one solution is

$$x^2 = (2ab - a^2)^{-1}.$$

19. If $\sin^{-1} \dfrac{x}{a} + \sin^{-1} \dfrac{y}{b} = \sin^{-1} \dfrac{c^2}{ab}$, then prove that

$$b^2x^2 + 2xy\,(a^2b^2 - c^4)^{\frac{1}{2}} + a^2y^2 = c^4.$$

20. Show that

$$\tan^{-1} \frac{x}{y} = \tan^{-1} \frac{c_1x - y}{c_1y + x} + \tan^{-1} \frac{c_2 - c_1}{c_2c_1 + 1} + \tan^{-1} \frac{c_3 - c_2}{c_3c_2 + 1} + \ldots$$

$$+ \tan^{-1} \frac{c_n - c_{n-1}}{c_nc_{n-1} + 1} + \tan^{-1} \frac{1}{c_n}.$$

where $c_1, c_2 \ldots\ldots c_n$ are any quantities whatever. (Alld. '28)

CHAPTER II

Complex Numbers

2·1. Definitions. If x and y are real numbers, and $i=\sqrt{-1}$, or $i^2=-1$, then the number $z=x+iy$ is called a complex number. The student has already come across such numbers while solving a quadratic equation. If $b^2<4ac$, then the roots of the equation $ax^2+bx+c=0$ are imaginary or complex. *

If $z=x+iy$, x is said to be the real part of z and expressed as $R(z)$; y is the imaginary part of z and expressed as $I(z)$. It will be assumed that the number i follows the laws of Algebra.

If $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ are two complex numbers, the following arithmetical operations on them will be assumed.

1. Equality : If $z_1=z_2$ or $x_1+iy_1=x_2+iy_2$

   then $x_1=x_2,\ y_1=y_2.$

2. Addition :

$$z_1+z_2=(x+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2).$$

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*The student should note that $i=\sqrt{-1}$ is *defined* to be a number such that $i^2=(\sqrt{-1}).(\sqrt{-1})=-1$, and must guard against the following fallacy :

$i^2=(\sqrt{-1})^2=\sqrt{(-1)(-1)}=\sqrt{1}$, and hence $i=\sqrt{+1}=\pm 1$.

Since $i^2=-1$, different integral powers of i can be reduced to $\pm i$ or ± 1.

e. g : $i^7=(i^2)^3.\ i=(-1)^3.\ i=-i$, $i^3=i^2.\ i=-i$, $i^4=(i^2)^2=1$

$i^{13}=(i^4)^3.\ i=i$, $i^{26}=(i^4)^6.\ i^2=-1.$

3. Subtraction :

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2).$$

4. Multiplication :

$$z_1 . z_2 = (x_1 + iy_1) . (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

5. Division :

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 x_2 + y_1 y_2) + i\ (x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2},$$

where x_2, y_2 are not simultaneously zero.

The last case can be deduced from (1) and (3).

Let $$\frac{x_1 + iy_1}{x_2 + iy_2} = \alpha + i\beta,$$

or $$x_1 + i\ y_1 = (\alpha + i\beta)\ (x_2 + iy_2)$$

or $$(x_1 + iy_1)\ (x_2 - iy_2) = (x_2^2 + y_2^2)\ (\alpha + i\beta)$$

Equating the real and imaginary parts from both sides α and β are obtained.

Squares of all real numbers are positive, but $i^2 = -1$, hence the complex number $x + iy$ is a new type of number, other than real.

2·2. Standard form of a complex number.

Let $x = r \cos \theta$, $y = r \sin \theta$.

Hence $x + iy = r\ (\cos \theta + i \sin \theta)$

Equating the real and imaginary parts

$$x = r \cos \theta, \quad \ldots(1)$$

and $$y = r \sin \theta. \quad \ldots(2)$$

Squaring and adding, $r^2 = x^2 + y^2$,

and $$r = +\sqrt{x^2 + y^2}. \quad \ldots(3)$$

By convention r is always taken positive.

From (1) and (2), we have

$$\cos\theta=\frac{x}{\sqrt{x^2+y^2}}, \text{ and } \sin\theta=\frac{y}{\sqrt{x^2+y^2}}, \qquad ...(4)^*$$

In the interval $-\pi<\theta\leqslant\pi$ there is always one and only one value of θ satisfying equations (4).

Thus r and θ are determined, so that $x+iy=z$ can always be expressed in the standard form

$$z=x+iy=r\,(\cos\theta+i\sin\theta).$$

The number $r\ (=+\sqrt{x^2+y^2})$ is said to be the **modulus** of z, and the angle θ is called the **argument** or **amplitude** of z. They are expressed in the following manner

$r=$mod. z or $r=|z|$, read as mod. z.

and $\theta=$arg. z.

Evidently infinite values of θ satisfy (4), but the value of θ lying in the interval $-\pi<\theta\leqslant\pi$ is called the **principal value** of the argument or simply the argument. The general value of the argument, clearly, is $\theta+2n\pi$, where n is any integer.

The standard form $r\ (\cos\theta+i\sin\theta)$ of a complex number is sometimes called the modulus-amplitude form.

The numbers $x+iy$ and $x-iy$ are said to be **conjugate** with respect to each other. Evidently each has the same modulus, and their product is the square of the modulus.

Examples

1. Express $\dfrac{2+3i}{3+4i}$ in the form $x+iy$.

$$\frac{2+3i}{3+4i}=\frac{(2+3i)(3-4i)}{(3+4i)(3-4i)}=\frac{6+12+i(9-8)}{3^2+4^2}$$

$$=\tfrac{18}{25}+i.\ \tfrac{1}{25}.$$

*θ in sometimes written as $\tan^{-1} y/x$, provided it satisfies equation (4).

2. Express $1-\sqrt{3}\,i$ in the form $r(\cos\theta+i\sin\theta)$

Let $1-\sqrt{3}\,i=r(\cos\theta+i\sin\theta)$

Hence $r\cos\theta=1,\ r\sin\theta=-\sqrt{3}$

$\therefore$ $r^2=1+3=4$ or $r=2$

Also $\cos\theta=\frac{1}{2}$ and $\sin\theta=-\frac{\sqrt{3}}{2}$

Hence $\theta=-\pi/3$

$\therefore$ $1-\sqrt{3}\,i=2\{\cos(-\pi/3)+i\sin(-\pi/3)\}$.

2·3. Argand Diagram.

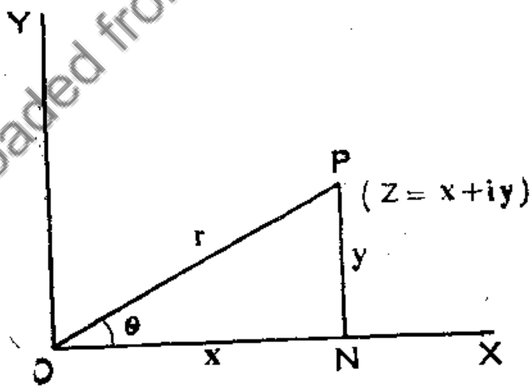
The complex number $z=x+iy$ can be uniquely represented by the point $P,(x,y)$ with reference to axis of coordinates OX and OY in a plane. All points on OX represent real numbers, and OX is called the real axis; all purely imaginary points, iy, lie on OY which is called the **imaginary** axis.

It is clear from the figure that

$OP=r=|z|$,

and $\angle PON=\theta=\arg. z.$

$=$angle OP makes with the positive direction of OX.



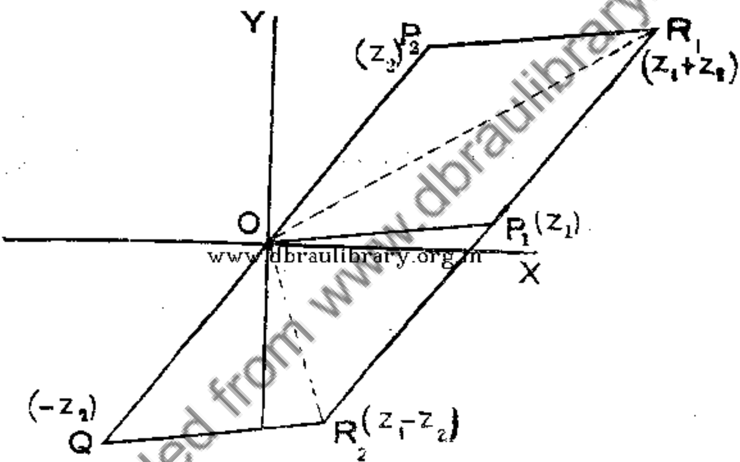
This manner of geometrical representation of a complex number is called Argand Diagram.

Evidently all real numbers have their arguments as 0 or π, and all purely imaginary numbers have their arguments as $\pm\pi/2$.

2·4. Geometrical representation of $z_1 \pm z_2$.

Let P_1 and P_2 represent the numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ respectively. Complete the parallelogram $OP_1R_1P_2$. The coordinates of R_1 are evidently $x_1 + x_2$ and $y_1 + y_2$ and hence it represents the number $x_1 + x_2 + i(y_1 + y_2) = z_1 + z_2$.

Hence $OR_1 = |z_1 + z_2|$, and $\angle R_1OX = \arg.(z_1 + z_2)$.



Produce P_2O to Q, making $OQ = OP_2$ and complete the parallelogram OQR_2P_1. Evidently Q is the point $(-x_2, -y_2)$ or $-z_2$, and hence R_2 represents $z_1 - z_2$, so that

$$OR_2 = |z_1 - z_2|.$$

Since $\quad OR_1 \leqslant OP_1 + P_1R_1$

we have $\quad |z_1 + z_2| \leqslant |z_1| + |z_2|$.

Thus the modulus of the sum of two complex quantities is less than or equal to the sum of their moduli. This result can easily be extended to more than two complex quantities.

2·5. Product of two complex quantities.

Let $\quad z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$,

$$z_2=x_2+iy_2=r_2(\cos\theta_2+i\sin\theta_2).$$

Hence $\quad z_1z_2=r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)$

$$=r_1r_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}$$

It follows that $\quad |z_1z_2|=r_1r_2=|z_1|.|z_2|$

and arg. $(z_1z_2)=\theta_1+\theta_2=$arg. $(z_1)+$arg. (z_2).

Evidently this result is true for the product of any number of complex quantities.

We may, therefore, conclude that

(1) *the modulus of the product any number of complex quantities is the product of their moduli, and*

(2) *the argument of the product of any number of complex quantities is the sum of their arguments.*

2·6. Quotient of two complex quantities.

Let $\quad z_1=x_1+iy_1=r_1(\cos\theta_1+i\sin\theta_1)$,

$$z_2=x_2+iy_2=r_2(\cos\theta_2+i\sin\theta_2).$$

Then $\quad \dfrac{z_1}{z_2}=\dfrac{r_1}{r_2}\cdot\dfrac{\cos\theta_1+i\sin\theta_1}{\cos\theta_2+i\sin\theta_2}$

$$=\frac{r_1}{r_2}\{\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\}.$$

Hence $\quad \left|\dfrac{z_1}{z_2}\right|=\dfrac{r_1}{r_2}=\dfrac{|z_1|}{|z_2|}$,

and $\quad$ arg. $\left(\dfrac{z_1}{z_2}\right)=\theta_1-\theta_2=$arg. z_1-arg. z_2.

It follows, therefore, that

(1) *the modulus of the quotient of two complex quantities is the quotient of their moduli, and*

(2) *the argument of the quotient of two complex quantities is the difference of their arguments.*

2·7. Geometrical representation of product and quotient of two complex quantities.

Let P and Q represent two complex quantities z_1 and z_2, where

$$z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1),$$
$$z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2),$$

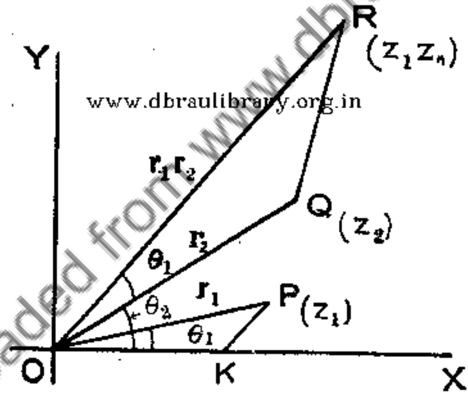
so that $\angle POX = \theta_1$, and $\angle QOX = \theta_2$.

Let K be a point on OX, so that

$$OK = 1 \quad \text{or} \quad 1 + 0.\, i$$

Construct the triangle OQR, so that

$$\angle QOR = \theta_1 \text{ and } \angle OQR = \angle OKP,$$



R and P being on opposite sides of OQ.

Evidently triangles OKP, OQR are similar,

Hence $\qquad \dfrac{OR}{OQ} = \dfrac{OP}{OK}$

or $\qquad OR = \dfrac{OP.OQ}{OK} = OP.OQ$

Also $\qquad \angle ROX = \angle QOX + \angle ROQ = \theta_1 + \theta_2$

Therefore, $\quad OR = r_1 r_2 = |z_1 z_2|$,

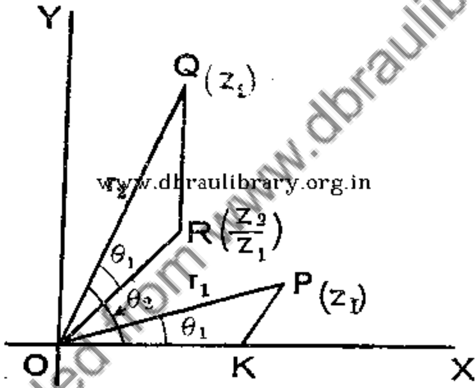
Hence R represents the number $z_1 z_2$.

If, however, triangle OQR is constructed so that R and P are on the same side of OQ, we have from the similar triangles OKP and ORQ,

$$\frac{OQ}{OR} = \frac{OP}{OK} \quad \text{or} \quad OR = \frac{OQ.OK}{OP}$$

Also $\quad \angle ROX = \angle QOX - \angle QOR$

$$= \theta_2 - \theta_1.$$



Hence $\quad OR = \frac{r_2}{r_1} = \left| \frac{z_2}{z_1} \right|$

Thus the point R represents the number $\frac{z_2}{z_1}$.

Examples

1. If $z = x + iy$ be a variable number, so that $|z - 1| = 2$, find its locus on the Argand Diagram.

Given $\quad |z-1| = 2$ or $|x + iy - 1| = 2$

or $\quad |(x-1) + iy|^2 = 4$

$\therefore \quad (x-1)^2 + y^2 = 4$, which is a circle.

2. Show that the points representing the numbers

$$1+i,\ -2+3\,i,\ 5i/3 \text{ are collinear.}$$

The points on the Argand Diagram are $(1, 1)$, $(-2, 3)$ and $(0, 5/3)$. Equation to the line joining the first two points is

$$y-1=\frac{3-1}{-2-1}(x-1)=-\tfrac{2}{3}(x-1)$$

or $$2x+3y-5=0.$$

Evidently the third point $(0, 5/3)$ lies on it.

3. A variable complex number $z=x+iy$ is such that the amplitude of the fraction $\frac{z-1}{z+1}$ is always equal to $\frac{\pi}{4}$. Show that $$x^2+y^2-2y=1.$$

Now $$\frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}=\frac{\{(x-1)+iy\}\,\{(x+1)-iy\}}{\{(x+1)+iy\}\,\{(x+1)-iy\}}$$

$$=\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$$

Hence $$\arg.\frac{z-1}{z+1}=\tan^{-1}\frac{2y}{x^2+y^2-1}=\frac{\pi}{4}$$

$$\therefore\quad \frac{2y}{x^2+y^2-1}=\tan\frac{\pi}{4}=1$$

or $$x^2+y^2-2y=1.$$

2·8. An important theorem :

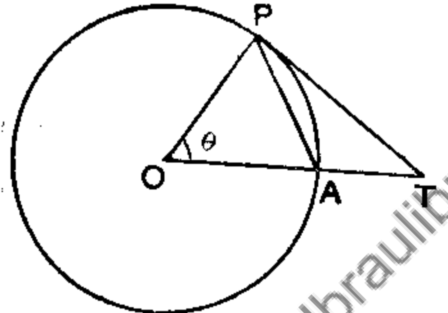
If θ tends to zero, then, in the limit, $\frac{\sin\theta}{\theta}=\frac{\tan\theta}{\theta}=1$.

Let θ be the circular measure of the angle POA, PT is the tangent at P to the circle. If $\theta<\pi/2$, triangle $OPA<$ sector $OAP<$ triangle OPT

or $\frac{1}{2}OA^2\ \sin\theta<\frac{1}{2}\,OA^2.\,\theta<\frac{1}{2}\,OA^2.\tan\theta$

or $\quad \sin\theta < \theta < \tan\theta$

or $\quad 1 < \dfrac{\theta}{\sin\theta} < \sec\theta$



As θ is indefinitely diminished to 0, $\sec\theta = 1$, and in the limit $\dfrac{\theta}{\sin\theta} = 1$ or $\dfrac{\sin\theta}{\theta} = 1$

Also $\dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta} \cdot \sec\theta = 1$, in the limit when $\theta \to 0$.

2·9. Relations between the roots of an equation and its coefficients.

It is known that the general equation of the n^{th} degree

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n = 0$$

has n roots. If the roots are $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$, then $(x - \alpha_1)$, $(x - \alpha_2) \ldots (x - \alpha_n)$ are the factors of the polynomial on the left hand side, and we can write

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n$$
$$\equiv a_0 (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$$
$$= a_0 [x^n - x^{n-1} \Sigma\alpha_1 + x^{n-2} \Sigma\alpha_1\alpha_2 - x^{n-3} \Sigma\alpha_1\alpha_2\alpha_3 + \ldots\ldots + (-1)^n \alpha_1\alpha_2 \ldots \alpha_n],$$

by multiplying all the factors.

$\Sigma\alpha_1$, $\Sigma\alpha_1\alpha_2$, $\Sigma\alpha_1\alpha_2\alpha_3 \ldots\ldots$ represent the sum of the roots taken once, twice, thrice...at a time respectively.

Equating the coefficients of different powers of x from both sides we have

$$a_1 = -a_0\Sigma\alpha_1,\ a_2 = a_0\Sigma\alpha_1\alpha_2,\ a_3 = -a_0\Sigma\alpha_1\alpha_2\alpha_3, \ldots\ldots$$

and $\quad a_n = (-1)^n a_0.\ \ \alpha_1\alpha_2\ldots\alpha_n.$

Hence
$$\Sigma\alpha_1 = -a_1/a_0,$$
$$\Sigma\alpha_1\alpha_2 = a_2/a_0,$$
$$\Sigma\alpha_1\alpha_2\alpha_3 = -a_3/a_0,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots$$
$$\alpha_1\alpha_2\ldots\ldots\alpha_n = (-1)^n a_n/a_0.$$

It is important to note these results as they will be used in the succeeding chapters.

Example

If α, β, γ are the roots of the equation $x^3 - ax^2 + bx - c = 0$,

(i) find $\alpha^2+\beta^2+\gamma^2$, and $\alpha^3+\beta^3+\gamma^3$,

(ii) find the condition that $\beta+\gamma=0$.

Since α, β, γ are the roots of the given equation, we have

$$\alpha+\beta+\gamma = a,\ \alpha\beta+\beta\gamma+\gamma\alpha = b, \text{ and } \alpha\beta\gamma = c$$

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$$= a^2 - 2b$$

Again, $\alpha^3+\beta^3+\gamma^3 - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$,

Hence $\quad \Sigma\alpha^3 - 3c = a\{(a^2-2b)-b\} = a^3 - 3ab$

or $$\Sigma\alpha^3 = a^3 - 3ab + 3c.$$

(ii) If $\beta+\gamma=0$, we have $\alpha = a$,

and since α is a root of the given equation

$$a^3 - a^3 + ab - c = 0$$

or $$ab = c,$$

which is the required condition.

Exercises II

1. Express the following in the form $A+iB$,

(i) $\dfrac{3+4i}{4-3i}$ (ii) $\dfrac{\cos\theta+i\sin\theta}{\cos\alpha-i\sin\alpha}$,

(iii) $\left(\dfrac{a+ib}{a-ib}\right)^2-\left(\dfrac{a-ib}{a+ib}\right)^2$.

2. Express in the modulus-amplitude form :

(i) $-6+8i$ (ii) $\cos\theta+\cos\phi+i(\sin\theta+\sin\phi)$.

(iii) $\sin\theta+i(1+\cos\theta)$.

3. Show that the points representing the complex numbers $-4+3i$, $2-3i$, and $-i$ are collinear.

4. If $z=x+iy=\dfrac{1}{1+\cos\theta+i\sin\theta}$, show that the locus of z is a straight line parallel to y-axis.

5. If $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$, $c=\cos\gamma+i\sin\gamma$, and $a+b+c=0$, prove that $a^{-1}+b^{-1}+c^{-1}=0$.

6. Two lines join the points $z=a$, $z=b$; $z=c$, $z=d$. Show that they are at right angles if $\dfrac{a-b}{c-d}$ is purely imaginary. When are the lines parallel ?

7. If $|z_1|=|z_2|$, and arg. z_1+arg. $z_2=0$, show that z_1 and z_2 are conjugate complex numbers.

8. If z_1, z_2 and z_3 are the vertices of an equilateral triangle, show that $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$.

CHAPTER III

De Moivre's Theorem

3·1. De Moivre's Theorem.

If n be a rational number, then the value, or one of the values, of $(\cos\theta+i\sin\theta)^n$ is $\cos n\theta+i\sin n\theta$.

Case I. Let n be a positive integer.

By actual multiplication

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)=$$

$$\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i(\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)$$

$$=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$$

Similarly

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3)$$

$$=\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}(\cos\theta_3+i\sin\theta_3).$$

$$=\cos(\theta_1+\theta_2+\theta_3)+i\sin(\theta_1+\theta_2+\theta_3)$$

In this way we can multiply n factors and obtain

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\ldots\ldots(\cos\theta_n+i\sin\theta_n)$$

$$=\cos(\theta_1+\theta_2+\ldots\ldots\theta_n)+i\sin(\theta_1+\theta_2+\ldots\ldots+\theta_n)$$

Putting $\theta_1=\theta_2=\ldots\ldots=\theta_n=\theta$, we have

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

which is De Moivre's Theorem for n to be a positive integer.

Case II. Let n be a negative integer.

Suppose $n=-m$, when m is a positive integer.

Now $(\cos\theta+i\sin\theta)^n=(\cos\theta+i\sin\theta)^{-m}$

$$=\frac{1}{(\cos\theta+i\sin\theta)^m}$$

$$=\frac{1}{\cos m\theta+i\sin m\theta}\text{, by Case I.}$$

$$=\frac{\cos m\theta-i\sin m\theta}{(\cos m\theta+i\sin m\theta)(\cos m\theta-i\sin m\theta)}$$

$$=\frac{\cos(-m)\,\theta+i\sin(-m)\,\theta}{\cos^2 m\,\theta+\sin^2 m\,\theta}$$

$$=\cos n\,\theta+i\sin n\,\theta.$$

This is De Moivre's Theorem when n is a negative integer.

Case III. Let n be a fraction, positive or negative.

Hence we can suppose that $n=p/q$ where q is a positive integer, and p is any integer, positive or negative.

Now $\left(\cos\frac{p\,\theta}{q}+i\sin\frac{p\,\theta}{q}\right)^q$

$$=\cos\left(\frac{p\,\theta}{q}.\,q\right)+i\sin\left(\frac{p\,\theta}{q}.\,q\right)\ldots\text{by Case I}$$

$$=\cos p\,\theta+i\sin p\,\theta$$

$$=(\cos\theta+i\sin\theta)^p,\text{ by Case I or II}$$

Hence $\cos\frac{p\,\theta}{q}+i\sin\frac{p\,\theta}{q}$ is one of the values of

$$(\cos\theta+i\sin\theta)^{\frac{p}{q}}$$

or $\cos n\,\theta+i\sin n\,\theta$ is one of the values of $(\cos\theta+i\sin\theta)^n$

Thus De Moivre's Theorem is proved completely.

It should be noted that when n is a positive or a negative integer then $\cos n\,\theta+i\sin n\,\theta$ is the only value of $(\cos\theta+i\sin\theta)^n$; when n is fractional, $\cos n\,\theta+i\sin n\,\theta$ is then *one* of the values of $(\cos\theta+i\sin\theta)^n$.

Examples

1. Prove $\cos n\,\theta-i\sin n\,\theta=(\cos\theta-i\sin\theta)^n$

$\cos n\,\theta-i\sin n\,\theta=\cos(-n)\,\theta+i\sin(-n)\,\theta$

$$=(\cos\theta+i\sin\theta)^{-n}=\left(\frac{1}{\cos\theta+i\sin\theta}\right)^n$$

$$=\frac{(\cos\theta-i\sin\theta)^n}{(\cos^2\theta+\sin^2\theta)^n}=(\cos\theta-i\sin\theta)^n.$$

2. Simplify $\dfrac{(\cos\theta+i\sin\theta)^7}{(\sin\theta+i\cos\theta)^4}$

$$\frac{(\cos\theta+i\sin\theta)^7}{(\sin\theta+i\cos\theta)^4}=\frac{(\cos\theta+i\sin\theta)^7(\sin\theta-i\cos\theta)^4}{(\sin^2\theta+\cos^2\theta)^4}$$

$$=(\cos\theta+i\sin\theta)^7.\ (-i)^4(\cos\theta+i\sin\theta)^4$$

$$=(\cos\theta+i\sin\theta)^{11}=\cos 11\,\theta+i\sin 11\,\theta.$$

3. Prove that

$$\left(\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n=\cos n\left(\frac{\pi}{2}-\phi\right)+i\sin n\left(\frac{\pi}{2}-\phi\right)$$

$$\left(\frac{1+\sin\phi+i\cos\phi}{1+\cos\phi-i\sin\phi}\right)^n=\left\{\frac{1+\cos\left(\frac{\pi}{2}-\phi\right)+i\sin\left(\frac{\pi}{2}-\phi\right)}{1+\cos\left(\frac{\pi}{2}-\phi\right)-i\sin\left(\frac{\pi}{2}-\phi\right)}\right\}^n$$

$$=\left\{\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)+2\,i\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)-2\,i\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}\right\}^n$$

$$=\left\{\frac{\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}\right\}^n$$

$$=\left\{\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right\}^{2n}$$

$$=\cos n\left(\frac{\pi}{2}-\phi\right)+i\sin n\left(\frac{\pi}{2}-\phi\right).$$

4. Prove that $(1+i)^n+(1-i)^n=2^{n/2+1}\cos\dfrac{n\pi}{4}$

$$(1+i)^n+(1-i)^n=(\sqrt{2})^n\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^n+(\sqrt{2})^n\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^n$$

$$=2^{n/2}\left[\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^n+\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^n\right]$$

$$= 2^{n/2}\left[\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}+\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right]$$

$$= 2^{n/2+1}\cos\frac{n\pi}{4}.$$

Exercises

Simplify :

1. $\dfrac{\cos 7\theta - i\sin 7\theta}{\cos 4\theta - i\sin 4\theta}$. 2. $\left(\sin\dfrac{\pi}{7}+i\cos\dfrac{\pi}{7}\right)^7$.

3. $\dfrac{(\cos\theta+i\sin\theta)(\sin\theta+i\cos\theta)}{\sin 3\theta+i\cos 3\theta}$.

4. $\dfrac{(1+i)^{-3}+(1-i)^{-3}}{(1+i)^{-3}-(1-i)^{-3}}$.

5. Express $(2+3i)^7$ in the forms $A+Bi$ and $r(\cos\theta+i\sin\theta)$.

6. Express $(1+7i)(2-i)^{-3}$ in the form $r(\cos\theta+i\sin\theta)$ and deduce or prove otherwise that its fourth power is a real negative number.

7. Show that $\dfrac{(\cos 3\theta+i\sin 3\theta)^4(\cos 7\theta-i\sin 7\theta)^2}{(\cos 5\theta+i\sin 5\theta)^3(\cos\theta-i\sin\theta)^{17}}=1$.

8. If $a=\cos 2\alpha+i\sin 2\alpha$, with similar expressions for b, c and d, prove that

(i) $a+b=2\cos(\alpha-\beta)\{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\}$.

(ii) $ab+cd=2\cos(\alpha+\beta-\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)+i\sin(\alpha+\beta+\gamma+\delta)\}$.

(iii) $(a+b)(c+d)=4\cos(\alpha-\beta)\cos(\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)+i\sin(\alpha+\beta+\gamma+\delta)\}$.

(vi) $\sqrt{abcd}+\dfrac{1}{\sqrt{abcd}}=2\cos(\alpha+\beta+\gamma+\delta)$. (Agra '56)

(v) $\sqrt{\dfrac{ab}{cd}}+\sqrt{\dfrac{cd}{ab}}=2\cos(\alpha+\beta-\gamma-\delta)$.

9. Prove that $(a+i\,b)^{\frac{m}{n}}+(a-i\,b)^{\frac{m}{n}}$

$$=2(a^2+b^2)^{\frac{m}{2n}}\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right). \text{ (Agra '53)}$$

10. If $x_r=\cos\frac{\pi}{2^r}+i\sin\frac{\pi}{2^r}$, prove that

$$x_1x_2x_3\ldots\ldots ad \text{ inf.}=-1. \qquad \text{(Agra '55)}$$

11. If $z^2-2z\cos\theta+1=0$ show that
$z^2+z^{-2}=2\cos 2\theta$, and $z^3-z^{-3}=2i\sin 3\theta$.

12. Prove that $(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n$

$$=2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}. \qquad \text{(Madras '36)}$$

13. If $(1+x)^n=p_0+p_1x+p_2x^2\ldots\ldots$, show that

$$p_0-p_2+p_4\ldots\ldots=2^{n/2}\cos\tfrac{1}{4}n\pi,$$

and $$p_1-p_3+p_5\ldots\ldots=2^{n/2}\sin\tfrac{1}{4}n\pi.$$

14. Using the expansion $(1+z)^n=1+{}^nC_1z+{}^nC_2z^2+\ldots\ldots$
$+{}^nC_nz^n$, prove that

(i) $1+{}^nC_1\cos\theta+{}^nC_2\cos 2\theta+\ldots\ldots+{}^nC_n\cos n\theta$
$=2^n\cos^n\theta/2\cos n\theta/2.$

(ii) ${}^nC_1\sin\theta+{}^nC_2\sin 2\theta+\ldots\ldots+{}^nC_n\sin n\theta$
$=2^n\cos^n\theta/2\sin n\theta/2.$

3·2. To find the q roots of $(\cos\theta+i\sin\theta)^{p/q}$ p and q being integers prime to each other.

It has been seen that $\cos n\theta+i\sin n\theta$ is *one* of the values of $(\cos\theta+i\sin\theta)^n$ when n is a fraction. If $n=p/q$, we are now going to obtain all the values of $(\cos\theta+i\sin\theta)^{p/q}$.

If r is any integer we know

$$\cos\theta+i\sin\theta=\cos(2r\pi+\theta)+i\sin(2r\pi+\theta)$$

Hence $(\cos\theta+i\sin\theta)^{p/q}=\{\cos(2r\pi+\theta)+i\sin(2r\pi+\theta)\}^{p/q}$

$$=\cos\left\{(2r\pi+\theta)\frac{p}{q}\right\}+i\sin\left\{(2r\pi+\theta)\frac{p}{q}\right\},$$

for all integral values of r.

Putting $r=0, 1, 2 \ldots\ldots q-1$, we get the following q values or roots of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}$

$$\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q},\ \cos\left\{(2\pi+\theta)\frac{p}{q}\right\}+i\sin\left\{(2\pi+\theta)\frac{p}{q}\right\},$$

$$\ldots\cos\left[\left\{2\pi(q-1)+\theta\right\}\frac{p}{q}\right]+i\sin\left[\left\{2\pi(q-1)+\theta\right\}\frac{p}{q}\right].$$

If $r=q$, $q+1\ldots\ldots$, these roots are repeated. Since no two angles involved in these roots are equal or differ by a multiple of 2π, these are the distinct q values of $(\cos\theta+i\sin\theta)^{p/q}$.

3·21. Any root of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}=\cos\frac{2r\pi+\theta}{q}p$

$$+i\sin\frac{2r\pi+\theta}{q}p, \text{ for integral value of } r.$$

Hence a root $=\cos\left(\frac{p\theta}{q}+\frac{2r\pi p}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi p}{q}\right)$

$$=\left(\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}\right)\left(\cos\frac{2r\pi p}{q}+i\sin\frac{2r\pi p}{q}\right)$$

$$=\left(\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}\right)\left(\cos\frac{2\pi p}{q}+i\sin\frac{2\pi p}{q}\right)^r$$

Putting $r=0, 1, 2\ldots\ldots q-1$, the q roots may be expressed as $\alpha, \alpha\beta, \alpha\beta^2\ldots\ldots\alpha\beta^{q-1}$, where

$$\alpha=\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q} \text{ and } \beta=\cos\frac{2\pi p}{q}+i\sin\frac{2\pi p}{q}.$$

Hence these roots are in geometric progression.

3·22. The above method can be conveniently used in finding the n roots of $(a+ib)^{m/n}$ by expressing $a+ib$ in the form $r(\cos\theta+i\sin\theta)$.

Examples

1. Solve $x^7=1$, by the help of De Moivre's Theorem. (Cal. '43)

$$x^7=1=\cos 2n\pi+i\sin 2n\pi$$

$$\therefore \quad x=\cos\frac{2n\pi}{7}+i\sin\frac{2n\pi}{7}$$

Putting x=0, 1, 2…6, the roots are

$$1,\ \cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7},\ \cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7},$$

$$\cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7},\ \sin\frac{8\pi}{7}+i\sin\frac{8\pi}{7},$$

$$\cos\frac{10\pi}{7}+i\sin\frac{10\pi}{7},\ \cos\frac{12\pi}{7}+i\sin\frac{12\pi}{7}$$

They can be expressed also as

$$1,\ \cos\frac{2\pi}{7}\pm i\sin\frac{2\pi}{7},\ \cos\frac{4\pi}{7}\pm i\sin\frac{4\pi}{7},$$

$$\cos\frac{6\pi}{7}\pm i\sin\frac{6\pi}{7}.$$

2. Find all the values of $(\sqrt{3}+i)^{\frac{1}{3}}$

$$\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

$$=2\left\{\cos\left(2\pi r+\frac{\pi}{6}\right)+i\sin\left(2\pi r+\frac{\pi}{6}\right.\right\}$$

$$\therefore \quad (\sqrt{3}+i)^{\frac{1}{3}} = 2^{\frac{1}{3}} \cos\left(\frac{2r\pi + \frac{\pi}{6}}{3}\right) + i \sin\left(\frac{2r\pi + \frac{\pi}{6}}{3}\right)$$

Putting $r=0, 1, 2$, the roots are

$$2^{\frac{1}{3}}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right),\ 2^{\frac{1}{3}}\left(\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{18}\right),$$

$$\text{and } 2^{\frac{1}{3}}\left(\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{18}\right).$$

3. Prove that, if n is a positive integer, and

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n,$$

then $C_0 + C_4 + C_8 + \ldots = 2^{n-2} + 2^{\frac{n}{2}-1} \cos\frac{n\pi}{4}$.

Putting $x=1, -1$, and i in the given relation, we get

$$2^n = C_0 + C_1 + C_2 + \ldots + C_n, \ldots\ldots \quad \ldots(1)$$

$$0 = C_0 - C_1 + C_2 - C_3 + \ldots\ldots \quad \ldots(2)$$

and $\quad (1+i)^n = C_0 + C_1 i + C_2 i^2 + C_3 i^3 + \ldots\ldots \quad \ldots(3)$

or $\quad (C_0 - C_2 + C_4 - \ldots) + i\,(C_1 - C_3 + C_5 - \ldots)$

$$= 2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

Equating the real parts from both sides.

$$C_0 - C_2 + C_4 - \ldots = 2^{n/2}\cos\frac{n\pi}{4} \quad \ldots(4)$$

From (2), $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$,

from (1) $\quad \ldots(5)$

Adding (4) and (5),

$$2(C_0 + C_4 + C_8 + \ldots) = 2^{n-1} + 2^{n/2} \cos\frac{n\pi}{4},$$

whence the result.

4. Prove that every root of the equation

$$(1+x)^6+x^6=0$$ (Baroda' 53)

has $-\frac{1}{2}$ for its real part.

$$(1+x)^6+x^6=0$$

or $$\left(\frac{1+x}{x}\right)^6=-1=\cos(2n+1)\pi+i\sin(2n+1)\pi$$

$$\therefore\quad \frac{1+x}{x}=\cos\frac{2n+1}{6}\pi+i\sin\frac{2n+1}{6}\pi,\ n \text{ being } 0, 1, 2, \ldots 5.$$

or $$-\frac{1}{x}=1-\cos\frac{2n+1}{6}\pi-i\sin\frac{2n+1}{6}\pi$$

$$=2\sin\frac{2n+1}{12}\pi\left(\sin\frac{2n+1}{12}\pi-i\cos\frac{2n+1}{12}\pi\right)$$

$$\therefore\quad x=-\frac{1}{2\sin\frac{2n+1}{12}\pi\left(\sin\frac{2n+1}{12}\pi-i\cos\frac{2n+1}{12}\pi\right)}$$

$$=-\frac{1}{2\sin\frac{2n+1}{12}\pi}\left(\sin\frac{2n+1}{12}\pi+i\cos\frac{2n+1}{12}\pi\right)$$

$$=-\tfrac{1}{2}\left(1+i\cot\frac{2n+1}{12}\pi\right)$$

The six roots, evidently, have each $-\frac{1}{2}$ as the real part.

Exercises III

1. Find the values of

(a) $(-1)^{\frac{1}{3}}$, (b) $(-1)^{\frac{1}{6}}$, (c) $(1+i)^{\frac{2}{3}}$.

2. Solve $x^{12}=1$, and determine which of the roots also satisfy

$$x^4+x^2+1=0.$$ (Lucknow' 44)

3. Express $$P=\frac{\sqrt{3}-1+i(\sqrt{3}+1)}{2\sqrt{2}}$$
in the form $r(\cos\theta+i\sin\theta)$ and derive all the values of $P^{\frac{1}{4}}$. (Cal. '42)

4. Find the cube roots of $1-\cos\phi-i\sin\phi$, where ϕ is real and state the argument and modulus of each. (London '46)

5. Use De Moivre's theorem to solve the equations :—

(i) $x^7+x^4+x^3+1=0$.

(ii) $x^5+x^4+x^3+x^2+1=0$.

6. Show that
$$\{\cos\theta+\cos\phi+i(\sin\theta+\sin\phi)\}^n+\{\cos\theta+\cos\phi-i(\sin\theta+\sin\phi)\}^n$$
$$=2^{n+1}\cos^n\frac{\theta+\phi}{2}\cos\frac{n}{2}(\theta+\phi).$$

7. Show that, if n be a positive integer,
$(\sqrt{3}+i)^n+(\sqrt{3}-i)^n=2^{n+1}\cos\frac{1}{6}n\pi$. (Dacca '50)

8. Show that if
$x=\cos\theta+i\sin\theta$, and $\sqrt{1-c^2}=nc-1$,
$$1+c\cos\theta=\frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right).$$ (Agra '54)

9. If $2\cos\alpha=a+1/a$, $2\cos\beta=b+1/b$, etc., prove that

(i) $2\cos(\alpha+\beta+\gamma+\ldots\ldots)=abc\ldots+\dfrac{1}{abc\ldots}$.

(ii) $2\cos(p\alpha+q\beta+r\gamma+\ldots)=a^pb^qc^r\ldots+\dfrac{1}{a^pb^qc^r\ldots}$.

(Agra '47)

10. If $p=\cos\theta+i\sin\theta$, $q=\cos\phi+i\sin\phi$,

show that $\dfrac{p-q}{p+q}=i\tan\dfrac{\theta-\phi}{2}$. (Agra '58)

11. Prove that the roots of the equation

$x^{10}+11x^5-1=0$ are $\dfrac{\pm\sqrt{5}-1}{2}\cos\left(\dfrac{2r\pi}{5}\pm i\sin\dfrac{2r\pi}{5}\right)$

12. Find the seven seventh roots of unity and prove that the sum of their n^{th} powers always vanishes unless n be a multiple of 7, n being an integer, and that then the sum is 7. (Delhi '46)

13. If $\left(1+\dfrac{ix}{a}\right)\left(1+\dfrac{ix}{b}\right)\left(1+\dfrac{ix}{c}\right)\ldots\ldots=A+iB$

then prove that

(i) $\left(1+\dfrac{x^2}{a^2}\right)\left(1+\dfrac{x^2}{b^2}\right)\left(1+\dfrac{x^2}{c^2}\right)\ldots\ldots=A^2+B^2.$

(ii) $\tan^{-1}\dfrac{x}{a}+\tan^{-1}\dfrac{x}{b}+\tan^{-1}\dfrac{x}{c}+\ldots=\tan^{-1}\dfrac{B}{A}.$

14. Find the value of x such that

$\dfrac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\dfrac{\sin n\theta}{\sin^n\theta}$, where α, β are the roots of

$t^2-2t+2=0.$ (Agra '52)

15. Prove by the use of De Moivre's theorem that the roots of the equation $(x-1)^n=x^n$ (n being a positive integer) are $\dfrac{1}{2}\left(1+i\cot\dfrac{r\pi}{n}\right)$, where r has the values $0, 1, 2, \ldots (n-1)$.

(Cambridge '39)

16. If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$, show that

(i) $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$

(ii) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma).$

(iii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma).$ (Lucknow '52)

17. If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\frac{3}{2}$, prove that $\cos n\alpha+\cos n\beta+\cos n\gamma=3\cos\frac{n}{3}(\alpha+\beta+\gamma)$ or 0, according as n is or is not a multiple of 3.

18. From the identity

$$\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{(x-a)(x-b)}$$

deduce that

$\cos(\theta+\alpha)\sin(\theta-\beta)-\cos(\theta+\beta)\sin(\theta-\alpha)=\cos 2\theta\sin(\alpha-\beta)$ and $\sin(\theta+\alpha)\sin(\theta-\beta)-\sin(\theta+\beta)\sin(\theta-\alpha)=\sin 2\theta\sin(\alpha-\beta)$.

CHAPTER IV

Applications of De Moivre's Theorem

4·1. Expansions of cos $n\theta$ and sin $n\,\theta$, n being a positive integer.

When n is a positive integer we have, from De Moivre's Theorem,

$$\cos n\,\theta + i \sin n\,\theta = (\cos\theta + i\sin\theta)^n$$
$$= \cos^n\theta + {}^nC_1 \cos^{n-1}\theta.\,(i\sin\theta) + {}^nC_2 \cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3 \cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots + {}^nC_{n-1}.\cos\theta\,(i\sin\theta)^{n-1} + {}^nC_n(i\sin\theta)^n, \qquad \ldots(1)$$

by the application of Binomial Theorem to the right hand side. Equating the real and imaginary parts, we have

$$\cos n\,\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \ldots$$

and $\sin n\,\theta = {}^nC_1 \sin\theta \cos^{n-1}\theta - {}^nC_3 \sin^3\theta \cos^{n-3}\theta + {}^nC_5 \sin^5\theta \cos^{n-5}\theta - \ldots\ldots$

It is evident that the last term in each will depend on whether n is odd or even. If n is odd the last terms in the expansions of $\cos n\theta$ and $\sin n\theta$ will be $(-1)^{\frac{n-1}{2}} n \cos\theta \sin^{n-1}\theta$ and $(-1)^{\frac{n-1}{2}} \sin^n\theta$ respectively. If n be even the corresponding terms are

$$(-1)^{\frac{n}{2}} \sin^n\theta \text{ and } (-1)^{\frac{n-2}{2}}\; n\cos\theta \sin^{n-1}\theta.$$

4·11. Expansion of tan $n\theta$.

Using the expansions of $\sin n\,\theta$ and $\cos n\,\theta$, we have

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{{}^nC_1 \sin\theta \cos^{n-1}\theta - {}^nC_3 \sin^3\theta \cos^{n-3}\theta + \ldots}{\cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \ldots\ldots}$$

Dividing the numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + \ldots\ldots}{1 - {}^nC_2 \tan^2\theta + \ldots\ldots}$$

If n be odd the last terms in the numerator and denominator are respectively $(-1)^{\frac{n-1}{2}} \tan^n\theta$ and $(-1)^{\frac{n-1}{2}} n \tan^{n-1}\theta$;

if n be even the corresponding terms are $(-1)^{\frac{n-2}{2}} n \tan^{n-1}\theta$ and $(-1)^{\frac{n}{2}} \tan^n\theta$.

4·2. Expansion of tan $(\theta_1+\theta_2+\ldots+\theta_n)$.

We have already seen that

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\ldots\ldots(\cos\theta_n+i\sin\theta_n)$$
$$=\cos(\theta_1+\theta_2+\ldots+\theta_n)+i\sin(\theta_1+\theta_2+\ldots+\theta_n)$$

Now, expressing

$$\cos\theta_1+i\sin\theta_1=\cos\theta_1(1+i\tan\theta_1),$$
$$\cos\theta_2+i\sin\theta_2=\cos\theta_2(1+i\tan\theta_2), \text{ and so on,}$$

we have

$$\cos(\theta_1+\theta_2+\ldots+\theta_n)+i\sin(\theta_1+\theta_2+\ldots+\theta_n)$$
$$=\cos\theta_1\cos\theta_2\ldots\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)\ldots\ldots(1+i\tan\theta_n)$$
$$=\cos\theta_1\cos\theta_2\ldots\cos\theta_n(1+iS_1-S_2-iS_3+S_4+\ldots\ldots),$$

where $S_1=\tan\theta_1+\tan\theta_2+\ldots\ldots+\tan\theta_n$,

$S_2=\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\ldots\ldots$,

$S_3=\tan\theta_1\tan\theta_2\tan\theta_3+\tan\theta_2\tan\theta_3\tan\theta_4+\ldots\ldots$,

and so on.

Hence equating the real and the imaginary parts from both sides, we have

$$\cos(\theta_1+\theta_2+\ldots+\theta_n)=\cos\theta_1\cos\theta_2\ldots\cos\theta_n(1-S_2+S_4-\ldots), \quad \ldots(1)$$

and $\sin(\theta_1+\theta_2+\ldots+\theta_n)=\cos\theta_1\cos\theta_2\ldots\cos\theta_n(S_1-S_3+S_5-\ldots) \quad \ldots(2)$

By division,

$$\tan(\theta_1+\theta_2+\ldots+\theta_n)=\frac{S_1-S_3+S_5-\ldots}{1-S_2+S_4-\ldots} \quad \ldots(3)$$

The student can now find out the last terms in the numerator and denominator accordingly as n is odd or even.

By putting $\theta_1=\theta_2=\ldots=\theta_n=\theta$ in (1), (2), (3) we get easily the corresponding expressions obtained in the last two articles.

Examples

1. Express $\cos 7\theta$ in terms of powers of $\cos\theta$.

$$\cos 7\theta=\cos^7\theta-{}^7C_2\cos^5\theta\sin^2\theta+{}^7C_4\cos^3\theta\sin^4\theta-{}^7C_6\cos\theta\sin^6\theta$$

$$=\cos^7\theta-21\cos^5\theta(1-\cos^2\theta)+35\cos^3\theta(1-\cos^2\theta)^2-7\cos\theta(1-\cos^2\theta)^3$$

$$=\cos^7\theta-21\cos^5\theta(1-\cos^2\theta)+35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta)-7\cos\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$$

$$=64\cos^7\theta-112\cos^5\theta+56\cos^3\theta-7\cos\theta.$$

2. Prove that

$$1+\cos 9\theta=(1+\cos\theta)(16\cos^4\theta-8\cos^3\theta-12\cos^2\theta+4\cos\theta+1)^2$$

$$\frac{1+\cos 9\theta}{1+\cos\theta}=\frac{2\cos^2\frac{9\theta}{2}}{2\cos^2\frac{\theta}{2}}=\left(\frac{2\cos\frac{9\theta}{2}\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\right)^2$$

$$=\left(\frac{\sin 5\theta-\sin 4\theta}{\sin\theta}\right)^2$$

Now $\sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

$$=\sin\theta\{5\cos^4\theta-10\cos^2\theta(1-\cos^2\theta)+1-2\cos^2\theta+\cos^4\theta\}.$$

$$=\sin\theta(16\cos^4\theta-12\cos^2\theta+1).$$

Again, $\sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta$

$$=4\sin\theta\cos\theta(2\cos^2\theta-1)$$

$$=\sin\theta(8\cos^3\theta-4\cos\theta)$$

Hence $\dfrac{\sin 5\theta-\sin 4\theta}{\sin\theta}=16\cos^4\theta-8\cos^3\theta-12\cos^2\theta+4\cos\theta+1,$

and the result follows.

3. Prove that, if n be odd,

$$\cot \alpha+\cot\left(\alpha+\frac{\pi}{n}\right)+\cot\left(\alpha+\frac{2\pi}{n}\right)+\ldots\ldots\text{to } n \text{ terms}$$
$$=n\cot n\alpha \qquad \text{(Cal. '37)}$$

Let $\theta=\alpha+\dfrac{r\pi}{n}$, $r=0,1,2,\ldots\ldots(n-1)$

Hence $\cot n\theta=\cot(r\pi+n\alpha)=\cot n\alpha$

or $\quad \cot n\alpha=\dfrac{\cos n\theta}{\sin n\theta}$

$$=\frac{\cos^n\theta-{}^nC_2\cos^{n-2}\theta\sin^2\theta+\ldots\ldots+{}^nC_1(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta}{{}^nC_1\cos^{n-1}\theta\sin\theta-{}^nC_3\cos^{n-3}\theta\sin^3\theta+\ldots+(-1)^{\frac{n-1}{2}}\sin^n\theta}$$

$$=\frac{\cot^n\theta-{}^nC_2\cot^{n-2}\theta+\ldots\ldots+{}^nC_1(-1)^{\frac{n-1}{2}}\cot\theta}{{}^nC_1\cot^{n-1}\theta-{}^nC_3\cot^{n-3}\theta+\ldots\ldots+(-1)^{\frac{n-1}{2}}}$$

or $\cot^n\theta-n\cot n\alpha\cot^{n-1}\theta+\ldots\ldots\ldots=0$,

which is an equation of the n^{th} degree in $\cot\theta$, having, evidently, the following n roots :

$$\cot\alpha,\ \cot\left(\alpha+\frac{\pi}{n}\right),\ldots\ldots\ldots\cot\left(\alpha+\frac{n-1}{n}\pi\right).$$

Hence, from Theory of Equations, the sum of the roots

$$=\cot\alpha+\cot\left(\alpha+\frac{\pi}{n}\right)+\ldots\ldots\ldots+\cot\left(\alpha+\frac{n-1}{n}\pi\right)$$

$=n\cot n\alpha$.

4. Prove that the equation

$$a^2\cos^2\theta+b^2\sin^2\theta+2ag\cos\theta+2fb\sin\theta+c=0$$

has four roots and that the sum of the values of θ which satisfy it is an even multiple of π radians.

Let $\tan\dfrac{\theta}{2}=t$, and hence $\cos\theta=\dfrac{1-t^2}{1+t^2}$, $\sin\theta=\dfrac{2t}{1+t^2}$.

Hence the equation becomes

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2+b^2\frac{4t^2}{(1+t^2)^2}+2ga\frac{1-t^2}{1+t^2}+2\,fb.\frac{2t}{1+t^2}+c=0$$

or $\quad a^2(1-t^2)^2+4\,b^2t^2+2ga(1-t^4)+4\,bft\,(1+t^2)+c(1+t^2)^2=0$

or $\quad t^4(c-2\,ga+a^2)+4\,bft^3+t^2(2c+4b^2-2\,a^2)+4\,bft+c+2\,ga+a^2=0$

This, being an equation of the fourth degree, has four roots,

$t_1, t_2, t_3, t_4,$ *i.e.* $\tan\frac{\theta_1}{2}$, $\tan\frac{\theta_2}{2}$, $\tan\frac{\theta_3}{2}$ and $\tan\frac{\theta_4}{2}$.

From Theory of Equations we have

$$\Sigma\, t_1=\Sigma\,\tan\frac{\theta_1}{2}=S_1=\frac{-4bf}{c-2\,ga+a^2},$$

$$\Sigma t_1 t_2=\Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=S_2=\frac{2\,c+4\,b^2-2\,a^2}{c-2\,g\,a+a^2},$$

$$\Sigma t_1\,t_2\,t_3=\Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}=S_3=\frac{-4bf}{c-2\,ga+a^2},$$

and $t_1\,t_2\,t_3\,t_4=\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2}$

$$=S_4=\frac{c+2\,g\,a+a^2}{c-2\,g\,a+a^2}.$$

Now $\tan\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}=\frac{S_1-S_3}{1-S_2+S_4}=0$

$$\therefore\quad \frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}=n\,\pi \quad\text{or}\quad \Sigma\,\theta_1=2\,n\,\pi.$$

5. Form the equation whose roots are

$$\cos\frac{\pi}{11},\ \cos\frac{3\pi}{11},\ \cos\frac{5\pi}{11},\ \cos\frac{7\pi}{11},\ \cos\frac{9\pi}{11};$$

and hence find

(i) $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$.

(ii) $\sec\frac{\pi}{11}+\sec\frac{3\pi}{11}+\sec\frac{5\pi}{11}+\sec\frac{7\pi}{11}+\sec\frac{9\pi}{11}$.

(Andhra '40)

The roots are $\cos\frac{2n+1}{11}\pi$, $n=0,1,2,3,4$.

Let $\frac{2n+1}{11}\pi=\theta$, or $11\theta=(2n+1)\pi$

Hence $\cos 6\theta=\cos\{(2n+1)\pi-5\theta\}=-\cos 5\theta$

or $\cos 6\theta+\cos\theta=\cos\theta-\cos 5\theta$

or $2\cos^2 3\theta-1+\cos\theta=2\sin 2\theta\sin 3\theta=4\sin^2\theta\cos\theta\,(3-4\sin^2\theta)$,

or $2(4\cos^3\theta-3\cos\theta)^2-1+\cos\theta=4\cos\theta\,(1-\cos^2\theta)(4\cos^2\theta-1)$,

or $2(4x^3-3x)^2-1+x=4x(1-x^2)(4x^2-1)$, if $x=\cos\theta$.

or $32x^6+16x^5-48x^4-20x^3+18x^2+5x-1=0$

or $(x+1)(32x^5-16x^4-32x^3+12x^2+6x-1)=0$

This equation has six roots of x or $\cos\theta$, including $x=-1$, which is the value of $\cos\frac{2n+1}{11}\pi$ when $n=5$. Hence the equation with the given roots is

$$32x^5-16x^4-32x^3+12x^2+6x-1=0 \quad ...(1)$$

The sum of the roots $=\Sigma x_1=\sum_0^4\cos\frac{2n+1}{11}\pi=\frac{16}{32}=\frac{1}{2}$.

If we put $x=\frac{1}{y}$, in the above equation, we get

$$y^5-6y^4-12y^3+32y^2+16y-32=0.$$

Its roots are the inverses of those of (1), *viz.* $\sec\frac{2n+1}{11}\pi$, $n=0, 1, 2, 3, 4$.

Hence $\sum_0^4\sec\frac{2n+1}{11}\pi=6$.

Hence the equation becomes

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2+b^2\frac{4t^2}{(1+t^2)^2}+2ga\frac{1-t^2}{1+t^2}+2fb.\frac{2t}{1+t^2}+c=0$$

or $\quad a^2(1-t^2)^2+4b^2t^2+2ga(1-t^4)+4bft(1+t^2)+c(1+t^2)^2=0$

or $\quad t^4(c-2ga+a^2)+4bft^3+t^2(2c+4b^2-2a^2)+4bft+c+2ga+a^2=0$

This, being an equation of the fourth degree, has four roots,

$t_1, t_2, t_3, t_4,$ *i.e.* $\tan\dfrac{\theta_1}{2}$, $\tan\dfrac{\theta_2}{2}$, $\tan\dfrac{\theta_3}{2}$ and $\tan\dfrac{\theta_4}{2}$.

From Theory of Equations we have

$$\Sigma t_1=\Sigma\tan\frac{\theta_1}{2}=S_1=\frac{-4bf}{c-2ga+a^2},$$

$$\Sigma t_1t_2=\Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=S_2=\frac{2c+4b^2-2a^2}{c-2ga+a^2},$$

$$\Sigma t_1t_2t_3=\Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}=S_3=\frac{-4bf}{c-2ga+a^2},$$

and $t_1t_2t_3t_4=\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_2}{2}\tan\dfrac{\theta_3}{2}\tan\dfrac{\theta_4}{2}$

$$=S_4=\frac{c+2ga+a^2}{c-2ga+a^2}.$$

Now $\tan\dfrac{\theta_1+\theta_2+\theta_3+\theta_4}{2}=\dfrac{S_1-S_3}{1-S_2+S_4}=0$

$$\therefore\quad \frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}=n\pi \quad\text{or}\quad \Sigma\theta_1=2n\pi.$$

5. Form the equation whose roots are

$$\cos\frac{\pi}{11},\ \cos\frac{3\pi}{11},\ \cos\frac{5\pi}{11},\ \cos\frac{7\pi}{11},\ \cos\frac{9\pi}{11};$$

and hence find

(i) $\cos\dfrac{\pi}{11}+\cos\dfrac{3\pi}{11}+\cos\dfrac{5\pi}{11}+\cos\dfrac{7\pi}{11}+\cos\dfrac{9\pi}{11}$.

(ii) $\sec\frac{\pi}{11}+\sec\frac{3\pi}{11}+\sec\frac{5\pi}{11}+\sec\frac{7\pi}{11}+\sec\frac{9\pi}{11}$.

(Andhra '40)

The roots are $\cos\frac{2n+1}{11}\pi$, $n=0,1,2,3,4$.

Let $\frac{2n+1}{11}\pi=\theta$, or $11\theta=(2n+1)\pi$

Hence $\cos 6\theta=\cos\{(2n+1)\pi-5\theta\}=-\cos 5\theta$

or $\cos 6\theta+\cos\theta=\cos\theta-\cos 5\theta$

or $2\cos^2 3\theta-1+\cos\theta=2\sin 2\theta\sin 3\theta=4\sin^2\theta\cos\theta(3-4\sin^2\theta)$,

or $2(4\cos^3\theta-3\cos\theta)^2-1+\cos\theta=4\cos\theta(1-\cos^2\theta)(4\cos^2\theta-1)$,

or $2(4x^3-3x)^2-1+x=4x(1-x^2)(4x^2-1)$, if $x=\cos\theta$.

or $32x^6+16x^5-48x^4-20x^3+18x^2+5x-1=0$

or $(x+1)(32x^5-16x^4-32x^3+12x^2+6x-1)=0$

This equation has six roots of x or $\cos\theta$, including $x=-1$, which is the value of $\cos\frac{2n+1}{11}\pi$ when $n=5$. Hence the equation with the given roots is

$$32x^5-16x^4-32x^3+12x^2+6x-1=0 \qquad \text{...(1)}$$

The sum of the roots $=\Sigma x_1=\sum_0^4\cos\frac{2n+1}{11}\pi=\frac{16}{32}=\frac{1}{2}$.

If we put $x=\frac{1}{y}$, in the above equation, we get

$$y^5-6y^4-12y^3+32y^2+16y-32=0.$$

Its roots are the inverses of those of (1), *viz.* $\sec\frac{2n+1}{11}\pi$,

$n=0, 1, 2, 3, 4$.

Hence $\sum_0^4\sec\frac{2n+1}{11}\pi=6$.

Exercises

Prove that

1. $\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

2. $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

3. $1 + \cos 10\theta = 2(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)^2$

4. $\tan 7\theta = \dfrac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$

5. If α, β and γ are the roots of the equation

$$x^3 + px^2 + qx + p = 0,$$

prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians, except in one particular case. (Lucknow '44)

6. Prove that the equation, $ah\sec\theta - bk\operatorname{cosec}\theta = a^2 - b^2$, has four roots and that the sum of the values of θ, which satisfy it, is equal to an odd multiple of π radians.

7. If θ_1, θ_2, θ_3 be the values of θ which satisfy the equation

$$\tan 2\theta = k\tan(\theta + \alpha),$$

and if no two of these values differ by a multiple of π, show that $\theta_1 + \theta_2 + \theta_3 + \alpha$ is a multiple of π. (Agra '54)

8. Prove that $\cos\dfrac{2\pi}{7}$, $\cos\dfrac{4\pi}{7}$, $\cos\dfrac{8\pi}{7}$ are the roots of the equation

$$8x^3 + 4x^2 - 4x - 1 = 0.$$ (Agra '57)

9. Show that $\cos\dfrac{\pi}{7}$, $\cos\dfrac{3\pi}{7}$ and $\cos\dfrac{5\pi}{7}$ are the roots of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0,$$

and deduce the equation whose roots are

$$\tan^2\frac{\pi}{7},\ \tan^2\frac{3\pi}{7} \text{ and } \tan^2\frac{5\pi}{7}$$ (Baroda '52)

10. Show that $\cos\dfrac{2\pi}{9}$, $\cos\dfrac{4\pi}{9}$, $\cos\dfrac{6\pi}{9}$, $\cos\dfrac{8\pi}{9}$ are the roots of

$$16x^4+8x^3-12x^2-4x+1=0.$$ (Travancore '40)

11. Solve the equation $\cot 3\theta=1$,

subject to the condition $0<\theta<\pi$. Hence using the formula for $\cot 3\theta$ in terms of $\cot\theta$, prove that the three roots of the cubic

$$x^3-3x^2-3x+1=0 \text{ are } \cot\frac{\pi}{12}, \cot\frac{5\pi}{12} \text{ and } \cot\frac{3\pi}{4}.$$

(Cal. '42)

12. Write down the five values of θ that lie in the interval $(0, \pi)$ and conform to the equation $\sin 5\theta=0$.

Deduce or prove directly that the four roots of the equation

$$16x^4-20x^2+5=0$$

are $\pm\sin\frac{\pi}{5}$ and $\pm\sin\frac{2\pi}{5}$.

13. If α, β, γ and δ are the roots of the equation

$\tan\left(\frac{\pi}{4}+\theta\right)=3\tan 3\theta$, no two of which have equal tangents, show that

$$\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=0.$$ (Andhra '37)

4·3. Series for sin α.

We know that, when n is a positive integer,

$$\sin n\theta=n\cos^{n-1}\theta\sin\theta-\frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta$$
$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{4!}\cos^{n-5}\theta\sin^5\theta$$
$$+\ldots\ldots$$

Let $n\theta=\alpha$,

$$\text{Hence }\sin\alpha=\frac{\alpha}{\theta}\cos^{n-1}\theta.\sin\theta-\frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)}{3!}$$
$$\cos^{n-3}\theta\sin^3\theta$$

$$+\frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)\left(\frac{\alpha}{\theta}-4\right)}{5!}\cos^{n-5}\theta\sin^5\theta\ldots\ldots$$

$$=\alpha.\cos^{n-1}\theta.\frac{\sin\theta}{\theta}-\frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{3!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3$$

$$+\frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)(\alpha-4\theta)}{5!}\cos^{n-5}\theta.\left(\frac{\sin\theta}{\theta}\right)^5-\ldots$$

Let n increase and θ diminish indefinitely, so that $n\,\theta$ remains constant and equal to α. We know that, as θ tends to zero,

$$\cos\theta=1,\ \text{and}\ \frac{\sin\theta}{\theta}=1.$$

Hence, in the limit,

$$\sin\alpha=\alpha-\frac{\alpha^3}{3!}+\frac{\alpha^5}{5!}-\ldots\ldots$$

This is an infinite series and is convergent and absolutely convergent for all values of α.

4·4. Series for cos α

If n be a positive integer, we know

$$\cos n\,\theta=\cos^n\theta-\frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta+\frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta-\ldots\ldots$$

Putting $n\,\theta=\alpha$, and proceeding as in Art. 4.3,

$$\cos\alpha=\cos^n\theta-\frac{\alpha(\alpha-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2$$

$$+\frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{4!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4-\ldots\ldots$$

As θ tends to zero, we have, in the limit,

$$\cos\alpha=1-\frac{\alpha^2}{2!}+\frac{\alpha^4}{4!}-\frac{\alpha^6}{6!}+\ldots\ldots$$

This is also an infinite series which is convergent and absolutely convergent for all values of α.

4·41. In the previous two articles α is the circular measure of the angle, given in radians. If, however, the angle is given in degrees, it has to be converted to radians first.

$$\text{Thus } \sin x^0 = \sin \frac{x\pi}{180}$$

$$= \frac{x\pi}{180} - \frac{1}{3!}\left(\frac{x\pi}{180}\right)^3 + \frac{1}{5!}\left(\frac{x\pi}{180}\right)^5 - \ldots\ldots$$

4·42. The above expansions for $\sin \alpha$ and $\cos \alpha$ are very important, and useful in various ways.

If x be a small quantity, then $x^2, x^3 ..$ are still smaller. Hence first approximations for $\sin x$ and $\cos x$ may be written as

$$\sin x = x, \cos x = 1.$$

The second approximations will be

$$\sin x = x - \frac{x^3}{6}, \text{ and } \cos x = 1 - \frac{x^2}{2}.$$

The following examples will illustrate the application of this principle.

Examples

1. If $\frac{\sin \theta}{\theta} = \frac{1013}{1014}$, prove that θ is equal to $4^0 24'$ nearly.

Since $\frac{\sin \theta}{\theta} = 1$ when θ tends to zero, evidently θ is small in the above equation.

Hence $\frac{1013}{1014} = \frac{1}{\theta}\left(\theta - \frac{\theta^3}{6}\right) = 1 - \frac{\theta^2}{6}$, neglecting θ^4 and higher powers of θ.

or $\frac{\theta^2}{6} = \frac{1}{1014}$, and $\theta = \frac{1}{13}$ radians.

or $\theta = \left(\frac{1}{13} \cdot \frac{180}{\pi}\right)^0 = 4^0\ 24'$ nearly.

2. Evaluate : $\text{Lim}_{\theta \to 0} \dfrac{\sin n\,\theta - n \sin \theta}{\theta\,(\cos n\,\theta - \cos \theta)}$

$$\frac{\sin n\theta - n \sin \theta}{\theta(\cos n\,\theta - \cos \theta)}$$

$$= \frac{\left(n\,\theta - \frac{n^3\,\theta^3}{6} + \frac{n^5\,\theta^5}{120} - \dots\right) - n\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots\right)}{\theta\left[\left(1 - \frac{n^2\,\theta^2}{2} + \frac{n^4\,\theta^4}{24} - \dots\right) - \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \dots\right)\right]}$$

$$= \frac{\frac{\theta^3}{6}(n - n^3) + \frac{\theta^5}{120}(n^5 - n) + \dots}{\theta\left[(1 - n^2)\frac{\theta^2}{2} + (n^4 - 1)\frac{\theta^4}{24} + \dots\right]}$$

$$= \frac{\frac{n - n^3}{6} + \frac{\theta^2}{120}(n^5 - n) + \dots}{\frac{1 - n^2}{2} + \frac{\theta^2}{24}(n^4 - 1) + \dots}$$

Hence the required limit, when θ tends to zero,

$$= \frac{n - n^3}{6} \cdot \frac{2}{1 - n^2} = \frac{n}{3}.$$

3. Show that $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15} x^5 + \dots,$

and hence calculate the limit of $\left(\dfrac{\tan\ x}{x}\right)^{3/x^2}$

when x becomes indefinitely small.

$$\tan x = \frac{\sin x}{\cos x} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^{-1}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left\{1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right)\right\}^{-1}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left\{1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right)\right.$$

$$\left. + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \dots\right)^2 + \dots\right\}$$

$$=\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\dots\right)\left(1+\frac{x^2}{2}+\frac{5}{24}x^4+\dots\right),$$

neglecting x^6 and higher powers.

$$=x+\frac{x^3}{3}+\frac{2}{15}x^5+\dots\dots$$

Hence $$\frac{\tan x}{x}=1+\frac{x^2}{3}+\frac{2}{15}x^4+\dots$$

and $$\text{Lt.}_{x\to 0}\left(\frac{\tan x}{x}\right)^{3/x^2}=\text{Lt.}_{x\to 0}\left(1+\frac{x^2}{3}\right)^{3/x^2}=e,$$

$$\text{as Lt.}_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

4. If $\tan x=a_1x+\frac{a_3x^3}{3!}+\frac{a_5x^5}{5!}+\dots\dots$, show that

$$a_{2n+1}=\frac{(2n+1)2n}{2!}a_{2n-1}\ \frac{(2n+1)2n(2n-1)(2n-2)}{4!}$$

$$a_{2n-3}+\dots\dots+(-1)^{n+1}(2n+1)a_1+(-1)^n. \qquad \text{(Cal. '46)}$$

We have $$\frac{\sin x}{\cos x}=a_1x+\frac{a_3x^3}{3!}+\frac{a_5x^5}{5!}+\dots\dots$$

or $$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots+\frac{(-1)^nx^{2n+1}}{(2n+1)!}+\dots$$

$$=\left\{a_1x+\frac{a_3x^3}{5!}+\frac{a_5x^5}{5!}+\dots+\frac{a_{2n+1}x^{2n+1}}{(2n+1)!}+\dots\right\}$$

$$\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\dots\frac{(-1)^nx^{2n}}{(2n)!}+\dots\right\}$$

Equating the coefficient of x^{2n+1} from both sides,

$$\frac{(-1)^n}{(2n+1)!}=\frac{a_{2n+1}}{(2n+1)!}-\frac{a_{2n-1}}{(2n-1)!}\cdot\frac{1}{2}+\frac{a_{2n-3}}{(2n-3)!}\cdot\frac{1}{4!}$$

$$-\dots\dots+\frac{(-1)^n}{(2n)!}a_1.$$

By transposing and multiplying by $(2n+1)!$, we have

$$a_{2n+1}=\frac{(2n+1)2n}{2!}\quad a_{2n-1}-\frac{(2n+1)2n(2n-1)(2n-2)}{4!}a_{2n-3}+\ldots$$
$$+(-1)^{n-1}(2n+1)a_1+(-1)^n.$$

Exercises IV

1. If $\frac{\sin\theta}{\theta}=\frac{2165}{2166}$, show that θ is nearly the circular measure of 3°.

2. If $\frac{\sin\theta}{\theta}=\frac{5765}{5766}$, show that $\theta=1^\circ 51'$ approx. (Dacca '50)

3. If $\sin(\pi/6+\theta)=\cdot 51$, prove that θ is nearly $40'$.

4. Show that $\frac{x\sin y-y\sin x}{x\cos y-y\cos x}=\tan(x-\tan^{-1}x)$, when y is equal to x.

5. Evaluate :

(i) $\lim_{x\to 0}\frac{\tan 2x-2\tan x}{2x}$. (Cal. '33)

(ii) $\lim_{x\to 0}\frac{\tan 2n-2\sin x}{x^3}$. (Annam '47)

(iii) $\lim_{x\to\pi/2}(\sec x-\tan x)$. (Annam '52)

(iv) $\lim_{x\to 0}(\cos x)^{1/x}$ (Bombay '47)

(v) $\lim_{x\to 0}\frac{3\sin x-\sin 3x}{x-\sin x}$. (Cal. '36)

6. Assuming the expansions of $\sin x$ and $\cos x$ in powers of x, adjust the constants a and b in such a way that

$$\lim_{x\to 0}\frac{a\cos x+b\,x\sin x-5}{x^4}$$

may exist. Also find the limit when a, b are so adjusted. (Cal '42)

7. Prove that

$$\frac{1}{6}\sin^3\theta=\frac{\theta^3}{3!}-\frac{(1+3^2)}{5!}\theta^5+\frac{(1+3^2+3^4)}{7!}\theta^7-\ldots \text{(Agra' 38)}$$

8. Prove that

$$\sin^2\theta\cos\theta=\theta^2-\tfrac{5}{6}\theta^4+\ldots\ldots+(-1)^{n+1}\frac{3^{2n}-1}{4}\cdot\frac{\theta^{2n}}{(2n)!}+\ldots$$

(Cal '48)

9. If $\theta\cot\theta=a_0+a_2\theta^2+a_4\theta^4+\ldots$,

show that $$a_{2n}=\frac{a_{2n-2}}{3!}-\frac{a_{2n-4}}{5!}+\ldots\ldots+\frac{(-1)^{n-1}a_0}{(2n+1)!}+\frac{(-1)^n}{(2n)!}$$

and hence find $\theta\cot\theta$ to four terms.

10. If $\sec\theta=a_0+a_2\theta^2+a_4\theta^4+\ldots+a_{2n}\theta^{2n}+\ldots$,

Show that $$a_{2n}=\frac{a_{2n-2}}{2\,!}-\frac{a_{2n-4}}{4\,!}+\ldots+\frac{(-1)^{n+1}}{(2n)!}a_0.$$

CHAPTER V

Expansions of Trigonometric Functions—I

5·1. Expansion of $\cos^n \theta$.

The expansion will be obtained in a series of cosines of multiples of θ, n being a positive integer.

Let $x=\cos\theta+i\sin\theta$, then $x^{-1}=\cos\theta-i\sin\theta$,

and $x^n=\cos n\theta+i\sin n\theta$, also $x^{-n}=\cos n\theta-i\sin\theta$.

Hence $x^n+x^{-n}=2\cos n\theta$, and $x^n-x^{-n}=2i\sin n\theta$.

In particular $x+x^{-1}=2\cos\theta$, $x-x^{-1}=2i\sin\theta$.

Now $(2\cos\theta)^n=(x+x^{-1})^n$

$$=x^n+{}^nC_1x^{n-2}+{}^nC_2x^{n-4}+\ldots+{}^nC_{n-1}x^{-(n-2)}+x^{-n}$$

$$=(x^n+x^{-n})+{}^nC_1\left(x^{n-2}+\frac{1}{x^{n-2}}\right)$$

$$+{}^nC_2\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+\ldots\ldots,$$

by rearranging terms, combining the first term with the last and so on. Hence

$$(2\cos\theta)^n=2\cos n\theta+n.\ 2\cos(n-2)\theta$$

$$+\frac{n(n-1)}{2!}2\cos(n-4)\theta+\ldots\ldots$$

or $$2^{n-1}\cos^n\theta=\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{2!}\cos(n-4)\theta$$

$$+\ldots\ldots$$

The last term of the series will assume different forms accordingly as n is odd or even, *viz.* ${}^nC_{\frac{n-1}{2}}\cos\theta$ or $\frac{1}{2}.{}^nC_{\frac{n}{2}}$ respectively.

5·2. Expansion of $\sin^n\theta$.

In this case $\sin^n \theta$ will be expressed in a series of sines or cosines of multiples of θ.

Following the same procedure as in the previous article,

$(2\, i \sin \theta)^n = (x - x^{-1})^n$

$$= x^n - n\, x^{n-2} + \frac{n(n-1)}{2\,!} x^{n-4} - \ldots + n(-1)^{n-1} \frac{1}{x^{n-2}}$$

$$+ (-1)^n \cdot \frac{1}{x^n}$$

Two cases arise, according as n is odd or even.

Case I, n is odd.

Here $i^n = i \cdot i^{n-1} = i\,(-1)^{\frac{n-1}{2}}$,

$(-1)^n = -1, (-1)^{n-1} = +1,$

Hence

$$2^n\, i\, (-1)^{\frac{n-1}{2}} \sin^n \theta = \left(x^n - \frac{1}{x^n}\right) - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right)$$

$$+ \frac{n(n-1)}{2!}\left(x^{n-4} - \frac{1}{x^{n-4}}\right) + \ldots$$

$$= 2\, i \sin n\,\theta - n \cdot 2\, i \sin (n-2)\theta$$

$$+ \frac{n(n-1)}{2!}\, 2\, i \sin (n-4)\, \theta \ldots\ldots$$

or $$2^{n-1} (-1)^{\frac{n-1}{2}} \sin^n \theta = \sin n\theta - n \sin (n-2)\, \theta + \frac{n(n-1)}{2\,!}$$

$$\sin (n-4\ \ \theta \ldots\ldots$$

The last term in this series will be $(-1)^{\frac{n-1}{2}}\, {}^nC_{\frac{n-1}{2}} \sin \theta$.

Case II, n is even,

Here $\quad i^n=(-1)^{\frac{n}{2}}$, and $(-1)^n=+1$, $(-1)^{n-1}=-1$.

Hence, $\quad (2\,i \sin\theta)^n=\left(x^n+\dfrac{1}{x^n}\right)-n\left(x^{n-2}+\dfrac{1}{x^{n-2}}\right)$

$$+\frac{n\,(n-1)}{2!}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)-\ldots\ldots$$

$$=2\cos n\,\theta-n.\,2\cos(n-2)\,\theta+\frac{n\,(n-1)}{2!}\,2\cos(n-4)\,\theta-\ldots\ldots$$

or $\quad 2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta=\cos n\,\theta-n\cos(n-2)\,\theta+\dfrac{n\,(n-1)}{2!}$

$$\cos(n-4)\,\theta-\ldots\ldots$$

The last term will be $\frac{1}{2}(-1)^{\frac{n}{2}}\;{}^nC_{\frac{n}{2}}$.

5.21. Expansion of $\sin^m\theta\cos^n\theta$.

The methods employed in Art. 5·1 and 5·2 can be conveniently used in expanding $\sin^m\theta\cos^n\theta$.

Examples.

1. Express $\sin^8\theta$ in a series of cosines of multiples of θ.

(Banaras '52)

Let $\quad x=\cos\theta+i\sin\theta$,

Hence $(2\,i\sin\theta)^8=(x-1/x)^8=x^8-8x^6+28x^4-56x^2+70$

$$-56x^{-2}+28x^{-4}-8x^{-6}+x^{-8}$$

$$=\left(x^8+\frac{1}{x^8}\right)-8\left(x^6+\frac{1}{x^6}\right)+28\left(x^4+\frac{1}{x^4}\right)$$

$$-56\left(x^2+\frac{1}{x^2}\right)+70$$

$$=2\cos 8\,\theta-16\cos 6\,\theta+56\cos 4\,\theta-112\cos 2\,\theta+70$$

$\therefore \quad \sin^8\theta=\dfrac{1}{2^7}(\cos 8\,\theta-8\cos 6\,\theta+28\cos 4\,\theta$

$$-56\cos 2\,\theta+35),\text{ since } i^8=1.$$

2. Express $\sin^7\theta$ in a series of sines of multiples of θ.

Let $x=\cos\theta+i\ \delta$ in θ.

Hence $(2\,i\sin\theta)^7=(x-x^{-1})^7$

$$=x^7-7x^5+21x^3-35x+35x^{-1}-21x^{-3}+7x^{-5}-x^{-7}$$

$$=\left(x^7-\frac{1}{x^7}\right)-7\left(x^5-\frac{1}{x^5}\right)+21\left(x^3-\frac{1}{x^3}\right)-35\left(x-\frac{1}{x}\right)$$

$$=2\,i\sin 7\theta-14\,i\sin 5\theta+42\,i\sin 3\theta-70\,i\sin\theta$$

$$\therefore\quad \sin^7\theta=-\frac{1}{2^6}(\sin 7\theta-7\sin 5\theta+21\sin 3\theta-35\sin\theta)$$

Since $i^6=i^4\,.\,i^2=-1.$

3. Expand $\sin^7\theta\cos^3\theta$ in a series of sines of multiples of θ. (Agra '56)

Let $x=\cos\theta+i\sin\theta.$

Hence $(2\,i\sin\theta)^7\,.\,(2\cos\theta)^3=\left(x-\frac{1}{x}\right)^7\left(x+\frac{1}{x}\right)^3$

To obtain the product on the right hand side we write down the coefficients of different powers of x in $\left(x-\frac{1}{x}\right)^7$ in descending order, and multiply by $x+\frac{1}{x}$ thrice in succession. The products obtained are shown in the following form :

1	−7	21	−35	35	−21	7	−1			
1	−6	14	−14	0	14	−14	6	−1		
1	−5	8	0	−14	14	0	−8	5	−1	
1	−4	3	8	−14	0	14	−8	−3	4	−1

This gives

$$(2\,i\sin\theta)^7(2\cos\theta)^3=x^{10}-4x^8+3x^6+8x^4-14x^2+14x^{-2}-8x^{-4}-3x^{-6}+4x^{-8}-x^{-10}$$

$$=\left(x^{10}-\frac{1}{x^{10}}\right)-4\left(x^{8}-\frac{1}{x^{8}}\right)+3\left(x^{6}-\frac{1}{x^{6}}\right)+8\left(x^{4}-\frac{1}{x^{4}}\right)$$
$$-14\left(x^{2}-\frac{1}{x^{2}}\right)$$

$$=2\,i \sin 10\,\theta-8\,i \sin 8\,\theta+6\,i \sin 6\,\theta+16\,i \sin 4\,\theta$$
$$-28\,i \sin 2\,\theta$$

Hence, $\sin^7\theta\cos^3\theta=-\dfrac{1}{2^9}(\sin 10\,\theta-4\sin 8\,\theta+3\sin 6\,\theta$
$+8\sin 4\,\theta-14\sin 2\,\theta)$

Note : It may be observed that the series for $\sin^m\theta\cos^n\theta$ will be in terms of cosines or sines of multiples of θ accordingly as m is even or odd.

Exercises

Prove that

1. $32\cos^6\theta=\cos 6\,\theta+6\cos 4\theta+15\cos 2\,\theta+10.$ (Madras '46)

2. $\cos^7\theta=2^{-6}(\cos 7\,\theta+7\cos 5\,\theta+21\cos 3\,\theta+35\cos\theta).$

3. $\cos^8\theta=2^{-7}(\cos 8\,\theta+8\cos 6\,\theta+28\cos 4\,\theta$
$+56\cos 2\,\theta+35.$ (Agra '28)

4. $\sin^9\theta=2^{-8}(\sin 9\,\theta-9\sin 7\,\theta+36\sin 5\,\theta$
$-84\sin 3\,\theta+126\sin\theta).$

5. $32\sin^4\theta\cos^2\theta=\cos 6\,\theta-2\cos 4\,\theta-\cos 2\,\theta+2.$ (Agra '57)

6. $64(\cos^8\theta+\sin^8\theta)=\cos 8\,\theta+28\cos 4\,\theta+35.$ (Andhra '35)

7. Express $\cos^5\theta\sin^3\theta$ in terms of series of multiples of θ. (Banaras '47)

8. Express $\cos^5\theta\sin^7\theta$ in a series of sines of multiples of θ. (Agra '41)

5·3. Two important expansions.

We will now establish two important expansions which will be found very useful in obtaining expansions for $\cos n\,\theta$ and $\sin n\,\theta$.

If $|z|<1$, we know that

$$\frac{1}{1-z}=1+z+z^2+\ldots\ldots\text{ad inf.}$$

Putting $z=r(\cos\theta+i\sin\theta)$, when $|z|=r<1$, we have

$$1+r(\cos\theta+i\sin\theta)+r^2(\cos\theta+i\cos\theta)^2+\ldots\ldots$$

$$=\frac{1}{1-r(\cos\theta+i\sin\theta)}=\frac{1}{1-r\cos\theta-i\,r\sin\theta}$$

or $\quad 1+r(\cos\theta+i\sin\theta)+r^2(\cos 2\theta+i\sin 2\theta)+\ldots\ldots$

$$=\frac{1-r\cos\theta+i\,r\sin\theta}{1-2r\cos\theta+r^2}.$$

Equating the real and imaginary parts from both sides, we establish the two expansions,

$$\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}=1+r\cos\theta+r^2\cos 2\theta+r^3\cos 3\theta+\ldots\ldots$$
$$+r^n\cos n\theta+\ldots \qquad \ldots(1)$$

and $$\frac{r\sin\theta}{1-2r\cos\theta+r^2}=r\sin\theta+r^2\sin 2\theta+r^3\sin 3\theta+\ldots\ldots$$
$$+r^n\sin n\theta+\ldots \qquad \ldots(2)$$

5·4. Expansion of $\cos n\theta$ in a series of descending powers of $\cos\theta$.

Equating the coefficient of r^n from both sides in (1) above, we have

$\cos n\theta=$coefficient of r^n in $(1-r\cos\theta)(1-2r\cos\theta+r^2)^{-1}$

$=$coefficient of r^n in $(1-2r\cos\theta+r^2)^{-1}-(\cos\theta)$

{coefficient of r^{n-1} in $(1-2r\cos\theta+r^2)^{-1}$}

Now $\quad (1-2r\cos\theta+r^2)^{-1}=\{1-r(2\cos\theta-r)\}^{-1}$

$$=1+r(2\cos\theta-r)+r^2(2\cos\theta-r)^2+\ldots+r^{n-2}(2\cos\theta-r)^{n-2}$$
$$+r^{n-1}(2\cos\theta-r)^{n-1}+r^n(2\cos\theta-r)^n+\ldots\ldots$$

Hence, coefficient of r^n in $(1-2r\cos\theta+r^2)^{-1}$

$$=(2\cos\theta)^n-(n-1)(2\cos\theta)^{n-2}+\frac{(n-2)(n-3)}{2\,!}(2\cos\theta)^{n-4}-\ldots\ldots,$$

the first term being coefficient of r^n in $r^n(2\cos\theta-r)^n$, the second term is coefficient of r^n in $r^{n-1}(2\cos\theta-r)^{n-1}$, and so on.

Similarly coefficient of r^{n-1} in $(1-2r\cos\theta+r^2)^{-1}$

$$=(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}+\frac{(n-3)(n-4)}{2\,!}(2\cos\theta)^{n-5}+\ldots\ldots\ldots$$

Hence, $\cos n\theta$

$$=(2\cos\theta)^n-(n-1)(2\cos\theta)^{n-2}+\frac{(n-2)(n-3)}{2\,!}(2\cos\theta)^{n-4}-\ldots\ldots\ldots$$

$$-\cos\theta\,\{(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}+\frac{(n-3)(n-4)}{2\,!}(2\cos\theta)^{n-5}-\ldots\ldots\ldots\}$$

$$=\frac{1}{2}\left[(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{2\,!}(2\cos\theta)^{n-4}-\ldots\ldots\ldots\right]$$

or $$2\cos n\theta=(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{2\,!}(2\cos\theta)^{n-4}-\ldots\ldots\ldots$$

The powers of $2\cos\theta$, on the right, will be positive; the last term will be $(-1)^{\frac{n-1}{2}}\cdot n(2\cos\theta)$ or $(-1)^{\frac{n}{2}}\cdot 2$ accordingly n is odd or even.

5·41. Expansion of $\sin n\theta/\sin\theta$ in a series of descending powers of $\cos\theta$.

The relation (2) in Article 5·3 can be written

$$(1-2r\cos\theta+r^2)^{-1}=1+r\frac{\sin 2\theta}{\sin\theta}+r^2\frac{\sin 3\theta}{\sin\theta}+\ldots\ldots\ldots\ldots$$

$$\ldots\ldots+r^{n-1}\frac{\sin n\theta}{\sin\theta}+\ldots\ldots$$

Hence $\dfrac{\sin n\theta}{\sin\theta}$ = coefficient of r^{n-1} in $(1-2r\cos\theta+r^2)^{-1}$

$$=(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}+\frac{(n-3)(n-4)}{2!}(2\cos\theta)^{n-5}-\ldots\ldots,$$

as found in the previous article. The last term in this expansion is $(-1)^{\frac{n}{2}-1}(n\cos\theta)$ or $(-1)^{\frac{n-1}{2}}$ according as n is even or odd.

5·42. To expand $\cos n\theta$ and $\sin n\theta/(\cos\theta)$ in a series in ascending powers of $\sin\theta$, n being a positive even integer.

We have seen in art. 4·1,

$$\cos n\theta=\cos^n\theta-{}^nC_2\cos^{n-2}\theta\sin^2\theta+{}^nC_4\cos^{n-4}\theta\sin^4\theta-\ldots$$

Since n is even, $\cos^n\theta$ can be replaced by $(1-\sin^2\theta)^{n/2}$ for all n, and, on simplification, we can assume that the above series will assume the form

$$\cos n\theta=A_0+A_2\sin^2\theta+A_4\sin^4\theta+A_6\sin^6\theta+\ldots\ldots+A_n\sin^n\theta \quad \ldots(1)$$

Putting $\theta=0$ on both sides, the first term A_0 is seen to be 1.

The coefficients $A_2, A_4,\ldots\ldots$ are determined in an indirect way.

Putting $\theta+h$ for θ, we have

$$\cos n(\theta+h)=\cos n\theta\cos nh-\sin n\theta\sin nh$$

$$=\cos n\theta\left(1-\frac{n^2h^2}{2}+\ldots\right)-\sin n\theta\left(nh-\frac{n^3h^3}{6}+\ldots\ldots\right)$$

$$=\cos n\theta-nh\sin n\theta-\frac{n^2h^2}{2}\cos n\theta+\ldots\ldots$$

Again $A_{2r}\sin^{2r}\theta$, the term on the right, becomes

$$A_{2r}[\sin(\theta+h)]^{2r}=A_{2r}(\sin\theta\cos h+\cos\theta\sin h)^{2r}$$

$$=A_{2r}\left(\sin\theta+h\cos\theta-\frac{h^2}{2}\sin\theta-\ldots\ldots\right)^{2r}$$

Hence $\quad \cos n\theta-nh\sin n\theta-\dfrac{n^2h^2}{2}\cos n\theta+\ldots\ldots$

$$=\Sigma A_{2r}\left(\sin\theta+h\cos\theta-\frac{h^2}{2}\sin\theta-\ldots\ldots\right)^{2r}$$

$$=\Sigma A_{2r}\left[\sin^{2r}\theta+2r\sin^{2r-1}\theta\left(h\cos\theta-\frac{h^2}{2}\sin\theta+\ldots\right)\right.$$

$$\left.+\frac{2r(2r-1)}{2}\sin^{2r-2}\theta\left(h\cos\theta-\frac{h^2}{2}\sin\theta+\ldots\ldots\right)^2+\ldots\right.$$

$$=\Sigma A_{2r}\left[\sin^{2r}\theta+2rh\sin^{2r-1}\theta\cos\theta\right.$$

$$\left.+h^2\left\{\frac{2r(2r-1)}{2}\sin^{2r-2}\theta\cos^2\theta-r\sin^{2r}\theta\right\}+\ldots\ldots\right] \quad \ldots\ldots(1)$$

Equating the coefficients of h^2 from both sides,

$$-\frac{n^2}{2}\cos n\theta=\Sigma A_{2r}\left\{\frac{2r(2r-1)}{2}\sin^{2r-2}\theta\cos^2\theta-r\sin^{2r}\theta\right\}$$

$$=A_2\{\cos^2\theta-\sin^2\theta\}+A_4\{2\cdot3\sin^2\theta\cos^2\theta-2\sin^4\theta\}+$$

$$\ldots\ldots+A_{2r}\left\{\frac{2r(2r-1)}{2}\sin^{2r-2}\theta\cos^2\theta-r\sin^{2r}\theta\right\}+\ldots\ldots$$

and this, from (1) $=-\dfrac{n^2}{2}\Sigma A_{2r}\sin^{2r}\theta$

Hence equating coefficients of $\sin^{2r}\theta$ from both sides

$$-\frac{n^2}{n}A_{2r}=-A_{2r}\left\{\frac{2r(2r-1)}{2}+r\right\}+A_{2r+2}\frac{(2r+2)(2r+1)}{2},$$

by replacing $\cos^2\theta$ by $1-\sin^2\theta$ and noting that $\sin^{2r}\theta$ is involved in the coefficient of A_{2r+2}, and A_{2r} only.

It follows that $\quad A_{2r+2}=-\dfrac{n^2-(2r)^2}{(2r+1)(2r+2)}A_{2r}.$

Since $A_0=1$, the coefficients A_2, A_4,......are obtained from the above.

$$A_2=-\frac{n^2}{1\cdot2}\ A_0=-\frac{n^2}{1\cdot2},$$

$$A_4=-\frac{n^2-2^2}{3.4}\ A_2=\frac{n^2\,(n^2-2^2)}{1.2.3.4}, \text{ and so on.}$$

Hence, from (1)

$$\cos n\,\theta=1-\frac{n^2}{2\,!}\sin^2\theta+\frac{n^2\,(n^2-2^2)}{4\,!}\sin^4\theta$$
$$-\frac{n^2\,(n^2-2^2)\,(n^2-4^2)}{6\,!}\sin^6\theta+\ldots \quad \ldots(A)$$

In the preceding process, equating the coefficients of h from both sides in (2), we have

$$-n\sin n\,\theta=\Sigma\,(2\,r\sin^{2r-1}\theta\cos\theta)\,A_{2r}$$
$$=\cos\theta\,[\,A_2\,.\,2\sin\theta+A_4\,.\,4\sin^3\theta+A_6\,.\,6\sin^5\theta+\ldots]$$

Substituting the values of A_2, A_4,......, we have

$$\sin n\,\theta=n\cos\theta\left[\sin\theta-\frac{n^2-2^2}{3\,!}\sin^3\theta\right.$$
$$\left.+\frac{(n^2-2^2)\,(n^2-4^2)}{5\,!}\sin^5\theta-\ldots\ldots\right] \quad \ldots(B)$$

These expansions (A) and (B) are true when n is even. When n is odd we may assume

$$\sin n\,\theta=A_1\sin\theta+A_3\sin^3\theta+\ldots\ldots+A_{2r+1}\sin^{2r+1}\theta+\ldots\ldots$$
$$\ldots\ldots+A_n\sin^n\theta.$$

Proceeding as before we may establish

$$\sin n\,\theta=n\sin\theta-\frac{n\,(n^2-1^2)}{3\,!}\sin^3\theta$$
$$+\frac{n\,(n^2-1^2)\,(n^2-3^2)}{5\,!}\sin^5\theta-\ldots \quad \ldots(C)$$

and
$$\cos n\,\theta=\cos\theta\left[1-\frac{n^2-1^2}{2\,!}\sin^2\theta\right.$$
$$\left.+\frac{(n^2-1^2)\,(n^2-3^2)}{4\,!}\sin^4\theta-\ldots\ldots\ldots\right] \quad \ldots(D)$$

5·43. If we write $\frac{\pi}{2}-\theta$ for θ in the expansions (*A*), (*B*), (*C*), (*D*) above we obtain the following formulae.

If n be even,

$$(-1)^{\frac{n}{2}}\cos n\theta=1-\frac{n^2}{2\,!}\cos^2\theta+\frac{n^2(n^2-2^2)}{4\,!}\cos^4\theta-\ldots\ldots \quad (1)$$

$$(-1)^{\frac{n}{2}+1}\sin n\theta=n\sin\theta\Big[\cos\theta-\frac{n^2-2^2}{3\,!}\cos^3\theta$$
$$+\frac{(n^2-2^2)(n^2-4^2)}{5\,!}\cos^5\theta-\ldots\ldots\Big] \quad \ldots(2)$$

If n be odd,

$$(-1)^{\frac{n-1}{2}}\cos n\theta=n\cos\theta-\frac{n(n^2-1^2)}{3\,!}\cos^3\theta$$
$$+\frac{n(n^2-1^2)(n^2-3^2)}{5\,!}\cos^5\theta-\ldots \quad \ldots(3)$$

and $$(-1)^{\frac{n-1}{2}}\sin n\theta=\sin\theta\Big[1-\frac{n^2-1^2}{2\,!}\cos^2\theta$$
$$+\frac{(n^2-1^2)(n^2-3^2)}{4\,!}\cos^4\theta-\ldots\ldots\Big] \quad \ldots(4)$$

5·5. The methods employed in obtaining expansions of $\cos n\theta$ or $\sin n\theta$ in ascending or descending powers of $\cos\theta$ or $\sin\theta$, as shown in the previous articles, are somewhat complicated, at least for a beginner. The student who is acquainted with Differential Calculus can, however, profitably employ the Maclaurin's series to get these expansions.

The following example will illustrate.

Let $\quad y=f(x)=\sin(m\sin^{-1}x)$

Maclaurin's series for $f(x)$ will be

$$f(x)=(y)_0+x\,(y_1)_0+\frac{x^2}{2\,!}(y_2)_0+\ldots\ldots,$$

where $(y)_0$, $(y_1)_0$, $(y_2)_0$...are the values of $f(x)$, $f'(x)$, $f''(x)$...... when $x=0$.

Now $\quad y_1=\cos(m\sin^{-1}x).\ \dfrac{m}{\sqrt{1-x^2}}$.

or $\quad y_1^2(1-x^2)=m^2(1-y^2)$

Differentiating, we have, after dividing out by $2y_1$,

$$(1-x^2)y_2-x\,y_1+m^2\,y=0 \qquad \text{...(1)}$$

Differentiating this equation n times, we get

$$(1-x^2)\,y_{n+2}-(2n+1)\,x\,y_{n+1}-(n^2-m^2)\,y_n=0$$

Putting $\quad x=0, \quad (y_{n+2})_0=(n^2-m^2)(y_n)_0$

Now $\quad (y)_0=0,\ (y_1)_0=m,\ (y_2)_0=0$ from (1)

Hence putting $n=2, 4,\ldots\ldots$,

$$(y_2)_0=(y_4)_0=(y_6)_0=\ldots\ldots=0$$

Putting $n=1, 3, 5,\ldots\ldots$

$(y_3)_0=(1^2-m^2)(y_1)_0=-m(m^2-1^2)$

$(y_5)_0=(3^2-m^2)(y_3)_0=m(m^2-1^2)(m^2-3^2)$, and so on.

Hence $\quad \sin(m\sin^{-1}x)=m\,x-\dfrac{m(m^2-1^2)}{3\,!}\,x^3$

$$+\frac{m(m^2-1^2)(m^2-3^2)}{5\,!}x^5-\ldots\ldots$$

Putting $\quad x=\sin\theta$, we have

$\sin m\theta=m\ \sin\ \theta-\dfrac{m(m^2-1^2)}{3\,!}\ \sin^3\theta$

$$+\frac{m(m^2-1^2)(m^2-3^2)}{5\,!}\ \sin^5\theta-\ldots \qquad \text{...(2)}$$

This is expansion (*C*) of art. 5·42. Moreover there is no restriction on the value of m. Hence (2) is true for all m, integral or otherwise.

The student may note that by suitably changing $f(x)$ the other expansions can be obtained.

5·6. Expanding $\sin n\theta$ or $\cos n\theta$ in ascending powers of $\sin\theta$ or $\cos\theta$ and equating coefficients of different powers of n from both sides in the expansions obtained above we can deduce various other expansions.

In the previous article, we have seen, that for all n,

$$\sin n\theta = n\sin\theta - \frac{n(n^2-1^2)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta - \ldots$$

Expanding $\sin n\theta$, equating the coefficients of n, we have

$$\theta = \sin\theta + \frac{1^2}{3!}\sin^3\theta + \frac{1^2.3^2}{5!}\sin^5\theta + \ldots\ldots$$

Putting $\sin\theta = x$, or $\theta = \sin^{-1}x$, we obtain

$$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\ldots$$

Examples

1. From the expansion of $\sin^{2n+1}\theta$ in term of the sines of multiples of θ, show that

$$1-(2n-1)+\frac{2n(2n-3)}{2!}-\frac{2n(2n-1)(2n-5)}{3!}$$
$$+\ldots\ldots\text{to } (n+1) \text{ terms} = 0.$$

If $x = \cos\theta + i\sin\theta$,

$$(2i\sin\theta)^{2n+1} = \left(x - \frac{1}{x}\right)^{2n+1}$$

$$= x^{2n+1} - (2n+1)\,x^{2n-1} + \frac{(2n+1)\,2n}{2!}x^{2n-3} - \ldots\ldots$$

$$\ldots\ldots - \frac{1}{x^{2n+1}}$$

$$= \left(x^{2n+1} - \frac{1}{x^{2n+1}}\right) - (2n+1)\left(x^{2n-1} - \frac{1}{x^{2n-1}}\right)$$

$$+\frac{(2n+1)\,2n}{2\,!}\left(x^{2n-3}-\frac{1}{x^{2n-3}}\right)-\ldots\ldots \text{to } (n+1) \text{ terms.}$$

$$=2i\Big[\sin(2n+1)\theta-(2n+1)\sin(2n-1)\theta$$

$$+\frac{(2n+1)\,2n}{2\,!}\sin(2n-3)\theta\ldots\ldots\text{to } (n+1) \text{ terms}\Big]$$

Hence $(-1)^n\, 2^{2n} \sin^{2n+1}\theta=\sin(2n+1)\theta$

$$-(2n+1)\sin(2n-1)\theta+\frac{(2n+1)\,2n}{2\,!}\sin(2n-3)\theta$$

$$-\ldots\ldots \text{to } (n+1) \text{ terms.}$$

Since the lowest power of θ in $\sin^{2n+1}\theta$ is $2n+1$, equating the coefficients of θ from both sides, we have

$$(2n+1)-(2n+1)(2n-1)+\frac{(2n+1)\,2n}{2\,!}(2n-3)$$

$$-\frac{(2n+1)\,2n\,(2n-1)}{3\,!}(2n-5)+\ldots\ldots \text{to } (n+1) \text{ terms}=0$$

Dividing by $2n+1$, the result follows.

2. Prove that

$$\theta\sec\theta=\sin\theta+\frac{2}{3}\sin^3\theta+\frac{2\cdot4}{3\cdot5}\sin^5\theta+\ldots\ldots$$

When n is even, we have

$$\frac{\sin n\theta}{\cos\theta}=n\left\{\sin\theta-\frac{n^2-2^2}{3\,!}\sin^3\theta\right.$$

$$\left.+\frac{(n^2-2^2)(n^2-4^2)}{3\,!}\sin^5\theta+\ldots\ldots\right\}$$

Equating the coefficients of n, we get

$$\theta\sec\theta=\sin\theta+\frac{2^2}{3\,!}\sin^3\theta+\frac{2^2\cdot4^2}{5\,!}\sin^5\theta+\ldots\ldots$$

$$=\sin\theta+\frac{2}{3}\sin^3\theta+\frac{2\cdot4}{3\cdot5}\sin^5\theta+\ldots\ldots$$

Exercises V

Prove that

1. $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$
2. $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$
3. $\cos 6\theta = 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta.$
4. $\sin^4\theta\cos^5\theta = 2^{-8}(\cos 9\theta + \cos 7\theta) - 2^{-6}(\cos 5\theta + \cos 3\theta) + 2^{-7}(3\cos\theta).$

5. Show that

$$\frac{1}{1+e\cos\theta} = \frac{1}{\sqrt{1-e^2}}(1 - 2n\cos\theta + 2n^2\cos 2\theta - \ldots),$$

where $2n = e(1+n^2)$.

6. Prove that

$$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \ldots\ldots,$$

from the identity

$$p^n + q^n = (p+q)^n - n(p+q)^{n-2}pq + \frac{n(n-3)}{2!}(p+q)^{n-4}p^2q^2 - \ldots$$

(Travancore '40)

Prove that

7. $$\frac{1}{2}\theta^2 = \frac{1}{2}\sin^2\theta + \frac{2}{3}\frac{\sin^4\theta}{4} + \frac{2\cdot 4}{3\cdot 5}\frac{\sin^6\theta}{6} + \ldots\ldots$$

8. $$\frac{1}{6}\theta^3 = \frac{1}{3!}\sin^3\theta + \frac{1^2+3^2}{5!}\sin^5\theta + \frac{1^2.3^2+3^2.5^2+5^2.1^2}{7!}\sin^7\theta + \ldots\ldots$$

9. $$\sec\theta = 1 + \frac{1}{2}\sin^2\theta + \frac{1\cdot 3}{2\cdot 4}\sin^4\theta + \ldots\ldots$$

10. Prove that $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \dfrac{2}{3}x^3 + \dfrac{2\cdot 4}{3\cdot 5}x^5 + \ldots\ldots$

and deduce that

$$\tan^{-1} y = \frac{y}{1+y^2}\left[1+\frac{2}{3}\frac{y^2}{1+y^2}+\frac{2\cdot 4}{3\cdot 5}\left(\frac{y^2}{1+y^2}\right)^2+\ldots\ldots\right]$$

11. From the expansion of $\sin^n \theta$ in multiple angles, show that

$$n^p - n(n-2)^p + \frac{n(n-1)}{2!}(n-4)^p - \ldots\ldots \text{to } \frac{n+1}{2} \text{ terms} = 0,$$

where n and p are odd positive integers and $p < n$.

12. Prove that $\pi^2 = 18 \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n+2)!}$ (Alld. '53)

CHAPTER VI

Trigonometrical and Exponential Functions

6·1. The student has already come across the series, the exponential function,

$$1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots\ldots+\frac{x^n}{n\,!}+\ldots\ldots$$

when x is real. This series is absolutely convergent for all values of x, and is equal to e^x, where

$$e=1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\ldots$$

Let us now consider the series

$$1+z+\frac{z^2}{2\,!}+\frac{z^3}{3\,!}+\ldots\ldots+\frac{z^n}{n\,!}+\ldots\ldots,$$

where the complex number $z=x+i\,y=r\,(\cos\,\theta+i\,\sin\,\theta)$.

If this series is represented by Exp. (z), we have

$$\text{Exp. } (z)=1+z+\frac{z^2}{2\,!}+\frac{z^3}{3\,!}+\ldots\ldots$$

$$=1+r\,(\cos\,\theta+i\,\sin\,\theta)+\frac{r^2}{2\,!}(\cos\,2\,\theta+i\,\sin\,2\,\theta)+\ldots\ldots$$

$$=1+r\,\cos\,\theta+\frac{r^2\cos\,2\theta}{2\,!}+\ldots\ldots$$

$$+i\left(r\,\sin\,\theta+\frac{r^2}{2\,!}\,\sin\,2\,\theta+\ldots\ldots\right)$$

By comparing each of the series

$$1+r\,\cos\,\theta+\frac{r^2}{2\,!}\,\cos\,2\,\theta+\ldots\ldots+\frac{r^n}{n\,!}\,\cos\,n\,\theta+\ldots\ldots \quad \ldots(1)$$

and $$r\,\sin\,\theta+\frac{r^2}{2\,!}\,\sin\,2\,\theta+\ldots\ldots+\frac{r^n}{n\,!}\,\sin\,n\,\theta+\ldots\ldots \quad \ldots(2)$$

with $$1+r+\frac{r^2}{2\,!}+\frac{r^3}{3\,!}+\ldots\ldots+\frac{r^n}{n}+\ldots\ldots,$$

which is absolutely convergent, we see that (1) and (2) are both absolutely convergent, and hence the series

$$\text{Exp. }(z)=1+z+\frac{z^2}{2!}+\ldots\ldots+\frac{z^n}{n!}+\ldots\ldots$$

is also absolutely convergent for all z.

6·11. Product of exponential functions.

From the preceding, we have

$$\text{Exp. }(z_1)\times\text{Exp. }(z_2)=\left(1+z_1+\frac{z_1^2}{2!}+\ldots\ldots+\frac{z_1^n}{n!}+\ldots\ldots\right)$$
$$\times\left(1+z_2+\frac{z_2^2}{2!}+\ldots\ldots+\frac{z_2^n}{n!}+\ldots\ldots\right)$$
$$=1+(z_1+z_2)+\left(\frac{z_1^2}{2!}+z_1\,z_2+\frac{z_2^2}{2!}\right)+\ldots\ldots$$
$$+\left\{\frac{z_1^n}{n!}+\frac{z_1^{n-1}}{(n-1)!}\,\frac{z_2}{1!}+\frac{z_1^{n-2}}{(n-2)!}\,\frac{z_2^2}{2!}+\ldots\ldots+\frac{z_2^n}{n!}\right\}+\ldots\ldots,$$

by multiplying the two series and grouping together terms of the same degree.

Hence $\text{Exp. }(z_1)\times\text{Exp. }(z_2)=1+\frac{(z_1+z_2)}{1!}+\frac{(z_1+z_2)^2}{2!}+\ldots\ldots$

$$\ldots\ldots+\frac{(z_1+z_n)^n}{n!}+\ldots\ldots$$

$$=\text{Exp. }(z_1+z_2).$$

6·12. Evidently the above process can be extended, and we have

$$\text{Exp. }(z_1)\times\text{Exp. }(z_2)\times\ldots\ldots\times\text{Exp. }(z_n)=\text{Exp. }(z_1+z_2+\ldots\ldots+z_n)$$

Putting $z_1=z_2=\ldots\ldots=z_n=z$, it follows

$$\{\text{Exp. }(z)\}^n=\text{Exp. }(nz).$$

6·13. On account of the similarities of properties of the exponential functions of real and complex numbers. *viz* e^x and Exp. (z) it is convenient and customary to use the same notation for them, Hence we represent Exp. (z) by e^z, and write

$$e^z = 1 + z + \frac{z^2}{2\,!} + \cdots\cdots + \frac{z^n}{n\,!} + \cdots\cdots$$

for all values of z, real or complex. It must, however, be remembered that it z if real,

$$e = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \frac{1}{3\,!} + \cdots\cdots$$

but it z if complex, e has no such meaning

6·2. Circular functions of complex quantities.

For all real values of x, we have seen that

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots\cdots + \frac{(-1)^n x^{2n}}{(2\,n)\,!} + \cdots\cdots$$

and $$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots\cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)\,!} + \cdots\cdots$$

If z be complex let us now *define* $\cos z$ and $\sin z$ as

$$\cos z = 1 - \frac{z^2}{2\,!} + \frac{z^4}{4\,!} - \cdots\cdots + \frac{(-1)^n z^{2n}}{(2\,n)\,!} + \cdots\cdots$$

and $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5\,!} - \cdots\cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)\,!} + \cdots\cdots$$

so that similarities of notation are maintained. If z be complex the other trigonometrical functions are also *defined* as follows :

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},$$

$$\sec z = \frac{1}{\cos z} \text{ and } \operatorname{cosec} z = \frac{1}{\sin z}.$$

6·3. Properties of exponential functions.

(*a*) Euler's Theorem : For all values of θ, real or complex,

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

From definition of $e^{i\theta}$, $\cos\theta$ and $\sin\theta$, we have

$$e^{i\theta}=1+(i\theta)+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\ldots\ldots$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\ldots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\ldots\right)$$

$$=\cos\theta+i\sin\theta.$$

(*b*) $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$, and $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Since $e^{i\theta}=\cos\theta+i\sin\theta$,

$$e^{-i\theta}=\cos(-\theta)+i\sin(-\theta)=\cos\theta-i\sin\theta$$

On adding and subtracting, the result follows.

(*c*) *Exp.* (z) *is a periodic function.*

If $z=x+iy$, with real values for x and y,

$$\text{Exp.}(z)=\text{Exp.}(x+iy)$$
$$=\text{Exp.}(x)\times\text{Exp.}(iy)$$
$$=e^x.e^{iy}=e^x(\cos y+i\sin y).$$

But since $\sin y=\sin(2n\pi+y)$ and $\cos y=\cos(2n\pi+y)$, when n is a positive or negative integer,

$$\text{Exp.}(z)=e^x\{\cos(2n\pi+y)+i\sin(2n\pi+y)\}$$

$$=e^x.e^{(2n\pi+y)i}$$

$$=e^{x+yi+2n\pi i}$$

$$=e^{z+2n\pi i}$$

Thus the *period* of Exp. (z) is $2\pi i$.

(*c*) $\cos^2 x+\sin^2 x=1$, x being real or complex.

Since $\cos x=\dfrac{e^{ix}+e^{-ix}}{2}$, and $\sin x=\dfrac{e^{ix}-e^{-ix}}{2i}$

for all values of x, real or complex, on squaring and a
we get the result.

(*d*) $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$, θ and ϕ being
or complex.

$$\cos(\theta+\phi)=\frac{e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}}{2}$$

$$=\frac{e^{i\theta}\cdot e^{i\phi}+e^{-i\theta}\cdot e^{-i\phi}}{2}$$

$$=\frac{(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)+(\cos\theta-i\sin\theta)(\cos\phi-}{2}$$

$$=\cos\theta\cos\phi-\sin\theta\sin\phi.$$

6·31. As a matter of fact, with the help of the expon
functions, we can prove that trigonometrical formulae tr
real values are also true for complex values.

Thus, if the variables are complex,

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi,$$
$$\sin 2z=2\sin z\cos z,$$
$$\cos 2z=2\cos^2 z-1=1-2\sin^2 z=\cos^2 z-\sin^2$$
$$\sin 3z=3\sin z-4\sin^3 z,$$
$$\cos 3z=4\cos^3 z-3\cos z,$$
$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi},$$

and so on.

Examples

1. Show that $\exp.\left(\pm\dfrac{\pi}{2}i\right)=\pm i$.

$$\exp.\left(\frac{\pi i}{2}\right)=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i$$

and $\exp.\left(\frac{-\pi i}{2}\right)=\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}=-i.$

2. Separate into real and imaginary parts : $\frac{\sin(i\theta)}{x+iy}$.

$$\frac{\sin i\theta}{x+iy}=\frac{e^{-\theta}-e^{\theta}}{2i(x+iy)}=\frac{e^{\theta}-e^{-\theta}}{2(y-ix)}$$

$$=\frac{(e^{\theta}-e^{-\theta})(y+ix)}{2(x^2+y^2)}.$$

Exercises

1. Use the exponential values of sine and cosine to prove that

(*a*) $\cos 2z=2\cos^2 z-1=1-2\sin^2 z.$

(*b*) $\cos\theta+\cos\phi=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$.

(*c*) $\frac{\sin 2\theta}{1-\cos 2\theta}=\cot\theta.$

2. Separate into real and imaginary parts :

(*i*) e^{z^3}, when $z=x+iy$.

(*ii*) $\sin(\theta+i\phi)$.

(*iii*) $\tan(\theta i).e^{\alpha i}$.

3. Prove that $\tan(\alpha+i\beta)=\frac{2\sin 2\alpha+i(e^{2\beta}-e^{-2\beta})}{e^{2\beta}+2\cos 2\alpha+e^{-2\beta}}$.

4. Prove that

(i) $\{\sin(\alpha+\theta)-e^{\alpha i}\sin\theta\}^n=\sin^n\alpha\; e^{-n\theta i}$

(ii) $\{\sin(\alpha-\theta)+e^{\alpha i}\sin\theta\}^n=\sin^{n-1}\alpha\,\{\sin(\alpha-n\,\theta)+e^{\alpha i}\sin n\,\theta.\}$

5. If α, β be the imaginary cube roots of unity, prove that

$$\alpha\, e^{\alpha x}+\beta\, e^{\beta x}=e^{-\frac{x}{2}}\left(\sqrt{3}\sin\frac{\sqrt{3}}{2}x+\cos\frac{\sqrt{3}}{2}x\right).$$

(Bombay '47)

6·4. Logarithm of a complex quantity : definition.

If $e^z=A$, where $z=x+i\,y$, and $A=a+i\,b$ are complex quantities, then z is said to be Napierian logarithm of A to the base e, and is expressed as

$$z=\log_e A.$$

But if $\qquad e^z=A,\ e^{z+2\,n\,\pi\,i}=A.$

Hence $\log_e A=z+2\,n\,\pi\,i$, n being an integer or zero.

Thus logarithm is a multiple valued function. If $n=0$ the value of logarithm is called the principal value. It is customary to express the general and principal values of the logarithm by the notations $\text{Log}_e A$ and $\log_e A$ respectively. Hence

$$\text{Log}_e A=\log_e A+2\,n\,\pi\,i.$$

6·41. To find $\text{Log}_e(a+i\,b)$.

Let $\qquad a+i\,b=r(\cos\theta+i\sin\theta)$,

where $\qquad r=+\sqrt{a^2+b^2}$, and $\theta=\tan^{-1}\left(\frac{b}{a}\right)$

Now $\text{Log}_e(a+i\,b)=\text{Log}_e\, r(\cos\theta+i\sin\theta)=\text{Log}\,(r\,e^{i\,\theta})$

$$=\text{Log}\left\{r\,e^{i(\theta+2\,n\,\pi)}\right\}$$

$$=\log_e r+i(\theta+2\,n\,\pi)$$

$$=\log_e\sqrt{a^2+b^2}+i\left(\tan^{-1}\frac{b}{a}+2\,n\,\pi\right).$$

The principal value is given by

$$\log_e (a+i\,b)=\log_e\sqrt{a^2+b^2}+i\tan^{-1}\frac{b}{a}$$

Cor. If $b=0$, then the number is real, and

$$\text{Log}_e\, a=\log_e a+2\,n\,\pi\,i,$$

which shows that the logarithm of a real quantity is also multiple valued ; the principal value is real, all the other values are imaginary.

Examples

1. $\text{Log}\, i=2\,n\,\pi\,i+\log i=2n\pi i+\log e^{\frac{\pi}{2}i}$

$$=2\,n\,\pi\,i+\frac{\pi\,i}{2}=\frac{\pi}{2}\,(4\,n+1)\,i.$$

2. Find the general value of Log (−3).

Let $-3=-3+i.\,o=r\,(\cos\theta+i\sin\theta)$

Hence $r\cos\theta=-3,\ r\sin\theta=0,\ r=3,\ \cos\theta=-1$

$\therefore\quad \theta=\pi$

Hence $\text{Log}\,(-3)=2\,n\,\pi\,i+\log\,(3\,e^{i\,\pi})$

$$=\log 3+i\,\pi+2\,n\,\pi\,i=\log 3+i\,\pi\,(2\,n+1)$$

The principal value is $\log(-3)=\log 3+i\,\pi$.

3. Prove that $\text{Log}_e\dfrac{a-b+i\,(a+b)}{a+b+i\,(a-b)}=i\left\{2\,n\,\pi+\tan^{-1}\dfrac{2\,a\,b}{a^2-b^2}\right\}$

(Dacca '51)

$$\text{Log}\,\frac{a-b+i\,(a+b)}{a+b+i\,(a-b)}=2\,n\,\pi\,i+\log\frac{a-b+i\,(a+b)}{a+b+i\,(a-b)}$$

$$=2\,n\,\pi\,i+\log\{(a-b)+i\,(a+b)\}-\log\{(a+b)+i\,(a-b)\}$$

$$=2n\pi i+\left[\log\sqrt{(a-b)^2+(a+b)^2}+i\tan^{-1}\frac{a+b}{a-b}\right.$$

$$\left.-\log\sqrt{(a+b)^2+(a-b)^2}-i\tan^{-1}\frac{a-b}{a+b}\right]$$

$$=2n\pi i+i\tan^{-1}\frac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{1+1}$$

$$=i\left\{2n\pi+\tan^{-1}\frac{2ab}{a^2-b^2}\right\}.$$

6·5. The general exponential function : definition.

If a and z are two quantities, real or complex, then the symbol a^z is defined to mean $e^{z\,\text{Log}_e a}$. This is the general value of a^z ; the principal value is $e^{z\log_e a}$. Since $\log_e a$ is a multiple valued function, a^z is also multiple valued.

The general value of a^z is given by

$$a^z=e^{z\,\text{Log}\,a}=1+\frac{z\,\text{Log}\,a}{1!}+\frac{(z\,\text{Log}\,a)^2}{2!}+\ldots\ldots$$

and the principal value is

$$e^{z\log a}=1+\frac{z\log a}{1!}+\frac{(z\log a)^2}{2!}+\ldots\ldots$$

6·51. To express $(\alpha+i\beta)^{x+iy}$ in the form $A+iB$.

Let $\alpha+i\beta=r(\cos\theta+i\sin\theta)=re^{i\theta}$

where $r=\sqrt{\alpha^2+\beta^2}$ and $\theta=\tan^{-1}\frac{\beta}{\alpha}$

Hence $(\alpha+i\beta)^{x+iy}=e^{(x+iy)\,\text{Log}\,(\alpha+i\beta)}$ by definition

$$=e^{(x+iy)\{2n\pi i+\log(re^{i\theta})\}}$$

$$=e^{(x+i\,y)\{\log r+i\,(\theta+2\,n\,\pi)\}}$$

$$=e^{x\log r-y\,(\theta+2\,n\,\pi)}\,.\,e^{i\,\{y\log r+x\,(\theta+2\,n\,\pi)\}}$$

$$=A+i\,B,\quad \text{where}$$

$$A=e^{x\log r-y\,(\theta+2\,n\,\pi)}\,.\cos\{y\log r+x\,(\theta+2\,n\,\pi)\}$$

and $$B=e^{x\log r-y\,(\theta+2\,n\,\pi)}\,.\sin\{y\log r+x\,(\theta+2\,n\,\pi)\}$$

Putting $n=0$, the principal value of

$$(\alpha+i\,\beta)^{x+i\,y}=e^{x\log r-y\,\theta}\{\cos(y\log r+x\,\theta)+i\sin(y\log r+x\,\theta)\}.$$

Examples

1. If $i^{i^{i\ldots\text{ad inf.}}}=A+i\,B$, principal values only being considered,

prove that $\tan\dfrac{\pi A}{2}=\dfrac{B}{A}$, and $A^2+B^2=e^{-\pi B}$.

From the given condition $i^{A+i\,B}=A+i\,B$

or $$e^{(A+i\,B)\log i}=A+i\,B$$

or $$e^{(A+i\,B)\left(\log 1+i\frac{\pi}{2}\right)}=A+i\,B$$

or $$e^{-\frac{B\pi}{2}}\,e^{\frac{A\pi i}{2}}=A+i\,B$$

Hence $e^{-\frac{B\pi}{2}}\cos\dfrac{A\pi}{2}=A$ and $e^{-\frac{B\pi}{2}}\sin\dfrac{A\pi}{2}=B$, by equating the real and imaginary parts ; whence the result.

2. If $u=\log_e \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=x+a_3\,x^3+a_5\,x^5+\ldots$,

prove that $\quad x=u-a_3\,u^3+a_5\,u^5-\ldots\ldots$ (Agra '37)

Since $\quad u=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right),$

$$e^u=\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

By Componendo and Dividendo,

$$\tan\frac{x}{2}=\frac{e^u-1}{e^u+1}=\frac{e^{\frac{u}{2}}-e^{-\frac{u}{2}}}{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}$$

$$=\frac{e^{i\left(\frac{iu}{2}\right)}-e^{-i\left(\frac{iu}{2}\right)}}{2i}\cdot\frac{2}{e^{i\left(\frac{iu}{2}\right)}+e^{-i\left(\frac{iu}{2}\right)}}\cdot(-i)$$

$$=-i\tan\frac{iu}{2}$$

or $\tan\frac{iu}{2}=i\tan\frac{x}{2}=\dfrac{i\sin\frac{x}{2}}{\cos\frac{x}{2}}.$

Again, by Componendo and Dividendo,

$$\frac{\cos\frac{x}{2}+i\sin\frac{x}{2}}{\cos\frac{x}{2}-i\sin\frac{x}{2}}=\frac{1+\tan\frac{iu}{2}}{1-\tan\frac{iu}{2}}$$

or $$e^{ix}=\tan\left(\frac{\pi}{4}+\frac{iu}{2}\right)$$

or $$i\,x=\log\tan\left(\frac{\pi}{4}+\frac{i\,u}{2}\right)$$

$$=(i\,u)+a_3\,(i\,u)^3+a_5\,(i\,u)^5+\ldots\ldots$$

$\therefore$ $$x=u-a_3\,u^3+a_5\,u^5-\ldots\ldots\ldots$$

3. Show that $\text{Log}_i\, i=\dfrac{4m+1}{4n+1}$, where m and n are integers.

(Andhra '40)

Since we know that $\log_b a=\dfrac{\log_e a}{\log_e b}$,

$$\text{Log}_i\, i=\frac{\text{Log}_e\, i}{\text{Log}_e\, i}=\frac{2\,m\,\pi\,i+i\dfrac{\pi}{2}}{2\,n\,\pi\,i+i\,\dfrac{\pi}{2}}=\frac{4\,m+1}{4\,n+1},$$

It may be noted that the general values are taken in the numerator and denominator, and $m=n$ is only a particular case.

Exercises VI

1. Prove that $\text{Log}_e\,(-1)=i\,(2\,n+1)\,\pi$.

2. If $A+Bi=\log\,(a+i\,b)$, show that

$$\tan B=\frac{b}{a}\ \text{ and } 2A=\log\,(a^2+b^2).$$

3. Show that $\text{Log}\,(1+\sqrt{3}\,i)=\log 2+\pi\left(2\,n+\dfrac{1}{3}\right)i$.

4. Prove that $\log\log\,(\alpha+i\,\beta)=\dfrac{1}{2}\log\left[\,\{\log\sqrt{\alpha^2+\beta^2}\,\}^2\right.$

$$\left.+\left(\tan^{-1}\frac{\beta}{\alpha}\right)^2\right]+i\tan^{-1}\left(\frac{\tan^{-1}\dfrac{\beta}{\alpha}}{\log\sqrt{\alpha^2+\beta^2}}\right).$$

5. Show that $i^i = e^{-(4n+1)\frac{\pi}{2}}$.

6. If $\dfrac{(1+i)^{p+qi}}{(1-i)^{p-qi}} = \alpha + i\beta$, prove that one value of

$$\tan^{-1}\frac{\beta}{\alpha} \text{ is } \frac{1}{2} p\pi + q \log_e 2.$$

7. If $\left[\dfrac{a+x+iy}{a-x-iy}\right]^{\alpha+i\beta} = X + iY$,

prove that one of the values of $\tan^{-1}\dfrac{Y}{X}$ is

$$\alpha \tan^{-1}\frac{2ay}{a^2-x^2-y^2} + \frac{\beta}{2}\log\frac{(a+x)^2+y^2}{(a-x)^2+y^2}.$$

8. Prove that $\tan\left[i \log \dfrac{a-bi}{a+bi}\right] = \dfrac{2ab}{a^2-b^2}$.

9. Show that $(a+ib)^{\alpha+i\beta}$ will be wholly real if

$\dfrac{\beta}{2}\log(a^2+b^2) + \alpha\tan^{-1}\dfrac{b}{a}$ is zero or an even multiple of $\dfrac{\pi}{2}$,

and wholly imaginary if this expression is an odd multiple of $\dfrac{\pi}{2}$.

10. If $a+bi = q^{x+iy}$, prove that $\dfrac{y}{x} = \dfrac{2\tan^{-1}\frac{b}{a}}{\log(a^2+b^2)}$.

(Mysore '39)

11. Prove that $(a+ia\tan\phi)^{\log(a\sec\phi)-i\phi}$ is a real number and find its value.

(Bombay '30)

12. Prove that the general value of $(1+i\tan\alpha)^{-i}$ is

$e^{\alpha+2m\pi}$ [cos (log cos α)$+i$ sin (log cos α)],

whose m is any integer. (Baroda '54)

13. If all the values of $(1+i\tan\alpha)^{1+i\tan\beta}$ are real, prove that one of them is $(\sec\alpha)^{\sec^2\beta}$.

(Lucknow '54)

CHAPTER VII

HYPERBOLIC FUNCTIONS

7·1. The expressions $\frac{1}{2}(e^{\theta}+e^{-\theta})$ and $\frac{1}{2}(e^{\theta}-e^{-\theta})$, where θ is real or complex, are called the *hyperbolic cosine* and *hyperbolic sine* of θ, and are indicated by the symbols $\cosh\theta$ and $\sinh\theta$ respectively.

Thus $\cosh\theta=\dfrac{e^{\theta}+e^{-\theta}}{2}$, $\sinh\theta=\dfrac{e^{\theta}-e^{-\theta}}{2}$.

The other hyperbolic functions are defined as under :

$$\tanh\theta=\frac{\sinh\theta}{\cosh\theta}=\frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}},\quad \coth\theta=\frac{\cosh\theta}{\sinh\theta}=\frac{e^{\theta}+e^{-\theta}}{e^{\theta}-e^{-\theta}},$$

$$\text{sech}\,\theta=\frac{1}{\cosh\theta}=\frac{2}{e^{\theta}+e^{-\theta}},\quad \text{cosech}\,\theta=\frac{1}{\sinh\theta}=\frac{2}{e^{\theta}-e^{-\theta}},$$

Similarity of notations with circular functions may be noted.

Since $\cos i\,\theta=\dfrac{e^{\theta}+e^{-\theta}}{2}$ and $\sin i\,\theta=\dfrac{e^{\theta}-e^{-\theta}}{2}\,i$,

we have $\cos i\,\theta=\cosh\theta$, and $\sin i\,\theta=i\sinh\theta$.

and hence $\tan i\,\theta=i\tanh\theta$.

From the above definitions we can easily deduce the following formulae :

$$\cosh\theta+\sinh\theta=e^{\theta},\quad \cosh\theta-\sinh\theta=e^{-\theta}.$$

$$\cosh^2\theta-\sinh^2\theta=1,$$

$$\text{sech}^2\theta+\tanh^2\theta=1,$$

and $$\coth^2\theta-\text{cosech}^2\theta=1.$$

Expressed in series, $\cosh \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots\ldots,$

$$\sinh \theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\ldots$$

7·11. Other formulae. We can deduce easily the following:

$$\cosh(\theta+\phi) = \cos(i\,\theta + i\,\phi)$$
$$= \cos i\,\theta \cos i\,\phi - \sin i\,\theta \sin i\,\phi$$
$$= \cosh \theta \cosh \phi + \sinh \theta \sinh \phi$$

From the definitions, too, this can also be easily established.

Similarly $\cosh(\theta-\phi) = \cosh \theta \cosh \phi - \sinh \theta \sinh \phi.$

$$\sinh(\theta \pm \phi) = \sinh \theta \cosh \phi \pm \cosh \theta \sinh \phi.$$

$$\tanh(\theta \pm \phi) = \frac{\tanh \theta \pm \tanh \phi}{1 \pm \tanh \theta \tanh \phi}.$$

$$\sinh 2\,\theta = 2 \sinh \theta \cosh \theta.$$

$$\cosh 2\,\theta = 2\cosh^2 \theta - 1 = 1 + 2\sinh^2 \theta = \cosh^2 \theta + \sinh^2 \theta.$$

$$\sinh \theta + \sinh \phi = 2 \sinh \frac{\theta+\phi}{2} \cosh \frac{\theta-\phi}{2}.$$

$$\sinh \theta - \sinh \phi = 2 \cosh \frac{\theta+\phi}{2} \sinh \frac{\theta-\phi}{2}.$$

$$\cosh \theta + \cosh \phi = 2 \cosh \frac{\theta+\phi}{2} \cosh \frac{\theta-\phi}{2}.$$

$$\cosh \theta - \cosh \phi = 2 \sinh \frac{\theta+\phi}{2} \sinh \frac{\theta-\phi}{2}.$$

7·2. Periodicity of hyperbolic functions

$$\cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2} = \frac{e^{\theta+2n\pi i} + e^{-(\theta+2n\pi i)}}{2}$$
$$= \cosh(\theta + 2\,n\,\pi\,i).$$

Similarly $\sinh \theta = \sinh(\theta + 2\,n\,\pi\,i)$, where n is an integer. Hence $\cosh \theta$ and $\sinh \theta$ are *periodic* functions, with $2\pi\,i$ as the period.

Since $e^{\theta+\pi i}=-e^{\theta}$ and $e^{-(\theta+\pi i)}=-e^{-\theta}$

we have $\cosh(\theta+\pi i)=-\cosh\theta$, and $\sinh(\theta+\pi i)=-\sinh\theta$,

so that $\tanh(\theta+\pi i)=\tanh\theta$,

and $\coth(\theta+\pi i)=\coth\theta$.

Hence $\tanh\theta$ and $\coth\theta$ are also periodic, but their periods are each πi.

7·3 Inverse circular Functions.

If $x+iy=\sin(\alpha+i\beta)$, whose α and β are real, then $\sin^{-1}(x+iy)$ is *defined* to be equal to $\alpha+i\beta$.

But $x+iy=\sin(\alpha+i\beta)=\sin\{n\pi+(-1)^n(\alpha+i\beta)\}$

Hence $\sin^{-1}(x+iy)$ is a multiple valued function and its *general* value indicated by $\text{Sin}^{-1}(x+iy)$, is given by

$$\text{Sin}^{-1}(x+iy)=n\pi+(-1)^n(\alpha+i\beta).$$

The *principal* value of this *inverse sine* is such that its real part, $n\pi+(-1)^n\alpha$, lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and is denoted by $\sin^{-1}(x+iy)$.

Hence $\text{Sin}^{-1}(x+iy)=n\pi+(-1)^n\sin^{-1}(x+iy)$.

Similarly if $x+iy=\cos(p+iq)$,

$$\text{Cos}^{-1}(x+iy)=2n\pi\pm\cos^{-1}(x+iy),$$
$$=2n\pi\pm(p+iq),$$

where the principal value, $\cos^{-1}(x+iy)$, is the value of the real part, $2n\pi\pm p$, lying between 0 and π. The principal part $\tan^{-1}(x+iy)$ of $\text{Tan}^{-1}(x+iy)$ is the real part of the latter lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and we have

$$\text{Tan}^{-1}(x+iy)=n\pi+\tan^{-1}(x+iy).$$

Similarly the other circular functions may be defined.

7·4. Inverse Hyperbolic Functions.

If $x=\sinh y$, then y is said to be the *inverse hyperbolic sine* of x and is denoted by $\sinh^{-1}x$. Similarly we may define

$\cosh^{-1}x$, $\tanh^{-1}x$, and so on.

If $\sinh^{-1} x = y$, $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$$\text{or } e^{2y} - 2\, e^y\, x - 1 = 0$$

solving this quadratic in e^y, $e^y = x \pm \sqrt{1+x^2}$

Hence $y = 2\, n\, \pi\, i + \log_e (x \pm \sqrt{1+x^2})$

Both these values of y can be included in the expression

$$y = n\, \pi\, i + (-1)^n \log (x + \sqrt{1+x^2}),$$

which is the *general value* of $\sinh^{-1} x$. The *principal value*, usually denoted by $\sinh^{-1} x$ is $\log (x + \sqrt{1+x^2})$.

Similarly, the general value of $\cosh^{-1} x$ is $2\, n\, \pi\, i \pm \log (x + \sqrt{x^2-1})$ and its *principal* value is $\log (x + \sqrt{x^2-1})$; $\tanh^{-1} x$ has $n\, \pi\, i + \dfrac{1}{2} \log \dfrac{1+x}{1-x}$ as its general value and its *principal* value is $\dfrac{1}{2} \log \dfrac{1+x}{1-x}$.

7·5. The following fundamental theorem of Algebra is taken for granted and is of wide use.

If $\qquad f(x + i\, y) = A + i\, B,$

then $\qquad f(x - i\, y) = A - i\, B.$

Thus if $\sin^{-1} (x + i\, y) = A + i\, B$, $\sin^{-1} (x - i\, y) = A - i\, B$.

Examples

1. Show that $\operatorname{Tanh}^{-1} x = n\, \pi\, i + \dfrac{1}{2} \log \dfrac{1+x}{1-x}$.

We know $\operatorname{Tanh}^{-1} x = n\, \pi\, i + \tanh^{-1} x$,

and let $\tanh^{-1} x = a$ or $\tanh a = x$

or $$\frac{e^{a}-e^{-a}}{e^{a}+e^{-a}}=x$$

or $$e^{2a}=\frac{1+x}{1-x},$$ by Componendo Dividendo.

Hence $$a=\tanh^{-1} x=\frac{1}{2}\ \log\ \frac{1+x}{1-x}$$

and $$\text{Tanh}^{-1} x=n\pi\ i+\frac{1}{2}\ \log\ \frac{1+x}{1-x}\ .$$

2. Express $\tan^{-1}(x+i\ y)$ as the sum of real and imaginary part. (Punjab '50)

Let $$\tan^{-1}(x+i\ y)=A+i\ B,$$

so that $$x+i\ y=\tan(A+i\ B),$$

and $$x-i\ y=\tan(A-i\ B).$$

Now $$\tan 2A=\tan\{(A+i\ B)+(A-i\ B)\}$$

$$=\frac{\tan(A+i\ B)+\tan(A-i\ B)}{1-\tan(A+iB)\tan(A-i\ B)}$$

$$=\frac{x+i\ y+x-i\ y}{1-x^2-y^2}$$

$$=\frac{2\ x}{1-x^2-y^2}$$

Hence $$A=\frac{1}{2}\ \tan^{-1}\ \frac{2\ x}{1-x^2-y}\ .$$

Also $$\tan 2\ i\ B=\tan\{(A+i\ B)-(A-i\ B)\}$$

$$=\frac{\tan(A+i\ B)-\tan(A-i\ B)}{1+\tan(A+i\ B)\tan(A-i\ B)}$$

or $$i\tanh 2\ B=\frac{(x+i\ y)-(x-i\ y)}{1+x^2+y^2}$$

$$=\frac{2\ i\ y}{1+x^2+y^2}$$

or, $\tanh 2B = \dfrac{2y}{1+x^2+y}$,

and $B = \dfrac{1}{2}\tanh^{-1}\dfrac{2y}{1+x^2+y^2}$

Hence, $\tan^{-1}(x+iy) = \dfrac{1}{2}\tan^{-1}\dfrac{2x}{1-x^2-y^2}$

$$+\frac{i}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}$$

If $x=\cos\alpha$, $y=\sin\alpha$, we deduce

$$\text{Tan}^{-1}(x+iy) = n\pi + \tan^{-1}(\cos\alpha + i\sin\alpha),$$

$$= n\pi + \frac{1}{2}\tan^{-1}\frac{2\cos\alpha}{0} + \frac{i}{2}\tanh^{-1}\frac{2\sin\alpha}{2}$$

$$= n\pi + \frac{\pi}{4} + \frac{i}{2}\tanh^{-1}\sin\alpha$$

$$= n\pi + \frac{\pi}{4} + \frac{i}{4}\log\frac{1+\sin\alpha}{1-\sin\alpha}$$

$$= n\pi + \frac{\pi}{4} + \frac{i}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

3. Express $\sin^{-1}(\cos\theta + i\sin\theta)$ in the form $x+iy$.

Let $\sin^{-1}(\cos\theta + i\sin\theta) = x+iy$

or $\sin(x+iy) = \cos\theta + i\sin\theta$

or $\sin x\cosh y + i\cos x\sinh y = \cos\theta + i\sin\theta$.

Separating the real and imaginary parts

$$\sin x\cosh y = \cos\theta \quad \ldots(1)$$

and $$\cos x\sinh y = \sin\theta \quad \ldots(2)$$

Eliminating y between (1) and (2), we have

$$\frac{\cos^2\theta}{\sin^2 x} - \frac{\sin^2\theta}{\cos^2 x} = \cosh^2 y - \sinh^2 y = 1$$

or $\cos^2 \theta \cos^2 x - \sin^2 \theta \sin^2 x = \cos^2 x \sin^2 x$

or $\cos^2 \theta \cos^2 x - (1-\cos^2 \theta)(1-\cos^2 x) = \cos^2 x (1-\cos^2 x)$

or $\cos^4 x = \sin^2 \theta$

or $\cos^2 x = \sin \theta$, and $x = \cos^{-1} \sqrt{\sin \theta}$.

Hence from (2), $\sinh y = \cos x = \sqrt{\sin \theta}$

and $y = \sinh^{-1} \sqrt{\sin \theta}$

$\therefore \quad \sin^{-1}(\cos \theta + i \sin \theta) = \cos^{-1} \sqrt{\sin \theta} + i \sinh^{-1} \sqrt{\sin \theta}$

$= \cos^{-1} \sqrt{\sin \theta}$

$+ i \log (\sqrt{\sin \theta} + \sqrt{1+\sin \theta})$.

4. Show that if $\cosh^{-1}(x+iy) + \cosh^{-1}(x-iy) = \cosh^{-1} a$,

then $2(a-1)x^2 + 2(a+1)y^2 = a^2 - 1$. (Bombay' 31)

Let $\cosh^{-1}(x+iy) = \alpha + i\beta$

and hence $\cosh^{-1}(x-iy) = \alpha - i\beta$,

so that, from the given condition, $2\alpha = \cosh^{-1} a$

or $\cosh 2\alpha = a$ ...(*i*)

Now $x+iy = \cosh(\alpha + i\beta)$

and $x-iy = \cosh(\alpha - i\beta)$

Hence $2x = \cosh(\alpha+i\beta) + \cosh(\alpha - i\beta) = 2 \cosh \alpha \cosh i\beta$,

and $2iy = \cosh(\alpha+i\beta) - \cosh(\alpha-i\beta) = 2 \sinh \alpha \sinh i\beta$

Eliminating β between them,

$$\frac{x^2}{\cosh^2 \alpha} + \frac{y^2}{\sinh^2 \alpha} = 1$$

or $\frac{2x^2}{a+1} + \frac{2y^2}{a-1} = 1$, from (*i*)

whence the result.

Exercises VII

Prove the following

1. $\sinh 2\theta = \dfrac{2\tanh\theta}{1-\tanh^2\theta}$.

2. $\cosh 2\theta = \dfrac{1+\tanh^2\theta}{1-\tanh^2\theta}$.

3. $\cos^2\alpha\cosh^2\beta + \sin^2\alpha\sinh^2\beta = \cosh^2\beta - \sin^2\alpha$.

4. $\cosh 3\theta = 4\cosh^3\theta - 3\cosh\theta$.

5. $(\cosh\theta \pm \sinh\theta)^n = \cosh n\theta \pm \sinh n\theta$.

Show that

6. $\sec(x+iy) = 2 \cdot \dfrac{\cos x\cosh y + i\sin x\sinh y}{\cos 2x + \cosh 2y}$.

7. $\cot(x+iy) = \dfrac{-\sin 2x + i\sinh 2y}{\cos 2x - \cosh 2y}$.

8. $\sin^{-1} i = 2n\pi - i\log(\sqrt{2}-1)$.

9. $\tanh^{-1} x = \sinh^{-1} \dfrac{x}{\sqrt{1-x^2}}$.

10. $\log\tan\left(\dfrac{\pi}{4} + \dfrac{ix}{2}\right) = i\tan^{-1}(\sinh x)$. (Baroda '54)

11. $\log\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right) = 2\tanh^{-1}\left(\tan\dfrac{\theta}{2}\right)$.

12. Solve $\tan^{-1}(e^{ix}) - \tan^{-1}(e^{-ix}) = \tan^{-1} i$. (Delhi '51)

13. If $\cos^{-1}(u+iv) = \alpha + i\beta$, prove that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

$$x^2 - x(1+u^2+v^2) + u^2 = 0.$$ (Cal. '38)

14. Break into real and imaginary parts

(*i*) $\cos^{-1}(\cos\theta + i\sin\theta)$.

(*ii*) $\cosh\log(x+iy) + \cosh\log(x-iy)$. (Andhra '37)

15. If $u=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, prove that

(*i*) $\cos\theta\cosh u=1$,

(*ii*) $\sinh u=\tan\theta$,

(*iii*) $\tanh u=\sin\theta$.

16. Show that $\tan^{-1}\left(i\,\frac{x-a}{x+a}\right)=\frac{1}{2}\,i\log\frac{a}{x}$.

17. If $\tan(\theta+i\,\phi)=\sin(x+i\,y)$, prove that

$$\coth y\sin 2\phi=\cot x\sin 2\theta. \qquad \text{(Agra '50)}$$

18. If $\tan(A+i\,B)=\tan\theta+i\sec\theta$, show that

$$e^{2B}=\pm\cot\frac{\theta}{2}, \text{ and } 2A=n\pi+\frac{\pi}{2}+\theta.$$

(Delhi '46)

19. If $\cos(x+i\,y)=\cos\theta+i\sin\theta$, show that

$$\cos 2x+\cosh 2y=2. \qquad \text{(Alld. '55)}$$

20. If $\tan(\alpha+i\,\beta)=x+i\,y$, prove that

(*i*) $x^2+y^2+2x\cot 2\alpha=1$,

(*ii*) $x^2+y^2-2y\coth 2\beta+1=0$, (Agra '49)

21. If $\cosh(u+i\,v)=\tan(x+i\,y)$,

show that $\tanh u\tan v=\operatorname{cosec} 2x\sinh 2y$. (Agra '31)

22. If $\cos(x+i\,y)=r(\cos\alpha+i\sin\alpha)$, show that

$$y=\frac{1}{2}\log\frac{\sin(x-\alpha)}{\sin(x+\alpha)}. \qquad \text{(Agra '32)}$$

23. If $\tan(\theta+i\,\phi)=\cos\alpha+i\sin\alpha$, prove that

$$\phi=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right),\ \theta=\frac{n\pi}{2}+\frac{\pi}{4}. \qquad \text{(Agra '38)}$$

24. If $\tan\theta=\tanh x\cot y$, and $\tan\phi=\tanh x\tan y$ prove that

$$\frac{\sin 2\theta}{\sin 2\phi}=\frac{\cosh 2x+\cos 2y}{\cosh 2x-\cos 2y}. \qquad \text{(Agra '35)}$$

25. If a be greater than unity, show that

$$\sin^{-1} a = \tfrac{1}{2}(4n+1)\pi - i \log(\sqrt{a^2-1}+a)$$

26. If $a \cos\theta + b\sin\theta = c$, where $c > \sqrt{a^2+b^2}$, show that

$$\theta = 2n\pi \pm i \log_e \frac{c+\sqrt{c^2-a^2-b^2}}{\sqrt{a^2+b^2}} + \tan^{-1}\frac{b}{a}.$$

(Rangoon '37)

27. If $[\cos(\alpha + i\beta)]^{p+iq} = A + iB$, show that

$$\tan^{-1}\frac{B}{A} = \frac{q}{2} \log(\cosh^2\beta - \sin^2\alpha)$$
$$+ p\tan^{-1}(\tan\alpha \tanh\beta).$$

(Mysore '32)

CHAPTER VIII

Expansions of Trigonometric Functions—II.

8·1. If $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$, to prove that

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots\dots$$

$$+\frac{(-1)^{n-1}}{2n-1}\tan^{2n-1}\theta + \dots\text{ad inf.}$$

we know $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$$=\cos\theta\,(1+i\tan\theta)$$

or $\quad \sec\theta .\, e^{i\theta} = 1 + i\tan\theta$

Hence $\quad \log\sec\theta + i\,\theta = \log(1+i\tan\theta)$

$$= i\tan\theta - \frac{(i\tan\theta)^2}{2} + \frac{(i\tan\theta)^3}{3} - \dots\dots\text{ad inf.}$$

if $\tan\theta$ is numerically less than 1 or $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$.

Equating the imaginary parts from both sides

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots\dots$$

$$+\frac{(-1)^{n-1}}{2n-1}\tan^{2n-1}\theta + \dots\dots\text{ad inf.}$$

This is known as *Gregory's Series.*

If $\tan\theta = x$, or $\theta = \tan^{-1}x$, and x numerically less than 1, another form of Gregory's series is

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\dots\dots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1} + \dots\text{ad inf.}$$

8·2. General form of Gregory's Series.

If θ does not lie between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, let $\theta = n\pi + \phi$,

where ϕ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, and n is an integer.

then $$\phi = \tan\phi - \frac{\tan^3\phi}{3} + \frac{\tan^5\phi}{5} - \dots\dots$$

or $$\theta - n\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots\dots,$$

since $\tan\phi = \tan\theta$.

Giving particular values to n, we obtain different series.

If $n = 3$, $\phi = \theta - 3\pi$, and we have

$$\theta - 3\pi = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \dots\dots,$$

where $-\frac{\pi}{4} \leqslant \theta - 3\pi \leqslant \frac{\pi}{4}$ or $3\pi - \frac{\pi}{4} \leqslant \theta \leqslant 3\pi + \frac{\pi}{4}$.

8·3. Several series, dependent on Gregory's series, have been used to calculate the value of π to desired places of decimals. Some of them are given below :

(*i*) Gregory's Series. Putting $\theta = \frac{\pi}{4}$ in this series, we obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots\dots$$

Since this series does not converge rapidly a large number of terms have to be taken to evaluate π fairly correctly.

(*ii*) Euler's Series. It is readily seen that

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

$$= \frac{1}{2} - \frac{1}{3} \cdot \frac{1}{2^3} + \frac{1}{5} \cdot \frac{1}{2^5} - \dots\dots$$

$$+\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}-\ldots\ldots$$

$$\text{or } \frac{\pi}{4}=\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right)\ldots\ldots$$

(*iii*) Machin's Series. This series is obtained from the result

$$\frac{1}{4}\pi=4\ \tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}.$$

To prove this, $4\tan^{-1}\frac{1}{5}-\frac{\pi}{4}$

$$=2\ \tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}-\tan^{-1}1=2\ \tan^{-1}\frac{5}{12}-\tan^{-1}1,$$

$$=\tan^{-1}\frac{\frac{5}{6}}{1-\frac{25}{144}}-\tan^{-1}1=\tan^{-1}\frac{120}{119}-\tan^{-1}1$$

$$=\tan^{-1}\frac{\frac{120}{119}-1}{1+\frac{120}{119}}=\tan^{-1}\frac{1}{239}.$$

Hence by applying Gregory's series,

$$\frac{\pi}{4}=4\left(\frac{1}{5}-\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}-\ldots\ldots\right)-\left(\frac{1}{239}-\frac{1}{3}\cdot\frac{1}{239^3}+\frac{1}{5}\cdot\frac{1}{239^5}-\ldots\ldots\right)$$

This series is more rapidly convergent than the previous ones.

(*iv*) Rutherford's Series. Here we use the identity

$$\frac{\pi}{4}=4\ \tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$$

Since $\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}=\tan^{-1}\frac{1}{239}$ this result is easily deduced from the one in (*iii*).

Hence
$$\frac{\pi}{4}=4\left(\frac{1}{5}-\frac{1}{3}\cdot\frac{1}{5^3}+\ldots\ldots\right)-\left(\frac{1}{70}-\frac{1}{3}\cdot\frac{1}{70^3}+\ldots\right)$$
$$+\left(\frac{1}{99}-\frac{1}{3}\cdot\frac{1}{99^3}+\ldots\ldots\right)$$

This series converges more rapidly than Machin's, and can be conveniently used. π has been evaluated correct to 707 places of decimals.

Example

If $x>0$, prove that

$$\tan^{-1}x=\frac{\pi}{4}+\frac{x-1}{x+1}-\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3$$
$$+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5-\ldots\ldots$$ (Travancore '44)

Since $x>0$, $-1<\frac{x-1}{x+1}<1$, and hence

$$\frac{x-1}{x+1}-\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5-\ldots\ldots$$
$$=\tan^{-1}\frac{x-1}{x+1}=\tan^{-1}x-\tan^{-1}1$$

$=\tan^{-1}x-\frac{\pi}{4}$, which proves the proportion.

8·4 If $-1<x<1$, to expand $\log(1-2x\cos\theta+x^2)$ in a series of cosines of multiples of θ. (Agra '45)

$$\log(1-2x\cos\theta+x^2)=\log\left\{1-x(e^{i\theta}+e^{-i\theta})+x^2\right\}$$
$$=\log\left(1-x\,e^{i\theta}\right)\left(1-x\,e^{-i\theta}\right),$$ and

since $|x|<1$

$$=-\left(x\,e^{ie}+\frac{x^2}{2}\,e^{2i\theta}+\frac{x^3}{3}\,e^{3\,i\theta}+\ldots\ldots\right)$$

$$-\left(xe^{-ie}+\frac{x^2}{2}\,e^{-2i\theta}+\frac{x^3}{3}\,e^{-3\,i\,\theta}+\ldots\right)$$

$$=-x\left(e^{i\,\theta}+e^{-i\theta}\right)-\frac{x^2}{2}\left(e^{2i\theta}+e^{-2i\theta}\right)-\frac{x^3}{3}\left(e^{3i\theta}+e^{-3\,i\,\theta}\right)$$

$$=-2\left(x\cos\theta+\frac{x^2}{2}\cos 2\,\theta+\frac{x^3}{3}\cos 3\,\theta+\ldots\right)$$

$$=-2\sum_{1}^{\infty}\frac{x^n}{n}\cos n\,\theta.$$

8·5. If $\sin x = n\sin(x+\alpha)$, to expand x in a series of ascending powers of n, where n is less than unity.

Given $\qquad \sin x = n\sin(x+\alpha),$

or $\qquad \sin x\,(1-n\cos\alpha)=n\cos x\sin\alpha$

or $\qquad \dfrac{\cos x}{i\sin x}=\dfrac{1-n\cos\alpha}{i\,n\sin\alpha}$

Hence $\qquad \dfrac{\cos x+i\sin x}{\cos x-i\sin x}=\dfrac{1-n\cos\alpha+i\,n\sin\alpha}{1-n\cos\alpha-i\,n\sin\alpha}$

or $\qquad \dfrac{e^{i\,x}}{e^{-i\,x}}=\dfrac{1-n\,e^{-i\,\alpha}}{1-n\,e^{i\,\alpha}}=e^{2\,i\,x}$

Since $|\,n\,|<1$, we have, on taking logarithm,

$$2\,i\,x=\log\left(1-n\,e^{-i\,\alpha}\right)-\log\left(1-n\,e^{i\,\alpha}\right)$$

$$=\left(n\,e^{i\,\alpha}+\frac{n^2}{2}e^{2\,i\,\alpha}+\frac{n^3}{3}e^{3i\alpha}+\ldots\ldots\right)$$

$$-\left(n\,e^{-i\,\alpha}+\frac{n^2}{2}e^{-2\,i\,\alpha}+\frac{n^3}{3}e^{-3\,i\,\alpha}+\ldots\ldots\right)$$

$$=n\left(e^{i\alpha}-e^{-i\alpha}\right)+\frac{n^2}{2}\left(e^{2i\alpha}-e^{-2i\alpha}\right)$$
$$+\frac{n^3}{3}\left(e^{3i\alpha}-e^{-3i\alpha}\right)+\ldots\ldots$$

Hence $x=n\sin\alpha+\frac{1}{2}\,n^2\sin 2\alpha+\frac{1}{3}\,n^3\sin 3\alpha+\ldots\ldots$

Here x has been assumed to lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$; if it does not, we will then write $e^{2ix}=e^{2ix+2k\pi i}$, and finally the left hand side will be $x+k\pi$, where k is such that $x+k\pi$ lies between $-\pi/2$ and $+\pi/2$.

8·6. If $\tan x=n\tan y$, to express x as a series in terms of sines of multiples of y.

Here $$\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}=n\,.\,\frac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}}$$

or $$\frac{e^{2xi}-1}{e^{2xi}+1}=n\,.\,\frac{e^{2iy}-1}{e^{2iy}+1}$$

or $$e^{2xi}=\frac{(1+n)\,e^{2iy}+1-n}{(1-n)\,e^{2iy}+1+n}=e^{2iy}.\,\frac{1+m\,e^{-2iy}}{1+m\,e^{2iy}},$$

where $m=\frac{1-n}{1+n}$.

If m is numerically less than unity, on taking logarithm,

we have $2\,x\,i=2\,i\,y+\log\left(1+m\,e^{-2iy}\right)-\log\left(1+m\,e^{2iy}\right)$

$$=2\,i\,y-m\left(e^{2iy}-e^{-2iy}\right)+\frac{m^2}{2}\left(e^{4iy}-e^{-4iy}\right)-\ldots\ldots$$

Hence $x=y-m\sin 2y+\frac{m^2}{2}\sin 4y-\frac{m^3}{3}\sin 6y+\ldots$

8·7 To expand $e^{ax}\cos bx$ in a series of ascending powers of x.

$$e^{ax}\cos bx = e^{ax}\cdot\frac{e^{bix}+e^{-bix}}{2} = \frac{1}{2}\{e^{(a+bi)x}+e^{(a-bi)x}\}$$

$$=\frac{1}{2}\Big[\Big\{1+(a+bi)x+(a+bi)^2\frac{x^2}{2!}+\ldots\ldots\Big\}$$
$$+\Big\{1+(a-bi)x+\frac{(a-bi)^2}{2!}x^2+\ldots\ldots\Big\}\Big]$$

$$=\frac{1}{2}\Big[2+(a+bi+a-bi)x+\{(a+bi)^2+(a-bi)^2\}\frac{x^2}{2!}+\ldots\ldots$$
$$+\{(a+bi)^n+(a-bi)^n\}\frac{x^n}{n!}+\ldots\ldots\Big]$$

If $a=r\cos\alpha$, $b=r\sin\alpha$, then $r=\sqrt{a^2+b^2}$ and $\alpha=\tan^{-1}\dfrac{b}{a}$ and the coefficient of x^n in the expansion

$$=\frac{1}{2.n!}[r^n(\cos\alpha+i\sin\alpha)^n+r^n(\cos\alpha-i\sin\alpha)^n]$$

$$=\frac{r^n}{2.n!}(\cos n\alpha+i\sin n\alpha+\cos n\alpha-i\sin n\alpha)$$

$$=\frac{r^n\cos n\alpha}{n!}=\frac{(a^2+b^2)^{\frac{n}{2}}}{n!}\cos\left(n\tan^{-1}\frac{b}{a}\right)$$

Hence $e^{ax}\cos bx=\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}(a^2+b^2)^{n/2}\cos\left(n\tan^{-1}\dfrac{b}{a}\right)$.

Examples

1. Prove that, if $x<1$,

$$\frac{1-x^2}{1-2x\cos\theta+x^2}=1+2x\cos\theta+2x^2\cos 2\theta$$
$$+2x^3\cos 3\theta+\ldots\ldots\text{ad inf.}\quad\text{(Agra '49)}$$

Now $\frac{1-x^2}{1-2x\cos\theta+x^2}=-1+\frac{2-2x\cos\theta}{1-2x\cos\theta+x^2}$

$$=-1+\frac{2-x(e^{i\theta}+e^{-i\theta})}{1-x(e^{i\theta}+e^{-i\theta})+x^2}$$

$$=-1+\frac{2-x(e^{i\theta}+e^{-i\theta})}{(1-xe^{i\theta})(1-xe^{-i\theta})}$$

$$=-1+\frac{1}{1-xe^{i\theta}}+\frac{1}{1-xe^{-i\theta}}$$

$$=-1+\left(1+x\,e^{i\theta}+x^2e^{2i\theta}+\ldots\right)+\left(1+xe^{-i\theta}+x^2e^{-2i\theta}+\ldots,\right) \text{ sine } x<1$$

$$=1+x\left(e^{i\theta}+e^{-i\theta}\right)+x^2\left(e^{2i\theta}+e^{-2i\theta}\right)+\ldots\ldots$$

$$=1+2x\cos\theta+2x^2\cos 2\theta+2x^3\cos 3\theta+\ldots\ldots\text{ad inf.}$$

2. If $\tan\theta=x+\tan\alpha$, show that

$$\theta=\alpha+x\cos^2\alpha-\frac{x^2}{2}\cos^2\alpha\sin 2\alpha-\frac{x^3}{3}\cos^3\alpha\cos^3\alpha$$

$$+\frac{x^4}{4}\cos^4\alpha\cos 4\alpha+\ldots\ldots \qquad \text{(Agra '41)}$$

Here $x+\tan\alpha=\tan\theta=\tan(\theta-\alpha+\alpha)$

$$=\frac{\tan(\theta-\alpha)+\tan\alpha}{1-\tan\alpha\tan(\theta-\alpha)}$$

or $x\,[\,1-\tan\alpha\tan(\theta-\alpha)\,]=\sec^2\alpha\tan(\theta-\alpha)$

or $\tan(\theta-\alpha)=\frac{x\cos^2\alpha}{1+x\sin\alpha\cos\alpha}$

or $\frac{i\sin(\theta-\alpha)}{\cos(\theta-\alpha)}=\frac{i\,x\cos^2\alpha}{1+x\sin\alpha\cos\alpha}$

or $\dfrac{\cos(\theta-\alpha)+i\sin(\theta-\alpha)}{\cos(\theta-\alpha)-i\sin(\theta-\alpha)}=\dfrac{1+x\cos\alpha\,(\sin\alpha+i\cos\alpha)}{1+x\cos\alpha\,(\sin\alpha-i\cos\alpha)}$

or $e^{2i(\theta-\alpha)}=\dfrac{1+i\,x\cos\alpha.e^{-i\alpha}}{1-i\,x\cos\alpha.e^{i\alpha}}$

Hence $2i(\theta-\alpha)=\log\left(1+i\,x\cos\alpha.e^{-i\alpha}\right)$

$$-\log\left(1-i\,x\cos\alpha.e^{i\alpha}\right)$$

$$=i\,x\cos\alpha.e^{-i\alpha}-\frac{i^2}{2}x^2\cos^2\alpha\,e^{-2i\alpha}$$

$$+\frac{i^3}{3}x^3\cos^3\alpha\,e^{-3i\alpha}-\ldots$$

$$+i\,x\cos\alpha\,e^{i\alpha}+\frac{i^2}{2}x^2\cos^2\alpha\,e^{2i\alpha}$$

 $+\dfrac{i^3}{3}x^3\cos^3\alpha\,e^{3i\alpha}+\ldots\ldots$

$$=2i\,x\cos\alpha.\cos\alpha-\frac{x^2}{2}\cos^2\alpha.\,2i\sin 2\alpha-\frac{i\,x^3}{3}\cos^3\alpha.$$

$$2\cos 3\alpha+\ldots\ldots,$$

whence the result.

3. If in a triangle ABC, B is in radians and is less than A prove that

$$B=\frac{b}{a}\sin C+\frac{b^2}{2a^2}\sin 2C+\frac{b^3}{3a^3}\sin 3C+\ldots\ldots$$

For the triangle $\dfrac{\sin B}{\sin A}=\dfrac{b}{a}$

Hence $\sin B=\dfrac{b}{a}\sin A=\dfrac{b}{a}\sin(B+C)$

Applying Art. 8·5, where $n=b/a$, we get the required result.

Exercises VIII

1. Prove that $\frac{1}{2}-\frac{\pi}{8}=\frac{1}{3\cdot5}+\frac{1}{7\cdot9}+\frac{1}{11\cdot13}+\ldots\ldots$

2. Prove that

(i) $\pi=2\sqrt{3}\left(1-\frac{1}{3\cdot3}+\frac{1}{5\cdot3^2}-\frac{1}{7\cdot3^3}+\ldots\ldots\right)$ (Andhra '50)

(ii) If $\theta<\frac{\pi}{4}$, $\log\sec\theta=\frac{1}{2}\tan^2\theta-\frac{1}{4}\tan^4\theta$

$+\frac{1}{6}\tan^6\theta\ldots\ldots$ (Cal. '46)

3. Show that

$\frac{\pi}{4}=\frac{17}{21}-\frac{713}{81\cdot343}+\ldots\ldots+\frac{(-1)^{n+1}}{2n-1}\left\{\frac{2}{3}9^{1-n}\right.$

$\left.+7^{1-2n}\right\}+\ldots$ (Agra '44)

4. If θ be a positive acute angle, show that

(i) $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=2\left(\sin\theta-\frac{\sin3\theta}{3}+\frac{\sin5\theta}{5}-\ldots\right)$

(ii) $\frac{\pi}{4}=\cos\theta-\frac{\cos3\theta}{3}+\frac{\cos5\theta}{5}-\ldots\ldots$

5. Prove that

(i) $\frac{\cos\theta-x}{1-2x\cos\theta+x^2}=\cos\theta+x\cos2\theta+x^2\cos3\theta+\ldots\ldots$

(ii) $\frac{\cos\theta-a\cos(\theta-\phi)}{1-2a\cos\phi+a^2}=\cos\theta+a\cos(\theta+\phi)$

$+a^2\cos(\theta+2\phi)+\ldots\ldots$

6. Expand $e^{a\cos\phi}\cos(\theta+a\sin\phi)$ in an infinite series. (Agra '51)

7. If $x < \sqrt{2}-1$, prove that

$$2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\ldots\right)=\frac{2x}{1-x^2}-\frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3+\frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5-\ldots\ldots$$

(Agra '50)

8. If $\cot y=\cot x+\operatorname{cosec}\alpha \operatorname{cosec} x$, show that

$$y=\sin x\sin\alpha-\frac{1}{2}\sin 2x\ \sin^2\alpha+\frac{1}{3}\sin 3x\ \sin^3\alpha-\ldots$$

9. Prove that

$$\log\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=4\left[c\sin^2\theta-\frac{1}{2}c^2\sin^2 2\theta+\frac{1}{3}c^3\sin^2 3\theta-\ldots\right],$$

where $c(a+b)=a-b$.

10. If $\cot\phi=\frac{1}{x}+\cot\theta$, prove that

$$\phi=\frac{x}{\sin\theta}.\sin\theta.-\frac{1}{2}\ \frac{x^2}{\sin^2\theta}.\ \sin 2\theta+\frac{1}{3}\ \frac{x^3}{\sin^3\theta}\ \sin 3\theta-\ldots\ldots$$

11. In any triangle, where $a<c$, show that

$$\frac{\cos nA}{b^n}=\frac{1}{c^n}\left\{1+\frac{na}{c}\cos B+\frac{n(n+1)}{2!}\ \frac{a^2}{c^2}\cos 2B+\ldots\ldots\right\},$$

and $$\frac{\sin nA}{b^n}=\frac{n}{c^n}\left\{\frac{a}{c}\sin B+\frac{n+1}{2!}\ \frac{a^2}{c^2}\sin 2B+\ldots\ldots\right\}.$$

(Lucknow '52)

12. If $\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\phi}{2}$, show that

$$\theta-\phi=2\sum_{\lambda=1}^{\infty}\frac{\lambda^{p}}{p}\sin p\phi$$

where $\lambda=\frac{1-\sqrt{1-e^2}}{e}$. (Agra '52)

CHAPTER IX

Summation of Series

9·1. In this chapter we shall deal with several methods used in the summation of trigonometric series of finite or infinite terms.

9·2 Sine and Cosine series with angles in arithmaectical progression.

To prove

(*a*) $\sin \alpha+\sin (\alpha+\beta)+\sin (\alpha+2\beta)+\ldots\ldots+\sin \{\alpha+(n-1)\beta\}$

$$=\frac{\sin \{\alpha+\frac{1}{2}(n-1)\beta\} \sin \frac{1}{2} n\beta}{\sin \frac{1}{2}\beta},$$

(*b*) $\cos \alpha+\cos (\alpha+\beta)+\cos (\alpha+2\beta)+\ldots+\cos \{\alpha+(n-1)\beta\}$

$$=\frac{\cos \{\alpha+\frac{1}{2}(n-1)\beta\} \sin \frac{1}{2} n\beta}{\sin \frac{1}{2}\beta}.$$

Each series consists of *n* terms; let the sum of the two series be S and C respectively.

We know $2 \sin \frac{\beta}{2} \sin \alpha=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right)$,

$$2 \sin \frac{\beta}{2} \sin (\alpha+\beta)=\cos\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right),$$

$$2 \sin \frac{\beta}{2} \sin (\alpha+2\beta)=\cos\left(\alpha+\frac{3\beta}{2}\right)-\cos\left(\alpha+\frac{5\beta}{2}\right),$$

....

....

$$2 \sin \frac{\beta}{2} \sin \{\alpha+(n-1)\beta\}=\cos\left\{\alpha+\left(n-\frac{3}{2}\right)\beta\right\}$$

$$-\cos\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\}$$

Adding up, we have

$$2\sin\frac{\beta}{2}\cdot S=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\}$$

$$=2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{1}{2}n\beta$$

Hence $S=\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots\ldots$

$$\ldots+\sin\{\alpha+(n-1)\beta\}$$

$$=\frac{\sin\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta} \quad \ldots(1)$$

For the cosine series, we have

$$2\sin\frac{\beta}{2}\cos\alpha=\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right),$$

$$2\sin\frac{\beta}{2}\cos(\alpha+\beta)=\sin\left(\alpha+\frac{3\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right),$$

$$2\sin\frac{\beta}{2}\cos(\alpha+2\beta)=\sin\left(\alpha+\frac{5\beta}{2}\right)-\sin\left(\alpha+\frac{3\beta}{2}\right),$$

....

....

$$2\sin\frac{\beta}{2}\cos\{\alpha+(n-1)\beta\}=\sin\left[\alpha+\left(n-\frac{1}{2}\right)\beta\right]$$

$$-\sin\left[\alpha+\left(n-\frac{3}{2}\right)\beta\right]$$

Adding up,

$$2\sin\frac{\beta}{2}\cdot C=\sin\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\}-\sin\left(\alpha-\frac{\beta}{2}\right)$$

$$=2\cos\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}$$

Hence $C=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots$

$$\ldots\ldots+\cos\{\alpha+(n-1)\beta\}$$

$$=\frac{\cos\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta} \quad \ldots(2)$$

9·21. Particular Cases. From (1) and (2) of the previous article we can easily deduce the following results :

(*a*) If $\alpha=0$, we have

$$\sin\beta+\sin 2\beta+\ldots\ldots+\sin(n-1)\beta=\frac{\sin\frac{1}{2}(n-1)\beta\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta} \qquad \ldots(3)$$

$$1+\cos\beta+\cos 2\beta+\ldots\ldots+\cos(n-1)\beta=\frac{\cos\frac{1}{2}(n-1)\beta\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta} \qquad \ldots(4)$$

(*b*) If $\alpha=\beta$,

$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots\ldots+\sin n\alpha=\frac{\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}, \qquad \ldots(5)$$

and

$$\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots\ldots+\cos n\alpha=\frac{\cos\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha} \qquad \ldots(6)$$

(*c*) If $\beta=\dfrac{2\pi}{n}$, since $\sin\dfrac{1}{2}n\beta=0$, we have

$$\sin\alpha+\sin\left(\alpha+\frac{2\pi}{n}\right)+\sin\left(\alpha+\frac{4\pi}{n}\right)+\ldots\ldots+\sin\left\{\alpha+\frac{2\pi}{n}(n-1)\right\}=0, \qquad \ldots(7)$$

and

$$\cos\alpha+\cos\left(\alpha+\frac{2\pi}{n}\right)+\cos\left(\alpha+\frac{4\pi}{n}\right)+\ldots\ldots+\cos\left[\alpha+\frac{2\pi}{n}(n-1)\right]=0 \qquad \ldots(8)$$

(*d*) Putting $\pi+\beta$ for β, we get

$$\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\ldots\ldots+(-1)^{n-1}\cos\{\alpha+(n-1)\beta\}$$

$$=\cos\left\{\alpha+\frac{1}{2}(n-1)(\pi+\beta)\right\}\sin\frac{1}{2}n(\pi+\beta)\sec\frac{\beta}{2}, \qquad \ldots(9)$$

and $\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)\ldots\ldots\ldots$

$$\ldots\ldots+(-1)^{n-1}\sin\{\alpha+(n-1)\beta\}$$

$$=\sin\left\{\alpha+\frac{1}{2}(n-1)(\pi+\beta)\right\}\sin\frac{1}{2}n(\pi+\beta)\sec\frac{\beta}{2}. \quad \ldots(10)$$

9·3. If m be a positive integer we have seen earlier than $\sin^m\alpha$ or $\cos^m\alpha$ can be expressed in terms of sines or cosines of mutliples of α. This transformation enables us to obtain the sum of series of the type

$$\sin^m\alpha+\sin^m(\alpha+\beta)+\sin^m(\alpha+2\beta)+\ldots\ldots+\sin^m\{\alpha+(n-1)\beta\}$$

or $$\cos^m\alpha+\cos^m(\alpha+\beta)+\ldots\ldots+\cos^m\{\alpha+(n-1)\beta\}.$$

In particular, when $\beta=2\pi/n$, and $m<n$, the sum will be independent of α, as in results (7) and (8) above.

The condition $m<n$ is necessary to ensure that the denomimater in the sum is not zero.

Examples

1. Sum to n terms the series

$$\sin\alpha\sin2\alpha+\sin2\alpha\sin3\alpha+\sin3\alpha\sin4\alpha+\ldots\ldots,$$

and deduce the sum of

$$1.2+2.3+3.4+\ldots\ldots+n(n+1)$$

The required sum

$$=\frac{1}{2}(\cos\alpha-\cos3\alpha)+\frac{1}{2}(\cos\alpha-\cos5\alpha)$$

$$+\frac{1}{2}(\cos\alpha-\cos7\alpha)+\ldots\ldots\text{to } n \text{ terms}$$

$$=\frac{n}{2}\cos\alpha-\frac{1}{2}(\cos3\alpha+\cos5\alpha+\cos7\alpha+\ldots\ldots\text{to } n \text{ terms})$$

$$=\frac{n}{2}\cos\alpha-\frac{\cos(n+2)\alpha\sin n\alpha}{2\sin\alpha}$$

$$=\frac{n}{2}\cos\alpha-\frac{\sin 2(n+1)\alpha-\sin 2\alpha}{4\sin\alpha}$$

$$=\frac{1}{4\sin\alpha}\{(n+1)\sin 2\alpha-\sin(2n+2)\alpha\}.$$

Hence $\sin\alpha(\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\sin 3\alpha\sin 4\alpha+\ldots\ldots\text{to } n \text{ terms})$

$$=\frac{1}{4}\{(n+1)\sin 2\alpha-\sin(2n+2)\alpha\}.$$

In this result expanding sines both sides and equating the coefficients of α^3, we have

$$1.2+2.3+3.4+\ldots\ldots+n(n+1)$$

$$=\frac{1}{4}\left[(n+1)\left(-\frac{8}{3!}\right)+\frac{(2n+2)^3}{3!}\right]=\frac{1}{3}n(n+1)(n+2).$$

2. Sum to n terms :

$$\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\ldots\ldots$$

Since $\sin^3\theta=\frac{1}{4}(3\sin\theta-\sin 3\theta)$,

the series $=\frac{1}{4}(3\sin\alpha-\sin 3\alpha)+\frac{1}{4}(3\sin 2\alpha-\sin 6\alpha)$

$$+\frac{1}{4}(3\sin 3\alpha-\sin 9\alpha)+\ldots\ldots$$

$$=\frac{3}{4}(\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots\ldots\text{to } n \text{ terms})$$

$$-\frac{1}{4}(\sin 3\alpha+\sin 6\alpha+\sin 9\alpha+\ldots\ldots\text{to } n \text{ terms})$$

$$=\frac{3}{4}\sin\frac{(n+1)}{2}\alpha\sin\frac{n\alpha}{2}\operatorname{cosec}\frac{\alpha}{2}-\frac{1}{4}\sin\frac{3}{2}(n+1)\alpha$$

$$\sin\frac{3n\alpha}{2}\operatorname{cosec}\frac{3\alpha}{2}.$$

Exercises

Find the sum to n terms of the series

1. $\sin^2 \alpha + \sin^2 (\alpha+\beta) + \sin^2 (\alpha + 2\beta) + \ldots\ldots$ (Agra '23)

2. $\cos^2 \alpha + \cos^2 2\alpha + \cos^2 3\alpha + \ldots\ldots$

3. $\sin^4 \alpha + \sin^4 2\alpha + \sin^4 3\alpha + \ldots\ldots$

4. $\cos\theta \cos(\theta+\alpha) + \cos(\theta+\alpha)\cos(\theta+2\alpha) + \cos(\theta+2\alpha) \cos(\theta+3\alpha) + \ldots\ldots$

5. $\sin 3\theta \sin\theta + \sin 6\theta \sin 2\theta + \sin 12\theta \sin 4\theta + \ldots\ldots$

6. $\sin^4 \alpha + \sin^4 \left(\alpha + \frac{2\pi}{n}\right) + \sin^4 \left(\alpha + \frac{4\pi}{n}\right) + \ldots\ldots$ (Agra '44)

Show that

7. $\tan n\theta = \dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \ldots \text{to } n \text{ terms}}{\cos\theta + \cos 3\theta + \cos 5\theta + \ldots \text{to } n \text{ terms}}$

8. $\dfrac{\sin\theta - \sin 2\theta + \sin 3\theta - \ldots \text{to } n \text{ terms}}{\cos\theta + \cos 2\theta + \cos 3\theta - \ldots \text{to } n \text{ terms}}$

$$= \tan \frac{n+1}{2}(\pi+\theta).$$

9·4 A general method : This method is very useful in many cases. Suppose it is required to find the sum of any one of the series,

$$C = a_1 \cos\alpha + a_2 \cos(\alpha+\beta) + a_3 \cos(\alpha+2\beta) + \ldots\ldots,$$

and

$$S = a_1 \sin\alpha + a_2 \sin(\alpha+\beta) + a_3 \sin(\alpha+2\beta) + \ldots\ldots,$$

the number of terms being finite or infinite.

Multiplying the second series by i and adding to the first,

$$C + iS = a_1 e^{i\alpha} + a_2 e^{i(\alpha+\beta)} + a_3 e^{i(\alpha+2\beta)} + \ldots\ldots$$

This series of complex terms can be usually summed up if it assumes any one of the following or allied forms :

(*i*) Series in geometric progression ;

(*ii*) Binomial series ;

(*iii*) Exponential series ;

(*iv*) Series of the forms of $\sin z$, $\cos z$, $\sinh z$ or $\cosh z$;

(*v*) Logarithmic expansion ;

(*vi*) Gregory's series.

If the sum is obtanied in the form $A+i\,B$, then equating the real and imaginary parts C and S, the sums of the two companion series, are obtained.

The following examples will illustrate the method.

1. Sum the series

$$\sin\alpha+\frac{1}{2}\sin 2\alpha+\frac{1}{2^2}\sin 3\alpha+\ldots\ldots\text{ad inf.}$$

(Luck. '45)

Let $\quad S=\sin\alpha+\dfrac{1}{2}\sin 2\alpha+\dfrac{1}{2^2}\sin 3\alpha+\ldots\ldots$

and $\quad C=\cos\alpha+\dfrac{1}{2}\cos 2\alpha+\dfrac{1}{2^2}\cos 3\alpha+\ldots\ldots$

Hence $\quad C+i\,S=(\cos\alpha+i\sin\alpha)+\dfrac{1}{2}(\cos 2\alpha+i\sin 2\alpha)$

$$+\frac{1}{2^2}(\cos 3\alpha+i\sin 3\alpha)+\ldots\ldots$$

$$=e^{i\alpha}+\frac{e^{2i\alpha}}{2}+\frac{1}{2^2}e^{3i\alpha}+\ldots\ldots$$

Which is a series in geometrical progression, and

$$=e^{i\alpha}\frac{1}{1-\frac{1}{2}e^{i\alpha}}=\frac{2e^{i\alpha}}{2-e^{i\alpha}}$$

$$=\frac{2e^{i\alpha}(2-e^{-i\alpha})}{(2-e^{i\alpha})(2-e^{-i\alpha})}=\frac{2(2e^{i\alpha}-1)}{4-4\cos\alpha+1}$$

$$=\frac{2(2\cos\alpha+2i\sin\alpha-1)}{5-4\cos\alpha}$$

Equating the imaginary parts, $S=\dfrac{4\sin\alpha}{5-4\cos\alpha}$.

2. Sum the series

$$1-\frac{1}{2}\cos\theta+\frac{1\cdot3}{2\cdot4}\cos2\theta-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos3\theta+\ldots.$$

(Alld. '55)

Let $C=1-\frac{1}{2}\cos\theta+\frac{1\cdot3}{2\cdot4}\cos2\theta-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos3\theta+\ldots\ldots$

......ad inf

and $S=-\frac{1}{2}\sin\theta+\frac{1\cdot3}{2\cdot4}\sin2\theta-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\sin3\theta+\ldots\ldots$

Then $C+iS=1-\frac{1}{2}e^{i\theta}+\frac{1\cdot3}{2\cdot4}e^{2i\theta}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}e^{3i\theta}+\ldots$

$$=\left(1+e^{i\theta}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{(1+\cos\theta+i\sin\theta)^{\frac{1}{2}}}=\frac{1}{\sqrt{2\cos\frac{\theta}{2}}}\cdot\frac{1}{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^{\frac{1}{2}}}$$

$$=\frac{\cos\frac{\theta}{4}-i\sin\frac{\theta}{4}}{\sqrt{2\cos\frac{\theta}{2}}}$$

Hence $C=\cos\frac{\theta}{4}\Big/\sqrt{2\cos\frac{\theta}{2}}$

3. Sum $c\sin\alpha+\frac{c^2}{2}\sin2\alpha+\frac{c^3}{3}\sin3\alpha+\ldots\ldots$ad inf.

(Agra '53)

Let $C=c\cos\alpha+\frac{c^2}{2}\cos2\alpha+\frac{c^3}{3}\cos3\alpha+\ldots\ldots$

and $S=c\sin\alpha+\frac{c^2}{2}\sin2\alpha+\frac{c^3}{3}\sin3\alpha+\ldots\ldots$

Hence $C+i\,S=ce^{i\alpha}+\frac{c^2}{2}\,e^{2i\alpha}+\frac{c^3}{3}\,e^{3i\alpha}+\ldots\ldots$

$$=-\log\left(1-c\,e^{i\alpha}\right),\text{ if } |c|<1$$

$$=-\log(1-c\cos\alpha-i\,c\sin\alpha)$$

$$=-\left[\log\sqrt{1-2c\cos\alpha+c^2}\right.$$

$$\left.-i\tan^{-1}\frac{c\sin\alpha}{1-c\cos\alpha}\right]$$

Equating the imaginary parts,

$$S=\tan^{-1}\frac{c\sin\alpha}{1-c\cos\alpha}.$$

4. Sum the series

$$\cos\theta+\frac{\sin\theta}{1\,!}\cos 2\theta+\frac{\sin^2\theta}{2\,!}\cos 3\theta+\ \ldots\text{ad inf.}$$

(Agra '41)

Let $\quad C=\cos\theta+\frac{\sin\theta}{1\,!}\cos 2\theta+\frac{\sin^2\theta}{2\,!}\cos 3\theta+\ldots\ldots$

and $\quad S=\sin\theta+\frac{\sin\theta}{1\,!}\sin 2\theta+\frac{\sin^2\theta}{2\,!}\sin 3\theta+\ldots\ldots$

Hence $C+i\,S=e^{i\theta}+\frac{\sin\theta}{1\,!}\,e^{2i\theta}+\frac{\sin^2\theta}{2\,!}\,e^{3i\theta}+\ldots\ldots$

$$=e^{i\theta}.\,e^{(\sin\theta)\,e^{i\theta}}=e^{\sin\theta\cos\theta}.\,e^{i(\theta+\sin^2\theta)}$$

$$=e^{\sin\theta\cos\theta}\{\cos(\theta+\sin^2\theta)+i\sin(\theta+\sin^2\theta)\}$$

Equating the real parts,

$$C=e^{\sin\theta\cos\theta}.\cos(\theta+\sin^2\theta).$$

5. Sum the series

$$\frac{2}{1.3}\sin 2x-\frac{4}{3\cdot5}\sin 4x+\frac{6}{5\cdot7}\sin 6x-\ldots\ldots\text{ad inf.}$$

Let $C=\frac{2}{1\cdot 3}\cos 2x-\frac{4}{3\cdot 5}\cos 4x+\frac{6}{5\cdot 7}\cos 6x-\ldots\ldots$

and $S=\frac{2}{1\cdot 3}\sin 2x-\frac{4}{3\cdot 5}\sin 4x+\frac{6}{5\cdot 7}\sin 6x-\ldots\ldots$

Hence $C+iS=\frac{2}{1\cdot 3}e^{2ix}-\frac{4}{3\cdot 5}e^{4ix}+\frac{6}{5\cdot 7}e^{6ix}-\ldots\ldots$

$$=\frac{1}{2}\left(1+\frac{1}{3}\right)e^{2ix}-\frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}\right)e^{4ix}+\frac{1}{2}\left(\frac{1}{5}+\frac{1}{7}\right)e^{6ix}-\ldots$$

$$=\frac{1}{2}\left[\left(e^{2ix}-\frac{e^{4ix}}{3}+\frac{e^{6ix}}{5}-\ldots\ldots\right)+\left(\frac{e^{2ix}}{3}-\frac{e^{4ix}}{5}+\frac{e^{6ix}}{7}-\ldots\right)\right]$$

$$=\frac{1}{2}\left[e^{ix}\left(e^{ix}-\frac{e^{3ix}}{3}+\frac{e^{5ix}}{5}-\ldots\right)+e^{-ix}\left(\frac{e^{3ix}}{3}-\frac{e^{5ix}}{5}+\frac{e^{7ix}}{7}-\ldots\right)\right]$$

$$=\tfrac{1}{2}\left[e^{ix}\tan^{-1}e^{ix}+e^{-ix}\left(e^{ix}-\tan^{-1}e^{ix}\right)\right]$$

$$=\tfrac{1}{2}+i\sin x\,.\,\tan^{-1}e^{ix}.$$

Let $\tan^{-1}e^{ix}=\alpha+i\beta$, or $e^{ix}=\tan(\alpha+i\beta)$

and $e^{-ix}=\tan(\alpha-i\beta)$

Now $\tan 2\alpha=\tan(\alpha+i\beta+\alpha-i\beta)$

$$=\frac{e^{ix}+e^{-ix}}{1-1}=\infty$$

Hence $2\alpha=\pi/2$ or $\alpha=\pi/4$, if $\pi>x\geqslant 0$

Here $\tan 2i\beta=\tan\{(\alpha+i\beta)-(\alpha-i\beta)\}$

$$=\frac{e^{ix}-e^{-ix}}{1+1}=i\sin x$$

or $\tanh 2\beta=\sin x$

Hence $\beta=\frac{1}{2}\tanh^{-1}\sin x.$

Now $C+iS=\frac{1}{2}+i\sin x\,.\,(\alpha+i\beta)$

Equating the imaginary parts,

$$S=\alpha\sin x=\frac{\pi}{4}\sin x.$$

9·5 Series of hyperbolic functions. A series of hyperbolic sines or cosines can be summed up by expressing them in terms of their exponential values or by other means as well.

Example

Sum the series

$$x \cosh y + \frac{x^2}{2!} \cosh 2y + \frac{x^3}{3} \cosh 3y + \ldots \text{ ad inf}$$

The series $= x \cdot \dfrac{e^y + e^{-y}}{2} + \dfrac{x^2}{2!} \cdot \dfrac{e^{2y} + e^{-2y}}{2} + \dfrac{x^3}{3!} \dfrac{e^{3y} + e^{-3y}}{2} + \ldots\ldots$

$$= \tfrac{1}{2}\left[\left(x\, e^y + \frac{x^2 e^{2y}}{2!} + \frac{x^3 e^{3y}}{3!} + \ldots\right) + \left(x\, e^{-y} + \frac{x^2}{2!} e^{-2y} + \frac{x^3 e^{-3y}}{3!} + \ldots\right)\right]$$

$$= \tfrac{1}{2}\left\{ e^{xe^y} - 1 + e^{xe^{-y}} - 1 \right\}$$

$$= \tfrac{1}{2}\left(e^{xe^y} + e^{xe^{-y}}\right) - 1.$$

Exercises

Sum the series

1. $1 + x\cos\theta + x^2 \cos 2\theta + \ldots + x^{n-1}\cos(n-1)\theta.$

(Agra '52)

2. $\sin\alpha + x\sin(\alpha+\beta) + \dfrac{x^2}{2!}\sin(\alpha + 2\beta) + \ldots$ad inf.

(Lucknow '46)

3. $\cos\theta - \dfrac{1}{2}\cos 2\theta + \dfrac{1}{3}\cos 3\theta - \ldots$ad inf.

(Banaras '52)

4. $1 - \dfrac{\cos 2\theta}{2!} + \dfrac{\cos 4\theta}{4!} - \ldots$ad inf.

5. $\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \ldots$ ad inf. (Alld. '50)

6. $\sin\alpha + \frac{\sin 2\alpha}{2!} + \frac{\sin 3\alpha}{3!} + \ldots$ ad inf. (Luck. '45)

7. $\frac{1}{2}\sin\alpha + \frac{1\cdot 3}{2\cdot 4}\sin 2\alpha + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin^3\alpha + \ldots$ ad inf. (Annam. '51)

8. $\cos n\theta + \cos\theta\cos(n-1)\theta + \cos^2\theta\cos(n-2)\theta + \ldots + \cos^n\theta$. (Agra '31)

9. $C\cos\alpha + \frac{C^3}{3}\cos 3\theta + \frac{C^5}{5}\cos 5\alpha + \ldots$ ad inf. (Travan. '50)

10. $x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{x^3}{3}\sin 3\theta - \ldots$ ad inf. (Alld. '54)

11. $\sin\theta\cos\theta + \frac{\sin 2\theta\cos^2\theta}{2!} + \frac{\sin 3\theta\cos^3\theta}{3!} + \ldots$ ad inf. (Alld. '40)

12. $\cos\frac{\pi}{3} + \frac{1}{3}\cos\frac{2\pi}{3} + \frac{1}{5}\cos\frac{3\pi}{3} + \frac{1}{7}\cos\frac{5\pi}{3} + \ldots$ ad inf. (Agra '36)

13. $\cos\theta + \frac{\cos\theta}{1!}\cos 2\theta + \frac{\cos^2\theta}{1!}\cos 3\theta + \ldots$ ad inf.

14. $\cosh\theta - \frac{1}{2}\cosh 2\theta + \frac{1}{3}\cosh 3\theta - \ldots$ ad inf.

15. $2\cos\theta + \frac{3}{2}\cos^2\theta + \frac{4}{3}\cos^3\theta + \frac{5}{4}\cos^4\theta + \ldots$ ad inf. (Panjab '46)

16. $\sinh x+\frac{1}{2!}\sinh 2x+\frac{1}{3!}\sinh 3x+\ldots$ad inf.

17. $\sin\theta-\frac{1}{3!}\sin 3\theta+\frac{1}{5!}\sin 5\theta-\ldots$ad inf.

18. $1+\cosh x+\frac{1}{2!}\cosh 2x+\frac{1}{3!}\cosh 3x+\ldots\ldots$ad inf.

(Agra '51)

19. Prove that $\sum_{n=1}^{m}\cos^n\theta\cos n\theta=\frac{\cos^{m+1}\theta\sin^m\theta}{\sin\theta}$

(Luck. '54)

20. Show that

$\cos 2\theta+\frac{1}{3}\cos 6\theta+\frac{1}{5}\cos 10\theta+\ldots$ad inf. $=\frac{1}{2}\log(\cot\theta)$.

21. Prove that

$$x\sin\theta-\frac{x^2\sin 2\theta}{2}+\frac{x^3\sin 3\theta}{3}-\ldots\text{ad inf.}$$

$$=\cot^{-1}\left(\frac{\operatorname{cosec}\theta}{x}+\cot\theta\right).$$

22. Sum the series

$$c\sin\alpha-\frac{c^3}{3}\sin 3\alpha+\frac{c^5}{5}\sin^5\alpha-\ldots\ldots\text{ad inf.}$$

9.6. The difference method. This method is applicable to a certain type of series. Consider the series

$$S_n=u_1+u_2+\ldots\ldots+u_r+\ldots\ldots u_n,$$

where the rth term, u_r, can be expressed as

$$u_r=f(r+1)-f(r).$$

Putting $r=1, 2, 3,\ldots\ldots n$, we have the sum to n terms,

$$S_n=\{f(2)-f(1)\}+\{f(3)-f(2)\}+\ldots+\{f(r+1)-f(r)\}+\ldots+\{f(n+1)-f(n)\}$$

$$=f(n+1)-f(1),$$

the other terms cancelling.

If the series be infinite and convergent, then the sum S is obtained as

$$S = \operatorname*{Lt}_{n\to\infty} S_n = \operatorname*{Lt}_{n\to\infty} \{f(n+1) - f(1)\}.$$

No rule can, however, be laid down as to how to break up u_r in the difference form; as a matter of fact it requires a certain amount of skill and experience which a student will acquire without much difficulty. The following examples illustrate the method.

Examples

1. Sum to n terms the series

$$\tan x + \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots\ldots, \qquad \text{(Alld. '54)}$$

Deduce its value when n increases indefinitely.

We know $\cot x - \tan x = \dfrac{\cos^2 x - \sin^2 x}{\cos x \sin x} = 2\cot 2x$

Hence $\tan x = \cot x - 2\cot 2x$, and we can deduce

$$\frac{1}{2}\tan\frac{x}{2} = \frac{1}{2}\cot\frac{x}{2} - \cot x,$$

$$\frac{1}{2^2}\tan\frac{x}{2^2} = \frac{1}{2^2}\cot\frac{x}{2^2} - \frac{1}{2}\cot\frac{x}{2},$$

$$\frac{1}{2^3}\tan\frac{x}{2^3} = \frac{1}{2^3}\cot\frac{x}{2^3} - \frac{1}{2^2}\cot\frac{x}{2^2},$$

....

....

$$\frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}} = \frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}} - \frac{1}{2^{n-2}}\cot\frac{x}{2^{n-2}}.$$

Adding up we obtain S_n, the sum to n term, as

$$S_n=\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}-\cot x.$$

Now $\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}=\frac{x/2^{n-1}}{\tan\frac{x}{2^{n-1}}}\cdot\frac{1}{x}=\frac{1}{x}$, when n is in-

definitely increased,

$$\text{Since} \underset{\theta\to 0}{\text{Lt}}\frac{\theta}{\tan\theta}=1$$

Hence $$S=\underset{n\to\infty}{\text{Lt}}\ S_n=\frac{1}{x}-\cot x.$$

2. Fix the sum of the series

$\cot^{-1}(2.1^2)+\cot^{-1}(2.2^2)+\cot^{-1}(2.3^2)+\ldots$ad inf.

(Luck. '56)

u_n, the nth term $=\cot^{-1}(2\ .\ n^2)=\tan^{-1}\frac{2}{4n^2}$

$$=\tan^{-1}\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}$$

$=\tan^{-1}(2n+1)-\tan^{-1}(2n-1)$

Hence S_n, the sum to n terms,

$=(\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}5-\tan^{-1}3)+(\tan^{-1}7-\tan^{-1}5)+$
$\ldots+\{\tan^{-1}(2n+1)+\tan^{-1}(2n-1)\}$

$=\tan^{-1}(2n+1)-\tan^{-1}1$

Hence $S=\underset{n\to\infty}{\text{Lt}}\ Sn=\tan^{-1}\infty-\tan^{-1}1.$

$$=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$

Exercises

Sum the series

1. $\frac{1}{\sin\theta\sin 2\theta}+\frac{1}{\sin 2\theta\sin 3\theta}+\frac{1}{\sin 3\theta\sin 4\theta}$ $+\ldots$to n terms. (Agra '26)

2. $\frac{1}{\cos\theta+\cos 3\theta}+\frac{1}{\cos\theta+\cos 5\theta}+\frac{1}{\cos\theta+\cos 7\theta}$ $+\ldots$to n terms. (Agra '39)

3. $\sin\alpha\ \sin(\alpha+\beta)-\sin(\alpha-\beta)\ \sin(\alpha+2\beta)+\sin(\alpha+2\beta)$ $\sin(\alpha+3\beta)-\ldots$to $2n$ terms. (Agra '34)

4. $\text{cosec}\ x+\text{cosec}\ 2x+\text{cosec}\ 4x+\ldots+\text{cosec}\ 2^{n-1}x.$ (Alld. '51)

5. $\tan\theta+2\tan 2\theta+2^2\tan 2^2\theta+2^3\tan 2^3\theta+\ldots$to n terms. (Agra '40)

6. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}+\tan^{-1}\frac{1}{21}+\ldots$to n terms (Agra '36)

7. $\tan^{-1}\frac{x}{1+1\cdot 2x^2}+\tan^{-1}\frac{x}{1+2\cdot 3x^2}+$ $\tan^{-1}\frac{x}{1+3\cdot 4x^2}+\ldots$to n terms. (Luck. '54)

8. $\cot^{-1}(2a^{-1}+a)+\cot^{-1}(2a^{-1}+3a)+\cot^{-1}(2a^{-1}+6a)$ $+\cot^{-1}(2a^{-1}+10a)+\ldots$to n terms. (Luck. '56)

9. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\ldots+\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$ $+\ldots$ad inf. (Luck. '58)

10. $\sin^{-1}\dfrac{1}{\sqrt{2}}+\sin^{-1}\dfrac{\sqrt{2}-1}{\sqrt{6}}+\sin^{-1}\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}+\dots$

$+\sin^{-1}\dfrac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n(n+1)}}+\dots$ ad inf. (Jabal. '61)

11. $\tan\dfrac{x}{2}\sec x+\tan\dfrac{x}{2^2}\sec\dfrac{x}{2}+\tan\dfrac{x}{2^3}\sec\dfrac{x}{2^2}\dots$

ad inf. (Luck. '58)

12. $\tan^2\alpha\cdot\tan 2\alpha+\frac{1}{2}\tan^2 2\alpha\cdot\tan 4\alpha+\dfrac{1}{2^2}\tan^2 4\alpha\cdot\tan 8\alpha$

$+\dots$ to n terms. (Luck. '62)

13. $\tan\theta\sec 2\theta+\tan 2\theta\sec 2^2\theta+\dots+\tan 2^{n-1}\theta\sec 2^n\theta$

(Madras '38)

14. If S_n be the sum to n terms of the series

$\sin x+\sin 2x+\sin 3x+\dots,$

prove that $\lim\limits_{n\to\infty}\dfrac{1}{n}(S_1+S_2+\dots+S_n)=\dfrac{1}{2}\cot\dfrac{x}{2}.$

(Luck. '58)

15. Show that

$$\cot^{-1}\left(2^2+\frac{1}{2}\right)+\cot^{-1}\left(2^3+\frac{1}{2^2}\right)+\cot^{-1}\left(2^4+\frac{1}{2^3}\right)$$

$$+\dots+\cot^{-1}\left(2^{n+1}+\frac{1}{2^n}\right)+\dots\text{ad inf.}$$

$$=\cot^{-1}2.$$

9.7. Use of Calculus. When the sum of an infinite series is known, we can obtain the sum of other series by differentiating or integrating this series. We shall assume that such differentiation or integration of the series term by term is permissible.

Examples

1. Prove that

$$\log\cos\frac{\theta}{2}+\log\cos\frac{\theta}{2^2}+\log\cos\frac{\theta}{2^3}+\ldots\ldots=\log\frac{\sin\theta}{\theta}$$

and deduce that

$$\frac{1}{2}\tan\frac{\theta}{2}+\frac{1}{2^2}\tan\frac{\theta}{2^2}+\frac{1}{2^3}\tan\frac{\theta}{2^3}+\ldots\ldots=\frac{1}{\theta}-\cot\theta.$$

Now $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2^2.\cos\frac{\theta}{2}.\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}.$

Repeating this process we have

$$\sin\theta=2^n\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}.\sin\frac{\theta}{2^n}$$

$$=\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\cos\frac{\theta}{2^n}\,2^n\sin\frac{\theta}{2^n}.$$

But $2^n\sin\frac{\theta}{2^n}=\theta\left(\sin\frac{\theta}{2^n}\Big/\frac{\theta}{2^n}\right)=\theta$

when n is increased indefinitely, as $\underset{\theta\to 0}{\text{Lt}}\frac{\sin\theta}{\theta}=1,$

Hence $\cos\frac{\theta}{2}.\cos\frac{\theta}{2^2}.\cos\frac{\theta}{2^3}\ldots$ to infinite factors $=\frac{\sin\theta}{\theta}$

or $\log\cos\frac{\theta}{2}+\log\cos\frac{\theta}{2^2}+\log\cos\frac{\theta}{2^3}+\ldots\ldots=\log\frac{\sin\theta}{\theta}$

Differentiating this, we have

$$-\frac{1}{2}\tan\frac{\theta}{2}-\frac{1}{2^2}\tan\frac{\theta}{2^2}-\ldots\ldots=\cot\theta-\frac{1}{\theta},$$

whence the required result follows.

2. Sum the series $\sin\theta+\frac{1}{2}\ \frac{\sin 3\theta}{3}+\frac{1}{2}\cdot\frac{3}{4}\ \frac{\sin 5\theta}{5}+\ldots$ ......ad inf.

where $$-\frac{\pi}{2}<\theta<+\frac{\pi}{2}.$$

Also find the sum when θ is outside the above limit. (Bombay '30)

Let $C=\cos\theta+\frac{1}{2}\cos 3\theta+\frac{1}{2}\cdot\frac{3}{4}\cos 5\theta+\ldots\ldots$

and $S=\sin\theta+\frac{1}{2}\sin 3\theta+\frac{1}{2}\cdot\frac{3}{4}\sin 5\theta+\ldots\ldots,$

where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}.$

Hence $C+iS=e^{i\theta}+\frac{1}{2}e^{3i\theta}+\frac{1}{2}\cdot\frac{3}{4}e^{5i\theta}+\ldots\ldots$

$$=e^{i\theta}\left(1-e^{2i\theta}\right)^{-\frac{1}{2}}$$

$$=\frac{e^{i\theta}}{\sqrt{1-\cos 2\theta-i\sin 2\theta}}$$

$$=\frac{e^{i\theta}}{\sqrt{2\sin\theta}\sqrt{\sin\theta-i\cos\theta}}$$

$$=\frac{e^{i\theta}}{\sqrt{2\sin\theta}\left\{\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)\right\}^{\frac{1}{2}}}$$

$$=\frac{(\cos\theta+i\sin\theta)\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}}{\sqrt{2\sin\theta}}$$

$$=\frac{\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{\sqrt{2\sin\theta}}$$

Equating real parts.

$$C=\cos\theta+\tfrac{1}{2}\cos 3\theta+\tfrac{1}{2}\cdot\tfrac{3}{4}\cos 5\theta+\ldots=\frac{\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{\sqrt{2\sin\theta}}$$

$$=\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{2\sqrt{\sin\theta}} \qquad \ldots \quad (1)$$

Integrating this result we obtain

$$\sin\theta+\frac{1}{2}\,\frac{\sin 3\theta}{3}+\frac{1}{2}\cdot\frac{3}{4}\,\frac{\sin 5\theta}{5}+\ldots\ldots$$

$$=\int\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)d\theta}{2\sqrt{\sin\theta}}=\int\frac{1\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)d\theta}{\sqrt{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2-1}}$$

$$=\cosh^{-1}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)+A,$$ where A is a constant.

Putting $\theta=0$ on both sides, we see that $A=0$, and hence

the sum of the series $=\cosh^{-1}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)$ when $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Replacing θ by $\theta-\pi$, we have, when $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$,

from (1) above,

$$\cos\theta+\frac{1}{2}\cos 3\theta+\frac{1}{2}\cdot\frac{3}{4}\cos 5\theta+\ldots\ldots$$

$$=-\frac{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}{2\sqrt{\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2-1}}$$

Integrating either side,

$$\sin\theta+\frac{1}{2}\,\frac{\sin 3\theta}{3}+\frac{1}{2}\cdot\frac{3}{4}\,\frac{\sin 5\theta}{5}+\ldots\ldots$$

$$=-\cosh^{-1}\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right),$$

the constant of integration vanishing by putting $\theta=\pi$.

Exercises

1. Show that the series

$$\cos x+\frac{1}{2\cdot3}\cos 3x+\frac{1}{2}\cdot\frac{3}{4}\frac{\cos 5x}{5}+\ldots\ldots\text{ad inf.}$$

has the sum $\sin^{-1}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)$, when $\pi>x>0$,

and $-\sin^{-1}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)$, when $2\pi>x>\pi$.

(Bombay '47)

2. Prove that

$2\operatorname{cosec}\theta\cot\theta+4\operatorname{cosec}2\theta\cot 2\theta+8\operatorname{cosec}4\theta\cot 4\theta+\ldots$ ......ad inf.

$$=\operatorname{cosec}^2\frac{\theta}{2}-2^n\operatorname{cosec}^2(2^{n-1}\theta).$$

Exercises IX

1. Show that

$$\cos\frac{\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}+\ldots\cos\frac{11\pi}{13}=\tfrac{1}{2}.$$

Sum the series

2. $\cos\alpha\cdot\cos\alpha+\cos^2\alpha\cos 2\alpha+\cos^3\alpha\cos 3\alpha-\ldots$ad inf.

(Lkw. '62)

3. $\sin\alpha+2\sin 2\alpha+3\sin 3\alpha+\ldots+n\sin n\alpha$

4. $\sin\alpha\sin\beta+\frac{1}{2}\sin 2\alpha\sin 2\beta+\frac{1}{3}\sin 3\alpha\sin 3\beta$ $+\ldots$ad inf. (Madras '38)

5 $1-\frac{\cos 2\theta}{2!}+\frac{\cos 4\theta}{4!}-\frac{\cos 6\theta}{6!}+\ldots$ ad inf. (Dacca '40)

6. $\dfrac{\sin 2\theta}{2!}+\dfrac{\sin 4\theta}{4!}+\dfrac{\sin 6\theta}{6!}+\ldots$ad inf. (Lkw. '6

7. $\tan\alpha\tan(\alpha+\beta)+\tan(\alpha+\beta)\tan(\alpha+2\beta)+\tan(\alpha+2\beta)$ $\tan(\alpha+3\beta)+\ldots$to n terms. (Lkw. '43)

8. $\dfrac{1}{\sin\theta\cos 2\theta}-\dfrac{1}{\sin 2\theta\cos 3\theta}+\dfrac{1}{\sin 3\theta\cos 4\theta}$

$-\ldots$to n terms (Agra '41)

9. $\sum\limits_{1}^{n}\sqrt{1+\sin rx}$. (Cal. '42)

10. $\dfrac{\sin 2\theta}{\sin\theta\sin 3\theta}-\dfrac{\sin 4\theta}{\sin 3\theta\sin 5\theta}+\dfrac{\sin 6\theta}{\sin 5\theta\sin 7\theta}$

$-\ldots$to $2n$ terms (Agra '33)

11. $\sin\theta\cos\phi+\dfrac{1}{3}\sin 3\theta\cos 3\phi+\dfrac{1}{5}\sin 5\theta\cos 5\phi$

$+\ldots$ad inf.

where θ and ϕ are positive acute angles. (Travan. '51)

12. $2\tan\theta-\dfrac{4}{3}\tan^3\theta+\dfrac{6}{5}\tan^5\theta-\dfrac{8}{7}\tan^7\theta+\ldots$ad inf.,

$-\dfrac{\pi}{4}<\theta<\dfrac{\pi}{4}$. (Cal '42)

13. $\cos^2 x-\dfrac{1}{2}\sin^2 2x+\dfrac{1}{3}\cos^2 3x-\dfrac{1}{4}\sin^2 4x+\ldots$ad inf.

(Bombay '25)

14. $\dfrac{1}{2^2}\tan\dfrac{\pi}{2^2}+\dfrac{1}{2^3}\tan\dfrac{\pi}{2^3}+\dfrac{1}{2^4}\tan\dfrac{\pi}{2^4}+\ldots\ldots$ad inf.

(Bombay '26)

15. $\dfrac{\tan^3\theta}{1-3\tan^2\theta}+\dfrac{1}{3}\dfrac{\tan^3 3\theta}{1-\tan^2 3\theta}+\dfrac{1}{3^2}\dfrac{\tan^3 3^2\theta}{1-3\tan^2 3^2\theta}+\ldots\ldots$

$\ldots\ldots$to n terms (Punjab '45)

16. $3 \sin \alpha + 5 \sin 2\alpha + 7 \sin 3\alpha + \ldots$ to n terms. (Agra '27)

17. If $\theta - \alpha = \tan^2 \frac{w}{2} \sin 2\theta - \frac{1}{2} \tan^4 \frac{w}{2} \sin 4\theta +$

$$\frac{1}{3} \tan^6 \frac{w}{2} \sin 6\theta - \ldots\ldots \text{ad inf.}$$

Show that $\tan \alpha = \tan \theta \cos w$. (Alld. '52)

18. Prove that

$$\sin \theta + \frac{1}{3} \sin^3 \theta + \frac{1}{5} \sin^5 \theta + \ldots\ldots \text{ad inf.}$$

$$= 2\left(\sin \theta - \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin^5 \theta - \ldots \text{ad inf.}\right)$$

where $\theta \neq (2n+1)\frac{\pi}{2}$.

19. Circles are inscribed in triangles, whose bases are the sides of a regular polygon of n sides, and whose vertices lie in one of the angular points : show that the sum of the radii of the circles is $2r(1 - n \sin^2 \pi/2n)$, where r is the radius of the circle circumscribing the polygon.

CHAPTER X

Factorisation : Infinite Products

10. Let $f(x)$ represent the polynomial $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n$, of degree n. If the value of $f(x)$ becomes zero when $x = \alpha$, then α is a root of the equation $f(x) = 0$, and $x - \alpha$ is a factor of $f(x)$.

It is known that

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n = 0,$$

an equation of the nth degree, has n roots, and if these roots are $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$, we, therefore, have

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n$$

$$\equiv a_0 (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n).$$

This is true whether the coefficient, $a_0, a_1, \ldots\ldots a_n$ are real or complex. We are going to use this property of polynomials to factorise certain algebraic and trigonometric functions.

10·1. Factors of $x^n - 1$

Consider the equation $x^n - 1 = 0$

or $x^n = 1 = \cos 2r\pi + i \sin 2r\pi$

Hence the roots of this equation are given by

$$x = \cos \frac{2r\pi}{2n} + i \sin \frac{2r\pi}{n}$$

Case I. n is even, or $n = 2m$, m being a positive integer.

Here the roots are $\cos \dfrac{2r\pi}{2m} + i \sin \dfrac{2r\pi}{2m}$ or $\cos \dfrac{r\pi}{m} + i \sin \dfrac{r\pi}{m}$.

where $r = 0, \pm 1, \pm 2, \ldots\ldots \pm (m-1), m$.

Corresponding to $r=0$ and m, the roots are ± 1, and the factors are x^2-1

For $r=\pm 1$, the roots are $\cos\frac{\pi}{m} \pm i \sin\frac{\pi}{m}$,

and the factors are $\left(x-\cos\frac{\pi}{m}-i \sin\frac{\pi}{m}\right)$

$$\left(x-\cos\frac{\pi}{m}+i\sin\frac{\pi}{m}\right)$$

$$=x^2-2x\cos\frac{\pi}{m}+1$$

For the general roots $\cos\frac{r\pi}{m} \pm i \sin\frac{r\pi}{m}$, the factors are

$$\left(x-\cos\frac{r\pi}{m}-i\sin\frac{r\pi}{m}\right)\left(x-\cos\frac{r\pi}{m}+i\sin\frac{r\pi}{m}\right)$$
$$=x^2-2x\cos\frac{r\pi}{m}+1$$

Hence putting $r=0, 1, 2, \ldots\ldots m-l, m$, we have

$$x^{2m}-1=(x^2-1)\left(x^2-2x\cos\frac{\pi}{m}+1\right)\left(x^2-2x\cos\frac{2\pi}{m}+1\right)$$
$$\ldots\ldots\ldots\left(x^2-2x\cos\frac{m-1}{m}\pi+1\right)$$

Putting $2m=n$ or $m=n/2$, we get

$$x^n-1=(x^2-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)$$
$$\ldots\ldots\ldots\left(x^2-2x\cos\frac{n-2}{n}\pi+1\right).$$

$$=(x^2-1)\prod_{r=1}^{r=\frac{n-2}{2}}\left(x^2-2x\cos\frac{2r\pi}{n}+1\right),$$

n even ... 1)

The symbol π, read as *pie* indicates continued product of factors.

Case II. n is odd or $n=2p+1$, p being a positive integer.

$$\text{The roots are } \cos\frac{2r\pi}{2p+1} \pm i\sin\frac{2r\pi}{2p+1},$$

where $r=0, \pm1, \pm2, \ldots\ldots \pm p$.

Corresponding to $r=0$, the root is 1, and the factor is $(x-1)$

For the general root $\cos\frac{2r\pi}{2p+1} \pm i\sin\frac{2r\pi}{2p+1}$, the factors are

$$\left(x-\cos\frac{2r\pi}{2p+1}-i\sin\frac{2r\pi}{2p+1}\right)\left(x-\cos\frac{2r\pi}{2p+1}+i\sin\frac{2r\pi}{2p+1}\right)=x^2-2x\cos\frac{2r\pi}{2p+1}+1$$

Putting $r=0, \pm1, \pm2, \ldots \pm p$, we have

$$x^{2p+1}-1=\left(x-1\right)\left(x^2-2x\cos\frac{2\pi}{2p+1}+1\right)$$

$$\left(x^2-2x\cos\frac{4\pi}{2p+1}+1\right)\ldots\ldots\left(x^2-2x\cos\frac{2px}{2p+1}+1\right)$$

Since $n=2p+1$, hence

$$x^n-1=(x-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)$$

$$\left\{x^2-2x\cos\frac{2(n-1)}{2n}\pi+1\right\}$$

$$=(x-1)\prod_{r=1}^{r=\frac{n-1}{2}}\left(x^2-2x\cos\frac{2p\pi}{n}+1\right)$$

n being odd ...(2)

10.11. Factors of x^n+1

Consider the equation $x^n+1=0$ or $x^n=-1$

$$=\cos(2r+1)\pi+i\sin(2r+1)\pi$$

The roots are given by $x=\cos\frac{2r+1}{n}\pi+i\sin\frac{2r+1}{n}\pi$,

when $r=0, 1, 2, \ldots\ldots n-1.$

Thus the roots are

$$\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}, \cos\frac{3\pi}{n}+i\sin\frac{3\pi}{n}, \ldots\ldots$$

$$\cos\frac{2n-1}{n}\pi+i\sin\frac{2n-1}{n}\pi.$$

Case I. n is even. There are $\frac{n}{2}$ pairs of conjugate imaginary roots

represented by $x=\cos\frac{2r+1}{n}\pi\pm i\sin\frac{2r+1}{n}\pi$,

where $r=0, 1, 2, \ldots\ldots, \frac{n}{2}-1$

The factors for these roots are

$$\left(x-\cos\frac{2r+1}{n}\pi-i\sin\frac{2r+1}{n}\pi\right)$$

$$\left(x-\cos\frac{2r+1}{n}\pi+i\sin\frac{2r+1}{n}\pi\right)$$

$$=x^2-2x\cos\frac{2r+1}{n}\pi+1$$

Hence $x^n+1=\left(x^2-2x\cos\frac{\pi}{n}+1\right)$

$$\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\ldots\ldots\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right)$$

$$\underset{r=0}{\overset{n/2-1}{\pi}}\left(x^2-2\ x\ \cos\ \frac{2\ r+1}{n}\ \pi+1\right) \qquad ...(3)$$

Case II. n is odd

In this case there is one real root, -1, corresponding to $r=\frac{1}{2}(n-1)$ and $\frac{1}{2}(n-1)$ pairs of conjugate imaginary roots represented by

$$x=\cos\ \frac{2\ r+1}{n}\ \pi \pm i\ \sin\ \frac{2\ r+1}{n} \text{ where } r=0,\ 1,\ 2\ .\ \frac{n-3}{2}$$

Proceeding as before, we have

$$n^n+1=(x+1)\left(x^2-2\ x\cos\ \frac{\pi}{n}+1\right)$$

$$\left(x^2-2\ x\ \cos\ \frac{3\ \pi}{n}+1\right)\cdots\left(x^2-2\ x\cos\ \frac{n-2}{n}\ \pi+1\right)$$

$$=(x+1)\underset{r=0}{\overset{(n-3)/2}{\pi}}\left(x^2-2\ x\cos\ \frac{2\ r+1}{n}\ \pi+1\right) \qquad ..(4)$$

10·2 Factors of $x^{2n}-2x^n\cos n\ \theta+1$

The roots of the equation $x^{2n}-2\ x^n\cos n\ \theta+1=0$ are

$$x^n=\cos n\ \theta \pm \sqrt{\cos^2 n\ \theta-1}$$

$$=\cos n\ \theta \pm i\ \sin n\ \theta$$

$$=\cos\ (n\ \theta+2\ r\ \pi)\pm i\ \sin\ (n\ \theta+2\ r\ \pi)$$

Hence $x=\cos\left(\theta+\frac{2\ r\ \pi}{n}\pm\right)i\ \sin\left(\theta+\frac{2\ r\ \pi}{n}\right)$

$r=0,\ 1,\ 2,\ .\ n-1$

The factors for any r are

$$\left\{x-\cos\left(\theta+\frac{2\ r\ \pi}{n}\right)-i\ \sin\left(\theta+\frac{2\ r\ \pi}{n}\right)\right\}$$

$$\left\{x-\cos\left(\theta+\frac{2\ r\ \pi}{n}\right)+i\ \sin\left(\theta+\frac{2\ r\ \pi}{n}\right)\right\}$$

$$=x^2-2\ x\ \cos\left(\theta+\frac{2\ r\ \pi}{n}\right)+1$$

Putting $r=0, 1, 2, \ldots\ldots n-1$, we obtain

$$x^{2n}-2x^n \cos n\,\theta+1$$

$$=\left\{x^2-2\,x \cos \theta+1\right\}\left\{x^2-2\,x \cos\left(\theta+\frac{2\pi}{n}\right)+1\right\}$$

$$\left\{x^2-2\,x \cos\left(\theta+\frac{4\pi}{n}\right)+1\right\}\ldots\left\{x^2-2\,x \cos\left(\theta+\frac{2\pi n-2\pi}{n}\right)+1\right\}$$

$$=\prod_{r=0}^{n-1}\left\{x^2-2\,x \cos\left(\theta+\frac{2\,r\,n}{n}\right)+1\right\} \quad \ldots(5)$$

10·3. We may deduce important corollaries from the preceding results.

It is easy to see that (5) may be put as

$$x^{2n}-2a^n x^n \cos n\theta+r^{2n}=\prod_{r=0}^{n-1}\left\{x^2-2ax \cos\left(\theta+\frac{2\,r\,\pi}{n}\right)+a^2\right\} \quad \ldots(6)$$

Similar results may be associated with (1), (2), (3) and (4).

Dividing both sides by x^n, we have from (5),

$$x^n+\frac{1}{x^n}-2 \cos\, n\,\theta=\prod_{0}^{n-1}\left\{x+\frac{1}{x}-2 \cos\left(\theta+\frac{2\,r\,\pi}{n}\right)\right\} \quad \ldots(7)$$

10·31. Factorisation of $\cos n\,\phi-\cos n\,\theta$

In (7), put $x=e^{i\,\phi}$, so that $x^n+x^{-n}=2 \cos n\,\phi$, and

$$2 \cos n\,\phi-2 \cos\, n\,\theta=\prod_{0}^{n-1}\left\{2 \cos \phi-2 \cos\left(\theta+\frac{2\,r\,\pi}{n}\right)\right\}$$

Hence $\cos n\,\phi-\cos n\,\theta=2^{n-1}\prod_{0}^{n-1}\left\{\cos \phi-\cos\left(\theta+\frac{2\,r\,\pi}{n}\right)\right\}$ $\ldots(8)$.

10·32. In (8) put $\phi=0$, and $\theta=2\,\alpha$, hence

$$1-\cos 2\,\alpha\, n=2^{n-1}\prod_{0}^{n-1}\left\{1-\cos\left(2\alpha+\frac{2\,r\,\pi}{n}\right)\right\}$$

or $\quad 2\sin^2 n\alpha = 2^{2n-1}\prod_{0}^{n-1} \sin^2\left(\alpha+\frac{r\pi}{n}\right)$

Hence $\quad \sin n\alpha = 2^{n-1}\prod_{0}^{n-1} \sin\left(\alpha+\frac{r\pi}{n}\right)$ ...(9)

10·33. In (8), put $\phi=0$, and $\theta=2\alpha+\frac{\pi}{n}$, hence

$$1-\cos(2n\alpha+\pi)=2^{n-1}\prod_{0}^{n-1}\left\{1-\cos\left(2\alpha+\frac{2r+1}{n}\pi\right)\right\}$$

or
$$2\cos^2 n\alpha = 2^{n-1}\prod_{0}^{n-1}\left\{2\sin^2\left(\alpha+\frac{2r+1}{2n}\pi\right)\right\}$$

Hence $\quad \cos n\alpha = 2^{n-1}\prod_{0}^{n-1} \sin\left(\alpha+\frac{2r+1}{2n}\pi\right)$ ...(10)

Examples

1. Prove that

$$2^{(n-1)/2}\sin\frac{\pi}{n}\ \sin\frac{2\pi}{n}\ldots\ldots\sin\frac{n-1}{2n}\pi\ \sqrt{n}$$

(Cal. '43)

From (9), $\sin n\alpha = 2^{n-1}\sin\alpha\,\sin\left(\alpha+\frac{\pi}{n}\right)\sin\left(\alpha+\frac{2\pi}{r}\right)$

$$\ldots\ldots\ldots\sin\left(\alpha+\frac{n-1}{n}\pi\right)$$

Since $\displaystyle \underset{\alpha\to 0}{\text{Lt}}\frac{\sin n\alpha}{\sin\alpha}=\underset{\alpha\to 0}{\text{Lt}}\left(\frac{\sin n\alpha}{n\alpha}\cdot\frac{\alpha}{\sin\alpha}\right)n=n,$

We have $\quad 2^{n-1}\sin\frac{\pi}{n}\ \sin\frac{2\pi}{n}\ldots\ldots\sin\frac{n-1}{n}\pi=n$

But since $\sin\frac{\pi}{n}=\sin\frac{n-1}{n}\pi$, the last factor, and so on,

$$2^{n-1}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{2n}\ldots\ldots\sin^2\frac{n-1}{n}\pi=n$$

Taking the square root the result follows.

2. If n be even, prove that

$$2^{(n-1)/2}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n}\ldots\ldots\sin\frac{n-1}{2n}\pi=1.$$ (Patna '35)

Putting $\alpha=0$ in (10), we have

$$1=2^{n-1}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n}\ldots\ldots\sin\frac{2n-1}{2n}\pi$$

$$=2^{n-1}\sin^2\frac{\pi}{2n}\sin^2\frac{3\pi}{2n}\ldots\ldots\sin^2\frac{n-1}{2n}\pi$$

Taking the square root the result follows.

3. Show that

$$2^{n-1}\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\ldots\ldots\cos\frac{n-1}{n}\pi=0,\ 1,\text{ or }-1$$

according as n is even or of the form $4p+1$ or $4p-1$. (Cal. '40)

From (9) we have,

$$\sin n\alpha=2^{n-1}\prod_{0}^{n-1}\sin\left(\alpha+\frac{r\pi}{r}\right)$$

Putting $\alpha=\frac{\pi}{2}$.

$$\sin\frac{n\pi}{2}=2^{n-1}\prod_{0}^{n-1}\sin\left(\frac{\pi}{2}+\frac{r\pi}{n}\right)=2^{n-1}\prod_{0}^{n-1}\cos\frac{r\pi}{r}$$

$$=2^{n-1}\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\ldots\ldots\cos\frac{n-1}{n}\pi.$$

If n is even $\sin\frac{n\pi}{2}=0$

If n is odd and $=4p+1$,

$$\sin\frac{n\pi}{2}=\sin\frac{\pi}{2}(4p+1)=\sin\left(2\pi p+\frac{\pi}{2}\right)=1$$

If n is odd and $=4p-1$,

$$\sin\frac{n\pi}{2}=\sin\frac{\pi}{2}(4p-1)=\sin\left(2\pi p-\frac{\pi}{2}\right)=-1.$$

4. Prove that

$$\frac{(1+x)^n-(1-x)^n}{2x}=A\left(x^2+\tan^2\frac{\pi}{n}\right)\left(\tan^2\frac{2\pi}{n}\right)\cdots\cdots\left(x^2+\tan^2\frac{r\pi}{n}\right),$$

where $r=\frac{n-1}{2}$ or $\frac{1}{2}n-1$, and A is 1 or n, according as n is even or odd. (Bombay '47)

Consider the equation

$$(1+x)^n-(1-x)^n=0$$

or $\quad (1+x)^n=(1-x)^n \cos(2r\pi+i\sin 2r\pi)$

Hence $\quad 1+x=(1-x)\cos\left(\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}\right)$

or $\quad \frac{1+x}{1-x}=\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}=\frac{\cos\frac{r\pi}{n}+i\sin\frac{r\pi}{n}}{\cos\frac{r\pi}{n}-i\sin\frac{r\pi}{n}}$ ...(A)

Hence $\quad x=i\tan\frac{r\pi}{n}$,

(1) n odd. The equation has n roots, given by

$$r=0,1,2,\ldots\ldots,n-1$$

or

$$r=0,\pm1,\pm2,\pm\left(\frac{n-1}{2}\right),$$

Hence $(1+x)^n-(1-x)^n=Ax\left(x+i\tan\frac{\pi}{n}\right)\left(x-i\tan\frac{\pi}{n}\right)$

$$\left(x+i\tan\frac{2\pi}{n}\right)\left(x-i\tan\frac{2\pi}{n}\right)\ldots\ldots\left(x+i\tan\frac{n-1}{2n}\pi\right)\left(x-i\tan\frac{n-1}{2n}\pi\right)$$

$$=Ax\left(x^2+\tan^2\frac{\pi}{n}\right)\left(x^2+\tan^2\frac{2\pi}{n}\right)\ldots\ldots\left(x^2+\tan^2\frac{n-1}{2n}\pi\right),$$

where A is a constant

Equating coefficients of X^n from both sides,

$A=1-(-1)^n=2$, since n is odd

Hence, in this case,

$$\frac{(1+x)^n-(1-x)^n}{2x}=\left(x^2+\tan^2\frac{\pi}{n}\right)\left(x^2+\tan^2\frac{2\pi}{n}\right)\ldots\ldots\left(x^2+\tan^2\frac{n-1}{2n}\pi\right)$$

(2) n even. In this case the equation is of the $(n-1)^{th}$ degree, and the $n-1$ roots are obtained from (A) where

$r=0, 1, 2, \quad n-2$

or $r=0, \pm 1, \pm 2, \ldots\ldots \pm\left(\frac{n}{2}-1\right)$

Hence $(1+x)^n-(1-x)^n=A\,x\left(x+i\tan\frac{\pi}{n}\right)\left(x-i\tan\frac{\pi}{n}\right)$

$$\left(x+i\tan\frac{2\pi}{n}\right)\left(x-i\tan\frac{2\pi}{n}\right)\ldots\ldots\left\{x+i\tan\left(\frac{n}{2}-1\right)\frac{\pi}{n}\right\}\left\{x-i\tan\left(\frac{n}{2}-1\right)\frac{\pi}{n}\right\}$$

$$=A\,x\left(x^2+\tan^2\frac{\pi}{2}\right)\left(x^2+\tan^2\frac{2\pi}{n}\right)\ldots\ldots\left\{x^2+\tan^2\left(\frac{n}{2}-1\right)\frac{\pi}{n}\right\}$$

Equating the coefficients of x^{n-1} from both sides,

$$A=n-(-n)=2n$$

Hence $$\frac{(1+x)^n-(1-x)^n}{2x}=n\left(x^2+\tan^2\frac{\pi}{n}\right)\left(x^2+\tan^2\frac{2\pi}{n}\right)\ldots\ldots\left\{x^2+\tan^2\left(\frac{n}{2}-1\right)\frac{\pi}{n}\right\}.$$

Exercises

1. Resolve x^7+1 into real and quadratic factors.
Deduce that

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cot\frac{5\pi}{7}=-\frac{1}{8}.$$ (Travan. '44)

2. Prove that

$$\cosh n\phi-\cos n\theta=2^{n-1}\prod_{s=0}^{n-1}\left\{\cosh\phi-\cos\left(\theta+\frac{2s\pi}{n}\right)\right\}.$$ (Utkal '50)

3. Prove that

(a) $8\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}=1.$

(b) $32\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}=1.$

(c) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{3\pi}{9}\sin\frac{4\pi}{9}=\frac{3}{16}.$

4. Show that $\sin n\theta=2^{n-1}\sin\theta\prod_{1}^{n-1}\left(\cos\theta-\cos\frac{r\pi}{n}\right)$

5. Prove that $\sin 2n\theta=2^{2n-2}\sin 2\theta\prod_{1}^{n-1}\left(\cos^2\theta-\cos^2\frac{r\pi}{2n}\right).$

Hence show that $\prod_{1}^{n-1} \sin \frac{r\pi}{2n} = \sqrt{n}/2^{n-1}$. (Madras '38)

6. rove that $n \cot n\theta = \sum_{0}^{n-1} \cot\left(\theta + \frac{r\pi}{n}\right)$.

7. If n be even, show that

$$\tan\theta \tan\left(\theta+\frac{\pi}{n}\right)\tan\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots$$

$$\tan\left(\theta+\frac{n-1}{n}\pi\right) = (-1)^{n/2}.$$ (Cal. '38)

8. If n be odd prove that

$$\tan\frac{\pi}{n}\tan\frac{2\pi}{n}\ldots\ldots\tan\frac{n-1}{2n}\pi = \sqrt{n}.$$ (Bombay '31)

9. Show that

$$2^{n-1}\prod_{0}^{n-1}\sin\left(\phi+\frac{2r\pi}{n}\right) = \cos\frac{n\pi}{2}$$

$$-\cos n\left(\phi+\frac{\pi}{2}\right).$$ (Delhi '33)

10. Show that

$$(x+a)^{2n}-(x-a)^{2n} = 4\ n\ x\ a\prod_{1}^{n-1}\left(x^2+a^2\cot^2\frac{r\pi}{2n}\right)$$

(Annamalai '39)

11. Establish

$$\sin n\theta = 2^{n-1}\prod_{0}^{n-1}\sin(\theta+r\alpha),\ \text{where } n\alpha=\pi,$$

and deduce that

$\tan^{-1}(\cot x \tanh y)+\tan^{-1}\{\cot(x+\alpha)\tanh y\}$
$+\tan^{-1}\{\cot(x+2\alpha)\tanh y\}+\ldots$ to n terms
$=\tan^{-1}(\cot nx \tanh ny)$.

12. If $A_1\ A_2\ A_3 \ldots A_n$ be a regular polygon of n sides inscribed in a circle of radius a, and P be any point in the plane of the cricle at a distance x from 0, the centre of the circle, show that

$$PA_1^2.PA_2^2.PA_3^2\ldots PA_n^2 = x^{2n} - 2\,x^n\,a^n \cos n\,\theta + a^{2n},$$

where $\angle POA_1 = \theta$. (Delhi '51)

Discuss the cases when $\theta = 0$ or π/n, and $x = a$.

(De-Moivre's and Cotes' Properties of a circle).

13. Prove that the product of all straight lines that can be drawn from one of the angles of a regular polygon of n sides inscribed in a circle whose radius is a to all the other angular points is $n\,a^{n-1}$.

10·4. Factors of sin θ

The equation $\sin\theta = 0$ is satisfied by $\theta = \pm r\,\pi$ where r is any integer, including zero.

Hence, factors of $\sin\theta$ can be written as

$$\theta\left(1+\frac{\theta}{\pi}\right)\left(1-\frac{\theta}{\pi}\right)\left(1+\frac{\theta}{2\pi}\right)\left(1-\frac{\theta}{2\pi}\right)\ldots\ldots \left(1+\frac{\theta}{r\,\pi}\right)\left(1-\frac{\theta}{r\,\pi}\right)\ldots\ldots$$

Therefore,

$$\sin\theta = A\ \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\,\pi^2}\right)\ldots\ldots\left(1-\frac{\theta^2}{r^2\,\pi^2}\right)\ldots\ldots$$

$$= A\,\theta \prod_{1}^{\infty}\left(1-\frac{\theta^2}{r^2\,\pi^2}\right), \text{ where } A \text{ is constant}$$

Since the infinite product $\prod_{1}^{\infty}\left(1-\frac{\theta^2}{r^2\,\pi^2}\right)$ is convergent and tends to the limit 1 when θ tends to zero,

$$\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin\theta}{\theta} = A \prod_{0}^{\infty}\left(1-\frac{\theta^2}{r^2\,\pi^2}\right)$$

or $A = 1$.

Hence $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots\left(1-\frac{\theta^2}{r^2\pi^2}\right)\ldots$

$$=\prod_{1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right).$$

10·5. Factors of cos θ.

The equation $\cos\theta=0$ is satisfied by $\theta=\pm\frac{2r+1}{2}\pi$, where $r=0$ or any integer. Hence the factors of $\cos\theta$ are

$$\left(1-\frac{2\theta}{\pi}\right)\left(1+\frac{2\theta}{\pi}\right)\left(1-\frac{2\theta}{3\pi}\right)\left(1+\frac{2\theta}{3\pi}\right)\ldots\ldots$$

$$\left\{1-\frac{2}{(2r-1)\pi}\right\}\left\{1+\frac{2}{(2r-1)\pi}\right\}\ldots\ldots$$

or $\quad \cos\theta=A\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\ldots\left\{1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right\}\ldots$

The infinite product $\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\ldots$is convergent, and tends to 1 as θ tends to zero. Hence putting $\theta=0$, we have $\quad 1=A$, and we have

$$\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\ldots\left\{1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right\}\ldots$$

$$=\prod_{1}^{\infty}\left\{1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right\}.$$

10·6. Sum of powers of the reciprocals of natural numbers.

From the two infinite products for $\sin\theta$ and $\cos\theta$ we can make the following important deductions.

We have

$$\frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1+\frac{\theta^2}{3^2\pi^2}\right)\ldots\text{ad inf.}$$

Taking logarithm,

$$\log\left(1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}\cdots\right)=\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)$$

$$+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\cdots$$

$$=\sum_1^{\infty}\log\left(1-\frac{\theta^2}{r^2\pi^2}\right)$$

$$=-\sum_1^{\infty}\left(\frac{\theta^2}{r^2\pi^2}+\frac{1}{2}\frac{\theta^4}{r^4\pi^4}+\frac{1}{3}\frac{\theta^6}{r^6\pi^6}+\cdots\cdots\right)$$

or $$\frac{\theta^4}{\pi^2}\sum_1^{\infty}\frac{1}{r_2}+\frac{\theta^4}{2\pi^4}\sum_2^{\infty}\frac{1}{r^4}+\frac{\theta^6}{3\pi^6}\sum_1^{\infty}\frac{1}{r^6}+\cdots\cdots\cdots$$

$$=-\log\left\{1-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right)\right\}$$

$$=\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right)+\frac{1}{2}\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\cdots\right)^2+\cdots\cdots$$

$$=\frac{\theta^2}{6}+\frac{\theta^4}{180}+\cdots\cdots$$

Equating the coefficients of θ^2 and θ^4 from both sides, we have

$$\frac{1}{6}=\frac{1}{\pi^2}\sum_1^{\infty}\frac{1}{r^3}\text{ and }\frac{1}{180}=\frac{1}{2\pi^4}\sum_1^{\infty}\frac{1}{r^4}$$

or $$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\text{ad inf.}=\frac{\pi^2}{6},$$

and $$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\text{an inf.}=\frac{\pi^4}{90}.$$

Evidently we can continue this process.

10·61. Similarly from the relation

$$\prod_{1}^{\infty}\left\{1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right\}=\cos\ \theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots$$

we can easily deduce

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots=\frac{\pi^2}{8}$$

and
$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\ldots=\frac{\pi^4}{96}.$$

10·7 Factors of cosh θ and sinh θ

Putting $\theta=i\ x$ in the infinite products for sin θ and cos θ,

we have
$$\sin i\ x=i\ x\prod_{1}^{\infty}\left(1-\frac{i^2x^2}{r^2\pi^2}\right)$$

and
$$\cos i\ x=\prod_{1}^{\infty}\left\{1-\frac{4i^2x^2}{(2r-1)^2\pi^2}\right\}$$

or
$$\sinh\ x=x\prod_{1}^{\infty}\left(1+\frac{x^2}{r^2\pi^2}\right),$$

and
$$\cosh\ x=\prod_{1}^{\infty}\left\{1+\frac{4x^2}{(2r-1)^2\pi^2}\right\}.$$

10·8. Since $\sin\theta=\theta\prod_{1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right)$, taking loga-rithms, we have $\log\sin\theta=\log\theta+\sum_{1}^{\infty}\log\frac{r^2\pi^2-\theta^2}{r^2\pi^2}$

Differentiating, $\cot \theta = \frac{1}{\theta} - 2 \sum_{1}^{\infty} \frac{\theta}{r^2\pi^2 - \theta^2}$.

Similarly we can establish

$$\tan \theta = \sum_{1}^{\infty} \frac{8\theta}{(2r-1)^2\pi^2 - 4\theta^2},$$

$$\coth x = \frac{1}{x} + \sum_{1}^{\infty} \frac{2x}{r^2\pi^2 + x^2},$$

$$\tanh x = \sum_{1}^{\infty} \frac{8x}{(2r-1)^2\pi^2 + 4x^2}$$

10·9 Wallis' Formula.

We know $\sin \theta = \theta \prod_{1}^{\infty} \left(1 - \frac{\theta^2}{r^2\pi^2}\right)$

Putting $\theta = \frac{\pi}{2}$, we have

$$\frac{2}{\pi} = \prod_{1}^{\infty} \left(1 - \frac{1}{2^2 r^2}\right) = \prod_{1}^{\infty} \left(1 - \frac{1}{2r}\right)\left(1 + \frac{1}{2r}\right)$$

$$= \operatorname*{Lt}_{n\to\infty} \left[\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{4}\right) \cdots\cdots \left(1 - \frac{1}{2n}\right)\left(1 + \frac{1}{2n}\right)\right]$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\right]$$

If n is large,

$$\frac{2}{\pi}=\frac{3^2.\ 5^2.\ 7^2.\ \ldots\ (2n-1)^2}{2^2.\ 4^2.\ 6^2\ \ldots(2n)^2}\cdot(2n+1)$$

or $$\frac{2.\ 4.\ 6\ \ldots\ 2n}{1.\ 3.\ 5.\ \ldots 2n-1}=\sqrt{\pi\left(n+\frac{1}{2}\right)}\text{ or }\sqrt{\pi n}.$$

Examples

1. Show that

$$\sin x+\cos x=\left(1+\frac{4x}{\pi}\right)\left(1-\frac{4}{3}\frac{x}{\pi}\right)\left(1+\frac{4}{5}\frac{x}{\pi}\right)\left(1-\frac{4}{7}\frac{x}{\pi}\right)\cdots\cdots,$$

and hence deduce

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots\cdots=\frac{\pi^3}{32}$$ (Nagpur '40)

Now $$\cos x+\sin x=\sqrt{2}\cos\left(\frac{x-\pi}{4}\right)=\frac{\cos\left(\frac{x-\pi}{4}\right)}{\cos\frac{\pi}{4}}$$

$$=\prod_1^\infty\left\{1-\frac{4\left(\frac{x-\pi}{4}\right)^2}{(2r-1)^2\pi^2}\right\}\div\prod_1^\infty\left\{1-\frac{4\ .\ \pi^2/16}{(2r-1)^2\pi^2}\right\}$$

$$=\prod_1^\infty\frac{(2r-1)^2\pi^2-4\left(\frac{x-\pi}{4}\right)^2}{(2r-1)^2\pi^2-\frac{\pi^2}{4}}$$

$$=\prod_{1}^{\infty}\left\{1+\frac{2x}{(2r-1)^2\pi^2-\frac{\pi^2}{4}}-\frac{4x^2}{(2r-1)^2\pi^2-\frac{\pi^2}{4}}\right\}$$

$$=\prod_{1}^{\infty}\left\{1+\frac{2x}{(2r+1)\pi-\frac{\pi}{2}}\right\}\left(1-\frac{2x}{(2r-1)\pi+\frac{\pi}{2}}\right)$$

$$=\left(1+\frac{4x}{\pi}\right)\left(1-\frac{4x}{3\pi}\right)\left(1+\frac{4x}{5\pi}\right)\left(1-\frac{4x}{7\pi}\right)\cdots$$

Taking lorigathms of both sides, we have

$$\log\left(1+\frac{4x}{\pi}\right)+\log\left(1-\frac{4x}{3\pi}\right)+\log\left(1+\frac{4x}{5\pi}\right)+\log\left(1-\frac{4x}{7\pi}\right)+\ldots$$

$$=\log(\sin x+\cos x)=\tfrac{1}{2}\log(1+\sin 2x)$$

$$=\frac{1}{2}\log\left\{1+\left(2x-\frac{8x^3}{6}+\ldots\right)\right\}$$

$$=\frac{1}{2}\left[\left(2x-\frac{8x^3}{6}+\ldots\right)-\frac{1}{2}\left(2x-\frac{8x^3}{6}+\ldots\right)^2+\frac{1}{3}\left(2x-\frac{8x^3}{6}+\ldots\right)^3-\ldots\right]$$

Equating the coefficients of x^3 from both sides,

$$\frac{1}{3}\cdot\frac{4^3}{\pi^3}\left(\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots\right)=\frac{1}{2}\left(-\frac{8}{6}+\frac{8}{3}\right)=\frac{2}{3},$$

whence the result.

2. Deduce the expression for $\sin x$ in factors from that for $\cos x$. (Cal. '53)

Given $\cos x = \prod_{1}^{\infty} \left\{ 1 - \frac{4x^2}{(2r-1)^2\pi^2} \right\}$

Putting $\frac{\pi}{2} - x$ for x, we have

$$\sin x = \prod_{1}^{\infty} \left\{ 1 - \frac{4(\pi/2 - x)^2}{(2r-1)^2\pi^2} \right\} = \prod_{1}^{\infty} \frac{(2r-1)^2\pi^2 - (\pi - 2x)^2}{(2r-1)^2\pi^2},$$

$$= \prod_{1}^{\infty} \frac{4\,(r\pi - x)\,\{(r-1)\,\pi + x\}}{(2r-1)^2\pi^2}$$

$$= x \prod_{1}^{\infty} \frac{4\,(r\pi - x)\,(r\pi - x)}{(2r-1)^2\pi^2},$$

the factor $(r-1)\pi + x$ for $r = 1$ being taken out.

or $$\frac{\sin x}{x} = \prod_{1}^{\infty} \frac{4\,(r^2\pi^2 - x^2)}{(2r-1)^2\pi^2}, \qquad \ldots(1)$$

If x tends to zero, since $\underset{x \to 0}{\mathrm{Lt}} \frac{\sin x}{x} = 1$, we have

$$1 = \prod_{1}^{\infty} \frac{4\,r^2\pi^2}{(2r-1)^2\pi^2} \qquad \ldots(2)$$

From (1) and (2), by division,

$$\frac{\sin x}{x} = \prod_{1}^{\infty} \frac{r^2\pi^2 - x^2}{r^2\pi^2} = \prod_{1}^{\infty} \left(1 - \frac{x^2}{r^2\pi^2} \right)$$

Exercises X

Prove that

1. $\frac{1}{1^2.2^2} + \frac{1}{2^2.3^2} + \frac{1}{3^2.4^2} + \ldots\ldots = \frac{\pi^2}{3} - 3.$

2. $\sum_{1}^{\infty} \frac{1}{\{n(n+1)(n+2)\}^2} = \frac{\pi^2}{4} - \frac{39}{16}$.

3. $\frac{1}{2}\pi = \frac{2.2.4.4.6.6.8.8\ldots}{1.3.3.5.5.7.7.9\ldots}$ (Cal. '50)

4. $\frac{4-\pi}{8} = \sum_{1}^{\infty} \frac{1}{16r^2-1}$.

5. $\sqrt{2} = \frac{4.36.100.196.324\ldots}{3.35.99.195.323\ldots}$

6. Prove that the sum of the products taken two at a time of the squares of the reciprocals of

(i) all positive integers is $\pi^4/120$,

(ii) all positive odd integers is $\pi^4/384$. (Utkal '54)

7. Show that

$$\sin \pi\theta + \cos \pi\theta = (1+4\theta) \prod_{1}^{\infty} \left(1 - \frac{4\theta}{4r-1}\right) \left[1 + \frac{4\theta}{4r+1}\right].$$ (Annam. '39)

8. Prove that

$$\cos\left[\frac{\pi}{2}\sin\theta\right] = \frac{1}{4}\pi\cos^2\theta\left[1 + \frac{\cos^2\theta}{2\cdot4}\right]\left[1 + \frac{\cos^2\theta}{4\cdot6}\right]$$

9. Show that

$$\cos a + \tan x \sin a = \left[1 + \frac{2a}{\pi - 2x}\right]\left[1 - \frac{2a}{\pi + 2x}\right] \left[1 + \frac{2a}{3\pi - 2x}\right]\left[1 - \frac{2a}{3\pi + 2x}\right]\ldots\ldots$$

Hence, or otherwise, prove that

$$\tan x = 8x \sum_{1}^{\infty} \frac{1}{(2n-1)^2 \pi^2 - 4x^2}. \qquad \text{(Cal. '31)}$$

10. Find the value of infinite product

$$\left[1+\frac{1}{1^2}\right]\left[1+\frac{1}{2^2}\right]\left[1+\frac{1}{3^2}\right]\ldots\ldots \qquad \text{(Delhi '31)}$$

11. Prove that

$$\left\{1+\frac{2}{1+1^2}+\frac{2}{1+2^2}+\frac{2}{1+3^2}+\ldots\right\}$$

$$\left\{\frac{1}{4+1^2}+\frac{1}{4+3^2}+\frac{1}{4+5^2}+\ldots\right\}=\frac{\pi^2}{8}. \qquad \text{(Delhi '53)}$$

12. Prove that

$$\frac{\sin(\alpha+\theta)}{\sin\alpha}=\pi\left[1+\frac{\theta}{\alpha+r\pi}\right]$$

where r is any positive integer, including zero.

(Bombay '47)

13. Prove that

$$\frac{1}{4\pi}\sec x=\frac{1}{\pi^2-4x^2}-\frac{3}{3^2\pi^2-4x^2}+\frac{5}{5^2\pi^2-4x^2}-\ldots$$

(Cal. '36)

14. Prove that $\dfrac{1}{\sin^2 x}=\sum_{-\infty}^{\infty}\dfrac{1}{(x+n\pi)^2}$. (Annam. '43)

15. Show that the sum of the infinite series

$$\frac{1}{1^2+x^2}+\frac{1}{3^2+x^2}+\frac{1}{5^2+x^2}+\ldots=\frac{\pi}{4x}\tanh\frac{\pi x}{2}.$$

(Banaras '47)

16. If 2, 3, 5,...are all the prime numbers show that

$$\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{3^2}\right]\left[1-\frac{1}{5^2}\right]\ldots=\left[\frac{6}{\pi^2}\right].$$ (Mysore '45)

17. Prove that the sum of the squares of the reciprocals of all numbers which are not divisible by the square of any prime numbers is $15/\pi^2$.

18. Show that

$$\tan^{-1}\frac{2a^2}{\pi^2}+\tan^{-1}\frac{2a^2}{2^2\pi^2}+\tan^{-1}\frac{2a^2}{3^2\pi^2}+\ldots\ldots$$

$$=\frac{\pi}{4}-\tan^{-1}\left[\frac{e^a-e^{-a}}{e^a+e^{-a}}\cot a\right].$$

19. Show that

$$\tan^{-1}x-\tan^{-1}\frac{x}{3}+\tan^{-1}\frac{x}{5}-\tan^{-1}\frac{x}{7}+\ldots$$

$$=\tan^{-1}e^{\pi x/2}-\pi/4$$

20. Show that

$$\frac{1}{1^2+x^2}+\frac{1}{2^2+x^2}+\frac{1}{3^2+x^2}+\ldots=\frac{\pi}{2x}\coth\pi x-\frac{1}{2x^2}.$$

ANSWERS

Exercises I (pages 5—9)

4. $-1, -\frac{15}{7}$ **5.** $\frac{1}{3}, -3$. **6.** $-461/9$. **7.** $0, 3, -\frac{2}{3}$.

8. $0, \pm 1/2$. **9.** ± 1 or $\pm(1 \pm \sqrt{2})$. **10.** ± 1.

11. $(a-b)/(1+ab)$ **12.** a, a^2-a+1.

Exercises II, (page 19)

1. (i) i (ii) $\cos(\theta+\alpha)+i\sin(\theta+\alpha)$ (iii) $\dfrac{8\,ab\,(a^2-b^2)\,i}{(a^2+b^2)^2}$

2. (i) $10(\cos\alpha+i\sin\alpha)$, where $\cos\alpha=-3/5$, $\sin\alpha=4/5$

(ii) $2\cos\left(\dfrac{\theta+\phi}{2}\right)\left(\cos\dfrac{\theta+\phi}{2}+i\sin\dfrac{\theta+\phi}{2}\right)$

(iii) $2\cos\dfrac{\theta}{2}\left\{\cos\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)+i\sin\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)\right\}$.

§ 3·1 (pages 23—24)

1. $\cos 3\theta - \sin 3\theta$. **2.** i **3.** $\cos 3\theta + i\sin 3\theta$ **4.** $-i$

5. $r^7(\cos 7\alpha + i\sin 7\alpha)$, where $r=\sqrt{13}$, $\cos\alpha=2/\sqrt{13}$, $\sin\alpha=3/\sqrt{13}$.

Exercises III (pages 28—31)

1. (a) $-1, \frac{1}{2}(1\pm i\sqrt{3})$. (b) $\cos\dfrac{2n+1}{6}\pi+i\sin\dfrac{2n+1}{6}\pi$, $n=0, 1, 2, \ldots 5$.

(c) $2^{\frac{1}{3}}\left(\pm\dfrac{1}{2}\sqrt{3}+\dfrac{i}{2}\right)$, $2^{\frac{1}{3}}\,i$.

2. $\pm 1, \pm i, \frac{1}{2}(\sqrt{3}\pm i), \frac{1}{2}(1\pm i\sqrt{3}), \frac{1}{2}(-1\pm i\sqrt{3}), \frac{1}{2}(-\sqrt{3}\pm i)$.

The common roots are $\frac{1}{2}(1\pm i\sqrt{3}), \frac{1}{2}(-1\pm i\sqrt{3})$,

3. $\cos\dfrac{5\pi}{12}+i\sin\dfrac{5\pi}{12}$; $\pm\left(\cos\dfrac{r\pi}{48}+i\sin\dfrac{r\pi}{48}\right)$, where $r=5, 29$.

11. $\sin\left(\frac{1}{2}\sin 2\theta\right) \cdot e^{\cos^2\theta}$.

12. $\frac{1}{8}\{2\sqrt{3}\log(2+\sqrt{3})-\pi\}$.

13. $e^{\cos^2\theta}\cos(\theta+\sin\theta\cos\theta)$.

14. $\log\left(2\cosh\frac{\theta}{2}\right)$.

15. $\frac{\cos\theta}{1-\cos\theta}-\log(1-\cos\theta)$. 16. $\frac{1}{2}\left(e^{e^x}-e^{e^{-x}}\right)$.

17. $\cos(\cos\theta) . \sinh(\sin\theta)$. 18. $\frac{1}{6}\left(e^{e^x}+e^{e^{-x}}\right)$.

22. $\frac{1}{4}\log\frac{1+2c\sin\alpha+c^2}{1-2c\sin\alpha+c^2}$.

§ 9·6. (pages 113—114)

1. $\frac{1}{\sin\theta}\{\cot\theta-\cot(n+1)\theta\}$

2. $\frac{1}{2}\operatorname{cosec}\theta\{\tan(n+1)\theta-\tan\theta\}$.

3. $-\frac{1}{4\cos\beta}\sin(2\alpha+2n\beta)\sin 2n\beta$.

4. $\cot\frac{x}{2}-\cot 2^{n-1}x$. 5. $\cot\theta-2^n\cot 2^n\theta$.

6. $\frac{\pi}{4}-\tan^{-1}\frac{1}{n+1}$. 7. $\tan^{-1}(n+1)x-\tan^{-1}x$.

8. $\cot^{-1}\frac{a}{2}-\cot^{-1}\left(\frac{n+1}{2}a\right)$. 9. $\frac{\pi}{4}$. 10. $\frac{\pi}{2}$.

11. $\tan x$. 12. $\frac{1}{2^{n-1}}\tan 2^n\alpha-2\tan\alpha$.

13. $\tan(2^n\theta)-\tan\theta$.

Exercises IX (pages 118—120)

2. 0. 3. $\frac{\sin n\alpha}{4\sin^2\alpha/2}-\frac{n}{2\sin\alpha}\cos\frac{2n+1}{2}\alpha$.

4. $\frac{1}{2}\log\frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}$, except when $\alpha\pm\beta$ is a multiple of 2π

5. $\cos(\cos\theta)\cosh(\sin\theta)$. 6. $\sin(\sin\theta)\sinh(\cos\theta)$.

7. $\cot\beta\{\tan(\alpha+n\beta)-\tan\alpha\}-n$.

8. $\dfrac{\tan(n+1)\left(\theta+\frac{\pi}{2}\right)-\tan\left(\theta+\frac{\pi}{2}\right)}{\sin\left(\theta+\frac{\pi}{2}\right)}$

9. $\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{n+1}{4}x\right)\sin\frac{nx}{4}\operatorname{cosec}\frac{x}{4}$.

10. $\frac{1}{2\cos\theta}\left\{\frac{1}{\sin\theta}-\frac{1}{\sin(4x+1)\theta}\right\}$.

11. $\frac{\pi}{2}$ or 0, according as $\theta>$ or $<\phi$.

12. $\theta+\sin\theta\cos\theta$. 13. $\frac{1}{3}\log\operatorname{coses}x$. 14. $\frac{1}{\pi}$.

15. $\frac{1}{8}\left(\frac{1}{3^{n-1}}\tan 3^n\theta-3\tan\theta\right)$

16. $\frac{1}{2\sin\frac{\theta}{2}}\left\{\cos\frac{\theta}{2}-(2n+1)\cos\frac{2n+1}{2}\theta\right\}$

$+\frac{\sin n\theta}{1-\cos\theta}$.